Dimensional Regularization and Non-Renormalizable Quantum Field Theories

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Chapter 1

Introduction

1.1 Preface

This book is a quantum field theory treatise that aims to simplify the subject by including some mathematical techniques devised in the 50's and 60's which have not yet percolated on the physicists' family. With them what one might call a Schwartz' distributions approach to quantum field theory can be devised. Our purpose is twofold: i) to make accessible the concomitant design, on the one hand, and ii) to spread it by making the construct more well known, on the other. The conjunction of these mathematics with the dimensional regularization approach of Giambiagi and Bollini vastly expands the outreach of quantum field theory (QFT), as we will show here.

The main difficulty of conventional QFT concerns infinities. QFT is plagued by puzzling infinities, that emerge when dealing with basic entities called quantum propagators. One is not exaggerating by asserting that quantum propagators (QP) are the very quantities around which quantum field theory revolves. QP constitute its main predictors. The central idea to be advanced here is that CP are Schwartz' distributions (SD). This book is mainly concerned with this last statement. We will in it step-by-step develop an SD approach to quantum field theory, which is absent from all extant QFT textbooks.

1.2 Why is this book needed?

As stated in the Preface, the main difficulty of quantum field theory (QFT) concerns infinities. QFT is plagued by puzzling infinities. For instance, a familiar example is posed by the energy density of a static electric field (EE) [1]. It is proportional to the square of its EE intensity E. The EE intensity at a distance r due to a charge Q uniformly distributed over a spherical surface of radius R is, in turn, proportional to Q/r^2 for $r > R$ and 0 for $r < R$ [1]. Therefore, the total energy V of the field is given by [1],

$$V = \frac{Q^2}{6\pi c^2}(1/R), \tag{1.2.0.1}$$

with c light's speed. This quantity is called the self-energy of the particle. It obviously diverges as $R \to 0$. Thus, an EE emanating from a point charge displays infinite energy. Theorizing with point particles, as in classical analysis, may produce valid results, but when it involves manipulations of the particles' fields, these results may be contaminated by the infinite energy of the field of a point charge [1]. The QFT infinities appear in dealing with quantum propagators. Quantum propagators are in a sense the quantities around which quantum field theory (QFT) revolves. They are its main predictors and can be regarded as Green functions and, crucially, also as *Schwartz' distributions (SD)*. This book is mainly concerned with this last statement. We will in it develop an SD approach to quantum field theory, which is absent from all extant QFT textbooks.

The infinities of QFT appear in the guise of products (or, more precisely, of convolutions) of SDs, We need these products to develop the QFT theory.

There are other ways of discussing QFT, but this book concentrates only on the SD approach to QFT. Why? Because it is the only one that permits an easy handling of non-renormalizable problems, the main QFT blockage, as we shall see in the next chapters. Thus, SDs will be one of our main protagonists here. Immediately below, we denote the propagators as ρ^{-1}. Ours is the only book, as far as we know, that discusses in detail the SD approach to QFT..

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Dimensional regularization (DR) is another main QFT issue, and our second leitmo in this effort. DR is a method for dealing with infinities that works by taking the system's dimension ν as a continuous variable. Calculations use this continuous variable ν, and at the end of the computations, the limit is taken $\nu \to d$, with $d = 1, 2, 3, 4$, as the case may be. Thus, this book is the first to detailedly discuss the two central themes of SD and DR, together with the intimate connection between them. If you read it, you will get easy access to the deepest secrets of QFT, that are not accessible elsewhere in all their gory details.

1.3 Prerequisites

What prior knowledge is needed to embark into this book? It treats very technical issues in theoretical physics, so that a kind of warning is the honest way to proceed. To make plain sailing of what follows the reader necessitates acquaintance with a variety of themes. We might say that one requires the experience equivalent to have followed two semesters of quantum mechanics (including quantum electrodynamics), one of quantum field theory, and one of functional analysis. We emphasize here familiarity with Feynman diagrams A very useful source of mathematical information (or physical one) is provided by the two mathematics/physics sections of Wikipedia [1], to which we will refer when some knowledge is required that we do not have the space here to discuss at length.

1.4 Book's organization

Chapters 2 and 3 are of a preparatory character. The first introduces the absolutely central concept of Schwartz distribution (SD) while the second provides a foretaste of the kind of infinities this book tries to deal with. Chapter 4 studies in detail the mathematical concept of Ultradistributions, a special kind of SD that will become our main weapon to face infinities. Chapter 5 discusses a more involved and historically important approach to avoiding infinities: Bollini-Giamgiaggi's dimensional regularization. In Chapter 6 we introduce our two most important physical quantities: the Feynman and Wheeler propagators, that can be regarded as special SDs. All infinities in quantum field theory can be shown to emerge as we face the convolution of ultradistibution (UD)s, that we analyze in Chapter 7. We specialize this subject to even tempered UDs in Chapter 8, to Lorentz invariant UDs in chapter 9, and to UDs of exponential type in chapter 10. Final words regarding all these UDs are found in chapter 11. The formidable mathematical apparatus developed in the preceding chapters is applied to two important physical problems in chapters 12 an 13. An epilogue closes the book in Chapter 14.

Chapter 2

Distribution Theory

2.1 Introduction to Schwartz' distributions

SDs are special linear functionals. They are continuous and one defines them over a space of infinitely differentiable functions. The pertinent derivatives are themselves generalized functions as well [1]. The most commonly encountered generalized function is the delta function. These mathematical objects are the main mathematical tools of this book. Schwartz' distributions (generalized functions) (SD) are thus mathematical objects devised with the intent of generalizing the concept of function that we learned at an early age [1].

Distribution theory (DT) constitutes a powerful tool for physics endeavors. In particular, for physicists the paradigmatic example is the Dirac's delta that we learned at college.

DT regards distributions as *linear functionals* acting on a space of so-called test functions [1]. Now, which set of test functions is appealed to constitutes the essential question for this book. A good choice is important in order to tackle difficult problems.

As stated, SD act by integration over a test function. Thus, the particular choice we make for the space of test functions, when we are given several options, is of crucial importance, because each choice leads to a different space of distributions. Selecting as test functions smooth functions with compact support leads to the conventional, standard Schwartz distributions one finds in textbooks.

One can instead appeal to a very nice space, that of smooth and rapidly diminishing (faster than any polynomial growth) test functions (also called Schwartz' test functions). This choice yields the so-called tempered distributions, that will play a protagonist's role below. They possess a well defined (distributional) Fourier transform.

We have above defined distributions as a special class of linear functionals: those that map a set of test functions into the set of complex numbers C. In the simplest situation, the set of test functions to have in mind is the set of functions $\mathcal{D} = \{\phi\}$ displaying two properties i) ϕ is *infinitely differentiable* (smooth) and 2) ϕ has *compact support.* A Schwartz' distribution (SD) T is the linear mapping $T : \mathcal{D} \to C$. For the delta function one writes $<\delta, \phi> = \phi(0)$, which entails that δ computes a test function at the origin [1]. A SD can be multiplied by complex numbers and added together, yielding another SD. They may also be multiplied by smooth functions and we keep looking at a SD [1]. We usually denote by T_f the distribution

$$T_f = < T, \phi > = \int_R f(x)\phi(x)dx, \tag{2.1.0.1}$$

for $\phi \in D$ and $x \in R$, with R the set of reals. One would like to be able to select a definition for the *derivative* of a distribution which displays the property that $T_f' = T_{f'}$ [1]. A distribution is called regular if f is an ordinary function [1]. In this case, integrating by parts one has

$$< f', \phi > = \int_R f'(x)\phi(x)dx =$$

$$[f(x)\phi(x)]_{-\infty}^{\infty} - \int_R f(x)\phi'(x)dx = - < f, \phi' >, \tag{2.1.0.2}$$

which leads us to define, for a SD in general,

$$< T', \phi > = - < T, \phi' > . \tag{2.1.0.3}$$

2.2 Explicitly Lorentz invariant distributions

In special relativity, physics equations and important quantities should have the same form in all inertial frames. This invariance of form is called Lorentz invariance and is usually expressed in Minkowski's space [1]. In this Section you will find important definitions. We consider first the case of the ν-dimensional *Minkowskian space* $\mathbf{M}_\nu$ of special relativity. Let $\mathbf{S}^{'}$ be the space of the above mentioned Schwartz *test tempered distributions*, belonging to the space of smooth and rapidly diminishing (faster than any polynomial growth) test functions [5, 6] and consider an element $g \in \mathbf{S}^{'}$. Focus attention now upon a new set $\mathbf{S}_L^{'}$ defined below. In such a context we say that $g \in \mathbf{S}_L^{'}$ if and only if

$$g(\rho) = \frac{d^l}{d\rho^l} f(\rho), \tag{2.2.0.1}$$

where the derivative is to be regarded in the sense of distributions discussed above, l *is a natural number*, $\rho = k^2 = k_0^2 - k_1^2 - k_2^2 - \cdots - k_{\nu-1}^2$, and our as yet unknown $f \in \mathbf{M}_\nu$ satisfies

$$\int\limits_{-\infty}^{\infty} \frac{|f(\rho)|}{(1+\rho^2)^n} d\rho < \infty, \tag{2.2.0.2}$$

and is also continuous in $\mathbf{M}_\nu$. *The exponent* n *is a natural number.* We assert then that f belongs to a new set $\mathbf{T}_{1L}$. This new set is also called in a different way $(\mathbf{S}_{LA}^{'})$ via the equality $\mathbf{T}_{1L} = \mathbf{S}_{LA}^{'}$.

In the case of the *Euclidean space* $\mathbf{R}_\nu$, let $g \in \mathbf{S}^{'}$. We say that $g \in \mathbf{S}_R^{'}$ if and only if

$$g(k) = \frac{d^l}{dk^l} f(k), \tag{2.2.0.3}$$

where $k^2 = k_0^2 + k_1^2 + k_2^2 + \cdots + k_{\nu-1}^2$, with $f(k)$ satisfying

$$\int\limits_{0}^{\infty} \frac{|f(k)|}{(1+k^2)^n} dk < \infty, \tag{2.2.0.4}$$

and $f(k)$ is continuous in $\mathbf{R}_\nu$. We assert then that $f \in \mathbf{T}_{1R} = \mathbf{S}_{RA}^{'}$, a new subset.

We will grandly, but also aptly call $\mathbf{S}'_{\mathsf{LA}}$ and $\mathbf{S}'_{\mathsf{RA}}$ the Fourier Anti-transformed spaces of $\mathbf{S}'_{\mathsf{L}}$ and $\mathbf{S}'_{\mathsf{R}}$, respectively.

Chapter 3

Analytical regularization

Let us have a taste of the problems to be tackled in this book. We will now speak of propagators, quantities that specify at time t the probability amplitudes for traveling from one site to another [1]. Precisely, one of the main problems in quantum field theory (QFT) is the *convolution* of propagators [2]. The purpose of this Section is to describe how the convolution of *two* propagators, our leit motif in this book, is to be calculated using *analytical regularization* (AR) [2, 3]. This is a sophisticated procedure employed to convert some kind of mathematical problems into other simpler ones [1]. Historically, one wished to transform those boundary value problems that can be written as Fredholm integral equations (of the first kind) involving singular operators into equivalent Fredholm integral equations of the second kind [1]. Why? Because the latter may be easier to analytically treat and can be studied with discretization schemes (like the finite differences method) because they are point-wise convergent [1]. A modified AR constituted the prerequisite step, developed by Bollini and Giambiagi (BG), that led a posteriori to the discovery of dimensional regularization [3]. All that the reader needs in this respect is explained below. BG confronted the convolution of propagators ρ^{-1} corresponding to a scalar field without mass, the simplest scenario, working in a Euclidean space, that is able to deal with spin zero particles [3]. One must consider the quantity $K(x, t; x', t') =< x|\hat{U}(t, t')|x' >$, where $\hat{U}(t, t')$ is the unitary time-evolution operator for the system taking states at time t' to states at time t. Then, the propagators' convolution

$$\rho^{-1} * \rho^{-1} = \int \frac{d^4p}{\vec{p}^2(\vec{p}-\vec{k})^2} \tag{3.1.0.1}$$

where $\rho^{-1} = K^{-2}$ is the *field propagator*. In these circumstances we define the analytic extension of the convolution by introducing a complex arbitrary number α and writing the α-generalized expression [3]

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \int \frac{d^4p}{\vec{p}^{2\alpha}(\vec{p}-\vec{k})^{2\alpha}}. \tag{3.1.0.2}$$

Introduce now two auxiliary quantities A and B, called generalized *Feynman's parameters* [1]

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax+B(1-x)]^{\alpha+\beta}} dx, \tag{3.1.0.3}$$

and cast the above convolution in the fashion

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int d^4p \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[(\vec{p}-\vec{k})^2x+\vec{p}^2(1-x)]^{2\alpha}} dx =$$

$$\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int \frac{d^4p}{[(\vec{p}-\vec{k})^2x+\vec{p}^2(1-x)]^{2\alpha}}, \tag{3.1.0.4}$$

or, in simpler manner

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha =$$

$$\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int \frac{d^4p}{[(\vec{p}-\vec{k}x)^2+\vec{k}^2x(1-x)]^{2\alpha}}. \tag{3.1.0.5}$$

Making now the change of variables $\vec{s} = \vec{p}-\vec{k}x$ and calling $a = \vec{k}^2x(1-x)$ we obtain

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int \frac{d^4s}{,}(\vec{s}^2+a)^{2\alpha}, \tag{3.1.0.6}$$

or

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = 2\pi^2 \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int_0^\infty \frac{s^3}{(s^2+a)^{2\alpha}} ds. \tag{3.1.0.7}$$

Making another change of variables $y = s^2$ we have

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \int_0^\infty \frac{y}{(y+a)^{2\alpha}} dy. \tag{3.1.0.8}$$

Using now the essential reference for any theorist [4], we can indeed calculate the previous integral, obtaining

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{[k^2x(1-x)]^{2\alpha-2}} dx, \tag{3.1.0.9}$$

or

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} k^{4-4\alpha} \int_0^1 [x(1-x)]^{1-\alpha} dx. \tag{3.1.0.10}$$

Employing again results given in [4] we have now

$$\left(\rho^{-1} * \rho^{-1}\right)_\alpha = \pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} k^{4-4\alpha} \frac{[\Gamma(2-\alpha)]^2}{\Gamma(4-2\alpha)}. \tag{3.1.0.11}$$

At this point we tell you, dear reader, that our four-dimensional convolution is to be obtained as the *residue* in the *pole* when α tends to one (see [3]), a fact that we ask you to graciously accept. Thus,

$$\rho^{-1} * \rho^{-1} = \lim_{\alpha \to 1} \frac{\partial}{\partial \alpha} \left\{ (\alpha-1)\pi^2 \frac{\Gamma(2\alpha-2)}{[\Gamma(\alpha)]^2} k^{4-4\alpha} \frac{[\Gamma(2-\alpha)]^2}{\Gamma(4-2\alpha)} \right\}. \tag{3.1.0.12}$$

Using once again [4] we have

$$\Gamma(4-2\alpha) = 2^{3-2\alpha} \pi^{-\frac{1}{2}} \Gamma(2-\alpha) \Gamma\left(\frac{5}{2}-\alpha\right). \tag{3.1.0.13}$$

Evaluating the limit, we get for the four-dimensional *convolution*, without much pain, the desired final result

$$\rho^{-1} * \rho^{-1} = -\pi^2 [\ln \rho - 1]. \tag{3.1.0.14}$$

We have seen that, in order to face Eq. (3.1.0.1), we made an α-detour that allowed for a simple solution. This detour was grandly called an *analytical regularization*, which sounds wise and sophisticated enough, but is essentially simple.

Chapter 4

Ultradistributions

4.1 Distributions of exponential type

For the benefit of the reader, we present here a brief description of the main properties of the so called tempered ultradistributions and of ultradistributions of *exponential type* (UET). We need for this purpose to recapitulate some well known ideas regarding Hilbert spaces [1].

Remember first from elementary quantum mechanics that a countable *Hilbert* $\mathcal{H}$ space is one possessing a countable basis [1]. One defines at this junction the related notion of nuclear spaces. These are spaces that retain some convenient features of finite-dimensional vector spaces, particularly with respect to their topology [1]. In this regard, we think of those special semi-norms whose unit balls' sizes diminish in rapid fashion. All finite-dimensional vector spaces are nuclear, of course.

Let us pass now to the important notion of rigged Hilbert space (RHS) [1]. A RHS consists of 1) a Hilbert space H plus 2) a subspace $\mathbf{\Phi}$ which carries a finer topology that of H [1]. Thus, $\mathbf{\Phi} \subset \mathsf{H}$. Then, we define the RHS in terms of the inequalities $\mathbf{\Phi} \subset \mathsf{H} \subset \mathbf{\Phi}'$, with $\mathbf{\Phi}'$ the dual space to $\mathbf{\Phi}$ [1], a triplet of symbols [1].

Notations. Our notation is almost textually taken from Ref. [9]. Let $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) be the real (respectively complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, ..., x_n)$ (resp. $z = (z_1, z_2, ..., z_n)$). We shall use the following notations

(i) $x+y = (x_1+y_1, x_2+y_2, ..., x_n+y_n)$; $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$
(ii)$x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, ..., x_n \geqq 0$
(iii)$x \cdot y = \sum_{j=1}^{n} x_j y_j$
(iV)$| x |= \sum_{j=1}^{n} | x_j |$

Consider the set of n-tuples of natural numbers $\mathbb{N}^n$. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, ..., p_n)$, where p_j is a natural number, $1 \leqq j \leqq n$. $p+q$ denote $(p_1+q_1, p_2+q_2, ..., p_n+q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, ..., p_n \geqq q_n$. x^p means $x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$. We denote by $| p |= \sum_{j=1}^{n} p_j$ and by D^p we understand the differential operator $\partial^{p_1+p_2+...+p_n}/\partial {x_1}^{p_1} \partial {x_2}^{p_2} ... \partial {x_n}^{p_n}$.
For any natural number k we define $x^k = x_1^k x_2^k ... x_n^k$ and $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \partial x_2^k ... \partial x_n^k$.

4.1.1 An important set of test functions

This is the space $\boldsymbol{\mathcal{H}}$ of test functions such that $e^{p|x|}|D^q \phi(x)|$ is bounded for any natural numbers p and q. This test-space is of great importance for our present purposes. It is defined (Ref. [7]) by means of the countable set of norms

$$\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p,\, x} e^{p|x|} \left|D^q \hat{\phi}(x)\right| \quad , \quad p = 0, 1, 2, ... \tag{4.1.1.1}$$

According to reference [8], $\boldsymbol{\mathcal{H}}$ is a *countable* and nuclear Hilbert space $\mathcal{K}\{\boldsymbol{M_p}\}$ with

$$M_p(x) = e^{(p-1)|x|} \quad , \quad p = 1, 2, ... \tag{4.1.1.2}$$

$\mathcal{K}\{\boldsymbol{e^{(p-1)|x|}}\}$ complies with a special mathematical demand called $(\mathcal{N})$ by *Guelfand* (Ref. [8]), whose details we do not really need to enter into. Let us insist on the fact that $\mathcal{K}\{\boldsymbol{e^{(p-1)|x|}}\}$ is a countable Hilbert and nuclear space

$$\mathcal{K}\{\boldsymbol{e^{(p-1)|x|}}\} = \boldsymbol{\mathcal{H}} = \bigcap_{p=1}^{\infty} \boldsymbol{\mathcal{H}_p} \tag{4.1.1.3}$$

where $\mathcal{H}_p$ is obtained by completing $\mathcal{H}$ with the norm induced by the scalar product

$$< \hat{\phi}, \hat{\psi} >_p = \int\limits_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q \overline{\hat{\phi}}(x) D^q \hat{\psi}(x)\ dx \quad ; \quad p = 1, 2, ... \tag{4.1.1.4}$$

where $dx = dx_1\ dx_2 ... dx_n$.

If we take the conventional *scalar product*

$$< \hat{\phi}, \hat{\psi} > = \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x) \hat{\psi}(x)\ dx, \tag{4.1.1.5}$$

then $\mathcal{H}_o$, completed with (4.1.1.5), is the familiar old Hilbert space H of *square integrable* functions.

4.1.2 The associated distributions

Consider the space generated by exponentials $e^{(p)|x|}$, with p real. Distributions of *exponential type* (Ref. [9]) are those belonging to the space of continuous linear functionals defined on $\mathcal{H} = \mathcal{K}\{e^{(p-1)|x|}\}$. The new space is itself a Hilbert space, the dual of $\mathcal{H}$, with the same dimension. We are speaking of the space Λ_∞. To repeat, this space can be identified with the set of continuous linear functionals. We will badly need such space, that we call the one of distributions of *exponential type* (Ref. [9]).

Let H_y stand for the Heaviside function. The Fourier transform of a distribution of exponential type $\hat{F}$ is called a tempered ultradistribution, being given by (see [9, 10])

$$F(k) = \int\limits_{-\infty}^{\infty} H_y[\Im(k)] H_y[\Re(x)] - H_y[-\Im(k)] H_y[-\Re(x)] \hat{F}(x) e^{ikx}\ dx =$$

$$H_y[\Im(k)] \int\limits_{0}^{\infty} \hat{F}(x) e^{ikx} - H_y[-\Im(k)] \int\limits_{-\infty}^{0} \hat{F}(x) e^{ikx} \tag{4.1.2.6}$$

where F is, as stated above, the corresponding *tempered ultradistribution* (see also the next section).

The triplet

$$\mathfrak{H} = (\mathcal{H}, H, \Lambda_\infty) \quad (4.1.2.7)$$

is a rigged Hilbert Space (or a Guelfand's triplet [8]). Moreover, we have $\mathcal{H} \subset \mathcal{S} \subset H \subset \mathcal{S}^{'} \subset \Lambda_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (Refs. [5, 6]).

Any rigged Hilbert space $\mathfrak{G} = (\boldsymbol{\Phi}, H, \boldsymbol{\Phi}^{'})$ displays the fundamental property that a linear and symmetric operator on $\boldsymbol{\Phi}$, which admits an extension to a self-adjoint operator in H, has a complete set of generalized eigenfunctions in $\boldsymbol{\Phi}^{'}$, with real eigenvalues [1].

4.2 Tempered ultradistributions

They are the Fourier transforms of distributions of exponential type. The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\; e^{iz\cdot x}\; dx \quad (4.2.0.1)$$

Here $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We call $\mathfrak{H}$ the set of all such functions.

$$\mathfrak{H} = \mathcal{F}\{\mathcal{H}\} \quad (4.2.0.2)$$

It is called a $\mathcal{Z}\{\mathbf{M_p}\}$ countably normed and complete space (Ref. [7]), with

$$M_p(z) = (1 + |z|)^p \quad (4.2.0.3)$$

$\mathfrak{H}$ is a *nuclear space* defined with the norms

$$\|\phi\|_{pn} = \sup_{z \in V_n} \; (1 + |z|)^p |\phi(z)| \quad (4.2.0.4)$$

where $V_k = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n \mid Im\, z_j \;| \leqq k, 1 \leqq j \leqq n\}$,

We can define the usual *scalar product*

$$< \phi(z), \psi(z) >= \int\limits_{-\infty}^{\infty} \phi(z)\psi_1(z)\; dz = \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\; dx \quad (4.2.0.5)$$

where

$$\psi_1(z) = \int_{-\infty}^{\infty} \hat{\psi}(x)\, e^{-iz\cdot x}\, dx$$

and $dz = dz_1\, dz_2 ... dz_n$. By completing $\mathfrak{H}$ with the norm induced by (4.2.0.5) we once again obtain the Hilbert space of square integrable functions. The dual of $\mathfrak{H}$ is the space $\boldsymbol{\mathcal{U}}$ of tempered ultradistributions (Refs. [9, 10]). Namely, a *tempered ultradistribution* is a continuous linear functional defined on the space $\mathfrak{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis. The set $\mathfrak{U} = (\mathfrak{H}, \mathsf{H}, \boldsymbol{\mathcal{U}})$ is also a rigged Hilbert space.

Moreover, we have $\mathfrak{H} \subset \boldsymbol{\mathcal{S}} \subset \mathsf{H} \subset \boldsymbol{\mathcal{S}}' \subset \boldsymbol{\mathcal{U}}$. Now, $\boldsymbol{\mathcal{U}}$ can also be characterized in the following way (Ref. [9]): let $\boldsymbol{\mathcal{A}}_\omega$ be the space of all special functions $F(z)$, that will become very important to us, such that

A) $F(z)$ is analytic on the set $\{z \in \mathbb{C}^n | \text{Im}(z_1)| > p, |\text{Im}(z_2)| > p, ..., |\text{Im}(z_n)| > p\}$.

B) $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n | \text{Im}(z_1)| \geqq p, |\text{Im}(z_2)| \geqq p, ..., |\text{Im}(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $F(z)$.

Let $\boldsymbol{\Pi}$ be the set of all z-dependent pseudo-polynomials (defined below) $z \in \mathbb{C}^n$. Then $\boldsymbol{\mathcal{U}}$ is the *quotient space*

C) $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{A}}_\omega / \boldsymbol{\Pi}$. Let us clarify that, by a *pseudo-polynomial*, we refer to a function of z of the form
$\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$ with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \boldsymbol{\mathcal{A}}_\omega$.

Due to these properties it is possible to represent any ultradistribution as (Ref. [9])

$$F(\phi) =< F(z), \phi(z) >= \oint_\Gamma F(z)\phi(z)\, dz, \tag{4.2.0.6}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup ... \Gamma_n$. The path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $\text{Im}(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $\text{Im}(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$). Recall now that a branch cut [1] is a special curve (with ends possibly open, closed, or half-open) in the complex plane [1]. Across it, an analytic multivalued function is discontinuous. For convenience, branch cuts are often taken as lines or line segments [1]. Branch cuts (even those consisting of curves) are also known as cut lines [1].

Formula (4.2.0.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of " *Dirac's formula*" for ultradistributions (Ref. [10])

$$F(z) = \frac{1}{(2\pi i)^n} \int\limits_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)}\, dt \tag{4.2.0.7}$$

where the "density" $f(t)$ is the cut of $F(z)$ along the real axis and satisfies

$$\oint\limits_{\Gamma} F(z)\phi(z)\, dz = \int\limits_{-\infty}^{\infty} f(t)\phi(t)\, dt. \tag{4.2.0.8}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (4.2.0.7) should be interpreted in the sense of Schwartz' distribution's theory. Another important property of this *analytic representation* is the fact that on Γ, $F(z)$ is bounded by a power of z (Ref. [9])

$$|F(z)| \leq C|z|^p, \tag{4.2.0.9}$$

where C and p depend on F. The representation (4.2.0.6) implies that the sum of a *pseudo-polynomial* $P(z)$ to $F(z)$ does not alter the ultradistribution

$$\oint\limits_{\Gamma}\{F(z) + P(z)\}\phi(z)\, dz = \oint\limits_{\Gamma} F(z)\phi(z)\, dz + \oint\limits_{\Gamma} P(z)\phi(z)\, dz$$

. However,

$$\oint\limits_{\Gamma} P(z)\phi(z)\, dz = 0$$

as $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing),

$$\therefore \quad \oint\limits_{\Gamma}\{F(z) + P(z)\}\phi(z)\, dz = \oint\limits_{\Gamma} F(z)\phi(z)\, dz. \tag{4.2.0.10}$$

The *inverse Fourier transform* of (4.1.2.6) is given by

$$\hat{F}(x) = \frac{1}{2\pi}\oint\limits_{\Gamma} F(k)e^{-ikx}\, dk = \int\limits_{-\infty}^{\infty} f(k)e^{-ikx}\, dx. \tag{4.2.0.11}$$

4.3 Ultradistributions of exponential type

Any vector space has associated to it what is called a dual space. It consists of linear functionals of the elements of the original space [1].

Consider the Schwartz space of rapidly decreasing test functions $\mathcal{S}$. Let Λ_j be the region of the complex plane defined as

$$\Lambda_j = \{z \in \mathbb{C} | \Im(z)| < j;\ j \in \mathbb{N}\} \tag{4.3.0.1}$$

According to Ref. [10, 11] the space of test functions $\hat{\phi} \in \mathfrak{V}_j$ is constituted by the set of all entire analytic functions of $\mathcal{S}$ for which

$$||\hat{\phi}||_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{(j|\Re(z)|)} |\hat{\phi}^{(k)}(z)| \right] \right\} \tag{4.3.0.2}$$

is finite.

The new space $\mathfrak{Z}$ is then defined as

$$\mathfrak{Z} = \bigcap_{j=0}^{\infty} \mathfrak{V}_j. \tag{4.3.0.3}$$

It is a complete countably normed space with the topology generated by the set of *semi-norms* $\{||\cdot||_j\}_{j\in\mathbb{N}}$. The *topological dual* of $\mathfrak{Z}$, denoted by $\mathfrak{B}$, is by definition the space of Ultradistributions of *exponential type* (Ref.[9, 10, 11]). Let $\mathfrak{S}$ be the space of rapidly decreasing sequences. According to Ref.[8] $\mathfrak{S}$ is a nuclear space. We consider now the space of sequences $\mathfrak{P}$ generated by the Taylor expansion of $\hat{\phi} \in \mathfrak{Z}$

$$\mathfrak{P} = \left\{ \mathfrak{Q} \left(\hat{\phi}(0), \hat{\phi}'(0), \frac{\hat{\phi}''(0)}{2}, ..., \frac{\hat{\phi}^{(n)}(0)}{n!}, ... \right); \hat{\phi} \in \mathfrak{Z} \right\}. \tag{4.3.0.4}$$

The norms that define the topology of $\mathfrak{P}$ are given by

$$||\hat{\phi}||'_p = \sup_n \frac{n^p}{n!} |\hat{\phi}^n(0)|. \tag{4.3.0.5}$$

$\mathfrak{P}$ is a subspace of $\mathfrak{S}$ and as consequence is a nuclear space. The norms $||\cdot||_j$ and $||\cdot||'_p$ are equivalent, the correspondence

$$\mathfrak{Z} \Longleftrightarrow \mathfrak{P} \tag{4.3.0.6}$$

being an isomorphism and, therefore, $\mathfrak{Z}$ is a countably normed nuclear space. We define now the set of scalar products

$$< \hat{\phi}(z), \hat{\psi}(z) >_n = \sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|z|} \overline{\hat{\phi}^{(q)}}(z) \hat{\psi}^{(q)}(z)\ dz =$$

$$\sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|x|} \overline{\hat{\phi}^{(q)}}(x) \hat{\psi}^{(q)}(x)\ dx. \tag{4.3.0.7}$$

They induce the norm

$$\|\hat{\phi}\|_n^{''} = [< \hat{\phi}(x), \hat{\phi}(x) >_n]^{\frac{1}{2}}. \tag{4.3.0.8}$$

The norms $\|\cdot\|_j$ and $\|\cdot\|_n^{''}$ are equivalent, and therefore $\mathfrak{Z}$ is a *countably hilbert* nuclear space. Thus, if we call now $\mathfrak{Z}_p$ the completion of $\mathfrak{Z}$ by the norm p given in (4.3.0.5), we have

$$\mathfrak{Z} = \bigcap_{p=0}^{\infty} \mathfrak{Z}_p, \tag{4.3.0.9}$$

where

$$\mathfrak{Z}_0 = H, \tag{4.3.0.10}$$

is the Hilbert space of square integrable functions.

As a consequence the triplet

$$\mathfrak{U} = (\mathfrak{Z}, H, \mathfrak{B}) \tag{4.3.0.11}$$

is also a Guelfand's triplet.

$\mathfrak{B}$ can also be characterized in the following way (Refs. [9],[10]): let $\mathfrak{C}_\omega$ be the space of all functions $\hat{F}(z)$ such that **A**) $\hat{F}(z)$ is an analytic function for $\{z \in \mathbb{C} | Im(z)| > p\}$. **B**)- $\hat{F}(z)e^{-p|\Re(z)|}/z^p$ is a bounded continuous function in $\{z \in \mathbb{C} | Im(z)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $\hat{F}(z)$.
Let further $\mathfrak{N}$ be $\mathfrak{N} = \{\hat{F}(z) \in \mathfrak{C}_\omega \hat{F}(z)$ is entire analytic$\}$. Then

$\mathfrak{B}$ is the *quotient space* **C**) $\mathfrak{B} = \mathfrak{C}_\omega / \mathfrak{N}$

Due to these properties it is possible to represent any ultradistribution of *exponential type* as [9, 10]

$$\hat{F}(\hat{\phi}) = <\hat{F}(z), \hat{\phi}(z)> = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz, \tag{4.3.0.12}$$

where the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $Im(z) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $\hat{F}(z)$).

Eq. (4.3.0.12) will be our fundamental representation for a ultradistribution of exponential type. The " *Dirac's formula*" for ultradistributions of exponential type is (Refs. [9, 10])

$$\hat{F}(z) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{t-z}\ dt \equiv \frac{\cosh(\lambda z)}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t-z)\cosh(\lambda t)}\ dt, \tag{4.3.0.13}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz = \int_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t)\ dt. \tag{4.3.0.14}$$

Eq. (4.3.0.13) should be used carefully. While $\hat{F}(z)$ is an analytic function on Γ, the density $\hat{f}(t)$ is in general singular, so that the right hand side of (4.3.0.14) should be interpreted again in the sense of distribution's theory.

Another important property of the analytic representation is the fact that, on Γ, $\hat{F}(z)$ is bounded by a exponential and a power of z (Ref. [9, 10])

$$|\hat{F}(z)| \leq C|z|^p e^{p|\Re(z)|}, \tag{4.3.0.15}$$

where C and p depend on $\hat{F}$.

The representation (4.3.0.12) implies that the addition of any entire function $\hat{G}(z) \in \mathfrak{H}$ to $\hat{F}(z)$ does not alter the ultradistribution

$$\oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\ dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz + \oint_{\Gamma} \hat{G}(z)\hat{\phi}(z)\ dz$$

. However,

$$\oint_{\Gamma} \hat{G}(z)\hat{\phi}(z)\ dz = 0$$

as $\hat{G}(z)\hat{\phi}(z)$ is an entire analytic function,

$$\therefore \quad \oint_{\Gamma}\{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\ dz = \oint_{\Gamma}\hat{F}(z)\hat{\phi}(z)\ dz. \tag{4.3.0.16}$$

Another very important property of $\mathfrak{B}$ is that $\mathfrak{B}$ is *reflexive* under the Fourier transform

$$\mathfrak{B} = \mathcal{F}_c\left\{\mathfrak{B}\right\} = \mathcal{F}\left\{\mathfrak{B}\right\}, \tag{4.3.0.17}$$

where the *complex Fourier transform* $F(k)$ of $\hat{F}(z) \in \mathfrak{B}$ is given by

$$F(k) = H_y[\Im(k)]\int_{\Gamma_+}\hat{F}(z)e^{ikz}\ dz - H_y[-\Im(k)]\int_{\Gamma_-}\hat{F}(z)e^{ikz}\ dz =$$

$$\oint_{\Gamma}\{H_y[\Im(k)H_y[\Re(z)] - H_y[-\Im(k)H_y[-\Re(z)]\}\hat{F}(z)e^{ikz}\ dz =$$

$$H[\Im(k)]\int_0^{\infty}\hat{f}(x)e^{ikx}\ dx - H[-\Im(k)]\int_{-\infty}^{0}\hat{f}(x)e^{ikx}\ dx \tag{4.3.0.18}$$

Here, Γ_+ is the part of Γ with $\Re(z) \geq 0$ and Γ_- is the part of Γ with $\Re(z) \leq 0$ Using (4.3.0.18) we can interpret Dirac's formula as

$$F(k) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)}{s-k}\ ds \equiv \mathcal{F}_c\left\{\mathcal{F}^{-1}\left\{f(s)\right\}\right\}. \tag{4.3.0.19}$$

The *inverse Fourier transform* corresponding to (4.3.0.18) is given by

$$\hat{F}(z) = \frac{1}{2\pi}\oint_{\Gamma}\{H_y[\Im(z)]H_y[-\Re(k)] - H_y[-\Im(z)]H_y[\Re(k)]\}F(k)e^{-ikz}\ dk. \tag{4.3.0.20}$$

The treatment of ultradistributions of exponential type defined on $\mathbb{C}^n$ is similar to that for the case of just one variable. Thus let Λ_j be given as

$$\Lambda_j = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n | \Im(z_k)| \leq j \quad 1 \leq k \leq n\}, \tag{4.3.0.21}$$

and

$$||\hat{\phi}||_j = \max_{k\leq j}\left\{\sup_{z\in\Lambda_j}\left[e^{j\left[\sum\limits_{p=1}^{n}|\Re(z_p)|\right]}\left|D^{(k)}\hat{\phi}(z)\right|\right]\right\}, \tag{4.3.0.22}$$

where $D^{(k)} = \partial^{(k_1)}\partial^{(k_2)}\cdots\partial^{(k_n)} \quad k = k_1 + k_2 + \cdots + k_n$.

$\mathfrak{B}^n$ is characterized as follows. Let $\mathfrak{C}^n_\omega$ be the space of all functions $\hat{F}(z)$ such that
$\mathbf{A}'$) $\hat{F}(z)$ is analytic for
$\{z \in \mathbb{C}^n | \mathrm{Im}(z_1)| > p, |\mathrm{Im}(z_2)| > p, ..., |\mathrm{Im}(z_n)| > p\}$.
$\mathbf{B}'$) $\hat{F}(z)e^{-\left[p\sum\limits_{j=1}^{n}|\Re(z_j)|\right]}/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n | \mathrm{Im}(z_1)| \geqq p, |\mathrm{Im}(z_2)| \geqq p, ..., |\mathrm{Im}(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $\hat{F}(z)$.
Let further $\mathfrak{N}^n$ be $\mathfrak{N}^n = \left\{\hat{F}(z) \in \mathfrak{C}^n_\omega \hat{F}(z)\right.$ is entire analytic function at minus in one of the variables $z_j \quad 1 \leq j \leq n\}$. Then $\mathfrak{B}^n$ is the quotient space
$\mathbf{C}'$) $\mathfrak{B}^n = \mathfrak{C}^n_\omega/\mathfrak{N}^n$ We have now

$$\hat{F}(\hat{\phi}) =< \hat{F}(z), \hat{\phi}(z) >= \oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\, dz, \tag{4.3.0.23}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup ...\Gamma_n$ and the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $\mathrm{Im}(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $\mathrm{Im}(z_j) < -\zeta$, $-\zeta < -p$. (Again the path Γ surrounds all the singularities of $\hat{F}(z)$). The n-dimensional Dirac's formula is now

$$\hat{F}(z) = \frac{1}{(2\pi i)^n}\int\limits_{-\infty}^{\infty}\frac{\hat{f}(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)}\, dt, \tag{4.3.0.24}$$

and the "density" $\hat{f}(t)$ is such that

$$\oint\limits_{\Gamma}\hat{F}(z)\hat{\phi}(z)\, dz = \int\limits_{-\infty}^{\infty}\hat{f}(t)\hat{\phi}(t)\, dt. \tag{4.3.0.25}$$

The modulus of $\hat{F}(z)$ is bounded by

$$|\hat{F}(z)| \leq C|z|^p e^{\left[p\sum\limits_{j=1}^{n}|\Re(z_j)|\right]}, \tag{4.3.0.26}$$

where C and p depend on $\hat{F}$.

Chapter 5

Dimensional regularization (DR)

5.1 Bollini-Giambiaggi dimensional regularization

Our present purpose is to compare a *dimensional regularization*-generalization to be obtained later in this book with the usual BG-DR, highlighting the differences between them. With this purpose we consider the convolution of two massless propagators in Euclidean space. We start then with the usual formula for the convolution in four dimensions that we saw above in Chapter 1

$$\rho^{-1} * \rho^{-1} = \int \frac{d^4 p}{\vec{p}^2 (\vec{p} - \vec{k})^2}. \tag{5.1.0.1}$$

The generalization of the previous convolution to ν dimensions is [12]

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \int \frac{d^\nu p}{\vec{p}^2 (\vec{p} - \vec{k})^2}. \tag{5.1.0.2}$$

We appeal now to the *Feynman's parameters*

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2}, \tag{5.1.0.3}$$

and rewrite the convolution as

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \int d^\nu p \int_0^1 \frac{dx}{[(\vec{p}-\vec{k})^2 x + \vec{p}^2(1-x)]^2} =$$

$$\int_0^1 dx \int \frac{d^\nu p}{[(\vec{p}-\vec{k})^2 x + \vec{p}^2(1-x)]^2}, \tag{5.1.0.4}$$

or

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \int_0^1 dx \int \frac{d^\nu p}{[(\vec{p}-\vec{k}x)^2 + \vec{k}^2 x(1-x)]^2}. \tag{5.1.0.5}$$

We effect the change of variable $\vec{s} = \vec{p} - \vec{k}x$, and calling $a = \vec{k}^2 x(1-x)$ we obtain

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \int_0^1 dx \int \frac{d^\nu s}{(\vec{s}^2 + a)^2}, \tag{5.1.0.6}$$

or

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^1 dx \int_0^\infty \frac{s^{\nu-1}}{(s^2+a)^2} ds. \tag{5.1.0.7}$$

As in Chapter 1 we make another change of variable, $y = s^2$, so that

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \frac{\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^1 dx \int_0^\infty \frac{y^{\frac{\nu}{2}-1}}{(y+a)^2} dy. \tag{5.1.0.8}$$

Using [4] we can calculate the previous integral as

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \pi^{\frac{\nu}{2}} \Gamma\left(2 - \frac{\nu}{2}\right) \int_0^1 a^{\frac{\nu}{2}-2} dx, \tag{5.1.0.9}$$

or

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \pi^{\frac{\nu}{2}} \Gamma\left(2 - \frac{\nu}{2}\right) \rho^{\frac{\nu}{2}-2} \int_0^1 [x(1-x)]^{\frac{\nu}{2}-2} dx. \tag{5.1.0.10}$$

By recourse again to results of [4] this yields

$$\left(\rho^{-1} * \rho^{-1}\right)_\nu = \frac{\pi^{\frac{\nu}{2}} \left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^2 \rho^{\frac{\nu}{2}-2}}{\Gamma(\nu-2)} \Gamma\left(2-\frac{\nu}{2}\right). \tag{5.1.0.11}$$

We realize that the four-dimensional convolution is not univocally obtained from the ν- *dimensional convolution*, due to the pole at $\nu = 4$. There are several ways to choose the finite part of the above equation, after effecting regularization in the manner described in previous Chapters.

Instead, we can resort to a new DR-generalization of ours, to be discussed below, and then keep only the independent term of the $\nu - 4$ calculation. In this way, we get for the four-dimensional convolution a nicer result. This is

$$\rho^{-1} * \rho^{-1} = -\pi^2[\ln\rho + \ln\pi - \psi(2)]. \tag{5.1.0.12}$$

Thus, we make it evident that the original BG-DR demands generalization, which should motivate the reader, we hope, to continue reading us.

5.2 DR in configuration space

In order to perform and explain the dimensional regularization in configuration space, Bollini and Giambiaggi [13] resorted to the *Bochner's formula* [14]

$$f(k) = \frac{(2\pi)^{\frac{\nu}{2}}}{k^{\frac{\nu-2}{2}}} \int_0^\infty \hat{f}(r) r^{\frac{\nu}{2}} \mathcal{J}_{\frac{\nu-2}{2}}(kr)\, dr \tag{5.2.0.1}$$

where $r^2 = x_0^2 + x_1^2 + \cdots + x_{\nu-1}^2$; $k^2 = k_0^2 + k_1^2 + \cdots + k_{\nu-1}^2$ and $\mathcal{J}_{\frac{\nu}{2}}$ is the *Bessel's function* of order $\nu - 2/2$. We can write

$$\hat{f}(r) = \frac{1}{(2\pi)^{\frac{\nu}{2}} r^{\frac{\nu-2}{2}}} \int_0^\infty f(k) k^{\frac{\nu}{2}} \mathcal{J}_{\frac{\nu-2}{2}}(kr)\, dk \tag{5.2.0.2}$$

By performing the change of variables $x = r^2$, $\rho = k^2$, (5.2.0.1) can be re-written as

$$f(\rho) = \pi \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\rho^{1/2} x^{1/2})\, dx \tag{5.2.0.3}$$

and (5.2.0.2) as

$$\hat{f}(x) = \frac{\pi}{(2\pi)^{\frac{\nu+2}{2}} x^{\frac{\nu-2}{4}}} \int_0^\infty f(\rho)\rho^{\frac{\nu-2}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\, d\rho \qquad (5.2.0.4)$$

Consider now the case in which f is the propagator of the *scalar field*

$$f(k) = \frac{1}{k^2 + m^2} \qquad (5.2.0.5)$$

Then,

$$\hat{f}(r) = \frac{1}{(2\pi)^{\frac{\nu}{2}} r^{\frac{\nu-2}{2}}} \int_0^\infty \frac{k^{\frac{\nu}{2}}}{k^2 + m^2} \mathcal{J}_{\frac{\nu-2}{2}}(kr)\, dk. \qquad (5.2.0.6)$$

Using here the result of Ref. [4]

$$\int_0^\infty \frac{x^{\nu-1}}{(x^2 + a^2)^{\mu+1}} \mathcal{J}_\nu(bx)\, dx = \frac{a^{\nu-\mu} b^\mu}{2^\mu \Gamma(\mu+1)} \mathcal{K}_{\nu-\mu}(ab), \qquad (5.2.0.7)$$

we obtain

$$\int_0^\infty \frac{k^{\frac{\nu}{2}}}{k^2 + m^2} \mathcal{J}_{\frac{\nu-2}{2}}(kr)\, dk = m^{\frac{\nu-2}{2}} \mathcal{K}_{\frac{\nu-2}{2}}(mr), \qquad (5.2.0.8)$$

and then

$$\hat{f}(r) = \frac{r^{\frac{2-\nu}{2}} m^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}} \mathcal{K}_{\frac{\nu-2}{2}}(mr). \qquad (5.2.0.9)$$

We consider now the formula

$$\hat{g}(r) = \frac{1}{(2\pi)^{\frac{\nu}{2}} r^{\frac{\nu-2}{2}}} \int_0^\infty k^{\frac{\nu}{2}-2} \mathcal{J}_{\frac{\nu-2}{2}}(kr)\, dk. \qquad (5.2.0.10)$$

So as to evaluate this integral we use the result (see [4])

$$\int_0^\infty x^\mu \mathcal{J}_\nu(ax)\, dx = \frac{2^\mu}{a^{\mu+1}} \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{2]} + \frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\nu}{2]} - \frac{\mu}{2}\right)}. \qquad (5.2.0.11)$$

Taking into account that in this case

$$g(k) = \frac{1}{k^2}, \tag{5.2.0.12}$$

we have

$$\hat{g}(r) = \frac{2^{\frac{\nu-4}{2}}}{(2\pi)^{\frac{\nu}{2}}}\Gamma\left(\frac{\nu-2}{2}\right) r^{2-\nu}. \tag{5.2.0.13}$$

We now use the well-known formulas for *Fourier transformations*

$$\mathcal{F}^{-1}\{f_1 * f_2 * \cdots * f_n\} = (2\pi)^{(n-1)\nu}\hat{f}_1\hat{f}_2\cdots\hat{f}_n, \tag{5.2.0.14}$$

$$f_1 * f_2 * \cdots * f_n = (2\pi)^{(n-1)\nu}\mathcal{F}(\hat{f}_1\hat{f}_2\cdots\hat{f}_n). \tag{5.2.0.15}$$

From (5.2.0.15) we have then

$$\{f * g\}_\nu(k) = \frac{(2\pi)^{\frac{\nu}{2}} 2^{\frac{\nu-4}{2}} m^{\frac{\nu-2}{2}} \Gamma\left(\frac{\nu-2}{2}\right)}{k^{\frac{\nu-2}{2}}} \int\limits_0^\infty r^{3-\nu}\mathcal{K}_{\frac{\nu-2}{2}}(kr)\mathcal{J}_{\frac{\nu-2}{2}}(kr)\,dr. \tag{5.2.0.16}$$

Using here the result of ref.[4] we find, in terms of hypergeometric function F,

$$\int\limits_0^\infty x^{-\lambda}\mathcal{K}_\mu(ax)\mathcal{J}_\nu(bx)\,dx = \frac{b^\nu\Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right)}{2^{\lambda+1}a^{\nu-\lambda+1}\Gamma(\nu+1)}\otimes$$

$$F\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1; -\frac{b^2}{a^2}\right). \tag{5.2.0.17}$$

We have now for the ν- *dimensional convolution*

$$\left\{\frac{1}{k^2} * \frac{1}{m^2+k^2}\right\}_\nu = \int \frac{d^\nu p}{\vec{p}^2[(\vec{p}-\vec{k})^2+m^2]} =$$

$$2^{\nu-2}\pi^{\frac{\nu}{2}} m^{\nu-4}\frac{\Gamma\left(\frac{\nu-2}{2}\right)\Gamma\left(\frac{4-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} F\left(1,,2-\frac{\nu}{2}, \frac{\nu}{2}; -\frac{k^2}{m^2}\right). \tag{5.2.0.18}$$

Now we use the equality

$$\Gamma\left(\frac{4-\nu}{2}\right) F\left(1, \frac{4-\nu}{2}; \frac{\nu}{2}; -\frac{k^2}{m^2}\right) =$$

$$\Gamma\left(\frac{4-\nu}{2}\right) - \frac{2}{\nu}\Gamma\left(\frac{6-\nu}{2}\right)\frac{k^2}{m^2} F\left(1, \frac{6-\nu}{2}; \frac{2+\nu}{2}; -\frac{k^2}{m^2}\right), \tag{5.2.0.19}$$

and then obtain

$$\left\{\frac{1}{k^2} * \frac{1}{m^2+k^2}\right\}_\nu = 2^{\nu-2}\pi^{\frac{\nu}{2}} m^{\nu-4} \frac{\Gamma\left(\frac{\nu-2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \otimes$$

$$\left[\Gamma\left(\frac{4-\nu}{2}\right) - \frac{2}{\nu}\Gamma\left(\frac{6-\nu}{2}\right)\frac{k^2}{m^2} F\left(1, \frac{6-\nu}{2}; \frac{2+\nu}{2}; -\frac{k^2}{m^2}\right)\right] \tag{5.2.0.20}$$

or, equivalently,

$$\left\{\frac{1}{k^2} * \frac{1}{m^2+k^2}\right\}_\nu = \frac{(4\pi)^{\frac{\nu}{2}} m^{\nu-4}}{2(\nu-2)}\Gamma\left(\frac{4-\nu}{2}\right) -$$

$$\frac{(4\pi)^{\frac{\nu}{2}} m^{\nu-4}}{\nu(\nu-2)}\Gamma\left(\frac{6-\nu}{2}\right)\frac{k^2}{m^2} F\left(1, \frac{6-\nu}{2}; \frac{2+\nu}{2}; -\frac{k^2}{m^2}\right). \tag{5.2.0.21}$$

Then we can write

$$\left\{\frac{1}{k^2} * \frac{1}{m^2+k^2}\right\}_\nu = A_\nu + \Sigma(p, \nu), \tag{5.2.0.22}$$

where

$$A_\nu = \frac{(4\pi)^{\frac{\nu}{2}} m^{\nu-4}}{2(\nu-2)}\Gamma\left(\frac{4-\nu}{2}\right), \tag{5.2.0.23}$$

$$\Sigma_f(p, \nu) = -\frac{(4\pi)^{\frac{\nu}{2}} m^{\nu-4}}{\nu(\nu-2)}\Gamma\left(\frac{6-\nu}{2}\right)\frac{k^2}{m^2} F\left(1, \frac{6-\nu}{2}; \frac{2+\nu}{2}; -\frac{k^2}{m^2}\right). \tag{5.2.0.24}$$

In four dimensions this reads

$$\Sigma_f(p, 4) = -2\pi^2 \frac{k^2}{m^2} F\left(1, 1; 3 - \frac{k^2}{m^2}\right), \tag{5.2.0.25}$$

a nice compact result. The usual ultraviolet divergences appear as poles of the resultant analytic functions of ν.

Chapter 6

Feynman and Wheeler propagators

6.1 Wheeler propagator

Consider now the quantity

$$G_F(x-y) = < 0|\hat{T}\Phi(x)\Phi(y)|0 > . \qquad (6.1.0.1)$$

This propagator is a *Green function* of the *Klein– Gordon* equation, and is discussed in almost any text-book on quantum mechanics. Not so well-known at all is the *Wheeler propagator*. In fact, to provide a fairly complete description of it constitutes one of the main goals of this book.

More than half a century ago, J. A. *Wheeler* and R. P. *Feynman* published a work [15] in which they represented electromagnetic interactions by means of *half advanced* and *half retarded* Green functions. The charged medium was supposed to be a *perfect absorber*, so that no radiation could possibly escape the system.

We are going to call this kind of *Green function* a “ *Wheeler function*” (or propagator), that had been used before by *P. A. M. Dirac* [16], when trying to avoid some run-away solutions, in which one finds rapid increases that cannot be controlled. Later on, in 1949, J. A. Wheeler and R. P. Feynman showed that, in spite of the fact that the Green function contains an advanced part, the results do not contradict causality [17].

Of course, the success of QED and renormalization theory made it soon unnecessary or not advisable to follow that line of research (at least for electromagnetism).

One of the distinctive characteristics of the Green function used in references [15, 16, 17] is its lack of asymptotic free waves. This is the reason behind the choice of a "perfect absorber" for the medium through which the field propagates. As the quantization of free waves is associated to free particles, the above mentioned feature of Wheeler-Green functions implies that no free quantum of the field can ever be observed. Nevertheless, we are now used to the existence of *confined particles*, as *quarks*. They do not manifest themselves as free entities. One might give some examples (outside QCD) for which such a behavior is displayed. A *Lorentz invariant* higher order equation can be decomposed into Klein-Gordon factors, but the corresponding mass parameters need not be real. For instance, the equation

$$\left(\Box^2 + m^4\right) \varphi = \left(\Box + im^2\right) \left(\Box - im^2\right) \varphi = 0, \qquad (6.1.0.2)$$

gives rise to a pair of *constituent fields* [18] obeying

$$\left(\Box \pm im^2\right) \varphi_{\pm} = 0. \qquad (6.1.0.3)$$

Any solution of (6.1.0.3) blows-up asymptotically. We can assert that the corresponding fields should then be forbidden to appear asymptotically as free waves. Therefore, they should have a Wheeler function as the propagator [18]. Equations similar to (6.1.0.2), or of a more general nature, might be of the form

$$\left(\Box^n \pm m^{2n}\right) \varphi = 0, \qquad (6.1.0.4)$$

and they appear in natural fashion in super-symmetric models for higher dimensional spaces [20].

Another example is provided by fields obeying Klein-Gordon equations with a "wrong" sign of the mass term (as it happens for *tachyons*). A careful analysis shows that the propagator should be a Wheeler function [21, 22]. Accordingly, no *tachyon* could ever be observed as a free particle. They can only exist as "mediators" of interactions.

To define propagators in a proper way we have to solve the equations for the Green functions with suitable boundary conditions. Thus, in the case of the wave equation

$$\Box\tilde{G}(x) = \delta(x) \tag{6.1.0.5}$$

a Fourier transformation gives

$$G(p) = \left(\vec{p}^{\,2} - p_0^2\right)^{-1} \equiv \left(p_\mu p^\mu\right)^{-1} \equiv P^{-1} \tag{6.1.0.6}$$

Of course, it is necessary to specify the nature of the accompanying singularity (when the denominator vanishes). Different determinations of them imply different types of Green functions. For the classical solution of (6.1.0.5) it is natural to use the *retarded function* ($\tilde{G}_{rt}$). It corresponds to the propagation towards the future of the effect produced by the sources. This function can be obtained by means of a Fourier transform of (6.1.0.6) in which the p_0 integration is taken along a path from $-\infty$ to $+\infty$, leaving the pertinent poles to the right. In practice, we add to p_0 a small positive imaginary part (this practice is called "the avoiding of singularities by the ϵ trick")

$$G_{rt}(p) = \left[\vec{p}^{\,2} - (p_0 + i0)^2\right]^{-1} =$$

$$\left(\vec{p}^{\,2} - p_0^2 - i0\,\mathrm{sign}\,p_0\right)^{-1} = \left(P - i0\,\mathrm{sign}\,p_0\right)^{-1}. \tag{6.1.0.7}$$

The *advanced solution* is the complex conjugate of (6.1.0.7)

$$G_{ad}(p) = \left(\vec{p}^{\,2} - p_0^2 + i0\,\mathrm{sign}\,p_0\right)^{-1} = \left(P + i0\;\,\mathrm{sign}\,p_0\right)^{-1}. \tag{6.1.0.8}$$

For the Feynman's propagator, instead, we have to add now a small imaginary part to P (not just to p_0)

$$G_\pm(p) = (P \pm i0)^{-1}, \tag{6.1.0.9}$$

and, in the massive case,

$$G_\pm(p) = \left(P + m^2 \pm i0\right)^{-1}. \tag{6.1.0.10}$$

The Cauchy principal value assigns values to special improper integrals [1]. Without it, one can not use these integrals. The Wheeler function is half advanced and half retarded. On the real axis, the Wheeler function coincides with Cauchy's "*principal value*" associated Green function, which is known to be zero on the mass-shell, entailing no free waves. Recall that physical configurations of a given system that happen to satisfy classical equations of motion (CEM) are usually called on shell, while those configurations that do not satisfy CEM are called off shell [1]. We can write

$$G_{\pm}(p) = G(p) \pm i\pi\delta\left(P + m^2\right), \tag{6.1.0.11}$$

where

$$i\pi\delta\left(P + m^2\right) = \frac{1}{2}G_{+}(p) - \frac{1}{2}G_{-}(p). \tag{6.1.0.12}$$

Equation (6.1.0.11) is a decomposition of the Feynman's function into two terms. The first one only contains virtual propagation. The second one is a Lorentz invariant solution of the homogeneous equation representing the free particle.

Recall now that the so called Hankel transform is an integral transform [1]. Its defining feature is that the transform becomes equivalent to a two-dimensional Fourier one, with a radially symmetric integral kernel [1]. Now, to perform convolution integrations in p-space, we will utilize the method presented in reference [13]. Essentially, it consists in the use of the *Bochner's theorem* for the reduction of the Fourier transform to a *Hankel transform* [1]. The nucleus of this transformation is made to correspond to an arbitrary number of dimensions ν, taken as a free parameter. In this way, starting with a given propagator in p-space, we get a function in x-space whose singularity at the origin depends analytically on ν . There exists then a range of values (of ν) such that the product of Green functions exists and is well determined. In x-space we define

$$Q = r^2 - x_0^2 = x_\mu x^\mu$$

The Fourier transform of the massless Feynman's function is

$$\mathcal{F}\{(P - i0)^{-1}\}(x) = \frac{1}{(2\pi)^{\frac{\nu}{2}}} \int d^\nu p \; (P - i0)^{-1} e^{ipx}.$$

Remember that a Wick rotation is a method for solving a problem in Minkowski space via a solution to a related problem in Euclidean space [1]. This is achieved by means of a transformation that substitutes an imaginary-number variable for a real-number variable [1]. Above then, by means of a " *Wick rotation*", the p_0-integration can be made to run along the imaginary axis, without crossing any pole. Mathematically, we perform a dilation

$$p_0 = ap_0^{'} \;\; ; \;\; x_0 = \frac{1}{a}x_0^{'} \;\; ; \;\; p_0 x_0 = p_0^{'} x_0^{'}. \qquad (6.1.0.13)$$

A subsequent continuation to a = i produces the transformation

$$P \Rightarrow \vec{p}^{\,2} + p_0^{'2} = P^{'}$$

$$Q \Rightarrow \vec{x}^{\,2} + x_0^{'2} = Q^{'}.$$

The new quadratic forms are Euclidean and *Bochner's theorem* [14] tells us that the pertinent Fourier transformation reduces to

$$\mathcal{F}\left\{(P - i0)^{-1}\right\}(x) = \frac{i}{x^{\frac{\nu}{2}}} \int\limits_0^{\infty} dy \, \frac{y^{\frac{\nu}{2}}}{y^2} \mathcal{J}_{\frac{\nu}{2}-1}(xy), \qquad (6.1.0.14)$$

where $\mathcal{J}_\alpha$ is a *Bessel's function* of the first kind and order α. In Eq. (6.1.0.13) we see that a *Wick rotation* in p-space ($a \to i$) implies an anti-Wick rotation in x-space ($a^{-1} \to -i$). We must then choose

$$x = (Q + i0)^{\frac{1}{2}}.$$

Note that the imaginary unit in Eq.(6.1.0.14) (r.h.s.), is due to the transformation $dp_0 \to idp_0^{'}$. From Ref. [4] we get

$$\int\limits_0^{\infty} dy \; y^{\mu} \mathcal{J}_{\rho}(ay) = 2^{\mu} a^{-\mu-1} \frac{\Gamma\left(\frac{1+\rho+\mu}{2}\right)}{\Gamma\left(\frac{1+\rho-\mu}{2}\right)},$$

i.e.,

$$\mathcal{F}\left\{(P - i0)^{-1}\right\}(x) = i2^{\frac{\nu}{2}-2}\Gamma\left(\frac{\nu}{2} - 1\right)(Q + i0)^{1-\frac{\nu}{2}}. \qquad (6.1.0.15)$$

More generally, for a function $f(P \pm i0)$, we obtain

$$\mathcal{F}\{f(P \pm i0)\}(x) = \mp \frac{i}{x^{\frac{\nu}{2}-1}} \int\limits_0^{\infty} dy \; y^{\frac{\nu}{2}} f\left(y^2\right) \mathcal{J}_{\frac{\nu}{2}-1}(xy), \qquad (6.1.0.16)$$

where $x = (Q \mp i0)^{\frac{1}{2}}$. The right hand side of eq.(6.1.0.16) is a *Hankel transform* of the function $f(y^2)$ [23].

6.2 Massless case

a) Fourier transforms

With the procedures described in section 1, we can obtain the Fourier transforms of general massless Feynman's functions, defined as $(P \pm i0)^{\alpha}$. From (6.1.0.16) and Ref. [4], we get

$$\mathcal{F}\{(P \pm i0)^{\alpha}\}(x) = \mp i 2^{2\alpha + \frac{\nu}{2}} \frac{\Gamma\left(\alpha + \frac{\nu}{2}\right)}{\Gamma(-\alpha)} (Q \mp i0)^{-\alpha - \frac{\nu}{2}}. \quad (6.2.0.1)$$

The exponent of $Q \mp i0$ can be deduced via dimensional considerations, as described previously above. Furthermore, if we interchange the quadratic forms $P \leftrightarrow Q$ and write $\mathcal{F}^{-1}$ for $\mathcal{F}$, then Eq. (6.2.0.1) is still valid. We define the *massless Wheeler propagator* as

$$P^{\alpha} = \frac{1}{2}(P + i0)^{\alpha} + \frac{1}{2}(P - i0)^{\alpha}. \quad (6.2.0.2)$$

(We will not use any index for Wheeler functions.) The Fourier transform of (6.2.0.2) is (Cf. Eq.(6.2.0.1))

$$\mathcal{F}\{P^{\alpha}\}(x) = i 2^{2\alpha + \frac{\nu}{2}} \frac{\Gamma\left(\alpha + \frac{\nu}{2}\right)}{\Gamma(-\alpha)} \left[\frac{1}{2}(Q + i0)^{-\alpha - \frac{\nu}{2}} - \frac{1}{2}(Q - i0)^{-\alpha - \frac{\nu}{2}}\right] \quad (6.2.0.3)$$

Note that we also have the relation (valid for any quadratic form [6])

$$(Q \pm i0)^{\lambda} = Q_{+}^{\lambda} + e^{\pm i\pi\lambda} Q_{-}^{\lambda}, \quad (6.2.0.4)$$

where

$$Q_{+}^{\lambda} = \begin{cases} Q^{\lambda} & Q > 0 \\ 0 & Q \leq 0 \end{cases}$$

$$Q_{-}^{\lambda} = \begin{cases} (-Q)^{\lambda} & Q < 0 \\ 0 & Q \geq 0 \end{cases}$$

Thus, we can rewrite (6.2.0.3) in the form

$$\mathcal{F}\{P^{\alpha}\}(x) = 2^{2\alpha + \frac{\nu}{2}} \frac{\Gamma\left(\alpha + \frac{\nu}{2}\right)}{\Gamma(-\alpha)} \sin\pi\left(\alpha + \frac{\nu}{2}\right) Q_{-}^{-\alpha - \frac{\nu}{2}}. \quad (6.2.0.5)$$

Equation (6.2.0.5) illustrates another interesting property of Wheeler functions. They are real and have support inside the light-cone of the coordinates. Furthermore, for $\alpha = -1$, the trigonometric function tends to zero for $\nu \to 4$, but $Q_-^{1-\frac{\nu}{2}}$ has a pole at $\nu = 4$ with residue $\delta(Q)$ [6]. Then,

$$\lim_{\nu \to 4} \mathcal{F}\left\{P^{-1}\right\}(x) = \delta(Q). \tag{6.2.0.6}$$

In four dimensions the massless Wheeler function is concentrated on the light cone. From Eq. (6.2.0.5) we obtain

$$\mathcal{F}^{-1}\left\{Q_-^{\lambda}\right\}(p) = -\frac{2^{2\lambda} + \frac{\nu}{2}}{\sin\pi\lambda}\frac{\Gamma\left(\lambda+\frac{\nu}{2}\right)}{\Gamma(-\lambda)}P^{-\lambda-\frac{\nu}{2}} =$$

$$2^{2\lambda+\frac{\nu}{2}}\Gamma(\lambda+1)\,\Gamma\left(\lambda+\frac{\nu}{2}\right)P^{-\lambda-\frac{\nu}{2}}, \tag{6.2.0.7}$$

where the relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z},$$

has been used. From (6.2.0.1) we can also get

$$\mathcal{F}^{-1}\left\{Q_+^{\lambda}\right\}(p) = -2^{2\lambda+\frac{\nu}{2}}\Gamma(\lambda+1)\,\Gamma\left(\lambda+\frac{\nu}{2}\right) \quad \times$$

$$\left[\cos\pi\lambda\; P_+^{-\lambda-\frac{\nu}{2}} + \cos\frac{\pi}{2}\nu\; P_-^{-\lambda-\frac{\nu}{2}}\right]. \tag{6.2.0.8}$$

Now, from relation (6.1.0.12) we find

$$2\pi i\delta(P) = (P - i0)^{-1} - (P + i0)^{-1}, \tag{6.2.0.9}$$

and

$$\mathcal{F}\{\delta(P)\}(x) = \frac{2^{\frac{\nu}{2}-2}}{2\pi}\Gamma\left(\frac{\nu}{2}-1\right)\left[(Q+i0)^{1-\frac{\nu}{2}} + (Q-i0)^{1-\frac{\nu}{2}}\right] \tag{6.2.0.10}$$

$$\mathcal{F}\{\delta(P)\}(x) = \frac{2^{\frac{\nu}{2}-2}}{2\pi}\Gamma\left(\frac{\nu}{2}-1\right)\left[Q_+^{1-\frac{\nu}{2}} - \cos\frac{\pi}{2}\nu Q_-^{1-\frac{\nu}{2}}\right]. \tag{6.2.0.11}$$

Note that Eq. (6.2.0.5) allows us to obtain the advanced and retarded components of the Wheeler functions in x-space.

$$\tilde{G}_{rt}(x) = 2\Theta(t)\mathcal{F}\{P^{\alpha}\}(x); \quad \tilde{G}_{ad}(x) = 2\Theta(-t)\mathcal{F}\{P^{\alpha}\}(x),$$

where $\Theta(t)$ is Heaviside's step function.

b) Convolutions

The convolution product pg is the product of the Fourier transform of each factor

$$f(p) * g(p) = c\mathcal{F}^{-1}\{\mathcal{F}\{f(p)\}(x)\mathcal{F}\{g(p)\}(x)\}(p),$$
$$\text{with } c = (2\pi)^{\frac{\nu}{2}}. \tag{6.2.0.12}$$

Formula (6.2.0.12) can be taken as a definition of the operation $*$. The product of distributions inside the curly brackets can be effected in a suitable range of ν, and analytically extended to other values. For Feynman's propagators, (6.2.0.12) gives

$$(P - i0)^{\alpha} * (P - i0)^{\alpha} = ic2^{-\frac{\nu}{2}} \frac{\Gamma^2\left(\alpha + \frac{\nu}{2}\right)}{\Gamma^2(-\alpha)} \times$$
$$\frac{\Gamma\left(-2\alpha - \frac{\nu}{2}\right)}{\Gamma(2\alpha + \nu)} (P - i0)^{2\alpha + \frac{\nu}{2}}. \tag{6.2.0.13}$$

An analogous equation can be obtained by changing the sign of the imaginary unit in (6.2.0.13). The convolution of two Feynman's functions of the same kind (+ or -) gives another Feynman's function of the same kind. For Wheeler propagators, Eqs.(6.2.0.5), (6.2.0.7), and (6.2.0.12) yield

$$P^{\alpha} * P^{\alpha} = 2^{-\frac{\nu}{2}-1} c \frac{\Gamma^2\left(\alpha + \frac{\nu}{2}\right)}{\Gamma^2(-\alpha)} \cdot \frac{\Gamma\left(-2\alpha - \frac{\nu}{2}\right)}{\Gamma(2\alpha + \nu)} \times$$
$$\operatorname{tg}\pi\left(\alpha + \frac{\nu}{2}\right) P^{2\alpha + \frac{\nu}{2}}, \tag{6.2.0.14}$$

so that the convolution of two Wheeler functions gives another Wheeler function. For the wave equation we choose $\alpha = -1$ and

$$(P - i0)^{-1} \cdot (P - i0)^{-1} = 2ia(\nu)(P - i0)^{\frac{\nu}{2}-2} \tag{6.2.0.15}$$

$$(P + i0)^{-1} \cdot (P + i0)^{-1} = -2ia(\nu)(P + i0)^{\frac{\nu}{2}-2} \tag{6.2.0.16}$$

$$P^{-1} * P^{-1} = a(\nu)\operatorname{tg}\pi\left(\frac{\nu}{2} - 1\right) P^{\frac{\nu}{2}-2}, \tag{6.2.0.17}$$

where

$$a(\nu) = c 2^{-\frac{\nu}{2}-1} \Gamma^2 \left(\frac{\nu}{2} - 1\right) \frac{\Gamma\left(2 - \frac{\nu}{2}\right)}{\Gamma\left(\nu - 2\right)}.$$

Eqs. (6.2.0.15) and (6.2.0.16) display a pole for $\nu \to 4$ (the usual ultraviolet divergence), while (6.2.0.17) is well determined in that limit. The convolution of two $\delta(P)$ functions can be found with the help of Eqs. (6.2.0.7), (6.2.0.8), (6.2.0.11), and (6.2.0.12).

$$\pi^2 \delta(P) * \delta(P) = a(\nu) \ \mathrm{tg}\pi \left(\frac{\nu}{2} - 1\right) \left[P_+^{\frac{\nu}{2}-2} - \cos\frac{\pi}{2}\nu P_-^{\frac{\nu}{2}-2}\right]. \quad (6.2.0.18)$$

A comparison with Eq.(6.2.0.15) shows that

$$\pi^2 \delta(P) * \delta(P) = \mathrm{sgn} P \cdot P^{-1} * P^{-1}. \quad (6.2.0.19)$$

This relation implies

$$(P - i0)^{-1} * (P + i0)^{-1} + \pi^2 \delta(P) * \delta(P) = 2\Theta(P)\ P^{-1} * P^{-1}.$$

The convolution of two Feynman's propagators of different kinds has a support outside the light-cone in p-space.

6.3 Massive case

Bradyons, also called tardyons, are particles that travel with velocities below the speed of light, as opposed to hypothetical tachyons [1]. All known massive subatomic particles are bradyons. A *bradyon field* obeys a normal Klein-Gordon equation. Its Feynman's propagator is given by Eq.(6.1.0.10). The Wheeler function is

$$\left(P + m^2\right)^{-1} = \frac{1}{2}\left(P + m^2 + i0\right)^{-1} + \frac{1}{2}\left(P + m^2 - i0\right)^{-1}. \quad (6.3.0.1)$$

To repeat: in quantum field theory, configurations of a physical system that satisfy classical equations of motion are called on shell while those that do not are called off shell [1]. For example, virtual particles are termed off shell because they do not satisfy the energy–momentum relation. The on-shell δ-function, solution of the homogeneous equation is

$$\delta\left(P+m^2\right)=\frac{1}{2\pi i}\left[\left(P+m^2+i0\right)^{-1}-\left(P+m^2-i0\right)^{-1}\right] \quad (6.3.0.2)$$

a) Fourier transforms

To find the Fourier transform of the Feynman propagators we use Eq. (6.1.0.16) and Ref. [4] (p. 687 - 6.566 - 2)

$$\mathcal{F}\left\{\left(P+m^2\pm i0\right)^{-1}\right\}(x)=$$
$$\mp i m^{\frac{\nu}{2}-1}\left(Q\mp i0\right)^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{K}_{\frac{\nu}{2}-1}\left[m\left(Q\mp i0\right)^{\frac{1}{2}}\right],$$

where $\mathcal{K}_\alpha$ is a *Bessel function* of the third kind and order α (for the definition of different Bessel's functions we follow Ref. [4], p.951 - 8.40). Note also that with the more general formula of Ref. [4] (p.687 - 6.565 - 4), we can find the Fourier transform of arbitrary powers of Feynman's propagator. Using the relation (6.2.0.4) we write

$$\mathcal{F}\left\{\left(P+m^2\pm i0\right)^{-1}\right\}(x)=\mp i m^{\frac{\nu}{2}-1}\left[Q_+^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{K}_{\frac{\nu}{2}-1}\left(mQ_+^{\frac{1}{2}}\right)+\right.$$
$$\left. i\frac{\pi}{2}Q_-^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{H}^{\beta}_{1-\frac{\nu}{2}}\left(mQ_-^{\frac{1}{2}}\right)\right], \quad (6.3.0.3)$$

where $\beta=1$ for the upper sign and $\beta=2$ for the lower sign. It is now easy to find the transforms of (6.3.0.1) and (6.3.0.2). Consider

$$\mathcal{F}\left\{\left(P+m^2\right)^{-1}\right\}(x)=\frac{\pi}{2}m^{\frac{\nu}{2}-1}Q_-^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{J}_{1-\frac{\nu}{2}}\left(mQ_-^{\frac{1}{2}}\right) \quad (6.3.0.4)$$

$$\mathcal{F}\left\{\delta\left(P+m^2\right)\right\}(x)=-\frac{1}{\pi}m^{\frac{\nu}{2}-1}\left[Q_+^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{K}_{\frac{\nu}{2}-1}\left(mQ_+^{\frac{1}{2}}\right)-\right.$$
$$\left.\frac{\pi}{2}Q_-^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{N}_{1-\frac{\nu}{2}}\left(mQ_-^{\frac{1}{2}}\right)\right]. \quad (6.3.0.5)$$

Also, for the massive case, the Wheeler function is zero outside the light-cone. With some slight changes in notation, one can verify that (6.3.0.3), and (6.3.0.5), coincides with the results of Ref. [4].

b) Convolutions

We are going to follow the procedure already used in Section 2. The convolution of a *massive Feynman propagator* with a massless one, Eqs. (6.2.0.1), (6.3.0.3), and (6.2.0.12) yields

$$\left(P+m^2-i0\right)^{-1}*(P-i0)^{-1}=-2^{\frac{\nu}{2}-2}cm^{\frac{\nu}{2}-1}\Gamma\left(\frac{\nu}{2}-1\right)\quad\times$$

$$\mathcal{F}^{-1}\left\{(Q+i0)^{\frac{3}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{K}_{\frac{\nu}{2}-1}\left[m\,(Q+i0)^{\frac{1}{2}}\right]\right\}(p).$$

and, with the help of Eq. (6.1.0.16) and Ref. [4] (p.643 - 6.576 - 3), we get

$$\left(P+m^2-i0\right)^{-1}*(P-i0)^{-1}=i2^{\frac{\nu}{2}}c\frac{m^{\nu-2}}{\Gamma\left(\frac{\nu}{2}\right)}\Gamma\left(\frac{\nu}{2}-1\right)\Gamma\left(2-\frac{\nu}{2}\right)\quad\times$$

$$F\left(1,2-\frac{\nu}{2},\frac{\nu}{2};-\frac{P-i0}{m^2}\right),\tag{6.3.0.6}$$

where $F(a,b,c;z)$ is Gauss' *hypergeometric function.*

For two Wheeler propagators and two on-shell δ-functions, we can follow the same method. It is then not difficult to prove the following relation (compare with (6.2.0.19))

$$\pi^2\delta\left(P+m^2\right)*\delta\left(P\right)=\mathrm{sgn}P\cdot\left(P+m^2\right)^{-1}*P^{-1},$$

or, more generally,

$$\pi^2\delta\left(P+m_1^2\right)*\delta\left(P+m_2^2\right)=\mathrm{sgn}P\cdot\left(P+m_1^2\right)^{-1}*\left(P+m_2^2\right)^{-1}.\tag{6.3.0.7}$$

6.4 Tachyons

A *tachyon field* obeys a Klein-Gordon equation with the wrong sign of the "mass" term. The Green function is an inverse of $P-\mu^2$ (we use $\mu^2=-m^2$ for the mass of the tachyon). Although it is not easy to see what a Feynman's propagator should be in that case in which the inverse of $P-\mu^2=\vec{p}^{\,2}-p_0^2-\mu^2$ has a pair of imaginary poles (if $\vec{p}^{\,2}<\mu^2$), we may nevertheless define

$$\left(P-\mu^2\pm i0\right)^{-1}=-\left(-P+\mu^2\mp i0\right)^{-1},$$

or, introducing the "dual" quadratic form, we arrive at

$$
\begin{gathered}
{}^{+}P = -P = p_0^2 - \vec{p}^{\,2} \\
\left(P - \mu^2 \pm i0\right)^{-1} = -\left({}^{+}P + \mu^2 \mp i0\right)^{-1} .
\end{gathered}
$$

In other words, instead of propagators with the wrong sign of the mass, we see that we have propagators with the wrong sign of the metric. To find the Fourier transform, we recall (see section 2) that we have an "i" for the Wick rotation of p_0. For the dual metric we have three Wick rotations. The three factors $d\vec{p}$ contribute with $i^3 = -i$. We also note that ${}^{+}Q_{+} = Q_{-}$ and ${}^{+}Q_{-} = Q_{+}$. Thus, (compare with (6.3.0.3))

$$
\begin{aligned}
\mathcal{F}\left\{\left(P - \mu^2 \pm i0\right)^{-1}\right\}(x) = \pm i\mu^{\frac{\nu}{2}-1} \Bigg[& \frac{\mathcal{K}_{\frac{\nu}{2}-1}\left(\mu Q_{-}^{\frac{1}{2}}\right)}{Q_{-}^{\frac{1}{2}\left(\frac{\nu}{2}-1\right)}} \mp \\
& i\frac{\pi}{2}\frac{\mathcal{H}^{\beta}_{1-\frac{\nu}{2}}\left(\mu Q_{+}^{\frac{1}{2}}\right)}{Q_{+}^{\frac{1}{2}\left(\frac{\nu}{2}-1\right)}}\Bigg],
\end{aligned} \tag{6.4.0.1}
$$

where $\beta = 2$ for the upper sign and $\beta = 1$ for the lower sign. The real part of (6.4.0.1) is

$$
\begin{gathered}
\frac{1}{2}\mathcal{F}\left\{\left(P - \mu^2 + i0\right)^{-1}\right\}(x) + \frac{1}{2}\mathcal{F}\left\{\left(P - \mu^2 - i0\right)^{-1}\right\}(x) = \\
\mathrm{Re}\mathcal{F}\left\{\left(P - \mu^2 \pm i0\right)^{-1}\right\}(x) = \frac{\pi}{2}\mu^{\frac{\nu}{2}-1} Q_{+}^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)} \mathcal{J}_{1-\frac{\nu}{2}}\left(\mu Q_{+}\right) .
\end{gathered} \tag{6.4.0.2}
$$

This real part has support outside the light-cone, while for bradyons, the real part of Feynman's propagator is zero for x^{μ} space like (Cf. Eq. (6.3.0.4)).

We will now show that (6.4.0.2) is not the Wheeler propagator for the tachyon. To see this we go back to the original definition in terms of a half retarded and a half advanced propagator.

$$
\left(P - \mu^2\right)^{-1} = \frac{1}{2}\left(P - \mu^2\right)^{-1}_{Ad} + \frac{1}{2}\left(P - \mu^2\right)^{-1}_{Rt} . \tag{6.4.0.3}
$$

The Fourier transform is

$$\mathcal{F}\left\{\left(P-\mu^2\right)^{-1}\right\}(x) = \frac{1}{(2\pi)^{\nu/2}}\frac{1}{2}\int d^\nu p \frac{e^{ipx}}{(P-\mu^2)_{Ad}} +$$

$$\frac{1}{(2\pi)^{\nu/2}}\frac{1}{2}\int d^\nu p \frac{e^{ipx}}{(P-\mu^2)_{Rt}}. \tag{6.4.0.4}$$

We will first evaluate the advanced part of (6.4.0.4), i.e.,

$$\mathcal{F}\left\{\left(P-\mu^2\right)^{-1}_{Ad}\right\}(x) = \frac{1}{(2\pi)^{\nu/2}}\int d^{\nu-1}p\; e^{i\vec{p}\cdot\vec{r}} \int\limits_{Ad} dp_0 \frac{e^{-ip_0x_0}}{\vec{p}^{\,2}-p_0^2-\mu^2}, \tag{6.4.0.5}$$

where the path of integration runs parallel to the real axis and below both poles of the integrand. For $x_0 < 0$ the path can be closed on the upper half plane of p_0. The contribution will be that of the residues at the poles

$$p_0 = \pm\omega = \pm\sqrt{\vec{p}^{\,2}-\mu^2} \quad ,\text{if} \quad \vec{p}^{\,2} \geq \mu^2$$

$$p_0 = \pm i\omega' = \pm i\sqrt{\mu^2-\vec{p}^{\,2}} \quad ,\text{if} \quad \vec{p}^{\,2} \leq \mu^2$$

$$\mathcal{F}\left\{\left(P-\mu^2\right)^{-1}_{Ad}\right\}(x) =$$

$$\frac{1}{(2\pi)^{\nu/2}}\int d^{\nu-1}p e^{i\vec{p}\cdot\vec{r}} 2\pi i \left[\left(\frac{e^{i\omega x_0}}{2\omega} - \frac{e^{-i\omega x_0}}{2\omega}\right)\Theta\left(\vec{p}^{\,2}-\mu^2\right)\right.$$

$$\left. + \left(\frac{-e^{\omega' x_0}}{2i\omega'} + \frac{e^{-\omega' x_0}}{2i\omega'}\right)\Theta\left(\mu^2-\vec{p}^{\,2}\right)\right] =$$

$$\frac{-2\pi}{(2\pi)^{\nu/2}}\int d^{\nu-1}p e^{i\vec{p}\cdot\vec{r}}\left[\frac{\sin\omega x_0}{\omega}\Theta\left(\vec{p}^{\,2}-\mu^2\right) + \frac{\mathrm{sh}\,\omega' x_0}{\omega'}\Theta\left(\mu^2-\vec{p}^{\,2}\right)\right] \tag{6.4.0.6}$$

Note that we can write the brackets as (Cf. Eq. (6.2.0.4))

$$\left[\frac{\sin\omega x_0}{\omega}\Theta\left(\vec{p}^{\,2}-\mu^2\right) + \frac{\mathrm{sh}\,\omega' x_0}{\omega'}\Theta\left(\mu^2-\vec{p}^{\,2}\right)\right] =$$

$$\frac{\sin\left[x_0\left(\vec{p}^{\,2}-\mu^2+i0\right)^{\frac{1}{2}}\right]}{\left(\vec{p}^{\,2}-\mu^2+i0\right)^{\frac{1}{2}}} = \frac{\sin x_0\Omega}{\Omega}. \tag{6.4.0.7}$$

For the spatial Fourier transform (6.4.0.6), we use *Bochner's theorem* (Cf. Eq. (6.1.0.14)))

$$\mathcal{F}\left\{\left(P-\mu^2\right)^{-1}_{Ad}\right\}(x) =$$

$$(2\pi)^{\frac{1}{2}}\,\frac{\Theta(-t)}{r^{\frac{\nu-3}{2}}}\int\limits_0^\infty dk\,k^{\frac{\nu-1}{2}}\,\frac{\sin|t|\,\Omega}{\Omega}\mathcal{J}_{\frac{\nu-3}{2}}(rk).$$

Eq. 6.737-5, p.761 of the table of Ref. [4] gives now ($b = i\mu + 0$)

$$\mathcal{F}\left\{\left(P-\mu^2\right)^{-1}_{Ad}\right\}(x) = \pi\mu^{\frac{\nu}{2}-1}Q_-^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{I}_{1-\frac{\nu}{2}}\left(\mu Q_-^{\frac{1}{2}}\right)\quad (x_0<0), \tag{6.4.0.8}$$

and, of course,

$$\mathcal{F}\left\{\left(P-\mu^2\right)^{-1}_{Ad}\right\}(x) = 0 \quad \text{for } x_0 \geq 0.$$

The retarded Fourier transform reproduces (6.4.0.6), with a change of sign, for $x_0 > 0$ and is zero for $x_0 \leq 0$. We then get, for the *Wheeler Green function* (6.4.0.4)

$$\mathcal{F}\left\{\left(P-\mu^2\right)^{-1}\right\}(x) = \frac{\pi}{2}\mu^{\frac{\nu}{2}-1}Q_-^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)}\mathcal{I}_{1-\frac{\nu}{2}}\left(\mu Q_-^{\frac{1}{2}}\right). \tag{6.4.0.9}$$

Again, the Wheeler propagator has support inside the light-cone. But, instead of a Bessel's function of the first kind, we have now a Bessel's function of the second kind. It is clear that by taking into account Eq. (6.2.0.12), we can evaluate convolutions of different Green functions, as we did in Section 5.3 for bradyon fields.

6.5 Fields with complex mass parameters

The decomposition in Klein-Gordon factors of a higher order equation often leads to *complex mass* parameters. Equation (6.1.0.2) is an example. The constituent fields obey Eq.(6.1.0.3). A simple higher order equation such as (6.1.0.4) exhibits the same behavior. Of course, for a real equation, the masses come in complex conjugate pairs. We consider

$$\left(\Box - M^2\right)\phi = 0 \;\;,\;\; M = m + i\mu \quad (m>0). \tag{6.5.0.1}$$

This type of equation has been analyzed elsewhere (Ref. [19]). The Green functions are inverses of $P + M^2 = \Omega^2 - p_0^2$, where $\Omega = (\vec{p}^{\,2} + M^2)^{1/2}$. The two poles at $p_0 \pm \Omega$ move when $\vec{p}^{\,2}$ varies from 0 to ∞, on a line contained in a band of width $\pm i\mu$, centered at the real axis.

The *retarded Green function* is obtained with a p_0-integration that runs parallel to the real axis, with $\mathrm{Im} p_0 > |\mu|$. For the advanced solution, the integration runs below both poles ($\mathrm{Im} p_0 < -|\mu|$)

$$\mathcal{F}\left\{\left(P + M^2\right)^{-1}_{Ad}\right\}(x) =$$

$$\frac{1}{(2\pi)^{\nu/2}} \int d^{\nu-1}p \; e^{i\vec{p}\cdot\vec{r}} \int_{Ad} dp_0 \; \frac{e^{ip_0 t}}{\Omega^2 - p_0^2} =$$

$$\frac{1}{(2\pi)^{\nu/2}} \int d^{\nu-1}p \; e^{i\vec{p}\cdot\vec{r}} \frac{2\pi i \Theta(-t)}{2\Omega} \left(e^{i\Omega t} - e^{-i\Omega t}\right) =$$

$$-2\pi \frac{\Theta(-t)}{(2\pi)^{\nu/2}} \int d^{\nu-1}p \; e^{i\vec{p}\cdot\vec{r}} \frac{\sin \Omega t}{\Omega}$$

$$\mathcal{F}\left\{\left(P + M^2\right)^{-1}_{Ad}\right\}(x) =$$

$$(2\pi)^{\frac{1}{2}} \frac{\Theta(-t)}{r^{\frac{\nu-3}{2}}} \int_0^{\infty} dk \; K^{\frac{\nu-1}{2}} \frac{\sin \Omega \mid t \mid}{\Omega} \mathcal{J}_{\frac{\nu-3}{2}}(rk),$$

and, according to Ref. [4] (6.735 - 5)

$$\mathcal{F}\left\{\left(P + M^2\right)^{-1}_{Ad}\right\}(x) = \pi \Theta(-t) M^{\frac{\nu}{2}-1} Q_{-}^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)} \mathcal{J}_{\frac{\nu-3}{2}}\left(M Q_{-}^{\frac{1}{2}}\right).$$

For the *retarded Green function* we get the same answer, with $\Theta(+t)$ replacing $\Theta(-t)$. The *Wheeler propagator* is then

$$\mathcal{F}\left\{\left(P + M^2\right)^{-1}\right\}(x) = \frac{\pi}{2} M^{\frac{\nu}{2}-1} Q_{-}^{\frac{1}{2}\left(1-\frac{\nu}{2}\right)} \mathcal{J}_{\frac{\nu-3}{2}}\left(M Q_{-}^{\frac{1}{2}}\right). \quad (6.5.0.2)$$

Now we finally have the general result. The Wheeler function propagates inside the light-cone for any value of the mass, real (bradyons), imaginary (tachyons) or complex ($M = m + i\mu$). In the case of complex masses, a natural definition for the Feynman propagator is obtained by a p_0-integration along the real axis. It is not difficult to see that

$$\mathcal{F}\left\{\left(P+M^2\right)_F^{-1}\right\}(x)=\sqrt{\frac{\pi}{2}}r^{\frac{3-\nu}{2}}\int_0^\infty dk\, k^{\frac{\nu-1}{2}}\left(\frac{\sin\Omega\mid t\mid}{\Omega}-\right.$$

$$\left. i\,\mathrm{sgn}\mu\,\frac{\cos\Omega\mid t\mid}{\Omega}\right)\mathcal{J}_{\frac{\nu-3}{2}}(rk). \qquad (6.5.0.3)$$

The first term on the right hand side is the Wheeler function. The second term corresponds to a positive loop around the pole in the upper half-plane, and a negative loop around the pole in the lower half-plane. The conjugate Feynman's function (**not** the complex conjugate one) is obtained by changing the sign of the second term, that is, by changing the sign of both loops. The term in $\cos\Omega|t|$ can be taken from Ref. [4] (6.735 - 6). With these definitions the Wheeler function is half Feynman and half its conjugate. Note that for a real mass, μ corresponds to the "small" negative imaginary part added to the mass in Feynman's definition. Accordingly, for real mass we take $\mathrm{sgn}\mu=-1$. Equation (6.5.0.3) can be employed to define

$$\mathcal{F}\left\{\delta\left(P+M^2\right)\right\}(x)=-$$

$$\frac{\mathrm{sgn}\mu}{\pi}\sqrt{\frac{\pi}{2}}r^{\frac{3-\nu}{2}}\int_0^\infty dk\, k^{\frac{\nu-1}{2}}\frac{\cos\Omega\mid t\mid}{\Omega}\mathcal{J}_{\frac{\nu-3}{2}}(rk) \qquad (6.5.0.4)$$

$$\left(P+M^2\right)_F^{-1}=\left(P+M^2\right)^{-1}+i\pi\delta\left(P+M^2\right). \qquad (6.5.0.5)$$

The last formula is valid for any mass, real or complex.

6.6 Associated vacuum

It is well known that the perturbative solution to the quantum equation of motion leads to a Green function which is the vacuum expectation value (VEV) of the chronological product of field operators. Furthermore, when the quanta are not allowed to have negative energies, the VEV turns out to be Feynman's propagator. However, when the energy-momentum vector is space-like, the sign of its energy component is not Lorentz invariant. It is then natural to look for symmetry between positive and negative energies. It has been shown in references [21] and [22] that under this premise, the VEV is a Wheeler propagator. Thus, as to clearly see the origin of the difference between both types of propagators, we are going to compare the usual procedure with the symmetric one. A quantized Klein-Gordon field can be written as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} \left[a(\vec{k})\, e^{ik\cdot x} + a^{+}(\vec{k})\, e^{-ik\cdot x} \right], \tag{6.6.0.1}$$

where

$$\left[a(\vec{k}), a^{+}(\vec{k}') \right] = \delta(\vec{k} - \vec{k}') \quad ; \quad \omega = \sqrt{\vec{k}^2 + m^2}.$$

For simplicity, we are going to consider a single (discretized) degree of freedom. The raising and lowering operators obey

$$\left[a, a^{+} \right] = 1. \tag{6.6.0.2}$$

The energy operator is

$$h = \frac{\omega}{2} \left(aa^{+} + a^{+}a \right) = \omega a^{+}a + \frac{\omega}{2} = h_0 + \frac{\omega}{2}.$$

Usually, the ground state energy-operator is called h_0. The vacuum then obeys

$$h_0 \mid 0 >= 0. \tag{6.6.0.3}$$

It is a consequence of (6.6.0.2) and (6.6.0.3) that

$$< 0|aa^{+}|0 >= 1 \quad , \quad < 0|a^{+}a|0 >= 0. \tag{6.6.0.4}$$

On the other hand, the symmetric vacuum is defined to be the state that has zero "true energy"

$$h|0 >= \frac{\omega}{2} \left(aa^{+} + a^{+}a \right) |0 >= 0. \tag{6.6.0.5}$$

Equations (6.6.0.2) and (6.6.0.5) imply

$$< 0|aa^{+}|0 >= \frac{1}{2} \;\; , \;\; < 0|a^{+}a|0 >= -\frac{1}{2}. \tag{6.6.0.6}$$

Let as assume, for the sake of argument, that we define a "ceiling" state (as opposed to a ground state)

$$a^+|0>= 0. \tag{6.6.0.7}$$

Equations (6.6.0.2) and (6.6.0.7) give

$$< 0|aa^+|0>= 0 \ , \ < 0|a^+a|0>= -1. \tag{6.6.0.8}$$

The usual normal case, Eq. (6.6.0.4), leads to the Feynman propagator. The "inverted" case, Eq. (6.6.0.8), leads to its complex conjugate. Then Eq. (6.6.0.6), which is one half of (6.6.0.4) and one half of (6.6.0.8), leads to one half of the Feynman function and one half of its conjugate. This is the Wheeler propagator defined in section 1. The space of states generated by successive applications of a and a^+ on the symmetric vacuum has an indefinite metric. The scalar product can be defined by means of the *holomorphic representation* [a holomorphic function is a complex-valued function of variables that is, at every point of its domain, complex differentiable in a neighborhood of the point [1]] [24]. The functional space is formed by analytic functions $f(z)$, with the scalar product

$$\langle f, g\rangle = \int dz\, d\bar{z} e^{-z\bar{z}} f(z)\, \overline{g(z)}, \tag{6.6.0.9}$$

or, in *polar coordinates*,

$$\langle f, g\rangle = \int_0^\infty d\rho\, \rho e^{-\rho^2} \int_0^{2\pi} d\phi f(z).\overline{g(z)} \tag{6.6.0.10}$$

The raising and lowering operators are represented by

$$a^+ = z \quad , \quad a = \frac{d}{dz}. \tag{6.6.0.11}$$

The *symmetric vacuum* obeys

$$\left(\frac{d}{dz}\, z + z\, \frac{d}{dz}\right) f_0 = \left(1 + 2z\, \frac{d}{dz}\right) f_0 = 0,$$

whose normalized solution is

$$f_0 = \left(2\pi^{3/2}\right)^{-1/2} z^{-1/2}.$$

The energy eigenfunctions are

$$f_n = \left[2\pi\Gamma\left(n+\frac{1}{2}\right)\right]^{-1/2} z^{-1/2} z^n \qquad (6.6.0.12)$$

6.7 Unitarity

In QFT, the equations of motions for the states of a system of interacting fields are formally solved by means of the evolution operator.

$$U(t, t_0)\,|t_0 >= |t > .$$

The interactions between the quanta of the fields is supposed to take place in a limited region of space-time. The initial and final times can be taken to be $t_0 \to -\infty$ and $t \to +\infty$. Thus we define the S-operator

$$S = U(+\infty, -\infty)\,.$$

We do not intend to discuss the possible problems of such a definition. Here we are only interested in its relation to Wheeler propagators. Usually, the initial and final states are represented by free particles. However, when Wheeler fields are present, their quanta either mediate interactions between other particles, or they end up at an absorber. This circumstance had been pointed out by J.A.Wheeler and R.P.Feynman in references [15] and [17]. As a consequence, the S-matrix not only contains the initial and final free particles, but it also contains the states of the absorbers. Through the latter we can determine the physical quantum numbers of the Wheeler virtual " *asymptotic particles*". For these reasons, even if the initial and final states do not contain any Wheeler free particle, for the verification of *perturbative unitarity* it is necessary to take them into account. We shall illustrate this point with some examples.

Let us consider four scalar fields ϕ_s (s=1,...,4) obeying Klein-Gordon equations with mass parameters m_s^2 and an interaction $\Lambda = \lambda\phi_1\phi_2\phi_3\phi_4$. They can be written as in Eq. (6.6.0.1). Unitarity implies

$$SS^+ = 1,$$

or, with $S = 1 - T$,

$$T + T^+ = TT^+.$$

We introduce the initial and final states and also a complete decomposition of the unity operator

$$< \alpha|T + T^{+}|\beta >= \int d\sigma_{\gamma} < \alpha|T|\gamma >< \gamma|T^{+}|\beta > .$$

For the perturbative expansion

$$T = \sum_{n} \lambda^{n} T_{n}$$

$$< \alpha|T_{n} + T_{n}^{+}|\beta >= \sum_{s=1}^{n-1} \int d\sigma_{\gamma} < \alpha|T_{n-s}|\gamma >< \gamma|T_{s}^{+}|\beta >, \quad (6.7.0.1)$$

In particular, $T_0 = 0$ and T_1=pure imaginary. For n=2,

$$< \alpha|T_2 + T_2^{+}|\beta >= \int d\sigma_{\gamma} < \alpha|T_1|\gamma >< \gamma|T_1^{+}|\beta >, \qquad (6.7.0.2)$$

where we will take $T_1 = i\phi_1\phi_2\phi_3\phi_4$. ϕ_1 and ϕ_2 are supposed to be normal fields whose states can be obtained from the usual vacuum.

$$|\alpha >= a_2^{+} a_1^{+}|0 > \ , \ |\beta >= a_{2'}^{+} a_{1'}^{+}|0 > .$$

On the other hand, for ϕ_3 and ϕ_4 we leave open the possibility of a choice between the usual vacuum or the symmetric one. The left hand side of (6.7.0.2) comes from the second order loop formed with the convolution of a propagator for ϕ_3 and another for ϕ_4. When both fields are normal, we have the convolution of two Feynman's propagators, where the real part is

$$\mathrm{Re}\left[\left(P + m_3^2 - i0\right)^{-1} * \left(P + m_4^2 - i0\right)^{-1}\right] = \left(P + m_3^2\right)^{-1} * \left(P + m_4^2\right) -$$

$$\pi^2 \delta\left(P + m_3^2\right) * \delta\left(P + m_4^2\right),$$

and, according to (6.3.0.7), we have in the *physical region* (P negative)

$$\mathrm{Re}\left[\left(P + m_3^2 - i0\right)^{-1} * \left(P + m_4^2 - io\right)^{-1}\right] =$$

$$2\left(P + m_3^2\right)^{-1} * \left(P + m_4^2\right)^{-1} (P < 0) . \qquad (6.7.0.3)$$

Equation (6.7.0.3) implies that the left hand side of (6.7.0.2) for Feynman's particles is twice the value corresponding to Wheeler particles. The relation (6.7.0.2) is known to be valid for normal fields. Thus, there is no point in proving each. We are going to show how the relative factor 2 arises. The decomposition of unity for normal fields is

$$\mathbf{I} = \int d\sigma_{\gamma}\ |\gamma><\gamma| = |0><0| + \int d^{\nu-1}q;a^{+}(\vec{q})\,|0><0|a(\vec{q}) +$$

$$\int d^{\nu-1}q_1\ d^{\nu-1}q_2 \frac{1}{\sqrt{2}} a^{+}(\vec{q}_1)\,a+(\vec{q}_2)\,|0><0|a(\vec{q}_1)\,a(\vec{q}_2)\,\frac{1}{\sqrt{2}} + .. \tag{6.7.0.4}$$

Then, for the T_1 matrix we have

$$<\alpha|T_1|\gamma>=<0|a_1(\vec{p})\,\phi_1(x)|0><0|a_2\left(\vec{p}^{\,'}\right)\phi_2(x)|0> \quad \times$$
$$<0|\phi_3(x)a_3^{+}(\vec{q}_3)\,|0><0|\phi_4(x)a_4^{+}(\vec{q}_4)\,|0>, \tag{6.7.0.5}$$

where an integration over x-space is understood. When the fields are expressed in terms of the operators $a(q)$ and $a^{+}(q)$, as in equation (6.6.0.3), we obtain

$$<\alpha|T_1|\gamma>= \frac{(2\pi)^{\nu}}{(2\pi)^{2(\nu-1)}} \frac{\delta(p-q_3-q_4)}{4\sqrt{\omega_1\omega_2\omega_3\omega_4}} \quad (p = p_1 + p_2), \tag{6.7.0.6}$$

and

$$<\gamma|T_1|\beta>= \frac{(2\pi)^{\nu}}{(2\pi)^{2(\nu-1)}} \frac{\delta\left(q_3+q_4-p^{'}\right)}{4\sqrt{\omega_1^{'}\omega_2^{'}\omega_3\omega_4}} \quad (p^{'} = p_1^{'}+p_2^{'}). \tag{6.7.0.7}$$

Multiplying together (6.7.0.6) and (6.7.0.7), and adding all possible $|\gamma><\gamma|$ (all $\vec{q}_3$ and $\vec{q}_4$), we get

$$\int d\sigma_{\gamma} <\alpha|T_1|\gamma><\gamma|T_1^{+}|\beta>= \frac{\delta\left(p-p^{'}\right)}{16\,(2\pi)^{2\nu-4}\sqrt{\omega_1\omega_2\omega_1^{'}\omega_2^{'}}}$$
$$\int d\vec{q}\ \frac{\delta\left(p^0-\omega_3(\vec{q})-\omega_4(\vec{p}-\vec{q})\right)}{\omega_3(\vec{q})\,\omega_4(\vec{p}-\vec{q})}. \tag{6.7.0.8}$$

This result coincides with (6.7.0.2) (l.h.s.) when the p^0-convolution is carried out. Suppose now that one of the fields, says ϕ_4, has a Wheeler function as propagator. Instead of Eq. (6.7.0.3) we have

$$Re\left[\left(P+m_3^2-i0\right)^{-1} * \left(P+m_4^2-io\right)^{-1}\right] =$$

$$\left(P+m_3^2\right)^{-1} * \left(P+m_4^2\right)^{-1}, \tag{6.7.0.9}$$

half the value of (6.7.0.3). So as to evaluate the matrix $< T_1 >$ for this case we note that the decomposition of unity for the states of ϕ_4 (with an indefinite metric) is now

$$\mathbf{I} = \int d\sigma_\gamma \; |\gamma >< \gamma| =$$

$$|0 >< 0| + \int d^{\nu-1}q \; \sqrt{2} a^+\left(\vec{q}\right)|0 >< 0| a\left(\vec{q}\right)\sqrt{2} -$$

$$\int d^{\nu-1}q \; \sqrt{2} a\left(\vec{q}\right)|0 >< 0| a^+\left(\vec{q}\right)\sqrt{2} + ... \tag{6.7.0.10}$$

The normalization factors come from the VEV quoted in section 5.6, Eq. (6.6.0.6) It is not necessary to evaluate again the matrix element (6.7.0.5). Its last vacuum expectation value acquires now a factor 1/2 from Eq. (6.6.0.6), and a factor $\sqrt{2}$ from normalization in (6.7.0.10). When the matrix for T_1 and T_1^+ are multiplied together, we get an extra factor $(\sqrt{2}/2)^2 = 1/2$, as it should be for unitarity to hold. When both fields ϕ_3 and ϕ_4 are of the Wheeler type, we get for the convolution of the respective Wheeler propagators the same result (6.7.0.9).

The *matrix element* of T_1 gains now two factors $\sqrt{2}/2$, i.e. a factor 1/2. When we multiply $< T_1 >< T_1^+ >$ we get a factor $1/2 \cdot 1/2 = 1/4$ and we seem to be in trouble with unitarity. However, in this case a new matrix contributes to $< T_1 >$. It is

$$< 0|a_1\left(\vec{p}_1\right)\phi_1(x) a_1^+\left(\vec{q}_1\right) a_1^+\left(\vec{q}_1^{\,'}\right)|0 > \times$$

$$< 0|a_2\left(\vec{p}_2\right)\phi_2(x) a_2^+\left(\vec{q}_2\right) a_2^+\left(\vec{q}_2^{\,'}\right)|0 >$$

$$< 0|\phi_3(x) a_3^+\left(\vec{q}_3\right)|0 >< 0|\phi_4(x) a_4^+\left(\vec{q}_4\right)|0 >, \tag{6.7.0.11}$$

(6.7.0.11) is only viable when both ϕ_3 and ϕ_4 are associated with symmetric vacua. For the first matrix factor we have

$$< 0|a_1(\vec{p}_1)\,\phi_1(x) a_1^+(\vec{q}_1)\,a_1^+\left(\vec{q}_1^{\,'}\right)|0>=$$

$$\delta(p_1 - q_1)\frac{e^{-iq_1^{'}x}}{\sqrt{2\omega_1(q_1^{'})}} + \delta\left(p_1 - q_1^{'}\right)\frac{e^{-iq_1x}}{\sqrt{2\omega_1(q_1)}}. \qquad (6.7.0.12)$$

A similar matrix factor from $< T_1^+ >$ gives

$$< 0|a_1(\vec{q}_1)\,a_1\left(\vec{q}_1^{\,'}\right)\phi_1(y) a_1\left(\vec{p}_1^{\,'}\right)|0>=$$

$$\delta\left(p_1^{'} - q_1^{'}\right)\frac{e^{iq_1y}}{\sqrt{2\omega_1(q_1)}} + \delta\left(p_1^{'} - q_1\right)\frac{e^{iq_1^{'}y}}{\sqrt{2\omega_1(q_1^{'})}}. \qquad (6.7.0.13)$$

When we multiply together (6.7.0.12) and (6.7.0.13), the crossed terms do not contribute ($\delta(p_1 - p_1^{'}) = 0$). The other two terms give equal contributions. A similar evaluation can be done for the second factor of (6.7.0.11) and the corresponding factor of $< T^+ >$. For this reason we are going to keep only the first terms from (6.7.0.12) and (6.7.0.13) (multiplied with the appropriate constants)

$$< \alpha|T_1|\gamma >= \frac{2}{(2\pi)^{2(\nu-1)}}\delta(p_1 - q_1)\frac{e^{-iq_1^x}}{\sqrt{2\omega_1 q_1}}\delta(p_2 - q_2) \quad \times$$

$$\frac{e^{-iq_2x}}{\sqrt{2\omega_2 q_2}}\frac{e^{iq_3x}}{2\sqrt{2\omega_3 q_3}}\frac{e^{iq_4x}}{2\sqrt{2\omega_4 q_4}}.$$

And after performing the x-integration we find

$$< \alpha|T_1|\gamma >= \frac{(2\pi)^\nu}{2(2\pi)^{2(\nu-1)}}\times$$

$$\frac{\delta\left(-q_1^{'} - q_2^{'} + q_3 + q_4\right)\delta(p_1 - q_1)\,\delta(p_2 - q_2)}{4\sqrt{\omega_1^{'}\omega_2^{'}\omega_3\omega_4}}.$$

Analogously,

$$< \gamma|T_1^+|\beta >= \frac{(2\pi)^\nu}{2(2\pi)^{2(\nu-1)}}\times$$

$$\frac{\delta\left(q_1+q_2-q_3-q_4\right)\delta\left(p_1^{'}-q_1^{'}\right)\delta\left(p_2^{'}-q_2^{'}\right)}{4\sqrt{\omega_1^{'}\omega_2^{'}\omega_3\omega_4}}.$$

The sum $\int d\sigma_\gamma \ <\alpha|T_1|\gamma><\gamma|T_1^{+}|\beta>$ corresponds to an integration on $\vec{q}_1, \vec{q}_1^{\,'}, \vec{q}_2, \vec{q}_2^{\,'}$. It is easy to see that, after these operations, we get one fourth of (6.7.0.8). Thus, we complete the proof of unitarity for the proposed example.

Let us now consider the case in which ϕ_3 and ϕ_4 are fields obeying Klein-Gordon equations with complex mass parameters (See section 5.5). The solution of Eq. (6.5.0.1)

$$\left(\Box - M^2\right)\phi = 0 \ \ , \ \ M = m + i\mu \quad (m > 0),$$

can be written as

$$\phi(x) = \frac{1}{(2\pi)^{\frac{\nu-1}{2}}}\int d^{\nu-1}p \ \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\Omega}}\left[a\left(\vec{p}\right)e^{-i\Omega t} + b\left(\vec{p}\right)e^{i\Omega t}\right], \tag{6.7.0.14}$$

where

$$\Omega = \left(\vec{p}^{\,2} + M^2\right)^{\frac{1}{2}}. \tag{6.7.0.15}$$

The quantization of ϕ leads to the rules (ref.[19])

$$\left[a\left(\vec{p}\right), b\left(\vec{p}^{\,'}\right)\right] = \delta\left(\vec{p} - \vec{p}^{\,'}\right),$$

and to the adoption of the symmetric vacuum, from which we get

$$<0|a\left(\vec{p}\right)b\left(\vec{p}^{\,'}\right)|0> = -<0|b\left(\vec{p}\right)a\left(\vec{p}^{\,'}\right)|0> = \frac{1}{2}\delta\left(\vec{p}-\vec{p}^{\,'}\right). \tag{6.7.0.16}$$

Accordingly, the decomposition of unity for complex mass fields is (compare with (6.7.0.10))

$$\mathbf{I} = |0><0| + \int d^{\nu-1}q \ \sqrt{2}b\left(\vec{q}\right)|0><0|a\left(\vec{q}\right)|0>\sqrt{2} -$$

$$\int d^{\nu-1}q \ \sqrt{2}a\left(\vec{q}\right)|0><0|b\left(\vec{q}\right)|0>\sqrt{2} + \ldots.$$

The matrix $<\alpha|T_1|\gamma>$ has the form given in Eq. (6.7.0.5), but now, instead of a_3^+ and a_4^+, we have to write the four possible operators a_3, a_4; a_3, b_4; b_3, a_4, and b_3, b_4. When multiplied with $<\gamma|T_1^+|\beta>$ as in (6.7.0.8) they give similar contributions except for the signs of ω_3 and ω_4 in the arguments of the δ-functions. Thus, the δ-function for each of the terms in the integral of (6.7.0.8) (r.h.s), should be $\delta(p_0-\Omega_3-\Omega_4)$, $\delta(p_0-\Omega_3+\Omega_4)$, $\delta(p_0+\Omega_3-\Omega_4)$ and $\delta(p_0+\Omega_3+\Omega_4)$, respectively. This is exactly one half of the convolution product of the Wheeler propagators for ϕ_3 and ϕ_4. The other half comes from the contribution of the matrix elements of the form (6.7.0.11). Of course, for every field corresponding to a complex mass M, there is another field corresponding to the complex conjugate mass M^*. We have

$$\phi^+(x) = \frac{1}{(2\pi)^{\frac{\nu-1}{2}}} \int d^{\nu-1}p \, \frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2\Omega^*}} \left[b^+(\vec{p})\, e^{-i\Omega^* t} + a^+(\vec{p})\, e^{i\Omega^* t} \right]$$

$$\left[b^+(\vec{p}), a^+\left(\vec{p}\,'\right) \right] = \delta\left(\vec{p} - \vec{p}\,'\right)$$

$$< 0|b^+(\vec{p})\, a^+\left(\vec{p}\,'\right)|0> = -< 0|a^+(\vec{p})\, b^+\left(\vec{p}\,'\right)|0> = \frac{1}{2}\delta\left(\vec{p} - \vec{p}\,'\right)$$

$$\mathbf{I} = |0><0| + \int d^{\nu-1}q\, \sqrt{2}a^+(\vec{q})\,|0><0|b^+(\vec{q})\,|0>\sqrt{2} -$$

$$\int d^{\nu-1}q\, \sqrt{2}b^+(\vec{q})\,|0><0|a^+(\vec{q})\,|0>\sqrt{2} + ...$$

The T-matrix can be constructed with the four possible contributions ϕ_3,ϕ_4; ϕ_3, ϕ_4^+; ϕ_3^+, ϕ_4 and ϕ_3^+, ϕ_4^+. Correspondingly, for each Wheeler propagator with mass M, there is another one with mass M^*. And of course, four possible convolution products. It is easy to see that the total convolution is real, as well as the total T-matrix.

The proof of unitarity for the chosen example, where the masses are complex numbers, contains as particular cases all fields with symmetric vacua. For bradyons $M = m$. For tachyons, the limit $M \to i\mu$ must be taken. Similarly, in other cases a proof of unitarity for Feynman's propagators, based on the decomposition (6.7.0.4) can be converted into a proof of unitarity for Wheeler propagators by using the corresponding decomposition (6.7.0.10).

6.8 Discussion

We have shown that the Wheeler propagator has several interesting properties. In the first place, note the fact that it implies only virtual propagation. The on-shell δ-function, solution of the free equation, is absent. No quantum of the field can be found in a free state. The propagator vanishes for space-like distances. The field propagation takes place inside the light-cone. This is true for bradyons, but is also true for fields that obey the Klein-Gordon equations with the wrong sign of the mass term, and even for complex mass fields. The convolution of two Wheeler functions gives another Wheeler function. In p-space, this convolution coincides for $P < 0$, with the convolution of the two on-shell δ-functions, in spite of the fact that each of the latter only contains free propagation, while each of the former only contains virtual propagation. The usual vacuum state is annihilated by the lowering operator a, and gives rise to the Feynman propagator. The Wheeler Green function is associated to the symmetric vacuum. This vacuum is not annihilated by a, but rather by the "true-energy" operator, a symmetric combination of annihilation and creation operators. The space of states generated by a and a^+ has an indefinite metric. There are known methods to deal with this kind of space. In particular we can define and handle all scalar products by means of the "holomorphic representation" [24]. Due to the absence of asymptotic free waves, no Wheeler particle will appear in external legs of the Feynman's diagrams. Only that propagator associated to internal lines will explicitly appear. Thus, the theoretical tools to deal with matrix elements in spaces with indefinite metric will not in fact be necessary for the evaluation of cross-sections. However, the decomposition of unity for spaces with indefinite metric is needed for the proof of another important issue, as seen above. The inclusion of Wheeler fields and the corresponding Wheeler propagators does not produce any violation of unitarity, if only normal particles are found in external legs of Feynman diagrams. To complete the theoretical framework for a rigorous mathematical analysis, it is perhaps convenient to notice that the propagators we have defined are continuous linear functionals on the space of the entire analytic functions rapidly decreasing on the real axis. They are known in general as "tempered ultradistributions" [9, 10, 25, 26]. The Fourier transformed space contains the usual distributions and also admits exponentially increasing functions (distributions of the exponential type), as can be seen in

Ref. [27]. We must also answer an important question: what are the possible uses of the Wheeler propagators?

In the first place we would like to stress the fact that the quanta of Wheeler fields cannot appear as free particles. They can only be detected as virtual mediators of interactions. It is in the light of this observation that we must look for probable applications. We will first take the case of a tachyon field. It is known that unitarity cnot be achieved, provided we accept the implicit premise that only Feynman's propagators are to be used, with the consequent presence of free tachyons. This can also be considered to be a proof of the incompatibility of unitarity and Feynman's propagator for tachyons. To this observation we add the fact that if the propagator is a Wheeler function, a tachyon cannot propagate freely. Consequently, we are led to the acceptance of the complete spectrum $\vec{p}^{\,2} < \mu^2$ and $\vec{p}^{\,2} \geq \mu^2$, with the caveat that the real exponential functions one gets for $\vec{p}^{\,2} < \mu^2$ are not eigenfunctions of the Hamiltonian (Ref. [22, 28]). Furthermore, this procedure fits naturally into the treatment for complex mass fields of section 5. To this case one could relate the Higgs boson problems. The scalar field of the standard model behaves as a tachyon field for low amplitudes. The fact that the Higgs has not yet been unambiguously observed without controversy, suggests the possibility that the corresponding propagator might be a Wheeler function [29]. It is easy to see that this assumption does not spoil any of the experimental confirmations of the model (on the contrary, it explains the non observation of the free Higgs boson).

Another possible application emerges in higher order equations. Those equations appear for example in some supersymmetric models for higher dimensional spaces [30]. They can be decomposed into Klein-Gordon factors with general mass parameters. The corresponding fields have Wheeler functions as propagators. It is interesting that there are models of higher order equations, coupled to electromagnetism, which can be shown to be unitary, no matter how high the order may be [31]. The acceptance of tachyons as Wheeler particles might be of interest for boson string theory. Using the symmetric vacuum we can show that the Virasoro algebra turns out to be free of anomalies in spaces of arbitrary number of dimensions [32]. The excitations of the string are Wheeler functions in this case.

Chapter 7

Convolution of ultradistributions

It is sometimes necessary to work with functions that grow exponentially in space or time. For those cases Schwartz' space of tempered distributions (see [5, 6]) is too restrictive. On the other hand, the space of test functions with bounded support allows the distributions to blow-up more rapidly than any exponential. In this sense, they should be considered to be too "permissive" for physical applications. What is needed is an equilibrium between the necessities in x-space and the possibility to work in the Fourier transformed space (p-space) with propagators. The latter are, from a mathematical point of view, analytic functionals defined on a space of entire test functions.

We shall see that the equilibrium sought above is achieved by working with tempered ultradistributions (see below). They also have the advantage of being representable by means of analytic functions, so that, in general, they are easy to work with and have interesting properties. One of these properties (as we shall see) is the possibility of defining a convolution product (CP). This possibility suffices in general for the CP to be valid for any two tempered ultradistributions, and of course, this automatically provides a definition for the product of distributions of exponential type in x-space.

7.1 The convolution product

If we try to define the *convolution product* by means of the naive relation

$$(\mathsf{F} * \mathsf{G})\{\phi\} = \oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} dk_1\ dk_2\ \mathsf{F}(k_1)\mathsf{G}(k_2)\phi(k_1 + k_2), \tag{7.1.0.1}$$

we will soon discover that it is not always defined. The reason is simple. The result of

$$\oint\limits_{\Gamma} dk\ \mathsf{F}(k)\phi(k + k^{'}) = \chi(k^{'}),$$

does not, in general, belong to Hilbert's space. However, if at least one of the ultradistributions F and G is rapidly decreasing (say G), then a convolution can be defined (see Ref. [10]) by

$$\mathsf{H}(k) = \int\limits_{-\infty}^{\infty} dt\ f(t)\mathsf{G}(k - t), \tag{7.1.0.2}$$

where $f(t)$ is the density associated to $\mathsf{F}(k)$ (Cf. (4.2.0.7), Dirac's formula for F). In order to eliminate the test function from (7.1.0.1) use can be made of the complex δ-function, which is an ultradistribution (Cauchy's theorem)

$$\delta_{z^{'}}\{\phi\} = -\frac{1}{2\pi i}\oint\limits_{\Gamma} dz\ \frac{\phi(z)}{z - z^{'}} = \phi(z^{'}), \tag{7.1.0.3}$$

where the point $z^{'}$ is enclosed by Γ. (This procedure was previously used in Ref. [8]). We can then recast (7.1.0.1) as

$$(\mathsf{F} * \mathsf{G})\{\phi\} = -\frac{1}{2\pi i}\oint\limits_{\Gamma} dz \oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} dk_1\ dk_2\ \frac{\mathsf{F}(k_1)\mathsf{G}(k_2)}{z - k_1 - k_2}\phi(z). \tag{7.1.0.4}$$

The path Γ must have

$$|\mathrm{Im}(z)| > |\mathrm{Im}(k_1)| + |\mathrm{Im}(k_2)|, \tag{7.1.0.5}$$

in order to embrace the point $k_1 + k_2$, ($k_1 \in \Gamma_1$, $k_2 \in \Gamma_2$). Equation (7.1.0.4) leads to

$$F * G = H = -\frac{1}{2\pi i} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} dk_1 \; dk_2 \; \frac{F(k_1)G(k_2)}{z - k_1 - k_2}. \qquad (7.1.0.6)$$

However, we do not expect (7.1.0.6) to define a tempered ultradistribution for every pair F, G. Note that in (7.1.0.1) F and G operate on $\phi(k)$, which is rapidly decreasing, while in (7.1.0.6) they act on $(z - k)^{-1}$ ($k = k_1 + k_2$). Furthermore, due to (7.1.0.5) and to the fact that Γ_1 and Γ_2 run outside a horizontal band containing all the singularities of F and G, the integrand in (7.1.0.6) is analytic at every point of the integration paths. Taking into account the property (4.2.0.9) of tempered ultradistributions, we come to the conclusion that the integrations in (7.1.0.6) have at most, a tempered singularity for $k \to \infty$. We define (see [33])

$$H_\lambda(z) = \frac{i}{2\pi} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} dk_1 \; dk_2 \; \frac{k_1^\lambda F(k_1) k_2^\lambda G(k_2)}{z - k_1 - k_2}. \qquad (7.1.0.7)$$

Now, if we have the bounds

$$|F(k_1)| \leq C_1 |k_1|^m \quad ; \quad |G(k_2)| \leq C_2 |k_2|^n, \qquad (7.1.0.8)$$

then (7.1.0.7) is convergent for

$$Re(\lambda) < -l - 1 \;\; ; \;\; l = max\{m, n\}, \qquad (7.1.0.9)$$

being also analytic in the region (7.1.0.9) of the λ plane. The derivative with respect to λ merely multiplies by a logarithmic factor the integrand of (7.1.0.7), without spoiling the convergence. According to the method of Ref. [34], H_λ can be analytically continued to other parts of the λ plane. In particular, near the origin we have the *Laurent* (or *Taylor*) expansion

$$H_\lambda = \sum_n H^{(n)}(z)\lambda^n, \qquad (7.1.0.10)$$

where the sum might have terms with negative n. We now define the convolution product as the λ-independent term of (7.1.0.10)

$$H(z) = H^{(0)}(z). \qquad (7.1.0.11)$$

Note that the derivatives of $H_\lambda(z)$ with respect to z can be obtained, from (7.1.0.7), by taking different powers of the denominator

$$\frac{d^p H_\lambda(z)}{dz^p} = (-1)^p p! \frac{i}{2\pi} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} dk_1 \; dk_2 \; \frac{k_1^\wedge F(k_1) k_2^\wedge G(k_2)}{(z - k_1 - k_2)^p}. \quad (7.1.0.12)$$

The convergence of (7.1.0.7) also guarantees that of (7.1.0.12). Therefore, also ensures analytic behavior in z outside the horizontal band defined by (7.1.0.5). We will now show that $|H_\lambda(z)|$ is bounded by a power of $|z|$ (Cf. (4.2.0.9)). To this end we take

$$\mathrm{Im}(\lambda) = 0 \;\; ; \;\; \lambda < -l - 1 \;\; ; \;\; z = x + iy$$

$$k_i = \kappa_i \pm i\sigma_i \;\; ; \;\; \sigma_i > 0 \;\; ; \;\; dk_i = d\kappa_i.$$

The integrals along Γ_i can be expressed as integrals on $d\kappa_i$ between $0 \to \infty$. Thus, we have

$$|H_\lambda| = \frac{1}{2\pi} \left| \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} dk_1 \; dk_2 \; \frac{k_1^\lambda F(k_1) k_2^\lambda G(k_2)}{z - k_1 - k_2} \right| \leq$$

$$\frac{1}{2\pi} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} \mathrm{sgn}\, \mathrm{Im}(k_1) \; dk_1 \; \mathrm{sgn}\, \mathrm{Im}(k_2) \; dk_2 \times$$

$$\frac{|k_1|^\lambda C_1 |k_1|^m |k_2|^\lambda C_2 |k_2|^n}{|z - k_1 - k_2|} \leq$$

$$\frac{C_1 C_2}{2\pi} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} \mathrm{sgn}\, \mathrm{Im}(k_1) \; dk_1 \; \mathrm{sgn}\, \mathrm{Im}(k_2) \; dk_2 |k_1|^{\lambda+m} |k_2|^{\lambda+n} =$$

$$\frac{8 C_1 C_2}{\pi} \int\limits_0^\infty \int\limits_0^\infty d\kappa_1 \; d\kappa_2 \left(\kappa_1^2 + \sigma_1^2\right)^{\frac{\lambda+m}{2}} \left(\kappa_2^2 + \sigma_2^2\right)^{\frac{\lambda+n}{2}}. \quad (7.1.0.13)$$

We perform now the change of variables $w_i = \kappa_i^2$ and obtain

$$(7.1.0.13) = \frac{2 C_1 C_2}{\pi} \int\limits_0^\infty dw_1 \; w_1^{-\frac{1}{2}} \left(w_1 + \sigma_1^2\right)^{\frac{\lambda+m}{2}} \times$$

$$\int\limits_0^\infty dw_2 \; w_2^{-\frac{1}{2}} \left(w_2 + \sigma_2^2\right)^{\frac{\lambda+m}{2}} = \quad (7.1.0.14)$$

$$\frac{2C_1C_2}{\pi}\mathcal{B}\left(\frac{1}{2},-\frac{\lambda+m+1}{2}\right)\mathcal{B}\left(\frac{1}{2},-\frac{\lambda+n+1}{2}\right)\sigma_1^{\frac{\lambda+m+1}{2}}\sigma_2^{\frac{\lambda+n+1}{2}} \leq$$
$$C(\lambda,m,n)|z|^{\lambda+m+n+1}, \tag{7.1.0.15}$$

where $\mathcal{B}(x,y)$ is a Gauss' beta function. It is to be noted that if $G(k)$ is a *rapidly decreasing* ultradistribution, then $H_\lambda(z)$ (Eq. (7.1.0.7)) coincides with $H_0(z)$

$$H_0(z) = \frac{i}{2\pi}\oint\limits_{\Gamma_1} dk_1\ F(k_1)\oint\limits_{\Gamma_2} dk_2\ \frac{G(k_2)}{z-k_1-k_2}. \tag{7.1.0.16}$$

In fact, near $\lambda = 0$ we have ($|k| > 1$)

$$|k^\lambda - 1| \leq \lambda(2\pi + |\ln|k||)|k|^\lambda, \tag{7.1.0.17}$$

$$H_\lambda - H_0(z) = \frac{i}{2\pi}\oint\limits_{\Gamma_1} dk_1\ k_1^\lambda F(k_1)\oint\limits_{\Gamma_2} dk_2\ (k_2^\lambda - 1)\frac{G(k_2)}{z-k_1-k_2} +$$
$$\frac{i}{2\pi}\oint\limits_{\Gamma_1} dk_1\ (k_1^\lambda - 1)F(k_1)\oint\limits_{\Gamma_2} dk_2\ \frac{G(k_2)}{z-k_1-k_2}. \tag{7.1.0.18}$$

In Eq. (7.1.0.18) the integrals are convergent, as $G(k)$ and $k^\lambda G(k)$ are both rapidly decreasing. Furthermore, due to (7.1.0.17), the difference $H_\lambda - H_0$ is proportional to λ, entailing

$$\lim_{\lambda\to 0}\ [H_\lambda - H_0] = 0. \tag{7.1.0.19}$$

Again, when $G(k)$ is rapidly decreasing, the convolution defined in Ref. [10] reads

$$H(z) = \int\limits_{-\infty}^{\infty} dt\ f(t)G(z-t), \tag{7.1.0.20}$$

(where $f(t)$ is given by (4.2.0.7), (4.2.0.8), and also coincides with (7.1.0.16). To show that (7.1.0.16) implies (7.1.0.20), we use (4.2.0.8) in (7.1.0.16)

$$H_0(z) = \frac{i}{2\pi}\int\limits_{-\infty}^{\infty} dt\ f(t)\oint\limits_{\Gamma_2} dk_2\ \frac{G(k_2)}{z-k_1-k_2}.$$

However, if $G(t)$ is the density associated to $G(z)$, then

$$\frac{i}{2\pi}\oint_{\Gamma_2} dk_2\ \frac{G(k_2)}{z-t-k_2} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dt_2\ \frac{g(t_2)}{t_2-(z-t)} = G(z-t),$$

i.e.,

$$H_0(z) = H(z). \tag{7.1.0.21}$$

7.1.1 Examples

Here we are going to use definition (7.1.0.7) to evaluate the convolution of tempered ultradistributions and, indirectly, product of distributions $\in \Lambda_\infty$. The convolution theorem tells us that

$$\mathcal{F}\{f_1(x)f_2(x)\} = \frac{1}{2\pi}\check{f}_1(k) * \check{f}_2(k), \tag{7.1.1.22}$$

where

$$\check{f} = \mathcal{F}\{f(x)\}(k).$$

I) As a first example, we shall consider the distribution $x_\pm^\alpha$ (Ref. [6], ch.1, §3.2, also Ref. [36], ch.4) whose Fourier transform is

$$\check{x}_\pm^\alpha = ie^{\mp i\frac{1}{2}\alpha}\Gamma(\alpha+1)k^{-\alpha-1}\Theta[\mp\epsilon(k)], \tag{7.1.1.23}$$

where $\Theta(x)$ is Heaviside's step function and $\epsilon(k) = \mathrm{sgn}\,\mathrm{Im}(k)$. The ultradistribution (7.1.1.23) has a line of singularities (a discontinuity) on the real axis. Accordingly, the path Γ (Cf. (4.2.0.6)) runs parallel to the real axis at a distance as small as we please.

$$\mathcal{F}\left\{x_+^\alpha x_+^\beta\right\} = \frac{i}{4\pi^2}\oint_{\Gamma_1} dk_1 \oint_{\Gamma_2} dk_2\ \frac{\check{x}_+^\alpha \check{x}_+^\beta}{z-k_1-k_2} =$$

$$\left[\frac{i}{4\pi^2} ie^{-i\frac{\pi}{2}\alpha}\Gamma(\alpha+1)ie^{-i\frac{\pi}{2}\beta}\Gamma(\beta+1)\right] \times$$

$$\oint_{\Gamma_1} dk_1 k_1^{-\alpha-1}\Theta[-\epsilon(k_1)]\oint_{\Gamma_2} dk_2\frac{k_2^{-\beta-1}\Theta[-\epsilon(k_2)]}{z-k_1-k_2}.$$

The functions $\Theta[\epsilon(k_1)]$ and $\Theta[\epsilon(k_2)]$ eliminate the branches of Γ_1 and Γ_2 (respectively) on the lower half plane of k_1 and k_2. By taking the remaining integration arbitrarily close to the real axis we get

$$\mathcal{F}\left\{x_+^\alpha x_+^\beta\right\} = -[\,]\oint_{\Gamma_1} dk_1 k_1^{-\alpha-1}\Theta[-\epsilon(k_1)]\int_{-\infty}^{\infty} dy\ \frac{(y-i0)^{-\beta-1}}{z-k_1-y} =$$

$$-[\,]\oint_{\Gamma_1} dk_1 k_1^{-\alpha-1}\Theta[-\epsilon(k_1)]\int_{-\infty}^{\infty} dy\ \frac{y_+^{-\beta-1}+e^{-i\pi(-\beta-1)}y_-^{-\beta-1}}{z-k_1-y} =$$

$$-[\,]\oint_{\Gamma_1} dk_1 k_1^{-\alpha-1}\Theta[-\epsilon(k_1)]\frac{\Gamma(-\beta)\Gamma(1+\beta)}{(z-k_1)^{\beta+1}}\ \times$$

$$\left[e^{-i\pi(-\beta-1)} - e^{-i\pi\epsilon(z)(-\beta-1)}\right] =$$

$$2i[\,]\Theta[-\epsilon(z)]\Gamma(-\beta)\Gamma(1+\beta)\sin\pi(-\beta-1)\ \times$$

$$\oint_{\Gamma_1} dk_1\ \frac{k_1^{-\alpha-1}}{(z-k_1)^{\beta+1}}\Theta[-\epsilon(k_1)] =$$

$$2i\pi\Theta[-\epsilon(z)][\,]\int_{-\infty}^{\infty} dx\ \frac{x_+^{-\alpha-1}+e^{-i\pi(-\alpha-1)}x_-^{-\alpha-1}}{(z-x)^{\beta+1}} =$$

$$2i\pi\Theta[-\epsilon(z)][\,]\mathcal{B}(-\alpha,\beta+\alpha+1)\left[e^{i\pi\epsilon(z)\alpha}-e^{i\pi\alpha}\right]z^{-\alpha-\beta-1} =$$

$$2i\pi\{\Theta[-\epsilon(z)]\}^2[\,]\frac{\Gamma(-\alpha)\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)}2i\sin\pi(-\alpha)z^{-\alpha-\beta-1} =$$

$$ie^{-i\frac{\pi}{2}(\alpha+\beta)}\Gamma(\alpha+\beta+1)z^{-\alpha-\beta-1}\Theta[-\epsilon(z)] =$$

$$\check{x}_+^{\alpha+\beta} = \mathcal{F}\left\{x_+^{\alpha+\beta}\right\} = \mathcal{F}\left\{x_+^\alpha x_+^\beta\right\}, \tag{7.1.1.24}$$

where use has been made of the relation

$$\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\pi}{\sin(\pi\lambda)}.$$

For the evaluation of the convolution $\check{x}_+^\alpha * \check{x}_-^\beta$ the procedure is entirely similar. However, in this case one of the integrations gives rise to a factor $\Theta[-\epsilon(z)]$ and the other to a factor $\Theta[\epsilon(z)]$. So that, instead of $\{\Theta[-\epsilon(z)]\}^2 = \Theta[-\epsilon(z)]$, we get $\Theta[-\epsilon(z)]\Theta[\epsilon(z)] = 0$. This entails

$$\check{x}_+^\alpha * \check{x}_-^\beta \equiv 0 \quad \therefore \quad x_+^\alpha \cdot x_-^\beta = 0. \tag{7.1.1.25}$$

II) As a second example we consider Dirac's δ-functions, whose Fourier transform is

$$\check{\delta}^{(m)} = i^m k^m \frac{\epsilon(k)}{2}. \tag{7.1.1.26}$$

For the convolution (7.1.0.7) we have

$$\check{\delta}^{(m)} * \check{\delta}^{(n)} = \frac{i}{4\pi} \oint\limits_{\Gamma_1} dk_1 \; i^m k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \oint\limits_{\Gamma_2} dk_2 \; \frac{i^n k_2^{\lambda+n} \epsilon(k_2)}{z - k_1 - k_2}.$$

In this case, the factors ϵ_1 and ϵ_2 change the sign of the integrations in the lower half plane of k_1 and k_2

$$\frac{i^{m+n+1}}{4\pi} \oint\limits_{\Gamma_1} dk_1 \; k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \int\limits_{-\infty}^{\infty} dy \; \frac{(y+i0)^{\lambda+n} + (y-i0)^{\lambda+n}}{z - k_1 - y} =$$

$$\frac{i^{m+n+1}}{2\pi} \oint\limits_{\Gamma_1} dk_1 \; k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \int\limits_{-\infty}^{\infty} dy \; \frac{y_+^{\lambda+n} + \cos\pi(\lambda+n) y_-^{\lambda+n}}{z - k_1 - y} =$$

$$\frac{i^{m+n+1}}{2\pi} \oint\limits_{\Gamma_1} dk_1 \; k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \frac{\Gamma(\lambda+n+1)\Gamma(-\lambda-n)}{z - k_1} \times$$

$$\left[\cos\pi(\lambda+n) - e^{-i\pi\epsilon(z)(\lambda+n)}\right] =$$

$$-\frac{i\pi\epsilon(z)}{2\pi} i^{m+n+1} \int\limits_{-\infty}^{\infty} dx \; \frac{x_+^{\lambda+m} + \cos\pi(\lambda+m) x_-^{\lambda+m}}{(z-x)^{-\lambda-n}} =$$

$$\frac{\epsilon(z)}{2} i^{m+n} \frac{\Gamma(\lambda+m+1)\Gamma(-2\lambda-m-n-1)}{\Gamma(-\lambda-n)} z^{2\lambda+m+n+1} \times$$

$$\left[e^{-i\pi\epsilon(z)(\lambda+m+1})+\cos\pi(\lambda+m)\right] =$$

$$\frac{[\epsilon(z)]^2}{2} i^{m+n+1} \frac{\Gamma(\lambda+m+1)\Gamma(-2\lambda-m-n-1)}{\Gamma(-\lambda-n)} \times$$

$$\sin\pi(\lambda+m)z^{2\lambda+m+n+1} =$$
$$_{\lambda\to 0} \longrightarrow 0 = \check{\delta}^{(m)} * \check{\delta}^{(n)}. \tag{7.1.1.27}$$

There are two reasons for this null result. The Γ functions have simple poles when their arguments are negative integers (or zero). Thus, the ratio of Γ functions has a finite limit. However, they are multiplied by $\sin\pi(\lambda+m)_{\lambda\to 0} \longrightarrow 0$. Furthermore, $[\epsilon(z)]^2 = 1$, and

$$z^{2\lambda+m+n+1} \quad _{\lambda\to 0}\longrightarrow \quad z^{m+n+1}$$

Thus, we can set (C = arbitrary constant)

$$\check{\delta}^{(m)} * \check{\delta}^{(n)} = Cz^{m+n+1}. \tag{7.1.1.28}$$

However, due to the property C, §3.2, the utradistribution (7.1.1.28) is equivalent to zero. We have then

$$\delta^{(m)}(x)\cdot\delta^{(n)}(x) = 0. \tag{7.1.1.29}$$

This result was previously obtained in Ref. [35] and can be summarized in general as by the statement "the product of two distributions with point support is zero".

III) We can combine examples I and II, to find the product $\delta^{(m)}\cdot \check{x}_+^{\alpha}$.

$$\frac{1}{2\pi}\check{\delta}^{(m)} * \check{x}_+^{\alpha} = \left[\frac{i}{4\pi^2} i^m i e^{-i\frac{\pi}{2}\alpha}\Gamma(\alpha+1)\right] \times$$

$$\oint\limits_{\Gamma_1} dk_1\; k_1^{\lambda+m}\frac{\epsilon(k_1)}{2}\oint\limits_{\Gamma_2} dk_2\;\frac{k_2^{-\alpha-1}\Theta[-\epsilon(k_2)]}{z-k_1-k_2} =$$

$$2\pi i\Theta[-\epsilon(z)][\,]\int\limits_{-\infty}^{\infty} dx\;\frac{x_+^{\lambda+m}+\cos\pi(\lambda+m)x_-^{\lambda+m}}{(z-x)^{\alpha+1}} =$$

$$2\pi i\Theta[-\epsilon(z)][\,]\frac{\Gamma(\lambda+m+1)\Gamma(\alpha-\lambda-m)}{\Gamma(\alpha+1)}z^{\lambda+m-\alpha}\times$$

$$\left[e^{-i\pi\epsilon(z)(\lambda+m+1)}+\cos\pi(\lambda+m)\right] =$$

$$2\pi i\Theta[-\epsilon(z)]\left(-\frac{i^m}{4\pi^2}e^{-i\frac{\pi}{2}\alpha}\right)\Gamma(\lambda+m+1)\Gamma(\alpha-\lambda-m)\times$$

$$i \sin \pi(\lambda + m)\ \epsilon(z) z^{\lambda+m-\alpha} \longrightarrow 0_{\lambda\to 0}\ , \tag{7.1.1.30}$$

if α is not an integer s such that $s \leq m$. When $0 \leq \alpha = s \leq m$ one has

$$\frac{1}{2\pi}\breve{\delta}^{(m)} * \breve{x}_+^s = -2i\pi\Theta[-\epsilon(z)]\frac{i^m}{4\pi^2}(-i)^s i\epsilon(z)z^{\lambda+m-s}\ \times$$

$$\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+m+1-s)}\frac{\sin\pi(\lambda+m)}{\sin\pi(\lambda+m-s)} =$$

$$\frac{i^m}{2}(-i)^s\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+m+1-s)}\frac{\sin\pi(\lambda+m)}{\sin\pi(\lambda+m-s)}\Theta[-\epsilon(z)]\epsilon(z)z^{\lambda+m-s}$$

$$_{\lambda\to 0}\longrightarrow\quad (-1)^s\frac{i^{m-s}}{2}\frac{m!}{(m-s)!}\frac{\epsilon(z)}{2}z^{m-s} =$$

$$\frac{(-1)^s}{2}\frac{m!}{(m-s)!}\breve{\delta}^{(m-s)}. \tag{7.1.1.31}$$

In particular, for $s = 0$ we get

$$\delta^{(m)}(x)\Theta(x) = \frac{1}{2}\delta^{(m)}(x). \tag{7.1.1.32}$$

If $\alpha = s$=negative number=$-n$ we must be careful as x_+^α has a pole for $\alpha = -n$. We shall deal with this case by the replacement $\alpha = -n - \lambda$ in (7.1.1.30). One has

$$\Gamma(\alpha-\lambda-m)\longrightarrow\Gamma(-2\lambda-m-n) =$$

$$-\frac{\pi}{\Gamma(2\lambda+m+n+1)\sin\pi(2\lambda+m+n)},$$

and, when taking the limit $\lambda \to 0$,

$$\frac{1}{2\pi}\breve{\delta}^{(m)} * x_+^{-n} = \frac{i^{m+n}}{2}\frac{m!}{(m+n)!}\frac{(-1)^n}{2}\frac{\epsilon(z)}{2}z^{m+n} =$$

$$\frac{(-1)^n}{4}\frac{m!}{(m+n)!}\breve{\delta}^{(m+n)}. \tag{7.1.1.33}$$

In Eqs. (7.1.1.31) and (7.1.1.33) we have used

$$\Theta[-\epsilon(z)]\epsilon(z) = -\Theta[-\epsilon(z)] = \frac{1}{2}(\epsilon(z)-1) = \frac{\epsilon(z)}{2}-\frac{1}{2}$$

$$\Theta[-\epsilon(z)]\epsilon(z)z^s = \frac{\epsilon(z)}{2}z^s - \frac{1}{2}z^s \approx \frac{\epsilon(z)}{2}z^s,$$

since Cz^s is equivalent to zero (Cf. (7.1.1.28)).

There are also similar expressions which originate in the use of $\check{x}_-^{\alpha}$ in (7.1.1.30). In particular, if we employ

$$\check{x}^{-n} = \check{x}_+^{-n} + (-1)^n \check{x}_-^{-n}, \tag{7.1.1.34}$$

then we easily find

$$\frac{1}{2\pi}\check{\delta}^{(m)} * \check{x}^{-n} = \frac{(-1)^n}{2}\frac{m!}{(m+n)!}\check{\delta}^{(m+n)}. \tag{7.1.1.35}$$

The case $m = 0$, $n = 1$, was first published in Ref. [37]. For $m = n$ and Eq. (7.1.1.35), we use Ref.[38].

IV). To illustrate the use of (7.1.0.10) and (7.1.0.11), we are now going to examine an interesting example. Let us take the ultradistribution (7.1.1.34), which is found to be

$$\check{x}^{-n} = \frac{(-i)^n\pi}{(n-1)!}\left[-\frac{1}{\pi i}\ln(k) + \frac{\epsilon(k)}{2}\right]k^{n-1}. \tag{7.1.1.36}$$

The convolution product is now

$$\check{x}^{-m} * \check{x}^{-n} = -\frac{(-i)^{m+n+1}}{4(m-1)!(n-1)!}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} dk_1\ dk_2$$

$$\left\{-\frac{1}{\pi^2}\frac{k_1^{\lambda+m-1}\ln(k_1)k_2^{\lambda+n-1}\ln(k_2)}{z-k_1-k_2}\right.$$

$$-\frac{1}{2\pi i}\frac{k_1^{\lambda+m-1}\ln(k_1)k_2^{\lambda+n-1}\epsilon(k_2)}{z-k_1-k_2} +$$

$$\left.-\frac{1}{2\pi i}\frac{k_1^{\lambda+m-1}\epsilon(k_1)k_2^{\lambda+n-1}\ln(k_2)}{z-k_1-k_2} + \frac{1}{4}\frac{k_1^{\lambda+m-1}\epsilon(k_1)k_2^{\lambda+n-1}\epsilon(k_2)}{z-k_1-k_2}\right\}. \tag{7.1.1.37}$$

The last term of (7.1.1.37) is null according to example II). We analyze now the first term. We shall use the identity

$$k^{\lambda+m-1}\ln(k) = D_\alpha k^{\alpha+m-1} \quad ; \quad D_\alpha = \left.\frac{\partial}{\partial\alpha}\right|_{\alpha=\lambda},$$

that yields

$$\frac{i}{4\pi^2}\frac{(-i)^{m+n}}{(m-1)!(n-1)!}\times$$

$$\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} dk_1\; dk_2\; \frac{k_1^{\lambda+m-1}\ln(k_1)k_2^{\lambda+n-1}\ln(k_2)}{z-k_1-k_2} =$$

$$\left[\frac{i}{4\pi^2}\frac{(-i)^{m+n}}{(m-1)!(n-1)!}\right]\oint\limits_{\Gamma_1} dk_1\; D_\alpha k^{\alpha+m-1}\times$$

$$\oint\limits_{\Gamma_2} dk_2\; \frac{D_\beta k_2^{\beta+n-1}\ln(k_2)}{z-k_1-k_2} =$$

$$[\,]D_\alpha D_\beta \oint\limits_{\Gamma_1} dk_1\; \frac{k_1^{\alpha+m-1}}{(z-k_1)^{1-\beta-n}}\times$$

$$2i\sin\pi(\beta+n-1)\Gamma(\beta+n)\Gamma(1-\beta-n) =$$

$$2\pi i[\,]D_\alpha D_\beta \oint\limits_{\Gamma_1} dk_1\; \frac{k_1^{\alpha+m-1}}{(z-k_1)^{1-\beta-n}} =$$

$$2\pi i[\,]D_\alpha D_\beta \frac{\Gamma(\alpha+m)\Gamma(1-\alpha-m-\beta-n)}{\Gamma(1-\beta-n}$$

$$2i\sin\pi(\alpha+m-1)\; z^{\alpha+\beta+m+n-1} =$$

$$4\pi[\,]D_\alpha D_\beta \frac{\Gamma(\alpha+m)\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n+m}\frac{\sin\pi\alpha\;\sin\pi\beta}{\sin\pi(\alpha+\beta)}z^{\alpha+\beta+m+n-1} =$$

$$-\frac{1}{\pi}\frac{(-i)^{m+n-1}}{(m+n-1)!}D_\alpha D_\beta\left\{\frac{\sin\pi\alpha\;\sin\pi\beta}{\sin\pi(\alpha+\beta)}z^{\alpha+\beta+m+n-1}\right\}, \quad (7.1.1.38)$$

where we have used the fact that any derivative, D_α or D_β, acting on a Γ function will lead to a null result for (7.1.1.38) through the substitutions $\alpha = \lambda$, $\beta = \lambda$, and $\lambda \to 0$. Now, the derivatives in (7.1.1.38) give rise essentially to two types of terms. The two derivatives acting on the *trigonometric functions* give rise to a pole term (in λ). If one takes a derivative of the trigonometric functions and a derivative of $z^{\alpha+\beta}$, a constant term is obtained. For the term $D_\alpha D_\beta z^{\alpha+\beta}$, the limit $\lambda \to 0$ of the trigonometric functions is zero. Thus, we get

$$= -\frac{(-i)^{m+n-1}}{(m+n-1)!}z^{m+n-1}\left\{\frac{1}{4}\frac{1}{\lambda}z^{2\lambda}+\frac{1}{2}\ln(z)\right\}.$$

The second and third terms of (7.1.1.38) give the same contribution, that can be evaluated by a similar procedure. This contribution is

$$\frac{1}{8\pi}\frac{(-i)^{m+n-2}}{(m-1)!(n-1)!}\oint_{\Gamma_1}\oint_{\Gamma_2} dk_1\ dk_2 \frac{k_1^{\lambda+m-1}\ln(k_1)k_2^{\lambda+n-1}\epsilon(k_2)}{z-k_1-k_2} =$$

$$\frac{(-i)^{m+n}}{(m+n-1)!}\frac{\pi}{4}\epsilon(z)\ z^{m+n-1}. \tag{7.1.1.39}$$

According, we finally get

$$(6.1.37) = \frac{(-i)^{m+n}}{(m+n-1)!}z^{m+n-1}\left\{\frac{i}{4}\frac{1}{\lambda}z^{2\lambda}+\frac{i}{2}\ln(z)+\frac{\pi}{2}\epsilon(z)\right\} =$$

$$\frac{(-i)^{m+n}}{(m+n-1)!}z^{m+n-1}\left\{\frac{i}{4}\frac{1}{\lambda}(1+2\lambda\ln(z))+\frac{i}{2}\ln(z)+\frac{\pi}{2}\epsilon(z)\right\} =$$

$$\frac{(-i)^{m+n}\pi}{(m+n-1)!}z^{m+n-1}\left\{-\frac{1}{\pi i}\ln(z)+\frac{1}{2}\epsilon(z)\right\}. \tag{7.1.1.40}$$

The λ-independent term is recognized to be $\check{x}^{-m-n}$ (Cf. 5.15). The pole term is equivalent to zero, according to §3.2 III.

V). Finally, we give a physical example. We consider a massless scalar $\frac{\lambda}{4!}\phi^4(x)$ theory in four dimensions. For this theory we shall evaluate the *self-energy* Green function. The propagator for the field $\phi(x)$ is

$$\Delta(x) = [-4\pi^2(u^2 - i0)]^{-1}. \tag{7.1.1.41}$$

From ref.[36] we have

$$\delta^{(m)}(u^2) = \delta^{(m)}(x^0+r)(x^0-r)^{-m-1}\mathrm{sgn}(x^0-r)+$$

$$\delta^{(m)}(x^0-r)(x^0+r)^{-m-1}\mathrm{sgn}(x^0+r),$$

where

$$u^2 = x_0^2 - x_1^2 - \cdots - x_{n-1}^2$$

$$r^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2$$

$$(u^2 \pm i0)^{-m} = u^{-2m} \pm \frac{(-1)^m}{(m-1)!} i\pi\delta^{(m-1)}(u^2)$$

$$x^{-m}\mathrm{sgn}(x) = \frac{(-1)^{m-1}}{(m-1)!}\{|x|^{-1}\}^{(m-1)},$$

—

$$x|^{-1} = \{sgn(x) \ln |x|\}' + C\delta(x)$$

with C an arbitrary constant. According to these equations, we can write

$$(u^2 - i0)^{-1} = (2x_0)^{-1} \left[(x_0 - r)^{-1} + (x_0 + r)^{-1}\right] +$$

$$(2r)^{-1} [\delta(x_0 - r) + \delta(x_0 + r)] + C\delta(x_0 - r)\delta(x_0 + r), \qquad (7.1.1.42)$$

(where C is an arbitrary constant appearing in the definition of some distributions (see Ref. citetp9, 8.8, 8.9). Using the results of I) to IV) it is easy show that

$$(u^2 - i0)^{-1}(u^2 - i0)^{-1} = (u^2 - i0)^{-2}.$$

Thus, we have for the self-energy

$$\Sigma(x) = (\Delta(x))^2 = \frac{1}{16\pi^4}(u^2 - i0)^{-2}, \qquad (7.1.1.43)$$

where $(u^2 - i0)^{-2}$ is defined in Ref. ([36], 8.8, 8.9).

7.2 Discussion

When we use the perturbative expansion in quantum field theory, we have to deal with products of distributions in configuration space, or else, with convolutions in the Fourier transformed p-space. Unfortunately, products or convolutions (of distributions) are in general ill-defined quantities. However, in physical applications one introduces some "regularization" scheme, which allows us to give sense to divergent integrals. Among these procedures, we would like to mention the dimensional regularization method (Refs. [12]). Essentially, the method consist in the separation of the volume element $(d^\nu p)$ into an angular factor $(d\Omega)$ and a radial factor $(p^{\nu-1}dp)$. First the angular integration is carried out and then the number of dimensions ν is taken as a free parameter, which can later be adjusted so as to yield a convergent integral that is an analytic function of ν. Our formula (7.1.0.7) is similar to the expression one obtains with dimensional regularization. However, the parameter λ is completely independent of any dimensional interpretation. All ultradistributions provide integrands ((7.1.0.7)) that are analytic functions along the integration path. The parameter λ permits us to control the possible tempered asymptotic behavior. The existence of a region of analytic character of λ, and a subsequent continuation to the point of interest (Ref. [6]), defines the convolution product. These properties show that tempered ultradistributions provide an appropriate framework for applications to physics. Furthermore, they can "absorb" arbitrary polynomials, thanks to Eq. (4.2.0.10), a property that is interesting for renormalization theory.

Chapter 8

Even tempered ultradistributions

Geometric quantization is a mathematical approach used to define a quantum theory corresponding to a given classical theory [1]. In another vein, loop integrals appear if one considers the Feynman's diagrams with one (or more) loops by integrating over the internal momenta [1]. As we have stressed above, an important issue in QFT is that of the product of distributions with coincident point singularities, related to the asymptotic behavior of loop integrals of propagators [1].

From a mathematical point of view, practically all definitions of such products lead to limitations on the set of distributions that can be multiplied together to give another distribution of the same kind [9, 10]. The properties of ultradistributions (Refs. [9, 10]) are well adapted for their use in field theory. In this respect we have shown (Ref. [33]) that it is possible to define, in one dimensional spaces, the convolution of any pair of tempered ultradistributions, yielding as a result another tempered ultradistribution. The next step is to consider the convolution of any pair of tempered ultradistribution in n-dimensional space [9, 10]. As we shall see, this follows from the formula obtained in Ref. [33] for one-dimensional spaces. However, the resultant formula is too involved to be used in practical applications and calculus. Thus, for applications, it is convenient to consider the convolution of any two tempered ultradistributions which are even in the variables k^0 y ρ (see section 5).

Ultradistributions also have the advantage of being representable by means of analytic functions. Accordingly, they are easier to work with and have interesting properties. One of those properties is that Schwartz tempered distributions are canonical and continuously injected into tempered ultradistributions.

8.1 Convolutions

The existence of the convolution product between tempered ultradistributions is demonstrated in Ref. [33]. We now define

$$H_\lambda(k) = \frac{i}{(2\pi)^n} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} \frac{k_1^\lambda F(k_1) k_2^\lambda G(k_2)}{k - k_1 - k_2} \, dk_1 \, dk_2, \tag{8.1.0.1}$$

with $(k - k_1 - k_2 = \prod_{i=1}^{n}(k_i - k_{1i} - k_{2i}))$. Let $\mathfrak{b}_i$ be a vertical band contained in the λ_i-plane $\mathfrak{p}_i$. Integral (8.1.0.1) is an *analytic function* of λ defined in a domain $\mathfrak{B}$ given by the *Cartesian product* of vertical bands $\prod \mathfrak{b}_i$ contained in the also Cartesian product $\mathfrak{P} = \prod \mathfrak{p}_i$ of the n λ-planes. Moreover, it is bounded by a power of $|k|$. Then, according to the method of Ref. [33], H_λ can be analytically continued to other parts of $\mathfrak{P}$. In particular, near the origin we have the *Laurent expansion*

$$H_\lambda(k) = \sum_n H^{(n)}(k)\lambda^n. \tag{8.1.0.2}$$

We now define the convolution product as the λ-independent term of (8.1.0.2)

$$H(k) = H^{(0)}(k). \tag{8.1.0.3}$$

The proof that $H^{(0)}(k)$ is a tempered ultradistribution is similar to the one given in Ref. [33] for the one-dimensional case. For an immediate application of (8.1.0.1)- (8.1.0.3), we can evaluate the product of two arbitrary derivatives of an n-dimensional δ distribution. By calculating the convolution product of the Fourier transforms of $\delta^{(m)}(x)$ and $\delta^{(n)}(x)$, and then antitransforming, we can show that

$$\delta^{(m)}(x) \cdot \delta^{(n)}(x) = 0, \tag{8.1.0.4}$$

extending the result obtained in Ref. [33] for the one-dimensional case. Likewise, we can obtain

$$(x_{1+}^{\alpha_1} \; x_{2+}^{\alpha_2} ... x_{n+}^{\alpha_n}) \cdot (x_{1+}^{\beta_1} \; x_{2+}^{\beta_2} ... x_{n+}^{\beta_n}) = (x_{1+}^{\alpha_1+\beta_1} \; x_{2+}^{\alpha_2+\beta_2} ... x_{n+}^{\alpha_n+\beta_n}), \tag{8.1.0.5}$$

generalizing again the result of Ref. [33]. As another example let us consider the product $(x^{-n_1}y^{-m_1})\cdot(x^{-n_2}y^{-m_2})$. We have

$$\mathcal{F}\{(x^{-n_1}y^{-m_1})\cdot(x^{-n_2}y^{-m_2})\} =$$

$$\frac{(-i)^{n_1+n_2}}{(n_1+n_2-1)!}z_1^{n_1+n_2-1}\left[\frac{i}{4}\frac{z_1^{2\lambda_1}}{\lambda_1}+\frac{i}{2}\ln(z_1)+\right.$$

$$\left.\frac{\pi}{2}Sgn[\Im(z_1)]\right]\frac{(-i)^{m_1+m_2}}{(m_1+m_2-1)!}z_2^{m_1+m_2-1}\times$$

$$\left[\frac{i}{4}\frac{z_2^{2\lambda_2}}{\lambda_2}+\frac{i}{2}\ln(z_2)+\frac{\pi}{2}Sgn[\Im(z_2)]\right]=$$

$$\frac{(-i)^{n_1+n_2}}{(n_1+n_2-1)!}z_1^{n_1+n_2-1}\left[\frac{i}{4\lambda_1}\times\right.$$

$$\left.[1+2\lambda_1\ln(z_1)]+\frac{i}{2}\ln(z_1)+\frac{\pi}{2}Sgn[\Im(z_1)]\right]\times$$

$$\frac{(-i)^{m_1+m_2}}{(m_1+m_2-1)!}z_2^{m_1+m_2-1}\times$$

$$\left[\frac{i}{4\lambda_2}[1+2\lambda_2\ln(z_2)]+\frac{i}{2}\ln(z_2)+\frac{\pi}{2}Sgn[\Im(z_2)]\right]. \tag{8.1.0.6}$$

The $(\lambda_1;\lambda_2)$-independent term is

$$\frac{(-i)^{n_1+n_2}\pi}{(n_1+n_2-1)!}z_1^{n_1+n_2-1}\left[\frac{1}{\pi i}\ln(z_1)-\frac{\pi}{2}Sgn[\Im(z_1)]\right]\times$$

$$\frac{(-i)^{m_1+m_2}}{(m_1+m_2-1)!}z_2^{m_1+m_2-1}\left[\frac{1}{\pi i}\ln(z_2)-\frac{\pi}{2}Sgn[\Im(z_2)]\right], \tag{8.1.0.7}$$

recognized to be $\mathcal{F}\{x^{-n_1-n_2}y^{-m_1-m_2}\}$.

8.2 Four dimensional ultradistributions

We pass now to consider the convolution of two even tempered ultradistributions. The Fourier transform of a distribution of exponential type, even in the variables x^0 and $|\vec{x}|$, is by definition an even tempered ultradistribution in the variables k^0 and $\rho = (k_1^2 + k_2^2 + \cdots + k_n^2)^{1/2}$. Taking into account the equality

$$\int\limits_{-\infty}^{+\infty} \hat{f}(x)\hat{\phi}(x)\, dx = \oint\limits_{\Gamma} F(k)\phi(k)\, dk = \int\limits_{-\infty}^{+\infty} f(k)\phi(k)\, dk, \qquad (8.2.0.1)$$

(where $F(k)$ and $f(k)$ are related by (4.2.0.7), we conclude that $f(k)$ is even in k^0 and ρ.

For most practical applications one has to deal with the convolution of two *Lorentz invariant ultradistributions.* They are particular cases of ultradistributions which are even in two relevant variables, one temporal and the other spatial, that we call *even ultradistributions.* Let as now consider $\hat{f} \in \mathbf{H}$ even. Then we can write

$$\hat{f}(x_0, r) = \frac{i}{(2\pi)^3 r} \iint\limits_{-\infty}^{+\infty} f(k_0, \rho) e^{-ik^0x^0} e^{-i\rho r}\, \rho\, d\rho dk^0 \qquad (8.2.0.2)$$

$$f(k_0, \rho) = -\frac{2\pi i}{\rho} \iint\limits_{-\infty}^{+\infty} \hat{f}(x_0, r) e^{ik^0x^0} e^{i\rho r}\, r\, dr dx^0. \qquad (8.2.0.3)$$

Let as now take $\hat{g} \in \mathbf{H}$. Then according to (8.2.0.2)

$$\hat{f}(x)\hat{g}(x) = -\frac{1}{(2\pi)^6 r^2} \iiiint\limits_{-\infty}^{+\infty} f(k_1^0, \rho_1) g(k_2^0, \rho_2) e^{-i(k_1^0 + k_2^0)x^0} e^{-i(\rho_1 + \rho_2)r} \times$$

$$\times\, \rho_1 \rho_2\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0, \qquad (8.2.0.4)$$

and Fourier transforming (8.2.0.4) we have

$$\mathcal{F}\{\hat{f}(x)\hat{g}(x)\}(k) = \frac{i}{(2\pi)^5 \rho} \int\limits_{-\infty}^{+\infty} \cdots \int$$

$$f(k_1^0, \rho_1) g(k_2^0, \rho_2) e^{i(k^0 - k_1^0 - k_2^0)x^0} e^{i(\rho - \rho_1 - \rho_2)r} \times$$

$$\times\, \rho_1 \rho_2\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0\, r^{-1} dr\, dx^0. \qquad (8.2.0.5)$$

Evaluating the integral in the variable x^0 and calling $h(k^0, \rho) = \mathcal{F}\{\hat{f}(x)\hat{g}(x)\}(k)$ in (8.2.0.5), we obtain

$$h(k^0,\rho) = i\int_{-\infty}^{+\infty}\cdots\int f(k_1^0,\rho_1)g(k_2^0,\rho_2)\delta(k^0-k_1^0-k_2^0)\frac{e^{i(\rho-\rho_1-\rho_2)r}}{\rho}\times$$

$$\times\ \rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0\ r^{-1}dr. \tag{8.2.0.6}$$

We want now to extend $h(k^0, \rho)$ to the complex plane as a tempered ultradistribution. For this we can use, for example, Eq. (4.2.0.7). First we consider the term

$$\frac{e^{i(\rho-\rho_1-\rho_2)r}}{\rho}. \tag{8.2.0.7}$$

The extension to the *complex plane* is

$$\{\Theta(r)\ \Theta[\Im(\rho)] - \Theta(-r)\ \Theta[-\Im(\rho)]\}\ \frac{e^{i(\rho-\rho_1-\rho_2)r}}{\rho}, \tag{8.2.0.8}$$

where Θ is the Heaviside's step function and $\Im$ denotes "Imaginary part". On the other hand, the extension of

$$\delta(k^0-k_1^0-k_2^0), \tag{8.2.0.9}$$

is

$$-\frac{1}{2\pi i(k^0-k_1^0-k_2^0)}. \tag{8.2.0.10}$$

Replacing [(8.2.0.8) and (8.2.0.10)] into (8.2.0.6), and then integrating out the variable r, we obtain

$$H(k^0,\rho) = \frac{1}{2\pi\rho}\iiiint_{-\infty}^{+\infty}\frac{f(k_1^0,\rho_1)g(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\times$$

$$\{\Theta[\Im(\rho)]\ln(\rho_1+\rho_2-\rho)+\Theta[-\Im(\rho)]\ \times$$

$$\ln(\rho-\rho_1-\rho_2)\}\ \rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0, \tag{8.2.0.11}$$

where $H(k^0, \rho)$ is the extension of $f(k^0, \rho)$. Taking into account that $f(k_1^0, \rho_1)$ and $g(k_2^0, \rho_2)$ are even functions in the first and second variables, (8.2.0.11) takes the form

$$H(k^0,\rho) = \frac{1}{4\pi\rho}\iiiint_{-\infty}^{+\infty}\frac{f(k_1^0,\rho_1)g(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\ln[\rho^2-(\rho_1+\rho_2)^2]\ \times$$

$$\rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0. \tag{8.2.0.12}$$

The expression (8.2.0.12) for $H(k^0,\rho)$ can be re-written in the form

$$H(k^0,\rho) = \frac{1}{4\pi\rho}\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\frac{F(k_1^0,\rho_1)G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\ln[\rho^2-(\rho_1+\rho_2)^2]\times$$

$$\rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0, \tag{8.2.0.13}$$

where $F(k_1^0,\rho_1)$ and $G(k_2^0,\rho_2)$ are, respectively, the extensions of $f(k_1^0,\rho_1)$ and $g(k_2^0,\rho_2)$, and where we have taken $|\Im(k^0)| > |\Im(k_1^0)| + |\Im(k_2^0)|$, $|\Im(\rho)| > |\Im(\rho_1)| + |\Im(\rho_2)|$. In addition Γ_1^0, Γ_2^0, Γ_1, and Γ_2 are, respectively, paths (as we have described in section 3) in the variables k_1^0, k_2^0, ρ_1, and ρ_2, enclosing all the singularities of the integrand in (8.2.0.13). The difference between

$$\int\frac{2\rho}{\rho^2-(\rho_1+\rho_2)^2}\ d\rho \qquad \text{and} \qquad \ln[\rho^2-(\rho_1+\rho_2)^2]$$

is an entire analytic function. With this substitution into (8.2.0.13) we obtain

$$H(k^0,\rho) = \frac{1}{2\pi\rho}\int\rho\ d\rho\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\frac{F(k_1^0,\rho_1)G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\frac{1}{\rho^2-(\rho_1+\rho_2)^2}\times$$

$$\rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0. \tag{8.2.0.14}$$

Now, we can use the method of Ref. [33] to define the convolution for the case in which $F(k_1^0,\rho_1)$ and $G(k_2^0,\rho_2)$ are tempered ultradistributions. We define

$$H_{\lambda_0\lambda}(k^0,\rho) = \frac{1}{2\pi\rho}\int\rho\ d\rho\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\times$$

$$\frac{k_1^{0\ \lambda_0}\rho_1^{\lambda+1}F(k_1^0,\rho_1)k_2^{0\ \lambda_0}\rho_2^{\lambda+1}G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\times$$

$$\frac{1}{\rho^2-(\rho_1+\rho_2)^2}\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0. \tag{8.2.0.15}$$

Integral (8.2.0.15) is an analytic function of (λ_0,λ) bounded by a power of $|k|$ and defined in a domain $\mathfrak{B}$, given by the Cartesian product of a vertical band $\mathfrak{b}_0$ contained in the λ_0-plane and vertical band $\mathfrak{b}$, contained in the λ-plane. We can again extend this domain using the method given in Ref. [33] and perform the Laurent expansion

$$H_{\lambda_0\lambda}(k^0,\rho) = \sum_{mn} H^{(m,n)}(k^0,\rho)\lambda_0^m\lambda^n. \tag{8.2.0.16}$$

We define the convolution product as the (λ_0,λ)- independent term of (8.2.0.16)

$$H(k) = H(k^0,\rho) = H^{(0,0)}(k^0,\rho). \tag{8.2.0.17}$$

The proof that $H(k)$ is an ultradistribution is similar to the one given in Ref. [33] for the one-dimensional case. To simplify the evaluation of (8.2.0.15) we define

$$L_{\lambda_0\lambda}(k^0,\rho) = \oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{k_1^{0\ \lambda_0}\rho_1^{\lambda+1}F(k_1^0,\rho_1)k_2^{0\ \lambda_0}\rho_2^{\lambda+1}G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\times$$

$$\frac{1}{\rho^2-(\rho_1+\rho_2)^2}\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0, \tag{8.2.0.18}$$

so that

$$H_{\lambda_0\lambda}(k^0,\rho) = \frac{1}{2\pi\rho}\int L_{\lambda_0\lambda}(k^0,\rho)\ \rho\ d\rho. \tag{8.2.0.19}$$

Now, we will show that the cut on the real axis of (8.2.0.17) $h_{\lambda_0\lambda}(k^0,\rho)$ is an even function of k^0 and ρ. For this purpose we consider

$$H_{\lambda_0\lambda}(k^0,\rho) = \frac{1}{4\pi\rho}\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{k_1^{0\ \lambda_0}\rho_1^{\lambda+1}F(k_1^0,\rho_1)k_2^{0\ \lambda_0}\rho_2^{\lambda+1}G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\times$$

$$\ln[\rho^2-(\rho_1+\rho_2)^2]\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0. \tag{8.2.0.20}$$

(8.2.0.20) is explicitly odd in ρ. For the variable k^0 we take into account that $e^{i\pi\lambda_0\{Sgn[\Im(k_1^0)]+Sgn[\Im(k_2^0)]\}} = 1$, and, as a consequence (8.2.0.20) is odd in k^0 too. We consider now the parity in the variable ρ

$$\oint\limits_{\Gamma_0}\oint\limits_{\Gamma} H_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\ dk^0\ d\rho =$$

$$-\iint\limits_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\ dk^0\ d\rho =$$

$$-\oint\limits_{\Gamma_0}\oint\limits_{\Gamma} H_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\ dk^0\ d\rho =$$

$$-\iint\limits_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\ dk^0\ d\rho. \tag{8.2.0.21}$$

Thus, we have

$$h_{\lambda_0\lambda}(k^0,-\rho) = h_{\lambda_0\lambda}(k^0,\rho). \tag{8.2.0.22}$$

The proof for the variable k^0 is similar.

8.3 Massless Wheeler propagators

The *massless Wheeler propagator* w_0 is given by

$$w_0(k) = \frac{i}{k_0^2 - \rho^2}. \tag{8.3.0.1}$$

It can be extended to the complex plane as a tempered ultradistribution in the variables k^0 and ρ

$$W_0(k) = -i\frac{Sgn\Im(k^0)}{8k^0}\left[\frac{Sgn\Im(\rho) - Sgn\Im(k^0)}{\rho - k^0} - \right.$$

$$\left.\frac{Sgn\Im(\rho) + Sgn\Im(k^0)}{\rho + k^0}\right], \tag{8.3.0.2}$$

where $Sgn(x)$ is the function sign for the variable x. We proceed now to evaluate the convolution of two massless Wheeler propagators. According to (8.2.0.18) and (8.3.0.2) we can write

$$L_{\lambda_0\lambda}(k^0,\rho) = -\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\frac{Sgn\Im(k_1^0)}{8k_1^0}\times$$

$$\left[\frac{Sgn\Im(\rho_1) - Sgn\Im(k_1^0)}{\rho_1 - k_1^0} - \frac{Sgn\Im(\rho_1) + Sgn\Im(k_1^0)}{\rho_1 + k_1^0}\right]$$

$$\frac{\mathrm{Sgn}\Im(k_2^0)}{8k_2^0}\left[\frac{\mathrm{Sgn}\Im(\rho_2)-\mathrm{Sgn}\Im(k_2^0)}{\rho_2-k_2^0}-\frac{\mathrm{Sgn}\Im(\rho_2)+\mathrm{Sgn}\Im(k_2^0)}{\rho_2+k_2^0}\right]\times$$

$$\frac{k_1^{0\ \lambda_0}\rho_1^{\lambda+1}k_2^{0\ \lambda_0}\rho_2^{\lambda+1}}{(k^0-k_1^0-k_2^0)[\rho^2-(\rho_1+\rho_2)^2]}\,d\rho_1\,d\rho_2\,dk_1^0\,dk_2^0. \qquad (8.3.0.3)$$

Eq. (8.3.0.3) can be written as

$$L_{\lambda_0\lambda}(k^0,\rho)=-\oint_{\Gamma_1^0}\oint_{\Gamma_2^0}\iint_{-\infty}^{+\infty}\left\{\frac{\mathrm{Sgn}\Im(k_1^0)}{8\rho_1}\left[\frac{1}{k_1^0-\rho_1}-\frac{1}{k_1^0+\rho_1}\right]\times\right.$$

$$\left[(\rho_1+i0)^{\lambda+1}+(\rho_1-i0)^{\lambda+1}\right]+\frac{1}{8k_1^0}\left[\frac{1}{k_1^0+\rho_1}-\frac{1}{k_1^0-\rho_1}\right]\times$$

$$\left.\left[(\rho_1+i0)^{\lambda+1}-(\rho_1-i0)^{\lambda+1}\right]\right\}\ \left\{\frac{\mathrm{Sgn}\Im(k_2^0)}{8\rho_2}\left[\frac{1}{k_2^0-\rho_2}-\frac{1}{k_2^0+\rho_2}\right]\times\right.$$

$$\left[(\rho_2+i0)^{\lambda+1}+(\rho_2-i0)^{\lambda+1}\right]+\frac{1}{8k_2^0}\left[\frac{1}{k_2^0+\rho_2}-\frac{1}{k_2^0-\rho_2}\right]\times$$

$$\left.\left[(\rho_2+i0)^{\lambda+1}-(\rho_2-i0)^{\lambda+1}\right]\right\}\frac{k_1^{0\ \lambda_0}k_2^{0\ \lambda_0}\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0}{(k^0-k_1^0-k_2^0)[\rho^2-(\rho_1+\rho_2)^2]}. \qquad (8.3.0.4)$$

Integrating (8.3.0.4) in the variable k_1^0 we obtain

$$L_\lambda(k^0,\rho)=-\oint_{\Gamma_2^0}\iint_{-\infty}^{+\infty}\left\{\frac{i\pi}{4\rho_1}\mathrm{Sgn}\Im(k^0)\right.$$

$$\left[\frac{1}{k_2^0-(k^0-\rho_1)}-\frac{1}{k_2^0-(k^0+\rho_1}\right]\times$$

$$\left[(\rho_1+i0)^{\lambda+1}+(\rho_1-i0)^{\lambda+1}\right]+$$

$$\frac{i\pi}{4\rho_1}\left[\frac{2}{k_2^0-k^0}-\frac{1}{k_2^0-(k^0-\rho_1)}-\frac{1}{k_2^0-(k^0-\rho_1)}\right]\times$$

$$\left.\left[(\rho_1+i0)^{\lambda+1}-(\rho_1-i0)^{\lambda+1}\right]\right\}\ \left\{\frac{\mathrm{Sgn}\Im(k_2^0)}{8\rho_2}\left[\frac{1}{k_2^0-\rho_2}-\frac{1}{k_2^0+\rho_2}\right]\times\right.$$

$$\left[(\rho_2+i0)^{\lambda+1}+(\rho_2-i0)^{\lambda+1}\right]+\frac{1}{8k_2^0}\left[\frac{1}{k_2^0+\rho_2}-\frac{1}{k_2^0-\rho_2}\right]\times$$

$$\left[(\rho_2 + i0)^{\lambda+1} - (\rho_2 - i0)^{\lambda+1}\right]\Big\} \frac{d\rho_1\ d\rho_2\ dk_2^0}{\rho^2 - (\rho_1 + \rho_2)^2} , \tag{8.3.0.5}$$

where we have selected $\lambda_0 = 0$ due to the fact the integral is convergent for this value $\lambda_0 = 0$. There is a sole term in (8.3.0.5) whose integral is not null. It is

$$L_\lambda(k^0, \rho) = -\oint_{\Gamma_2^0} \iint_{-\infty}^{+\infty} \frac{i\pi}{4\rho_1} \mathrm{Sgn}\Im(k^0) \left[\frac{1}{k_2^0 - (k^0 - \rho_1)} - \frac{1}{k_2^0 - (k^0 + \rho_1)}\right] \times$$

$$\left[(\rho_1 + i0)^{\lambda+1} + (\rho_1 - i0)^{\lambda+1}\right] \frac{\mathrm{Sgn}\Im(k_2^0)}{8\rho_2} \left[\frac{1}{k_2^0 - \rho_2} - \frac{1}{k_2^0 + \rho_2}\right] \times$$

$$\left[(\rho_2 + i0)^{\lambda+1} + (\rho_2 - i0)^{\lambda+1}\right] \frac{d\rho_1\ d\rho_2\ dk_2^0}{\rho^2 - (\rho_1 + \rho_2)^2} . \tag{8.3.0.6}$$

Evaluation of (8.3.0.6) gives

$$L_\lambda(k^0, \rho) = \frac{\pi^2 k^0}{2} \iint_{-\infty}^{+\infty} \left[(\rho_1 + i0)^{\lambda+1} + (\rho_1 - i0)^{\lambda+1}\right] \times$$

$$\left[(\rho_2 + i0)^{\lambda+1} + (\rho_2 - i0)^{\lambda+1}\right]$$

$$\frac{d\rho_1\ d\rho_2}{\left[(k_0^2 + {\rho_1}^2 - {\rho_2}^2)^2 - 4k_0^2 {\rho_1}^2\right]\left[\rho^2 - (\rho_1 + \rho_2)^2\right]}. \tag{8.3.0.7}$$

We can evaluate now the integral in the variable ρ_2 in (8.3.0.7). The result is

$$L_\lambda(k^0, \rho) = \frac{\pi^3}{16\rho} \frac{(1 + \cos \pi\lambda)^2}{\sin \frac{\pi(\lambda+1)}{2}} \int_0^\infty d\rho_1\ \ \rho_1^\lambda \times$$

$$\left\{ \frac{e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(k^0 + \rho_1)^{\lambda+1} - e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho + \rho_1)^{\lambda+1}}{(\rho - k^0)\left(\frac{\rho+k^0}{2} + \rho_1\right)} - \right.$$

$$\frac{e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(k^0 + \rho_1)^{\lambda+1} - e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho_1 - \rho)^{\lambda+1}}{(\rho + k^0)\left(\frac{\rho-k^0}{2} - \rho_1\right)} -$$

$$\frac{e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(\rho_1 - k^0)^{\lambda+1} - e^{-\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho_1 + \rho)^{\lambda+1}}{(\rho + k^0)\left(\frac{\rho-k^0}{2} + \rho_1\right)} +$$

$$\left. \frac{e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(k^0)}(\rho_1 - k^0)^{\lambda+1} - e^{\frac{i\pi}{2}(\lambda+1)\mathrm{Sgn}\Im(\rho)}(\rho_1 - \rho)^{\lambda+1}}{(\rho - k^0)\left(\frac{\rho+k^0}{2} - \rho_1\right)} \right\}. \tag{8.3.0.8}$$

The evaluation of (8.3.0.8) is a tedious task. Fortunately, lim $\lambda \to 0$ can be taken without any problems in the final steps of the calculation. The result is

$$L(k^0,\rho) = \frac{\pi^3}{4\rho}\left[\frac{\pi}{2}\mathrm{Sgn}\Im(k^0)\mathrm{Sgn}\Im(k^0+\rho) + \frac{\pi}{2}\mathrm{Sgn}\Im(\rho)\mathrm{Sgn}\Im(k^0+\rho)\ + \right.$$
$$\left.\frac{\pi}{2}\mathrm{Sgn}\Im(k^0)\mathrm{Sgn}\Im(\rho-k^0) - \mathrm{Sgn}\Im(\rho-k^0)\right] - \tag{8.3.0.9}$$

Eq. (8.3.0.9) can be cast as

$$L(k^0,\rho) = \frac{\pi^4}{8\rho}\left[\left(\mathrm{Sgn}\Im(k^0) + \mathrm{Sgn}\Im(\rho)\right)\mathrm{Sgn}\Im(\rho+k^0)\ + \right.$$
$$\left.\left(\mathrm{Sgn}\Im(k^0) - \mathrm{Sgn}\Im(\rho)\right)\mathrm{Sgn}\Im(\rho-k^0)\right]\ =$$
$$\frac{\pi^4}{4\rho}\,\mathrm{Sgn}\Im(k^0)\mathrm{Sgn}\Im(\rho). \tag{8.3.0.10}$$

Taking into account that

$$H(k^0,\rho) = \frac{1}{2\pi\rho}\int L(k^0,\rho)\rho\, d\rho$$

we obtain

$$H(k^0,\rho) = \frac{\pi^3}{8}\mathrm{Sgn}\Im(k^0)\mathrm{Sgn}\Im(\rho) = [W_0 * W_0](k^0,\rho), \tag{8.3.0.11}$$

where the symbol $*$ indicates the convolution product. Thus, the cut of $H(k^0,\rho)$ along the real axis, i.e., the distribution $h(k^0,\rho)$ is

$$h(k^0,\rho) = \frac{\pi^3}{2} = [w_0 * w_0](k^0,\rho). \tag{8.3.0.12}$$

8.4 Complex-mass Wheeler propagators

The *complex mass Wheeler propagator* is

$$w_\mu(x) = -\frac{i\pi}{2}\frac{\mu^{n/2-1}}{(2\pi)^{n/2}}Q_-^{1/2(1-n/2)}J_{1-n/2}(\mu Q_-^{1/2}), \tag{8.4.0.1}$$

and its Fourier transform adopts the form

$$W_{\mu}(k^0,\rho) = -\frac{iSgn[\Im(k^0)]}{8\sqrt{k_0^2-\mu^2}}\left[\frac{Sgn[\Im(\rho)] - Sgn[\Im(\sqrt{k_0^2-\mu^2})]}{\rho-\sqrt{k_0^2-\mu^2}} - \right.$$
$$\left.\frac{Sgn[\Im(\rho)] + Sgn[\Im(\sqrt{k_0^2-\mu^2})]}{\rho+\sqrt{k_0^2-\mu^2}}\right]. \qquad (8.4.0.2)$$

Using (8.4.0.2) we have now

$$L(k^0,\rho) = -\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\frac{Sgn[\Im(k_1^0)]}{8\sqrt{k_1^{02}-\mu_1^2}}\left[\frac{Sgn[\Im(\rho_1)] - Sgn[\Im(\sqrt{k_1^{02}-\mu_1^2})]}{\rho_1-\sqrt{k_1^{02}-\mu_1^2}} - \right.$$
$$\left.\frac{Sgn[\Im(\rho_1)] + Sgn[\Im(\sqrt{k_1^{02}-\mu_1^2})]}{\rho+1+\sqrt{k_1^{02}-\mu_1^2}}\right]\frac{Sgn[\Im(k_2^0)]}{8\sqrt{k_2^{02}-\mu_2^2}}\times$$
$$\left[\frac{Sgn[\Im(\rho_2)] - Sgn[\Im(\sqrt{k_2^{02}-\mu_2^2})]}{\rho_2-\sqrt{k_2^{02}-\mu_2^2}} - \right.$$
$$\left.\frac{Sgn[\Im(\rho_2)] + Sgn[\Im(\sqrt{k_2^{02}-\mu_2^2})]}{\rho_2+\sqrt{k_2^{02}-\mu_2^2}}\right]\frac{\rho_1\rho_2 d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0}{(k^0-k_1^0-k_2^0)[\rho^2-(\rho_1+\rho_2)^2]}, \qquad (8.4.0.3)$$

where we have selected $\lambda_0 = \lambda = 0$ due to the fact that (8.4.0.3) is convergent at this point (this is due to the definition of $L(k^0,\rho)$). Now, (8.4.0.3) is equal to

$$L(k^0,\rho) = -\frac{1}{4}\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\iint\limits_{-\infty}^{+\infty}\frac{Sgn[\Im(k_1^0)]}{\rho_1^2+\mu_1^2-k_1^{02}}\ \frac{Sgn[\Im(k_2^0)]}{\rho_2^2+\mu_2^2-k_2^{02}}\times$$
$$\frac{\rho_1\rho_2}{(k^0-k_1^0-k_2^0)[\rho^2-(\rho_1+\rho_2)^2]}d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0, \qquad (8.4.0.4)$$

and can be re-written as

$$L(k^0,\rho) = -\frac{1}{16}\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\iint\limits_{-\infty}^{+\infty}\frac{Sgn[\Im(k_1^0)]}{\sqrt{\rho_1^2+\mu_1^2}}\times$$

$$\left[\frac{1}{k_1^0-\sqrt{\rho_1^2+\mu_1^2}}-\frac{1}{k_1^0+\sqrt{\rho_1^2+\mu_1^2}}\right]\times$$

$$\frac{Sgn[\Im(k_2^0)]}{\sqrt{\rho_2^2+\mu_2^2}}\left[\frac{1}{k_2^0-\sqrt{\rho_2^2+\mu_2^2}}-\frac{1}{k_2^0+\sqrt{\rho_2^2+\mu_2^2}}\right]\frac{1}{(k^0-k_1^0-k_2^0)}\times$$

$$\frac{\rho_1\rho_2}{\rho^2-(\rho_1+\rho_2)^2}d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0. \tag{8.4.0.5}$$

Consider now

$$\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\frac{Sgn[\Im(k_1^0)]Sgn[\Im(k_2^0)]}{k^0-k_1^0-k_2^0}\left[\frac{1}{k_1^0-\sqrt{\rho_1^2+\mu_1^2}}-\frac{1}{k_1^0+\sqrt{\rho_1^2+\mu_1^2}}\right]\times$$

$$\left[\frac{1}{k_2^0-\sqrt{\rho_2^2+\mu_2^2}}-\frac{1}{k_2^0+\sqrt{\rho_2^2+\mu_2^2}}\right]dk_1^0\ dk_2^0=$$

$$-\frac{32\pi^2k^0\sqrt{\rho_1^2+\mu_1^2}\sqrt{\rho_2^2+\mu_2^2}}{[k_0^2+(\rho_2^2+\mu_2^2)-(\rho_1^2+\mu_2^2)]^2-4k_0^2(\rho_2^2+\mu_2^2)}. \tag{8.4.0.6}$$

Replacing this result into (8.4.0.5) we obtain

$$L(k^0,\rho)=2\pi^2k^0\iint\limits_{-\infty}^{+\infty}\frac{1}{[k_0^2+(\rho_2^2+\mu_2^2)-(\rho_1^2+\mu_2^2)]^2-4k_0^2(\rho_2^2+\mu_2^2)}\times$$

$$\frac{\rho_1\rho_2}{\rho^2-(\rho_1+\rho_2)^2}\ d\rho_1\ d\rho_2. \tag{8.4.0.7}$$

Taking into account that

$$\int\frac{\rho\ d\rho}{\rho^2-(\rho_1+\rho_2)^2}=\Theta[\Im(\rho)]\ln(\rho_1+\rho_2-\rho)+\Theta[-\Im(\rho)]\ln(\rho-\rho_1-\rho_2), \tag{8.4.0.8}$$

and using the result (8.4.0.7) we obtain

$$H(k^0,\rho)=\frac{\pi k^0}{\rho}\iint\limits_{-\infty}^{+\infty}\frac{1}{[k_0^2+(\rho_2^2+\mu_2^2)-(\rho_1^2+\mu_2^2)]^2-4k_0^2(\rho_2^2+\mu_2^2)}\times$$

$$\Theta[\Im(\rho)] \ln(\rho_1 + \rho_2 - \rho) + \Theta[-\Im(\rho)] \ln(\rho - \rho_1 - \rho_2)\ d\rho_1\ d\rho_2. \quad (8.4.0.9)$$

Equation (8.4.0.9) can be written in the real ρ-axis as

$$H(k^0, \rho) = \frac{i\pi^2 k^0}{\rho} \iint\limits_{-\infty}^{+\infty} \frac{Sgn(\rho_1 + \rho_2 - \rho)\ \rho_1\rho_2\ d\rho_1\ d\rho_2}{[k_0^2 + (\rho_2^2 + \mu_2^2) - (\rho_1^2 + \mu_2^2)]^2 - 4k_0^2(\rho_2^2 + \mu_2^2)}. \quad (8.4.0.10)$$

After evaluation of the double integral of (8.4.0.10) we find

$$H(k^0, \rho) = \frac{\pi^3 Sgn[\Im(k^0)]}{4(k_0^2 - \rho^2)} \sqrt{(k_0^2 - \rho^2 + \mu_2^2 - \mu_1^2)^2 - 4(k_0^2 - \rho^2)\mu_2^2} =$$

$$[W_{\mu_1} * W_{\mu_2}](k^0, \rho). \quad (8.4.0.11)$$

8.5 Discussion

In an earlier work [33] its authors demonstrated the existence of the convolution of two one-dimensional tempered ultradistributions. Here we have extended that procedure to an n-dimensional space. In four-dimensional space we have obtained an expression for the convolution of two tempered ultradistributions that are even in the variables k^0 and ρ. When we use the perturbative expansion in quantum field theory, we have to deal with either products of distributions in configuration space, or with convolutions in the Fourier transformed p-space. Unfortunately, products or convolutions (of distributions) are, in general, ill-defined quantities. However, in physical applications one introduces some "regularization" scheme that allows us to give sense to divergent integrals. Amongst these procedures we would like to mention the dimensional regularization method (Ref. [12]). Essentially, the method consists in the separation of the volume element ($d^\nu p$) into an angular factor ($d\Omega$) and a radial factor ($p^{\nu-1}\,dp$). First the angular integration is carried out and then the number of dimensions ν is taken as a free parameter that can be adjusted to yield a convergent integral, which is an analytic function of ν. Our Eq. (8.1.0.1) is similar to the expression one obtains with dimensional regularization. However, the parameters λ are completely independent of any dimensional interpretation. All ultradistributions provide integrands (in (8.1.0.1)) that are analytic functions along the integration paths. The parameters λ permit us to control the possible tempered asymptotic behavior. Since the existence of a region of analyticity for each λ

is guaranteed, a subsequent continuation to the point of interest (Ref. [33]) defines the convolution product. For tempered ultradistributions (even in the variables k^0 and ρ) we have obtained Eq. (8.2.0.15), for which similar considerations to those given for (7.1.0.7) are valid. The properties above described show that tempered ultradistributions provide an appropriate framework for applications in physics. Furthermore, they can "absorb" arbitrary pseudo-polynomials, thanks to Eq. (4.2.0.10), a property that is interesting for renormalization theory. For this reason, we began this chapter with a summary of the main characteristics of n-dimensional tempered ultradistributions and their Fourier transformed distributions of exponential type.

Chapter 9

Lorentz ultradistributions

As previously stated, the question of the product of distributions with coincident point singularities is related, in field theory (FT), to the asymptotic behavior of loop integrals of propagators. From a mathematical point of view, practically all definitions lead to limitations on the set of distributions that can be multiplied together to give another distribution of the same kind. The properties of ultradistributions (Ref. [9, 10]) are well adapted for their use in FT. In this respect, it was shown (see Ref. [33]) that, in one dimensional space, it is possible to define the convolution of any pair of tempered ultradistributions, giving as a result another tempered ultradistribution. The next step is to consider the convolution of any pair of tempered ultradistribution in n-dimensional space, that follows from the formula obtained in Ref. [33] for one dimensional space (See Ref. [39].) However, the resultant equation is rather involved to be used in practical applications. Thus, for applications, it is convenient to consider the convolution of any two tempered ultradistributions which are even in the variables k^0 and ρ (See Ref. [39]). A further step is to consider the convolution of two Lorentz invariant tempered ultradistributions (See Section 8.2).

Ultradistributions also have the advantage of being representable by means of analytic functions. In general, they are easier to work with and have interesting properties. One of these properties is that

Schwartz' tempered distributions are canonical and continuously injected into tempered ultradistributions and, as a consequence, the rigged Hilbert space with tempered distributions is canonical and continuously included in the rigged Hilbert space with tempered ultradistributions.)

9.1 Fourier transform in Euclidean space

The Fourier transform of a *spherically symmetric function* $\hat{f} \in \mathsf{H}$ is given, according to *Bochner's theorem*, by

$$f(k) = \frac{(2\pi)^{\frac{\nu}{2}}}{k^{\frac{\nu-2}{2}}} \int\limits_0^\infty \hat{f}(r) r^{\frac{\nu}{2}} \mathcal{J}_{\frac{\nu-2}{2}}(kr)\, dr, \tag{9.1.0.1}$$

where $r = x_0^2 + x_1^2 + \cdots + x_{\nu-1}^2$, $k = k_0^2 + k_1^2 + \cdots + k_{\nu-1}^2$, and $\mathcal{J}_{\frac{\nu}{2}}$ is the Bessel's function of order $\nu - 2/2$. Appealing to the equality

$$\pi \mathcal{J}_{\frac{\nu-2}{2}}(z) = e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-iz) + e^{i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(iz), \tag{9.1.0.2}$$

where $\mathcal{K}$ is the *modified Bessel's function*, (9.1.0.1) takes the form

$$f(k) = 2\frac{(2\pi)^{\frac{\nu-2}{2}}}{k^{\frac{\nu-2}{2}}} \int\limits_0^\infty \hat{f}(r) r^{\frac{\nu}{2}} \Big[e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-ikr) +$$

$$e^{i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(ikr) \Big]\, dr. \tag{9.1.0.3}$$

Perform now the change of variables $x = r^{\frac{1}{2}}$, $\rho = k^{\frac{1}{2}}$ (9.1.0.1), and then (9.1.0.3) can be re-written as

$$f(\rho) = \pi \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int\limits_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\rho^{1/2} x^{1/2})\, dx, \tag{9.1.0.4}$$

$$f(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int\limits_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \Big[e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) +$$

,

$$e^{i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \Big]\, dx. \tag{9.1.0.5}$$

Here we have taken $\rho = \gamma + i\sigma$ and

$$\rho^{1/2} = \sqrt{\frac{\gamma + \sqrt{\gamma^2 + \sigma^2}}{2}} + iSgn(\sigma)\sqrt{\frac{-\gamma + \sqrt{\gamma^2 + \sigma^2}}{2}}. \quad (9.1.0.6)$$

We can extend (9.1.0.4) to the complex plane and obtain the corresponding ultradistribution. As a first step we calculate the Fourier anti-transform of $\rho^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})$. We have

$$\frac{1}{2\pi}\int_0^\infty \rho^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2}) e^{-i\rho t}\, d\rho =$$

$$\frac{e^{\frac{i\pi(\nu-4)}{8}}(t-i0)^{\frac{\nu-4}{4}}}{\pi x^{1/2}\Gamma(\frac{\nu}{2})} e^{\frac{ix}{8t}} \mathcal{M}_{\frac{4-\nu}{4},\frac{\nu-2}{4}}\left(-\frac{ix}{4t}\right). \quad (9.1.0.7)$$

We have used above Eq. (6.631-1) of Ref. [4] ($\mathcal{M}$ is the *Whittaker's function*). Now we can employ also Eqs. (9.233- 1-2) of Ref. [4] and write

$$\mathcal{M}_{\frac{4-\nu}{4},\frac{\nu-2}{4}}\left(-\frac{ix}{4t}\right) = \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu-2}{2})} e^{\frac{i\pi(4-\nu)}{4}} \mathcal{W}_{\frac{\nu-4}{4},\frac{\nu-2}{4}}\left(\frac{ix}{4t}\right) +$$

$$\Gamma\left(\frac{\nu}{2}\right) e^{\frac{i\pi(2-\nu)}{2}} \mathcal{W}_{\frac{4-\nu}{4},\frac{\nu-2}{4}}\left(-\frac{ix}{4t}\right) \quad t > 0.$$

$$\mathcal{M}_{\frac{4-\nu}{4},\frac{\nu-2}{4}}\left(-\frac{ix}{4t}\right) = \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu-2}{2})} e^{\frac{i\pi(\nu-4)}{4}} \mathcal{W}_{\frac{\nu-4}{4},\frac{\nu-2}{4}}\left(\frac{ix}{4t}\right) +$$

$$\Gamma\left(\frac{\nu}{2}\right) e^{\frac{i\pi(\nu-2)}{2}} \mathcal{W}_{\frac{4-\nu}{4},\frac{\nu-2}{4}}\left(-\frac{ix}{4t}\right) \quad t < 0. \quad (9.1.0.8)$$

As a second step, we calculate the *complex Fourier transform* of the second term of (9.1.0.7) using (9.1.0.8). We obtain

$$\mathcal{F}_c\left[\frac{e^{\frac{i\pi(\nu-4)}{8}}(t-i0)^{\frac{\nu-4}{4}}}{\pi x^{1/2}\Gamma(\frac{\nu}{2})} e^{\frac{ix}{8t}} \mathcal{M}_{\frac{4-\nu}{4},\frac{\nu-2}{4}}\left(-\frac{ix}{4t}\right)\right](\rho) =$$

$$\rho^{\frac{2-\nu}{4}}\left\{\Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) - \right.$$

$$\Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) +$$

$$\frac{2^{\frac{4-\nu}{2}} i}{\Gamma(\frac{\nu-2}{2})} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\Bigg\}, \tag{9.1.0.9}$$

where we have used Eqs. (7.629- 1, 2) of Ref. [4], with $\mathcal{S}$ the so-called *Lommel function* (Ref. [4], page 349, formula 3). The corresponding ultradistribution is then defined as

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \Big\{\Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) -$$

$$\Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\Big\}\ dx +$$

$$\frac{2\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\ dx. \tag{9.1.0.10}$$

When $\nu = 2n$, n an entire number, $\rho^{\frac{2-\nu}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}$ is equivalent to zero. In fact

$$\rho^{\frac{2-\nu}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}} = \sum_{m=0}^{\frac{\nu-4}{2}} \frac{(\frac{\nu}{2}-m)!}{m!} 4^{\frac{\nu-2-4m}{4}} x^{\frac{4m+2-\nu}{4}} \rho^{\frac{2m+2-\nu}{2}}, \tag{9.1.0.11}$$

so that (9.1.0.11) is a polynomial in ρ^{-1}. However, when the volume element is taken into account, such expression is transformed into a polynomial in ρ, which according to (4.2.0.10) leaves us with a null Ultradistribution. Thus, in this case the second integral in (9.1.0.10) vanishes so that it becomes

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \Big[\Theta[\Im(\rho)] e^{-i\frac{\pi}{4}\nu} \boldsymbol{\mathcal{K}}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})$$

$$-\Theta[-\Im(\rho)] e^{i\frac{\pi}{4}\nu} \boldsymbol{\mathcal{K}}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\Big]\ dx. \tag{9.1.0.12}$$

Note that the complex Fourier transform (9.1.0.12) is not merely the Fourier transform (9.1.0.5) in which the variable ρ is considered to be a complex number. Instead, (9.1.0.12) gives the ultradistribution associated to $f(\rho)$. In the next section we shall see that formulae (9.1.0.5) and (9.1.0.12) can be generalized to Minkowskian space.

When $\hat{f}$ is a spherically symmetric distribution of exponential type, we can use (9.1.0.10) to define its Fourier transform. In addition, we can follow the treatment of Ref. [6] to define the Fourier transform. Thus, we have

$$\int_0^\infty f(\rho)\phi(\rho)\rho^{\frac{\nu-2}{2}}\, d\rho = (2\pi)^\nu \int_0^\infty \hat{f}(x)\hat{\phi}(x)x^{\frac{\nu-2}{2}}\, dx. \tag{9.1.0.13}$$

The corresponding tempered ultradistribution in the one-dimensional complex variable ρ is obtained in the following way: let $\hat{g}(t)$ be defined as

$$\hat{g}(t) = \frac{1}{(2\pi)^\nu}\int_0^\infty f(\rho)e^{-i\rho t}\, d\rho. \tag{9.1.0.14}$$

Then,

$$F(\rho) = \Theta[\Im(\rho)]\int_0^\infty \hat{g}(t)e^{i\rho t}\, dt - \Theta[-\Im(\rho)]\int_{-\infty}^0 \hat{g}(t)e^{i\rho t}\, dt, \tag{9.1.0.15}$$

or, if we use *Dirac's formula*,

$$F(\rho) = \frac{1}{2\pi i}\int_0^\infty \frac{f(t)}{t-\rho}\, dt. \tag{9.1.0.16}$$

The inversion formula ($\nu = 2n$) for $F(\rho)$ is given by

$$\hat{f}(x) = \frac{\pi}{(2\pi)^{\frac{\nu+2}{2}} x^{\frac{\nu-2}{4}}}\oint_\Gamma F(\rho)\rho^{\frac{\nu-2}{4}}\mathcal{J}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\, d\rho. \tag{9.1.0.17}$$

Note that the factor multiplying $F(\rho)$ is an entire function of ρ for $\nu = 2n$ (recall that in complex analysis, an entire function, also called an integral function, is a complex-valued function that is holomorphic at all finite points over the whole complex plane [1]). In this case the first term of (9.1.0.13) takes the form

$$\oint_\Gamma F(\rho)\phi(\rho)\rho^{\frac{\nu-2}{2}}\, d\rho = (2\pi)^\nu \int_0^\infty \hat{f}(x)\hat{\phi}(x)x^{\frac{\nu-2}{2}}\, dx \tag{9.1.0.18}$$

We can now define a spherically symmetric tempered ultradistribution as the complex Fourier transform of a spherically symmetric distribution of exponential type. Note that a spherically symmetric ultradistribution is not necessarily spherically symmetric in an explicit way. We will now look at some examples of the use of Fourier transforms.

9.2 Examples

As a first example we calculate the complex Fourier transform of e^{ar} (where a is a complex number) for $\nu = 2n$. From (9.1.0.12) we write

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty e^{ax^{1/2}} x^{\frac{\nu-2}{4}} \left\{ \Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) - \right.$$

$$\left. \Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\, dx \right\}. \tag{9.2.0.1}$$

Now,

$$\int_0^\infty e^{ax^{1/2}} x^{\frac{\nu-2}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) =$$

$$2\sqrt{\pi}\, e^{\frac{i\pi(\nu+2)}{4}} \frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{\rho^{\frac{\nu-2}{4}}}{(\rho^{1/2} - ia)} \times$$

$$\mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a - i\rho^{1/2}}{a + i\rho^{1/2}}\right) \quad \Im(\rho) > 0$$

$$\int_0^\infty e^{ax^{1/2}} x^{\frac{\nu-2}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) =$$

$$2\sqrt{\pi}\, e^{-\frac{i\pi(\nu+2)}{4}} \frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{\rho^{\frac{\nu-2}{4}}}{(\rho^{1/2} + ia)} \times$$

$$\mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a + i\rho^{1/2}}{a - i\rho^{1/2}}\right) \quad \Im(\rho) < 0. \tag{9.2.0.2}$$

To obtain (9.2.0.2) we have used (6.621 - 3) of Ref. [4] (here $\mathbf{F}$ is the hypergeometric function). Thus we have

$$F(\rho) = (4\pi)^{\frac{\nu-2}{2}} i \frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \left\{ \frac{\Theta[\Im(\rho)]}{(\rho^{1/2} - ia)} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a - i\rho^{1/2}}{a + i\rho^{1/2}}\right) + \right.$$

$$\left. \frac{\Theta[-\Im(\rho)]}{(\rho^{1/2} + ia)} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a + i\rho^{1/2}}{a - i\rho^{1/2}}\right) \right\}. \tag{9.2.0.3}$$

As a second example, we evaluate the Fourier antitransform of $[-2\pi i(\rho-\mu^2)]^{-1}$, where μ is a complex number and $\nu = 2n$. Using (9.1.0.17) we have

$$\hat{f}(x) = -\frac{\pi}{(2\pi)^{\frac{\nu+2}{2}} x^{\frac{\nu-2}{4}}} \oint_{\Gamma} \frac{\rho^{\frac{\nu-2}{4}}}{2\pi i(\rho-\mu^2)} \mathcal{J}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\, d\rho =$$

$$\frac{\pi\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu+2}{2}}} x^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\mu x^{1/2}). \tag{9.2.0.4}$$

We can test the result (9.2.0.4) by transforming it. For this we take into account that, for ν even, $\mathcal{J}_{\frac{\nu-2}{2}} = e^{\frac{i\pi(\nu-2)}{2}} \mathcal{J}_{\frac{2-\nu}{2}}$. Thus

$$F(\rho) = \frac{\mu^{\frac{\nu-2}{2}}}{4\pi} e^{\frac{i\pi(\nu-2)}{2}} \rho^{\frac{2-\nu}{4}} \int_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \times$$

$$\left\{ \Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) - \right.$$

$$\left. \Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \right\}\, dx. \tag{9.2.0.5}$$

Now,

$$\int_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})\, dx =$$

$$e^{\frac{i\pi(6-\nu)}{4}} \mu^{\frac{2-\nu}{2}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2}\ ;\ \Im(\rho) > 0$$

$$\int_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\, dx =$$

$$e^{-\frac{i\pi(6-\nu)}{4}} \mu^{\frac{2-\nu}{2}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2}\ ;\ \Im(\rho) < 0, \tag{9.2.0.6}$$

where we have used (6.576 -3) of Ref. [4] and thus we have

$$F(\rho) = -\frac{1}{2\pi i(\rho-\mu^2)}. \tag{9.2.0.7}$$

As a third example, we consider the Fourier transform of $\delta(x-a)$ for all ν. Using (9.1.0.10) we obtain

$$F(\rho)=\frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}}a^{\frac{\nu-2}{4}}\left\{\Theta[\Im(\rho)]e^{-\frac{i\pi\nu}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(-ia^{1/2}\rho^{1/2})-\right.$$

$$\left.\Theta[-\Im(\rho)]e^{\frac{i\pi\nu}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(ia^{1/2}\rho^{1/2})\right\}\ +$$

$$\frac{2\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})\rho^{\frac{\nu-2}{4}}}a^{\frac{\nu-2}{4}}\mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}(a^{1/2}\rho^{1/2}). \tag{9.2.0.8}$$

The reader can verify that the cut of (9.2.0.8) along the negative real axis is zero.

9.3 Fourier transform in Minkowskian space

For the Minkowskian case we begin with the formula

$$f(k_0,k)=\frac{(2\pi)^{\frac{\nu-1}{2}}}{k^{\frac{\nu-3}{2}}}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\hat{f}(x_0,r)r^{\frac{\nu-1}{2}}\mathcal{J}_{\frac{\nu-3}{2}}(kr)e^{ik_0x^0}\ dx^0\ dr, \tag{9.3.0.1}$$

that can be re-written as

$$f(k_0^2-k^2)=\frac{(2\pi)^{\frac{\nu-3}{2}}}{k^{\frac{\nu-3}{2}}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\hat{f}(x)e^{it(x-s_0^2+s^2)}s^{\frac{\nu-1}{2}}\mathcal{J}_{\frac{\nu-3}{2}}(ks)\times$$

$$e^{ik_0s^0}\ dt\ dx\ ds^0\ ds. \tag{9.3.0.2}$$

Now,

$$\int\limits_{0}^{\infty}e^{its^2}s^{\frac{\nu-1}{2}}\mathcal{J}_{\frac{\nu-3}{2}}(ks)\ ds=\frac{1}{2}\left(\frac{k}{2}\right)^{\frac{\nu-3}{2}}(t+i0)^{\frac{1-\nu}{2}}e^{i\left[\frac{\pi}{2}\left(\frac{\nu-1}{2}\right)-\frac{k^2}{4t}\right]} \tag{9.3.0.3}$$

$$\int\limits_{-\infty}^{\infty}e^{-its_0^2}e^{ik_0s^0}\ ds^0=\sqrt{\pi}(t-i0)^{-\frac{1}{2}}e^{i\left(\frac{k_0^2}{4t}-\frac{\pi}{4}\right)}. \tag{9.3.0.4}$$

We have used (6.631 -4) and (3.462- 3) of Ref. [4]. With the results (9.3.0.3) - (9.3.0.4) we obtain for (9.3.0.2)

$$f(k_0^2-k^2)=\frac{(2\pi)^{\frac{\nu-3}{2}}}{2^{\frac{\nu-1}{2}}}\sqrt{\pi}\,e^{\frac{i\pi(\nu-2)}{4}}\int_{-\infty}^{\infty}\int_0^{\infty}\hat{f}(x)\left[e^{itx}e^{\frac{i(k_0^2-k^2)}{4t}}t^{-\frac{\nu}{2}}+\right.$$

$$\left.e^{\frac{i\pi(2-\nu)}{2}}e^{-itx}e^{-\frac{i(k_0^2-k^2)}{4t}}t^{-\frac{\nu}{2}}\right]\,dx\,dt. \tag{9.3.0.5}$$

We can evaluate the integral in the variable t

$$\int_0^{\infty}e^{itx}e^{\frac{i\rho}{4t}}t^{-\frac{\nu}{2}}\,dt=2^{\frac{\nu}{2}}\frac{(x+i0)^{\frac{\nu-2}{4}}}{(\rho+i0)^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}(\rho+i0)^{1/2}]$$

$$\int_0^{\infty}e^{-itx}e^{-\frac{i\rho}{4t}}t^{-\frac{\nu}{2}}\,dt=2^{\frac{\nu}{2}}\frac{(x-i0)^{\frac{\nu-2}{4}}}{(\rho-i0)^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}(\rho-i0)^{1/2}], \tag{9.3.0.6}$$

where $\rho=k_0^2-k^2$ (here we have used (3.471- 9) of Ref. [4]). Thus, (9.3.0.5) transforms into

$$f(\rho)=(2\pi)^{\frac{\nu-2}{2}}\int_{-\infty}^{\infty}\hat{f}(x)\times$$

$$\left\{e^{\frac{i\pi(\nu-2)}{4}}\frac{(x+i0)^{\frac{\nu-2}{4}}}{(\rho+i0)^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}(\rho+i0)^{1/2}]+\right.$$

$$\left.+\,e^{\frac{i\pi(2-\nu)}{4}}\frac{(x-i0)^{\frac{\nu-2}{4}}}{(\rho-i0)^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}(\rho-i0)^{1/2}]\right\}\,dx. \tag{9.3.0.7}$$

The corresponding inversion formula is then given by

$$\hat{f}(x)=\frac{1}{(2\pi)^{\frac{\nu+2}{2}}}\int_{-\infty}^{\infty}f(\rho)\times$$

$$\left\{e^{\frac{i\pi(\nu-2)}{4}}\frac{(\rho+i0)^{\frac{\nu-2}{4}}}{(x+i0)^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}(\rho+i0)^{1/2}]+\right.$$

$$+ e^{\frac{i\pi(2-\nu)}{4}} \frac{(\rho - i0)^{\frac{\nu-2}{4}}}{(x - i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x - i0)^{1/2}(\rho - i0)^{1/2}] \Bigg\} \, d\rho. \quad (9.3.0.8)$$

Eq. (9.3.0.7) is the generalization of Bochner's formula (9.1.0.1) to the Minkowskian Space. In this case, the extension as ultradistribution of $f(\rho)$ to the complex ρ-plane is immediate

$$F(\rho) = (2\pi)^{\frac{\nu-2}{2}} \int_{-\infty}^{\infty} \hat{f}(x) \times$$

$$\Bigg\{ \Theta[\Im(\rho)] e^{\frac{i\pi(\nu-2)}{4}} \frac{(x + i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x + i0)^{1/2}\rho^{1/2}] -$$

$$\Theta[-\Im(\rho)] e^{\frac{i\pi(2-\nu)}{4}} \frac{(x - i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x - i0)^{1/2}\rho^{1/2}] \Bigg\} \, dx. \quad (9.3.0.9)$$

Here we have taken $\rho = \gamma + i\sigma$ and

$$\rho^{1/2} = \sqrt{\frac{\gamma + \sqrt{\gamma^2 + \sigma^2}}{2}} + i\mathrm{Sgn}(\sigma)\sqrt{\frac{-\gamma + \sqrt{\gamma^2 + \sigma^2}}{2}}. \quad (9.3.0.10)$$

It is convenient here to define a Lorentz invariant tempered ultradistribution as the Fourier transform of a Lorentz invariant distribution of exponential type. Note that a Lorentz invariant tempered ultradistribution is not necessarily explicitly Lorentz invariant. When $\hat{f}$ is a Lorentz invariant distribution of exponential type, we can use (9.3.0.9) to effect the treatment to be discussed below, starting from

$$\iiiint_{-\infty}^{\infty} f(\rho)\phi(\rho, k^0) \, d^4k = (2\pi)^\nu \iiiint_{-\infty}^{\infty} \hat{f}(x)\hat{\phi}(x, x^0) d^4x, \quad (9.3.0.11)$$

from which we can deduce the equality

$$\iint_{-\infty}^{\infty} f(\rho)\phi(\rho, k^0)(k_0^2 - \rho)_+^{\frac{\nu-3}{2}} \, d\rho \, dk^0 =$$

$$\iint_{-\infty}^{\infty} \hat{f}(x)\hat{\phi}(x, x^0)(x - x_0^2)_+^{\frac{\nu-3}{2}} \, dx \, dx^0. \quad (9.3.0.12)$$

Let $g(t)$ be defined as

$$\hat{g}(t) = \frac{1}{(2\pi)^{\nu}} \int_{-\infty}^{\infty} f(\rho) e^{-i\rho t} \, d\rho. \tag{9.3.0.13}$$

Then,

$$F(\rho) = \Theta[\Im(\rho)] \int_{0}^{\infty} \hat{g}(t) e^{i\rho t} \, dt - \Theta[-\Im(\rho)] \int_{-\infty}^{0} \hat{g}(t) e^{i\rho t} \, dt, \tag{9.3.0.14}$$

or, if we use *Dirac's formula*

$$F(\rho) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-\rho} \, dt. \tag{9.3.0.15}$$

The inverse of the Fourier transform can also be evaluated in the following way: first, we define

$$\hat{G}(x, \Lambda) = \frac{1}{(2\pi)^{\frac{\nu+2}{2}}} \oint_{\Gamma} F(\rho) \times$$

$$\left\{ e^{\frac{i\pi(\nu-2)}{4}} \frac{(\rho+\Lambda)^{\frac{\nu-2}{4}}}{(x+i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}} [-i(x+i0)^{1/2}(\rho+\Lambda)^{1/2}] + \right.$$

$$\left. + \, e^{\frac{i\pi(2-\nu)}{4}} \frac{(\rho-\Lambda)^{\frac{\nu-2}{4}}}{(x-i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}} [i(x-i0)^{1/2}(\rho-\Lambda)^{1/2}] \right\} d\rho, \tag{9.3.0.16}$$

and then

$$\hat{f}(x) = \hat{G}(x, i0^{+}). \tag{9.3.0.17}$$

9.4 Examples

As a first example we consider the Fourier transform of the function $e^{a\sqrt{|x_0^2 - r^2|}}$, where a is a complex number. The Fourier transform is

$$F(\rho) = (2\pi)^{\frac{\nu-2}{2}} \int_{-\infty}^{\infty} e^{|x|^{\frac{1}{2}}} \times$$

$$\left\{\Theta[\Im(\rho)]e^{\frac{i\pi(\nu-2)}{4}}\frac{(x+i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}]-\right.$$

$$\left.\Theta[-\Im(\rho)]e^{\frac{i\pi(2-\nu)}{4}}\frac{(x-i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}]\right\}\;dx. \tag{9.4.0.1}$$

Now,

$$e^{\frac{i\pi(\nu-2)}{4}}\int\limits_{-\infty}^{\infty}e^{a|x|^{\frac{1}{2}}}(x+i0)^{\frac{\nu-2}{4}}\mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}]=$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}-ia)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)-$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}+a)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)\quad\Im(\rho)>0 \tag{9.4.0.2}$$

$$e^{\frac{i\pi(2-\nu)}{4}}\int\limits_{-\infty}^{\infty}e^{a|x|^{\frac{1}{2}}}(x-i0)^{\frac{\nu-2}{4}}\mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}]=$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{-\frac{i\pi\nu}{2}}}{(\rho^{1/2}+ia)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)-$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}+a)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)\quad\Im(\rho)<0. \tag{9.4.0.3}$$

To obtain (9.4.0.3) and (9.4.0.3) we have used (6.621 - 3) of Ref. [4]. With these results we have

$$F(\rho)=(4\pi)^{\frac{\nu-1}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\left\{\Theta[\Im(\rho)]e^{\frac{i\pi\nu}{2}}\left[\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^{\nu}}\right.\right.-$$

$$\left.\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}+a)^{\nu}}\right]-$$

$$\Theta[-\Im(\rho)]e^{-\frac{i\pi\nu}{2}}\left[\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^{\nu}}\right.-$$

$$\left.\left.\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}+a)^{\nu}}\right]\right\}. \tag{9.4.0.4}$$

As a second example, we evaluate the Fourier transform of the complex mass Wheeler propagator. We start with

$$w_{\mu}(x)=-\frac{i\pi}{2}\frac{\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}}x_{+}^{\frac{2-\nu}{4}}\mathcal{J}_{\frac{2-\nu}{2}}(\mu x_{+}^{1/2}). \tag{9.4.0.5}$$

Then, according to (9.3.0.9),

$$\mathcal{W}_{\mu}(\rho)=-\frac{i(\mu)^{\frac{\nu-2}{4}}}{4}\int\limits_{0}^{\infty}\mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2})\times$$

$$\left[\Theta[\Im(\rho)]\frac{e^{\frac{i\pi(\nu-2)}{4}}}{\rho^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})-\right.$$

$$\left.\Theta[-\Im(\rho)]\frac{e^{\frac{i\pi(2-\nu)}{4}}}{\rho^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\right]\,dx. \tag{9.4.0.6}$$

Taking into account that (See (6.576- 3), ref.[4])

$$\int\limits_{0}^{\infty}\mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})\,dx=$$

$$2\mu^{\frac{2-\nu}{2}}e^{\frac{i\pi(6-\nu)}{4}}\frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^{2}}\;\Im(\rho)>0$$

$$\int\limits_{0}^{\infty}\mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\,dx=$$

$$2\mu^{\frac{2-\nu}{2}}e^{\frac{i\pi(\nu-6)}{4}}\frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^{2}}\;\Im(\rho)<0, \tag{9.4.0.7}$$

we obtain

$$\mathcal{W}_{\mu}(\rho)=\frac{i}{2}\frac{Sgn[\Im(\rho)]}{\rho-\mu^{2}}. \tag{9.4.0.8}$$

As a third example, we evaluate the transform of $\delta(x_0^2 - r^2)$. From (9.3.0.12) we obtain

$$\iint\limits_{-\infty}^{\infty} f(\rho)\phi(\rho,k^0)(k_0^2-\rho)_+^{\frac{\nu-3}{2}}\, d\rho\, dk^0 = (2\pi)^\nu \int\limits_{-\infty}^{\infty} \phi(0,x^0)|x^0|^{\nu-3}\, dx^0. \tag{9.4.0.9}$$

According to (9.3.0.1), we can write

$$\hat{\phi}(x,x^0) = 2^{-1}(2\pi)^{-\frac{\nu+1}{2}}(x_0^2-x)_+^{\frac{3-\nu}{4}} \times$$
$$\iint\limits_{-\infty}^{\infty} \phi(\rho,k^0)\mathcal{J}_{\frac{\nu-3}{2}}[(x_0^2-x)_+^{1/2}(k_0^2-\rho)_+^{1/2}] \times$$
$$(k_0^2-\rho)_+^{\frac{\nu-3}{4}} e^{ik_0x^0}\, dk^0\, d\rho, \tag{9.4.0.10}$$

and consequently,

$$\hat{\phi}(0,x^0) = 2^{-1}(2\pi)^{-\frac{\nu+1}{2}}|x^0|^{\frac{3-\nu}{4}} \iint\limits_{-\infty}^{\infty} \phi(\rho,k^0)\mathcal{J}_{\frac{\nu-3}{2}}[|x^0|^{1/2}(k_0^2-\rho)_+^{1/2}] \times$$
$$(k_0^2-\rho)_+^{\frac{\nu-3}{4}} e^{ik_0x^0}\, dk^0\, d\rho. \tag{9.4.0.11}$$

Then,

$$(2\pi)^\nu \int\limits_{-\infty}^{\infty} \phi(0,x^0)|x^0|^{\nu-3}\, dx^0 =$$
$$2^{-1}(2\pi)^{\frac{\nu-1}{2}} \int\limits_{-\infty}^{\infty} \phi(\rho,k^0)(k_0^2-\rho)_+^{\frac{\nu-3}{4}} \left[\int\limits_{-\infty}^{\infty} |x^0|^{\frac{\nu-3}{2}} \times\right.$$
$$\left.\mathcal{J}_{\frac{\nu-3}{2}}[|x^0|^{1/2}(k_0^2-\rho)_+^{1/2}]e^{ik_0x^0}\, dx^0\right]\, dk^0\, d\rho. \tag{9.4.0.12}$$

Note that

$$\int\limits_{-\infty}^{\infty} |x^0|^{\frac{\nu-3}{2}}\mathcal{J}_{\frac{\nu-3}{2}}[|x^0|^{1/2}(k_0^2-\rho)_+^{1/2}]e^{ik_0x^0}\, dx^0 =$$
$$\frac{2^{\frac{\nu-3}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{\nu-2}{2}\right)\left[e^{\frac{i\pi(\nu-2)}{2}}(\rho+i0)^{\frac{2-\nu}{2}} + e^{\frac{i\pi(2-\nu)}{2}}(\rho-i0)^{\frac{2-\nu}{2}}\right], \tag{9.4.0.13}$$

(see (6.623- 1) in Ref [4]), from which we deduce that

$$f(\rho) = \frac{(4\pi)^{\frac{\nu-2}{2}}}{2}\Gamma\left(\frac{\nu-2}{2}\right)\left[\frac{e^{\frac{i\pi(\nu-2)}{2}}}{(\rho+i0)^{\frac{\nu-2}{2}}} + \frac{e^{\frac{i\pi(2-\nu)}{2}}}{(\rho-i0)^{\frac{\nu-2}{2}}}\right]. \tag{9.4.0.14}$$

Using then [(9.3.0.13) - (9.3.0.14)] or (9.3.0.15), the corresponding ultradistribution is

$$F(\rho) = 2^{-1}(4\pi)^{\frac{\nu-2}{2}}\Gamma\left(\frac{\nu-2}{2}\right)Sgn[\Im(\rho)](-\rho)^{\frac{2-\nu}{2}}. \tag{9.4.0.15}$$

If we wish, we can proceed now to the calculation of the convolution of two spherically symmetric tempered ultradistributions.

9.5 Convolution in Euclidean Space

The expression for the convolution of two spherically symmetric functions was deduced in Ref. [13] ($h(k) = (f * g)(k)$). Consider

$$h(k) = \frac{2^{4-\nu}\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu-1}{2})k^{\nu-2}}\iint\limits_0^\infty f(k_1)g(k_2) \times$$

$$[4k_1^2k_2^2 - (k^2-k_1^2-k_2^2)^2]_+^{\frac{\nu-3}{2}} k_1k_2\, dk_1\, dk_2, \tag{9.5.0.1}$$

and, with the change of variables $\rho = k^2$,$\rho_1 = k_1^2$, $\rho_2 = k_2^2$ takes the form

$$h(\rho) = \frac{2^{2-\nu}\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu-1}{2})\rho^{\frac{\nu-2}{2}}}\iint\limits_0^\infty f(\rho_1)g(\rho_2) \times$$

$$[4\rho_1\rho_2 - (\rho-\rho_1-\rho_2)^2]_+^{\frac{\nu-3}{2}}\, d\rho_1\, d\rho_2. \tag{9.5.0.2}$$

In particular, when $\nu = 4$ one has

$$h(\rho) = \frac{\pi}{2\rho}\iint\limits_0^\infty f(\rho_1)g(\rho_2)[4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2]_+^{\frac{1}{2}}\, d\rho_1\, d\rho_2. \tag{9.5.0.3}$$

$h(\rho)$ can be extended to complex plane as an ultradistribution, thus generalizing the procedure of Ref. [33]. According to (9.3.0.12) we can then write

$$\hat{f}(x)\hat{g}(x) = \frac{\pi^2}{(2\pi)^6 x}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} F(\rho_1)G(\rho_2)\times$$

$$\rho_1^{1/2}\rho_2^{1/2}\mathcal{J}_1(x^{1/2}\rho_1^{1/2})\mathcal{J}_1(x^{1/2}\rho_2^{1/2})d\rho_1\ d\rho_2, \tag{9.5.0.4}$$

and, Fourier transforming,

$$\mathcal{F}\left\{\hat{f}(x)\hat{g}(x)\right\}(\rho) = \frac{-\pi^2}{(2\pi)^5\rho^{1/2}}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} F(\rho_1)G(\rho_2)\rho_1^{1/2}\rho_2^{1/2}\times$$

$$\left\{\int\limits_0^\infty x^{-1/2}\mathcal{J}_1(x^{1/2}\rho_1^{1/2})\mathcal{J}_1(x^{1/2}\rho_2^{1/2})\right.$$

$$\left.\left[\Theta[\Im(\rho)]\mathcal{K}_1(-ix^{1/2}\rho^{1/2}) - \Theta[-\Im(\rho)]\mathcal{K}_1(ix^{1/2}\rho^{1/2})\right]dx\right\}d\rho_1\ d\rho_2. \tag{9.5.0.5}$$

The x-integration can also be performed with the result

$$\int\limits_0^\infty \mathcal{J}_1(x^{1/2}\rho_1^{1/2})\mathcal{J}_1(x^{1/2}\rho_2^{1/2})\mathcal{K}_1(-ix^{1/2}\rho^{1/2})dx =$$

$$-i(\rho\rho_1\rho_2)^{-1}\left[\rho-\rho_1-\rho_2-\sqrt{(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2}\right]\quad \Im(\rho)>0, \tag{9.5.0.6}$$

$$\int\limits_0^\infty \mathcal{J}_1(x^{1/2}\rho_1^{1/2})\mathcal{J}_1(x^{1/2}\rho_2^{1/2})\mathcal{K}_1(ix^{1/2}\rho^{1/2})dx =$$

$$i(\rho\rho_1\rho_2)^{-1}\left[\rho-\rho_1-\rho_2-\sqrt{(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2}\right]\quad \Im(\rho)<0, \tag{9.5.0.7}$$

where we have used (6.578 -2) of [4] and (7) in page 238 of [23]. Thus,

$$H(\rho) = \frac{i\pi}{4\rho}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} F(\rho_1)G(\rho_2)\times$$

$$\left[\rho-\rho_1-\rho_2-\sqrt{(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2}\right]d\rho_1\ d\rho_2, \tag{9.5.0.8}$$

where $|\Im(\rho)| > |\Im(\rho_1)| + |\Im(\rho_2)|$ We notice here that, in Ref. [33], the existence of the convolution product between to arbitrary one dimensional tempered ultradistributions always exists. Analogously, for spherically symmetric ultradistributions, we now define

$$H_\lambda(\rho) = \frac{i\pi}{4\rho} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} F(\rho_1) G(\rho_2) \rho_1^\lambda \rho_2^\lambda \times$$

$$\left[\rho - \rho_1 - \rho_2 - \sqrt{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}\right] d\rho_1 \; d\rho_2. \qquad (9.5.0.9)$$

Let $\mathfrak{B}$ be a vertical band contained in the complex λ-plane $\mathfrak{P}$. Integral (9.5.0.9) is an analytic function of λ defined in the domain $\mathfrak{B}$. Moreover, it is bounded by a power of $|\rho|$. Thus, according to the method of Ref. [33], H_λ can be analytically continued to other parts of $\mathfrak{P}$. In particular, near the origin, we have the *Laurent expansion*

$$H_\lambda(\rho) = \sum_{n=-m}^{\infty} H^{(n)}(\rho)\lambda^n. \qquad (9.5.0.10)$$

We now define the convolution product as the λ-independent term of (9.5.0.10)

$$H(\rho) = H^{(0)}(\rho). \qquad (9.5.0.11)$$

The proof that $H(\rho)$ is a tempered ultradistribution is similar to the one given in Ref. [33] for the one-dimensional case. The Fourier antitransform of (9.5.0.11) defines the product of two distributions of exponential type. Let $\hat{H}_\lambda(x)$ be the Fourier antitransform of $H_\lambda(\rho)$. Then,

$$\hat{H}_\lambda(x) = \sum_{n=-m}^{\infty} \hat{H}^{(n)}(x)\lambda^n. \qquad (9.5.0.12)$$

If we define

$$\hat{f}_\lambda(x) = \mathcal{F}^{-1}\{\rho^\lambda F(\rho)\}$$
$$\hat{g}_\lambda(x) = \mathcal{F}^{-1}\{\rho^\lambda G(\rho)\}, \qquad (9.5.0.13)$$

then

$$\hat{H}_\lambda(x) = (2\pi)^4 \hat{f}_\lambda(x)\hat{g}_\lambda(x), \qquad (9.5.0.14)$$

and taking into account the Laurent expansion of $\hat{f}$ and $\hat{g}$

$$\hat{f}_\lambda(x) = \sum_{n=-m_f}^{\infty} \hat{f}^{(n)}(x)\lambda^n$$

$$\hat{g}_\lambda(x) = \sum_{n=-m_g}^{\infty} \hat{g}^{(n)}(x)\lambda^n, \tag{9.5.0.15}$$

we can finally write

$$\sum_{n=-m}^{\infty} \hat{H}^{(n)}(x)\lambda^n = (2\pi)^4 \sum_{n=-m}^{\infty} \left(\sum_{k=-m_f}^{n+m_g} \hat{f}^{(k)}(x)\hat{g}^{(n-k)}(x) \right) \lambda^n. \tag{9.5.0.16}$$

$(m = m_f + m_g)$
Consequently,

$$\hat{H}^{(0)}(x) = \sum_{k=-m_f}^{m_g} \hat{f}^{(k)}(x)\hat{g}^{(-k)}(x). \tag{9.5.0.17}$$

We will give now some examples of the use of (9.5.0.11) and (9.5.0.17).

9.6 Examples

As a first example, we evaluate the convolution of two Dirac's deltas for complex mass. We have

$$\left(\delta(\rho - \mu^2) = -\frac{1}{2\pi i(\rho - \mu^2)} \right)$$

According to (9.5.0.9), (9.5.0.10), and (9.5.0.11), we have

$$\delta(\rho-\mu_1^2)*\delta(\rho-\mu_2^2) = \frac{i\pi}{4\rho}\left[\rho - \mu_1^2 - \mu_2^2 - \sqrt{(\rho - \mu_1^2 - \mu_2^2)^2 - 4\mu_1^2\mu_2^2}\right].$$

As an ultradistribution, only the term containing the square root is different from zero (Cf. (9.3.0.11)). Thus,

$$\delta(\rho - \mu_1^2) * \delta(\rho - \mu_2^2) = -\frac{i\pi}{4\rho}\sqrt{(\rho - \mu_1^2 - \mu_2^2)^2 - 4\mu_1^2\mu_2^2}. \tag{9.6.0.1}$$

When $\mu_1 = \mu_2 = m$ (m real) we obtain,

$$\delta(\rho - m^2) * \delta(\rho - m^2) = -\frac{i\pi}{4\rho^{1/2}}\sqrt{\rho - 4m^2}. \tag{9.6.0.2}$$

As a second example, we evaluate the convolution of two massless Feynman's propagators. One has

$$f(\rho) = \frac{1}{\rho}$$

$$F(\rho) = -\frac{1}{2\pi i \rho} \ln(-\rho)$$

$$F_\lambda(\rho) = -\frac{1}{2\pi i} \rho^{\lambda-1} \ln(-\rho)$$

$$\hat{f}_\lambda(x) = \frac{1}{8\pi^2 x^{1/2}} \oint_\Gamma \left(-\frac{1}{2\pi i} \rho^{\lambda-1} \ln(-\rho) \right) \rho^{1/2} \mathcal{J}_1(x^{1/2}\rho^{1/2})\, d\rho =$$

$$\frac{2^{2\lambda}\Gamma(1+\lambda)}{4\pi^2\Gamma(1-\lambda)} x^{-\lambda-1} - e^{i\pi\lambda} \sin(\pi\lambda) \frac{2^{2\lambda}\Gamma(1+\lambda)}{4\pi^2\Gamma(1-\lambda)} x^{-\lambda-1} \left[i\pi + \right.$$

$$\left. 2\ln(2) + \psi(1+\lambda) + \psi(1-\lambda) - \ln(x) \right], \qquad (9.6.0.3)$$

where $\psi(z) = \Gamma^{'}(z)/\Gamma(z)$. From (9.6.0.3) we have

$$\hat{f}_\lambda(x) = (2\pi)^{-2} x^{-1} + S_\lambda(x), \qquad (9.6.0.4)$$

with

$$\lim_{\lambda \to 0} S_\lambda(x) = 0$$

. Then,

$$\hat{f}^2_\lambda(x) = (2\pi)^{-4} x^{-2} + T_\lambda(x), \qquad (9.6.0.5)$$

with

$$\lim_{\lambda \to 0} T_\lambda(x) = 0$$

, so that, as a consequence,

$$\hat{f}^2(x) = (2\pi)^{-4} x^{-2}. \qquad (9.6.0.6)$$

Taking into account that

$$\mathcal{F}\{x^{-2}\} = -\pi^2 \ln(\rho),$$

we obtain

$$\frac{1}{\rho} * \frac{1}{\rho} = -\pi^2 \ln(\rho). \qquad (9.6.0.7)$$

9.7 DR-generalization to Minkowskian one

The convolution of two Lorentz invariant functions is given by

$$\{f * g\}(p_\mu^2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f[(p_\mu - k_\mu)^2] g(k_\mu^2)\, d^\nu k, \tag{9.7.0.1}$$

that can be re-written as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\eta_1) g(\eta_2) \delta[\eta_1 - (p_\mu - k_\mu)^2] \delta(\eta_2 - k_\mu^2)\, d\eta_1\, d\eta_2\, d^\nu k. \tag{9.7.0.2}$$

We select the axis of coordinates in a way such that the spatial component of p_μ, $\vec{p}$ coincides with the first spatial coordinate ($p_\mu^2 = p_0^2 - p_1^2$). Then, we have

$$\frac{\pi^{\frac{\nu-2}{2}}}{2|p_0|} \iiint_{-\infty}^{\infty} \frac{f(\eta_1) g(\eta_2)}{\Gamma\left(\frac{\nu-2}{2}\right)}$$

$$\left[\frac{(p_\mu^2 - \eta_1 + \eta_2 + 2p_1 k_1)^2}{4p_0^2} - k_1^2 - \eta_2\right]^{\frac{\nu-4}{2}} d\eta_1\, d\eta_2\, dk_1. \tag{9.7.0.3}$$

Using

$$x_+^{\frac{\nu-4}{2}} = \frac{\Gamma\left(\frac{\nu-2}{2}\right) e^{i\pi\left(\frac{2-\nu}{4}\right)}}{2\pi} \int_{-\infty}^{\infty} (t - i0)^{\frac{3-\nu}{2}} e^{itx}\, dt \tag{9.7.0.4}$$

with

$$x = -4p_\mu^2 k_1^2 + 4p_1 k_1 (p_\mu^2 - \eta_1 + \eta_2) + (p_\mu^2 - \eta_1 + \eta_2)^2 - 4p_0^2 \eta_2, \tag{9.7.0.5}$$

we can evaluate the integral in the variable k_1 using (2.462 - 1) of Ref. [4]. The result is

$$\sqrt{2\pi}[i(8tp_\mu^2 - i0)]^{-\frac{1}{2}} e^{\frac{itp_1^2(p_\mu^2 - \eta_1 + \eta_2)}{p_\mu^2}}. \tag{9.7.0.6}$$

We can now perform the t integration

$$I = \lim_{\epsilon \to 0} \frac{\Gamma\left(\frac{\nu-2}{2}\right) e^{\frac{i\pi(1-\nu)}{4}}}{4\sqrt{\pi}} \times$$

$$\int_{-\infty}^{\infty} (t-i\epsilon)^{\frac{2-\nu}{2}} (tp_\mu^2 - i\epsilon)^{-\frac{1}{2}} e^{\frac{itp_0^2[(p_\mu^2-\eta_1+\eta_2)^2-4p_\mu^2\eta_2]}{p_\mu^2}} dt. \quad (9.7.0.7)$$

Formula (9.7.0.7) is defined for $\nu = 2n$. In this case (9.7.0.7) is proportional to the derivative of the same order of the Dirac's formula for

$$(tp_\mu^2 - i0)^{-\frac{1}{2}} e^{\frac{itp_0^2[(p_\mu^2-\eta_1+\eta_2)^2-4p_\mu^2\eta_2]}{p_\mu^2}}.$$

Then, we have

$$I = \frac{\Gamma\left(\frac{\nu-2}{2}\right) e^{\frac{i\pi(1-\nu)}{4}}}{4\sqrt{\pi}} \int_{-\infty}^{\infty} (p_\mu^2 - i0)^{-\frac{1}{2}} t_+^{\frac{1-\nu}{2}} e^{\frac{itp_0^2[(p_\mu^2-\eta_1+\eta_2)^2-4p_\mu^2\eta_2]}{p_\mu^2}}$$

$$+(p_\mu^2 + i0)^{-\frac{1}{2}} t_+^{\frac{1-\nu}{2}} e^{-\frac{itp_0^2[(p_\mu^2-\eta_1+\eta_2)^2-4p_\mu^2\eta_2]}{p_\mu^2}} dt. \quad (9.7.0.8)$$

The result of (9.7.0.8) is immediate (a Fourier transform). We consider first the case $\nu \neq 2n+1$

$$I = \frac{e^{\frac{i\pi(2-\nu)}{2}}}{4\sqrt{\pi}} \Gamma\left(\frac{\nu-2}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right) |p_0|^{\nu-3} \quad \times$$

$$\left\{ (p_\mu^2 - i0)^{-\frac{1}{2}} \left[\frac{(p_\mu^2-\eta_1+\eta_2)^2 - 4p_\mu^2\eta_2}{p_\mu^2} + i0 \right]^{\frac{\nu-3}{2}} \right.$$

$$\left. + e^{i\pi(\nu-2)} (p_\mu^2 + i0)^{-\frac{1}{2}} \left[\frac{(p_\mu^2-\eta_1+\eta_2)^2 - 4p_\mu^2\eta_2}{p_\mu^2} - i0 \right]^{\frac{\nu-3}{2}} \right\}. \quad (9.7.0.9)$$

With this result we have for (9.7.0.3)

$$h(\rho) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right) \iint_{-\infty}^{\infty} f(\rho_1) g(\rho_2) \quad \times$$

$$\left\{ (\rho - i0)^{-\frac{1}{2}} \left[\frac{(\rho-\rho_1-\rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0 \right]^{\frac{\nu-3}{2}} + e^{i\pi(\nu-2)} \quad \times \right.$$

$$\left. (\rho + i0)^{-\frac{1}{2}} \left[\frac{(\rho-\rho_1-\rho_2)^2 - 4\rho_1\rho_2}{\rho} - i0 \right]^{\frac{\nu-3}{2}} \right\} d\rho_1 \, d\rho_2, \quad (9.7.0.10)$$

where $\rho = p_\mu^2$ and $h = f * g$.
When $\nu = 4$, we have

$$h(\rho) = \frac{\pi}{2\rho} \iint\limits_{-\infty}^{\infty} f(\rho_1) g(\rho_2) \left[(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2\right]_+^{\frac{1}{2}} \, d\rho_1 \, d\rho_2. \tag{9.7.0.11}$$

When $\nu = 2n + 1$, we obtain

$$h(\rho) = -\frac{i\pi^{n-1}}{2^{2n}(n-1)!} \iint\limits_{-\infty}^{\infty} f(\rho_1) g(\rho_2) \times$$

$$\left[\frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho}\right]^{n-1} \Big\{(\rho - i0)^{-\frac{1}{2}} \times$$

$$\left[\psi(n) + \frac{i\pi}{2} + \ln\left[\frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0\right]\right] - (\rho + i0)^{-\frac{1}{2}}$$

$$\left[\psi(n) + \frac{i\pi}{2} + \ln\left[-\frac{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0\right]\right]\Big\} \, d\rho_1 \, d\rho_2. \tag{9.7.0.12}$$

As an example., we will evaluate the convolution of $\delta(\rho - m_1^2)$ with $\delta(\rho - m_2^2)$ for $\nu \neq 2n + 1$. In this case we have

$$h(\rho) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \times$$

$$\Big\{(\rho - i0)^{-\frac{1}{2}} \left[\frac{(\rho - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{\rho} + i0\right]^{\frac{\nu-3}{2}} +$$

$$e^{i\pi(\nu-2)} (\rho + i0)^{-\frac{1}{2}} \left[\frac{(\rho - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{\rho} - i0\right]^{\frac{\nu-3}{2}}\Big\}. \tag{9.7.0.13}$$

When $\nu = 4, m_1 = 0, m_2 = m$, we obtain

$$\delta(\rho) * \delta(\rho - m^2) = \frac{\pi}{2\rho} |\rho - m^2|. \tag{9.7.0.14}$$

If we use the dimension ν as a regularizing parameter, we can define the product of two tempered distributions as

$$\hat{h}(x, \nu) = (2\pi)^\nu \hat{f}(x, \nu) \hat{g}(x, \nu) = (2\pi)^\nu \mathcal{F}^{-1}\{f(\rho, \nu)\} \mathcal{F}^{-1}\{g(\rho, \nu)\} =$$

$$\mathcal{F}^{-1}\{f(\rho,\nu) * g(\rho,\nu)\} = \mathcal{F}^{-1}\{h(\rho,\nu)\}, \tag{9.7.0.15}$$

where $\mathcal{F}^{-1}$ was defined in section 5 by means of (9.3.0.8) and where (9.7.0.10) should be re-interpreted as

$$h(\rho,\nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right) \iint\limits_{-\infty}^{\infty} f(\rho_1,\nu) g(\rho_2,\nu) \quad \times$$

$$\left\{ (\rho - i0)^{-\frac{1}{2}} \left[\frac{(\rho-\rho_1-\rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0 \right]^{\frac{\nu-3}{2}} + e^{i\pi(\nu-2)} \quad \times \right.$$

$$\left. (\rho + i0)^{-\frac{1}{2}} \left[\frac{(\rho-\rho_1-\rho_2)^2 - 4\rho_1\rho_2}{\rho} - i0 \right]^{\frac{\nu-3}{2}} \right\} d\rho_1\, d\rho_2. \tag{9.7.0.16}$$

The same procedure is valid when $\hat{f}(x,\nu)$ and $\hat{g}(x,\nu)$ are distributions of exponential type. Here $f(\rho,\nu)$ and $g(\rho,\nu)$ are defined by

$$F(\rho,\nu) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f(t,\nu)}{t-\rho} dt$$

$$G(\rho,\nu) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{g(t,\nu)}{t-\rho} dt,$$

where F and G are the tempered ultradistributions given by

$$F(\rho,\nu) = \mathcal{F}\{\hat{f}(x,\nu)\} \qquad G(\rho,\nu) = \mathcal{F}\{\hat{g}(x,\nu)\}.$$

A significant result has been achieved at this point, whose importance must be emphasized. The above described procedure generalizes to Minkowskian space the dimensional regularization in configuration space defined in Ref. [13] for Euclidean space.

As an example of the use of this method, we give the evaluation of the convolution product of two complex mass Wheeler propagators. From (9.4.0.5) and (9.3.0.9), we have

$$\mathcal{F}\{w_{\mu_1}(x,\nu) w_{\mu_2}(x,\nu)\} =$$

$$-\frac{\pi^2}{2\rho} \frac{(\mu_1\mu_2)^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu+2}{2}}} \int\limits_0^{\infty} x^{\frac{4-\nu}{2}} \mathcal{J}_{\frac{2-\nu}{2}}(\mu_1 x) \mathcal{J}_{\frac{2-\nu}{2}}(\mu_2 x) \times$$

$$\Big\{\Theta[\Im(\rho)]e^{\frac{i\pi(\nu-2)}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(-ix\rho^{1/2})-$$

$$\Theta[-\Im(\rho)]e^{\frac{i\pi(2-\nu)}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(ix\rho^{1/2})\Big\}\,dx. \tag{9.7.0.17}$$

To evaluate (9.7.0.17) we use

$$\int_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu_1 x)\mathcal{J}_{\frac{2-\nu}{2}}(\mu_2 x)\mathcal{K}_{\frac{\nu-2}{2}}(xz)\,dx =$$

$$\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3-\nu}{2}\right)}{2^{\frac{3\nu-6}{2}}}\frac{z^{\frac{2-\nu}{2}}}{(\mu_1\mu_2)^{\frac{\nu-2}{2}}}\left[(z^2+\mu_1^2+\mu_2^2)^2-4\mu_1^2\mu_2^2\right]^{\frac{\nu-3}{2}}, \tag{9.7.0.18}$$

where to deduce (9.7.0.18) we have used

$$\mathcal{K}_{\frac{\nu-2}{2}}(xz)=\frac{1}{2}\left(\frac{zx}{2}\right)^{\frac{\nu-2}{2}}\int_0^\infty t^{-\frac{\nu}{2}}e^{-t-\frac{z^2x^2}{4t}}\,dt,$$

(see (8.432- 6) of Ref. [4]). Thus, from (9.7.0.18) we have

$$\mathcal{F}\{w_{\mu_1}(x,\nu)w_{\mu_2}(x,\nu)\}=\frac{(2\pi)^{\frac{1-\nu}{2}}}{2^{\frac{3\nu-1}{2}}}\Gamma\left(\frac{3-\nu}{2}\right)e^{\frac{i\pi(\nu-2)}{2}}\quad\times$$

$$\rho^{\frac{\nu-2}{2}}Sgn[\Im(\rho)]\left[(\rho-\mu_1^2-\mu_2^2)^2-4\mu_1^2\mu_2^2\right]^{\frac{\nu-3}{2}}, \tag{9.7.0.19}$$

and consequently,

$$\{W_{\mu_1}(\rho,\nu)*W_{\mu_2}(\rho,\nu)\}=\frac{(2\pi)^{\frac{\nu+1}{2}}}{2^{\frac{3\nu-1}{2}}}\Gamma\left(\frac{3-\nu}{2}\right)e^{\frac{i\pi(\nu-2)}{2}}\quad\times$$

$$\rho^{\frac{\nu-2}{2}}Sgn[\Im(\rho)]\left[(\rho-\mu_1^2-\mu_2^2)^2-4\mu_1^2\mu_2^2\right]^{\frac{\nu-3}{2}}. \tag{9.7.0.20}$$

9.8 Convolution of L.I. ultradistributions

To obtain an expression for the convolution of two tempered ultradistributions we focus attention upon Eq. (9.7.0.11). As a first step, we extend $h(\rho)$ as a tempered ultradistribution. For this purpose we consider the function

$$l(\rho,\rho_1,\rho_2)=\left[(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2\right]_+^{\frac{1}{2}}. \tag{9.8.0.1}$$

The Fourier anti-transform of (9.8.0.1) is

$$\hat{l}(x,\rho_1,\rho_2) = \frac{e^{-i(\rho_1+\rho_2)x}}{|x|}\Big\{(\rho_1\rho_2+i0)^{\frac{1}{2}}\mathcal{N}_1\left[2(\rho_1\rho_2+i0)^{\frac{1}{2}}|x|\right]+$$

$$\Theta(-\rho_1\rho_2)\sqrt{-\rho_1\rho_2}\,\mathcal{J}_1(2i\sqrt{-\rho_1\rho_2}\;|x|)\Big\}, \qquad (9.8.0.2)$$

where $\mathcal{N}_1$ is the *Newman function.* If we consider now the distribution

$$m(\rho,\rho_1,\rho_2) = \rho^{-1}\left[(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2\right]_+^{\frac{1}{2}}, \qquad (9.8.0.3)$$

the corresponding tempered ultradistribution is

$$M(\rho,\rho_1,\rho_2) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{t^{-1}\left[(t-\rho_1-\rho_2)^2-4\rho_1\rho_2\right]_+^{\frac{1}{2}}}{t-\rho}\,dt, \quad (9.8.0.4)$$

which can also be written as

$$M(\rho,\rho_1,\rho_2) = \frac{1}{\rho}\Big\{\mathcal{F}\{\hat{l}\}(\rho,\rho_1,\rho_2) \quad -$$

$$\frac{1}{2}\left[\mathcal{F}\{\hat{l}\}(i0,\rho_1,\rho_2)+\mathcal{F}\{\hat{l}\}(-i0,\rho_1,\rho_2)\right]\Big\}. \qquad (9.8.0.5)$$

Thus, the extension to the complex plane of $h(\rho)$, $N(\rho)$ is

$$N(\rho) = \frac{\pi}{2}\iint\limits_{-\infty}^{\infty} f(\rho_1)g(\rho_2)M(\rho,\rho_1,\rho_2)\;d\rho_1\;d\rho_2. \qquad (9.8.0.6)$$

To obtain M in explicit manner we use the following Laplace transforms

$$\mathcal{L}\left\{t^{-1}\mathcal{N}_1(at)\right\}(s) = -\frac{2}{\pi a}\sqrt{s^2+a^2}\ln\left(\frac{\sqrt{s^2+a^2}+s}{a}\right) \;+$$

$$\frac{2s}{a\pi}\left(\ln(2)+1-\gamma\right), \qquad (9.8.0.7)$$

$$\mathcal{L}\left\{t^{-1}\mathcal{J}_1(at)\right\}(s) = \frac{\sqrt{s^2+a^2}-s}{a}, \qquad (9.8.0.8)$$

(see [4] pags. 310 and 313). Thus, we have for the Fourier transforms

$$\mathcal{F}\left\{|t|^{-1}\mathcal{N}_1(a|t|)\right\}(\rho) =$$

$$-\frac{2}{\pi a}\left\{\Theta[\Im(\rho)]\left[\sqrt{a^2-\rho^2}\ln\left(\frac{\sqrt{a^2-\rho^2}-i\rho}{a}\right)+\right.\right.$$

$$i\rho\left(\ln(2)+1-\gamma\right)\Big]-\Theta[-\Im(\rho)]\left[\sqrt{a^2-\rho^2}\ln\left(\frac{\sqrt{a^2-\rho^2}+i\rho}{a}\right)-\right.$$

$$\left.\left.i\rho\left(\ln(2)+1-\gamma\right)\right]\right\}, \qquad (9.8.0.9)$$

$$\mathcal{F}\left\{|t|^{-1}\mathcal{J}_1(a|t|)\right\}(\rho)=\Theta[\Im(\rho)]\frac{\sqrt{a^2-\rho^2}-i\rho}{a}\ -$$

$$\Theta[-\Im(\rho)]\frac{\sqrt{a^2-\rho^2}+i\rho}{a}. \qquad (9.8.0.10)$$

With these results, we obtain

$$M(\rho)=\Theta[\Im(\rho)]\left\{\Theta(\rho_1\rho_2)\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}\ \times\right.$$

$$\ln\left[\frac{\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2\sqrt{\rho_1\rho_2}}\right]\ +$$

$$\Theta(-\rho_1\rho_2)\left\{\frac{i\pi}{2}\left[\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]\ +\right.$$

$$\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}$$

$$\left.\left.\ln\left[\frac{\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-\rho_1\rho_2}}\right]\right\}\right\}-$$

$$\Theta[-\Im(\rho)]\left\{\Theta(\rho_1\rho_2)\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}\times\right.$$

$$\ln\left[\frac{\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}+i(\rho-\rho_1-\rho_2)}{2\sqrt{\rho_1\rho_2}}\right]\ +$$

$$\Theta(-\rho_1\rho_2)\left\{\frac{i\pi}{2}\left[\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}+i(\rho-\rho_1-\rho_2)\right]\ +\right.$$

$$\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}$$

$$\left.\left.\ln\left[\frac{\sqrt{4\rho_1\rho_2-(\rho-\rho_1-\rho_2)^2}+i(\rho-\rho_1-\rho_2)}{2i\sqrt{-\rho_1\rho_2}}\right]\right\}\right\}-$$

$$\frac{i}{2}\left\{\Theta(\rho_1\rho_2)(\rho_1-\rho_2)\ln\left(\frac{\rho_1}{\rho_2}\right)+\Theta(-\rho_1\rho_2)(\rho_1-\rho_2)\ln\left(-\frac{\rho_1}{\rho_2}\right)+\right.$$
$$\Theta(-\rho_1)\Theta(\rho_2)\left[i\pi(\rho_1-\rho_2)\right.$$
$$Sgn(\rho_1+\rho_2)+2i\pi\rho_2\Theta(\rho_1+\rho_2)+2i\pi\rho_1\Theta(-\rho_1-\rho_2)]+$$
$$\Theta(\rho_1)\Theta(-\rho_2)\left[-i\pi(\rho_1-\rho_2)Sgn(\rho_1+\rho_2)+2i\pi\rho_1\Theta(\rho_1+\rho_2)\right.\quad+$$
$$\left.\left.2i\pi\rho_2\Theta(-\rho_1-\rho_2)\right].\right\} \tag{9.8.0.11}$$

So as to obtain an expression for the convolution of two ultradistributions, we use for the Heaviside function the identity

$$\Theta(xy)=\Theta(x)\Theta(y)+\Theta(-x)\Theta(-y). \tag{9.8.0.12}$$

Taking into account that

$$\Theta(\rho)=\lim_{\Lambda\to i0^+}\frac{1}{2\pi i}[\ln(-\rho+\Lambda)-\ln(-\rho-\Lambda)], \tag{9.8.0.13}$$

a conceptually simple by rather lengthy expression is obtained for Lorentz invariant tempered ultradistributions:

$$H_\lambda(\rho,\Lambda)=$$

$$\frac{1}{8\pi^2\rho}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}F(\rho_1)G(\rho_2)\rho_1^\lambda\rho_2^\lambda\{\Theta[\Im(\rho)]\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)]\times$$
$$[\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]$$
$$+[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]$$
$$+[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\Bigg\}$$

$$+[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}\Bigg\}-$$

$$\Theta[-\Im(\rho)]\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]$$

$$+[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]$$

$$+[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\Bigg\}\Bigg\}$$

$$-\frac{i}{2}\times$$

$$\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(i\sqrt{\frac{\rho_1+\Lambda}{\rho_2+\Lambda}}\right)+\ln\left(-i\sqrt{\frac{\rho_1-\Lambda}{\rho_2-\Lambda}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(-i\sqrt{\frac{\Lambda-\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(i\sqrt{\frac{\Lambda+\rho_1}{\Lambda+\rho_2}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)\right]+\right.$$

$$\frac{(\rho_1-\rho_2)}{2}[\ln(-\rho_1-\rho_2+\Lambda)-\ln(-\rho_1-\rho_2-\Lambda)-$$

$$\ln(\rho_1+\rho_2+\Lambda)+\ln(\rho_1+\rho_2-\Lambda)]+\rho_2[\ln(-\rho_1-\rho_2+\Lambda)-$$

$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\}$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)\right]+\right.$$

$$\frac{(\rho_1-\rho_2)}{2}[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)-$$

$$\ln(-\rho_1-\rho_2+\Lambda)+\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1[\ln(-\rho_1-\rho_2+\Lambda)-$$

$$\ln(-\rho_1-\rho_2-\Lambda)]+$$

$$\rho_2[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\}\}\}\}\ d\rho_1\ d\rho_2 \tag{9.8.0.14}$$

which defines an ultradistribution in the variables ρ and Λ for $|\Im(\rho)|>\Im(\Lambda)>|\Im(\rho_1)|+|\Im(\rho_2)|$.

Let $\mathfrak{B}$ be a vertical band contained in the complex λ-plane $\mathfrak{P}$. The integral (9.8.0.14) is an analytic function of λ defined in the domain $\mathfrak{B}$. Moreover, it is bounded by a power of $|\rho\Lambda|$. Thus, according to Ref. [33], $H_\lambda(\rho,\Lambda)$ can be analytically continued to other parts of $\mathfrak{P}$. We define

$$H(\rho) = H^{(0)}(\rho, i0^+), \tag{9.8.0.15}$$

$$H_\lambda(\rho, i0^+) = \sum_{-m}^{\infty} H^{(n)}(\rho, i0^+)\lambda^n. \tag{9.8.0.16}$$

As in other cases, we further define

$$\{F * G\}(\rho) = H(\rho), \tag{9.8.0.17}$$

as the convolution of two Lorentz invariant tempered ultradistributions. The proof that $H(\rho)$ is a tempered ultradistribution is similar to the one given in Ref. [33] for the one-dimensional case. Starting with (9.8.0.14), we can write

$$H_\lambda(\rho, i0^+) = -\frac{1}{2\rho}\iint_{-\infty}^{\infty} f_\lambda(\rho_1)g_\lambda(\rho_2)M(\rho.\rho_1,\rho_2)\, d\rho_1\, d\rho_2, \tag{9.8.0.18}$$

where $f_\lambda(\rho)$ and $g_\lambda(\rho)$ are defined by Dirac's formula

$$\rho^\lambda F_\lambda(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f_\lambda(t)}{t-\rho}\, dt \;\; ; \;\; \rho^\lambda G_\lambda(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g_\lambda(t)}{t-\rho}\, dt. \tag{9.8.0.19}$$

Let $\hat{H}_\lambda(x)$ be the Fourier anti-transform of $H_\lambda(\rho, i0^+)$. Then, according to (9.5.0.12) - (9.5.0.17), we can express $H^{(0)}(x)$ as a function of the Laurent expansions of $\hat{f}_\lambda(x)$ and $\hat{g}_\lambda(x)$.

9.9 Examples

As an example of the use of (9.8.0.15), we will evaluate the convolution product of $\delta(\rho)$ and $\delta(\rho-\mu^2)$, with $\mu = \mu_R + i\mu_I$ a complex number such that $\mu_R^2 > \mu_I^2$, $\mu_R\mu_I > 0$. Thus, from (9.8.0.14) we obtain

$$H_0(\rho,\Lambda) = -$$

$$i\pi[\ln(-\mu^2+\Lambda) - \ln(-\mu^2+\lambda)]\left\{\frac{i(\rho-\mu^2)}{8\pi^2\rho}\left[\ln\left(\frac{\rho-\mu^2}{\sqrt{\Lambda(\mu^2+\Lambda)}}\right)+\right.\right.$$

$$\ln\left(\frac{\mu^2-\rho}{\sqrt{-\Lambda(\mu^2+\Lambda)}}\right)\Bigg] + \frac{\mu^2-\rho}{16\pi\rho}\Bigg\}$$

$$-i\pi[\ln(-\mu^2+\Lambda)-\ln(-\mu^2+\lambda)] \;\times$$

$$\left\{\frac{-i\mu^2)}{8\pi^2\rho}\left[\ln\left(\sqrt{\frac{\Lambda}{\mu^2+\Lambda}}\right)+\ln\left(\sqrt{\frac{\Lambda}{\Lambda-\mu^2}}\right)\right]-\frac{\mu^2}{16\pi\rho}\right\}. \quad (9.9.0.1)$$

Simplifying terms, (9.9.0.1) leads to

$$H_0(\rho,\Lambda) = -$$

$$i\pi[\ln(-\mu^2+\Lambda)-\ln(-\mu^2+\Lambda)]\left\{\frac{i(\rho-\mu^2)}{8\pi^2\rho}\left[\ln\left(\rho-\mu^2\right)+\right.\right.$$

$$\left.\left.\ln\left(\mu^2-\rho\right)\right]+\frac{i\mu^2}{8\pi^2\rho}\left[\ln(\mu^2+\Lambda)+\ln(\mu^2-\Lambda)\right]\right\}. \quad (9.9.0.2)$$

Now, if

$$F_1(\mu,\Lambda) = \ln(-\mu^2+\Lambda)-\ln(-\mu^2-\Lambda),$$

then

$$F_1(\mu, i0^+) = 2i\pi \;;\; \mu_R^2 > \mu_I^2 \;;\; \mu_R\mu_I > 0.$$

Also, if

$$F_2(\mu,\Lambda) = \ln(\mu^2+\Lambda)-\ln(\mu^2-\Lambda),$$

then

$$F_2(\mu, i0^+) = 0 \;;\; \mu_R^2 > \mu_I^2 \;;\; \mu_R\mu_I > 0.$$

Using these results we find

$$H(\rho) = \frac{i(\rho-\mu^2)}{4\rho}\left[\ln(\rho-\mu^2)+\ln(\mu^2-\rho)\right]+\frac{i\mu^2}{2\rho}\ln(\mu^2). \quad (9.9.0.3)$$

As an example of the use of (9.5.0.17), we will evaluate the convolution product of two Dirac's deltas $\delta(\rho) * \delta(\rho)$. In this case, we have

$$F_\lambda(\rho) = -\frac{\rho^{\lambda-1}}{2\pi i}, \quad (9.9.0.4)$$

and as a consequence

$$f_\lambda(\rho) = \frac{\sin(\pi\lambda)}{\pi}\rho_-^{\lambda-1}. \quad (9.9.0.5)$$

The Fourier anti-transform of (9.9.0.5) is

$$\hat{f}_\lambda(x) - \frac{2^{2\lambda}}{4\pi^3}\frac{\Gamma(1+\lambda)}{\Gamma(1-\lambda)}\left[x_+^{-\lambda-1} - \cos(\pi\lambda)x_-^{-\lambda-1}\right], \tag{9.9.0.6}$$

which can be written as

$$\hat{f}_\lambda(x) - \frac{2^{2\lambda}}{4\pi^3}\frac{\Gamma(1+\lambda)}{\Gamma(1-\lambda)}\left[\frac{\cos(\pi\lambda)-1}{\lambda}\delta(x) + x_+^{-1} - \cos(\pi\lambda)x_-^{-1} \quad + \right.$$

$$\left. S_+^{-\lambda-1} - \cos(\pi\lambda)S_-^{-\lambda-1}\right]. \tag{9.9.0.7}$$

Thus, we have

$$\hat{f}^2_\lambda(x) - \frac{2^{4\lambda}}{16\pi^6}\frac{\Gamma^2(1+\lambda)}{\Gamma^2(1-\lambda)}\left\{\frac{(\cos(\pi\lambda)-1)^2}{\lambda^2}\delta^2(x) + x_+^{-2} + \cos^2(\pi\lambda)x_-^{-2} + \right.$$

$$\left[S_+^{-\lambda-1} - \cos(\pi\lambda)S_-^{-\lambda-1}\right]^2 +$$

$$2[x_+^{-1} - \cos(\pi\lambda)x_-^{-1}][S_+^{-\lambda-1} - \cos(\pi\lambda)S_-^{-\lambda-1}] +$$

$$\left. 2\left[\frac{\cos(\pi\lambda)-1}{\lambda}\delta(x)\right]\left[x_+^{-1} - \cos(\pi\lambda)x_-^{-1} + S_+^{-\lambda-1} - \cos(\pi\lambda)S_-^{-\lambda-1}\right]\right\}. \tag{9.9.0.8}$$

From (9.9.0.8) we then obtain

$$\lim_{\lambda\to 0}\hat{f}^2_\lambda(x) = \frac{4}{(2\pi)^6}x^{-2}, \tag{9.9.0.9}$$

and taking into account that

$$\mathcal{F}\{x^{-2}\} = \frac{\pi^3}{2}Sgn(\rho), \tag{9.9.0.10}$$

we find

$$\delta(\rho) * \delta(\rho) = \frac{\pi}{2}Sgn(\rho). \tag{9.9.0.11}$$

9.10 N massless Feynman propagators

Let us now calculate the convolution of n massless Feynman propagators ($n \geq 2$). For this purpose, we take into account that

$$\mathcal{F}^{-1}\{f_1 * f_2 * \cdots * f_n\} = (2\pi)^{(n-1)\nu}\hat{f}_1\hat{f}_2\cdots\hat{f}_n. \tag{9.10.0.1}$$

According to Reference [6], we have

$$\mathcal{F}^{-1}\left\{(\rho+i0)^{\lambda-1}\right\} = \frac{i2^{2\lambda}}{(2\pi)^2}\frac{\Gamma(\lambda+1)}{\Gamma(1-\lambda}(x-i0)^{-\lambda-1}, \tag{9.10.0.2}$$

and therefore,

$$\mathcal{F}^{-1}\left\{(\rho+i0)^{\lambda-1} * (\rho+i0)^{\lambda-1} * \cdots * (\rho+i0)^{\lambda-1}\right\} =$$

$$(2\pi)^{4(n-1)}\frac{i^n 2^{2n\lambda}}{(2\pi)^{2n}}\left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-1)}\right]^n (x-i0)^{-n(\lambda+1)}. \tag{9.10.0.3}$$

Using again Reference [6] we face now

$$\mathcal{F}\left\{(x-i0)^{-n(\lambda+1)}\right\} =$$

$$-i2^{4-2n(\lambda+1)}\pi^2\frac{\Gamma(2-n\lambda-n)}{\Gamma(n\lambda+n)}(\rho+i0)^{n\lambda+n-2}, \tag{9.10.0.4}$$

with which we obtain

$$(\rho+i0)^{\lambda-1} * (\rho+i0)^{\lambda-1} * \cdots * (\rho+i0)^{\lambda-1} =$$

$$i^{n-1}\pi^{2(n-1)}\frac{\Gamma^n(\lambda+1)}{\Gamma^n(1-\lambda)}\frac{\Gamma(2-n\lambda-n)}{\Gamma(n\lambda+n)}(\rho+i0)^{n\lambda+n-2}. \tag{9.10.0.5}$$

We have then, for the convolution of n massless Feynman propagators, the result

$$i(\rho+i0)^{\lambda-1} * i(\rho+i0)^{\lambda-1} * \cdots * i(\rho+i0)^{\lambda-1} =$$

$$i(-1)^{n+1}\pi^{2(n-1)}\frac{\Gamma^n(\lambda+1)}{\Gamma^n(1-\lambda)}\frac{\Gamma(2-n\lambda-n)}{\Gamma(n\lambda+n)}(\rho+i0)^{n\lambda+n-2}. \tag{9.10.0.6}$$

After a tedious calculation we find the corresponding Laurent expansion around $\lambda = 0$

$$i(\rho+i0)^{\lambda-1} * i(\rho+i0)^{\lambda-1} * \cdots * i(\rho+i0)^{\lambda-1} = \frac{i\pi^{2(n-1)}\rho^{n-2}}{n\lambda\Gamma(n)\Gamma(n-1)}+$$

$$\frac{i\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)}\left[\ln(\rho+i0)+2\psi(1)-\psi(n-1)-\psi(n)\right]+$$

$$\sum_{m=1}^{\infty} a_m(\rho)\lambda^m. \tag{9.10.0.7}$$

The independent of λ term is the result of the convolution

$$i(\rho+i0)^{-1} * i(\rho+i0)^{-1} * \cdots * i(\rho+i0)^{-1} =$$

$$\frac{i\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)}\left[\ln(\rho+i0)+2\psi(1)-\psi(n-1)-\psi(n)\right]. \tag{9.10.0.8}$$

9.11 Discussion

The existence of the convolution of two one-dimensional tempered ultradistributions has been previously demonstrated in [33]. Later, the efforts of Ref. [39] have extended the procedure to an n-dimensional space. Here we dealt with a four-dimensional space and have given an expression for the convolution of two tempered ultradistributions that are even in the variables k^0 and ρ. In this chapter we also found an expression for the convolution of two Lorentz invariant tempered ultradistributions in both Euclidean and Minkowskian spaces. In an intermediate step of the associated deduction [see the surroundings of Eq. (9.7.0.16)] we obtained also a very important result, namely the generalization to Minkowskian space of the dimensional regularization technique devised originally for configuration space (Ref. [13]).

When we use the perturbative expansion in quantum field theory, we often face products of distributions in configuration space, or else, convolutions in the Fourier transformed p-space. Unfortunately, products or convolutions (of distributions) are in general ill-defined quantities. In such unfortunate circumstances however, in physical applications, it is customary to introduce some "regularization" scheme, so as to give sense to divergent integrals. Amongst these procedures, we would like to mention the dimensional regularization method (Ref. [12]). Essentially, the method consists in the separation of the volume element ($d^\nu p$) into an angular factor ($d\Omega$) and a radial factor ($p^{\nu-1}dp$). First, the angular integration is carried out and then, the number of dimensions ν is taken as a free parameter. It can be adjusted to yield a convergent integral, which is an analytic function of ν.

Our formula (7.34) is similar to the expression one obtains with dimensional regularization. However, the parameter λ is now completely independent of any dimensional interpretation.

All ultradistributions provide integrands (in (7.34)) that are analytic functions along the integration path. The parameter λ permits us to control the possible tempered asymptotic behavior (Cf. Eq. (3.9)). The existence of a region of analyticity in λ, and a subsequent continuation to the point of interest (Ref. [33] , defines the convolution product.

The properties just described show that tempered ultradistributions provide an appropriate framework for applications to physics. Furthermore, they can "absorb" arbitrary pseudo-polynomials, thanks to

Eq. (3.10), a feature that is interesting for renormalization theory. For this reason, and also for the benefit of the reader, we began this chapter with a summary of the main characteristics of n-dimensional tempered ultradistributions and their Fourier transformed distributions of exponential type.

As a final remark, we would like to point out that our formula for convolutions is a definition, and not a regularization technique.

Chapter 10

Exponential ultradistributions

We begin by insisting on recalling the following facts. The question of the product of distributions with coincident point singularities is related, in field theory, to the asymptotic behavior of loop integrals of propagators. From a mathematical point of view, the question reduces to the possibility of defining a product in a ring with zero-factors. As it is well known, the usual definitions lead to limitations on the set of distributions that can be multiplied together to give another distribution of the same kind. The properties of tempered ultradistributions (Refs. [9, 10]) are well adapted for their use in field theory. In this respect, it has been shown (Refs. [33, 39, 41]) that it is indeed possible to define the convolution of any pair of tempered ultradistributions, giving as a result another tempered ultradistribution.

Ultradistributions have the further advantage of being representable by means of analytic functions, so that, in general, they are easier to work with and, as we saw above, have interesting properties. One of those properties is that Schwartz's tempered distributions are canonical and continuously injected into tempered ultradistributions and, as a consequence, the rigged Hilbert space with tempered distributions is canonical and continuously included in the rigged Hilbert space with tempered ultradistributions.

A further advance is that of considering ultradistributions of *exponential type* (Res. [10, 11]) and to define a convolution product for any pair of them. This is also made in this chapter, together with some

examples and applications. It should be remembered that Schwartz's tempered distributions and Sebastiao e Silva's tempered ultradistributions are canonical and continuously injected into ultradistributions of exponential type and, as a consequence, the rigged Hilbert spaces with tempered distributions and tempered ultradistributions are canonical and continuously included in the rigged Hilbert space with ultradistributions of exponential type.
Furthermore, ultradistributions of exponential type are adequate to describe Gamow states and exponentially increasing fields in quantum field theory, as we will show in this chapter.

10.1 Fourier transform in Euclidean space

We define a spherically symmetric ultradistribution of exponential type $\hat{F}(z)$ as a ultradistribution of exponential type such that $\hat{f}(t)$ in (4.3.0.24) is spherically symmetric (note that a spherically symmetric ultradistribution is not necessarily spherically symmetric in explicit fashion). In this case, we can use for the Fourier transform of $\hat{f}(t)$ the formula obtained in Ref. [41]

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int\limits_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \Big\{ \Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) -$$

$$\Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \Big\} \; dx \; +$$

$$\frac{2\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})\rho^{\frac{\nu-2}{4}}} \int\limits_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2}) \; dx. \tag{10.1.0.1}$$

When $\nu = 2n$, n an integer number, $\rho^{\frac{2-\nu}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}$ is null. In fact

$$\rho^{\frac{2-\nu}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}} = \sum_{m=0}^{\frac{\nu-4}{2}} \frac{(\frac{\nu}{2} - m)!}{m!} 4^{\frac{\nu-2-4m}{4}} x^{\frac{4m+2-\nu}{4}} \rho^{\frac{2m+2-\nu}{2}} = 0, \tag{10.1.0.2}$$

that is null in the complex variable ρ in a space of dimension $\nu = 2n$. Thus, in this case the second integral in (10.1.0.1) vanishes so that

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int\limits_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \Big[\Theta[\Im(\rho)] e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})$$

$$-\Theta[-\Im(\rho)]e^{i\frac{\pi}{4}\nu}\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\Big]\ dx. \tag{10.1.0.3}$$

In the next section we shall see that formulae (10.1.0.2) and (10.1.0.3) can be generalized to Minkowskian space. When $\hat{f}$ is spherically symmetric, we can use (10.1.0.3) to define its Fourier transform. In addition, for $\nu = 2n$ we can follow the treatment of Ref. [6] to define the Fourier transform. Thus, we have

$$\int_{-\infty}^{\infty} f(\rho)\phi(\rho)\rho^{\frac{\nu-2}{2}}\ d\rho = (2\pi)^{\nu}\int_{0}^{\infty}\hat{f}(x)\hat{\phi}(x)x^{\frac{\nu-2}{2}}\ dx. \tag{10.1.0.4}$$

The corresponding ultradistribution of exponential type in the one-dimensional complex variable ρ is obtained in the following way. Let $\hat{g}(t)$ be defined as

$$\hat{g}(t) = \frac{1}{(2\pi)^{\nu}}\int_{-\infty}^{\infty} f(\rho)e^{-i\rho t}\ d\rho. \tag{10.1.0.5}$$

Then,

$$F(\rho) = \Theta[\Im(\rho)]\int_{0}^{\infty}\hat{g}(t)e^{i\rho t}\ dt - \Theta[-\Im(\rho)]\int_{-\infty}^{0}\hat{g}(t)e^{i\rho t}\ dt, \tag{10.1.0.6}$$

or, if we use Dirac's formula,

$$F(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-\rho}\ dt. \tag{10.1.0.7}$$

The inversion formula ($\nu = 2n$) for $F(\rho)$ is given by

$$\hat{f}(x) = \frac{\pi}{(2\pi)^{\frac{\nu+2}{2}}x^{\frac{\nu-2}{4}}}\oint_{\Gamma} F(\rho)\rho^{\frac{\nu-2}{4}}\mathcal{J}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\ d\rho. \tag{10.1.0.8}$$

Note that the part of the integrand that multiplies $F(\rho)$ is an entire function of ρ for $\nu = 2n$. In this case, the first term of (10.1.0.4) takes the form

$$\oint_{\Gamma} F(\rho)\phi(\rho)\rho^{\frac{\nu-2}{2}}\ d\rho = (2\pi)^{\nu}\int_{0}^{\infty}\hat{f}(x)\hat{\phi}(x)x^{\frac{\nu-2}{2}}\ dx. \tag{10.1.0.9}$$

We give now same examples of the use of Fourier transforms.

10.2 Examples

As a first example, we calculate the Fourier transform of

$$2^{-\nu}\Theta[\Im(z_1)]\Theta[\Im(z_2)]\cdots\Theta[\Im(z_\nu)]\cosh\left(a\sqrt{z_1^2+z_2^2+\cdots+z_\nu^2}\right), \tag{10.2.0.1}$$

where a is a complex number for $\nu=2n$. From (10.1.0.3)

$$F(\rho)=\frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}}\int\limits_0^\infty \cosh(ax^{\frac{1}{2}})x^{\frac{\nu-2}{4}}\left\{\Theta[\Im(\rho)]e^{-\frac{i\pi\nu}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})-\right.$$

$$\left.\Theta[-\Im(\rho)]e^{\frac{i\pi\nu}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\;dx\right\}. \tag{10.2.0.2}$$

Now,

$$\int\limits_0^\infty e^{ax^{1/2}}x^{\frac{\nu-2}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})=2\sqrt{\pi}\,e^{\frac{i\pi(\nu+2)}{4}}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{\rho^{\frac{\nu-2}{4}}}{(\rho^{1/2}-ia)}\times$$

$$\mathsf{F}\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)\quad \Im(\rho)>0$$

$$\int\limits_0^\infty e^{ax^{1/2}}x^{\frac{\nu-2}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})=2\sqrt{\pi}\,e^{-\frac{i\pi(\nu+2)}{4}}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{\rho^{\frac{\nu-2}{4}}}{(\rho^{1/2}+ia)}\times$$

$$\mathsf{F}\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)\quad \Im(\rho)<0. \tag{10.2.0.3}$$

To obtain (10.2.0.3) we have used **6.621.**(10) of Ref. [4]. Here F is the hypergeometric function. Thus, we have

$$F(\rho)=(4\pi)^{\frac{\nu-2}{2}}i\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\left\{\frac{1}{(\rho^{1/2}-ia)}\mathsf{F}\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)+\right.$$

$$\left.\frac{1}{(\rho^{1/2}+ia)}\mathsf{F}\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}.\right)\right\}. \tag{10.2.0.4}$$

As a second example, we evaluate the Fourier transform of

$$2^{-\nu}\Theta[\Im(z_1)]\Theta[\Im(z_2)]\cdots\Theta[\Im(z_\nu)]\frac{\pi\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu+2}{2}}}(z_1^2+z_2^2+\cdots+z_\nu^2)^{\frac{2-\nu}{2}}\times$$

$$\mathcal{J}_{\frac{\nu-2}{2}}[\mu(z_1^2 + z_2^2 + \cdots + z_\nu^2)^{\frac{1}{2}}]. \qquad (10.2.0.5)$$

We take into account that for ν even, $\mathcal{J}_{\frac{\nu-2}{2}} = e^{\frac{i\pi(\nu-2)}{2}} \mathcal{J}_{\frac{2-\nu}{2}}$. Thus

$$F(\rho) = \frac{\mu^{\frac{\nu-2}{2}}}{4\pi} e^{\frac{i\pi(\nu-2)}{2}} \rho^{\frac{2-\nu}{4}} \times$$

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \left\{ \Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) - \right.$$

$$\left. \Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \right\} \; dx. \qquad (10.2.0.6)$$

Now,

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) \; dx =$$

$$e^{\frac{i\pi(6-\nu)}{4}} \mu^{\frac{2-\nu}{2}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho - \mu^2} \; ; \; \Im(\rho) > 0$$

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \; dx =$$

$$e^{-\frac{i\pi(6-\nu)}{4}} \mu^{\frac{2-\nu}{2}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho - \mu^2} \; ; \; \Im(\rho) < 0, \qquad (10.2.0.7)$$

where we have used **6.576,** (3) of Ref. [4]. Thus, we have

$$F(\rho) = -\frac{1}{2\pi i(\rho - \mu^2)}. \qquad (10.2.0.8)$$

10.3 Minkowskian space

We define a Lorentz invariant ultradistribution of exponential type $\hat{F}(z)$ as an ultradistribution of exponential type such that $\hat{f}(t)$ in (4.3.0.24) is Lorentz invariant (Note that a Lorentz invariant Ultra-distribution is not necessarily Lorentz invariant in an explicit way). In this case, we can use for the Fourier transform of $\hat{f}(t)$ the formula obtained in Ref. [41]

$$F(\rho) = (2\pi)^{\frac{\nu-2}{2}} \int_{-\infty}^{\infty} \hat{f}(x) \times$$

$$\Big\{ \Theta[\Im(\rho)] e^{\frac{i\pi(\nu-2)}{4}} \frac{(x+i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}] -$$

$$\Theta[-\Im(\rho)] e^{\frac{i\pi(2-\nu)}{4}} \frac{(x-i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}] \Big\} \, dx. \tag{10.3.0.1}$$

Here, we have taken $\rho = \gamma + i\sigma$ and

$$\rho^{1/2} = \sqrt{\frac{\gamma + \sqrt{\gamma^2 + \sigma^2}}{2}} + i\mathrm{Sgn}(\sigma)\sqrt{\frac{-\gamma + \sqrt{\gamma^2 + \sigma^2}}{2}}. \tag{10.3.0.2}$$

When $\hat{f}$ is Lorentz invariant, we can use (10.3.0.1) or adopt the following treatment, starting from

$$\iiiint_{-\infty}^{\infty} f(\rho)\phi(\rho, k^0) \; d^4k = (2\pi)^\nu \iiiint_{-\infty}^{\infty} \hat{f}(x)\hat{\phi}(x, x^0) d^4x, \tag{10.3.0.3}$$

so that we can deduce the equality

$$\iint_{-\infty}^{\infty} f(\rho)\phi(\rho, k^0)(k_0^2 - \rho)_+^{\frac{\nu-3}{2}} \; d\rho \; dk^0 =$$

$$\iint_{-\infty}^{\infty} \hat{f}(x)\hat{\phi}(x, x^0)(x - x_0^2)_+^{\frac{\nu-3}{2}} \; dx \; dx^0. \tag{10.3.0.4}$$

Let $g(t)$ be defined as

$$\hat{g}(t) = \frac{1}{(2\pi)^\nu} \int_{-\infty}^{\infty} f(\rho) e^{-i\rho t} \; d\rho. \tag{10.3.0.5}$$

Then,

$$F(\rho) = \Theta[\Im(\rho)] \int_{0}^{\infty} \hat{g}(t) e^{i\rho t} \; dt - \Theta[-\Im(\rho)] \int_{-\infty}^{0} \hat{g}(t) e^{i\rho t} \, , \tag{10.3.0.6}$$

or, if we use Dirac's formula,

$$F(\rho) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-\rho}\, dt. \tag{10.3.0.7}$$

The inverse of the Fourier transform can be evaluated in the following way: we define

$$\hat{G}(x,\Lambda) = \frac{1}{(2\pi)^{\frac{\nu+2}{2}}} \oint_{\Gamma} F(\rho) \times$$

$$\left\{ e^{\frac{i\pi(\nu-2)}{4}} \frac{(\rho+\Lambda)^{\frac{\nu-2}{4}}}{(x+i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}(\rho+\Lambda)^{1/2}] + \right.$$

$$\left. + e^{\frac{i\pi(2-\nu)}{4}} \frac{(\rho-\Lambda)^{\frac{\nu-2}{4}}}{(x-i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}(\rho-\Lambda)^{1/2}] \right\} d\rho, \tag{10.3.0.8}$$

and then

$$\hat{f}(x) = \hat{G}(x, i0^+). \tag{10.3.0.9}$$

We present now same examples of the use of Fourier transform in Minkowskian space.

10.4 Examples

As a first example, we consider the Fourier transform of the ultradistribution

$$2^{\nu}\Theta;\Im(z_0)]\Theta[\Im(z_2)]\cdots\Theta[\Im(z_{\nu-1})]\left[\cosh\left(a\sqrt{|z_0^2-z_1^2-\cdots-z_{\nu-1}^2|}\right)\right.$$

$$\left. + \cos\left(a\sqrt{|z_0^2-z_1^2-\cdots-z_{\nu-1}^2|}\right)\right], \tag{10.4.0.1}$$

where a is a complex number. The cut along the real axis of (10.4.0.1) is

$$2^{-1}\left[e^{a\sqrt{|x_0^2-r^2|}} + e^{-a\sqrt{|x_0^2-r^2|}} + e^{ia\sqrt{|x_0^2-r^2|}} + e^{-ia\sqrt{|x_0^2-r^2|}}\right]. \tag{10.4.0.2}$$

The Fourier transform is

$$F(\rho) = (2\pi)^{\frac{\nu-2}{2}} \int\limits_{-\infty}^{\infty} e^{|x|^{\frac{1}{2}}} 2^{-1} \times$$

$$\left[e^{a\sqrt{|x_0^2-r^2|}} + e^{-a\sqrt{|x_0^2-r^2|}} + e^{ia\sqrt{|x_0^2-r^2|}} + e^{-ia\sqrt{|x_0^2-r^2|}}\right]$$

$$\left\{\Theta[\Im(\rho)] e^{\frac{i\pi(\nu-2)}{4}} \frac{(x+i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}] - \right.$$

$$\left. \Theta[-\Im(\rho)] e^{\frac{i\pi(2-\nu)}{4}} \frac{(x-i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}]\right\} \, dx. \tag{10.4.0.3}$$

Now,

$$e^{\frac{i\pi(\nu-2)}{4}} \int\limits_{-\infty}^{\infty} e^{a|x|^{\frac{1}{2}}} (x+i0)^{\frac{\nu-2}{4}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}] =$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}-ia)^{\nu}} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right) -$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}+a)^{\nu}} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right) \;\; \Im(\rho) > 0 \tag{10.4.0.4}$$

$$e^{\frac{i\pi(2-\nu)}{4}} \int\limits_{-\infty}^{\infty} e^{a|x|^{\frac{1}{2}}} (x-i0)^{\frac{\nu-2}{4}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}] =$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{e^{-\frac{i\pi\nu}{2}}}{(\rho^{1/2}+ia)^{\nu}} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right) -$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}+a)^{\nu}} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right) \;\; \Im(\rho) < 0 \tag{10.4.0.5}$$

So as to obtain (10.4.0.4) and (10.4.0.5) we have used **6.621**, (3) of Ref. [4]. With these results we have

$$F(\rho) = \frac{(4\pi)^{\frac{\nu-1}{2}}}{2} \frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \left\{\Theta[\Im(\rho)] e^{\frac{i\pi\nu}{2}} \left[\frac{\mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^{\nu}} + \right.\right.$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^{\nu}}+\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}+a)^{\nu}}+$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}-a)^{\nu}}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}+a)^{\nu}}-$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}-a)^{\nu}}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^{\nu}}-$$

$$\left.\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2}\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^{\nu}}\right]-$$

$$\Theta[-\Im(\rho)]e^{-\frac{i\pi\nu}{2}}\left[\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^{\nu}}+\right.$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^{\nu}}+\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}-a)^{\nu}}+$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}+a)^{\nu}}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}+a)^{\nu}}-$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}-a)^{\nu}}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^{\nu}}-$$

$$\left.\left.\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^{\nu}}\right]\right\}. \tag{10.4.0.6}$$

As a second example, we evaluate the Fourier transform of the ultra-distribution ($\nu = 2n$)

$$\hat{F}(z) = -\frac{(-1)^{\frac{\nu}{2}}i\mu^{\nu-2}}{2^{\nu}\pi^{\frac{\nu-2}{2}}}\Theta[\Im(z_0)]\Theta[\Im(z_2)]\cdots-\Theta[\Im(z_{\nu-1})]\times$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k\mu^{2k}(z_0^2-z_1^2-\cdots-z_{\nu-1}^2)^{kk}}{2^{2k}(k)!\Gamma(\nu+k)}. \tag{10.4.0.7}$$

The cut along the real axis of $\hat{F}(z)$ is

$$\hat{f}(x) = \hat{f}_{\mu}(x_+) - \hat{f}_{\mu}(x_-), \tag{10.4.0.8}$$

where

$$\hat{f}_\mu(x) = -\frac{i\pi}{2}\frac{\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}}x^{\frac{2-\nu}{4}}\mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}). \tag{10.4.0.9}$$

Note that

$$\hat{f}_\mu(x_+) = w_\mu(x) = -\frac{i\pi}{2}\frac{\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}}x_+^{\frac{2-\nu}{4}}\mathcal{J}_{\frac{2-\nu}{2}}(\mu x_+^{1/2}), \tag{10.4.0.10}$$

is the complex mass Wheeler propagator. Thus, according to (10.3.0.1)

$$F(\rho) = -\frac{i(\mu)^{\frac{\nu-2}{2}}}{4}\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2})$$

$$\left\{\frac{\Theta[\Im(\rho)]}{\rho^{\frac{\nu-2}{4}}}\left[e^{\frac{i\pi(\nu-2)}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) + \right.\right.$$

$$\left. e^{\frac{i\pi(\nu-2)}{2}}\mathcal{K}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\right] - \Theta[-\Im(\rho)]\left[e^{\frac{i\pi(2-\nu)}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) + \right.$$

$$\left.\left. e^{\frac{i\pi(2-\nu)}{2}}\mathcal{K}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\right]\right\} dx. \tag{10.4.0.11}$$

Taking into account that (see **6.576,** (3), Ref. [4])

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2})\, dx =$$

$$2\mu^{\frac{2-\nu}{2}}e^{\frac{i\pi(6-\nu)}{4}}\frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2}\ \Im(\rho) > 0$$

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\, dx =$$

$$2\mu^{\frac{2-\nu}{2}}e^{\frac{i\pi(\nu-6)}{4}}\frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2}\ \Im(\rho) < 0$$

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\, dx = 2\mu^{\frac{2-\nu}{2}}\frac{\rho^{\frac{\nu-2}{4}}}{\rho+\mu^2}, \tag{10.4.0.12}$$

we obtain

$$F(\rho) = \frac{i}{2}Sgn[\Im(\rho)]\left[\frac{1}{\rho-\mu^2} + \frac{\cosh\pi(\frac{\nu-2}{2})}{\rho+\mu^2}\right]. \tag{10.4.0.13}$$

10.5 UET

Let $\hat{F}(z)$ and $\hat{G}(z)$ be ultradistributions of exponential type (UET) such that their cuts along the real axis are $\hat{f}(x)$ and $\hat{g}(x)$. We suppose that $\hat{F}(z)$ and $\hat{G}(z)$ are spherically symmetric in the Euclidean case or Lorentz invariant in Minkowskian space. If we use the dimension ν as a regularizing parameter we can define the convolution of $F(\rho)$ and $G(\rho)$ as

$$F(\rho,\nu) * G(\rho,\nu) = (2\pi)^{\nu}\mathcal{F}\left\{\hat{f}(x,\nu)\hat{g}(x,\nu)\right\}. \quad (10.5.0.1)$$

10.6 The Euclidean case

As an example of the use of (10.1.0.3) in Euclidean space, we consider an UET $\hat{F}(z)$ such that $\hat{f}(x)$ is defined at the point $a > 0$ of the real axis and takes the value $\hat{f}(a)$, together with the ultradistribution $\hat{G}(z)$ whose cut along the real axis is $\delta(x-a)$. According to (10.3.0.1), we have

$$\mathcal{F}\left\{\hat{F}\right\}(\rho) = F(\rho), \quad (10.6.0.1)$$

$$\mathcal{F}\{\delta(x-a)\} =$$

$$G(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} a^{\frac{\nu-2}{4}} \left\{\Theta[\Im(\rho)]e^{-\frac{i\pi\nu}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(-ia^{1/2}\rho^{1/2}) - \right.$$

$$\left.\Theta[-\Im(\rho)]e^{\frac{i\pi\nu}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(ia^{1/2}\rho^{1/2})\right\} +$$

$$\frac{2\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})\rho^{\frac{\nu-2}{4}}} a^{\frac{\nu-2}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}(a^{1/2}\rho^{1/2}). \quad (10.6.0.2)$$

Due to

$$\mathcal{F}\left\{\hat{f}(x)\delta(x-a)\right\} = \hat{f}(a)\mathcal{F}\{\delta(x-a)\} = \hat{f}(a)G(\rho), \quad (10.6.0.3)$$

we have

$$F(\rho) * G(\rho) = (2\pi)^{\nu}\hat{f}(a)G(\rho). \quad (10.6.0.4)$$

10.7 The Minkowskian case

We consider now

$$F(\rho) = G(\rho) = \frac{i}{2} Sgn[\Im(\rho)] \left[\frac{1}{\rho - \mu^2} + \frac{\cosh \pi(\frac{\nu-2}{2})}{\rho + \mu^2} \right]. \quad (10.7.0.1)$$

From (10.4.0.8) we have

$$\hat{f}(x) = \hat{g}(x) =$$

$$-\frac{i\pi}{2} \frac{(-\mu)^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}} \left[x_+^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\mu x_+^{1/2}) - x_-^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\mu x_-^{1/2}) \right]. \quad (10.7.0.2)$$

Then,

$$F(\rho) * G(\rho) =$$

$$\frac{(2\pi)^{\frac{\nu+1}{2}}}{2^{\frac{3\nu-1}{2}}} \Gamma\left(\frac{3-\nu}{2}\right) e^{i\pi(\frac{\nu-2}{2})} \rho^{\frac{2-\nu}{2}} Sgn[\Im(\rho)] \times$$

$$\left[(\rho^2 - 2\rho\mu^2)^{\frac{\nu-3}{2}} + (\rho^2 + 2\rho\mu^2)^{\frac{\nu-3}{2}} \right]. \quad (10.7.0.3)$$

To obtain (10.7.0.3) we use

$$\int_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu_1 x) \mathcal{J}_{\frac{2-\nu}{2}}(\mu_2 x) \mathcal{K}_{\frac{\nu-2}{2}}(xz)\, dx =$$

$$-\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3-\nu}{2}\right)}{2^{\frac{3\nu-6}{2}}} \frac{z^{\frac{2-\nu}{2}}}{(\mu_1\mu_2)^{\frac{\nu-2}{2}}} \left[(z^2 + \mu_1^2 + \mu_2^2)^2 - 4\mu_1^2\mu_2^2 \right]^{\frac{\nu-3}{2}}, \quad (10.7.0.4)$$

and to deduce (10.7.0.4) we have employed

$$\mathcal{K}_{\frac{\nu-2}{2}}(xz) = \frac{1}{2} \left(\frac{zx}{2}\right)^{\frac{\nu-2}{2}} \int_0^\infty t^{-\frac{\nu}{2}} e^{-t-\frac{z^2x^2}{4t}}\, dt, \quad (10.7.0.5)$$

see (8.432-6) of Ref. [4]. We proceed now to the calculation of the convolution of two UETs.

10.8 The convolution of two UETs

The convolution of two UETs can be defined with a change in the formula obtained in Ref. [41] for tempered ultradistributions. Let

$$H_{\gamma\lambda}(k) = \frac{i}{2\pi}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{[2\cosh(\gamma k_1)]^{-\lambda}F(k_1)[2\cosh(\gamma k_2)]^{-\lambda}G(k_2)}{k-k_1-k_2}\, dk_1\, dk_2, \tag{10.8.0.1}$$

$$| \Im(k) |>| \Im(k_1) | + | \Im(k_2 | \,;\, \gamma < \min\left(\frac{\pi}{2 \mid \Im(k_1) \mid}; \frac{\pi}{2 \mid \Im(k_2) \mid}\right).$$

With this value of γ, the hyperbolic functions has no singularities in the integration zone. Again, we have the Laurent (or Taylor) expansion

$$H_{\gamma\lambda}(k) = \sum_n H_\gamma^{(n)}(k)\lambda^n, \tag{10.8.0.2}$$

where the sum might have terms with negative n. We now define the convolution product as the λ-independent term of (10.8.0.2)

$$(F * G)(k) = H(k) = H_\gamma^{(0)}(k) = H^{(0)}(k), \tag{10.8.0.3}$$

that is γ-independent. To see this we consider a typical integral term in (10.4.0.1)

$$I = \int\limits_c^\infty \frac{F(k+i\sigma)}{[\cosh\gamma(k+i\sigma)]^\lambda}\, dk, \tag{10.8.0.4}$$

with

$$|F(k)| \le A\,|k|^p\, e^{p|\Re(k)|}. \tag{10.8.0.5}$$

Then, I has the value

$$I = e^{i(p+1)}\left[\sum_{n=1;n\neq(p-1,p+1)}^{\infty} a_n(p,\sigma)\frac{e^{-c(n-p-1)}}{n-p-1} - a_{p-1}(p,\sigma)\frac{e^{2c}}{2} + \frac{a_{p+1}(p,\sigma)}{\lambda\gamma}\right]. \tag{10.8.0.6}$$

Thus, the λ-independent term of I does not depend on γ. As (10.4.0.1) is composed of sums and products of integrals of the type (10.4.0.4), we conclude that (10.4.0.3) is true.

10.9 Examples

As a first example, we consider the convolution of two exponentials. Let

$$F(k) = Sgn[\Im(k)]\frac{e^{ak}}{2} \quad ; \quad G(k) = Sgn[\Im(k)]\frac{e^{bk}}{2}, \tag{10.9.0.1}$$

(a and b complex). Then,

$$H_{\gamma\lambda}(k) =$$

$$\frac{i}{8\pi}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{Sgn[\Im(k_1)]e^{ak_1}Sgn[\Im(k_2)]e^{bk_2}}{[2\cosh(\gamma k_1)]^\lambda[2\cosh(\gamma k_2)]^\lambda(k-k_1-k_2)}dk_1\, dk_2 =$$

$$\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{e^{ak_1}e^{bk_2}}{[2\cosh(\gamma k_1)]^\lambda[2\cosh(\gamma k_2)]^\lambda(k-k_1-k_2)}dk_1\, dk_2, \tag{10.9.0.2}$$

or

$$H_{\gamma\lambda}(k) = \frac{\Theta[\Im(k)]}{2\pi}\int\limits_{0}^{\infty}\iint\limits_{-\infty}^{\infty} \frac{e^{ak_1}e^{bk_2}e^{i(k-k_1-k_2)t}}{(e^{\gamma k_1}+e^{-\gamma k_1})^\lambda(e^{\gamma k_2}+e^{-\gamma k_2})^\lambda}\, dk_1\, dk_2\, dt$$

$$-\frac{\Theta[-\Im(k)]}{2\pi}\int\limits_{-\infty}^{0}\iint\limits_{-\infty}^{\infty} \frac{e^{ak_1}e^{bk_2}e^{i(k-k_1-k_2)t}}{(e^{\gamma k_1}+e^{-\gamma k_1})^\lambda(e^{\gamma k_2}+e^{-\gamma k_2})^\lambda}\, dk_1\, dk_2\, dt. \tag{10.9.0.3}$$

To evaluate (10.9.0.3) we take into account that

$$\int\limits_{-\infty}^{\infty} \frac{e^{(a-it)k_1}}{(e^{\gamma k_1}+e^{-\gamma k_1})^\lambda}\, dk_1 = \frac{1}{2\gamma}\int\limits_{0}^{\infty} \frac{y^{\frac{\gamma\lambda+a-it}{2\gamma}-1}}{(1+Y)^\lambda}\, dy =$$

$$\frac{1}{2\gamma}\frac{\Gamma\left(\frac{\gamma\lambda+a-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-a+it}{2\gamma}\right)}{\Gamma(\lambda)}. \tag{10.9.0.4}$$

Then,

$$H_{\gamma\lambda}(k) = \frac{1}{8\pi\gamma^2\Gamma^2(\lambda)}\left\{\Theta[\Im(k)]\int\limits_{0}^{\infty}\Gamma\left(\frac{\gamma\lambda+a-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-a+it}{2\gamma}\right)\times\right.$$

$$\Gamma\left(\frac{\gamma\lambda+b-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-b+it}{2\gamma}\right)e^{ikt}\,dt\quad-$$

$$\Theta[-\Im(k)]\int\limits_{-\infty}^{0}\Gamma\left(\frac{\gamma\lambda+a-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-a+it}{2\gamma}\right)\times$$

$$\Gamma\left(\frac{\gamma\lambda+b-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-b+it}{2\gamma}\right)e^{ikt}\,dt\Bigg\},\qquad(10.9.0.5)$$

and using the equality ([4], **3.381**, 4),

$$\int\limits_{0}^{\infty}x^{\nu-1}e^{-\mu x}\,dx=\mu^{-\nu}\Gamma(\nu),\qquad(10.9.0.6)$$

so that performing the integral in the variable t, we have, for (10.9.0.5),

$$H_{\gamma\lambda}(k)=-\frac{1}{8\pi i\gamma^2\Gamma^2(\lambda)}\iiiint\limits_{0}^{\infty}\times$$

$$s_1^{\frac{\gamma\lambda+a}{2\gamma}-1}e^{-s_1}s_2^{\frac{\gamma\lambda-a}{2\gamma}-1}e^{-s_2}s_3^{\frac{\gamma\lambda+b}{2\gamma}-1}e^{-s_3}s_4^{\frac{\gamma\lambda-b}{2\gamma}-1}e^{-s_4}\times$$

$$\frac{1}{k+\frac{1}{2\gamma}\ln(\frac{s_2s_4}{s_1s_3})}\,ds_1\,ds_2\,ds_3\,ds_4.\qquad(10.9.0.7)$$

As

$$\frac{1}{k+\frac{1}{2\gamma}\ln(\frac{s_2s_4}{s_1s_3})}=\sum_{n=0}^{\infty}\frac{(ik_I)^n}{n!}\frac{\partial^n}{\partial k_R^n}\delta\left[k_R+\frac{1}{2\gamma}\ln\left(\frac{s_2s_4}{s_1s_3}\right)\right],\qquad(10.9.0.8)$$

where $(k=k_R+ik_I)$ and

$$\delta\left[k_R+\frac{1}{2\gamma}\ln\left(\frac{s_2s_4}{s_1s_3}\right)\right]=\frac{s_1s_3}{s_2}e^{-2\gamma k_R}\delta\left(s_4-\frac{s_1s_3}{s_2}e^{-2\gamma k_R}\right),\qquad(10.9.0.9)$$

we obtain

$$H_{\gamma\lambda}(k)=-\sum_{n=0}^{\infty}\frac{(ik_I)^n}{n!}\frac{\partial^n}{\partial k_R^n}\frac{e^{k_R(b-\gamma\lambda)}}{4\pi i\gamma\Gamma^2(\lambda)}\iiiint\limits_{0}^{\infty}s_1^{\frac{2\gamma\lambda+a-b}{2\gamma}-1}e^{-s_1}\times$$

$$s_2^{\frac{b-a}{2\gamma}-1}e^{-s_2}s_3^{\lambda-1}e^{-s_3}e^{-\left(\frac{s_1s_3}{s_2}e^{-2\gamma k_R}\right)}\,ds_1\,ds_2\,ds_3\,ds_4.\qquad(10.9.0.10)$$

After evaluating the four-fold integral, $H_{\gamma\lambda}$ takes the form

$$H_{\gamma\lambda}(k) = -\frac{e^{k(b+\gamma\lambda)}}{4\pi i\gamma\Gamma(2\lambda)}\Gamma\left(\frac{a-b+2\gamma\lambda}{2\gamma}\right)\Gamma\left(\frac{b-a+2\gamma\lambda}{2\gamma}\right)\times$$

$$F\left(\lambda+\frac{b-a}{2\gamma},\lambda,2\lambda;1-e^{2\gamma k}\right). \tag{10.9.0.11}$$

When $a \neq b$

$$\lim_{\lambda\to 0} H_{\gamma\lambda}(k) = 0. \tag{10.9.0.12}$$

When $a = b$,

$$\lim_{\lambda\to 0} H_{\gamma\lambda}(k) = \frac{ke^{ka}}{2\pi i} \equiv 0, \tag{10.9.0.13}$$

and then $H(k)$ is the *null ultradistribution* Thus, we have finally

$$Sgn[\Im(k)]\frac{e^{ak}}{2} * Sgn[\Im(k)]\frac{e^{bk}}{2} = 0,$$

and Fourier anti-transforming

$$\delta(z-a)\delta(z-b) = 0. \tag{10.9.0.14}$$

As a second example, we consider the convolution of two complex Dirac deltas

$$F(k) = -\frac{1}{2\pi i(k-a)} \quad ; \quad G(k) = -\frac{1}{2\pi i(k-b)}. \tag{10.9.0.15}$$

We have

$$H_{\gamma\lambda}(k) = \frac{i}{2\pi}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\frac{i}{2\pi(k_1-a)}\frac{i}{2\pi(k_2-b)}\times$$

$$\frac{[2\cosh(\gamma k_1)]^{-\lambda}[2\cosh(\gamma k_2)]^{-\lambda}}{k-k_1-k_2}dk_1\ dk_2 = \tag{10.9.0.16}$$

$$\frac{1}{2\pi}\frac{[2\cosh(\gamma a)]^{-\lambda}[2\cosh(\gamma b)]^{-\lambda}}{k-a-b}, \tag{10.9.0.17}$$

and, as a consequence,

$$H(k) = -\frac{1}{2\pi i}\frac{1}{k-a-b}, \tag{10.9.0.18}$$

or

$$\delta(k-a) * \delta(k-b) = \delta(k-a-b), \tag{10.9.0.19}$$

and, in configuration space,

$$\frac{Sgn[\Im(z)]}{2}e^{az}\ \ \frac{Sgn[\Im(z)]}{2}e^{bz}=\frac{Sgn[\Im(z)]}{2}e^{(a+b)z}. \qquad (10.9.0.20)$$

Formula (10.3.0.1) can be generalized to ν dimensions

$$H_{\gamma\lambda}(k)=\frac{i^{\nu}}{(2\pi)^{\nu}}\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{\prod_{j=1}^{\nu}[2\cosh(\gamma_j k_{1j}]^{-\lambda_j}[2\cosh(\gamma_j k_{2j})]^{-\lambda_j}}{\prod_{j=1}^{\nu}(k_j-k_{1j}-k_{2j})}\times$$

$$F(k_1)G(k_2)\ \ d^{\nu}k_1\ d^{\nu}k_2. \qquad (10.9.0.21)$$

$$\gamma_j<\min\left(\frac{\pi}{2\mid\Im(k_{1j})\mid};\frac{\pi}{2\mid\Im(k_{2j})\mid}\right)$$

As in the one-dimensional case,

$$H_{\gamma\lambda}(k)=\sum_{n_1,n_2,..,n_\nu}\lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_\nu^{n_\nu}H^{(n_1+n_2+\cdots+n_\nu)}(k), \quad (10.9.0.22)$$

and again,

$$(F*G)(k)=H(k)=H^{(0)}(k). \qquad (10.9.0.23)$$

10.10 Normalization of Gamow states

As an application of the results of the previous discussions we give in this section a solution to the question of normalization of Gamow states in quantum mechanics. If we have a *Gamow state* that depends on $l+m$ variables $\phi(k_1,k_2,...,k_l;\rho_1,\rho_2,...\rho_m)$, and we wish to calculate

$$I(k_1,k_2,..,k_l)=$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\phi|^2(k_1,k_2,..,k_l;\rho_1,\rho_2,..,\rho_m)\ d\rho_1\ d\rho_2\cdot\cdot d\rho_m, \quad (10.10.0.1)$$

we define

$$\Phi(k_1,k_2,...,k_l;z_1,z_2,...,z_m)=\frac{1}{(2\pi i)^m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\quad\times$$

$$\frac{|\phi|^2(k_1,k_2,...,k_l;\rho_1,\rho_2,...,\rho_m)}{(\rho_1-z_1)(\rho_2-z_2)\cdots(\rho_m-z_m)}\ d\rho_1\ d\rho_2\cdots d\rho_m, \qquad (10.10.0.2)$$

and

$$H_{\gamma_1\gamma_2...\gamma_m\lambda_1\lambda_2...\lambda_m}(k_1,k_2,...,k_l) = \oint\limits_{\Gamma_1}\cdots\oint\limits_{\Gamma_m} \times$$

$$\frac{\Phi(k_1,k_2,...,k_l;z_1,z_2,...,z_m)}{[\cosh(\gamma_1 z_1)]^{\lambda_1}[\cosh(\gamma_2 z_2)]^{\lambda_2}...[\cosh(\gamma_m z_m)]^{\lambda_m}}\ dz_1\ dz_2\cdots dz_m. \qquad (10.10.0.3)$$

We have again the Laurent's expansion

$$H_{\gamma_1\gamma_2...\gamma_m\lambda_1\lambda_2...\lambda_m}(k_1,k_2,...,k_l) = \sum_{n_1,n_2,...n_m} \times$$

$$\lambda_1^{n_1}\lambda_2^{n_2}...\lambda_m^{n_m}H^{(n_1+n_2+...+n_m)}_{\gamma_1\gamma_2...\gamma_m}(k_1,k_2,...,k_l), \qquad (10.10.0.4)$$

and as a consequence of Section 9.9 we define

$$I(k_1,k_2,...,k_l) = H(k_1,k_2,...,k_l) = H^{(0)}(k_1,k_2,...,k_l) =$$

$$H^{(0)}_{\gamma_1\gamma_2...\gamma_m}(k_1,k_2,...,k_l). \qquad (10.10.0.5)$$

As an example of application of (10.10.0.1-10.10.0.5) we evaluate

$$I(k) = \int\limits_0^\infty \phi_0^2(k,r)\ dr = \int\limits_{-\infty}^\infty \Theta(r)\phi_0^2(k,r)\ dr, \qquad (10.10.0.6)$$

where $\phi_0(k,r)$ is the $l=0$ function corresponding to the *square-well potential* used in Ref. [43]

$$\phi_0(k,r) = \begin{cases} \frac{\sin(qr)}{q} & \text{if } r<a \\ \frac{\sin(qa)}{q}e^{ik(a-r)} & \text{if } r>a. \end{cases} \qquad (10.10.0.7)$$

Here q is given by

$$q^2 = \frac{2m}{\hbar^2}[E-V(r)] = k^2 - \frac{2m}{\hbar^2}V(r), \qquad (10.10.0.8)$$

and

$$V(r) = \begin{cases} 0 & \text{if } r>a \\ -V_0 & \text{if } r\geq a. \end{cases} \qquad (10.10.0.9)$$

We can write

$$\phi(k,r) = [\Theta(r) - \Theta(r-a)]\,\frac{\sin(qr)}{q} + \Theta(r-a)\frac{\sin(qa)}{q}e^{ik(a-r)}, \tag{10.10.0.10}$$

$$\phi^2(k,r) = [\Theta(r) - \Theta(r-a)]\,\frac{\sin^2(qr)}{q^2} + \Theta(r-a)\frac{\sin^2(qa)}{q^2}e^{2ik(a-r)}, \tag{10.10.0.11}$$

and, according to (10.10.0.2),

$$\Phi(k,z) = \frac{1}{2\pi i}\,[\ln(a-z) - \ln(z)]\,\frac{\sin^2(qz)}{q^2} \quad -$$

$$\frac{1}{2\pi i}\frac{\sin^2(qa)}{q^2}\ln(a-z).e^{2ik(a-z)} \tag{10.10.0.12}$$

Thus, we have

$$H_{\gamma\lambda}(k) = \frac{1}{2\pi i q^2}\oint_\Gamma \frac{\ln(a-z)-\ln(z)}{[\cosh(\gamma z)]^\lambda}\sin^2(qz)\,dz \quad -$$

$$\frac{\sin^2(qa)}{2\pi i q^2}e^{2ika}\oint_\Gamma \frac{\ln(a-z)}{[\cosh(\gamma z)]^\lambda}e^{-2ikz}\,dz \;=$$

$$\frac{1}{q^2}\int_0^a \frac{\sin^2(qr)}{[\cosh(\gamma r)]^\lambda}\,dr + \frac{\sin^2(qa)}{q^2}e^{2ika}\int_a^\infty \frac{e^{-2ikr}}{[\cosh(\gamma r)]^\lambda}\,dr. \tag{10.10.0.13}$$

We can evaluate the second integral in (10.10.0.13)

$$\int_a^\infty \frac{e^{-2ikr}}{[\cosh(\gamma r)]^\lambda}\,dr \;=$$

$$\frac{e^{-a(\gamma\lambda+2ik)}}{\gamma\lambda+2ik}F\left(\lambda, \frac{\gamma\lambda+2ik}{2\gamma}, \frac{\gamma\lambda+2ik}{2\gamma}+1; -e^{4\gamma a}\right). \tag{10.10.0.14}$$

Taking into account that

$$\lim_{\lambda\to 0}\frac{1}{\gamma\lambda+2ik} = \begin{cases} \frac{-i}{2(k-i0)} & \text{if } \Im(k)=0 \\ \frac{-i}{2k} & \text{if } \Im(k)\neq 0, \end{cases} \tag{10.10.0.15}$$

we obtain

$$I(k) = H(k) = \frac{1}{q^2}\int_0^a \sin^2(qr)\, dr + \frac{\sin^2(qa)}{2ikq^2} =$$

$$\frac{a}{2q^2} - \frac{\sin(2qa)}{4q^3} + \frac{\sin^2(qa)}{2ikq^2}. \tag{10.10.0.16}$$

Using now the equality

$$\cos(qa) = -i\frac{k}{q}\sin(qa), \tag{10.10.0.17}$$

we have

$$I(k) = \frac{1+ika}{2ik}\;\frac{q^2-k^2}{q^4}\;\sin^2(qa). \tag{10.10.0.18}$$

This result coincides with the result obtained in Ref. [43].

10.11 Four-dimensional even UETs

The convolution of two even UETs can be defined with a change in the formula obtained in Ref. [41] for tempered ultradistributions. Let

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho) =$$

$$\frac{1}{4\pi\rho}\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{[2\cosh(\gamma_0 k_1^0)]^{-\lambda_0}[2\cosh(\gamma\rho_1)]^{-\lambda}F(k_1^0,\rho_1)}{k^0-k_1^0-k_2^0}\times$$

$$[2\cosh(\gamma_0 k_2^0)]^{-\lambda_0}[2\cosh(\gamma\rho_2)]^{-\lambda}G(k_2^0,\rho_2)\;\times$$

$$\ln[\rho^2-(\rho_1+\rho_2)^2]\rho_1\rho_2\; d\rho_1\; d\rho_2\; dk_1^0\; dk_2^0, \tag{10.11.0.1}$$

$$\mid \Im(k^0)\mid > \mid \Im(k_1^0\mid + \mid \Im(k_2^0)\mid\; ;\; \mid \Im(\rho)\mid > \mid \Im(\rho_1\mid + \mid \Im(\rho_2)\mid$$

$$\gamma_0 < \min\left(\frac{\pi}{2\mid \Im(k_1^0)\mid};\frac{\pi}{2\mid \Im(k_2^0)\mid}\right)\;;\;\gamma < \min\left(\frac{\pi}{2\mid \Im(\rho_1)\mid};\frac{\pi}{2\mid \Im(\rho_2)\mid}\right).$$

The difference between

$$\int \frac{2\rho}{\rho^2-(\rho_1+\rho_2)^2}\, d\rho \qquad \text{and} \qquad \ln[\rho^2-(\rho_1+\rho_2)^2]$$

is an entire analytic function. Substitution now in (10.11.0.1) yields

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho) = \frac{1}{2\pi\rho}\int \rho\ d\rho \oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{F(k_1^0,\rho_1)G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\times$$

$$[2\cosh(\gamma_0 k_1^0)]^{-\lambda_0}[2\cosh(\gamma\rho_1)]^{-\lambda}[2\cosh(\gamma_0 k_2^0)]^{-\lambda_0}[2\cosh(\gamma\rho_2)]^{-\lambda}\times$$

$$\frac{1}{\rho^2-(\rho_1+\rho_2)^2}\quad \rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0. \qquad (10.11.0.2)$$

We can again perform the Laurent expansion

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho) = \sum_{mn} H^{(m,n)}_{\gamma_0\gamma}(k^0,\rho)\lambda_0^m\lambda^n, \qquad (10.11.0.3)$$

and define the convolution product as the (λ_0,λ)-independent term of (10.11.0.3)

$$H(k) = H(k^0,\rho) = H^{(0,0)}_{\gamma_0\gamma}(k^0,\rho) = H^{(0,0)}(k^0,\rho). \qquad (10.11.0.4)$$

If we define

$$L_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho) = \oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{F(k_1^0,\rho_1)G(k_2^0,\rho_2)}{k^0-k_1^0-k_2^0}\times$$

$$[2\cosh(\gamma_0 k_1^0)]^{-\lambda_0}[2\cosh(\gamma\rho_1)]^{-\lambda}[2\cosh(\gamma_0 k_2^0)]^{-\lambda_0}[2\cosh(\gamma\rho_2)]^{-\lambda}\times$$

$$\frac{1}{\rho^2-(\rho_1+\rho_2)^2}\quad \rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0, \qquad (10.11.0.5)$$

then

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho) = \frac{1}{2\pi\rho}\int L_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho)\ \rho\ d\rho. \qquad (10.11.0.6)$$

Now, we will show that the cut on the real axis of (10.11.0.1), $h_{\lambda_0\lambda}(k^0,\rho)$, is an even function of k^0 and ρ, but it is explicitly odd in ρ. For the variable k^0 we take into account that $e^{i\pi\lambda_0\{Sgn[\Im(k_1^0)]+Sgn[\Im(k_2^0)]\}} = 1$, and, as a consequence, (10.11.0.1) is odd in k^0 too. We consider now the parity in the variable ρ.

$$\oint\limits_{\Gamma_0}\oint\limits_{\Gamma} H_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\ dk^0\ d\rho =$$

$$-\iint\limits_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\;dk^0\;d\rho =$$

$$-\oint\limits_{\Gamma_0}\oint\limits_{\Gamma} H_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\;dk^0\;d\rho =$$

$$-\iint\limits_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\;dk^0\;d\rho. \tag{10.11.0.7}$$

Thus, we have

$$h_{\lambda_0\lambda}(k^0,-\rho) = h_{\lambda_0\lambda}(k^0,\rho). \tag{10.11.0.8}$$

The proof for the variable k^0 is similar.

10.12 Examples

As a first example, we shall calculate the convolution between $F(k_0,\rho) = \delta(k_0^2-a^2)\delta(\rho-b)$ and $G(k_0,\rho) = \delta(k_0^2-c^2)\delta(\rho-d)$. We have

$$H(k_0,\rho) = \frac{bd}{16\pi|a||b|\rho}\times$$

$$\left(\frac{1}{k_0-a-c}+\frac{1}{k_0-a+c}+\frac{1}{k_0+a-c}+\frac{1}{k_0+a+c}\right)\times$$

$$\ln[\rho^2-(b+d)^2], \tag{10.12.0.1}$$

and simplifying the last expression,

$$H(k_0,\rho) = \frac{bd}{8\pi|a||b|\rho}\left[\frac{k_0-a}{(k_0-a)^2-c^2}+\frac{k_0+a}{(k_0+a)^2-c^2}\right]\times$$

$$\ln[\rho^2-(b+d)^2]. \tag{10.12.0.2}$$

As a second example, we evaluate the convolution of $F(k_0,\rho) = \delta(k_0)\delta(\rho-a)$ and $G(k_0,\rho) = \frac{1}{2}Sgn[\Im(k_0)]e^{ibk_0}\delta(\rho-c)$. We have

$$H_{\gamma_0\gamma\lambda_0\lambda}(k_0,\rho) =$$

$$\frac{ac}{8\pi\rho}\frac{\ln[\rho^2-(a+c)^2]}{[\cosh(\gamma a]^{\lambda}[\cosh(\gamma c)]^{\lambda}}\oint\limits_{\Gamma_{02}}\frac{Sgn[\Im(k_{02})]e^{ibk_{02}}}{[\cosh(\gamma_0 k_{02})]^{\lambda_0}(k_0-k_{02})}\;dk_{02}$$

$$= \frac{ac}{4\pi\rho} \frac{\ln[\rho^2 - (a+c)^2]}{[\cosh(\gamma a]^\lambda [\cosh(\gamma c)]^\lambda} \int\limits_{-\infty}^{\infty} \frac{e^{ibk_{02}}}{[\cosh(\gamma_0 k_{02})]^{\lambda_0} (k_0 - k_{02})} \, dk_{02}. \tag{10.12.0.3}$$

Taking into account that

$$\lim \lambda_0 \to 0 \int\limits_{-\infty}^{\infty} \frac{e^{ibk_{02}}}{[\cosh(\gamma_0 k_{02})]^{\lambda_0} (k_0 - k_{02})} \, dk_{02} =$$

$$-\pi i Sgn[\Im(k_0)] e^{ibk_0}, \tag{10.12.0.4}$$

we obtain

$$H(k_0, \rho) = \frac{ac}{4\pi i \rho} Sgn[\Im(k_0)] e^{ibk_0} \ln[\rho^2 - (a+d)^2]. \tag{10.12.0.5}$$

10.13 Convolution of spherically symmetric UET

The convolution of two spherically symmetric UETs can be defined with a change of the formula obtained in Ref. [41] for tempered ultradistributions. Let

$$H_{\gamma\lambda}(\rho) = \frac{i\pi}{4\rho} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} [2\cosh(\gamma\rho_1)]^{-\lambda} F(\rho_1) [2\cosh(\gamma\rho_2)]^{-\lambda} G(\rho_2) \times$$

$$\left[\rho - \rho_1 - \rho_2 - \sqrt{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}\right] d\rho_1 \, d\rho_2. \tag{10.13.0.1}$$

Again, we have the Laurent expansion

$$H_{\gamma\lambda}(\rho) = \sum_{n=-m}^{\infty} H_\gamma^{(n)}(\rho) \lambda^n. \tag{10.13.0.2}$$

We now define the convolution product as the λ-independent term of (10.13.0.2). One has

$$H(\rho) = H_\gamma^{(0)}(\rho) = H^{(0)}(\rho). \tag{10.13.0.3}$$

Let $\hat{H}_{\gamma\lambda}(x)$ be the Fourier anti-transform of $H_{\gamma\lambda}(\rho)$. Then,

$$\hat{H}_{\gamma\lambda}(x) = \sum_{n=-m}^{\infty} \hat{H}_{\gamma}^{(n)}(x)\lambda^n. \tag{10.13.0.4}$$

If we define

$$\hat{f}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{[2\cosh(\gamma\rho)]^{-\lambda}F(\rho)\}$$

$$\hat{g}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{[2\cosh(\gamma\rho)]^{-\lambda}G(\rho)\}, \tag{10.13.0.5}$$

then,

$$\hat{H}_{\gamma\lambda}(x) = (2\pi)^4\hat{f}_{\gamma\lambda}(x)\hat{g}_{\gamma\lambda}(x), \tag{10.13.0.6}$$

and with the use of the Laurent's expansion of $\hat{f}$ together with $\hat{g}$ we write

$$\hat{f}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{f}_{\gamma}^{(n)}(x)\lambda^n$$

$$\hat{g}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{g}_{\gamma}^{(n)}(x)\lambda^n, \tag{10.13.0.7}$$

so that

$$\sum_{n=-m}^{\infty} \hat{H}_{\gamma}^{(n)}(x)\lambda^n = (2\pi)^4 \sum_{n=-m}^{\infty} \left(\sum_{k=-m}^{n} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x) \right) \lambda^n, \tag{10.13.0.8}$$

$(m = m_f + m_g$, and as a consequence,

$$\hat{H}^{(0)}(x) = \sum_{k=-m}^{0} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x). \tag{10.13.0.9}$$

We shall now give some examples.

10.14 Examples

The first example that we shall tackle is the convolution between $F[\rho) = \delta(\rho - a)$ and $G(\rho) = \delta(\rho - b)$. We have

$$H_{\gamma\lambda}(\rho) = \frac{i\pi}{4\rho} \oint_{\Gamma_1} \oint_{\Gamma_2} [2\cosh(\gamma\rho_1)]^{-\lambda}\delta(\rho_1 - a)[2\cosh(\gamma\rho_2)]^{-\lambda}\delta(\rho_2 - b) \times$$

$$\left[\rho - \rho_1 - \rho_2 - \sqrt{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}\right] d\rho_1\, d\rho_2, \qquad (10.14.0.1)$$

whose result is

$$H(\rho) = \frac{i\pi}{4\rho}\left[\rho - a - b - \sqrt{(\rho - a - b)^2 - 4ab}\right]. \qquad (10.14.0.2)$$

When a and b are real numbers, from (10.14.0.2) we obtain in the real ρ-axis

$$h(\rho) = \frac{\pi}{2\rho}\left[(\rho - a - b)^2 - 4ab\right]_+^{\frac{1}{2}}. \qquad (10.14.0.3)$$

As a second example, we evaluate the convolution between $F(\rho) = E_i(-ia\rho)e^{ia\rho}/2\pi i$ and $G(\rho) = \delta^{'}(\rho) = (2\pi i\rho^2)^{-1}$, where $E_i(z)$ is the exponential integral function. We have

$$H_{\gamma\lambda}(\rho) = \frac{1}{8\rho}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\frac{E_i(-ia\rho_1)e^{ia\rho_1}\delta^{'}(\rho_2)}{[2\cosh(\gamma\rho_1)]^\lambda[2\cosh(\gamma\rho_2)]^\lambda}$$

$$\left[\rho - \rho_1 - \rho_2 - \sqrt{(\rho - \rho_1 - \rho_1)^2 - 4\rho_1\rho_2}\right] d\rho_1\, d\rho_2. \qquad (10.14.0.4)$$

After integration in the variable ρ_2 we have

$$H_{\gamma\lambda}(\rho) = \frac{1}{4\rho}\oint\limits_{\Gamma_1}\frac{\rho_1 E_i(-ia\rho_1)e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^\lambda(\rho_1 - \rho)}\, d\rho_1 =$$

$$\frac{1}{4\rho}\int\limits_0^\infty\frac{\rho_1 e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^\lambda(\rho_1 - \rho)}\, d\rho_1 =$$

$$\frac{1}{4\rho}\int\limits_0^\infty\frac{e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^\lambda}\, d\rho_1 + \frac{1}{4}\int\limits_0^\infty\frac{e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^\lambda(\rho_1 - \rho)}\, d\rho_1. \qquad (10.14.0.5)$$

For the integrals in (10.14.0.5) $(\lambda \to 0)$ we obtain

$$H(\rho) = \frac{i}{4a\rho} - \frac{i}{8\pi}e^{ia\rho}.E_i(-ia\rho). \qquad (10.14.0.6)$$

10.15 Convolution of Lorentz invariant UETs

For Lorentz invariant UETs, following Ref. [41] we have,

$$H_{\gamma\lambda}(\rho,\Lambda) = \frac{1}{8\pi^2\rho}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}[2\cosh(\gamma\rho_1)]^{-\lambda}F(\rho_1)[2\cosh(\gamma\rho_2)]^{-\lambda}G(\rho_2)$$

$$\{\Theta[\Im(\rho)]\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)]\times$$

$$[\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}\right\}-$$

$$\Theta[-\Im(\rho)]\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}\right\}-\frac{i}{2}\times$$

$$\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(i\sqrt{\frac{\rho_1+\Lambda}{\rho_2+\Lambda}}\right)+\ln\left(-i\sqrt{\frac{\rho_1-\Lambda}{\rho_2-\Lambda}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(-i\sqrt{\frac{\Lambda-\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(i\sqrt{\frac{\Lambda+\rho_1}{\Lambda+\rho_2}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)\right]+\right.$$

$$\frac{(\rho_1-\rho_2)}{2}\left[\ln(-\rho_1-\rho_2+\Lambda)-\ln(-\rho_1-\rho_2-\Lambda)-\right.$$

$$\ln(\rho_1+\rho_2+\Lambda)+\ln(\rho_1+\rho_2-\Lambda)]+\rho_2\left[\ln(-\rho_1-\rho_2+\Lambda)-\right.$$

$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\right\}$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)\right]+\right.$$

$$\frac{(\rho_1-\rho_2)}{2}\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)-\right.$$

$$\ln(-\rho_1-\rho_2+\Lambda)+\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1\left[\ln(-\rho_1-\rho_2+\Lambda)-\right.$$

$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_2\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\right\}\}\}\ d\rho_1\ d\rho_2, \tag{10.15.0.1}$$

$$|\Im(\rho)|>\Im(\Lambda)>|\Im(\rho_1)|+|\Im(\rho_2)|\ ;\ \gamma<\min\left(\frac{\pi}{2\mid\Im(\rho_1)\mid};\frac{\pi}{2\mid\Im(\rho_2)\mid}\right).$$

We define

$$H(\rho)=H^{(0)}(\rho,i0^+)=H^{(0)}_\gamma(\rho,i0^+), \tag{10.15.0.2}$$

$$H_{\gamma\lambda}(\rho,i0^+)=\sum_{-m}^{\infty}H^{(n)}_\gamma(\rho,i0^+)\lambda^n. \tag{10.15.0.3}$$

If we take into account that singularities (in the variable Λ) are contained in a horizontal band of width $|\sigma_0|$, we have

$$H_{\gamma\lambda}(\rho,i0^+)=\sum_{-m}^{\infty}H^{(n)}_{\gamma\lambda}(\rho,i\sigma)\frac{(-i\sigma)^n}{n!}\qquad \sigma>|\sigma_0|. \tag{10.15.0.4}$$

As in the other cases discussed above, we define now

$$\{F*G\}(\rho)=H(\rho), \tag{10.15.0.5}$$

as the convolution of two Lorentz invariant UETs. Let $\hat{H}_{\gamma\lambda}(x)$ be the Fourier anti-transform of $H_{\gamma\lambda}(\rho,i0^+)$

$$\hat{H}_{\gamma\lambda}(x)=\sum_{n=-m}^{\infty}\hat{H}^{(n)}_\gamma(x)\lambda^n. \tag{10.15.0.6}$$

If we define

$$\hat{f}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{F_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}F(\rho)\}$$

$$\hat{g}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{G_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}G(\rho)\}, \qquad (10.15.0.7)$$

then,

$$\hat{H}_{\gamma\lambda}(x) = (2\pi)^4\hat{f}_{\gamma\lambda}(x)\hat{g}_{\gamma\lambda}(x), \qquad (10.15.0.8)$$

and taking into account the Laurent expansion of $\hat{f}$ together with $\hat{g}$

$$\hat{f}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{f}^{(n)}_{\gamma}(x)\lambda^n$$

$$\hat{g}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{g}^{(n)}_{\gamma}(x)\lambda^n, \qquad (10.15.0.9)$$

we can write

$$\sum_{n=-m}^{\infty} \hat{H}^{(n)}_{\gamma}(x)\lambda^n = (2\pi)^4 \sum_{n=-m}^{\infty} \left(\sum_{k=-m}^{n} \hat{f}^{(k)}_{\gamma}(x)\hat{g}^{(n-k)}_{\gamma}(x)\right)\lambda^n, \qquad (10.15.0.10)$$

$(m = m_f + m_g)$
and as a consequence,

$$\hat{H}^{(0)}(x) = \sum_{k=-m}^{0} \hat{f}^{(k)}_{\gamma}(x)\hat{g}^{(n-k)}_{\gamma}(x). \qquad (10.15.0.11)$$

10.16 Examples

As a first example of the use of (10.15.0.1), we shall evaluate the convolution product of $\delta(\rho)$ with $\delta(\rho-\mu^2)$ with $\mu = \mu_R + i\mu_I$ a complex number such that $\mu_R^2 > \mu_I^2$, $\mu_R\mu_I > 0$. Thus, from (10.15.0.1) we obtain

$$H_{\gamma\lambda}(\rho,\Lambda) = -i\pi\frac{\ln(-\mu^2+\Lambda) - \ln(-\mu^2+\lambda)}{[2\cosh(\gamma\mu^2)]^{\lambda}}\times$$

$$\left\{\frac{i(\rho-\mu^2)}{8\pi^2\rho}\left[\ln\left(\frac{\rho-\mu^2}{\sqrt{\Lambda(\mu^2+\Lambda)}}\right)+\right.\right.$$

$$\ln\left(\frac{\mu^2-\rho}{\sqrt{-\Lambda(\mu^2+\Lambda)}}\right)\Bigg] + \frac{\mu^2-\rho}{16\pi\rho}\Bigg\} - i\pi\frac{\ln(-\mu^2+\Lambda)-\ln(-\mu^2+\lambda)}{[2\cosh(\gamma\mu^2)]^\lambda} \times$$

$$\left\{\frac{-i\mu^2)}{8\pi^2\rho}\left[\ln\left(\sqrt{\frac{\Lambda}{\mu^2+\Lambda}}\right)+\ln\left(\sqrt{\frac{\Lambda}{\Lambda-\mu^2}}\right)\right]-\frac{\mu^2}{16\pi\rho}\right\}. \tag{10.16.0.1}$$

Simplifying terms, and taking the limit $\lambda \to 0$, (10.16.0.1) becomes

$$H^{(0)}(\rho,\Lambda) = -i\pi[\ln(-\mu^2+\Lambda)-\ln(-\mu^2+\lambda)]\left\{\frac{i(\rho-\mu^2)}{8\pi^2\rho}\left[\ln\left(\rho-\mu^2\right)+\right.\right.$$

$$\left.\left.\ln\left(\mu^2-\rho\right)\right]+\frac{i\mu^2}{8\pi^2\rho}\left[\ln(\mu^2+\Lambda)+\ln(\mu^2-\Lambda)\right]\right\}. \tag{10.16.0.2}$$

Now, if

$$F_1(\mu,\Lambda) = \ln(-\mu^2+\Lambda)-\ln(-\mu^2-\Lambda)$$

then,

$$F_1(\mu, i0^+) = 2i\pi \ ; \ \mu_R^2 > \mu_I^2 \ ; \ \mu_R\mu_I > 0,$$

and, if

$$F_2(\mu,\Lambda) = \ln(\mu^2+\Lambda)-\ln(\mu^2-\Lambda)$$

then,

$$F_2(\mu, i0^+) = 0 \ ; \ \mu_R^2 > \mu_I^2 \ ; \ \mu_R\mu_I > 0.$$

Using these results we obtain

$$H(\rho) = \frac{i(\rho-\mu^2)}{4\rho}\left[\ln(\rho-\mu^2)+\ln(\mu^2-\rho)\right]+\frac{i\mu^2}{2\rho}\ln(\mu^2). \tag{10.16.0.3}$$

As a second example, we will evaluate the convolution of $\Theta[\Im(\rho)]e^{ia\rho}$ (a real) with $\delta(\rho)$. The convolution can be performed on the real ρ-axis to obtain

$$h_{\gamma\lambda}(\rho) = \frac{\pi}{2^{\lambda+1}\rho}\int\limits_{-\infty}^{\infty}\frac{e^{ia\rho_2}|\rho-\rho_2|}{[2\cosh(\gamma\rho_2)]^\lambda}\,d\rho_2, \tag{10.16.0.4}$$

which can be written as

$$h_{\gamma\lambda}(\rho) = \frac{\pi}{2^{\lambda+1}}\left[\frac{i}{\rho}\frac{d}{da}\int\limits_{-\infty}^{\rho}\frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^\lambda}\,d\rho_2+\right.$$

$$\int_{-\infty}^{\rho} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^{\lambda}}\, d\rho_2 -$$

$$\frac{i}{\rho}\frac{d}{da}\int_{\rho}^{\infty} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^{\lambda}}\, d\rho_2 - \int_{\rho}^{\infty} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^{\lambda}}\, d\rho_2 \Bigg]. \quad (10.16.0.5)$$

With the use of the results

$$\int_{-\infty}^{\rho} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^{\lambda}}\, d\rho_2 = \frac{e^{(ia+\gamma\lambda)\rho}}{ia+\gamma\lambda} \times$$

$$F\left(\lambda, \frac{ia+\gamma\lambda}{2\gamma}, \frac{ia+\gamma\lambda}{2\gamma}+1; -e^{-2\gamma\rho}\right), \quad (10.16.0.6)$$

$$\int_{\rho}^{\infty} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^{\lambda}}\, d\rho_2 = \frac{e^{(ia-\gamma\lambda)\rho}}{\gamma\lambda - ia} \times$$

,

$$F\left(\lambda, \frac{\gamma\lambda - ia}{2\gamma}, \frac{\gamma\lambda - ia}{2\gamma}+1; -e^{2\gamma\rho}\right), \quad (10.16.0.7)$$

in the limit $\lambda \to 0$ we obtain

$$h(\rho) = -\frac{\pi}{a^2}\frac{e^{ia\rho}}{\rho}, \quad (10.16.0.8)$$

and therefore, in the complex ρ-plane, the corresponding UET is

$$H(\rho) = -\frac{\pi}{a^2\rho}\left\{\Theta[\Im(\rho)]e^{ia\rho} - \frac{1}{2}\right\}. \quad (10.16.0.9)$$

As the final example, we evaluate the convolution between $F(\rho) = (1/2)Sgn[\Im(\rho)]\, e^{ia\rho}\cosh(\rho^{1/2})$ (a real) and $G(\rho) = \delta(\rho)$. We perform the calculation of the convolution in the real ρ-plane and then we pass to the complex ρ-plane. By the use of the Taylor's expansion of $\cosh(\rho^{1/2})$

$$\cosh(\rho^{1/2}) = \sum_{n=0}^{\infty} \frac{\rho^n}{2n!} \quad (10.16.0.10)$$

we obtain

$$h_{\gamma\lambda}(\rho) = \frac{\pi}{2\rho}\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\int_{-\infty}^{\infty} e^{ia\rho_1}\delta(\rho_2) \times$$

$$\frac{[(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2]_+^{\frac{1}{2}}}{[\cosh(\gamma\rho_1)]^\lambda[\cosh(\gamma\rho_2)]^\lambda}\;d\rho_1\;d\rho_2\;=$$

$$\frac{\pi}{2\rho}\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\int\limits_{-\infty}^{\infty}\frac{e^{ia\rho_1}|\rho-\rho_1|}{[\cosh(\gamma\rho_1)]^\lambda}\,d\rho_1. \tag{10.16.0.11}$$

By means of the use of equations (10.16.0.6) and (10.16.0.7), in the limit $\lambda\to 0$ we obtain

$$h(\rho)=-\pi\left(1+\frac{i}{\rho}\frac{\partial}{\partial a}\right)\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\left(\frac{e^{ia\rho}}{a}\right), \tag{10.16.0.12}$$

and consequently,

$$H(\rho)=\pi\left[\left(\frac{\Theta[\Im(\rho)]}{\rho}\frac{\partial}{\partial a}-\frac{i}{2}\mathrm{Sgn}[\Im(\rho)]\right)\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\left(\frac{e^{ia\rho}}{a}\right)\right]+$$

$$\frac{\pi}{2\rho}\sum_{n=0}^{\infty}\frac{i^n}{2n!}\frac{(n+1)!}{a^{n+2}}. \tag{10.16.0.13}$$

As an example of the use of (10.15.0.11), we will evaluate the convolution product of two Dirac's deltas $\delta(\rho)*\delta(\rho)$. In this case we have

$$F_{\gamma\lambda}(\rho)=-\frac{[\cosh(\gamma\rho)]^\lambda}{2\pi i\rho}=-\frac{1}{2\pi i\rho}, \tag{10.16.0.14}$$

and as a consequence

$$f_{\gamma\lambda}(\rho)=\delta(\rho). \tag{10.16.0.15}$$

The Fourier anti-transform of (10.16.0.15) is

$$\hat{f}_{\gamma\lambda}(x)=\frac{2}{(2\pi)^3}x^{-1}. \tag{10.16.0.16}$$

Thus, we have

$$\hat{f}^2_{\gamma\lambda}(x)=\frac{4}{(2\pi)^6}x^{-2}. \tag{10.16.0.17}$$

From (10.16.0.17) we obtain

$$\lim_{\lambda\to 0}\hat{f}^2_{\gamma\lambda}(x)=\frac{4}{(2\pi)^6}x^{-2}, \tag{10.16.0.18}$$

and taking into account that

$$\mathcal{F}\{x^{-2}\} = \frac{\pi^3}{2} Sgn(\rho), \tag{10.16.0.19}$$

we obtain

$$\delta(\rho) * \delta(\rho) = \frac{\pi}{2} Sgn(\rho). \tag{10.16.0.20}$$

Chapter 11

A final word

As we have been insisting above, the issue of defining the product of two distributions (a product in a ring with divisors of zero) is an old problem of functional analysis. We have already seen that, in quantum field theory, the problem of evaluating the product of distributions with coincident point singularities is related to the asymptotic behavior of loop integrals of propagators.

We also have seen that practically all definitions of that product lead to limitations on the set of distributions that can be multiplied by each other to give another distribution of the same type.

In fact, Laurent Schwartz himself was not able to define a product of distributions regarded as an algebra, instead of as a ring with divisors of zero.

In References [33, 39, 41, 42], it was demonstrated that it is indeed possible to define a general convolution between two ultradistributions of Sebastiao e Silva (ultrahyperfunctions), resulting in another ultradistribution, and, therefore, a general product in a ring with zero divisors. Such a ring is the space of distributions of exponential type, or UETs, obtained by applying the anti-Fourier transform to the space of tempered ultradistributions. We must clarify at this point that ultrahyperfunctions are the generalization and extension to the complex plane of the Schwartz tempered distributions and of the UETs.

The problem we then faced, and that was answered in this book, is that of formulating the convolution between ultradistributions. This is an involved issue, difficult to manage, even if it has the immense advantage of allowing one to discuss non-renormalizable quantum field theories, something no one has yet achieved, as far as we know.

Fortunately, we have found in the present book that a method similar to that used to define the convolution of ultradistributions can also be used to define the convolution of Lorentz invariant distributions using the dimensional regularization (DR) of Bollini and Giambiagi in momentum space. Taking advantage of such DR we can also work in configuration space [13]. Thus, one can obtain a convolution of Lorentz invariant tempered distributions in momentum space and the corresponding product in configuration space [2].

DR is one of the most important advances in theoretical physics and is used in several disciplines of it [44]-[98]. With it DR happens to be a convolution of special distributions in momentum space and a product in a ring with divisors of zero in configuration space. It is our hope that this convolution can then be used to treat non-renormalizable quantum field theories.

Thus, a quite significant result is at hand here. One has *generalized* Bollini and Giambiagi's dimensional regularization to *all Schwartz tempered, explicitly Lorentz invariant, distributions (STDELI)*, something that Bollini-Giambiagi were unable to achieve. This generalization would permit one to deal with non-renormalizable QFT, a monumental feat indeed, that allows us to forget about so called counterterms.

The vocable **counterterm** is often used to denote a special term added to an equation or formula that exactly cancels the contribution of another term which diverges [1]. This is often done in quantum field theory, sometimes without adequate justification, regrettably enough. *This should not be done, since a non-renormalizable theory involves an infinite number of counterterms.* The central purpose of the book was to define a STDELI-convolution in order to avoid counterterms. The STDELI convolution, once obtained, converts configuration space into a ring with zero-divisors [2]. In it, one possesses a product between the ring-elements. Thus, any unitary-causal-Lorentz invariant theory quantified in such a manner can be said to have become "predictive" [2], if we assume knowledge of the pertinent experimental results on which the theory is based to begin with [2]. The distinction between renormalizable and not-renormalizable QFTs becomes unnecessary now [2].

With our Bollini and Giambiagi generalization, that uses Laurent expansions in the dimension, all finite constants of the convolutions become completely determined, eliminating arbitrary choices of finite constants [2]. This is tantamount to eliminating all finite renormalizations of the theory [2]. What is the importance of using *the term independent of the dimension* in Laurent's expansion? That the result obtained for finite convolutions will coincide with such a term. This translates to configuration space of the product-operation in a ring with divisors of zero [2][2].

As further examples, we will calculate below some convolutions of distributions used in quantum field theory. In particular, the convolution of n massless Feynman propagators and the convolution of n massless Wheeler propagators. For a full discussion about definition and properties of Wheeler propagators see [99, 100], works which, in turn, are based on Wheeler and Feynman's papers [15, 17]. The results obtained below have already allowed the present authors to calculate the classical partition function of Newtonian gravity, for the first time ever, in the Gibbs' formulation and in the Tsallis' one [98].

11.1 First generalization of DR

The expression for the convolution of two spherically symmetric functions was deduced in Ref. [41] $(h(k,\nu) = (f * g)(k,\nu))$

$$h(k,\nu) = \frac{2^{4-\nu}\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu-1}{2})k^{\nu-2}} \iint\limits_0^\infty f(k_1,\nu)g(k_2,\nu) \times$$

$$[4k_1^2k_2^2 - (k^2 - k_1^2 - k_2^2)^2]_+^{\frac{\nu-3}{2}} k_1 k_2 \, dk_1 \, dk_2. \tag{11.1.0.1}$$

However, Bollini and Giambiagi did not obtain a product in a *ring with divisors of zero* [2], something that we will do now below and constitutes an essential step.

Consider here that f and g belong to $\mathbf{S}_R^{'}$. With the change of variables

$\rho = k^2$, $\rho_1 = k_1^2$, $\rho_2 = k_2^2$ takes the form

$$h(\rho,\nu) = \frac{2^{2-\nu}\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu-1}{2})\rho^{\frac{\nu-2}{2}}} \iint_0^\infty f(\rho_1,\nu)g(\rho_2,\nu) \times$$

$$[4\rho_1\rho_2 - (\rho-\rho_1-\rho_2)^2]_+^{\frac{\nu-3}{2}}\ d\rho_1\ d\rho_2. \tag{11.1.0.2}$$

Let $\mathfrak{V}$ be a vertical band contained in the complex ν-plane $\mathfrak{W}$. Integral (11.1.0.2) is an analytic function of ν defined in the domain $\mathfrak{V}$. Then, according to the methodology of Ref. [33], $h(\nu,\rho)$ can be analytically continued to other parts of $\mathfrak{W}$. In particular, near the dimension ν_0 we have the Laurent expansion

$$h(\rho,\nu) = \sum_{m=-1}^{\infty} h^{(m)}(\rho)(\nu-\nu_0)^m. \tag{11.1.0.3}$$

Here, ν_0 is the dimension of the considered space. In particular, $\nu_0 = 4$ is the dimension that we will focus on. Define now the convolution product as the $(\nu-\nu_0)$-independent term of the Laurent's expansion (11.1.0.3)

$$h_{\nu_0}(\rho) = h^{(0)}(\rho). \tag{11.1.0.4}$$

Thus, in the ring with zero divisors $\mathcal{S}'_{RA}$, we have indeed able to define a product of distributions. This is exactly what is needed to start thinking about a non-renormalizable field theory.

11.2 Example

As an example of the use of (11.1.0.4), we evaluate the convolution of a massless propagator with a propagator corresponding to a scalar particle of mass m. The result of this convolution, using this formula, is given in [13]. It is

$$h(k,\nu) = 2^{\nu-2}\pi^{\frac{\nu}{2}} m^{\nu-4} \frac{\Gamma\left(\frac{\nu-2}{2}\right)\Gamma\left(\frac{4-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} F\left(1, \frac{4-\nu}{2}; \frac{\nu}{2}; -\frac{k^2}{m^2}\right). \tag{11.2.0.1}$$

Now, we use the equality

$$\Gamma\left(\frac{4-\nu}{2}\right) F\left(1, \frac{4-\nu}{2}; \frac{\nu}{2}; -\frac{k^2}{m^2}\right) =$$

$$\Gamma\left(\frac{4-\nu}{2}\right)-\frac{2}{\nu}\Gamma\left(\frac{6-\nu}{2}\right)\frac{k^2}{m^2}F\left(1,\frac{6-\nu}{2};\frac{2+\nu}{2};-\frac{k^2}{.}m^2\right). \tag{11.2.0.2}$$

After a tedious calculation, we obtain the corresponding Laurent expansion of $h(k,\nu)$ as

$$h(k,\nu)=-\frac{8\pi^2}{\nu-4}+4\pi^2\left(\mathbf{C}+2-\ln 4-\ln\pi-\ln m^2\right)-$$
$$2\pi^2\frac{k^2}{m^2}F\left(1,1;3;-\frac{k^2}{m^2}\right)+\sum_{s=1}^{\infty}a_s(\nu-4)^s, \tag{11.2.0.3}$$

where $\mathbf{C}$ is Euler's constant with the sign changed, $\mathbf{C}=-0.57721566490$. Thus, we have

$$\frac{1}{k^2}*\frac{1}{k^2+m^2}=4\pi^2\left(\mathbf{C}+2-\ln 4-\ln\pi-\ln m^2\right)-$$
$$2\pi^2\frac{k^2}{m^2}F\left(1,1;3;-\frac{k^2}{m^2}\right). \tag{11.2.0.4}$$

11.3 Second generalization of DR

In this sub-section we repeat the efforts of the preceding one for Minkowskian space. The generalization of the Bochner's theorem to Minkowskian space has been obtained in Reference [41]. The corresponding expression for $\nu=2n$ is

$$h(\rho,\nu)=\frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}}e^{\frac{i\pi(2-\nu)}{2}}\Gamma\left(\frac{3-\nu}{2}\right)\iint_{-\infty}^{\infty}f(\rho_1,\nu)g(\rho_2,\nu)\quad\times$$
$$\left\{(\rho-i0)^{-\frac{1}{2}}\left[\frac{(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2}{\rho}+i0\right]^{\frac{\nu-3}{2}}+e^{i\pi(\nu-2)}\quad\times\right.$$
$$\left.(\rho+i0)^{-\frac{1}{2}}\left[\frac{(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2}{\rho}-i0\right]^{\frac{\nu-3}{2}}\right\}d\rho_1\,d\rho_2, \tag{11.3.0.1}$$

$h(\rho,\nu)=(f*g)(\rho,\nu)$.
When $\nu=2n+1$ we obtain

$$h(\rho,\nu)=-\frac{i\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}\Gamma\left(\frac{\nu-3}{2}\right)}\iint_{-\infty}^{\infty}f(\rho_1,\nu)g(\rho_2,\nu)\left[\frac{(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2}{\rho}\right]^{\frac{\nu-3}{2}}$$

$$\Big\{(\rho - i0)^{-\frac{1}{2}} \times$$

$$\left[\psi\left(\frac{\nu-1}{2}\right) + \frac{i\pi}{2} + \ln\left[\frac{(\rho-\rho_1-\rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0\right]\right] - (\rho + i0)^{-\frac{1}{2}}$$

$$\left[\psi\left(\frac{\nu-1}{2}\right) + \frac{i\pi}{2} + \ln\left[-\frac{(\rho-\rho_1-\rho_2)^2 - 4\rho_1\rho_2}{\rho} + i0\right]\right]\Big\} \, d\rho_1 \, d\rho_2. \tag{11.3.0.2}$$

For the Minkowskian case one can also employ Laurent's expansion

$$h(\rho, \nu) = \sum_{m=-1}^{\infty} h^{(m)}(\rho)(\nu - \nu_0)^m \tag{11.3.0.3}$$

and therefore, again, we have for the convolution the result

$$h_{\nu_0}(\rho) = h^{(0)}(\rho). \tag{11.3.0.4}$$

Thus, in the ring with zero divisors $\mathbf{S}^{'}_{LA}$ we have introduced a product of distributions.

11.4 Applications

As an application of the use of (11.3.0.1), we will consider the convolution of two Dirac's δ-Distributions, $\delta(\rho)$. The result is

$$h(\rho, \nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right)$$

$$\left[(\rho - i0)^{-\frac{1}{2}}(\rho + i0)^{\frac{\nu-3}{2}} + e^{i\pi(\nu-2)}(\rho + i0)^{-\frac{1}{2}}(\rho - i0)^{\frac{\nu-3}{2}}\right]. \tag{11.4.0.1}$$

Simplifying terms we obtain

$$h(\rho, \nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right)\left[\rho_+^{\frac{\nu-4}{2}} + e^{\frac{i\pi(\nu-2)}{2}} \rho_-^{\frac{\nu-4}{2}}\right]. \tag{11.4.0.2}$$

Thus, in four dimensions we have

$$h_4(\rho) = \delta(\rho) * \delta(\rho) = \frac{\pi}{2} Sgn(\rho). \tag{11.4.0.3}$$

Note that this convolution does not make sense in a four-dimensional Euclidean space, since in such a case $\delta(\rho) \equiv 0$.

As a second application, we calculate the convolution $\delta(\rho - m^2) * \delta(\rho - m^2)$. In this case we have

$$h(\rho,\nu) = \frac{\pi^{\frac{\nu-3}{2}}}{2^{\nu-1}} e^{\frac{i\pi(2-\nu)}{2}} \Gamma\left(\frac{3-\nu}{2}\right)$$

$$\Big[(\rho - i0)^{-\frac{1}{2}} (\rho - 2m^2 + i0)^{\frac{\nu-3}{2}} +$$

$$e^{i\pi(\nu-2)} (\rho + i0)^{-\frac{1}{2}} (\rho - 2m^2 - i0)^{\frac{\nu-3}{2}}\Big] \qquad (11.4.0.4)$$

When $\nu = 4$ we then obtain

$$\delta(\rho - m^2) * \delta(\rho - m^2) =$$

$$\frac{\pi}{4}\Big[(\rho - i0)^{-\frac{1}{2}} (\rho - 2m^2 + i0)^{\frac{1}{2}} +$$

$$e^{i\pi(\nu-2)} (\rho + i0)^{-\frac{1}{2}} (\rho - 2m^2 - i0)^{\frac{1}{2}}\Big]. \qquad (11.4.0.5)$$

N massless Feynman's propagators

11.5 The Minkowskian space case

This is a significant result. Why? Because it has never been achieved before. Let us calculate the convolution of n massless Feynman propagators ($n \geq 2$). For this purpose we take into account that

$$\mathcal{F}^{-1}\{f_1 * f_2 * \cdots * f_n\} = (2\pi)^{(n-1)\nu} \hat{f}_1 \hat{f}_2 \cdots \hat{f}_n. \qquad (11.5.0.1)$$

According to Reference [6], we have

$$\mathcal{F}^{-1}\left\{(\rho + i0)^{-1}\right\} = \frac{e^{-\frac{i\pi}{2}(\nu-1)}}{(2\pi)^\nu} 2^{(\nu-2)} \pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2} - 1\right)(x - i0)^{1-\frac{\nu}{2}}, \qquad (11.5.0.2)$$

and therefore,

$$\mathcal{F}^{-1}\left\{(\rho + i0)^{-1} * (\rho + i0)^{-1} * \cdots * (\rho + i0)^{-1}\right\} =$$

$$(2\pi)^{(n-1)\nu} \frac{e^{-\frac{i\pi}{2}(\nu-1)n}}{(2\pi)^{\nu n}} 2^{(\nu-2)n} \pi^{\frac{\nu n}{2}} \left[\Gamma\left(\frac{\nu}{2} - 1\right)\right]^n (x - i0)^{n(1-\frac{\nu}{2})}. \qquad (11.5.0.3)$$

Using again Reference [6] we have now

$$\mathcal{F}\left\{(x-i0)^{n(1-\frac{\nu}{2})}\right\} =$$

$$\frac{e^{-\frac{i\pi}{2}(\nu-1)}}{\Gamma\left[n\left(\frac{\nu}{2}-1\right)\right]}2^{\nu+2n\left(1-\frac{\nu}{2}\right)}\pi^{\frac{\nu}{2}}\Gamma\left[\frac{\nu}{2}+n\left(1-\frac{\nu}{2}\right)\right](\rho+i0)^{n\left(\frac{\nu}{2}-1\right)-\frac{\nu}{2}}, \tag{11.5.0.4}$$

with which we obtain

$$(\rho+i0)^{-1}*(\rho+i0)^{-1}*\cdots*(\rho+i0)^{-1} =$$
$$\frac{e^{-\frac{i\pi}{2}(n-1)(\nu-1)}}{\Gamma\left[n\left(\frac{\nu}{2}-1\right)\right]}\pi^{\frac{\nu}{2}(n-1)}\left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^n$$
$$\Gamma\left[\frac{\nu}{2}+n\left(1-\frac{\nu}{2}\right)\right](\rho+i0)^{n\left(\frac{\nu}{2}-1\right)-\frac{\nu}{2}}. \tag{11.5.0.5}$$

We have then, for the convolution of n massless Feynman's propagators, the result

$$i(\rho+i0)^{-1}*i(\rho+i0)^{-1}*\cdots*i(\rho+i0)^{-1} =$$
$$\frac{e^{\frac{i\pi}{2}[n-(n-1)(\nu-1)]}}{\Gamma\left[n\left(\frac{\nu}{2}-1\right)\right]}\pi^{\frac{\nu}{2}(n-1)}\left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^n$$
$$\Gamma\left[\frac{\nu}{2}+n\left(1-\frac{\nu}{2}\right)\right](\rho+i0)^{n\left(\frac{\nu}{2}-1\right)-\frac{\nu}{2}}. \tag{11.5.0.6}$$

After a tedious calculation we obtain the corresponding Laurent expansion around $\nu=4$

$$i(\rho+i0)^{-1}*i(\rho+i0)^{-1}*\cdots*i(\rho+i0)^{-1} = \frac{2i\pi^{2(n-1)}\rho^{n-2}}{[\Gamma(n)]^2(\nu-4)}+$$
$$\frac{i\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)}\left[\ln(\rho+i0)-i\pi+\ln(\pi)+\frac{n}{n-1}\psi(1)-\frac{n}{n-1}\psi(n)-\right.$$
$$\left.\psi(n-1)\right]+\sum_{m=1}^{\infty}a_m(\rho)(\nu-4)^m. \tag{11.5.0.7}$$

The independent $\nu-4$ term is the result of the convolution in four dimensions

$$[i(\rho+i0)^{-1}*i(\rho+i0)^{-1}*\cdots*i(\rho+i0)^{-1}]_{\nu_0=4} =$$
$$\frac{i\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)}\left[\ln(\rho+i0)-i\pi+\ln(\pi)+\frac{n}{n-1}\psi(1)-\frac{n}{n-1}\psi(n)-\right.$$
$$\left.\psi(n-1)\right]. \tag{11.5.0.8}$$

11.6 The Euclidean space case

Let us now calculate the convolution of n massless Feynman propagators ($n \geq 2$) in Euclidean space, using again (11.1.0.1). According to reference [6], we obtain

$$\mathcal{F}^{-1}\left\{k^{-2}\right\} = \frac{1}{(2\pi)^{\nu}} 2^{(\nu-2)} \pi^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}-1\right).r^{2-\nu}. \tag{11.6.0.1}$$

For n propagators we have then

$$\mathcal{F}^{-1}\left\{k^{-2} * k^{-2} * \cdots * k^{-2}\right\} =$$

$$\frac{(2\pi)^{(n-1)\nu}}{(2\pi)^{\nu n}} 2^{(\nu-2)n} \pi^{\frac{\nu n}{2}} \left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^{n} r^{n(2-\nu)}. \tag{11.6.0.2}$$

Appealing again to Reference [6], we can evaluate the corresponding Fourier transform

$$\mathcal{F}\left\{r^{n(2-\nu)}\right\} =$$

$$\frac{1}{\Gamma\left[n\left(\frac{\nu}{2}-1\right)\right]} 2^{\nu+2n\left(1-\frac{\nu}{2}\right)} \pi^{\frac{\nu}{2}} \Gamma\left[\frac{\nu}{2}+n\left(1-\frac{\nu}{2}\right)\right] k^{n(\nu-2)-\nu}. \tag{11.6.0.3}$$

Thus,

$$k^{-2} * k^{-2} * \cdots * k^{-2} =$$

$$\frac{\pi^{\frac{\nu}{2}(n-1)}}{\Gamma\left[n\left(\frac{\nu}{2}-1\right)\right]} \left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^{n} \Gamma\left[\frac{\nu}{2}+n\left(1-\frac{\nu}{2}\right)\right] k^{n(\nu-2)-\nu}. \tag{11.6.0.4}$$

Let $\rho = k^2$. We have then for the convolution of n massless Feynman propagators the result

$$\rho^{-1} * \rho^{-1} * \cdots * \rho^{-1} =$$

$$\frac{\pi^{\frac{\nu}{2}(n-1)}}{\Gamma\left[n\left(\frac{\nu}{2}-1\right)\right]} \left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^{n} \Gamma\left[\frac{\nu}{2}+n\left(1-\frac{\nu}{2}\right)\right] \rho^{n\left(\frac{\nu}{2}-1\right)-\frac{\nu}{2}}. \tag{11.6.0.5}$$

By recourse to Laurent expansion we then obtain

$$\rho^{-1} * \rho^{-1} * \cdots * \rho^{-1} = \frac{2(-1)^{n}\pi^{2(n-1)}\rho^{n-2}}{[\Gamma(n)]^{2}(\nu-4)} +$$

$$\frac{(-1)^{n}\pi^{2(n-1)}\rho^{n-2}}{\Gamma(n)\Gamma(n-1)} \left[\ln(\rho) + \ln(\pi) + \frac{n}{n-1}\psi(1) - \frac{n}{n-1}\psi(n) - \right.$$

$$\psi(n-1)] + \sum_{m=1}^{\infty} a_m(\rho)(\nu-4)^m. \tag{11.6.0.6}$$

The result of the convolution in four dimensions is then

$$[\rho^{-1} * \rho^{-1} * \cdots * \rho^{-1}]_{\nu_0=4} =$$

$$\frac{(-1)^n \pi^{2(n-1)} \rho^{n-2}}{\Gamma(n)\Gamma(n-1)} \left[\ln(\rho) + \ln(\pi) + \frac{n}{n-1}\psi(1) - \frac{n}{n-1}\psi(n) - \right.$$
$$\left. \psi(n-1)\right]. \tag{11.6.0.7}$$

We emphasize that the results of this section are completely original.

Massless Wheeler propagators

11.7 Two massless Wheeler propagators

The Wheeler massless propagator is given by (note that this propagator can not be defined in Euclidean space)

$$W(\rho) = \frac{i}{2}\left[\frac{1}{\rho + i0} + \frac{1}{\rho - i0}\right], \tag{11.7.0.1}$$

and can be written in the form

$$W(\rho) = \frac{i}{\rho + i0} - \pi\delta(\rho). \tag{11.7.0.2}$$

Therefore, we have

$$W(\rho) * W(\rho) = \frac{i}{\rho + i0} * \frac{i}{\rho + i0} - 2\pi\delta(\rho) * \frac{i}{\rho + i0} + \pi^2\delta(\rho) * \delta(\rho). \tag{11.7.0.3}$$

After a long and tedious calculation, using (11.3.0.1) we obtain

$$-2\pi\delta(\rho) * \frac{i}{\rho + i0} = \frac{-i\pi^{\frac{\nu-1}{2}}}{2^{\nu-2}} e^{i\pi\left(\frac{2-\nu}{2}\right)} \Gamma\left(\frac{3-\nu}{2}\right)\Gamma(\nu-2)\Gamma(3-\nu)\times$$
$$\left\{\left[1 + e^{i\pi(\nu-2)}\right]\left[1 - e^{-i\pi(3-\nu)}\right] H(\rho)\rho^{\frac{\nu}{2}-1} + \right.$$
$$\left. 2e^{i\pi\left(\frac{\nu-2}{2}\right)}\left[e^{i\pi(\nu-2)} - 1\right] H(-\rho)(-\rho)^{\frac{\nu}{2}-2}\right\}. \tag{11.7.0.4}$$

This last equation can be re-written in the form

$$-2\pi\delta(\rho) * \frac{i}{\rho + i0} = \frac{\pi^{\frac{\nu+3}{2}} e^{\frac{i\pi}{2}(3-\nu)} \cos\pi\left(\frac{\nu-2}{2}\right)}{2^{\nu-4}\Gamma\left(\frac{\nu-1}{2}\right)\sin\pi\left(\frac{\nu-3}{2}\right)\sin\pi\nu}$$

$$\left\{\cos\pi\left(\frac{\nu-2}{2}\right)H(\rho)\rho^{\frac{\nu}{2}-1} - e^{i\pi(\nu-2)}H(-\rho)(-\rho)^{\frac{\nu}{2}-2}\right\}. \quad (11.7.0.5)$$

For the first convolution of (11.7.0.3), we have from (11.5.0.8), with $n = 2$,

$$\frac{i}{\rho + i0} * \frac{i}{\rho + i0} = \frac{e^{i\frac{\pi}{2}(3-\nu)}\pi^{\frac{\nu}{2}}}{\Gamma(\nu-2)}\left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^2\Gamma\left(2-\frac{\nu}{2}\right)(\rho + i0)^{\frac{\nu}{2}-2}. \quad (11.7.0.6)$$

This equation can be recast in the form

$$\frac{i}{\rho + i0} * \frac{i}{\rho + i0} = \frac{e^{i\frac{\pi}{2}(3-\nu)}\pi^{\frac{\nu-3}{2}}\cos\left(\frac{\nu-3}{2}\right)}{2^{\nu}\Gamma\left(\frac{\nu-1}{2}\right)\sin\pi\nu}(\rho + i0)^{\frac{\nu}{2}-2}. \quad (11.7.0.7)$$

When $\nu = 4$, the sum of (11.7.0.5) and (11.7.0.7) has as a result

$$\frac{i}{\rho + i0} * \frac{i}{\rho + i0} - 2\pi\delta(\rho) * \frac{i}{\rho + i0} = \pi^3 H(-\rho). \quad (11.7.0.8)$$

Using now (11.4.0.3), we find

$$W(\rho) * W(\rho) = \frac{\pi^3}{2}. \quad (11.7.0.9)$$

This result was obtained in Chapter 7, formula (8.3.0.12) using the convolution of even tempered ultradistributions. The coincidence of (11.7.0.9) (8.3.0.12) confirms the validity of the results obtained in section 6 of this chapter. We emphasize that the present results are obtained in a manner considerably simpler to that of Chapter 7.

11.8 N Massless Wheeler propagators

According to reference [6], we have

$$\mathcal{F}^{-1}\left\{(\rho + i0)^{-1}\right\} = \frac{e^{-\frac{i\pi}{2}(\nu-1)}}{(2\pi)^{\nu}}2^{(\nu-2)}\pi^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}-1\right)(x - i0)^{1-\frac{\nu}{2}}, \quad (11.8.0.1)$$

$$\mathcal{F}^{-1}\left\{(\rho - i0)^{-1}\right\} = \frac{e^{\frac{i\pi}{2}(\nu-1)}}{(2\pi)^\nu} 2^{(\nu-2)}\pi^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}-1\right)(x+i0)^{1-\frac{\nu}{2}}. \tag{11.8.0.2}$$

Thus,

$$\mathcal{F}^{-1}\{W(\rho)\} = \frac{i\pi^{\frac{\nu}{2}}}{(2\pi)^\nu} 2^{(\nu-2)}\Gamma\left(\frac{\nu}{2}-1\right)\sin\left(\frac{\pi\nu}{2}\right)x_+^{1-\frac{\nu}{2}}. \tag{11.8.0.3}$$

As a consequence, we obtain for n Wheeler propagators

$$\mathcal{F}^{-1}\{W(\rho) * W(\rho) * \cdots * W(\rho)\} =$$

$$\frac{i^n\pi^{n\frac{\nu}{2}}}{(2\pi)^{n\nu}} 2^{n(\nu-2)}\left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^n \sin^n\left(\frac{\pi\nu}{2}\right)x_+^{n\left(1-\frac{\nu}{2}\right)}. \tag{11.8.0.4}$$

Resorting again to Reference [6] we have

$$\mathcal{F}\left\{x_+^{n\left(1-\frac{\nu}{2}\right)}\right\} = \pi^{\frac{\nu}{2}-1}2^{(1-n)\nu+2n}\Gamma\left(n+1-\frac{n\nu}{2}\right)\Gamma\left[n-\frac{(n-1)\nu}{2}\right]\otimes$$

$$\frac{1}{2}\left\{e^{-i\pi\left[n-(n-1)\frac{\nu}{2}\right]}(\rho - i0)^{(n-1)\frac{\nu}{2}-n} + e^{i\pi\left[n-(n-1)\frac{\nu}{2}\right]}(\rho + i0)^{(n-1)\frac{\nu}{2}-n}\right\}. \tag{11.8.0.5}$$

Using (11.8.0.5) we finally arrive at

$$W(\rho) * W(\rho) * \cdots * W(\rho) =$$

$$\frac{i^n\pi^{\frac{\nu}{2}(n-1)-1}}{2}\left[\Gamma\left(\frac{\nu}{2}-1\right)\right]^n \times$$

$$\Gamma\left(n+1-\frac{n\nu}{2}\right)\Gamma\left[n-\frac{(n-1)\nu}{2}\right]\sin^n\left(\frac{\pi\nu}{2}\right)\times$$

$$\left\{e^{i\pi\left[n-(n-1)\frac{\nu}{2}\right]}(\rho + i0)^{(n-1)\frac{\nu}{2}-n} + \right.$$

$$\left. e^{-i\pi\left[n-(n-1)\frac{\nu}{2}\right]}(\rho - i0)^{(n-1)\frac{\nu}{2}-n}\right\}. \tag{11.8.0.6}$$

We see that formula (11.8.0.6) has a zero of order $n-2$ for $\nu \geq 4$, ν even, and consequently cancels for those dimensions when $n \geq 3$. So we can assert that, for $\nu = 4$

$$W(\rho) * W(\rho) * \cdots * W(\rho) = 0 \tag{11.8.0.7}$$

when $n \geq 3$.

11.9 Electron self energy

11.9.1 Original BG electron's self-energy

The *self-energy* of the electron (to one loop) is defined as

$$\Sigma(p,\nu) = \frac{e^2}{(2\pi)^\nu}\int \gamma_\mu \frac{[i\gamma\cdot(p-k)-m]}{[(p-k)^2+m^2]k^2}\gamma^\mu d^\nu k \qquad (11.9.1.1)$$

BG evaluated this integral for the first time [12]. The result they obtained using their definition of DR is

$$\Sigma(p,\nu) = \frac{e^2}{(4\pi)^{\frac{\nu}{2}}}(m^2\rho)^{\frac{\nu}{2}-2}\Gamma\left(2-\frac{\nu}{2}\right)$$
$$\left\{[(i\gamma\cdot p+m)(2-\nu)-2m]\frac{2}{\nu-2}\cdot\right.$$
$$F\left(2-\frac{\nu}{2},\frac{\nu}{2}-1,\frac{\nu}{2};1-\frac{1}{\rho}\right)+[m-(i\gamma\cdot p+m)]\left(\frac{4}{\nu}-2\right)\cdot$$
$$\left.F\left(2-\frac{\nu}{2},\frac{\nu}{2},\frac{\nu}{2}+1;1-\frac{1}{\rho}\right)\right\} \qquad (11.9.1.2)$$

where the variable ρ is defined as $\rho = (p^2+m^2)/m^2$. To obtain the finite part of self-energy, they used the following method. They decomposed it the form

$$\Sigma(p,\nu) = A + (i\gamma\cdot p+m)B + (i\gamma\cdot p+m)^2\Sigma_f(p,\nu), \qquad (11.9.1.3)$$

where A, B, and $\Sigma_f(p,\nu)$ have been defined as

$$A = [\Sigma(p,\nu)]_{i\gamma\cdot p+m=0}$$
$$B = \left[\frac{\Sigma(p,\nu)-A}{i\gamma\cdot p+m}\right]_{i\gamma\cdot p+m=0}$$
$$\Sigma_f(p,\nu) = \left[\frac{\Sigma(p,\nu)-A-(i\gamma\cdot p+m)B}{(i\gamma\cdot p+m)^2}\right]_{i\gamma\cdot p+m=0}, \qquad (11.9.1.4)$$

with $(i\gamma\cdot p+m)^{-1} = (m-i\gamma\cdot p)/(p^2+m^2)$. $\Sigma_f(p,\nu)$ thus turns out to be the finite part of self energy. As a result of these definitions we get

$$A = -\frac{e^2m^{\nu-3}}{(4\pi)^{\frac{\nu}{2}}}\frac{\nu-1}{\nu-3}\Gamma\left(2-\frac{\nu}{2}\right) \qquad (11.9.1.5)$$

$$B = \frac{A}{m} \tag{11.9.1.6}$$

$$\Sigma_f(p,\nu) = \frac{e^2 m^{\nu-4}}{(4\pi)^{\frac{\nu}{2}}}\Gamma\left(2-\frac{\nu}{2}\right)\times$$

$$\left\{\left[(2-\nu)\frac{m-i\gamma\cdot p}{m^2\rho} - 4\frac{m-i\gamma\cdot p}{m^2\rho^2} + \frac{2}{m\rho}\right]\cdot\right.$$

$$\frac{2}{\nu-2}F\left(1, 2-\frac{\nu}{2}, \frac{\nu}{2}; 1-\rho\right) +$$

$$\left[2\frac{m-i\gamma\cdot p}{m^2\rho^2} - \frac{m-i\gamma\cdot p}{m^2\rho} - \frac{1}{m\rho}\right]\left(\frac{4}{\nu}-2\right)$$

$$F\left(1, 2-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\rho\right) +$$

$$\left.\frac{\nu-1}{\nu-3}\left[2\frac{m-i\gamma\cdot p}{m^2\rho^2} - \frac{1}{m\rho} + \frac{m-i\gamma\cdot p}{m^2\rho}\right]\right\} \tag{11.9.1.7}$$

It should be noted that A and B are independent of p. This result has not been modified up to the present time. However, that decomposition has an unwanted aspect. If we closely look at it, (11.9.1.2) has no quadratic dependence on $i\gamma\cdot p + m$, as seen in (11.9.1.4). This is the consequence of extant products of the gamma function with the hypergeometric function at a pole of the gamma function. We must therefore isolate the pole in a proper way to avoid this problem. The most rigorous way to do this is to use the convolution of Lorentz invariant distributions obtained in [2]. If this is done directly, the result obtained turns out to be very complex to calculate numerically, since Kamp de Friet (KdF) functions appear, which are very difficult to evaluate. These functions arise when we derive the hypergeometric functions (that appear in (11.9.1.2)) with respect to the dimension ν, so as to perform the corresponding Laurent's series expansion. Instead of doing this directly, before differentiating we must isolate the gamma function at the pole, from the hypergeometric function, and subsequently perform the Laurent expansion. We will do that in the next section.

11.9.2 Vacuum polarization

BG also calculated the *vacuum polarization* in QED [12]. The integral that defines it is given by

$$\Pi_{\mu,\nu}(k,\nu)=\frac{ie^2}{(2\pi)^\nu}\mathrm{Tr}\int\gamma_\mu\frac{i\gamma\cdot p-m}{p^2+m^2}\gamma_\nu\frac{i\gamma\cdot(p-k)-m}{(p-k)^2+m^2}d^\nu p \tag{11.9.2.8}$$

To evaluate it, the following results should be used

$$\mathrm{Tr}\gamma_\mu\gamma_\nu=d(\nu)\eta_{\mu\nu}, \tag{11.9.2.9}$$

$$\mathrm{Tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma=d(\nu)(\eta_{\mu\nu}\eta\rho\sigma-\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}), \tag{11.9.2.10}$$

where $d(\nu)$ is an analytic function of the dimension ν, which, for ν positive integer, matches the number of components of the pertinent spinor in a ν dimensional space. The result of the integral (11.9.2.8) is

$$\Pi_{\mu\nu}(k,\nu)=\frac{e^2}{(4\pi)^{\frac{\nu}{2}}}\frac{d(\nu)}{3}\Gamma\left(2-\frac{\nu}{2}\right)m^{\nu-4}\times$$

$$(k_\mu k_\nu-\eta_{\mu\nu}k^2)F\left(2-\frac{\nu}{2},2,\frac{5}{2};-\frac{k^2}{4m^2}\right). \tag{11.9.2.11}$$

In this result, all terms appear save for the first one, the multiplication of the gamma function by a zero of the hypergeometric function from which the unit has been subtracted. Therefore, the finite part of the vacuum polarization is given by

$$\Pi_{\mu\nu F}(k,\nu)=\frac{e^2}{(4\pi)^{\frac{\nu}{2}}}\frac{d(\nu)}{3}\Gamma\left(2-\frac{\nu}{2}\right)m^{\nu-4}(k_\mu k_\nu-\eta_{\mu\nu}k^2)$$

$$\left[F\left(2-\frac{\nu}{2},2,\frac{5}{2};-\frac{k^2}{4m^2}\right)-1\right], \tag{11.9.2.12}$$

and the vacuum polarization is written as

$$\Pi_{\mu\nu}(k,\nu)=\frac{e^2}{(4\pi)^{\frac{\nu}{2}}}\frac{d(\nu)}{3}\Gamma\left(2-\frac{\nu}{2}\right)m^{\nu-4}(k_\mu k_\nu-\eta_{\mu\nu}k^2)+\Pi_{\mu\nu F}(k,\nu) \tag{11.9.2.13}$$

11.10 Exact results of this book

11.10.1 The electron self-energy to one loop

To evaluate the self-energy we use the following equalities

$$F\left(2-\frac{\nu}{2},\frac{\nu}{2}-1,\frac{\nu}{2};1-\frac{1}{\rho}\right)=\rho^{2-\frac{\nu}{2}}F\left(1,2-\frac{\nu}{2},\frac{\nu}{2};1-\rho\right), \tag{11.10.1.1}$$

$$F\left(2-\frac{\nu}{2},\frac{\nu}{2},\frac{\nu}{2}+1;1-\frac{1}{\rho}\right)=\rho^{2-\frac{\nu}{2}}F\left(1,2-\frac{\nu}{2},\frac{\nu}{2}+1;1-\rho\right). \tag{11.10.1.2}$$

We then write (11.10.1.2) in the form

$$\Sigma(p,\nu)=\frac{e^2}{(4\pi)^{\frac{\nu}{2}}}m^{\nu-4}\Gamma\left(2-\frac{\nu}{2}\right)\left\{[(i\gamma\cdot p+m)(2-\nu)-2m]\frac{2}{\nu-2}\cdot\right.$$
$$F\left(1,2-\frac{\nu}{2},\frac{\nu}{2};1-\rho\right)+[m-(i\gamma\cdot p+m)]\left(\frac{4}{\nu}-2\right)\cdot$$
$$\left.F\left(1,2-\frac{\nu}{2},\frac{\nu}{2}+1;1-\rho\right)\right\}. \tag{11.10.1.3}$$

Using the following formulas, we can isolate the gamma function at the pole:

$$\Gamma\left(2-\frac{\nu}{2}\right)F\left(1,2-\frac{\nu}{2},\frac{\nu}{2};1-\rho\right)=$$
$$\Gamma\left(2-\frac{\nu}{2}\right)+\frac{2(1-\rho)}{\nu}\Gamma\left(3-\frac{\nu}{2}\right)F\left(1,3-\frac{\nu}{2},\frac{\nu}{2}+1;1-\rho\right), \tag{11.10.1.4}$$

$$\Gamma\left(2-\frac{\nu}{2}\right)F\left(1,2-\frac{\nu}{2},\frac{\nu}{2}+1;1-\rho\right)=$$
$$\Gamma\left(2-\frac{\nu}{2}\right)+\frac{2(1-\rho)}{\nu+2}\Gamma\left(3-\frac{\nu}{2}\right)F\left(1,3-\frac{\nu}{2},\frac{\nu}{2}+2;1-\rho\right). \tag{11.10.1.5}$$

Replacing these results in (11.10.1.3) we obtain

$$\Sigma(p,\nu)=\frac{e^2}{(4\pi)^{\frac{\nu}{2}}}m^{\nu-4}\left\{\left\{[(i\gamma\cdot p+m)(2-\nu)-2m]\frac{2}{\nu-2}+\right.\right.$$
$$\left.[m-(i\gamma\cdot p+m)]\frac{4-2\nu}{\nu}\right\}\Gamma\left(2-\frac{\nu}{2}\right)+$$

$$\left\{[(i\gamma \cdot p + m)(2-\nu) - 2m]\frac{4(1-\rho)}{\nu(\nu-2)}\cdot\right.$$

$$F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\rho\right) + [m - (i\gamma \cdot p + m)]\frac{(8-4\nu)(1-\rho)}{\nu(\nu+2)}\cdot$$

$$\left.F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+2; 1-\rho\right)\right\}\Gamma\left(3-\frac{\nu}{2}\right)\Big\}, \qquad (11.10.1.6)$$

or, equivalently,

$$\Sigma(p,\nu) = \frac{e^2}{(4\pi)^{\frac{\nu}{2}}} m^{\nu-4}\left\{\left\{\left(\frac{4}{\nu-2} - \frac{2\nu-4}{\nu}\right)m - \right.\right.$$

$$\left.\frac{4}{\nu}(i\gamma \cdot p + m)\right\}\Gamma\left(2-\frac{\nu}{2}\right) +$$

$$\left\{[(i\gamma \cdot p + m)(2-\nu) - 2m]\frac{4(1-\rho)}{\nu(\nu-2)}\cdot\right.$$

$$F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\rho\right) + [m - (i\gamma \cdot p + m)]\frac{(8-4\nu)(1-\rho)}{\nu(\nu+2)}\cdot$$

$$\left.F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+2; 1-\rho\right)\right\}\Gamma\left(3-\frac{\nu}{2}\right)\Big\}. \qquad (11.10.1.7)$$

Note that $1-\rho = -p^2/m^2$. We can decompose the self-energy in the form

$$\Sigma(p,\nu) = A + B(i\gamma \cdot p + m) + \Sigma_f(p,\nu), \qquad (11.10.1.8)$$

where A, B, and $\Sigma_f(p,\nu)$ are given by

$$A = \frac{e^2}{(4\pi)^{\frac{\nu}{2}}} m^{\nu-3}\left(\frac{4}{\nu-2} - \frac{2\nu-4}{\nu}\right)\Gamma\left(2-\frac{\nu}{2}\right), \qquad (11.10.1.9)$$

$$B = \frac{e^2}{(4\pi)^{\frac{\nu}{2}}} m^{\nu-4}\Gamma\left(2-\frac{\nu}{2}\right) \qquad (11.10.1.10)$$

$$\Sigma_f(p,\nu) = \frac{e^2}{(4\pi)^{\frac{\nu}{2}}} m^{\nu-4}\left\{[(i\gamma \cdot p + m)(2-\nu) - 2m]\frac{4(1-\rho)}{\nu(\nu-2)}\cdot\right.$$

$$F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\rho\right) + [m - (i\gamma \cdot p + m)]\frac{(8-4\nu)(1-\rho)}{\nu(\nu+2)}\cdot$$

$$\left.F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+2; 1-\rho\right)\right\}\Gamma\left(3-\frac{\nu}{2}\right). \qquad (11.10.1.11)$$

This decomposition is consistent, since both the self-energy and its *finite part* depend linearly on $i\gamma \cdot p + m$. Note that $\Sigma_f(p.\nu)$ can be re-written as

$$\Sigma_f(p,\nu) = \Sigma_{f1}(p,\nu) + (i\gamma \cdot p + m)\Sigma_{f2}(p,\nu), \tag{11.10.1.12}$$

with

$$\begin{aligned}\Sigma_{f1}(p,\nu) = -\frac{e^2}{(4\pi)^{\frac{\nu}{2}}} m^{\nu-3} \left\{ \frac{8(1-\rho)}{\nu(\nu-2)} \cdot \right.\\ F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\rho\right) - \frac{(8-4\nu)(1-\rho)}{\nu(\nu+2)} \cdot \\ \left. F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+2; 1-\rho\right)\right\} \Gamma\left(3-\frac{\nu}{2}\right),\end{aligned} \tag{11.10.1.13}$$

$$\begin{aligned}\Sigma_{f2}(p,\nu) = \frac{e^2}{(4\pi)^{\frac{\nu}{2}}} m^{\nu-4} \left\{ (2-\nu)\frac{4(1-\rho)}{\nu(\nu-2)} \cdot \right.\\ F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\rho\right) - \frac{(8-4\nu)(1-\rho)}{\nu(\nu+2)} \cdot \\ \left. F\left(1, 3-\frac{\nu}{2}, \frac{\nu}{2}+2; 1-\rho\right)\right\} \Gamma\left(3-\frac{\nu}{2}\right).\end{aligned} \tag{11.10.1.14}$$

Note that $\Sigma_{f1}(p,\nu)$ and $\Sigma_{f2}(p,\nu)$ are independent of $i\gamma \cdot p + m$. Although makes sense, to have the true convolution we must perform the Laurent expansion To that end we define

$$f(\nu) = \frac{m^{\nu-4}}{(4\pi)^{\frac{\nu}{2}}} \left[\left(\frac{4}{\nu-2} - \frac{2\nu-4}{\nu} \right) m - \frac{4}{\nu}(i\gamma \cdot p + m) \right]. \tag{11.10.1.15}$$

We then use the following expression for the gamma function

$$\Gamma\left(2-\frac{\nu}{2}\right) = -\frac{2}{\nu-4} - C + \sum_{k=1}^{\infty} c_k(\nu-4)^k, \tag{11.10.1.16}$$

and then obtain the corresponding Laurent expansion

$$\begin{aligned}f(\nu)\Gamma\left(2-\frac{\nu}{2}\right) = \frac{i\gamma \cdot p}{(4\pi)^2(\nu-4)} + \\ \frac{1}{(4\pi)^2} \left\{ i\gamma \cdot p \left[C + 2\ln(m) - \ln(4\pi) - \frac{1}{2} \right] + 2m + \sum_{k=1}^{\infty} b_k(\nu-4)^k \right\}.\end{aligned} \tag{11.10.1.17}$$

Using the previous result, we have the following expression for the self energy

$$\Sigma(\nu, p) = \frac{e^2}{(4\pi)^2} \left\{ \frac{i\gamma \cdot p}{\nu - 4} + \right.$$

$$\left. i\gamma \cdot p \left[C + 2\ln(m) - \ln(4\pi) - \frac{1}{2} \right] + 2m + \sum_{k=1}^{\infty} b_k (\nu - 4)^k \right\} +$$

$$\frac{e^2}{(4\pi)^{\frac{\nu}{2}}} \left\{ [(i\gamma \cdot p + m)(2 - \nu) - 2m] \frac{4(1-\rho)}{\nu(\nu - 2)} \cdot \right.$$

$$F\left(1, 3 - \frac{\nu}{2}, \frac{\nu}{2} + 1; 1 - \rho\right) + [m - (i\gamma \cdot p + m)] \frac{(8 - 4\nu)(1-\rho)}{\nu(\nu + 2)} \cdot$$

$$\left. F\left(1, 3 - \frac{\nu}{2}, \frac{\nu}{2} + 2; 1 - \rho\right) \right\} \Gamma\left(3 - \frac{\nu}{2}\right) \Bigg\}. \qquad (11.10.1.18)$$

Completing the Laurent expansion, we have

$$\Sigma(\nu, p) = \frac{e^2}{(4\pi)^2} \left\{ \frac{i\gamma \cdot p}{\nu - 4} + \right.$$

$$i\gamma \cdot p \left[C + 2\ln(m) - \ln(4\pi) - \frac{1}{2} \right] + 2m + \sum_{k=1}^{\infty} b_k (\nu - 4)^k +$$

$$\frac{e^2}{(4\pi)^{\frac{\nu}{2}}} i\gamma \cdot p (1 - \rho) \left[\frac{1}{3} F(1, 1, 4; 1 - \rho) + \right.$$

$$\left. F(1, 1, 3; 1 - \rho) \right] + 2m(1-\rho) F(1, 1, 3; 1 - \rho) +$$

$$\left. \sum_{k=1}^{\infty} c_k (\nu - 4)^k \right\}, \qquad (11.10.1.19)$$

or, equivalently,

$$\Sigma(\nu, p) = \frac{e^2}{(4\pi)^2} \left\{ \frac{i\gamma \cdot p}{\nu - 4} + \right.$$

$$i\gamma \cdot p \left[C + 2\ln(m) - \ln(4\pi) - \frac{1}{2} \right] + 2m +$$

$$\frac{e^2}{(4\pi)^{\frac{\nu}{2}}} i\gamma \cdot p (1 - \rho) \left[\frac{1}{3} F(1, 1, 4; 1 - \rho) + \right.$$

$$\left. F(1, 1, 3; 1 - \rho) \right] + 2m(1-\rho) F(1, 1, 3; 1 - \rho) +$$

$$\sum_{k=1}^{\infty} a_k(\nu-4)^k\Bigg\}. \qquad (11.10.1.20)$$

This last result can be re-written as

$$\Sigma(\nu,p) = \frac{e^2}{(4\pi)^2}\left\{\frac{i\gamma\cdot p+m}{\nu-4} - \frac{m}{\nu-4} + \right.$$

$$i\gamma\cdot p\left[C+2\ln(m)-\ln(4\pi)-\frac{1}{2}\right]+2m+$$

$$\frac{e^2}{(4\pi)^{\frac{\nu}{2}}}i\gamma\cdot p(1-\rho)\left[\frac{1}{3}F\left(1,1,4;1-\rho\right)+\right.$$

$$\left.F\left(1,1,3;1-\rho\right)\right]+2m(1-\rho)F\left(1,1,3;1-\rho\right)+$$

$$\left.\sum_{k=1}^{\infty} a_k(\nu-4)^k\right\}. \qquad (11.10.1.21)$$

We can effect, then, the following decomposition

$$\Sigma(p,\nu) = A + (i\gamma\cdot p+m)B + \Sigma_f(p,\nu), \qquad (11.10.1.22)$$

where the constants A and B are given by

$$A = -\frac{e^2 m}{(4\pi)^2(\nu-4)}, \qquad (11.10.1.23)$$

$$B = \frac{e^2}{(4\pi)^2(\nu-4)}, \qquad (11.10.1.24)$$

and the finite part of the self energy is

$$\Sigma_f(\nu,p) = \frac{e^2}{(4\pi)^2}\left\{\, i\gamma\cdot p\left[C+2\ln(m)-\ln(4\pi)-\frac{1}{2}\right]+2m+\right.$$

$$\frac{e^2}{(4\pi)^{\frac{\nu}{2}}}i\gamma\cdot p(1-\rho)\left[\frac{1}{3}F\left(1,1,4;1-\rho\right)+\right.$$

$$\left.\frac{1}{2}F\left(1,1,3;1-\rho\right)\right]+2m(1-\rho)F\left(1,1,3;1-\rho\right)+$$

$$\left.\sum_{k=1}^{\infty} a_k(\nu-4)^k\right\}. \qquad (11.10.1.25)$$

We have then the four-dimensional result

$$\Sigma_P(p) = \Sigma_f(p,4) =$$

$$\frac{e^2}{(4\pi)^2}\left\{i\gamma\cdot p\left[C + 2\ln(m) - \ln(4\pi) - \frac{1}{2}\right] + 2m + \right.$$

$$i\gamma\cdot p(1-\rho)[\frac{1}{3}F(1,1,4.1-\rho) + \frac{1}{2}F(1,1,3,1-\rho)] -$$

$$\left. 2m(1-\rho)F(1,1,3,1-\rho)\right\}, \tag{11.10.1.26}$$

which is the exact result of the convolution (see [2]). $\Sigma_P(p,\nu)$ is then the true physical self energy.

11.10.2 Vacuum polarization evaluation

The vacuum polarization can be written in the form

$$\Pi_{\mu\nu}(k,\nu) = \frac{e^2}{(4\pi)^{\frac{\nu}{2}}}\frac{d(\nu)}{3}\Gamma\left(2-\frac{\nu}{2}\right)m^{\nu-4}(k_\mu k_\nu - \eta_{\mu\nu}k^2) + \Pi_{\mu\nu F}(k,\nu) \tag{11.10.2.27}$$

where $\Pi_{\mu\nu F}(k,\nu)$ is given by (11.9.2.12). To perform the Laurent expansion we define

$$f(\nu) = \frac{1}{(4\pi)^{\frac{\nu}{2}}}\frac{d(\nu)}{3}m^{\nu-4}(k_\mu k_\nu - \eta_{\mu\nu}k^2), \tag{11.10.2.28}$$

Thus, we obtain

$$f(\nu)\Gamma\left(2-\frac{\nu}{2}\right) = -\frac{1}{(4\pi)^2}\left\{\frac{d(\nu)}{3}\frac{k_\mu k_\nu - \eta_{\mu\nu}k^2}{4-\nu} + (k_\mu k_\nu - \eta_{\mu\nu}k^2)\cdot\right.$$

$$\left.\left\{\frac{d(4)}{3}[2\ln(m) - \ln(4\pi) - C] + \frac{d'(4)}{3}\right\} + \sum_{k=1}^{\infty} b_k(\nu-4)^k\right\}. \tag{11.10.2.29}$$

Using this result, we obtain for the vacuum polarization

$$\Pi_{\mu\nu}(k,\nu) = -\frac{e^2}{(4\pi)^2}\frac{d(\nu)}{3}\frac{k_\mu k_\nu - \eta_{\mu\nu}k^2}{4-\nu} - \frac{e^2}{(4\pi)^2}\left\{(k_\mu k_\nu - \eta_{\mu\nu}k^2)\cdot\right.$$

$$\left\{\frac{d(4)}{3}[2\ln(m) - \ln(4\pi) - C] + \frac{d'(4)}{3}\right\} +$$

$$\sum_{k=1}^{\infty} b_k(\nu-4)^k \Bigg\} + \Pi_{\mu\nu F}(k,\nu) \tag{11.10.2.30}$$

The finite part is now

$$\Pi_{\mu\nu f}(k,\nu) = -\frac{e^2}{(4\pi)^2}\left\{(k_\mu k_\nu - \eta_{\mu\nu}k^2)\cdot\right.$$

$$\left\{\frac{d(4)}{3}(2\ln(m) - \ln(4\pi) - C] + \frac{d'(4)}{3}\right\}\Bigg\} + \Pi_{\mu\nu F}(k,4) + \sum_{k=1}^{\infty} a_k(\nu-4)^k \tag{11.10.2.31}$$

Consequently, we have in four-dimensions the convolution result

$$\Pi_{\mu\nu P}(k) = \Pi_{\mu\nu f}(k,4) = -\frac{e^2}{(4\pi)^2}\left\{(k_\mu k_\nu - \eta_{\mu\nu}k^2)\cdot\right.$$

$$\left\{\frac{d(4)}{3}[2\ln(m) - \ln(4\pi) - C] + \frac{d'(4)}{3}\right\}\Bigg\} + \Pi_{\mu\nu F}(k,4). \tag{11.10.2.32}$$

11.11 Discussion

In quantum field theory (QFT), when we use perturbative expansions, we are dealing with products of distributions in configuration space or, what is the same, with convolutions of distributions in momentum space. Four earlier papers [33, 39, 41, 42] have demonstrated the existence of the convolution of Sebastiao e Silva ultradistributions. This convolution allows us to treat non- renormalizable QFTs, but has the disadvantage of being extremely involved.

Following a procedure similar to that of the previously mentioned papers, we defined here *the convolution of Lorentz invariant tempered distributions*, using the dimensional regularization (DR) of Bollini and Giambiagi. [2]. Appealing to this convolution we have obtained, for example, the convolution of n massless Feynman propagators both in Minkowskian and Euclidean spaces and the convolution of two massless Wheeler propagators, all of them original results.

We conclude these considerations by asserting that convolutions pave the way to the treatment of non-renormalizable quantum field theories, a significant advance indeed.

Chapter 12

Non-relativistic QFT

12.1 Introduction

We will work here with non-relativistic quantum field theory propagators (NRP). They are valuable tools in nuclear physics and in the theory of condensed matter [101]. We will apply NRPs here to gravitation, in the so-called Verlinde's emergent scenario [127].

Imagine a cube whose sides are labeled the three particular quantities: 1) Newton's gravitation constant G, 2) light-velocity's c, and 3) Planck's $\hbar$. Here we will situate ourselves at the corner with $c^{-1} = 0$, with the other two quantities being finite [127]. Since since we will appeal in this chapter to potentials entering Schrödinger Equations (SE), the associated treatment will necessarily be non-relativistic, as such is the character of SE, of course.

12.1.1 Emergent entropy

In 2011, Verlinde conceived the audacious notion of linking gravity to an emerging entropic force [103]. Such conjecture was actually proved valid (in a classical context) afterwards, in [104].

According to [103], gravity is imagined to emerge as a result of information concerning the position of material bodies [104]. This great idea conjoins a "thermal" treatment of gravitation with 't Hooft's celebrated holographic principle, which leads to view gravitation as an emergent phenomenon, the key Verlinde's notion, that generated a lot of attention. For example, see [105, 106, 108]. For an excellent review of the statistical mechanics of gravity consult Padmanabhan [107]. Verlinde's initiative motivated works on cosmology, the dark energy hypothesis, cosmological acceleration, cosmological inflation, and loop quantum gravity [104]. The related literature is quite large indeed [106].

The notion of emerging entropic gravity (EEG) is a nice idea that we will here exploit so as to discuss gravity in a quantum environment, which has been the dream of many physicist during decades. In a first manuscript [104], we demonstrated that Verlinde's emerging gravity is certainly an entropic force at the classical level, In two posterior papers [109, 110], we repeated the feat at a quantum level, for both bosons and fermions. Additionally, in two additional efforts [111, 112], we effected a first quantization of EEG for bosons and fermions by solving the associated Schrödinger equations. Now, we will face a non-relativistic (NR) Quantum Field Theory (QFT) associated to the EEG for both type of particles, by using

- the results previously obtained in the five papers just cited, plus

- the formulation of the NR QFT described in the classical text book of Fetter and Walecka's.

We have certainly taken into account the fact that the NR QFT of the EEG can be non-renormalizable both for bosons and for fermions (for the latter above the Fermi level). Such strong inconvenience can be overcome, however, by recourse to

1. to techniques explained in precedent Chapters of this book and in either [29, 32, 39, 41], in which a complete treatment on the quantization of non-renormalizable QFT using ultrahyperfunctions is made, or

2. to the approach of [113], also discussed earlier in this book, in which one generalizes the Dimensional Regularization (DR) of Bollini and Giambiagi (BG) [114], showing that this generalization is quite apt to quantize non-renormalizable QFTs.

12.2 A quantum entropic force

12.2.1 Fermionic entropic Force

In this chapter, gravity is regarded as an emergent phenomenon. Verlinde states that it derives from the quantum entanglement between small bits of space-time information [116]. This Verlinde-Gravitation differs at short distances from Newton's one. The associated (emergent) gravitation-potential, after introduction into the Schrödinger equation (SE), will of course yield quantized states with definite energies. This is our (not Verlinde's) central observation. These associated energies are to be regarded as new, not yet reported energy-sources. To repeat, we speak of sources that have not been taken into account till now, with the exception of our previous treatments of Refs. [111, 112]. Thus, the association Verlinde-Schroedinger provides the Universe with a new source of energy!
We now proceed, as we did in [110], to make use of a statistical treatment of fermion gases. In [110] we encountered a fermion-fermion gravitational force from such a treatment, specifically, a baryon-baryon one, that turned out to be proportional to $1/r^2$ for r larger than one micron. For smaller r's, however, more involved, new contributions arose. Accordingly, the pertinent potential $V_F(r)$ differs from Newton's at such short distances. Looking at [110] we see that the associated entropic force F_{eF} reads

$$F_{eF} = \frac{4\pi\lambda k_B\left(\pi e m E\right)^{\frac{3}{2}}}{(3N)^{\frac{3}{2}}\,h^3} r \times$$
$$\left\{\ln\left[32\pi r^3\left(\pi e m E\right)^{\frac{3}{2}} - (3N)^{\frac{5}{2}} h^3\right] - \ln\left[32\pi r^3\left(\pi e m E\right)^{\frac{3}{2}}\right]\right\}. \tag{12.2.1.1}$$

Such is our emergent Verlinde's gravity force between a couple of fermions, derived in reference [110], that we will employ in what follows.

12.2.2 Boson entropic force

An identical procedure to that described above for fermions was also made for bosons [109], yielding a boson-boson gravitational force, that again resulted proportional to $1/r^2$ for distances larger than one micron, while for smaller distances, novel and more involved contributions emerged [109]. Thus, the pertinent potential $V_B(r)$will differ from the Newtonian one at short distances. The boson-boson entropic force of [109] reads

$$F_{eB} = \frac{4\pi\lambda k_B\left(\pi emE\right)^{\frac{3}{2}}}{(3N)^{\frac{3}{2}}\, h^3} r\times$$
$$\left\{\ln\left[32\pi r^3\left(\pi emE\right)^{\frac{3}{2}} + (3N)^{\frac{5}{2}} h^3\right] - \ln\left[32\pi r^3\left(\pi emE\right)^{\frac{3}{2}}\right]\right\}, \tag{12.2.2.2}$$

the emergent Verlinde's gravity force between a couple of bosons [109].

12.3 Quantum gravitational potential

12.3.1 Gravitational potential for N fermions

We enumerate first some important quantities (constants) introduced in [110]:

1. a and b of the form
2. $a = (3N)^{\frac{5}{2}} h^3$ and
3. $b = 32\pi(\pi emK)^{\frac{3}{2}}$, with
4. K the total energy of the N fermions (examples will be given below),
5. $r_2 = (a/b)^{1/3}$.
6. $A = Gm^2/r_2$,

where G is the gravitational constant and k_B Boltzmann's constant. $\frac{\lambda 3 N k_B}{8\pi} = \frac{2}{3}Gm^2$, so that the ensuing potential energy $V_F(r)$ reads

$$V_F(r) = -Gm^2\frac{2b}{3a}\left\{\frac{r^2}{2}\ln\left(1-\frac{a}{br^3}\right)\Theta\left[r-\left(\frac{a}{b}\right)^{\frac{1}{3}}\right] - \right.$$
$$\frac{a^{\frac{2}{3}}}{2b^{\frac{2}{3}}}\left\{\frac{1}{2}\ln\left[\frac{\left[r-\left(\frac{a}{b}\right)^{\frac{1}{3}}\right]^2}{r^2+\left(\frac{a}{b}\right)^{\frac{1}{3}} r+\left(\frac{a}{b}\right)^{\frac{2}{3}}}\right] + \right.$$
$$\left.\left.\sqrt{3}\left[\arctan\left[\frac{2r+\left(\frac{a}{b}\right)^{\frac{1}{3}}}{\sqrt{3}\left(\frac{a}{b}\right)^{\frac{1}{3}}}\right] - \frac{\pi}{2}\right]\right\}\right\}, \tag{12.3.1.1}$$

an important result for us here (note that $\Theta(x)$ is the Heaviside step function).

12.3.2 Boson gravitational potential

The boson-boson gravity's potential $V_B(r)$, with masses $m_1 = m$ and $m_2 = M$, for an $N-$boson gas, was given in [109] employing the microcanonical treatment of Lemons [117]. There, we dealt with identical bosons so that $m = M$. In [109], the entropy S for N bosons of total energy K was obtained. From S we can find an entropic force F_e, associated, à la Verlinde, to emerging gravity. The associated boson-boson potential $V(r)$ [109] is to be discussed below.
For deriving $V(r)$ in [109], one needs two constants, a and b, for N bosons of total energy K, with k_B Boltzmann's constant, e Euler's number, and h Planck's constant. Then,

$$a = (3N)^{\frac{5}{2}} h^3; \qquad b = 32\pi(\pi e m K)^{\frac{3}{2}}. \tag{12.3.2.2}$$

The relation that defines the proportionality constant λ between F_e and the entropic gradient [109] becomes now

$$\lambda = 8\pi G m^2 / 3 N k_B. \tag{12.3.2.3}$$

It is then shown in [109] that $V(r)$ acquires the form.

$$V_B(r) = G m^2 \frac{b}{a} \left\{ \frac{r^2}{2} \ln\left(1 + \frac{a}{b r^3}\right) - \right.$$

$$\frac{a^{\frac{2}{3}}}{2 b^{\frac{2}{3}}} \left\{ \frac{1}{2} \ln \left[\frac{\left[r + \left(\frac{a}{b}\right)^{\frac{1}{3}}\right]^2}{r^2 - \left(\frac{a}{b}\right)^{\frac{1}{3}} r + \left(\frac{a}{b}\right)^{\frac{2}{3}}} \right] + \right.$$

$$\left.\left. \sqrt{3} \left[\frac{\pi}{2} - \arctan \left[\frac{2r - \left(\frac{a}{b}\right)^{\frac{1}{3}}}{\sqrt{3} \left(\frac{a}{b}\right)^{\frac{1}{3}}} \right] \right] \right\} \right\}. \tag{12.3.2.4}$$

12.3.3 Estimates for r_2 and A

Let us give now some numerical estimates for the values of r_2 and A.

Fermions

For them we can estimate that, in the Universe, one has [112] $N = 6.25 \times 10^{79}$, $K = mc^2 10^{53}Kg$, $m = 1.63 \times 10^{-27}Kg$, h being the Planck's constant, and G the gravitation constant. So they turn out to be: $r_2 = 5.8 \times 10^{35}m$ and $A = 8.2 \times 10^{-124}Nm$ This shows that the discontinuity of the potential corresponding to two baryons is absolutely negligible.

Bosons

Focus attention on axions, the putative dark particle elementary particles. For them we estimate that, in the Universe [111] the numbers are: $N = 6 \times 10^{79}$, $K = mc^2 5.47 \times 10^{53}Kg$, and $m = 2.67 \times 10^{-39}Kg$ Thus, $r_2 = 7.4 \times 10^{24}m$ and $A = 2.3 \times 10^{-91}Nm$

12.4 Taylor expansion for $V(r)$

12.4.1 Fermions

It is impossible to analytically deal with $V_F(r)$. Instead, we will appeal to the rigorous approximation for $V_F(r)$ concocted in Ref. [112]. One subdivides the r axis into four different zones: $0 < r < r_0$, $r_0 < r < r_1$, $r_1 < r < r_2$, and $r > r_2$. We set r_0 as 10^{-10} meters (a typical Hydrogen-atom's length), $r_1$as 25 microns, and $r_2 = (a/b)^{\frac{1}{3}}$. Remark that there is empiric evidence for selecting $r_1 = 25$ micrometers [118]. Thus,

$$V_F(r) \approx V_{F1}(r) + V_{F2}(r) + V_{F3}(r). + V_{F4}(r). \tag{12.4.1.1}$$

We proved in [112] that the present approximation is quite good.

For convenience we define

$$V_{F0} = -Gm^2 \left(\frac{b}{a}\right)^{\frac{1}{3}} \frac{7\pi}{6\sqrt{3}} \tag{12.4.1.2}$$

and call V_{F1} the Taylor polynomial (TA), at zeroth order, for very small r.

$$V_{F1}(r) = -Gm^2 \left(\frac{b}{a}\right)^{\frac{1}{3}} \frac{7\pi}{6\sqrt{3}}\Theta(r_0 - r) = V_0\Theta(r_0 - r). \tag{12.4.1.3}$$

For large r one has

$$V_{F3}(r) = -\frac{Gm^2}{r}[\Theta(r - r_1) - \Theta(r - r_2)]. \tag{12.4.1.4}$$

For intermediate r—values, $r_0 < r < r_1$, we selected the form $W(r) = 0$ in [110] for interpolating between the fixed points $r_1 - r_0$. Thus, as explained in [110],

$$V_{F2}(r) = 0. \tag{12.4.1.5}$$

Finally, for $V_4(r)$, in [110] we chose

$$V_{F4(r)} = -\frac{2Gm^2}{3r}\Theta(r - r_2). \tag{12.4.1.6}$$

12.4.2 Bosons

Schrödinger's boson equation is, again, not amenable of analytic treatment. In the exploratory analysis of [109], a Taylor expansion was used that yielded a suitable, rigorous approximation to $V_B(r)$. One had in [111]

$$V_B(r) \approx V_{B1}(r) + V_{B2}(r) + V_{B3}(r), \tag{12.4.2.7}$$

It was proven in [111] that this approximate potential is very good V_{B1} is the first order Taylor approach for r small enough, that one uses for $0 < r < r_0$, with $r_0 = 10^{-10}$m.

$$V_{B1}(r) = -\frac{\pi G m^2}{\sqrt{3}}\left(\frac{b}{a}\right)^{\frac{1}{3}}\Theta(r_0 - r) = V_{B0}\Theta(r_0 - r), \tag{12.4.2.8}$$

where $r_1 = 25.0$ micron [118], the minimum known distance at which Newton's force still works. For large distances one has [111]

$$V_{B3}(r) = -\frac{Gm^2}{r}\Theta(r - r_1). \tag{12.4.2.9}$$

For the intermediate range $r_0 < r < r_1$ we call, first of all, $W(r \propto)r^2$, i.e., we make an harmonic interpolating-form between r_1 and r_0. Then (see details in [111]),

$$V_{B2}(r) = W(r) = 0 \tag{12.4.2.10}$$

12.5 FW's book's materials

12.5.1 Preliminaries

We compute next self-energies. In QFT, the energy that a given particle gains as the result of modifications that it itself generates in its environment is called the self-energy Σ, that represents the contribution to the particle's energy, or effective mass, due to interactions between this particle and its surroundings. In a condensed matter scenario corresponding to electrons moving in a certain medium, Σ stands for the potential felt by the electron due to the surrounding's interactions with itself. Since electrons repel each other, the moving electron polarizes those electrons in its neighborhood and then changes the potential of the moving electron fields.

12.5.2 Dressed fermion propagators

We will calculate here dressed propagators. For an accessible discussion of the concept see, for instance, [101]. In Fetter and Walecka's book [115], the authors comprehensively derived a fermion's non-relativistic quantum field theory. For free fermions, they introduce the following (current) propagator:

$$iG^0_{\alpha\beta}(x,t;x',t') =< 0|T[\psi_\alpha(x,t)\psi^+_\beta(x',t')]|0>, \tag{12.5.2.1}$$

i.e.,

$$iG^0_{\alpha\beta}(x,t;x',t') =$$

$$\frac{\delta_{\alpha\beta}}{(2\pi)^3}\int e^{ik\cdot(x-x')}e^{-\omega_k(t-t')}\times$$

$$\left[\Theta(t-t')\Theta(k-k_F)-\Theta(t'-t)\Theta(k_F-k)\right]d^3k, \tag{12.5.2.2}$$

with Θ the Heaviside's step function. We appeal here to the well known relation

$$\Theta(t-t') = -\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{e^{-i\omega(t-t')}}{\omega+i0}d\omega, \tag{12.5.2.3}$$

and find

$$iG^0_{\alpha\beta}(x,t;x',t') =$$

$$\frac{\delta_{\alpha\beta}}{(2\pi)^3}\int\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-\omega_k(t-t')}$$
$$\left[\frac{\Theta(k-k_F)}{\omega-\omega_k+i0}-\frac{\Theta(k_F-k)}{\omega-\omega_k-i0}\right]d^3k d\omega. \quad (12.5.2.4)$$

Thus, the associated expression in momentum space becomes

$$\hat{G}^0_{F\alpha\beta}(\mathbf{k},\omega)=\delta_{\alpha\beta}\left[\frac{\Theta(k-k_F)}{\omega-\omega_k+i0}+\frac{\Theta(k_F-k)}{\omega-\omega_k-i0}\right], \quad (12.5.2.5)$$

where

$$\frac{1}{\omega-\omega_k\pm i0}=PV\frac{1}{\omega-\omega_k}\mp i\pi\delta(\omega-\omega_k), \quad (12.5.2.6)$$

with $k=|\mathbf{k}|$ and $\omega_k=\sqrt{k^2/2m}$ (PV stands for "principal value of the given function"). The system's interaction's Hamiltonian is defined by a two bodies V_F potential

$$V_F(\mathbf{x}_1-\mathbf{x}_2)=V_F(|\mathbf{x}_1-\mathbf{x}_2|)\mathbf{1}(1)\mathbf{1}(2), \quad (12.5.2.7)$$

with $\mathbf{1}$ the unity matrix. The dressed propagator of the theory verifies

$$\hat{G}_{F\alpha\beta}=\delta_{\alpha\beta}\hat{G}_F, \quad (12.5.2.8)$$

i.e., the dressed propagator is diagonal. Then we get ($\hat{G}^0_F(\mathbf{k},\omega)\equiv\hat{G}^0_F(k)$)

$$\hat{G}_F(k)=\hat{G}^0_F(k)+\hat{G}^0_F(k)\Sigma_F(k)\hat{G}^0_F(k), \quad (12.5.2.9)$$

where $\Sigma_F(k)$ is the self-energy of the system (see Preliminaries above). Thus, one obtains its perturbative expansion, that at first order is

$$\Sigma^{(1)}_F(k)\equiv\Sigma^{(1)}(\mathbf{k})=\frac{n}{\hbar}\hat{V}(0)-\frac{1}{(2\pi)^3\hbar}\int\hat{V}_F(\mathbf{k}-\mathbf{k}')\Theta(k_F-k')d^3k', \quad (12.5.2.10)$$

where $n=N/V$ and

$$\hat{V}_F(\mathbf{k})=\int V_F(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x. \quad (12.5.2.11)$$

Accordingly, up to first order,

$$\hat{G}^{(1)}_F(k)=\hat{G}^0_F(k)+\hat{G}^0_F(k)\Sigma^{(1)}_F(k)\hat{G}^0_F(k). \quad (12.5.2.12)$$

12.5.3 Bosonic dressed propagators

Fetter and Walecka define for free bosons the following propagator in momentum space [115]:

$$iG^0(x,t;x',t') =< 0|T[\phi(x,t)\phi^+(x',t')]|0> . \qquad (12.5.3.13)$$

It reads

$$\hat{G}^0_B(k) = \frac{1}{k_0 - \omega_k + i0}, \qquad (12.5.3.14)$$

where $\omega_k = \sqrt{k^2/2m}$. Then,

$$\hat{G}_B(k) = -(2\pi)^4 n_0 i\delta(k) + \hat{G}'_B(k), \qquad (12.5.3.15)$$

were the primed part refers to the non-condensate ($n_0 = N_0/V$)

$$\hat{G}'^{(1)}_B(k) = \frac{n_0}{h}\hat{G}^0_B(k)[\hat{V}_B(0) + \hat{V}_B(\mathbf{k})]\hat{G}^0_B(k), \qquad (12.5.3.16)$$

and

$$\hat{V}_B(\mathbf{k}) = \int V_B(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x. \qquad (12.5.3.17)$$

12.6 Non-relativistic QFT of emergent gravity

have seen above how quantum potentials enter the Schrödinger equation, a non-relativistic relation.

12.6.1 Fermions

Above, our goal in getting the NR QFT of the EEG was to compute $\Sigma^{(1)}$ for the potential of (12.3.1.1), with $\mathbf{1}$ the unity matrix,

$$V_F(r) = \{V_{F1}(r)\Theta(r_0 - r) -$$

$$\left.\frac{Gm^2}{r}[\Theta(r - r_1) - \Theta(r - r_2)] - \frac{2Gm^2}{3r}\Theta(r - r_2)\right\}\mathbf{1}. \qquad (12.6.1.1)$$

Thus, we must find the Fourier transform of $V_F(r)$. For V_{F1} one has

$$\hat{V}_{F1}(0) = V_{F0}\int d^3x = 4\pi V_{F0}\int_0^\infty r^2 dr = 0, \qquad (12.6.1.2)$$

an integral for which we have employed the results of [6] regarding the regularization of integrals dependent on a power of x.
We now calculate I_1 defined as

$$I_1 = \int_0^{2\pi}\int_0^{\pi}\int_0^{r_0} e^{-ik\cdot x} r^2 \sin(\theta) d\phi d\theta dr =$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_2} e^{-ikr\cos(\theta)} r^2 \sin(\theta) d\phi d\theta dr =$$

$$4\pi \left[\sin(kr_0) PV\frac{1}{k^3} - \cos(kr_0) PV\frac{1}{k}\right]. \tag{12.6.1.3}$$

It satisfies

$$PV\frac{1}{k^n}|_{k=0} = 0. \tag{12.6.1.4}$$

I_1 is the Fourier transform of the first term of (12.6.1.1). Analogously, we pass to the integral I_2

$$I_2 = \int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_2} r^{-1} e^{-ik\cdot x} r^2 \sin(\theta) d\phi d\theta dr =$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_2} e^{-ikr\cos(\theta)} r \sin(\theta) d\phi d\theta dr =$$

$$4\pi[\cos(kr_1) - \cos(kr_2)] PV\frac{1}{k^2}. \tag{12.6.1.5}$$

I_2 is the Fourier transform of the second term of (12.6.1.1).

$$I_3 = \int_0^{2\pi}\int_0^{\pi}\int_{r_2}^{\infty} r^{-1} e^{-ik\cdot x} r^2 \sin(\theta) d\phi d\theta dr =$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_2}^{\infty} e^{-ikr\cos(\theta)} r \sin(\theta) d\phi d\theta dr =$$

$$4\pi \cos(kr_2) PV\frac{1}{k^2}. \tag{12.6.1.6}$$

Similarly, I_3 is the Fourier transform of the third term of (12.6.1.1).. We then write

$$\hat{V}_F(\mathbf{k}) = 4\pi \left\{ V_{F0}\left[\sin(kr_0)PV\frac{1}{k^3} - \cos(kr_0)PV\frac{1}{k}\right] - \right.$$

$$\left. Gm^2[\cos(kr_1) - \frac{1}{3}\cos(kr_2)]PV\frac{1}{k^2} \right\}, \qquad (12.6.1.7)$$

and, as a consequence,

$$\hat{V}_F(0) = 0. \qquad (12.6.1.8)$$

Note that

$$\frac{1}{(2\pi)^3\hbar}\int \hat{V}_F(\mathbf{k}-\mathbf{k}')\Theta(k_F - k')d^3k' \simeq V_F(0) = -\frac{Gm^2}{r_2}\frac{7\pi}{4\sqrt{3}}, \qquad (12.6.1.9)$$

where we took $k_F \to \infty$ to simplify the computation. We thus find for the self-energy, up to first order

$$\Sigma_F^{(1)}(k) \simeq \frac{Gm^2}{r_2}\frac{7\pi}{4\sqrt{3}}\mathbf{1}. \qquad (12.6.1.10)$$

The unity matrix is included with the goal of emphasizing the matrix nature. Correspondingly, we also find, up to first order, for the dressed propagator, the relation

$$\hat{G}_F^{(1)}(k) \simeq \hat{G}_F^0(k) + \frac{Gm^2}{r_2}\frac{7\pi}{4\sqrt{3}}[\hat{G}_F^0(k)]^2. \qquad (12.6.1.11)$$

If $k_F \to \infty$ we have

$$\hat{G}^0(k) = \frac{1}{\omega - \omega_k - i0}. \qquad (12.6.1.12)$$

In [29] it was been proven that

$$PV\frac{1}{x^n}\delta^{(m)}(x) = \frac{(-1)^n}{2}\frac{m!}{(m+n)!}\delta^{(m+n)}(x). \qquad (12.6.1.13)$$

Employing then the result

$$PV\frac{1}{x^n}PV\frac{1}{x^m} = PV\frac{1}{x^{(n+m)}}, \qquad (12.6.1.14)$$

we find

$$\frac{1}{\omega - \omega_k - i0}\frac{1}{\omega - \omega_k - i0} = \frac{1}{(\omega - \omega_k - i0)^2}, \qquad (12.6.1.15)$$

so that

$$[\hat{G}^0(k)]^2 = \frac{1}{(\omega - \omega_k - i0)^2}. \qquad (12.6.1.16)$$

For $V \to \infty$, n finite, we get

$$\Sigma_F^{(1)}(k) \simeq 0, \qquad (12.6.1.17)$$

and

$$\hat{G}_F^{(1)}(k) \simeq \hat{G}_F^0(k). \qquad (12.6.1.18)$$

12.6.2 Bosons

Now we have

$$V_B(r) = V_{B1}(r)\Theta(r_0 - r) - \frac{Gm^2}{r}\Theta(r - r_1). \qquad (12.6.2.19)$$

We need to calculate the Fourier transform of that potential. We proceed in steps as done for fermions above. For V_{B1} we have

$$\hat{V}_{B1}(0) = V_{B0}\int d^3x = 4\pi V_{B0}\int_0^\infty r^2 dr = 0. \qquad (12.6.2.20)$$

This integral is found using the result of [6] regarding the regularization of integrals dependent on a power of x.
We now evaluate the integral I_1

$$I_1 = \int_0^{2\pi}\int_0^{\pi}\int_0^{r_0} e^{-ik\cdot x} r^2 \sin(\theta) d\phi d\theta dr =$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_2} e^{-ikr\cos(\theta)} r^2 \sin(\theta) d\phi d\theta dr =$$

$$4\pi\left[\sin(kr_0)PV\frac{1}{k^3} - \cos(kr_0)PV\frac{1}{k}\right], \qquad (12.6.2.21)$$

It verifies

$$PV\frac{1}{k^n}|_{k=0} = 0. \tag{12.6.2.22}$$

I_1 is the Fourier transform of the first term of (12.6.1.1). We pass now to compute the integral I_2 defined as

$$I_2 = \int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{\infty} r^{-1}e^{-ik\cdot x}r^2\sin(\theta)d\phi d\theta dr =$$

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{\infty} e^{-ikr\cos(\theta)}r\sin(\theta)d\phi d\theta dr =$$

$$4\pi\cos(kr_1)PV\frac{1}{k^2}. \tag{12.6.2.23}$$

I_2 is the Fourier transform of the second term of (12.6.1.1). For the full potential we thus have

$$\hat{V}_B(\mathbf{k}) = 4\pi\left\{V_{B0}\left[\sin(kr_0)PV\frac{1}{k^3} - \cos(kr_0)PV\frac{1}{k}\right] - \right.$$

$$\left. Gm^2\cos(kr_1)PV\frac{1}{k^2}\right\}. \tag{12.6.2.24}$$

As a consequence, we get again

$$\hat{V}_B(0) = 0. \tag{12.6.2.25}$$

For the dressed propagator we find, up to 1st order

$$\hat{G}_B^{\prime(1)}(k) = 4\pi\frac{n_0}{h}\left\{V_{B0}\left[\sin(kr_0)PV\frac{1}{k^3} - \cos(kr_0)PV\frac{1}{k}\right] - \right.$$

$$\left. Gm^2\cos(kr_1)PV\frac{1}{k^2}\right\}[\hat{G}^0(k)]^2 \tag{12.6.2.26}$$

Proceeding as above for the fermion propagator, we now face

$$[\hat{G}_B^0(k)]^2 = \frac{1}{(k_0 - \omega_k + i0)^2}. \tag{12.6.2.27}$$

12.7 Discussion

We have in this Chapter constructed the non-relativistic quantum field theory (NR QFT) of emergent entropic gravitation (EEG), for pairs of either fermions or bosons that interact amongst themselves via EEG. Our dealings were based on

- the results of the prior parts of this book [115],
- the Verlinde gravitational potentials found in [109, 110].

These potentials coincide from large distances down to atomic ones with Newton's one. They do not diverge at the origin.
Our present discussion generalizes the 1st. quantization methodology of Refs. [111, 112]. As examples, we have computed the dressed propagator of the system up to first order in perturbation theory, and also the self-energy for fermions.

The examples make it clear that we have now at our disposal a viable non-relativistic quantum field theory of gravitation.

Note that we here spoke just of gravitation à la Verlinde, an emergent gravitational force, not an elementary one. If we were to regard our Verlinde-potentials as phenomenological ones (not deriving from an underlying theory), these potentials could be viewed as quantum-generalized versions of Newton's classical one, that coincide with classical gravitation at macroscopic distances [109, 110].

Chapter 13

Einstein's gravity QFT

13.1 Introduction

Quantifying Einstein gravity (EG) is the holy grail for many physicists. During the last decades, variegated attempts to construct a quantum field theory (QFT) of gravitation have failed [128]. Why? We are convinced that the failure is due to three factors: 1) use of rigged Hilber spaces (RHS) with undefined metric, 2) non-unitarity troubles, and also 3) non-renormalizability based issues [128].

We recount here ways of building up an unitary EG's QFT [128], in the wake of related efforts by S. N. Gupta [119], although we will deviate from such reference by appealing to a different EG-constraint. This deviation leads to a problem similar to that presented by Quantum Electrodynamics (QED), that people know how to deal with. So as to quantize the associated non-renormalizable variational problem we use mathematics' tools developed in [33, 39, 41, 42, 113]. These tools derive from the theory of ultradistributions de J. Sebastiao e Silva (JSS) [10], also known, as we saw above, as ultrahyperfunctions. The above cited tools were specifically concocted to quantify non-renormalizable field theories, successfully culminating in [113]. Thus, one faces a theoretical structure similar to that of QED and endowed with unitarity at all finite orders in a power expansion in the gravitation constant G of the EG Lagrangian. Note that this task was attempted earlier, without success, fist by Gupta, followed afterwards by Feynman in his famous Acta Physica Polonica paper [121].

The secret of quantifying a non-renormalizable field theory, a feat that has eluded theoretical physicists for decades, is that of possessing a suitable definition of the two distributions' product, in a ring with zero-divisors. Where? In configuration space. The issue was tackled successfully in [33, 39, 41, 42, 113], but remained mostly ignored by high energy physicists.
Note that the problem of evaluating the product of distributions with coincident point singularities is connected to the asymptotic behavior of loop integrals of propagators [33, 39, 41, 42, 113]. Such is the mathematics $\rightarrow$ physics link uncovered in [128].

In [33, 39, 41, 42, 113] their authors showed that it is possible to define a (general) convolution between the ultradistributions of JSS [10] (Ultrahyperfunctions) that in turn yields another Ultrahyperfunction, confirming that we have a product in a ring with zero divisors. This a ring is the space of distributions of exponential type, or ultradistributions of exponential type. How do we obtain them? By applying the anti-Fourier transform to the space of tempered ultradistributions or ultradistributions of exponential type. We insist on the fact that the ultrahyperfunctions are the generalization and extension to the complex plane of the Schwartz tempered distributions and the distributions of exponential type. The tempered distributions and those of exponential type are a subset of the ultrahyprefunctions [113]. The pertinent mathematics were extensively discussed in precedent Chapters of this book.

As we saw in previous Chapters, the convolution, once obtained, converts configuration space into a ring with zero-divisors in which we defined a product between the ring-elements. Any unitary-causal-Lorentz invariant theory quantified in such a manner becomes predictive. The divide between renormalizable on non-renormalizable QFT's is not operative now [113].

This all-important convolution employs Laurent's expansions in the parameter employed to define it, as we have seen in precedent Chapters. All finite constants of the convolutions become completely determined, eliminating arbitrary choices of finite constants. This means that we eliminate all finite renormalizations from the theory and that the independent term in the Laurent expansion gives the convolution value. This fact transfers to configuration space the product-operation in a ring with divisors of zero [113], as extensively discussed above in this book.

We will proceed as follows [128]

1. Sect. 2 presents preliminary materials.

2. Sect. 3 is devoted to the QFT Lagrangian for EG.

3. In Sect. 4 we quantize the ensuing theory.

4. In Sect. 5 the graviton's self-energy is evaluated up to second order.

5. In Sect. 6 we introduce axions into our picture and deal with the axions-gravitons interaction.

6. In Sect. 7 we calculate the graviton's self-energy in the presence of axions.

7. In Sect. 8 we evaluate, up to second order, the axion's self-energy.

13.2 Preliminaries

We consider the most general quantification technique, namely Schwinger-Feynman's variational principle [122], which can deal even with high order supersymmetric theories , as done by [123, 124]. These theories cannot be quantized with the customary Dirac-brackets approach [128].

The action for a set of fields is given by [128]

$$\boldsymbol{\mathcal{S}}[\sigma(x), \sigma_0, \phi_A(x)] = \int\limits_{\sigma_0}^{\sigma(x)} \mathcal{L}[\phi_A(\xi), \partial_\mu \phi_A(\xi), \xi] d\xi, \qquad (13.2.0.1)$$

where $\sigma(x)$ is a space-like surface passing through the point x. σ_0 is the surface that at the remote past, and there, all field variations vanish. The Schwinger-Feynman variational principle asserts that

"Any Hermitian infinitesimal variation $\delta\mathcal{S}$ of the action induces a canonical transformation of the vector space in which the quantum system is defined, and the generator of this transformation is this same operator $\delta\mathcal{S}$" [122].

Thus, the following equality emerges thereof

$$\delta\phi_A = i[\delta\mathcal{S}, \phi_A]. \tag{13.2.0.2}$$

For a Poincare transformation one has now

$$\delta\mathcal{S} = a^\mu \mathcal{P}_\mu + \frac{1}{2} a^{\mu\nu} \mathcal{M}_{\mu\nu}, \tag{13.2.0.3}$$

where the field variation is

$$\delta\phi_a = a^\mu \hat{P}_\mu \phi_A + \frac{1}{2} a^{\mu\nu} \hat{M}_{\mu\nu} \phi_A. \tag{13.2.0.4}$$

From (13.2.0.2) one can appreciate that

$$\partial_\mu \phi_A = i[\mathcal{P}_\mu, \phi_A]. \tag{13.2.0.5}$$

More specifically,

$$\partial_0 \phi_A = i[\mathcal{P}_0, \phi_A]. \tag{13.2.0.6}$$

We will use this last result to quantize EG.

13.3 Lagrangian of Einstein's QFT

The EG Lagrangian is [119]

$$\mathcal{L}_G = \frac{1}{\kappa^2} R\sqrt{|g|} - \frac{1}{2} \eta_{\mu\nu} \partial_\alpha h^{\mu\alpha} \partial_\beta h^{\nu\beta}, \tag{13.3.0.1}$$

with $\eta^{\mu\nu} = diag(1, 1, 1, -1)$, $h^{\mu\nu} = \sqrt{|g|} g^{\mu\nu}$ The 2nd. therm in (13.3.0.1) establishes the gauge. We consider the linear approximation

$$h^{\mu\nu} = \eta^{\mu\nu} + \kappa\phi^{\mu\nu}, \tag{13.3.0.2}$$

where κ is the gravitation's constant and $\phi^{\mu\nu}$ the graviton field. Then,

$$\mathcal{L}_G = \mathcal{L}_L + \mathcal{L}_I, \tag{13.3.0.3}$$

where

$$\mathcal{L}_L = -\frac{1}{4}[\partial_\lambda \phi_{\mu\nu}\partial^\lambda \phi^{\mu\nu} - 2\partial_\alpha \phi_{\mu\beta}\partial^\beta \phi^{\mu\alpha} + 2\partial^\alpha \phi_{\mu\alpha}\partial_\beta \phi^{\mu\beta}]. \tag{13.3.0.4}$$

Up to 2nd order, we have [119]

$$\mathcal{L}_I = -\frac{1}{2}\kappa\phi^{\mu\nu}[\frac{1}{2}\partial_\mu \phi^{\lambda\rho}\partial_\nu \phi_{\lambda\rho} + \partial_\lambda \phi_{\mu\beta}\partial^\beta \phi^\lambda_\nu - \partial_\lambda \phi_{\mu\rho}\partial^\lambda \phi^\rho_\nu], \tag{13.3.0.5}$$

having employed the constraint

$$\phi^\mu_\mu = 0. \tag{13.3.0.6}$$

This constraint is needed so as to satisfy gauge invariance [125] For the graviton on has now

$$\Box \phi_{\mu\nu} = 0, \tag{13.3.0.7}$$

whose solution is

$$\phi_{\mu\nu} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[\frac{a_{\mu\nu}(\vec{k})}{\sqrt{2k_0}} e^{ik_\mu x^\mu} + \frac{a^+_{\mu\nu}(\vec{k})}{\sqrt{2k_0}} e^{-ik_\mu x^\mu} \right] d^3k, \tag{13.3.0.8}$$

with $k_0 = |\vec{k}|$.

13.4 Quantization of the theory

Start by defining the energy-momentum tensor [128]

$$T^\lambda_\rho = \frac{\partial \mathcal{L}}{\partial \partial^\rho \phi^{\mu\nu}} \partial^\lambda \phi^{\mu\nu} - \delta^\lambda_\rho \mathcal{L}, \tag{13.4.0.1}$$

and the time-component of the four-momentum [128]

$$\mathcal{P}_0 = \int T^0_0 \; d^3x. \tag{13.4.0.2}$$

Employing (13.3.0.4) one has

$$T_0^0 = \frac{1}{4}[\partial_0\phi_{\mu\nu}\partial^0\phi^{\mu\nu} + \partial_j\phi_{\mu\nu}\partial^j\phi^{\mu\nu} - 2\partial_\alpha\phi_{\mu 0}\partial^0\phi^{\mu\alpha} - 2\partial_\alpha\phi_{\mu j}\partial^j\phi^{\mu\alpha} +$$

$$2\partial_\alpha\phi^{\mu\alpha}\partial_0\phi_\mu^0 + 2\partial_\alpha\phi^{\mu\alpha}\partial_j\phi_\mu^j]. \quad (13.4.0.3)$$

Thus,

$$\mathcal{P}_0 = \frac{1}{4}\int |\vec{k}| \left[a_{\mu\nu}(\vec{k})a^{+\mu\nu}(\vec{k}) + a^{+\mu\nu}(\vec{k})a_{\mu\nu}(\vec{k})\right] d^3k. \quad (13.4.0.4)$$

From (13.2.0.6) we deduce

$$[\mathcal{P}_0, a_{\mu\nu}(\vec{k})] = -k_0 a_{\mu\nu}(\vec{k})$$

$$\mathcal{P}_0, a^{+\mu\nu}(\vec{k})] = k_0 a^{+\mu\nu}(\vec{k}) \quad (13.4.0.5)$$

and from the last relation in (13.4.0.5) one has

$$|\vec{k}|a^{+\rho\lambda}(\vec{k'}) = \frac{1}{2}\int |\vec{k}|[a_{\mu\nu}(\vec{k}), a^{+\rho\lambda}(\vec{k'})]a^{+\mu\nu}(\vec{k})\ d^3k. \quad (13.4.0.6)$$

Solving this integral equation we find

$$[a_{\mu\nu}(\vec{k}), a^{+\rho\lambda}(\vec{k'})] = \left[\delta_\mu^\rho\delta_\nu^\lambda + \delta_\nu^\rho\delta_\mu^\lambda\right]\delta(\vec{k} - \vec{k'}). \quad (13.4.0.7)$$

As usual, the physical state $|\psi>$ is given by the relation

$$\phi_\mu^\mu|\psi >= 0. \quad (13.4.0.8)$$

Appeal to the known definition leads to

$$\Delta_{\mu\nu}^{\rho\lambda}(x-y) =< 0|T[\phi_{\mu\nu}(x)\phi^{\rho\lambda}(y)]|0 > . \quad (13.4.0.9)$$

The graviton's propagator then becomes

$$\Delta_{\mu\nu}^{\rho\lambda}(x-y) = \frac{i}{(2\pi)^4}(\delta_\mu^\rho\delta_\nu^\lambda + \delta_\nu^\rho\delta_\mu^\lambda)\int \frac{e^{ik_\mu(x^\mu - y^\mu)}}{k^2 - i0}d^4k. \quad (13.4.0.10)$$

Thus,

$$\mathcal{P}_0 = \frac{1}{4}\int |\vec{k}| \left[a_{\mu\nu}(\vec{k})a^{+\mu\nu}(\vec{k'}) + a^{+\mu\nu}(\vec{k'})a_{\mu\nu}(\vec{k})\right]\delta(\vec{k} - \vec{k'})d^3kd^3k', \quad (13.4.0.11)$$

or, alternatively,

$$\mathcal{P}_0 = \frac{1}{4}\int |\vec{k}| \left[2a^{+\mu\nu}(\vec{k}')a_{\mu\nu}(\vec{k}) + \delta(\vec{k}-\vec{k}')\right]\delta(\vec{k}-\vec{k}')d^3kd^3k'. \tag{13.4.0.12}$$

Thus, we obtain

$$\mathcal{P}_0 = \frac{1}{2}\int |\vec{k}|a^{+\mu\nu}(\vec{k})a_{\mu\nu}(\vec{k})d^3k, \tag{13.4.0.13}$$

where we appealed to the well known fact that a product of two deltas with the same argument is zero [33], i.e., $\delta(\vec{k}-\vec{k}')\delta(\vec{k}-\vec{k}') = 0$. We see that using ultrahyperfunctions is here equivalent to adopting the normal order in defining the time-component of the four-momentum [128]

$$\mathcal{P}_0 = \frac{1}{4}\int |\vec{k}| : \left[a_{\mu\nu}(\vec{k})a^{+\mu\nu}(\vec{k}) + a^{+\mu\nu}(\vec{k})a_{\mu\nu}(\vec{k})\right] : d^3k. \tag{13.4.0.14}$$

We insist upon the fact that the physical state verifies not only Eq. (13.4.0.8) but the relation (see [119])

$$\partial_\mu\phi^{\mu\nu}|\psi >= 0, \tag{13.4.0.15}$$

as well. The ensuing theory is analogous to the QED-one gotten via the quantization approach of Gupta-Bleuler. This entails that the theory is unitary for any (finite) perturbative order. From this theory just one type of graviton arises, ϕ^{12}, while in Gupta's one two kinds of graviton emerge. Of course, this happens for a non-interacting theory, as stated by Gupta [128].

13.4.1 Effects of not using our constraint

If we do appeal to the constraint (13.4.0.8), one has

$$\mathcal{P}_0 = \frac{1}{2}\int |\vec{k}| \left[a^{+\mu\nu}(\vec{k})a_{\mu\nu}(\vec{k}) - \frac{1}{2}a^{+\mu}_{\mu}(\vec{k})a^{\nu}_{\nu}(\vec{k})\right] d^3k, \tag{13.4.1.16}$$

and, from the Schwinger-Feynman variational principle, we reach

$$|\vec{k}|a^{+}_{\rho\lambda}(\vec{k}') =$$

$$\frac{1}{2}\int |\vec{k}| \left\{ a^{+\mu\nu}(\vec{k})[a_{\mu\nu}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k'})] - \frac{1}{2} a^{+\mu}_{\mu}(\vec{k})[a^{\nu}_{\nu}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k'})] \right\} d^3k, \tag{13.4.1.17}$$

whose solution becomes

$$[a_{\mu\nu}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k'})] = [\eta_{\mu\rho}\eta_{\nu\lambda} + \eta_{\nu\rho}\eta_{\mu\lambda} - \eta_{\mu\nu}\eta_{\rho\lambda}]\,\delta(\vec{k} - \vec{k'}). \tag{13.4.1.18}$$

The above is the usual graviton's quantification, that leads to a theoretical framework whose S matrix is NOT unitary [119, 121].

13.5 Graviton's self energy

To compute the self-energy (SF) we begin with the interaction Hamiltonian $\mathcal{H}_I$. Remark that the Lagrangian has derivative interaction terms.

$$\mathcal{H}_I = \frac{\partial \mathcal{L}_I}{\partial \partial^0 \phi^{\mu\nu}} \partial^0 \phi^{\mu\nu} - \mathcal{L}_I. \tag{13.5.0.1}$$

A typical term is

$$\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k) = k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{-1}. \tag{13.5.0.2}$$

where $\rho = k_1^2 + k_2^2 + k_3^2 - k_0^2$
In ν dimensions, the Fourier transform of (13.5.0.2) becomes

$$\mathcal{F}\{[k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{-1}]_\nu\} =$$

$$\frac{2^{2\nu-2}}{(2\pi)^\nu} \pi^\nu \left[\Gamma\left(\frac{\nu}{2}\right)\right]^2 \eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}(x + i0)^{-\nu} +$$

$$\frac{2^{2\nu-1}}{(2\pi)^\nu} \pi^\nu \Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\nu}{2} + 1\right) (\eta_{\alpha_1\alpha_2} x_{\alpha_3} x_{\alpha_4} + \eta_{\alpha_3\alpha_4} x_{\alpha_1} x_{\alpha_2})(x + i0)^{-\nu-1}$$

$$+ \frac{2^{2\nu}}{(2\pi)^\nu} \pi^\nu \left[\Gamma\left(\frac{\nu}{2} + 1\right)\right]^2 x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} x_{\alpha_4} (x + i0)^{-\nu-2}. \tag{13.5.0.3}$$

where $x = x_1^2 + x_2^2 + x_3^2 - x_0^2$
Anti-transforming (ep12.5.3) one has

$$[k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{-1}]_\nu =$$

$$\left\{ -i \frac{\pi^{\frac{\nu}{2}}}{2} \frac{\left[\Gamma\left(\frac{\nu}{2}\right)\right]^2}{\Gamma(\nu)} \eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \right.$$

$$-i\frac{\pi^{\frac{\nu}{2}}}{2}\frac{\left[\Gamma\left(\frac{\nu}{2}+1\right)\right]^2}{\Gamma(\nu+2)}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\Bigg\}$$

$$\Gamma\left(-\frac{\nu}{2}\right)(\rho-i0)^{\frac{\nu}{2}}+$$

$$\Bigg\{i\frac{\pi^{\frac{\nu}{2}}}{2}\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}+1\right)}{\Gamma(\nu+1)}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})+$$

$$i\frac{\pi^{\frac{\nu}{2}}}{2}\frac{\left[\Gamma\left(\frac{\nu}{2}+1\right)\right]^2}{\Gamma(\nu+1)}\times$$

$$(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}+$$

$$\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\}\Gamma\left(1-\frac{\nu}{2}\right)(\rho-i0)^{\frac{\nu}{2}-1}-$$

$$i\pi^{\frac{\nu}{2}}\frac{\left[\Gamma\left(\frac{\nu}{2}+1\right)\right]^2}{\Gamma(\nu+2)}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\Gamma\left(2-\frac{\nu}{2}\right)(\rho-i0)^{\frac{\nu}{2}-2}. \quad (13.5.0.4)$$

13.5.1 Self-energy computation for $\nu = 4$

Let us consider the ν-Laurent expansion and keep there the $\nu - 4$ independent term [113]. We Laurent-expand (13.5.0.4) around $\nu = 4$ and obtain [128]

$$[k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{-1}*k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{-1}]_\nu=$$

$$i\frac{\pi^2}{\nu-4}\Bigg\{\frac{1}{5!}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\rho^2-$$

$$\Bigg[\frac{2}{4!}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})-$$

$$\frac{1}{6!}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}+\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+$$

$$\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\Bigg]\rho+\frac{8}{5!}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\Bigg\}-$$

$$\frac{i\pi^2}{5!2}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})$$

$$\left[\ln(\rho-i0)+\ln\pi+C-\frac{46}{15}\right]\rho^2+$$

$$i\frac{\pi^2}{4!}\left\{(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln(\rho-i0)+\ln\pi+C-\frac{8}{3}\right]-\right.$$

$$\frac{1}{24}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}+\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+$$

$$\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+$$

$$\left.\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\left[\ln(\rho-i0)+\ln\pi+2C-\frac{101}{15}\right]\right\}\rho-$$

$$i\frac{\pi^2}{30}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln(\rho-i0)+\ln\pi+C-\frac{47}{30}\right]+\left.\sum_{n=1}^{\infty}a_n(\nu-4)^n\right\}. \tag{13.5.1.5}$$

The exact value of the convolution that interests us, i.e., the left hand side of (5.5), is that yielded by the independent term in the precedent expansion. If the reader is not familiar with this scenario, see for instance [113]. We arrive at

$$\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k)=k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{-1}*k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{-1}=-$$

$$\frac{i\pi^2}{5!2}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})$$

$$\left[\ln(\rho-i0)+\ln\pi+C-\frac{46}{15}\right]\rho^2-$$

$$i\frac{\pi^2}{4!}\left\{(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln(\rho-i0)+\ln\pi+C-\frac{8}{3}\right]-\right.$$

$$\frac{1}{24}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}+\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+$$

$$\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+$$

$$\left.\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\left[\ln(\rho-i0)+\ln\pi+2C-\frac{101}{15}\right]\right\}\rho-$$

$$\left.i\frac{\pi^2}{30}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln(\rho-i0)+\ln\pi+C-\frac{47}{30}\right]\right\}. \tag{13.5.1.6}$$

Here we must tackle 1296 diagrams of this sort.

13.6 Inserting axions into the picture

These are hypothetical elementary particles postulated by Peccei–Quinn in 1977 to deal with the strong CP problem in quantum chromodynamics [1]. Should they exist and have low enough mass, they might be of interest as putative components of cold dark matter [126].

We consider now a massive scalar field (axions) interacting with the graviton and the Lagrangian is

$$\mathcal{L}_{GM} = \frac{1}{\kappa^2} R\sqrt{|g|} - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha h^{\mu\alpha}\partial_\beta h^{\nu\beta} - \frac{1}{2}[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2]. \tag{13.6.0.1}$$

Recast at this stage the Lagrangian in the form

$$\mathcal{L}_{GM} = \mathcal{L}_L + \mathcal{L}_I + \mathcal{L}_{LM} + \mathcal{L}_{IM}, \tag{13.6.0.2}$$

where

$$\mathcal{L}_{LM} = -\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2]. \tag{13.6.0.3}$$

$\mathcal{L}_{IM}$ has become the Lagrangian for the axion-graviton action

$$\mathcal{L}_{IM} = -\frac{1}{2}\kappa\phi^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{13.6.0.4}$$

The new term in the interaction Hamiltonian is then

$$\mathcal{H}_{IM} = \frac{\partial\mathcal{L}_{IM}}{\partial\partial^0\phi}\partial^0\phi - \mathcal{L}_{IM}. \tag{13.6.0.5}$$

13.7 Graviton's complete self energy

The presence of axions creates a novel contribution to the graviton's self energy

$$\Sigma_{GM\mu r\nu s}(k) = k_\mu k_r(\rho + m^2 - i0)^{-1} * k_\nu k_s(\rho + m^2 - i0)^{-1}. \tag{13.7.0.1}$$

To evaluate it we consider the usual ν dimensional integral together with Feynman-parameters that we denote with the letter x. After a Wick rotation one finds [128]

$$[k_\mu k_r(\rho + m^2 - i0)^{-1} * k_\nu k_s(\rho + m^2 - i0)^{-1}]_\nu =$$

$$\int_0^1 \int \frac{k_\mu k_r (p_\nu - k_\nu)(p_s - k_s)}{(k-px)^2 + a} d^\nu k dx, \tag{13.7.0.2}$$

where

$$a = p^2 x - p^2 x^2 + m^2. \tag{13.7.0.3}$$

Effect the variables-change $u = k - px$ and get

$$[k_\mu k_r (\rho + m^2 - i0)^{-1} * k_\nu k_s (\rho + m^2 - i0)^{-1}]_\nu =$$

$$i \int_0^1 \int \frac{f(u, x, \mu, r, \nu, s)}{u^2 + a} d^\nu u dx, \tag{13.7.0.4}$$

where

$$f(u, x, \mu, r, \nu, s) =$$

$$u_\mu u_r p_\nu p_s (1-x)^2 + u_\mu u_r u_\nu u_s - u_\mu u_s p_r p_\nu x(1-x) -$$

$$u_\mu u_\nu p_r p_s x(1-x) - u_r u_s p_\mu p_\nu x(1-x) - u_r u_\nu p_\mu p_s x(1-x) +$$

$$p_\mu p_r p_\nu p_s x^2 (1-x)^2 + u_\nu u_s p_\mu p_r x^2. \tag{13.7.0.5}$$

After computing the pertinent integrals we have

$$[k_\mu k_r (\rho + m^2 - i0)^{-1} * k_\nu k_s (\rho + m^2 - i0)^{-1}]_\nu =$$

$$i \frac{(\eta_{\mu r} k_\nu k_s + \eta_{\nu s} k_\mu k_r) m^{\nu-2} \pi^{\frac{\nu}{2}}}{8} \Gamma\left(1 - \frac{\nu}{2}\right) \times$$

$$\left[F\left(1, 1 - \frac{\nu}{2}, \frac{3}{2}; -\frac{\rho}{4m^2}\right) + \frac{1}{3} F\left(1, 1 - \frac{\nu}{2}, \frac{5}{2}; -\frac{\rho}{4m^2}\right) \right] +$$

$$i(\eta_{\mu r} \eta_{\nu s} + \eta_{\mu \nu} \eta_{rs} + \eta_{\mu s} \eta_{\nu r}) \frac{\pi^{\frac{\nu}{2}} m^\nu}{4}$$

$$\Gamma\left(\frac{\nu}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) F\left(1, -\frac{\nu}{2}, \frac{3}{2}; -\frac{\rho}{4m^2}\right) -$$

$$i(\eta_{\mu s} k_r k_\nu + \eta_{\mu \nu} k_r k_s + \eta_{rs} k_\mu k_\nu + \eta_{r\nu} k_\mu k_s) \frac{m^{\nu-2} \pi^{\frac{\nu}{2}}}{48} \times$$

$$\Gamma\left(1 - \frac{\nu}{2}\right) F\left(2, 1 - \frac{\nu}{2}, \frac{5}{2}; -\frac{\rho}{4m^2}\right) +$$

$$i k_\mu k_r k_\nu k_s \frac{m^{\nu-4} \pi^{\frac{\nu}{2}}}{12} \Gamma\left(2 - \frac{\nu}{2}\right) F\left(2, 2 - \frac{\nu}{2}, \frac{5}{2}; -\frac{\rho}{4m^2}\right). \tag{13.7.0.6}$$

13.7.1 Computing the Self-Energy for $\nu = 4$

We face once again a Laurent's expansion

$$[k_\mu k_r(\rho + m^2 - i0)^{-1} * k_\nu k_s(\rho + m^2 - i0)^{-1}]_\nu =$$

$$-i\frac{\pi^2}{\nu-4}\left\{m^2(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\left[\frac{1}{3} + \frac{1}{5}\frac{\rho}{4m^2}\right] - \right.$$

$$2m^4(\eta_{\mu r}\eta_{\nu s} + \eta_{\mu\nu}\eta_{rs} + \eta_{\mu s}\eta_{r\nu})\times$$

$$\left[\frac{1}{8} + \frac{1}{6}\frac{\rho}{4m^2} + \frac{1}{15}\left(\frac{\rho}{4m^2}\right)^2\right] -$$

$$\frac{m^2}{4m^2 + k^2 - i0}(\eta_{\mu s}k_r k_\nu + \eta_{\mu\nu}k_r k_s + \eta_{rs}k_\mu k_\nu + \eta_{r\nu}k_\mu k_s)\times$$

$$\left.\frac{k^2 - m^2}{12} + \frac{m^2}{4} + \frac{k^2 - m^2}{30}\frac{\rho}{4m^2} - \frac{1}{6}k_\mu k_r k_\nu k_s\right\} +$$

$$i\frac{m^2\pi^2}{2}(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\times$$

$$\left[\frac{1}{3}(\ln m^2 + \ln\pi + C - 1) + \frac{1}{5}\frac{\rho}{4m^2}\left(\ln m^2 + \ln\pi + C\right)\right] +$$

$$i\frac{m^2\pi^2}{30}(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\frac{\rho}{4m^2}\times$$

$$\left[F\left(1, 1, \frac{7}{2}; -\frac{\rho}{4m^2}\right) + \frac{1}{7}F\left(1, 1, \frac{9}{2}; -\frac{\rho}{4m^2}\right)\right] +$$

$$-i2\pi^2 m^4(\eta_{\mu r}\eta_{\nu s} + \eta_{\mu\nu}\eta_{rs} + \eta_{\mu s}\eta_{\nu r})\times$$

$$\left\{\left[\frac{1}{8} - \frac{1}{6}\frac{\rho}{4m^2} - \frac{1}{15}\left(\frac{\rho}{4m^2}\right)^2 + \right]\times\right.$$

$$\left.\left(\ln m^2 + \ln\pi + 1\right) - \frac{1}{2}\left[\frac{3}{32} - \frac{1}{3}\left(\frac{\rho}{4m^2}\right)\right]\right\} -$$

$$i\frac{2\pi^2 m^4}{105}(\eta_{\mu r}\eta_{\nu s} + \eta_{\mu\nu}\eta_{rs} + \eta_{\mu s}\eta_{\nu r})\left(\frac{\rho}{4m^2}\right)^3 F\left(1, 1, \frac{9}{2}; -\frac{\rho}{4m^2}\right) -$$

$$i\frac{\pi^2 m^2(k^2 - m^2)}{12(4m^2 + k^2 - i0}(\eta_{\mu s}k_r k_\nu + \eta_{\mu\nu}k_r k_s + \eta_{rs}k_\mu k_\nu + \eta_{r\nu}k_\mu k_s)\times$$

$$\left[\frac{1}{2}\left(\ln m^2 + \ln\pi + C - \frac{1}{4}\right) + \frac{1}{5}\left(\ln m^2 + \ln\pi + C\right)\frac{k^2}{4m^2}\right] -$$

$$i\frac{\pi^2 m^2}{8(4m^2+k^2-i0}(\eta_{\mu s}k_r k_\nu + \eta_{\mu\nu}k_r k_s + \eta_{rs}k_\mu k_\nu + \eta_{r\nu}k_\mu k_s)\times$$

$$m^2\left[\left(\ln m^2 + \ln\pi + C - \frac{1}{4}\right) + \frac{k^2}{6} + \frac{k^2}{15}\frac{k^2}{4m^2}\right] -$$

$$i\frac{\pi^2 m^2}{10}(\eta_{\mu s}k_r k_\nu + \eta_{\mu\nu}k_r k_s + \eta_{rs}k_\mu k_\nu + \eta_{r\nu}k_\mu k_s)\times$$

$$\frac{k^2-m^2}{21(4m^2+k^2-i0)}F\left(1,1,\frac{9}{2};-\frac{\rho}{4m^2}\right)\left(\frac{k^2}{4m^2}\right)^2 -$$

$$i\frac{\pi^2}{12}k_\mu k_r k_\nu k_s\left[(\ln m^2 + \ln\pi) + \frac{k^2}{4m^2+k^2-i0}\right] -$$

$$i\frac{\pi^2 m^2}{30}k_\mu k_r k_\nu k_s\frac{k^2-m^2}{4m^2+k^2-i0}\frac{k^2}{4m^2}F\left(1,1,\frac{7}{2};-\frac{k^2}{4m^2}\right) +$$

$$\sum_{n=0}^{\infty} a_n(\nu-4)^n. \tag{13.7.1.7}$$

Thus, the exact result for our four-dimensional convolution is

$$\Sigma_{GM\mu\nu rs}(k) = k_\mu k_r(\rho + m^2 - i0)^{-1} * k_\nu k_s(\rho + m^2 - i0)^{-1} =$$

$$i\frac{m^2\pi^2}{2}(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\times$$

$$\left[\frac{1}{3}(\ln m^2 + \ln\pi + C - 1) + \frac{1}{5}\frac{\rho}{4m^2}\left(\ln m^2 + \ln\pi + C\right)\right] +$$

$$i\frac{m^2\pi^2}{30}(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\frac{\rho}{4m^2}\times$$

$$\left[F\left(1,1,\frac{7}{2};-\frac{\rho}{4m^2}\right) + \frac{1}{7}F\left(1,1,\frac{9}{2};-\frac{\rho}{4m^2}\right)\right] +$$

$$-i2\pi^2 m^4(\eta_{\mu r}\eta_{\nu s} + \eta_{\mu\nu}\eta_{rs} + \eta_{\mu s}\eta_{\nu r})\times$$

$$\left\{\left[\frac{1}{8} - \frac{1}{6}\frac{\rho}{4m^2} - \frac{1}{15}\left(\frac{\rho}{4m^2}\right)^2 + \right]\times\right.$$

$$\left.(\ln m^2 + \ln\pi + 1) - \frac{1}{2}\left[\frac{3}{32} - \frac{1}{3}\left(\frac{\rho}{4m^2}\right)\right]\right\} +$$

$$i\frac{2\pi^2 m^4}{105}(\eta_{\mu r}\eta_{\nu s} + \eta_{\mu\nu}\eta_{rs} + \eta_{\mu s}\eta_{\nu r})\left(\frac{\rho}{4m^2}\right)^3 F\left(1,1,\frac{9}{2};-\frac{\rho}{4m^2}\right) -$$

$$i\frac{\pi^2 m^2(k^2-m^2)}{12(4m^2+k^2-i0}(\eta_{\mu s}k_r k_\nu+\eta_{\mu\nu}k_r k_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$

$$\left[\frac{1}{2}\left(\ln m^2+\ln\pi+C-\frac{1}{4}\right)+\frac{1}{5}\left(\ln m^2+\ln\pi+C\right)\frac{k^2}{4m^2}\right]-$$

$$i\frac{\pi^2 m^2}{8(4m^2+k^2-i0}(\eta_{\mu s}k_r k_\nu+\eta_{\mu\nu}k_r k_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$

$$m^2\left[\left(\ln m^2+\ln\pi+C-\frac{1}{4}\right)+\frac{k^2}{6}+\frac{k^2}{15}\frac{k^2}{4m^2}\right]-$$

$$i\frac{\pi^2 m^2}{10}(\eta_{\mu s}k_r k_\nu+\eta_{\mu\nu}k_r k_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$

$$\frac{k^2-m^2}{21(4m^2+k^2-i0)}F\left(1,1,\frac{9}{2};-\frac{\rho}{4m^2}\right)\left(\frac{k^2}{4m^2}\right)^2-$$

$$i\frac{\pi^2}{12}k_\mu k_r k_\nu k_s\left[\left(\ln m^2+\ln\pi\right)-\frac{k^2}{4m^2+k^2-i0}\right]-$$

$$i\frac{\pi^2 m^2}{30}k_\mu k_r k_\nu k_s\frac{k^2-m^2}{4m^2+k^2-i0}\frac{k^2}{4m^2}F\left(1,1,\frac{7}{2};-\frac{k^2}{4m^2}\right)\quad(13.7.1.8)$$

Accordingly, our desired self-energy total is a combination of $\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k)$ and $\Sigma_{GM\alpha_1\alpha_2\alpha_3\alpha_4}(k)$.

13.8 Axion's self energy

It is given by

$$\Sigma^{\mu s}(k)=(\eta^{\mu r}\eta^{\nu s}+\eta^{\mu s}\eta^{\nu r})k_\nu k_r(\rho+m^2-i0)^{-1}*(\rho-i0)^{-1}.\quad(13.8.0.1)$$

In ν dimensions we have

$$[k_\nu k_r(\rho+m^2-i0)^{-1}*(\rho-i0)^{-1}]_\nu=\int\frac{k_\nu k_r}{(k^2+m^2-i0)[(p-k)^2-i0]}d^\nu k.\quad(13.8.0.2)$$

Using the above Feynman parameters we get

$$[k_\nu k_r(\rho+m^2-i0)^{-1}*(\rho-i0)^{-1}]_\nu=i\int_0^1\int\frac{k_\nu k_r}{(k-px)^2+a}d^\nu k dx,\quad(13.8.0.3)$$

where

$$a = (p^2 + m^2)x - p^2x^2. \tag{13.8.0.4}$$

We compute the integral (13.8.0.3), obtaining

$$[k_\nu k_r(\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}]_\nu =$$

$$i\frac{\eta_{\nu r} m^{\nu-2}\pi^{\frac{\nu}{2}}}{\nu}\Gamma\left(1 - \frac{\nu}{2}\right) F\left(1, 1 - \frac{\nu}{2}, \frac{\nu}{2} + 1; -\frac{\rho}{m^2}\right) +$$

$$\frac{2ik_\nu k_r m^{\nu-4}\pi^{\frac{\nu}{2}}}{\nu+2}\Gamma\left(2 - \frac{\nu}{2}\right) F\left(1, 2 - \frac{\nu}{2}, \frac{\nu}{2} + 2; -\frac{\rho}{m^2}\right). \tag{13.8.0.5}$$

13.8.1 Self-Energy computation for $\nu = 4$

One Laurent-expands now (13.8.0.5) around $\nu = 4$, finding

$$[k_\nu k_r(\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}]_\nu =$$

$$i\pi^2 \left\{\frac{1}{\nu - 4}\left(\frac{\eta_{\nu r} m^2}{2} - 2k_\nu k_r\right) + \right.$$

$$\frac{\eta_{\nu r} m^2}{4}\left[\left(1 + \frac{1}{3}\frac{\rho}{m^2}\right)\left(\ln m^2 + \ln \pi + C - \frac{1}{2}\right) - \right.$$

$$\left.\left(1 + \frac{1}{9}\frac{\rho}{m^2}\right)\right] - \frac{k_\nu k_r}{3}\left(\ln m^2 + \ln \pi + C - \frac{1}{2}\right) +$$

$$\frac{1}{4}\left(\frac{\rho}{m^2}\right)\left[\frac{\eta_{\nu r} m^2}{12}\frac{\rho}{m^2} - \frac{k_\nu k_r}{3}\right] F\left(1, 1, 5; -\frac{\rho}{m^2}\right) +$$

$$\left.\sum_{n=1}^{\infty} a_n(\nu - 4)^n\right\} \tag{13.8.1.6}$$

As usual, the ν-independent term gives the exact convolution result:

$$\Sigma_{\nu r}(k) = k_\nu k_r(\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1} =$$

$$i\pi^2 \left\{\frac{\eta_{\nu r} m^2}{4}\left[\left(1 + \frac{1}{3}\frac{\rho}{m^2}\right)\left(\ln m^2 + \ln \pi + C - \frac{1}{2}\right) - \right.\right.$$

$$\left.\left(1 + \frac{1}{9}\frac{\rho}{m^2}\right)\right] - \frac{k_\nu k_r}{3}\left(\ln m^2 + \ln \pi + C - \frac{1}{2}\right) +$$

$$\left.\frac{1}{4}\left(\frac{\rho}{m^2}\right)\left[\frac{\eta_{\nu r} m^2}{12}\frac{\rho}{m^2} - \frac{k_\nu k_r}{3}\right] F\left(1, 1, 5; -\frac{\rho}{m^2}\right)\right\} \tag{13.8.1.7}$$

13.9 Discussion

We have developed above a QFT of Eintein's gravity (EG), that is both unitary and finite. These results critically depend upon the employment of a new constraint that we introduced in defining the EG-Lagrangian. Laurent expansions were an absolutely necessary tool here [128].

So as to quantify our theory we used the variational principle of Schwinger-Feynman's. This led to only one graviton type ϕ^{12}.

The underlying mathematics employed here was developed in [33, 39, 41, 42, 113] and is powerful enough so as to be tackle non-renormalizable field theories, a fact that remained mostly ignored until it was successfully exploited in [128].

We have computed in finite and exact fashion

- a graviton's self-energy in the EG-field,
- such self-energy in the added presence of a massive scalar field (axions). Two types of diagram appear: the original ones of the pure EG field plus the ones that emerge on account of the addition of a scalar field.
- The axion's self-energy.

Chapter 14

Further Generalization

14.1 Propagators as Tempered Ultradistributions

The Feynman propagators corresponding to a massless particle F and a massive (m) particle G are, respectively, the following ultrahyperfunctions

$$F(\rho) = -\Theta[-\Im(\rho)]\rho^{-1}$$

$$G(\rho) = -\Theta[-\Im(\rho)](\rho + m^2)^{-1}, \tag{14.1.0.1}$$

where ρ is a complex variable such that on the real axis one has $\rho = k_1^2 + k_2^2 + k_3^2 - k_0^2$. For F and G the following equalities are satisfied

$$\rho^\lambda F(\rho) = -\Theta[-\Im(\rho)]\rho^{\lambda-1}$$

$$\rho^\lambda G(\rho) = -\Theta[-\Im(\rho)](\rho + m^2)^{\lambda-1}, \tag{14.1.0.2}$$

where one uses $(\rho + m^2)^\lambda \simeq \rho^\lambda$, since we have chosen m to be very small. On the real axis, the previously defined propagators are given by

$$F(\rho) = F(\rho + i0) - F(\rho - i0) = (\rho - i0)^{-1}$$

$$g(\rho) = G(\rho + i0) - G(\rho - i0) = (\rho + m^2 - i0)^{-1}. \tag{14.1.0.3}$$

These are the usual expressions for Feynman propagators.

Consider first the convolution of two massless propagators. We use (14.1.0.2), since here the corresponding ultrahyperfunctions do not have singularities in the complex plane. We obtain, from (9.8.0.14), a simplified expression for the pertinent convolution

$$h_\lambda(\rho) = \frac{\pi}{2\rho} \iint\limits_{-\infty}^{\infty} (\rho_1 - i0)^{\lambda-1} (\rho_2 - i0)^{\lambda-1}$$

$$\left[(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2\right]_+^{\frac{1}{2}} \, d\rho_1 \, d\rho_2. \tag{14.1.0.4}$$

This expression is nothing but the usual convolution

$$h_\lambda(\rho) = (\rho - i0)^{\lambda-1} * (\rho - i0)^{\lambda-1}. \tag{14.1.0.5}$$

In the same manner we obtain for massive propagators

$$h_\lambda(\rho) = (\rho + m^2 - i0)^{\lambda-1} * (\rho - m^2 - i0)^{\lambda-1}. \tag{14.1.0.6}$$

These last two expressions are the ones we will use later to evaluate the graviton's self-energy.

14.2 Self energy of the graviton

To evaluate the graviton's self-energy (SF) we start again with the interaction Hamiltonian $\mathcal{H}_I$. Note that the Lagrangian contains derivative interaction terms.

$$\mathcal{H}_I = \frac{\partial \mathcal{L}_I}{\partial \partial^0 \phi^{\mu\nu}} \partial^0 \phi^{\mu\nu} - \mathcal{L}_I. \tag{14.2.0.1}$$

A typical term reads

$$\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k) = k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{\lambda-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{\lambda-1}. \tag{14.2.0.2}$$

where $\rho = k_1^2 + k_2^2 + k_3^2 - k_0^2$
The Fourier transform of (14.2.0.2) is

$$\mathcal{F}[k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{\lambda-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{\lambda-1}] =$$

$$-\frac{2^{4(\lambda+1)} \left[\Gamma(2+\lambda)^2\right]}{4\Gamma(1-\lambda)^2} \eta_{\alpha_1\alpha_2} \eta_{\alpha_3\alpha_4} (x + i0)^{-2\lambda-4} +$$

$$\frac{2^{4(\lambda+1)} \Gamma(2+\lambda)\Gamma(3+\lambda)}{2\Gamma(1-\lambda)^2} (\eta_{\alpha_1\alpha_2} x_{\alpha_3} x_{\alpha_4} + \eta_{\alpha_3\alpha_4} x_{\alpha_1} x_{\alpha_2})(x+i0)^{-2\lambda-5} -$$

$$\frac{2^{4(\lambda+1)}\Gamma(3+\lambda)^2}{\Gamma(1-\lambda)} x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} x_{\alpha_4} (x+i0)^{-\nu-2}, \qquad (14.2.0.3)$$

where $x = x_1^2 + x_2^2 + x_3^2 - x_0^2$.

Anti-transforming the above equation we have

$$k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{\lambda-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{\lambda-1} =$$

$$\frac{i\pi^2}{4\Gamma(1-\lambda)^2} \left\{ \Gamma(\lambda+2) \left[\frac{\Gamma(2+\lambda)}{\Gamma(2\lambda+4)} - 2\frac{\Gamma(3+\lambda)}{\Gamma(2\lambda+5)} \right] \eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \right.$$

$$\left. \frac{\Gamma(\lambda+3)^2}{\Gamma(2\lambda+6)} (\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4} + \eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3}) \right\}$$

$$\Gamma(-2\lambda-2)(\rho-i0)^{2\lambda+2} +$$

$$\frac{i\pi^2\Gamma(\lambda+3)}{2\Gamma(1-\lambda)^2} \left\{ \frac{\Gamma(2+\lambda}{\Gamma(2\lambda+5)} \Gamma(\nu+1)(\eta_{\alpha_1\alpha_2} k_{\alpha_3} k_{\alpha_4} + \eta_{\alpha_3\alpha_4} k_{\alpha_1} k_{\alpha_2}) - \right.$$

$$\frac{\Gamma(\lambda+3)}{\Gamma(2\lambda+6)}$$

$$(\eta_{\alpha_1\alpha_2} k_{\alpha_3} k_{\alpha_4} + \eta_{\alpha_1\alpha_3} k_{\alpha_2} k_{\alpha_4} + \eta_{\alpha_1\alpha_4} k_{\alpha_2} k_{\alpha_3} + \eta_{\alpha_3\alpha_4} k_{\alpha_1} k_{\alpha_2} +$$

$$\left. \eta_{\alpha_2\alpha_3} k_{\alpha_1} k_{\alpha_4} + \eta_{\alpha_2\alpha_4} k_{\alpha_1} k_{\alpha_3}) \right\} \Gamma(-2\lambda-1)(\rho-i0)^{2\lambda+1} +$$

,

$$\frac{i\pi^2\Gamma(\lambda+3)^2}{\Gamma(1-\lambda)^2\Gamma(2\lambda+6)} k_{\alpha_1} k_{\alpha_2} k_{\alpha_3} k_{\alpha_4} \Gamma(-2\lambda)(\rho-i0)^{2\lambda}. \qquad (14.2.0.4)$$

14.2.1 Self-Energy evaluation for $\lambda = 0$

We appeal now to a λ-Laurent expansion and retain there the $\lambda = 0$ independent term.. Thus, we Laurent-expand (14.2.0.4) around $\lambda = 0$ and find

$$k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{\lambda-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{\lambda-1} =$$

$$-i\frac{\pi^2}{4\lambda} \left\{ \frac{1}{5!} (\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4} + \eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\rho^2 - \right.$$

$$\left[\frac{2}{4!} (\eta_{\alpha_1\alpha_2} k_{\alpha_3} k_{\alpha_4} + \eta_{\alpha_3\alpha_4} k_{\alpha_1} k_{\alpha_2}) - \right.$$

$$\frac{1}{6!}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4} + \eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2} + \eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4} + \eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3} +$$

$$\left.\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4} + \eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})]\ \rho + \frac{8}{5!}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\right\} -$$

$$\frac{i\pi^2}{5!2}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4} + \eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})$$

$$\left[\ln(\rho - i0) - \frac{137}{60}\right]\rho^2 +$$

$$i\frac{\pi^2}{4!}\left\{(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4} + \eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln(\rho - i0) - \frac{11}{6}\right] -\right.$$

$$\frac{1}{24}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4} + \eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2} + \eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4} +$$

$$\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3} + \eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4} +$$

$$\left.\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\left[\ln(\rho - i0) + \ln\pi + 2C - \frac{101}{30}\right]\right\}\rho -$$

$$\left. i\frac{\pi^2}{30}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln(\rho - i0) - \frac{47}{60}\right] + \sum_{n=1}^{\infty} a_n\lambda^n\right\}. \tag{14.2.1.5}$$

The exact value of the convolution we are interested in, i.e., the left hand side of (14.2.1.5), is given by the independent term in the above expansion, as it is well-known. We then arrive at

$$\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k) = k_{\alpha_1}k_{\alpha_2}(\rho - i0)^{-1} * k_{\alpha_3}k_{\alpha_4}(\rho - i0)^{-1} = -$$

$$\frac{i\pi^2}{5!2}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4} + \eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\left[\ln(\rho - i0) - \frac{137}{60}\right]\rho^2 +$$

$$i\frac{\pi^2}{4!}\left\{(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4} + \eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln(\rho - i0) - \frac{11}{6}\right] -\right.$$

$$\frac{1}{24}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4} + \eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2} + \eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4} +$$

$$\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3} + \eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4} +$$

$$\left.\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\left[\ln(\rho - i0) + \ln\pi + 2C - \frac{101}{30}\right]\right\}\rho -$$

$$\left. i\frac{\pi^2}{30}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln(\rho - i0) - \frac{47}{60}\right]\right\}. \tag{14.2.1.6}$$

We have to deal with 1296 diagrams of this kind.

14.3 Complete Graviton's Self Energy

We include now a massive scalar field (axions) interacting with the graviton. The Lagrangian is given by (13.6.0.1). The presence of axions generates a new contribution to the graviton's self energy

$$\Sigma_{GM\mu r\nu s}(k) = k_\mu k_r(\rho+m^2-i0)^{-1} * k_\nu k_s(\rho+m^2-i0)^{-1}. \quad (14.3.0.1)$$

So as to compute it, we appeal to the usual integral together with the generalized Feynman-parameters. After a Wick rotation we obtain

$$k_\mu k_r(\rho+m^2-i0)^{\lambda-1} * k_\nu k_s(\rho+m^2-i0)^{\lambda-1} =$$

$$i\int_0^1 x^{-\lambda}(1-x)^{-\lambda}\int \frac{k_\mu k_r(p_\nu-k_\nu)(p_s-k_s)}{[(k-px)^2+a]^{2-2\lambda}}d^4k dx, \quad (14.3.0.2)$$

where

$$a = p^2x - p^2x^2 + m^2. \quad (14.3.0.3)$$

After the variables-change $u = k - px$ we find

$$k_\mu k_r(\rho+m^2-i0)^{\lambda-1} * k_\nu k_s(\rho+m^2-i0)^{\lambda-1} =$$

$$i\int_0^1 x^{-\lambda}(1-x)^{-\lambda}\int \frac{f(u,x,\mu,r,\nu,s)}{(u^2+a)^{2-2\lambda}}d^4u dx \quad (14.3.0.4)$$

where

$$f(u,x,\mu,r,\nu,s) = u_\mu u_r p_\nu p_s(1-x)^2 + u_\mu u_r u_\nu u_s - u_\mu u_s p_r p_\nu x(1-x) -$$

$$u_\mu u_\nu p_r p_s x(1-x) - u_r u_s p_\mu p_\nu x(1-x) - u_r u_\nu p_\mu p_s x(1-x) +$$

$$p_\mu p_r p_\nu p_s x^2(1-x)^2 + u_\nu u_s p_\mu p_r x^2. \quad (14.3.0.5)$$

After evaluation of the pertinent integrals, we arrive at

$$k_\mu k_r(\rho+m^2-i0)^{\lambda-1} * k_\nu k_s(\rho+m^2-i0)^{\lambda-1} =$$

$$\frac{i\pi^{\frac{5}{2}}2^{2\lambda}m^{2+4\lambda}}{16}\frac{\Gamma(-1-2\lambda)}{\Gamma(1-\lambda)}(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\times$$

$$\left[\frac{F\left(1-2\lambda,-1-2\lambda,\frac{3}{2}-\lambda;-\frac{\rho}{4m^2}\right)}{\Gamma\left(\frac{3}{2}-\lambda\right)}+\right.$$

$$\left.\frac{F\left(1-\lambda,-1-2\lambda,\frac{5}{2}-\lambda;-\frac{\rho}{4m^2}\right)}{2\Gamma\left(\frac{5}{2}-\lambda\right)}\right]+$$

$$i\frac{i\pi^{\frac{5}{2}}2^{2\lambda-1}m^{4+4\lambda}}{4}(\eta_{\mu r}\eta_{\nu s}+\eta_{\mu\nu}\eta_{rs}+\eta_{\mu s}\eta_{\nu r})$$

$$\frac{\Gamma(-2-2\lambda)}{\Gamma(1-\lambda)\Gamma\left(\frac{3}{2}-\lambda\right)}F\left(-2-2\lambda,1-\lambda,\frac{3}{2}-\lambda;-\frac{\rho}{4m^2}\right)-$$

$$i\frac{i\pi^{\frac{5}{2}}2^{2\lambda}m^{2+4\lambda}}{64}(\eta_{\mu s}k_r k_\nu+\eta_{\mu\nu}k_r k_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$

$$\frac{\Gamma(2-\lambda)\Gamma(-1-2\lambda)}{\Gamma(1-\lambda)^2\Gamma\left(\frac{5}{2}-\lambda\right)}F\left(-1-2\lambda,2-\lambda,\frac{5}{2}-\lambda;-\frac{\rho}{4m^2}\right)+$$

$$i\frac{i\pi^{\frac{5}{2}}2^{2\lambda}m^{4\lambda}}{32}k_\mu k_r k_\nu k_s[\frac{\Gamma(3-\lambda)\Gamma(-2\lambda)}{\Gamma(1-\lambda)^2\Gamma\left(\frac{5}{2}-\lambda\right)}$$

$$F\left(-2\lambda,2-\lambda,\frac{5}{2}-\lambda;-\frac{\rho}{4m^2}\right). \tag{14.3.0.6}$$

14.3.1 Self-Energy evaluation for $\lambda = 0$

We need again a Laurent's expansion and face

$$k_\mu k_r(\rho+m^2-i0)^{\lambda-1}*k_\nu k_s(\rho+m^2-i0)^{\lambda-1}=$$

$$i\frac{\pi^2}{4\lambda}\left\{m^2(\eta_{\mu r}k_\nu k_s+\eta_{\nu s}k_\mu k_r)\left[\frac{1}{3}+\frac{1}{5}\frac{\rho}{4m^2}\right]-\right.$$

$$m^4(\eta_{\mu r}\eta_{\nu s}+\eta_{\mu\nu}\eta_{rs}+\eta_{\mu s}\eta_{r\nu})\times$$

$$\left[\frac{1}{4}+\frac{1}{3}\frac{\rho}{4m^2}+\frac{4}{15}\left(\frac{\rho}{4m^2}\right)^2\right]-$$

$$\frac{m^2}{4m^2+k^2-i0}(\eta_{\mu s}k_r k_\nu+\eta_{\mu\nu}k_r k_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$

$$\left.\frac{k^2-m^2}{12}+\frac{m^2}{4}+\frac{k^2-m^2}{30}\frac{\rho}{4m^2}-\frac{1}{6}k_\mu k_r k_\nu k_s\right\}+$$

$$i\frac{m^2\pi^2}{2}(\eta_{\mu r}k_\nu k_s+\eta_{\nu s}k_\mu k_r)\times$$
$$\left[\frac{1}{3}(\ln m^2+\frac{1}{12})+\frac{1}{5}\frac{\rho}{4m^2}\left(\ln m^2+\frac{13}{15}\right)\right]+$$
$$i\frac{m^2\pi^2}{30}(\eta_{\mu r}k_\nu k_s+\eta_{\nu s}k_\mu k_r)\frac{\rho}{4m^2}\times$$
$$\left[F\left(1,1,\frac{7}{2};-\frac{\rho}{4m^2}\right)+\frac{1}{7}F\left(1,1,\frac{9}{2};-\frac{\rho}{4m^2}\right)\right]+$$
$$-i\frac{\pi^2m^4}{4}(\eta_{\mu r}\eta_{\nu s}+\eta_{\mu\nu}\eta_{rs}+\eta_{\mu s}\eta_{\nu r})\times$$
$$\left\{\left[\frac{1}{2}-\frac{2}{3}\frac{\rho}{4m^2}-\frac{8}{15}\left(\frac{\rho}{4m^2}\right)^2+\right]\times\right.$$
$$\left.\left(\ln m^2+1\right)-\frac{1}{2}\left[\frac{3}{2}-\frac{1}{9}\left(\frac{\rho}{4m^2}\right)+\frac{52}{225}\left(\frac{\rho}{4m^2}\right)^2\right]\right\}-$$
$$i\frac{2\pi^2m^4}{105}(\eta_{\mu r}\eta_{\nu s}+\eta_{\mu\nu}\eta_{rs}+\eta_{\mu s}\eta_{\nu r})\left(\frac{\rho}{4m^2}\right)^3F\left(1,1,\frac{9}{2};-\frac{\rho}{4m^2}\right)-$$
$$i\frac{\pi^2m^2(k^2-m^2)}{12(4m^2+k^2-i0}(\eta_{\mu s}k_rk_\nu+\eta_{\mu\nu}k_rk_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$
$$\left[\frac{1}{2}\left(\ln m^2+\frac{1}{3}\right)+\frac{1}{5}\left(\ln m^2+\frac{5}{6}\right)\frac{k^2}{4m^2}\right]-$$
$$i\frac{\pi^2m^2}{8(4m^2+k^2-i0}(\eta_{\mu s}k_rk_\nu+\eta_{\mu\nu}k_rk_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$
$$m^2\left[\left(\ln m^2+\frac{2}{3}\right)+\frac{k^2}{12}+\frac{k^2}{30}\frac{k^2}{4m^2}\right]-\frac{i\pi^2m^4}{40(4m^2+k^2-i0)}\frac{k^2}{4m^2}-$$
$$i\frac{\pi^2m^2}{10}(\eta_{\mu s}k_rk_\nu+\eta_{\mu\nu}k_rk_s+\eta_{rs}k_\mu k_\nu+\eta_{r\nu}k_\mu k_s)\times$$
$$\frac{k^2-m^2}{21(4m^2+k^2-i0)}F\left(1,1,\frac{9}{2};-\frac{\rho}{4m^2}\right)\left(\frac{k^2}{4m^2}\right)^2-$$
$$i\frac{\pi^2}{12}k_\mu k_r k_\nu k_s\left[\left(\ln m^2+\frac{3}{4}\right)+\frac{k^2-4m^2}{2(4m^2+k^2-i0)}\right]-$$
$$i\frac{\pi^2m^2}{30}k_\mu k_r k_\nu k_s\frac{k^2-m^2}{4m^2+k^2-i0}\frac{k^2}{4m^2}F\left(1,1,\frac{7}{2};-\frac{k^2}{4m^2}\right)+$$

$$\sum_{n=0}^{\infty} a_n \lambda^n. \tag{14.3.1.7}$$

Again, the exact result for our four-dimensional convolution becomes

$$\Sigma_{GM\mu\nu rs}(k) = k_\mu k_r (\rho + m^2 - i0)^{-1} * k_\nu k_s (\rho + m^2 - i0)^{-1} =$$

$$i\frac{m^2\pi^2}{2}(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\times$$

$$\left[\frac{1}{3}(\ln m^2 + \frac{1}{12}) + \frac{1}{5}\frac{\rho}{4m^2}\left(\ln m^2 + \frac{13}{15}\right)\right] +$$

$$i\frac{m^2\pi^2}{30}(\eta_{\mu r}k_\nu k_s + \eta_{\nu s}k_\mu k_r)\frac{\rho}{4m^2}\times$$

$$\left[F\left(1, 1, \frac{7}{2}; -\frac{\rho}{4m^2}\right) + \frac{1}{7}F\left(1, 1, \frac{9}{2}; -\frac{\rho}{4m^2}\right)\right] +$$

$$-i\frac{\pi^2 m^4}{4}(\eta_{\mu r}\eta_{\nu s} + \eta_{\mu\nu}\eta_{rs} + \eta_{\mu s}\eta_{\nu r})\times$$

$$\left\{\left[\frac{1}{2} - \frac{2}{3}\frac{\rho}{4m^2} - \frac{8}{15}\left(\frac{\rho}{4m^2}\right)^2 +\right]\times\right.$$

$$\left.(\ln m^2 + 1) - \frac{1}{2}\left[\frac{3}{2} - \frac{1}{9}\left(\frac{\rho}{4m^2}\right) + \frac{52}{225}\left(\frac{\rho}{4m^2}\right)^2\right]\right\} -$$

$$i\frac{2\pi^2 m^4}{105}(\eta_{\mu r}\eta_{\nu s} + \eta_{\mu\nu}\eta_{rs} + \eta_{\mu s}\eta_{\nu r})\left(\frac{\rho}{4m^2}\right)^3 F\left(1, 1, \frac{9}{2}; -\frac{\rho}{4m^2}\right) -$$

$$i\frac{\pi^2 m^2 (k^2 - m^2)}{12(4m^2 + k^2 - i0}(\eta_{\mu s}k_r k_\nu + \eta_{\mu\nu}k_r k_s + \eta_{rs}k_\mu k_\nu + \eta_{r\nu}k_\mu k_s)\times$$

$$\left[\frac{1}{2}\left(\ln m^2 + \frac{1}{3}\right) + \frac{1}{5}\left(\ln m^2 + \frac{5}{6}\right)\frac{k^2}{4m^2}\right] -$$

$$i\frac{\pi^2 m^2}{8(4m^2 + k^2 - i0}(\eta_{\mu s}k_r k_\nu + \eta_{\mu\nu}k_r k_s + \eta_{rs}k_\mu k_\nu + \eta_{r\nu}k_\mu k_s)\times$$

$$m^2\left[\left(\ln m^2 + \frac{2}{3}\right) + \frac{k^2}{12} + \frac{k^2}{30}\frac{k^2}{4m^2}\right] - \frac{i\pi^2 m^4}{40(4m^2 + k^2 - i0)}\frac{k^2}{4m^2} -$$

$$i\frac{\pi^2 m^2}{10}(\eta_{\mu s}k_r k_\nu + \eta_{\mu\nu}k_r k_s + \eta_{rs}k_\mu k_\nu + \eta_{r\nu}k_\mu k_s)\times$$

$$\frac{k^2 - m^2}{21(4m^2 + k^2 - i0)} F\left(1, 1, \frac{9}{2}; -\frac{\rho}{4m^2}\right)\left(\frac{k^2}{4m^2}\right)^2 -$$
$$i\frac{\pi^2}{12} k_\mu k_r k_\nu k_s \left[\left(\ln m^2 + \frac{3}{4}\right) + \frac{k^2 - 4m^2}{2(4m^2 + k^2 - i0)}\right] -$$
$$i\frac{\pi^2 m^2}{30} k_\mu k_r k_\nu k_s \frac{k^2 - m^2}{4m^2 + k^2 - i0} \frac{k^2}{4m^2} F\left(1, 1, \frac{7}{2}; -\frac{k^2}{4m^2}\right) \quad (14.3.1.8)$$

We have to deal with 9 diagrams of this kind.
Accordingly, our desired self-energy total is a combination of $\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k)$ and $\Sigma_{GM\alpha_1\alpha_2\alpha_3\alpha_4}(k)$.

14.4 Self Energy of the Axion

Here, a typical term of the self-energy is

$$\Sigma_{\nu r}(k) = k_\nu k_r(\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}. \quad (14.4.0.1)$$

In four dimensions one has

$$k_\nu k_r(\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}] = \int \frac{k_\nu k_r}{(k^2 + m^2 - i0)[(p - k)^2 - i0]} d^4k. \quad (14.4.0.2)$$

With the Feynman parameters used above we obtain

$$k_\nu k_r(\rho + m^2 - i0)^{\lambda-1} * (\rho - i0)^{\lambda-1} =$$
$$i\int_0^1 x^{-\lambda}(1 - x)^{-\lambda} \int \frac{k_\nu k_r}{[(k - px)^2 + a]^{2-\lambda}} d^4k dx, \quad (14.4.0.3)$$

where

$$a = (p^2 + m^2)x - p^2x^2. \quad (14.4.0.4)$$

We evaluate the integral (14.4.0.3) and find

$$k_\nu k_r(\rho + m^2 - i0)^{\lambda-1} * (\rho - i0)^{\lambda-1} =$$
$$i\frac{\eta_{\nu r} m^{2+4\lambda}\pi^2}{4} \frac{\Gamma(2 + \lambda)}{\Gamma(1 - \lambda} \Gamma(-1 - 2\lambda) F\left(-1 - 2\lambda, 1 - \lambda, 3; -\frac{\rho}{m^2}\right) +$$
$$\frac{ik_\nu k_r m^{4\lambda}\pi^2}{6} \frac{\Gamma(3 + \lambda)}{\Gamma(1 - \lambda} \Gamma(-2\lambda) F\left(-2\lambda, 1 - \lambda, 4; -\frac{\rho}{m^2}\right). \quad (14.4.0.5)$$

14.4.1 Self-energy evaluation for $\lambda = 0$

Once again, we Laurent-expand, this time (14.4.0.5) around $\lambda = 0$, encountering

$$[k_\nu k_r(\rho + m^2 - i0)^{\lambda-1} * (\rho - i0)^{\lambda-1} =$$

$$i\pi^2 \left\{ \frac{1}{2\lambda} \left(\frac{\eta_{\nu r} m^2}{4} - \frac{1}{3} k_\nu k_r \right) + \right.$$

$$\frac{\eta_{\nu r} m^2}{4} \left[\left(1 + \frac{1}{3} \frac{\rho}{m^2} \right) \left(\ln m^2 + \frac{1}{2} \right) - \right.$$

$$\left. \left(1 + \frac{1}{6} \frac{\rho}{m^2} \right) \right] - \frac{k_\nu k_r}{3} \left(\ln m^2 + \frac{3}{4} \right) +$$

$$\frac{1}{4} \left(\frac{\rho}{m^2} \right) \left[\frac{\eta_{\nu r} m^2}{12} \frac{\rho}{m^2} - \frac{k_\nu k_r}{3} \right] F\left(1, 1, 5; -\frac{\rho}{m^2} \right) +$$

$$\left. \sum_{n=1}^{\infty} a_n \lambda^n \right\}. \tag{14.4.1.6}$$

The λ-independent term gives the exact convolution result we are looking for

$$\Sigma_{\nu r}(k) = [k_\nu k_r(\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}] =$$

$$i\pi^2 \left\{ \frac{\eta_{\nu r} m^2}{4} \left[\left(1 + \frac{1}{3} \frac{\rho}{m^2} \right) \left(\ln m^2 + \frac{1}{2} \right) \right] - \right.$$

$$\left(1 + \frac{1}{6} \frac{\rho}{m^2} \right) \right] - \frac{k_\nu k_r}{3} \left(\ln m^2 + \frac{3}{4} \right) +$$

$$\left. \frac{1}{4} \left(\frac{\rho}{m^2} \right) \left[\frac{\eta_{\nu r} m^2}{12} \frac{\rho}{m^2} - \frac{k_\nu k_r}{3} \right] F\left(1, 1, 5; -\frac{\rho}{m^2} \right) \right\} \tag{14.4.1.7}$$

14.5 Discussion

We have developed above a quantum field theory (QFT) of Eintein's gravity (EG), that is both unitary and finite, by appeal to the Schwinger-Feyman variational principle. We emphatically avoid the functional integral method. Our results critically depend on the use of a rather novel constraint the we introduced in defining the EG-Lagrangian. Laurent expansions were also an indispensable tool for us.

As stated above, in order to quantify the theory we appealed to the variational principle of Schwinger-Feynman's. This process leads to just one graviton type ϕ^{12}.

We have evaluated here in finite and exact fashion, for the first time as far as we know, several quantities:

- the graviton's self-energy in the EG-field. This requires full use of the theory of distributions, appealing to the possibility of creating with them a ring with divisors of zero.
- the above self-energy in the added presence of a massive scalar field (axions, for instance). Two types of diagram ensue: the original ones of the pure EG field plus the ones originated by the addition of a scalar field.
- The axion's self-energy.

Chapter 15

Epilogue

In this book we discussed and solved the overarching dilemma of defining the product of two distributions (a product in a ring with divisors of zero), which is an old problem of functional analysis. All infinities in quantum field theory (QFT) can be traced back to such products [113]. In (QFT), when we use perturbative expansions, we need to deal with products of distributions in configuration space or, what is the same, with convolutions of distributions in momentum space. We have here concentrated efforts on the convolution of Sebastiao e Silva ultradistributions, that allow us to treat non- renormalizable QFTs, but has the disadvantage of being extremely involved. We proposed and illustrated a simpler way of dealing with them.

We appealed for this purpose to the convolution of Lorentz invariant tempered distributions, using an extension of the dimensional regularization (DR) of Bollini and Giambiagi. With this convolution we have obtained above, for example, the convolution of n massless Feynman propagators both in Minkowskian and Euclidean spaces and the convolution of two massless Wheeler propagators, all of them original results at the time of their publication.

As a final step of this book we told the reader about the Non-relativistic quantum field theory of Newton's gravity and the quantum field theory of Einstein's gravity. Both theories turn out to be finite ones, a rather important achievement.

Chapter 16

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