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SYSTEMS RELIABILITY ENGINEERING

MODELING AND PERFORMANCE IMPROVEMENT

Edited by Amit Kumar and Mangey Ram

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Systems Reliability Engineering

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Preface

The term “reliability” is often used by researchers and academicians having a keen interest in performance evaluation and improvement of various industrial as well as other systems in terms of functioning of the same. Many times the same can be done by converting the real-time system in a feasible mathematical model and then investigate it by different techniques and methods for optimal suggestions regarding the improvement of the system. An add-on redundancy also plays a crucial role in the improvement of various performance measures of the system, for example, reliability, availability, and excepted number of failures. This book focused on these aspects of different industrial systems associated with the system’s reliability and provides a concrete knowledge linked with the same, which will be very useful for researchers, academicians, and also for the concerned industry.

The book consists of 11 chapters. Chapter 1 presents exact approaches in terms of minimal cuts and minimal paths to assess the reliability of a multistate flow network. Chapter 2 investigates a rice manufacturing plant under fuzzy environment through lambda–tau methodology and obtains mean time to failure, mean time to returns, mean time between failure, reliability, and availability for the same. Chapter 3 investigates the reliability characteristics of well-known structures with two components. Different bivariate distribution models, such as the Freund’s family and the Gumbel Type III family, shed light on several reliability attributes of series and parallel systems. Also closed formulae for evaluating the reliability function, the mean residual lifetime, or the failure rate are derived. Chapter 4 describes the factors influencing the reliability of the equipment/system including intrinsic and extrinsic factors. It includes the types of reliability and how we can make improvements in each case. Chapter 5 investigates a system of two nonidentical units with the concept of mathematical modeling along with degradation, fault detection, inspection, and replacement policy by incorporating the concept of repairmen. Chapter 6 gives a brief description of models with varying demand and comparison between the models taken two at a time. It also analyzed how varying demand affects the reliability in a single-unit and two-unit redundant systems. Chapter 7 investigates a gas production unit in a yarn plant with equipment renewal policy by using mathematical modeling, Markov process, and supplementary variable technique to find various performance measures associated with the same. The investigation of a paper mill plant, which is handled by a human operator, through reworking/degradation of its components has been discussed in Chapter 8. In this chapter, the authors implemented the concept of mathematical modeling and Markov process and evaluated the different reliability characteristics of the paper mill plant. Chapter 9 discusses the design and evaluation of coherent redundant system reliability. Chapter10 discusses the concept of mathematical modeling for a delayed innovation diffusion model with media coverage. Shuffle exchange networks have been analyzed in Chapter 11. In this chapter, the authors have used the concept

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of using the universal generating function and find the reliability and signature indices for the same.

Hence, this book will be very much helpful and can be considered as a support book for many undergraduate courses, for example, mechanical, mechatronics, industrial, computer science, information technology, and reliability. The same can be considered as an elite reference for researchers/scholars from the above-mentioned domains.

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About the editors

Dr. Amit Kumar is working as an assistant professor in the Department of Mathematics at Lovely Professional University, Punjab, India. He has taught several core courses in pure and applied mathematics at undergraduate and postgraduate levels. He has done his bachelor's and master's degrees from Chaudhary Charan Singh University, Meerut, India, in 2006 and 2009, respectively. In 2016, he completed his doctorate in applied mathematics from Graphic Era (Deemed to be University), Dehradun, Uttarakhand, India, in the field of Reliability theory. He has published several research papers/books/book chapters in various esteemed international journals/books in Taylor & Francis, Springer, Emerald, World Scientific, InderScience, and many other national and international journals of repute and also presented his works at national and international conferences. He is a reviewer of many international journals published in Elsevier, Springer, Emerald, John Wiley, Taylor & Francis, and many more. His fields of research are operations research, reliability theory, fuzzy reliability, and system engineering. He is a lifetime member of the Indian Science Congress. He received the *research appreciation award,* for his research contribution, for the year 2017 from Lovely Professional University, Punjab, India.

Dr. Mangey Ram received his Ph.D. major in mathematics and minor in computer science from G. B. Pant University of Agriculture and Technology, Pantnagar, India. He has been a faculty member for around 12 years and has taught several core courses in pure and applied mathematics at undergraduate, postgraduate, and doctorate levels. He is currently a research professor at Graphic Era (Deemed to be University), Dehradun, India. Before joining the Graphic Era, he was a deputy manager (probationary officer) with Syndicate Bank for a short period. He is editor in chief of the *International Journal of Mathematical, Engineering and Management Sciences* and the guest editor and member of the editorial board of various journals. He is a regular reviewer for international journals in IEEE, Elsevier, Springer, Emerald, John Wiley, Taylor & Francis, and many other publishers. He has published 200 plus research publications in IEEE, Taylor & Francis, Springer, Elsevier, Emerald, World Scientific, and many other national and international journals of repute and also presented his works at national and international conferences. His fields of research are reliability theory and applied mathematics. Dr. Ram is a senior member of the IEEE, life member of Operational Research Society of India, Society for Reliability Engineering, Quality and Operations Management in India, Indian Society of Industrial and Applied Mathematics, member of International Association of Engineers in Hong Kong, and Emerald Literati Network in the UK. He has been a member of the organizing committee of a number of international and national conferences, seminars, and workshops. He has been conferred with "Young Scientist Award" by the Uttarakhand State Council for Science and Technology, Dehradun, in 2009. He has been awarded the "Best Faculty Award" in 2011; "Research Excellence Award" in 2015; and recently the "Outstanding Researcher Award" in 2018 for his significant contribution to academics and research at Graphic Era Deemed to be University, Dehradun, India.

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1 Exact reliability evaluation of multistate flow networks

Abstract: The problem of reliability evaluation of multistate flow networks (MFN) has been very attractive over the past decades, and many researchers from various fields of research have studied this problem. Generally, the proposed approaches in the literature are based on minimal cuts (MC) and minimal paths (MP), including the approximate and exact algorithms. This chapter states and discusses some important exact approaches that are in terms of MCs and MPs. To this end, it is first explained how the system reliability can be computed in terms of MCs and MPs. Then the MC-based and MP-based approaches are discussed in detail in separate sections. To have a more comprehensive study, the complexity results of the stated algorithms are also provided.

Keywords: Multistate flow networks, reliability, minimal cuts, minimal paths, d-MC, d-MP

1.1 Introduction

Multistate flow networks (MFNs) have been applied in various fields such as social networks [1], power systems [2], communication networks [3], satellite systems [4], manufacturing systems [5], and transportation networks [6]. This has made the problem of assessing the reliability of such networks to be of great importance and has been very attractive over the past decades [7, 8]. This chapter discusses some exact approaches based on minimal cuts (MCs) and minimal paths (MPs) to assess the reliability of such networks. In fact, these approaches are known as the solutions of d-MC and d-MP problems. First, some preliminaries on MFNs are stated in Section 1.2. Then Section 1.3 describes how the reliability of an MFN for a given demand level d is evaluated. Section 1.4 states the d-MC problem along with some solutions and complexity results. Afterward, the d-MP problem along with certain solutions and its computational complexity are discussed in Section 1.5. Ultimately, the conclusions are given in Section 1.6.

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1.2 Preliminaries

Here, some of the required notations, nomenclature, and definitions are stated. More details on each problem are given in the related section.

1.2.1 Notations, nomenclature, and assumptions

G	$G(N, A, M)$ is a *multistate network flow* with the set of nodes $N = \{1, 2, \ldots, n\}$, the set of arcs $A = \{a_1, a_2, \ldots, a_m\}$, and maximum capacity vector $M = (M_1, M_2, \ldots, M_m)$ in which M_r denotes the maximum capacity of arc a_r, for $r = 1, 2, \ldots, m$. Hence, n and m are, respectively, the number of nodes and arcs in the network. Moreover, nodes 1 and n in the network are, respectively, the source and destination (sink) nodes.
X	$= (x_1, x_2, \ldots, x_m)$ is the current system state vector in which $x_r \in \{0, 1, \ldots, M_r\}$, for $r = 1, 2, \ldots, m$, denotes the current capacity of arc a_r being a random integer-valued number. We note that M is a state vector by itself, and all the other state vectors are less than M.
$V(X)$	is the maximum flow of the network under X, which is the maximum amount of commodity (flow or data), which can be sent from source node, which is node 1, to sink node, which is node n, through the network under X.
$U(X)$	$= \{a_r \in A \| x_r < M_r\}$ is the set of all the unsaturated arcs under X.
$Z(X)$	$= \{a_i \| x_i \rangle 0\}$ is the set of arcs with positive capacity under X.
e_r	This is a special system state vector in which the rth component is 1 and the others are zero.
U_r	This is a system state vector in which the rth component is 0 and the lth component is M_l, for $l = 1, \ldots, m$, $l \neq r$. In fact, $U_r = M - M_r e_r$.
C_i	This is the ith MC, for $i = 1, \ldots, q$. So, q is the number of MCs.
$CC_i(X)$	$= \sum_{r: a_r \in C_i} x_r$ is the capacity of MC C_i under system state vector X. For simplicity, the capacity of C_i under M, which is its maximum capacity, is denoted by CC_i instead of $CC_i(M)$.
P_j	This is the jth MP, for $j = 1, \ldots, p$. So, p is the number of MPs.
$CP_j(X)$	$= \min\{x_r \| a_r \in P_j\}$ is the capacity of MP P_j under system state vector X. For simplicity, the capacity of P_j under M, which is its maximum capacity, is denoted by CP_j instead of $CP_j(M)$.
F	$= (f_1, f_2, \ldots, f_p)$ is a feasible flow vector in which f_j signifies the amount of flow on the MP P_j, for $j = 1, \ldots, p$..
d	This is a nonnegative integer-valued number that denotes the demanded units of commodity, which should be transmitted from the source node to the destination node through the network. We note that $d \leq V(M)$.

(continued)

R_d	This is the system reliability of the network at demand level of d which is the probability of successfully sending a minimum of d units of commodity from node 1 to node n. In other words, $R_d = \Pr\{X \mid V(X) \geq d\}$.
η	This is an upper bound for the number of all the obtainable d-MC candidates from every MC.
γ	This is the number of all the obtained d-MCs (counting the duplicates) from all the MCs.
β	This is the number of all the d-FVs (see Definition 4).
ζ	This is the number of all the obtained d-MPs (counting the duplicates) from all the MPs.

To have a better understanding of the notations, let us review some of stated notations considering the network depicted in Figure 1.1. In fact, Figure 1.1 is an MFN with $N = \{1,2,3,4,5,6\}$ and $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$. In this network, nodes 1 and 6 are the source and destination, respectively. Now, considering $M = (6,8,2,4,4,4,3,7,7)$, we have for example $M_1 = 6,$ which shows that not more than six units of commodity (data or flow) can be sent via arc a_1. Any nonnegative vector $X \leq M$, for example, $X^1 = (6,3,0,2,3,4,0,5,4)$ can be considered as a current system state vector for the given network in Figure 1.1. We note that $V(X^1) = 9$, $U(X^1) = \{a_2, a_3, a_4, a_5, a_7, a_8, a_9\}$, and $Z(X^1) = \{a_1, a_2, a_4, a_5, a_6, a_8, a_9\}$. Moreover, we have $e_1 = (1,0,0,0,0,0,0,0,0)$ and $U_1 = M - M_1 e_1 = (0,8,2,4,4,4,3,7,7)$ in this network.

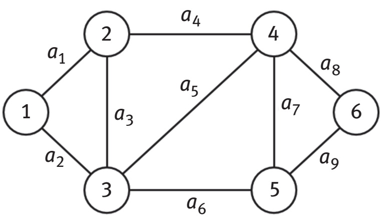

Figure 1.1: A benchmark network example with six nodes.

A *cut* is a subset of A after elimination of which there is no path from the source to the destination in G. An *MC* is a cut that is not cut anymore if any arc is removed from it. For instance, $C = \{a_1, a_2, a_3\}$ is a cut for the given network in Figure 1.1 but not an MC while $C_1 = \{a_1, a_2\}$ is an MC. The capacity of C_1 under X^1 is equal to $CC_1(X^1) = x_1 + x_2 = 6 + 3 = 9$, and its maximum capacity is $CC_1 = M_1 + M_2 = 6 + 8 = 14$. A *path* is a set of adjacent arcs from the source to the destination. An *MP* is a path whose proper subsets are not paths anymore. For instance, $P = \{a_1, a_4, a_5, a_6, a_7, a_8\}$ is a path for the given network in Figure 1.1 but not an MP while $P_1 = \{a_1, a_4, a_8\}$ is an MP. The capacity of P_1 under X^1 is $CP_1(X^1) = \min\{x_1, x_4, x_8\} = \min\{6,2,5\} = 2$, and its maximum capacity is $CP_1 = \min\{M_1, M_4, M_8\} = \min\{6,4,7\} = 4$.

It is said that $X=(x_1,x_2,\ldots,x_m)\le Y=(y_1,y_2,\ldots,y_m)$ if $x_i\le y_i$, for $i=1,\ldots,m$. Moreover, it is said that $X<Y$ if $X\le Y$ and $x_j<y_j$ for at least one $j=1,\ldots,m$. A vector X is called a *minimal* (*maximal*) *vector* in set Ψ if there is no other vector, say Y, in this set so that $Y<X$ ($Y>X$). The set of all the minimal (maximal) vectors in Ψ is denoted by $\Psi_{\min}$ ($\Psi_{\max}$). For instance, assuming $\Psi=\{(2,3,4),(3,2,4),(1,2,3),(3,2,1),(1,4,3),(1,5,1)\}$, we have $\Psi_{\min}=\{(1,2,3),(3,2,1),(1,5,1)\}$ and $\Psi_{\max}=\{(2,3,4),(3,2,4),(1,4,3),(1,5,1)\}$. In fact, as it is seen that, for a vector being a minimal (maximal) vector in a set of vectors, it is not necessary to be less (greater) than all the other vectors, rather it is enough to be not greater (less) than any other vector in the desired set. Hence, it is possible that a vector could belong to both sets of minimal vectors and maximal vectors simultaneously.

The following assumptions are considered throughout this chapter:
1. The capacity of each arc $a_r\in A$ is modeled as a nonnegative integer-valued random number not greater than M_r, for any $r=1,2,\ldots,m$.
2. The capacities of different arcs are statistically independent.
3. Flow in the network obeys the conservation law [9].
4. All the nodes are perfectly reliable, namely, deterministic.

1.3 System reliability

It is reminded that the *reliability* of network G at a demand level of d is defined as the probability that one can transmit at least d units of commodity or flow from node 1 to node n through the network, that is, $R_d=\Pr\{X|V(X)\ge d\}$. Therefore, one needs to determine the set $\{X|V(X)\ge d\}$ of the system state vectors in order to calculate R_d. However, determining such a set of vectors directly by checking all the system state vectors is too computationally expensive and somehow impractical. Rather, one can first find $\Psi=\{X|V(X)=d\}$ and determine $\Psi_{\min}$. Then letting $\Psi_{\min}=\{X^1,X^2,\ldots,X^\sigma\}$ and $A_r=\{X|X\ge X^r\}$, for $r=1,2,\ldots,\sigma$, it can be simply proven that $\cup_{r=1}^{\sigma}A_r=\{X|V(X)\ge d\}$. Note that according to the definition of $\Psi_{\min}$, if Y is a system state vector which is less than X^r, for every $r=1,2,\ldots,\sigma$, then it does not belong to Ψ and $V(Y)<d$. Thus, $R_d=\Pr\cup_{r=1}^{\sigma}A_r$, which can be calculated by applying the inclusion–exclusion method as follows [11]:

$$R_d=\Pr\bigcup_{r=1}^{\sigma}A_r=\sum_{r=1}^{\sigma}\Pr(A_r)-\sum_{l=2}^{\sigma}\sum_{r=1}^{l-1}\Pr(A_r\cap A_l)+\cdots+(-1)^{\sigma+1}\Pr\left(\bigcap_{r=1}^{\sigma}A_r\right),\qquad(1.1)$$

where $\Pr(A_r)=\sum_{X\in A_r}\Pr(X)$ and $\Pr(X)=\prod_{i=1}^{m}\Pr(x_i)$. Equation (1.1) states how the exact amount of system reliability can be computed with the set $\Psi_{\min}$ at hand. We note that the system reliability can be approximated by removing some of the last terms in this equation. Hence, finding the set $\Psi_{\min}$ is of great importance. However, determination of such a set of vectors directly by checking all the system state vectors is almost impossible in large enough networks. To address this problem, that is, finding $\{X|V(X) \geq d\}$, several practical methods in terms of MCs and MPs have been presented in the literature, which are discussed in the next sections.

1.4 MC-based approaches

Here, several results are stated and discussed based on which some approaches are provided to determine the set $\Psi = \{X|V(X) \geq d\}$. It is noted that this section is mainly based on [11–15, 17] from the literature. To this end, let us start with a definition.

Definition 1. A system state vector X is called a d-MC iff $V(X) = d$, and $V(X + e_r) = d + 1$, for any $r = 1, 2, \ldots, m$ with $a_r \in U(X)$.

For example, let $d = 10$ be a demand level for the network depicted in Figure 1.2 with $M = (5, 4, 6, 4, 3, 6)$ and $X = (5, 4, 1, 4, 3, 6)$. It can be simply seen that $V(X) = 10$, $U(X) = \{a_3\}$, and $V(X + e_3) = 11 > 10$. Hence, X is a $10-\mathrm{MC}$. In fact, there are several $10-\mathrm{MCs}$ for this network such as $(5, 3, 2, 4, 3, 6)$, $(5, 2, 3, 4, 3, 6)$, $(5, 1, 4, 4, 3, 6)$, $(5, 0, 5, 4, 3, 6)$, and $(4, 4, 2, 4, 3, 6)$. In this network, there are 29,400 system state vectors, and so even in such a small network it is unrealistic to check all the system state vectors for determining all the d-MCs for any given demand level of d. In order to address such a problem, Jane et al. [14] defined a new notion called d-MC candidate. A system state vector $X = (x_1, x_2, \ldots, x_m)$ is called a d-MC candidate if there exists at least one MC, say C_i, satisfying the following system:

$$\begin{cases} \text{(I)}\ CC_i(X) = d, \\ \text{(II)}\ 0 \leq x_r \leq M_r \quad \forall a_r \in C_i, \\ \text{(III)}\ x_l = M_l \qquad \forall a_l \notin C_i. \end{cases} \tag{1.2}$$

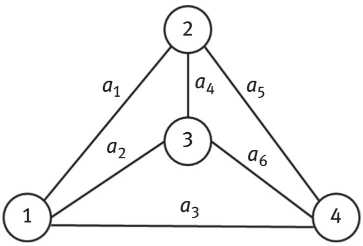

Figure 1.2: A benchmark network example with four nodes.

Jane et al. [14] then proved that each d-MC is a d-MC candidate, and accordingly for finding all the d-MCs, it is enough to determine all the candidates first, and thereafter check each candidate separately for being a d-MC.

Lemma 1. *All the d-MCs are d-MC candidates.*

Example 1 It can be observed that the obtained 10 – MC candidates in the example above contain all the producible d-MCs from MC C_1, which confirms Lemma 1. According to this lemma, the general stages for determining all the d-MCs can be considered as (i) to solve system (1.2) for all the MCs in order to determine all the d-MC candidates and (ii) to check each candidate to be a d-MC. Several different techniques have been presented in the literature for the later stage. For instance, based on the definition, one can use a maximum flow algorithm [9, 16] to check each candidate, say X, for (1) $V(X) = d$ as well as (2) $V(X + e_r) = d + 1$, for any $r = 1, 2, \ldots, m$ with $a_r \in U(X)$. This technique has been used in several algorithms [11, 12]. The main steps of such algorithms are stated further.

Algorithm 1 (determination of all the d-MCs in an MFN)

Step 1. Solve system (1.2) for all MCs for determination of all the candidates. Let Λ be the set of all the obtained solutions.

Step 2. Choose an $X \in \Lambda$ and remove it from Λ. If there is no $X \in \Lambda$, then go to Step 3.
2.1. If $V(X) \neq d$, then it is not a d-MC and go to Step 2.
2.2. If $V(X + e_r) = d$ for some r with $a_r \in U(X)$, then it is not a d-MC and go to Step 2.
2.3. Add X to the set Λ_{sol} which is the set of d-MCs and go to Step 2.

Step 3. Remove the duplicate vectors from set Λ_{sol} and stop. Λ_{sol} is the set of all the solutions.

Another approach that arises from the following result has been proposed in [14].

Theorem 1. *Suppose that Λ is the set of d-MC candidates. Then $\Lambda_{max} = \{X \in \Lambda | X$ is a maximal vector$\}$, which is the set of all the d-MCs.*

Proof. Supposing that X is a d-MC, Lemma 1 guarantees that it is a d-*MC* candidate. Contrariwise, suppose that X is not a maximal vector among all the candidates, and thus there exists at least one d-MC candidate that is greater than X, say $Y > X$. Since Y is a d-MC candidate, we have $V(Y) \leq d$ and as $X < Y$, there should be at least one $r \in \{1, \ldots, m\}$ so that $x_r < y_r$. Consequently, $X + e_r \leq Y$, and therefore $V(X + e_r) \leq V(Y) \leq d$ contradicting X to be a d-MC. Hence, X should be a maximal vector. Now, to prove the other side, assume that X is a maximal vector among all the d-MC candidates. As it is a candidate, we have $V(X) \leq d$. If $V(X) = d_1 < d$, then there exists a candidate $Y > X$ with $V(Y) = d_1 + 1 \leq d$, which contradicts X being a maximal vector, and hence $V(X) = d$. Now, let a_r be an arbitrary unsaturated arc under X, that is, $a_r \in U(X)$ or $x_r < M_r$ and that $X' = X + e_r$. If $V\left(X'\right) = d$, then as $X' > X$, it is a contradiction to X being a maximal vector. Thus, $V\left(X'\right) > d$, which completes the proof.
The main steps of such algorithms are as follows.

Algorithm 2 (determination of all d-MCs in an MFN)

Step 1. Determine all the candidates by solving system (1.2) for all MCs. Let Λ be the set of obtained candidates.

Step 2. Compare all the candidates to remove the duplicates and nonmaximal vectors from Λ in order to obtain Λ_{max} which is the set of all the d-MCs.

Theorem 1 shows that the sets Λ_{max} in Algorithm 2 and Λ_{sol} in Algorithm 1 are equal. Removing the nonmaximal d-MC candidates is a very time-consuming step as one needs to compare all the candidates, which are m – tuple vectors. This shows that Algorithm 2 is not so practical for large enough networks, and in fact it is less efficient than Algorithm 1. Now, let us turn our attention to Algorithm 1 and focus on improving it.

First of all, we note that even by using the max-flow algorithm, checking every candidate to be a d-MC is costly, and thus one can decrease the work in the algorithm by determining some d-MCs without using the checking process. The following result from [11] helps in this respect.

Theorem 2. *If $CC_i = V(M)$, then all the obtainable candidate from MC C_i are d-MCs.*

Proof. It is first reminded that $CC_i = CC_i(M)$ is the capacity of MC C_i under M. Now, supposing X as an arbitrary producible d-MC candidate from C_i, we know $CC_i(X) = d$ from the definition, and $V(X) \leq d$ according to *max-flow min-cut theorem* [9]. To demonstrate that $V(X) = d$, on the contrary, assume that $V(X) < d$. Therefore, there should be another MC $C_j(\neq C_i)$ such that $CC_j(X) < d$. Since $x_r = M_r$ for any r with $a_r \notin U(X)$, then we have

$$\sum_{a_r \in U(X)} (M_r - x_r) = CC_i(M) - CC_i(X) = V(M) - d$$

Thus, we have

$$CC_j(M) = CC_j(X) + \sum_{r: a_r \in U(X) \cap C_j} (M_r - x_r) \leq$$

$$CC_j(X) + \sum_{r: a_r \in U(X)} (M_r - x_r) < d + V(M) - d = V(M)$$

with the specification $CC_j(M) < V(M)$ which contradicts the max-flow min-cut theorem. Hence, $V(X) = d$. Now, it is easy to see that for any r with $x_r < M_r$, $X' = X + e_r$ is a $(d + 1)$-MC candidate obtained from C_i. Similarly, by replacing d with $d + 1$, it is concluded that $V(X + e_i) = d + 1 > d$. This completes the proof.

For example, as $CC_1(M) = CC_4(M) = 15 = V(M)$ in the network depicted in Figure 1.2, every producible d-MC from these two MCs is always a *d-MC* without the need for being checked. This way, one can conclude that all the 20 mentioned as 10 – MC candidates in Example 1 are indeed 10 – MCs. Moreover, there are eighteen 10 – MC candidates from C_4 which are indeed 10 – MCs according to Theorem 2. This example shows clearly how usage of this theorem can decrease the work in the algorithm.

As the number of all the candidates in a general MFN is quite greater than the number of all the d-MCs, another place for improving the solution is to propose certain techniques for lessening the number of candidates without loss of any d-MC. For instance, one can increase some of lower bounds in system (1.2), inequalities (II), which may lead to decrease the solutions for the system. Theorem 3 is useful in this regard and can be deduced from Theorem 1 in [15] or Theorem 2 in [19].

Theorem 3. *Let* $l_r = max\{0, d - V(U_r)\}$, *for* $r = 1, \ldots, m$, $L = (l_1, l_2, \ldots, l_m)$, *and* X *be an arbitrary* d*-MC. Then* $X \geq L$.

Proof. Assuming $X = (x_1, \ldots, x_m)$ as a d-MC, we need to show that $x_r \geq l_r$, for any $r = 1, 2, \ldots, m$. Let us remind that $U_r = M - M_r e_r$. If $V(U_r) \geq d$, then $l_r = 0$, and there is nothing to prove. Now, assume that $V(U_r) < d$, for some $r = 1, \ldots, m$. Thus, by removing arc a_r from the network, the maximum flow from nodes 1 to n is $V(U_r)$. As $V(U_r) < d \leq V(M)$, one observes that for transmitting d units of flow from nodes 1 to n (source to destination), a minimum of $l_r = d - V(U_r)$ units must be sent on arc a_r. Hence, $x_r \geq l_r = d - V(U_r)$, and consequently $X \geq L$, which completes the proof.

For instance, by applying Theorem 3 on the network depicted in Figure 1.2 for $d = 10$ and $M = (5, 4, 6, 4, 3, 6)$, we have $l_1 = \max\{0, d - V(U_1)\} = 0$, and similarly $l_2 = l_4 = l_5 = 0$ and $l_3 = l_6 = 1$. We note that the lower bounds may change due to the change of d. For example, by changing $d = 10$ to $d = 12$, we have $l_1 = 2$, $l_2 = 1$, $l_3 = 3$, $l_4 = 0$, $l_5 = 0$, and $l_6 = 3$. Therefore, it is observed that this theorem can play a helpful role in lessening the amount of candidates.

Let us now turn our attention to another notable point which is the possibility of generating duplicate candidates from different MCs [11, 15]. For instance, in the network given in Figure 1.2, $X = (5, 4, 1, 4, 3, 6)$ is a 10-MC which can be obtained from both $C_1 = \{a_1, a_2, a_3\}$ and $C_4 = \{a_3, a_5, a_6\}$. Therefore, after determination of all the d-MCs, it is necessary to remove the duplicates, which is an expensive stage (see Step 3 in Algorithm 1 and Step 2 in Algorithm 2). In fact, to remove the duplicates, one needs to compare all the d-MCs, which are m – tuple vectors, and hence this would be an expensive stage for large enough networks. Theorem 4, taken from [15], provides an approach to efficiently remove the duplicate candidates.

Theorem 4. *Let* C_r *and* C_j *be two different MCs. Any* d*-MC obtained from* C_r*, say* X*, can be also obtained from* C_j *if and only if* $CC_r = CC_j$ *and* $U(X) \subseteq C_j$.

To see how one can apply this theorem, let X be a candidate produced from C_r. Now, to avoid generating this candidate from other MCs, one can check the conditions $CC_r = CC_j$ and $U(X) \subseteq C_j$ for MC C_j, for any $j = 1, \ldots, r - 1$. This work even can be reduced by arranging MCs ascending due to CC_j values, and then grouping the MCs by putting the MCs with the same capacities to the same group. For instance, in the network depicted in Figure 1.2 with $M = (5, 4, 6, 4, 3, 6)$, it can be calculated that $CC_1 = 5 + 4 + 6 = 15$, $CC_2 = 21$, $CC_3 = 17$, and $CC_4 = 15$, which lead to have three groups $B_1 = \{C_1, C_4\}$, $B_2 = \{C_2\}$, and $B_3 = \{C_3\}$. Applying Theorem 4 in this example shows that the duplicate candidates can only be produced from C_4, and hence there is no need to check the obtained candidates from other MCs to detect the duplicates. For instance, as we mentioned earlier, $X = (5, 4, 1, 4, 3, 6)$ is a 10-MC, which can be obtained from both C_1 and C_4. In fact, applying this theorem, when such an X is obtained from C_4, it is considered as a duplicate one and is removed. This way, no more duplicate candidates are generated, and hence it is not necessary to compare all the solutions at the end. Niu et al. [15] showed that the time complexity of using this theorem in the algorithm is less than the time complexity of comparing all the solutions.

As it has been explained, removing all the nonmaximal candidates to obtain the set of all the maximal vectors is really time-consuming, and indeed it is somehow impractical for large enough networks. Therefore, Algorithm 1 is more efficient than Algorithm 2 in general, and consequently we employ all the preceding results on Algorithm 1 for providing a more efficient solution, Algorithm 3, to the d-MC problem.

Algorithm 3 (determination of all the d-MCs in an MFN.)
Input: An MFN $G(N, A, M)$, all MCs, and a demand level d.
Output: All the d-MCs.
Step 1. Let $D = \phi$, $i = 1$, $t = 1$, and q be the number of MCs. Then compute l_r, for $r = 1, \ldots, m$.
Step 2. Compute CC_k, for $k = 1, 2, \ldots, q$, rearrange the MCs ascending due to the values of CC_k, for $k = 1, \ldots, q$, as $C'_1, \ldots, C'_q$, and group them in α groups $B_1, \ldots, B_\alpha$ such that MCs with the same capacities belong to the same group.
Step 3. Find a solution, say X, of the following system:

$$\begin{cases} \text{(I)} \;\; CC_i(X) = d, & \\ \text{(II)} \;\; l_r \le x_r \le M_r & \forall a_r \in C_i, \\ \text{(III)} \;\; x_l = M_l & \forall a_l \notin C_i. \end{cases}$$

If there is no solution, then pass to Step 8.
Step 4. If $CC_i = V(M)$, then pass to Step 6.
Step 5. If $V(X) = d$, and $V(X + e_r) > d$, for every $a_r \in U(X)$, then pass to Step 6. Otherwise, pass to Step 3 for the next solution.
Step 6. If $j < i$ where $C'_j \in B_t$ and $U(X) \subseteq C'_j$, then X is a duplicate d-MC candidate, and pass to Step 3 for the next solution.
Step 7. Add X into D (it is a d-MC), then pass to Step 3 for the next solution.
Step 8. If $i = q$, halt (D is the set of d-MCs). Else, increase i by one, if $C'_i \notin B_t$, increase t by one, then pass to Step 3.

We note that Algorithm 3, which is an improved version of Algorithm 1, first calculates the lower bounds on the arcs' capacities, and then groups the MCs in α groups by computing the capacities. Moreover, as MCs are sorted in ascending order due to the capacities, the sets B_t are ordered, for $t = 1, \ldots, \alpha$. Thus, when C'_{i+1} is not in B_t, then it is in B_{t+1}. In Step 3, the d-MC candidates are found. Theorem 2 is applied in

Step 4 for relieving the need to check the d-MC candidates produced from the MCs that satisfy $CC_i = V(M)$. The max-flow algorithm [16] is used in Step 5 to check other d-MCs in accordance with the definition. The duplicate candidates are removed in Step 6 according to Theorem 4. Consequently, Theorem 5 is at hand.

Theorem 5. *Algorithms* 1–3 *determine all the d-MCs without duplicates for every MFN.*

Now, to have more intuition, let us do an example.

1.4.1 A descriptive example

Considering the flow network depicted in Figure 1.2 with the maximum capacity vector of $M = (5, 4, 6, 4, 3, 6)$, assume that $d = 13$ and use Algorithm 3 to find all the $13-$MCs.

Step 1. Let $D = \phi$, $i = 1$, $t = 1$, and $q = 4$. Then the lower bounds $l_1 = 3$, $l_2 = 2$, $l_3 = 4$, $l_4 = 0$, $l_5 = 1$, and $l_6 = 4$ are calculated by applying Theorem 3.

Step 2. We have $CC_1 = 15$, $CC_2 = 21$, $CC_3 = 17$, $CC_4 = 15$. Then $C'_1 = C_1$, $C'_2 = C_4$, $C'_3 = C_3$, and $C'_4 = C_2$. Also, $B_1 = \{C'_{11}, C'_2\}$, $B_2 = \{C'_3\}$, and $B_3 = \{C'_4\}$, and hence $\alpha = 3$.

Step 3. Considering $C'_1 = \{a_1, a_2, a_3\}$, the solution $X = (3, 4, 6, 4, 3, 6)$ of the system is found.

Step 4. As $C'C_1 = 15$, go to Step 6.

Step 6. $i = 1$, and so there is no $j < i$.

Step 7. $D = D \cup \{X\} = \{(3, 4, 6, 4, 3, 6)\}$.

Step 3. $X = (4, 3, 6, 4, 3, 6)$ is found.

Step 4. As $C'C_1 = 15$, go to Step 6.

Step 6. $i = 1$, and so there is no $j < i$.

Step 7. $D = D \cup \{X\} = \{(3, 4, 6, 4, 3, 6), (4, 3, 6, 4, 3, 6)\}$.

As $i = 1$ and $C'C_1 = 15$, every producible candidates from this MC is a $13-$MC and added to D, and consequently $D = \{(3, 4, 6, 4, 3, 6), (4, 3, 6, 4, 3, 6), (4, 4, 5, 4, 3, 6), (5, 2, 6, 4, 3, 6), (5, 3, 5, 4, 3, 6), (5, 4, 4, 4, 3, 6)\}$. Then as there is no solution obtained from C'_1, the transfer is made to Step 8.

Step 8. As $i = 1 \neq q = 4$, we let $i = i + 1 = 2$, and as $C'_2 \in B_1$, go to Step 3.

Step 3. Considering $C'_2 = \{a_3, a_5, a_6\}$, the solution $X = (5, 4, 4, 4, 3, 6)$ of the system is found.

Step 4. As $C'C_2 = 15$, go to Step 6.

Step 6. As $C'_1 \in B_1$ and $U(X) = \{a_3\} \subset C'_1$, then X is duplicate and transfer is made to Step 3.

There are five other $13-$MCs producible from C'_2 which are not duplicates, and hence we have $D = \{(3, 4, 6, 4, 3, 6), (4, 3, 6, 4, 3, 6), (4, 4, 5, 4, 3, 6), (5, 2, 6, 4, 3, 6), (5, 3, 5, 4, 3, 6), (5, 4, 4, 4, 3, 6), (5, 4, 5, 4, 2, 6), (5, 4, 5, 4, 3, 5), (5, 4, 6, 4, 1, 6), (5, 4, 6, 4, 2, 5), (5, 4, 6, 4, 3, 4)\}$. Then as there is no solution producible from C'_2, the transfer is made to Step 8.

Step 8. As $i = 2 \neq q = 4$, we let $i = i + 1 = 3$, and as $C'_3 \notin B_1$, we let $t = t + 1 = 2$, and go to Step 3.
Step 3. Considering $C'_3 = \{a_2, a_3, a_4, a_5\}$, the solution $X = (5, 2, 4, 4, 3, 6)$ is found.
Step 4. $C'C_3 = 17 \neq 15$.
Step 5. As $V(X) = 11 \neq 13$, go to Step 3.
Step 3. $X = (5, 2, 5, 3, 3, 6)$ is found.
Step 4. $C'C_3 = 17 \neq 15$.
Step 5. As $V(X) = 12 \neq 13$, go to Step 3.
In fact, twenty-three 13-MC candidates are generated from C'_3 among which there are only five real 13-MCs, which are (5, 3, 6, 1, 3, 6), (5, 3, 6, 2, 2, 6), (5, 4, 5, 1, 3, 6), (5, 4, 6, 0, 3, 6), (5, 4, 6, 1, 2, 6), and consequently we have D = {(3, 4, 6, 4, 3, 6), (4, 3, 6, 4, 3, 6), (4, 4, 5, 4, 3, 6), (5, 2, 6, 4, 3, 6), (5, 3, 5, 4, 3, 6), (5, 4, 4, 4, 3, 6), (5, 4, 5, 4, 2, 6), (5, 4, 5, 4, 3, 5), (5, 4, 6, 4, 1, 6), (5, 4, 6, 4, 2, 5), (5, 4, 6, 4, 3, 4), (5, 3, 6, 1, 3, 6), (5, 3, 6, 2, 2, 6), (5, 4, 5, 1, 3, 6), (5, 4, 6, 0, 3, 6), (5, 4, 6, 1, 2, 6)}. Then as there is no solution producible from C'_3, the transfer is made to Step 8.
Step 8. As $i = 3 \neq q = 4$, we let $i = i + 1 = 4$, and as $C'_4 \notin B_2$, we let $t = t + 1 = 3$, and go to Step 3.
Step 4. Considering $C'_4 = \{a_1, a_3, a_4, a_6\}$, the solution $X = (0, 4, 3, 4, 3, 6)$ is found.

There is no newly producible 13-MCs from C'_4, and consequently the algorithm stops with the previous set of determined 13-MCs. This example shows that

1. How usage of Theorem 2 helps to decrease the work in the algorithm. In fact, we determined eleven 13-MCs with no checking process.
2. How using Theorem 3 can lead to a considerable decrease in the number of generated candidates. For instance, without considering the lower bounds, thirty-four 13-MC candidates are obtained from C'_3 while by using the lower bounds only 23 candidates are generated.
3. How applying Theorem 4 can lead to a final solution with no duplicates.

1.4.2 Complexity results

Here, the complexity results of all the stated algorithms in this section are computed. We remind that the number of obtainable d-MC candidates from a MC is bounded by η and it is assumed to have q MCs in the network. Moreover, we use the simple efficient method proposed in [12] for solving system (1.2) knowing that there are several approaches to do so in the literature. Now, let us provide the complexity results on the stated algorithms in this section.

Algorithm 1: The time complexity of finding all the d-MC candidates produced from each MCs is $O(m\eta)$ (see Theorem 1 in [12]), and hence finding all the candidates generated from all the MCs is of the order of $O(mq\eta)$ (Step 1 in Algorithm 1). The max-flow algorithm is of the order of $O(mn)$ [16], and as there are m arcs in the network, Step 2 in this algorithm is of the order of $O(m^2nq\eta)$. Now, letting γ be the number of all the obtained d-MCs with the duplicates, comparing all the d-MCs and removing the duplicates in Step 3 are of the order of $O(m\gamma^2)$. Consequently, Algorithm 1 is of order of $O(m^2nq\eta + m\gamma^2)$. By increasing the size of network, the number γ grows exponentially, and therefore for large enough networks, the time complexity of Algorithm 1 can be considered as $O(m\gamma^2)$.

Algorithm 2: Step 1 in both Algorithms 1 and 2 are the same, and thus Step 1 in Algorithm 2 is also of the order of $O(mq\eta)$. As the number of all the generated d-MC candidates is bounded by $q\eta$ and the candidates are m – tuple vectors, comparing all the candidates to eliminate the nonmaximal vectors and duplicates in Step 2 is of the order of $O(m(q\eta)^2)$. Therefore, Algorithm 2 is of the order of $O(mq^2\eta^2)$.

Algorithm 3: This algorithm is an improved version of Algorithm 1. The max-flow algorithm is of the order of $O(mn)$ [16], and thus the time complexity of computation of l_is in Step 1 is $O(m^2n)$. For each $i = 1, \ldots, q$, the computation of CC_i is of the order of $O(m)$, and hence computing all the CC_is is of order $O(mq)$. The rearrangement of MCs is of the order of $O(q\log q)$, and as $q \le 2^n \le 2^m$, Step 2 is of the order of $O(mq)$. The time complexity of finding all the candidates produced from all MCs in Step 3 is $O(mq\eta)$. For every candidate, the time complexities of Steps 4 and 5 are $O(1)$ and $O(m^2n)$, respectively. The complexity of Step 6 is of the order of $O(m\rho_t^2)$, where ρ_t is cardinality of the set B_t [20]. Steps 7 and 8 are of the order of $O(1)$. Assuming that the number of all the obtainable candidates from an MC is bounded by η, as Steps 3–8 are run for all the d-MC candidates, it is vividly observed that the complexity of Steps 3–8 is of the order of $O\left(\left(mq + \sum_{t=1}^{\alpha}\rho_t^2\right)\eta\right) = O\left(m\eta \sum_{t=1}^{\alpha}\rho_t^2\right)$, for all the candidates. As these steps are executed in parallel with Steps 1 and 2, one can conclude that the time complexity of Algorithm 3 is $O\left(m\eta \sum_{t=1}^{\alpha}\rho_t^2\right)$. For convenience, the complexity results of Algorithms 1–3 are provided in Table 1.1.

Table 1.1: The time complexity of the algorithms on the d-MC problem.

Algorithms	Time complexities
Algorithm 1	$O(m^2nq\eta + m\gamma^2)$
Algorithm 2	$O(mq^2\eta^2)$
Algorithm 3	$O\left(m\eta \sum_{t=1}^{\alpha} \rho_t^2\right)$

It is recalled that the number of arcs is m, the number of nodes is n, the number of MCs is q, the number of producible d-MC candidates from each MC is bounded by η, the number of all the obtained d-MCs (including the duplicates) is γ, and the number of MCs in B_t is ρ_t. Now, first of all, as $\sum_{t=1}^{\alpha} \rho_t = q$ and that all the ρ_t are positive, it is clear that $\sum_{t=1}^{\alpha} \rho_t^2 \ll q^2$ for large enough networks. Moreover, it can be seen that for large enough networks, $\eta \ll \gamma$, and consequently Algorithm 3 is superior to Algorithm 1. Furthermore, as $m\eta \sum_{t=1}^{\alpha} \rho_t^2 \ll m\eta q^2 \ll mq^2\eta^2$, Algorithm 3 outperforms Algorithm 2 considerably. As a result, one can conclude that the best algorithm among the stated algorithms here is Algorithm 3 and the worst one is Algorithm 2.

1.4.3 System reliability based on the *d*-MCs

We have already discussed about system reliability in Section 1.3, and hence here it is briefly explained how one can evaluate the reliability of an MFN by using the d-MCs. In fact, for evaluating the reliability of an MFN in order to transmit d units of commodity (flow or data) from a source node to a destination node, it is required to determine the set $\{X|V(X) \geq d\}$. Now, according to Definition 1, it is seen that if X is a $(d-1)$-MC, then for every $a_r \in U(X)$, we have $V(X+e_r) = d > d-1$. Therefore, if X is a $(d-1)$-MC, then for any $X < Y \leq M$, we have $V(Y) \geq d$. In fact, assuming $\{X^1, X^2, \ldots, X^\sigma\}$ as the set of all the $(d-1)$-MCs, and letting $A_i = \{X|X\rangle X^i\}$, for $i = 1, 2, \ldots, \sigma$, it can be seen that $\{X|V(X) \geq d\} = \cup_{i=1}^{\sigma} A_i$, and consequently the reliability can be computed by using eq. (1.1). The important point is that we should find the set of $(d-1)$-MCs not the d-MCs.

1.4.4 More discussion

So far in this section, we discussed about the d-MC problem, stated three algorithms to solve the problem, and explained how to assess the reliability of an MFN by using the obtained d-MCs. Here, it is worth to mention that usually there are two important factors *time* and *cost* in reality, which should be considered. Accordingly, the definition of the *reliability* of the system is changed. Taking into account a *limited budget*, that is, b (*limited time*, i.e., T), the reliability of a network for a demand level of d can be defined as the probability of sending a minimum of d units of commodity from a source node to a destination node within b units of budget (T units of time). For sure, in such cases, some cost (transmitting time) is considered for each unit of capacity on each arc, which can be statistically dependent or independent for different arcs. In this way, some other problems such as (d, c)-MC have been arisen and worked out in the literature [20]. Moreover, some researchers have worked on determining an approximate solution for d-MC problem [21].

1.5 MP-based approaches

In addition to the MC-based approaches for reliability evaluation of MFNs, there are various methods based on *MPs* to do so. Here, some results and related algorithms in this regard are stated and discussed. It is noted that this section is mainly based on [19, 20, 22–25, 30] from the literature. First, let us remind that the number of MPs is p, the jth MP is denoted by P_j, for $j = 1, 2, \ldots, p$, $CP_j(X)$ is the capacity of P_j under system state vector X, and that $CP_j(M)$ is simply denoted by CP_j. Moreover, $Z(X) = \{a_i | x_i \rangle 0\}$ is the set of arcs with positive capacity under X.

Definition 2. Let the amount of flow on MP P_j be denoted by f_j, for $1 \le j \le p$.

The vector $F = (f_1, \ldots, f_p)$, where f_j is the flow on P_j, for $1 \le j \le p$ is called a *feasible flow vector* for demand level of d if the following system is fulfilled:

$$\begin{cases} (i)\ f_1 + f_2 + \cdots + f_p = d, \\ (ii)\ \sum\limits_{j:\ a_i \in P_j} f_j \le \min\{M_i, d\}, \quad i = 1, 2, \ldots, m, \\ (iii)\ 0 \le f_j \le \min\{CP_j, d\}, \quad j = 1, 2, \ldots, p. \end{cases} \tag{1.3}$$

As an example, consider the network depicted in Figure 1.3. We have four MPs, $P_1 = \{a_1, a_5\}$, $P_2 = \{a_1, a_3, a_6\}$, $P_3 = \{a_2, a_6\}$, and $P_4 = \{a_2, a_4, a_5\}$ in this network. The capacities of MPs are $CP_1 = 2$, $CP_2 = 1$, $CP_3 = 2$, and $CP_4 = 1$. Now, assuming $d = 3$, it is observed that $F = (0, 1, 1, 1)$ is a feasible flow vector in this network.

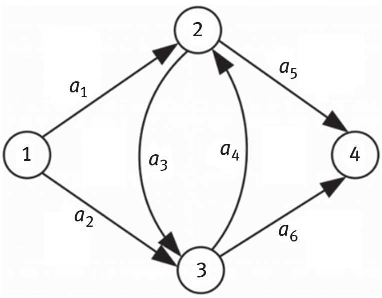

Figure 1.3: The most cited benchmark network example.

Assuming that $F = (f_1, \ldots, f_p)$ is a feasible flow vector for demand level of d, the state vector $X = (x_1, x_2, \ldots, x_m)$ is named a d-MP candidate if and only if the following equation holds:

$$x_i = \sum_{j:\, a_i \in P_j} f_j \quad \forall i = 1, 2, \ldots, m. \tag{1.4}$$

For example, $X = (1, 2, 1, 1, 1, 2)$ is the 3-MP candidate associated with the feasible flow vector $F = (0, 1, 1, 1)$ in the network depicted in Figure 1.3. According to the definition, in order to determine all the d-MP candidates, it is required to first solve system (1.3) for determining the feasible flow vectors, and then to use eq. (1.4) for computing the associated system state vector with each feasible flow vector. We note that to find all the feasible flow vectors, it is required to have all MPs in advance. In fact, usually the proposed solutions for d-MP problem have assumed that MPs are at hand in advance [19, 20, 22–25]. We note that there are several approaches in the literature to find all MPs [9].

Definition 3. A system state vector X is a d-MP if and only if (I) $V(X) = d$ and (II) $V(X -_i) < d$, for each $a_i \in Z(X)$.

For instance, $X = (1, 2, 1, 1, 1, 2)$ is not a 3-MP because $V(X - e_3) = 3$. Lin et al. [24] proved that each d-MP is a candidate, and then proposed an approach that first finds the d-MP candidates, and then checks every candidate to be a d-MP. Indeed, nearly all the presented algorithms in [19, 20, 22–25] consist of two general phases: (1) finding the d-MP candidates and (2) checking every candidate to be a d-MP. Theorem 6, proven in [24], can be used for determination of the d-MPs among the candidates.

Theorem 6. *If* Ψ *is the set of the d-MP candidates, then* $\Psi_{min} = \{X | X$ *is a minimal vector*$\}$*, which equals the set of the d-MPs.*

Based on the definitions and Theorem 6, the first algorithm to find all the d-MPs is stated, which is proposed by Lin et al. [24].

Algorithm 4: (determination of all the d-MPs in an MFN)

Step 1. Solve system (1.3).

Step 2. Use eq. (1.4) for computing the system state vectors associated with the solutions obtained in Step 1.

Step 3. Remove the duplicate and nonminimal vectors by comparing all the vectors obtained in the preceding step.

Algorithm 4 is easy to be understood and implemented, which is its main advantage; however it compares all the candidates in order to find all the d-MPs which is its main disadvantage. In fact, growing exponentially the number of candidates with the size of network shows that the comparative technique is not so practical for large enough flow networks. There is another technique to check the candidates, which is called *cycle-checking method*, proposed by Yeh [25]. This technique, which has been demonstrated to be considerably more efficient than the comparative approach, is based on Theorem 7.

Theorem 7. *A d-MP candidate, say X, is a d-MP if we have no directed cycle in the network under system state vector X, and* $V(X) = d$.

For instance, $X = (1, 2, 1, 1, 1, 2)$ is not a 3-MP for the network depicted in Figure 1.3 because $\{a_3, a_4\}$ is a directed cycle in this network under system state vector X. Based on the new checking process, the following algorithm is stated, which has originally been proposed in [25].

Algorithm 5: (determination of all the d-MPs in an MFN)
Step 1. Solve system (1.3).
Step 2. Use eq. (1.4) for computing the system state vectors associated with the solutions obtained in Step 1.
Step 3. For each vector obtained in Step 2, say X, if we have no directed cycle in the network under it and $V(X) = d$, then it is a d-MP; otherwise, remove it.
Step 3. Eliminate the duplicates by comparing all the d-MPs.

It is noted that Step 4 in Algorithm 5 has not been stated in the proposed algorithm in [25], which can be considered as a drawback. In fact, according to eq. (1.4), it is observed that a d-MP candidate can be generated from certain different feasible flow vectors if the number of MPs in the network is sufficiently greater than the number of arcs. Therefore, it is required to compare the candidates or d-MPs with each other for eliminating the duplicates. This is why we considered this step in Algorithm 5. Comparing Algorithms 4 and 5, it is seen that the main difference between these two algorithms is the checking process in them. Based on the complexity results, it is later proven that Algorithm 5 is superior to Algorithm 4. Next, some results are provided in order to still improve Algorithm 5.

Although Theorem 7 needs to check $V(X) = d$, for each d-MP candidate, say X, the next result from [19] shows that this condition always holds for every candidate and there is no need to check it.

Lemma 2. *Assume that F is a solution of system (1.3) and $X_F = (x_1, x_2, \ldots, x_m)$ satisfies eq. (1.4). Then $V(X_F) = d$.*

Proof. As X_F is calculated by using eq. (1.4), we have $V(X_F) = f_1 + \cdots + f_h$. Hence, from eq. (1.3) (i) the result is at hand.

A practical point to notice is that system (1.3) is a constrained Diophantine system including an equation given in (i), and several inequalities given in (ii). Noticing on the inequalities (ii) in the system, it is seen that we are in fact checking the condition $x_i \leq \min\{M_i, d\}$. As it is required to calculate the associated vector X with every solution F of the system, one can remove the inequalities (ii) from the system and after calculating X, check it for the condition $X \leq M$. This way, system (1.3) is reduced to a constrained equation, which would be really less complicated for solving. Also, it is not necessary to do some repetitive calculations several times. As a result, although generally the solution set for the constrained Diophantine system is smaller than the solution set for the constrained Diophantine equation, it is more efficient in practice to remove the inequalities (ii) from the system and check it later with $X \leq M$ [19]. Therefore, a new notion is defined and denoted by d-FV as follows.

Definition 4. A flow vector $F = (f_1, f_2, \ldots, f_p)$ is said to be a d-FV, if it satisfies the following system:

$$\begin{cases} (i)\ f_1 + f_2 + \cdots + f_p = d, \\ (ii)\ 0 \leq f_j \leq \min\{CP_j, d\}, \quad j = 1, 2, \ldots, p. \end{cases} \tag{1.5}$$

We note that the checking process in Step 3 is indeed the main part of Algorithm 5, and consequently as far as one can lessen the amount of candidates without losing any real d-MP, the algorithm is improved. Hence, one can apply Theorem 3 to calculate some lower bounds on the arcs' capacities. This way, letting $L = (l_1, l_2, \ldots, l_m)$ be the vector of lower bounds, one can check the condition $L \leq X \leq M$ instead of $X \leq M$ after calculating the system state vectors associated with the d-FVs. It is reminded that to calculate the lower bounds introduced in Theorem 3, one should use the max-flow algorithm. As the max-flow algorithm is of the order of $O(mn)$ [16] and there are m arcs in the network, the time complexity of calculation of vector L is $O(m^2 n)$. Moreover, noting that the time complexity of the other steps in the main algorithm is more than the one for determination of the vector L and that this vector can be calculated beforehand, it is observed that it does not influence the time complexity of the principal algorithm. As a result, using the lower bounds will improve the algorithm. Based on the preceding discussions, Algorithm 6 is stated as an improved version of Algorithm 5.

Algorithm 6: (determination of all the d-MPs in an MFN)
Input: An MFN, a demand level d, and the set of all the MPs.
Output: All the d-MPs.

Step 1. Compute $CP_j = \min\{M_i | a_i \in P_j\}$, for $j = 1, 2, \ldots, p$, and $l_i = \max\ \{0, d - V(M - M_i \times e_i)\}$, for $i = 1, 2, \ldots, m$. Let $L = (L_1, L_2, \ldots, L_m)$, and $Q = \phi$.

Step 2. Find a d-FV, say F, by solving the following system:

$$\begin{cases} (i)\ f_1 + f_2 + \cdots + f_p = d, \\ (ii)\ 0 \leq f_j \leq \min\{CP_j, d\}, \quad j = 1, 2, \ldots, p. \end{cases}$$

If there is no solution, pass to Step 5.

Step 3. Use eq. (1.4) to calculate the vector X corresponding to F. If $L \leq X \leq M$, then pass to Step 4, else pass to Step 2 for the next d-FV.

Step 4. Check the network under X for directed cycles. If there is any, then it is not a d-MP, else add it to set Q. Pass to Step 2 for the next d-FV.

Step 5. Compare all d-MPs in Q_1 to remove the duplicates and halt (Q is the set of all d-MPs with no duplicates).

According to the preceding discussions, Theorem 8 is at hand.

Theorem 8. *Algorithms* 4–6 *calculate all d-MPs without any duplicates in an MFN.*

Let us first do an example to have a better understanding of Algorithm 6, and afterward provide the complexity results to compare the algorithms.

1.5.1 A descriptive example

Consider the network depicted in Figure 1.3 with the maximum capacity vector of $M = (3, 2, 1, 1, 2, 2)$. We have four MPs $P_1 = \{a_1, a_5\}$, $P_2 = \{a_1, a_3, a_6\}$, $P_3 = \{a_2, a_6\}$, and $P_4 = \{a_2, a_4, a_5\}$ in this network from nodes 1 (source) to 4 (destination). The maximum flow of the network is $V(M) = 4$, and so the demand value $d = 3$ is reasonable. Now, we utilize Algorithm 6 to find all the 3-MPs in this network.

Step 1. It is computed that $CP_1 = 2$, $CP_2 = 1$, $CP_3 = 2$, and $CP_4 = 1$. Also, we have $L = (1, 0, 0, 0, 1, 1)$ and $Q = \phi$.
Step 2. The solution $F = (0, 0, 2, 1)$ is obtained.
Step 3. The associated system state vector is $X = (0, 3, 0, 1, 1, 2)$. As $X \nleq M$, go to Step 2.
Step 2. $F = (0, 1, 1, 1)$ is obtained.
Step 3. $X = (1, 2, 1, 1, 1, 2)$ is calculated. As $L \leq X \leq M$, go to Step 4.
Step 4. It can be seen that $\{a_3, a_4\}$ is a directed cycle in $G(N, A, X)$, the network flow under system state vector of X. Hence, it is not a 3-MP and go to Step 2.
Step 2. $F = (0, 1, 2, 0)$ is obtained.
Step 3. $X = (1, 2, 1, 0, 0, 3)$ is calculated. As $X \nleq M$, go to Step 2.
Step 2. $F = (1, 0, 1, 1)$ is obtained.
Step 3. $X = (1, 2, 0, 1, 2, 1)$ is calculated. As $L \leq X \leq M$, go to Step 4.
Step 4. As no directed cycle can be detected in the network with the arcs' capacities given by X, it is a 3-MP and is added to Q. Thus, we have $Q = \{(1, 2, 0, 1, 2, 1)\}$ and go to Step 2.
Step 2. $F = (1, 0, 2, 0)$ is obtained.
Step 3. $X = (1, 2, 0, 0, 1, 2)$ is calculated. As $L \leq X \leq M$, go to Step 4.
Step 4. As no directed cycle can be detected in the network with the arcs' capacities given by X, it is a 3-MP and is added to Q. Thus, we have $Q = \{(1, 2, 0, 1, 2, 1), (1, 2, 0, 0, 1, 2)\}$ and go to Step 2.
Step 2. $F = (1, 1, 0, 1)$ is obtained.
Step 3. $X = (2, 1, 1, 1, 2, 1)$ is calculated. As $L \leq X \leq M$, go to Step 4.
Step 4. As $\{a_3, a_4\}$ is a directed cycle in $G(N, A, X)$, so X is not a 3-MP and go to Step 2.
Step 2. $F = (1, 1, 1, 0)$ is obtained.

Step 3. $X = (2, 1, 1, 0, 1, 2)$ is calculated. As $L \leq X \leq M$, go to Step 4.
Step 4. As no directed cycle can be detected in the network with the arcs' capacities given by X, it is a 3-MP and is added to Q. Thus, we have Q = {(1, 2, 0, 1, 2, 1), (1, 2, 0, 0, 1, 2), (2, 1, 1, 0, 1, 2)} and go to Step 2.
Step 2. $F = (2, 0, 0, 1)$ is obtained.
Step 3. $X = (2, 1, 0, 1, 3, 0)$ is calculated. As $X \nleq M$, go to Step 2.
Step 2. $F = (2, 0, 1, 0)$ is obtained.
Step 3. $X = (2, 1, 0, 0, 2, 1)$ is calculated. As $X \nleq M$, go to Step 2.
Step 2. $F = (2, 1, 0, 0)$ is obtained.
Step 3. $X = (3, 0, 1, 0, 2, 1)$ is calculated. As $L \leq X \leq M$, go to Step 4.
Step 4. As no directed cycle can be detected in the network with the arcs' capacities given by X, it is a 3-MP and is added to Q. Thus, we have Q = {(1, 2, 0, 1, 2, 1), (1, 2, 0, 0, 1, 2), (2, 1, 1, 0, 1, 2), (3, 0, 1, 0, 2, 1)} and go to Step 2.
Step 2. As no more solution can be found, go to Step 5.
Step 5. As no duplicate 3-MP exists in Q = {(1, 2, 0, 1, 2, 1), (1, 2, 0, 0, 1, 2), (2, 1, 1, 0, 1, 2), (3, 0, 1, 0, 2, 1)}, it is the final set of all the 3-MPs.

1.5.2 Complexity results

Here, the time complexities of all the three stated algorithms to find all the d-MPs are calculated, based on which the algorithms are compared together. First, we note that to solve system (1.3) in Algorithms 4 and 5 or system (1.5) in Algorithm 6, the efficient approach proposed in [12] is used. As this approach solves a constrained Diophantine equation (only one equation and not a system of equations and inequalities), it is assumed that in both Algorithms 4 and 5, system (1.5) is considered and afterward the condition $X \leq M$ is checked instead of considering system (1.3) with inequalities (II). We also note that this change improves Algorithms 4 and 5. Hence, in fact, the complexity results are computed here for somehow improved versions of Algorithms 4 and 5.

We remind that the number of d-FVs, which is the number of solutions for system (1.5), is β, and the number of d-MPs with the duplicates is ζ. Moreover, n, m, and p are, respectively, the number of nodes, arcs, and MPs.

Algorithm 4: To solve system (1.5), one first needs to calculate the CP_js, for $j = 1, 2, \ldots, p$, which is of the order of $O(mp)$, and as β is the number of the solutions for system (1.5), solving the system in Step 1 is of the order of $O(p\beta)$. Hence, as $m < < \beta$, Step 1 is of the order of $O(p\beta)$. According to eq. (1.4), the time complexity of calculating all the candidates in Step 2 is $O(mp\beta)$. Moreover, checking the condition $X \leq M$ is of the order of $O(m\beta)$, and consequently, the time complexity of Step 2 is $O(mp\beta)$. It is noted that checking the condition $X \leq M$ does not influence the time complexity of Step 2. In Step 3, as the candidates are m – tuple vectors, removing the nonminimal candidates by comparing the candidates to each other is of the order of $O\left(m\beta^2\right)$. Now, as $p \ll \beta$ in large enough networks, and that all the three steps in this algorithm are executed in parallel, the time complexity of the algorithm is $O\left(m\beta^2\right)$. As it is seen, the worst part of this algorithm is its checking technique to detect the d-MPs among all the candidates.

Algorithm 5: The first two steps in this algorithm are the same as the first two steps in Algorithm 4, and so the time complexities of Steps 1 and 2 are, respectively, $O(p\beta)$ and $O(mp\beta)$. The max-flow algorithm is of the order of $O(mn)$ [16] and the time complexity of checking if there is any directed cycle in a multistate network is $O(n)$ [25]. Therefore, Step 3 is of the order of $O(mn\beta)$. As we have ζ-generated d-MPs, counting the duplicates and eliminating the duplicates in Step 4 are of the order of $O\left(m\zeta^2\right)$. It can be observed that all the steps in this algorithm are executed in parallel, and hence Algorithm 5 is of the order of $O\left(mp\beta + m\zeta^2\right)$.

We know that in any network, there are many d-MP candidates that are not d-MP, and consequently $\zeta \ll \beta$. Thus, $mp\beta + m\zeta^2 \ll m\beta^2$, which clearly shows the superiority of Algorithm 5 to Algorithm 4. In fact, the checking process employed in Algorithm 5 is the main reason of its superiority to Algorithm 4.

Algorithm 6: The calculation of all CP_js is of the order of $O(mp)$, and as the max-flow algorithm is of the order of $O(mn)$ [16], computing all the l_is is of the order of $O(m^2n)$. Therefore, the time complexity of Step 1 is $O(m^2n)$. As β is the number of solutions for system (1.5), that is the number of d-FVs, solving the system in Step 2 is of the order of $O(p\beta)$. It is observed that calculating the associated vectors with the d-FVs in Step 3 is of the order of $O(mp\beta)$, and checking the conditions $L \le X \le M$ is of the order of $O(m\beta)$. Therefore, this step is of the order of $O(mp\beta)$. Checking the candidates in Step 4 is of the order of $O(n\beta)$. Finally, the time complexity of removing the duplicate d-MPs in Step 5 is $O(m\zeta^2)$. Consequently, Algorithm 6 is of the order of $O(mp\beta + m\zeta^2)$. For convenience, the time complexities of all the three stated algorithms in this section are given in Table 1.2.

Table 1.2: The time complexity of the algorithms on the d-MP problem.

Algorithms	Time complexities
Algorithm 4	$O(m\beta^2)$
Algorithm 5	$O(mp\beta + m\zeta^2)$
Algorithm 6	$O(mp\beta + m\zeta^2)$

Although the time complexities of Algorithms 5 and 6 are the same, we note that Algorithm 6 does not check the candidates for $V(X) = d$, and also that it was assumed that Algorithm 5 solves first system (1.5), and then checks the solutions for $X \leq M$, which is an improvement. By the way, several numerical results have been provided in [19], which demonstrate that Algorithm 6 is superior to Algorithm 5 in practice.

1.5.3 System reliability based on the *d*-MPs

Similarly, as it was explained in Section 1.4.3 on how one can evaluate the reliability of an MFN by using the d-MCs, here it is described how the d-MPs can be used to assess the reliability of an MFN. It is again reminded that to assess the reliability of the system for transmitting d units of commodity (flow or data) from a source node to a sink node, it is required to determine the set $\{X|V(X) \geq d\}$. Now, let us define *unreliability* of a system at demand level d, denoted by U_d, as the probability of failure in transmitting at least d units of commodity from a source node to a sink node. In fact, we have $U_d = 1 - R_d$. It can be seen that to compute the unreliability of a system, one needs to determine the set $\{X|V(X) < d\}$. To this end, according to Definition 3, it is seen that if X is a d-MP, then for every $a_r \in Z(X)$, we have $V(X - e_r) = d - 1 < d$. Therefore, if X is a d-MP, then for any $0 \leq Y < X$, we have $V(Y) < d$. In fact, assuming that the set of all the d-MPs is $\{X^1, X^2, \ldots, X^\sigma\}$, and letting $B_i = \{X|X < X^i\}$, for $i = 1, 2, \ldots, \sigma$, one can observe that $\{X|V(X) < d\} = \cup_{i=1}^{\sigma} B_i$, and consequently the unreliability can be computed according to eq. (1.1). It is worth to notice that this time we first calculate the unreliability, and then we should use the equation $R_d = 1 - U_d$ to assess the reliability.

1.5.4 More discussion

Similar to the d-MC problem (see Section 1.4.4), one can consider *time* and/or *budget* constraints in d-MP problem to make it closer to real-world problems. This way, some problems such as (d, b)-MP [22] or *quickest path reliability* [26, 27, 31] have been worked in the literature. As our focus in this chapter was on the main cases of d-MC and d-MP problems, we considered neither the time nor the budget constraints in d-MC and d-MP problems. The readers can refer to [20, 22, 26, 27] on these problems. Moreover, some researchers have proposed some solutions for the d-MP problem for all the demand values d [10, 28] and some others proposed approximate solutions [29].

1.6 Conclusions

This chapter provided and discussed various exact approaches for assessing the reliability of the MFNs. To this end, first, it described how one can compute the system reliability of an MFN for a given demand level *d*. Then several methods based on MCs were discussed for determination of all the so-called *d*-MCs, which are the system state vectors with sensitivity to increasing the arcs' capacities and under them *d* units of commodity or flow can be sent from a source node to a destination node. The complexity results on the stated methods were provided and it was explained how the reliability can be calculated by using the *d*-MCs. Afterward, many approaches based on MPs were investigated for determination of the so-called *d*-MPs, which are the system state vectors with sensitivity to decreasing the arcs' capacities and under them *d* units of flow can be sent from a source node to a destination node. Similarly, the complexity results on these approaches were provided and it was explained how the reliability can be calculated by using the *d*-MPs.

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Pooja Dhiman and Amit Kumar

2 Behavior exploration of a rice manufacturing plant under fuzzy environment

Abstract: In this period of worldwide race and quicker conveyance times, it has become basic for all production structures to perform sufficiently during their normal life expectancy. In any case, failure is an unavoidable certainty-related mind-innovative items and frameworks utilized in the business. There is developing enthusiasm for the usage and examination of the reliability of industrial structures. This chapter presents a concrete case of a rice manufacturing plant. The whole mechanism includes components such as depot container, cleaner apparatus, destoner apparatus, bleach apparatus, backhoe, sortex apparatus, and packing apparatus. The mathematical formulation of this structure has been evolved by adopting the generalized trapezoidal method. In addition to this, the level of satisfaction has been associated with the parameter of each component. The lambda–tau mode has been used to find the combined parameter of the structure using parameters of each component, and the behavior of the structure has been examined. The final result of the reliability parameters has been analyzed with the aid of tables and graphs. Finally, the behavior of the system has been discussed, and a conclusion has been drawn.

Keywords: Reliability, availability, failure rate, generalized trapezoidal number, rice manufacturing plant

2.1 Introduction

Under the system's growing complexity, troubles in the real world frequently get complicated due to confusion of parameters in the problem or of the circumstances in which the problem occurs. However, mathematical intervention is arduous in the case of applications where the data available are inadequate. As the approach to probability has been widely extended to many real-world technological challenges, there are also certain drawbacks to the probabilistic system. For example, probabilistic approaches are based on mass data processing, which is arbitrary, to obtain the required level of satisfaction. But on a wide scale, the complex method has a vast fuzzy complexity due to which the precise outcome of the events is difficult to

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achieve. Associative and epistemic ambiguity usually influences the evaluation of the programs. Aleatory confusion emerges from variability or from the spontaneous existence of natural systems, while epistemic confusion emerges from the incomplete existence of our natural world information. By more examination, epistemic uncertainty can be minimized, whereas aleatory uncertainty cannot be minimized. Because of the disadvantage of being capable to manage only quantitative data, tests based on probability theory do not necessarily supply the practitioners with valuable knowledge. Because of these limitations, the results based on probability theory do not always provide practitioners with useful information, and thus the probabilistic approach is insufficient to account for such built-in data ambiguity. To solve these problems, techniques that rest on fuzzy set theory and logic are valuable in addition to the probability theory to deal with the uncertainties.

Aware of the importance of the topic, value has been produced and many authors have made a strong contribution. If the structure consists of more than one unit, there is a risk of a total system collapse in relation to a single trigger. These errors are referred to as general cause errors and are particularly useful in evaluating the reliability study of the structure. Choudhary et al. [1] have demonstrated the use of RAM examination to assess preventive maintenance programs for many cement plant subsystems. Gagloft [2] has defined the common cause failure for the system containing numerous subsystems or components at any prompt failure because of the single common cause? Gupta et al. [3] have also analyzed the reliability and availability of the serial order system of plastic pipe manufacturing plant by adopting the matrix calculus method, and also discussed the numerical examination of the reliability of serial processes in the butter oil processing plant. Gupta et al. [4] scrutinized the behavioral study of the manufacturing plant of cement with the help of a numerical approach. Kaur and Lal [5] have found the availability of the considered system, that is, the rice manufacturing plant with various effects of failure rates and repair rates. Lata et al. [6] have scrutinized the behavioral examination of piston manufacturing plant with the help of stochastic models. Gupta and Vinodiya [7] have examined the reliability of two noncorresponding cold standby repair structures with two forms of breakdown. Parashar and Bhardwaj [8] scrutinized a correlative profit examination of two reliability models adopting a two-unit PLC network. Gupta [9] has considered the stochastic modeling of numerous state systems based on failure arrangement dependence and availability examination of a critical engineering arrangement. Shakuntla et al. [10] have also examined the non-Markovian measures by the involvement of supplementary variables.

Knezevic and Odoom [11] have defined a technique that employs Petri net technique over fault-tree technique and also resolved persistent indices by adopting the lambda–tau fuzzy system. It is an efficient generation of minimal cut and path sets, and has introduced human errors also. The operator is also responsible for the working of the system. In this chapter, the human error rate and recovery time are also considered. Adamyan and David [12] have used the Petri net simulation technique for defining the sequence of failures because the system's assurance and reliability do not only depend in consequence to all the failed system unit phases. Srinath [13] has shown the evaluation of reliability parameters for many structures with failure and reparation levels after exponential distribution. Adamyan and Dravid [12] and Bhamare et al. [14] have used the Markovian technique. It helps the manager to plan the design alteration in the structure according to the specifications. Adhikary et al. [15] have suggested that the RAM examination is very successful to find a fretful unit of the structure (having low reliability than other units) that requires more improvement in sustainability. Ahmed et al. [16] addressed the challenges of sectors to achieve projected investment returns and output goals. They have tried a stochastic reproduction approach dependent on hazard by utilizing a dynamic procedure from Markov to decide the accessibility of a preparing unit known as the Markov technique risk-based accessibility. Sharma and Garg [17] carried out the various parameters of reliability by adopting lambda–tau methodology and its conditions for modeling and suggested the system. Dhiman and Garg [18] have presented improved operations on generalized fuzzy numbers by defeating the inadequacy of specific tasks that exist. They introduced the approach that utilizes the equivocal framework information and investigates its activities with a decreased vulnerability, which settles on a choice increasingly down to earth and progressively broad for additional utilization.

The outline of the chapter has been presented in the following manner. The preliminaries have been given in Section 2.2. The methodology for the reliability parameters of the rice manufacturing plant has been discussed in Section 2.3. Brief details on the rice production system along with the flow diagram in Figure 2.2, assumptions, and numerical computation have been addressed in Section 2.4. In addition to tables and graphs, results and discussion have been covered in Section 2.5. Some conclusions hinge on this study have been described using tables and graphs in Section 2.6.

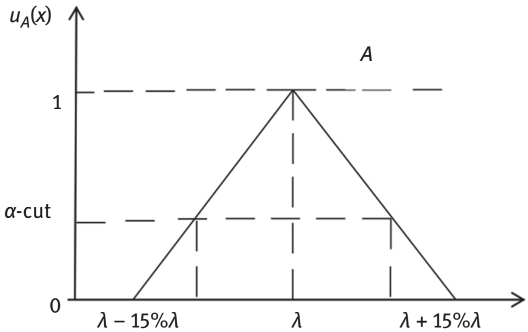

Figure 2.1: 15% spread of the crisp value.

2.2 Preliminaries

2.2.1 Reliability

Under the specified operating conditions, the potential to failure-free service over a defined period and the fixed time of repair, control, and prevention care is called structure reliability.

Reliability is a trait of a structure that has exhibited the potential, just in specific circumstances, to execute the basic elements of the structure over a measure of time. Reliability might be characterized as the ability to stay proficient from a subjective perspective. Quantitatively, it expresses that reliability does not cause operational interruptions during the predetermined time. The capacity of reliability can be characterized based on the capacity of failure rate:

$$R(t) = \exp\left[-\int_0^t \lambda(u)du\right],\ \lambda(t) \text{ is the instantaneous failure rate.} \tag{2.1}$$

2.2.2 Availability

This is the probability that the apparatus will almost definitely function beyond tolerance at a given time.

The probability that the framework will work effectively under such conditions is comprehended to be accessible consistently. At any time t, if these conditions are fulfilled, the framework will work as follows:

$$A_S(t) = \frac{u_S}{\lambda_S + u_S} + \frac{\lambda_S}{\lambda_S + u_S} e^{-(\lambda_S + u_S)t}, \tag{2.2}$$

where λ_S is the failure rate of the structure, $\mu_S = 1/\lambda_S$, and τ_S is the repair time of the structure.

2.2.3 Failure rate (hazard rate)

Failure rate (peril rate) is characterized to be the measure of failures per unit time, for instance, failures every year. The failure rate is estimated as the proportion of the number of failures of the test components to the test time.

2.2.4 Repair rate

Repair rate is characterized to be as far as the measure of repair per unit time, for example, redresses every year. The repair rate will be resolved as the proportion of the measure of adjustments of the items experiencing the trial.

2.2.5 Mean time to failures (MTTF)

The mean time to failure (MTTF) is the major parameter in the definition of reliability. MTTF is the cumulative average time before the mechanism fails.

2.2.6 Generalized fuzzy number

"A" is said that a fuzzy number is a generalized fuzzy number if the following properties are complied with the respective membership function.

If $A = \langle (a, b, c;\ \omega) | a, b, c \in R \rangle$

(i) it is continuous;
(ii) it is zero for all $x \in (-\infty, a) \cup (c, \infty)$;
(iii) it is strictly increasing on $[a, b]$ and strictly decreasing on $[b, c]$; and
(iv) $u_A(x) = \omega$ for all $x = b$ where $0 < \omega \le 1$.

If $\omega = 1$ then it is the normal fuzzy number, otherwise the generalized fuzzy number. A fuzzy number $A = \langle (a_1, a_2, a_3;\ \omega) | a_i \in R \rangle$ is called a generalized triangular fuzzy number (TFN) if its membership function is defined as

$$u_A = \begin{cases} \left(\frac{x - a_1}{a_2 - a_1}\right)\omega; a_1 \le x < a_2, \\ \omega; x = a_2, \\ \left(\frac{a_3 - x}{a_3 - a_2}\right)\omega; a_2 \le x < a_3, \\ 0; \text{otherwise.} \end{cases} \tag{2.3}$$

2.2.7 Alpha cut

If $A = (a_1, a_2, a_3;\ \omega)$, the alpha cut of the fuzzy number is the closed range as shown Figure 2.1:

$$A_\alpha = \left[A_\alpha^{(L)}, A_\alpha^{(R)}\right] = \left[a_1 + \frac{\alpha}{\omega}(a_2 - a_1), c - \frac{\alpha}{\omega}(a_3 - a_2)\right], \quad \alpha \in [0, \omega]. \tag{2.4}$$

2.3 Methodology

2.3.1 Steps to carry out the process

Step 1: Data collection

The process starts at the data collection point, where data related to the various components of the structure are obtained as failure rates (λ) and repair times (τ).

Step 2: Fuzzification of the data

The data assembled from Step 1 is normally obsolete or imprecise because of a missing update or human mistakes. To gauge the instability of the information, this data is in this way changed over into the diverse type of the breaker numbers, whereby the leaders conclude that the information is circulated. For instance, if the chief spreads the information by ±15% to cause it to generalize as shown in Figure 2.1:

$$(\lambda_{1i}, \lambda_{2i}, \lambda_{3i}; \omega_i) = (\lambda_{2i} - 15\%\lambda_{2i}, \lambda_{2i}, \lambda_{2i} + 15\%\lambda_{2i}; \omega_i),$$

where λ_i is the failure rate of the structure and ω_i is the level of satisfaction corresponding to the ith component of the structure.

The same protocol for repair rates will be followed:

$$(\tau_{1i}, \tau_{2i}, \tau_{3i}; \omega_i) = (\tau_{2i} - 15\%\tau_{2i}, \tau_{2i}, \tau_{2i} + 15\%\tau_{2i}; \omega_i).$$

Step 3:

Adopting the transformed fuzzy numbers and their corresponding alpha cuts from the above step, the complete method fuzzy numbers were verified by adopting improved fuzzy arithmetic operations with conventional AND/OR expressions in Table 2.1.

Table 2.1: Expressions for lambda–tau $(\lambda - \tau)$.

Logic gates	λ_{AND}	λ_{OR}	τ_{AND}	τ_{OR}
Expressions for $\lambda - \tau$	$\prod_{j=1}^{n} \lambda_j \left[\sum_{i=1}^{n} \prod_{j=1}^{n} \tau_j\right]$	$\sum_{i=1}^{n} \lambda_i$	$\frac{\prod_{i=1}^{n} \tau_i}{\sum_{j=1}^{n} \left[\prod_{i=1}^{n} \tau_i\right]}$	$\frac{\sum_{i=1}^{n} \lambda_i \tau_i}{\sum_{i=1}^{n} \lambda_i}$

Based on these expressions, different reliability indices are calculated in Table 2.2.

2.3.2 Improved arithmetic operations adopting generalized fuzzy numbers

Suppose that $A = (a_1, b_1, c_1, d_1; \omega_1)$ and $B = (a_2, b_2, c_2, d_2; \omega_2)$ are two generalized fuzzy numbers with different levels of satisfaction ω_1 and ω_2, where $0 \leq \omega_1,\ \omega_2 \leq 1$. If we take $\omega_1 \neq \omega_2$ and $\omega_1 < \omega_2$, then $\omega -$cut of the fuzzy number B transforms into new

Table 2.2: Reliability indices in lambda–tau expressions.

Parameters for structure	Mathematical expressions
MTTF (mean time to failure)	$\mathrm{MTTF}_S = \frac{1}{\lambda_S}$
MTTR (mean time to repair)	$\mathrm{MTTR}_S = \frac{1}{\mu_S} = \tau_S$
MTBF (mean time between failure)	$\mathrm{MTTF}_S + \mathrm{MTTR}_S$
Availability of the structure	$A_S = \frac{\mu_S}{\lambda_S + \mu_S} + \frac{\lambda_S}{\lambda_S + \mu_S} e^{-(\lambda_S + \mu_S)t}$
Reliability of the structure	$R_S = e^{-\lambda_S t}$

generalized fuzzy number B^*: $B^* = (a_2, b_2^*, c_2^*, d_2; \omega)$, where $b_2^* = a_2 + \frac{\omega}{\omega_2}(b_2 - a_2)$ and $c_2^* = d_2 - \frac{\omega}{\omega_2}(d_2 - c_2)$, where $\omega = \min(\omega_1, \omega_2) = \omega_1$.

It was suggested to implement alpha cuts between standard fuzzy numbers and better arithmetic operations.

2.3.2.1 Fuzzy arithmetic operations

Consider two generalized trapezoidal fuzzy numbers $A = (a_1, b_1, c_1, d_1; \omega_1)$ and $B^* = (a_2, b_2^*, c_2^*, d_2; \omega)$.

The following are some basic arithmetic operations between them:

- Addition of two fuzzy numbers A and B^* with two different confidence levels generates trapezoidal fuzzy numbers:

$$(a_1, b_1, c_1, d_1) \oplus (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2^*, c_1 + c_2^*, d_1 + d_2; \min(\omega_1, \omega_2)). \tag{2.5}$$

- Subtraction of two fuzzy numbers A and B^* with two different confidence levels generates trapezoidal fuzzy numbers:

$$(a_1, b_1, c_1, d_1) \ominus (a_2, b_2, c_2, d_2) = (a_1 - d_2, b_1 - c_2^*, c_1 - b_2^*, d_1 - a_2; \min(\omega_1, \omega_2)). \tag{2.6}$$

- Multiplication of two fuzzy numbers A and B^* with two different confidence levels generates trapezoidal fuzzy numbers:

$$(a_1, b_1, c_1, d_1) \otimes (a_2, b_2, c_2, d_2) = (a_1 \times a_2, b_1 \times b_2^*, c_1 \times c_2^*, d_1 \times d_2; \min(\omega_1, \omega_2)). \tag{2.7}$$

- Division of two fuzzy numbers A and B^* with two different confidence levels generates trapezoidal fuzzy numbers:

$$(a_1, b_1, c_1, d_1) / (a_2, b_2, c_2, d_2) = (a_1 / d_2, b_1 / c_2^*, c_1 / b_2^*, d_1 / a_2; \min(\omega_1, \omega_2)). \tag{2.8}$$

2.4 Case study

Throughout the years, we have manufactured goods such as cotton, wheat, and oil as our main business, but now we have diverted our focus toward rice. Husk and brown rice are the primary components of the rice crop. Earthy colored rice includes the external layer and the eatable bit. To extract the consumable part of rice, rice processing is carried out to which husk and grain are expelled. The procedure must be done cautiously to maintain a strategic distance from the breakage of the centers of rice. As indicated by different rice processes, the side effects develop in blended structure. The fundamental motivation behind the rice plant is to expel husk from the grain, a spotless and clean consumable segment.

An illustrated example of a rice manufacturing plant [6] has been considered for analyzing the behavior using the proposed methodology.

2.4.1 System description

A concise depiction of different parts of this plant is given:

1. Depot container
2. Cleaner apparatus
3. Destoner apparatus
4. Bleach apparatus
5. Length backhoe
6. Sortex apparatus
7. Packing apparatus

A concise overview of these components is as follows (Figure 2.2):

1. Depot container: The depot container is a box holding raw materials. Also it is used to stock the paddy grain. It is used in order to store grains safely for a long time.

2. Cleaner apparatus: Cleaner system is mostly fit for washing and filtering the grain. Cleaner system removes rice and extracts big or minor impurities and unnecessary stones from grain. It is the ideal tool for a grain refining facility, that is, vibration vacuum, hybrid vacuum, and locomotive paddy cleaner.

3. Destoner apparatus: A destoner mechanism is a system used to detach stones, dolts from soil ridges, as well as other contaminants from grains. It eliminates bulk content from rice lesser than 3 mm in size and is used in numerous mills, including the wheat mill and rice mill. The process results in a huge loss.

4. Bleach apparatus: Bleach process is used to shine the centers of the rice to improve their color and transform from brown rice to white rice, which also eliminates the part of the brain coating and the toxins from the rice. It helps prevent contamination and improves shelf life. After scrubbing the grain container, the appearance is soft and supple.

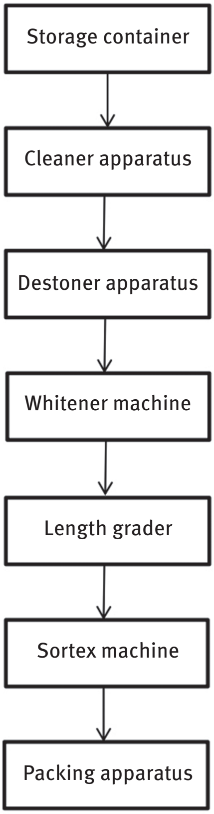

Figure 2.2: Flowchart of rice manufacturing plant.

5. Backhoe: Two kinds of backhoes are as follows:

(i) **Thickness backhoe:** Generally, backhoe is used to distinguishing grains of varying thicknesses. To extract excessive and overmatched grains, the sample is passed through 10 cylindrical monitors. This system is quite good for this sort of activity.

(ii) **Length backhoe:** Generally, length backhoe is being used to differentiate grains of varying lengths. Usually, this apparatus is used for every type of rice like basmati or steam or raw rice. The substance is pushed through the rotating ring, but can also be used to detach broken grains from maximum grains.

6. Sortex apparatus: Sortex system is usually used in processing grains. The primary function of the compactor is to detach the grains of rice in their colors. In the color sorter apparatus, we placed the rice blend through the hopper and in this hopper it ran through the elevator belt on the top of a tower, from which it ran straight through the color sorter pods, streamlining the flow from which the CCD sensors were screened. Afterward, the camera noticed some color error and directed the ejectors equipped to the system to unlock the nozzle.

7. Packing apparatus: Packing systems are also used to carry a variety of grains of rice. The packing process involves paper, dye, and printing content. Throughout this process, first, the substance is placed in plastic wrap; the strength of the bags depends on the volume of rice. Throughout this process, first, the substance is placed in plastic wrap, and the strength of the bags depends on the volume of rice. The heat seal is made with the help of an apparatus in the last.

2.4.2 Assumptions

The following hypotheses are known for the study of the reliability of the rice plant.

Component parameters such as failure rate and the repair rate are isolated from each.

1. The maintenance facility shall be separate for each component.
2. After repair, the elements are considered as new components.
3. There are no mutual faults between the subsystems.

2.4.3 Numerical computations

Step 1: In the input extraction process, the data related to the major components of the structure are collected in the form of failure rate and repair time and outlined in Table 2.3 along with their level of satisfaction.

Table 2.3: Notations of components and their parameters.

S. no.	Components	Notations	Failure rate (h^{-1})	Repair Time (h)
1	Depot container	X_1	0.005	8
2	Cleaner apparatus	X_2	0.0013	3.8
3	Destoner apparatus	X_3	0.0027	48
4	Bleach apparatus	X_4	0.0027	1
5	Backhoe	X_5	0.0054	12
6	Sortex apparatus	X_6	0.0027	12
7	Packing apparatus	X_7	0.2	2.1238

Step 2: To resolve the uncertainty in the existing data, the data collected (crisp) are fuzzified into TFNs with any known distribution or assistance, that is, with ±15% on both sides of the data (Table 2.4).

Table 2.4: Generalized triangular fuzzified data for the major components of the structure with 15% spread.

Components	Failure rate (h^{-1})	Repair Time (h)
X_1	(0.00425, 0.005, 0.00575; 0.7)	(6.8, 8, 9.2; 0.7)
X_2	(0.001105, 0.0013, 0.001495; 0.7)	(3.23, 3.8, 4.37; 0.7)
X_3	(0.002295, 0.0027, 0.003105; 0.7)	(40.8, 48, 55.2; 0.7)
X_4	(0.002295, 0.0027, 0.003105; 0.8)	(0.85, 1, 1.15; 0.8)
X_5	(0.00459, 0.0054, 0.00621; 0.8)	(10.2, 1,213.8; 0.8)
X_6	(0.002295, 0.0027, 0.003105; 0.8)	(10.2, 1,213.8; 0.8)
X_7	(0.17, 0.2, 0.23; 0.9)	(1.80523, 2.1238, 2.44237; 0.9)

Step 3: Though there is a different level of relevance and a minimum out of the values of the level of satisfaction, the data obtained are converted into a new generalized trapezoidal fuzzy number corresponding to the failure and repair time, and their values are listed in Table 2.5:

$$\omega = \min(0.7, 0.7, 0.7, 0.8, 0.8, 08, 0.9) = 0.7.$$

Table 2.5: Parameters of components in generalized trapezoidal fuzzy numbers with level of satisfaction.

Components	*Failure rate (h^{-1})	*Repair Time (h)
X_1	(0.00425, 0.005, 0.005, 0.005, 0.00575; 0.7)	(6.8, 8, 9.2; 0.7)
X_2	(0.001105, 0.0013, 0.0013, 0.001495; 0.7)	(3.23, 3.8, 4.37; 0.7)
X_3	(0.002295, 0.0027, 0.0027, 0.003105; 0.7)	(40.8, 48, 55.2; 0.7)
X_4	(0.002295, 0.0026494, 0.00308813, 0.003105; 0.8)	(0.85, 1, 1.15; 0.8)
X_5	(0.00459, 0.0052988, 0.00617625, 0.00621; 0.8)	(10.2, 1,213.8; 0.8)
X_6	(0.0022950.0026494, 0.00308813, 0.003105; 0.8)	(10.2, 1,213.8; 0.8)
X_7	(0.17, 0.1933333, 0.20666667, 0.23; 0.9)	(1.80523, 2.1238, 2.44237; 0.9)

Step 4: This is based on such transformed data and the minimum cut set of the structure obtained as $\{X_1, X_2, X_3, X_5, X_6, X_7\}$. Using the table, expression (2.1) for the failure rate of the structure and repair time of the structure is given as follows:

$$\text{The failure rate of the structure} = \lambda_S = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4^* + \lambda_5^* + \lambda_6^* + \lambda_7^*. \tag{2.9}$$

$$\text{The repair time of the structure} = \tau_S = \frac{\lambda_1\tau_1 + \lambda_2\tau_2 + \lambda_3\tau_3 + \lambda_4^*\tau_4^* + \lambda_5^*\tau_5^* + \lambda_6^*\tau_6^* + \lambda_7^*\tau_7^*}{\lambda_S}. \tag{2.10}$$

2.5 Results and discussion

Focused on all these values, the various reliability parameters were estimated at various satisfaction rates varying from 0 to 0.7 at mission time $t = 2$ units through left and right spreads. The parameters of the considered structure have been calculated using Tables 2.6 and 2.7.

Table 2.6: MTTF, MTTR (mean time to repair), and MTBF (mean time between failure) of the structure.

Parameters→	MTTF		MTTR		MTBF	
Alpha cut↓	Left alpha cut	Right alpha cut	Left alpha cut	Right alpha cut	Left alpha cut	Right alpha cut
0	3.9562	5.3525	1.9985	4.9494	5.9547	10.3019
0.1	4.017543	5.258771	2.131286	4.741829	6.1488286	10.0006
0.2	4.078886	5.165043	2.264071	4.534257	6.3429571	9.6993
0.3	4.140229	5.071314	2.396857	4.326686	6.5370857	9.398
0.5	4.262914	4.883857	2.662429	3.911543	6.9253429	8.7954
0.6	4.324257	4.790129	2.795214	3.703971	7.1194714	8.4941
0.7	4.3856	4.6964	2.928	3.4964	7.3136	8.1928

In Figures 2.3–2.7, the reliability parameters such as MTTF, MTTR (mean time to repair), MTBF (mean time between failure), reliability, and availability have been shown. The values of each parameter are given in the interval at 70% satisfaction level.

MTTF has values in the range (4.3856, 4.6964) with 70% satisfaction level.

MTTR has values in the range (2.928, 3.4964) with 70% satisfaction level.

MTBF has values in the range (7.3136, 8.1928) with 70% satisfaction level.

Table 2.7: Reliability and availability of the structure.

Parameters→	Reliability		Availability	
Alpha cut↓	Left alpha cut	Right alpha cut	Left alpha cut	Right alpha cut
0	0.6032	0.6882	0.4439	1.0518
0.1	0.607571	0.6832	0.468857	1.006971
0.2	0.611943	0.6782	0.493814	0.962143
0.3	0.616314	0.6732	0.518771	0.917314
0.5	0.625057	0.6632	0.568686	0.827657
0.6	0.629429	0.6582	0.593643	0.782829
0.7	0.6338	0.6532	0.6186	0.738

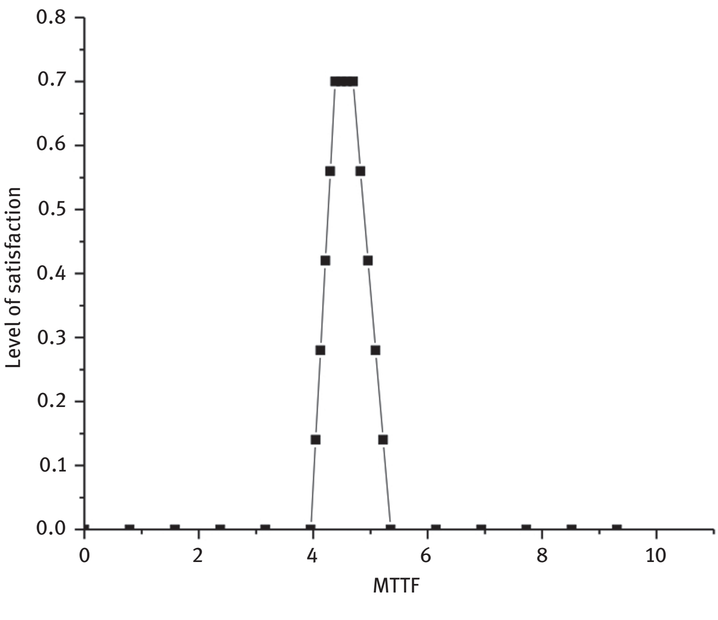

Figure 2.3: MTTF of the structure.

The reliability has values in the range (0.6338, 0.6532) with 70% satisfaction level.

The availability has values in the range (0.6186, 0.7380) with 70% satisfaction level.

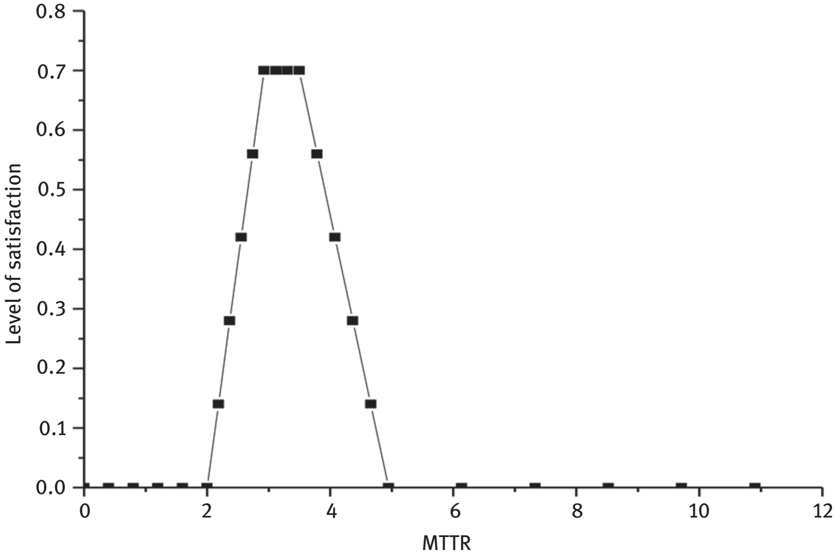

Figure 2.4: MTTR of the structure.

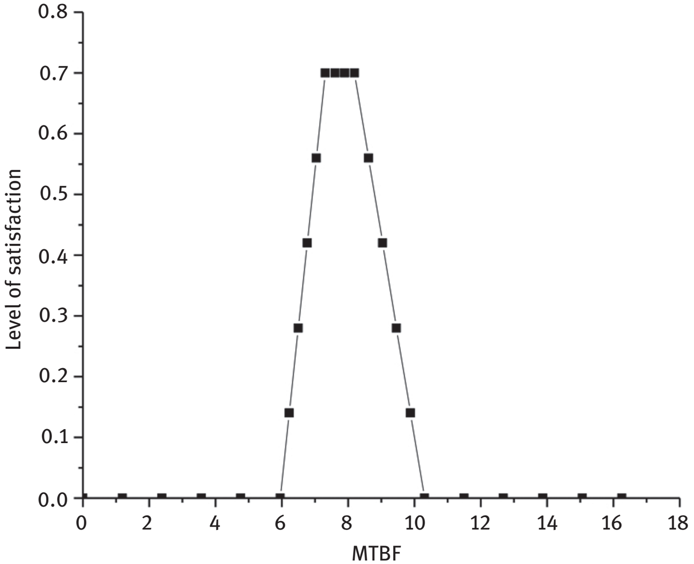

Figure 2.5: MTBF of the structure.

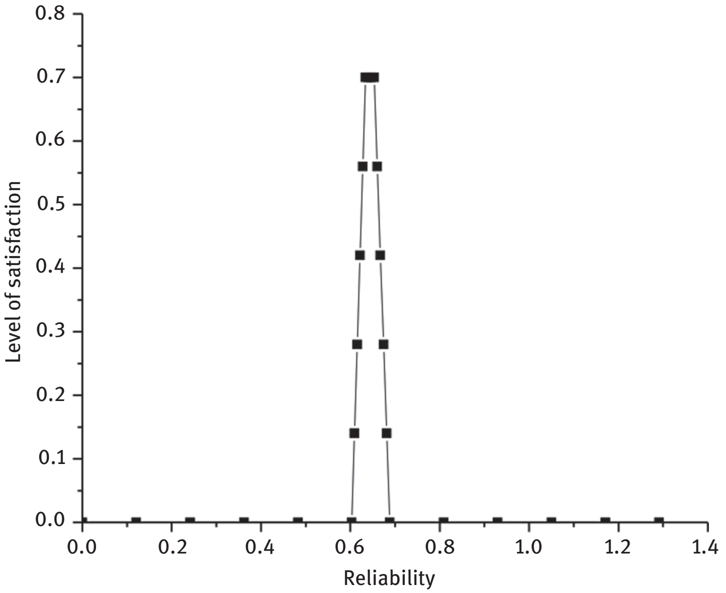

Figure 2.6: Reliability of the structure.

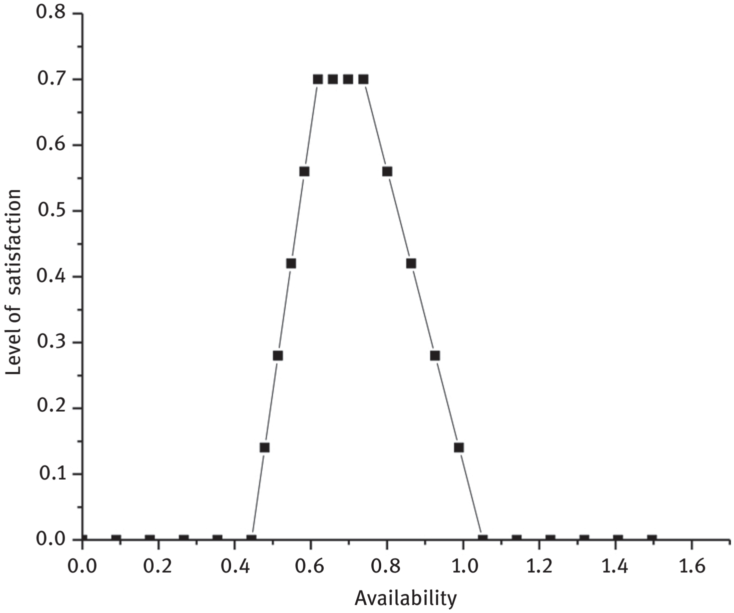

Figure 2.7: Availability of the structure.

2.6 Conclusion

This chapter takes into account the specific level of satisfaction to the data obtained, and thus, based on that, the fuzzified data are converted to the generalized trapezoidal fuzzy numbers to maintain the flatness of the number. An investigation has been carried out in evaluating the performance of the structure in a rice manufacturing plant by adopting an improved arithmetic operation between the generalized trapezoidal fuzzy numbers. The various reliability parameters at each spread rate have been calculated, adopting enhanced arithmetic operations. These system findings can allow executives who are concerned to prepare and adjust effective maintenance practices/strategies in enhancing system performance.

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3 On the bivariate modeling of reliability systems: some advances

Abstract: In this chapter, we study the reliability characteristics of well-known structures using two components. More precisely, we consider different bivariate distribution models such as the Freund's family or the Gumbel Type III family and shed light on several reliability attributes of series and parallel systems. Among other general results, we derive closed formulae for evaluating the reliability function, the mean residual lifetime, or the failure rate of the aforementioned reliability structures. Several figure-based remarks are pointed out, while some open issues referring to the bivariate reliability models are also highlighted.

Keywords: Continuous bivariate modeling, lifetime distributions, mean residual lifetime, failure rate, series and parallel systems

3.1 Introduction

Many researchers assume that the components that formulate a reliability structure operate independently to each other. This assumption leads practically to the acceptance that the corresponding lifetimes are distributed as independent random variables. However, this supposition is not always fulfilled in practice, since it seems quite often to deal with systems that operate under dependency. To that direction, some research activity has been already recorded in the literature. For example, Navarro and Lai [1] provided a study referring to the role of dependency to the performance of reliability structure. In addition, Gupta [2] and Navarro et al. [3] considered the bivariate Pareto distribution for modeling two-component reliability systems, while Joo and Mi [4] investigated the properties of parallel structures consisting of two dependent units under the Gumbel Type II bivariate exponential model.

In this chapter, we consider series and parallel structures consisting of two (possibly) dependent components. Two specific distribution models are taken into account in order to investigate several reliability properties of the underlying systems. Section 3.2 provides a reliability study under the Freund's bivariate model. Some explicit expressions for determining reliability characteristics of series and parallel structures consisting of two units are delivered. Several diagrams are constructed in order to clarify the importance of the design parameters of the Freund's

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bivariate model. Section 3.3 is devoted to an alternative distribution model, which is called Gumbel Type III exponential. Finally, Section 3.4 summarizes the results, while some concluding remarks are also provided.

3.2 The Freund's bivariate reliability modeling

Let us consider a reliability structure consisting of two (possibly) dependent components with lifetimes X_1, X_2. We then assume that the random vector (X_1, X_2) is exchangeable, namely, the components are supposed to be identical while the structure is symmetric. In this section, the vector (X_1, X_2) is assumed to follow the well-known bivariate Freund's distribution, which was originally introduced by Freund [5] and studied afterward by many researchers, such as Biswas and Nair [6], Adachi and Kodama [7], Klein and Moeschberger [8], or Khodr et al. [9]. Under the Freund's bivariate model, the joint probability density function (p.d.f.) of the vector (X_1, X_2) is given by (see, e.g., [10, 11])

$$h(x_1, x_2) = \begin{cases} a\beta' \exp\left(-(a+\beta-\beta')x_1 - \beta' x_2\right), & \text{for } x_1 \le x_2 \\ a'\beta \exp\left(-(a+\beta-a')x_2 - a' x_1\right), & \text{for } x_1 \ge x_2 \end{cases}, \tag{3.1}$$

while the corresponding cumulative density function (c.d.f.) is expressed as

$$H(x_1, x_2) = \begin{cases} \frac{1}{a+\beta-\beta'}\left(a \exp\left(-(a+\beta-\beta')x_1 - \beta' x_2\right) + (\beta-\beta')\exp(-(a+\beta)x_2)\right), & \text{for } x_1 \le x_2 \\ \frac{1}{a+\beta-a'}\left(\beta \exp\left(-(a+\beta-a')x_2 - a' x_1\right) + (a-a')\exp(-(a+\beta)x_1)\right), & \text{for } x_1 \ge x_2 \end{cases}. \tag{3.2}$$

It is evident that the marginals of the Freund's bivariate distribution correspond to mixtures of exponential univariate distributions. For this reason, Kotz et al. [10] argued that the aforementioned family of distributions should be termed as *bivariate mixture exponential distribution*. The p.d.f. of its marginal distribution is determined as (see, e.g., [11])

$$f(x_i) = \frac{(a-a')(a+\beta)}{a+\beta-a'}\exp(-(a+\beta)x_i) + \frac{a'\beta}{a+\beta-a'}\exp(-a' x_i), \quad i = 1, 2. \tag{3.3}$$

In fact, the Freund's bivariate model is closely related to systems, for which the breakdown of a single unit transfers a supplementary load to the others. In the sequel, we shall investigate several distribution properties of reliability structures consisting of two components under the Freund's bivariate model. We first consider the *odds for survival* of X or alternatively the so-called ratio $P(X \ge x)/P(X \le x)$ and the corresponding *bivariate survival odds ratio* of two random variables X_1, X_2 defined as $P(X_1 \ge x_1 \text{ or } X_2 \ge x_2)/P(X_1 \le x_1, X_2 \le x_2)$. The following proposition provides a formula for determining the bivariate survival odds ratio of lifetimes X_1, X_2 mentioned earlier.

Proposition 1. *Let us denote by X_1, X_2 the lifetimes of two units of a reliability structure, which follow the bivariate Freund's distribution (defined in eq. (3.1)). The bivariate survival odds ratio of X_1, X_2 is given as*

$$\frac{P(X_1 \geq x_1 \text{ or } X_2 \geq x_2)}{P(X_1 \leq x_1,\ X_2 \leq x_2)} = \begin{cases} \dfrac{(\alpha+\beta-\beta')\exp\big((\alpha+\beta-\beta')x_1+(\alpha+\beta)x_2\big)}{(\beta-\beta')\exp\big((\alpha+\beta-\beta')x_1\big)+\alpha\exp\big((\alpha+\beta-\beta')x_2\big)}-1, & \text{for } x_1 \leq x_2 \\ \dfrac{(\alpha+\beta-\alpha')\exp((\alpha+\beta)(x_1+x_2))}{(\alpha-\alpha')\exp((\alpha+\beta)x_1)+\beta\exp\big((\alpha+\beta-\alpha')x_2+\alpha' x_2\big)}-1, & \text{for } x_1 \leq x_2 \end{cases} . \tag{3.4}$$

Proof. The *bivariate survival odds ratio* of lifetimes X_1, X_2 can be written as

$$\frac{P(X_1 \geq x_1 \text{ or } X_2 \geq x_2)}{P(X_1 \leq x_1,\ X_2 \leq x_2)} = \frac{1-H(x_1, x_2)}{H(x_1, x_2)}.$$

We then substitute formula (3.2) into this equality and the result is straightforward. □

The following proposition deals with the conditional survival function of lifetime X_1 under the condition $X_2 = x_2$.

Proposition 2. *Let us denote by X_1, X_2 the lifetimes of two components of a reliability structure, which follow the bivariate Freund's distribution (defined in eq. (3.1)). The conditional survival function of X_1 given that $X_2 = x_2$ can be written as*

$$P(X_1 > x_1 | X_2 = x_2) = \frac{\alpha\alpha'(\alpha+\beta-\alpha')\exp\big(-(\alpha+\beta-\beta')(x_1-x_2)\big)}{(\alpha+\beta-\beta')\big((\alpha+\beta)(\alpha-\alpha')+\alpha'\beta\exp((\alpha+\beta-\alpha')x_2\big)}, \quad \alpha+\beta-\beta' > 0. \tag{3.5}$$

Proof. The conditional survival function of lifetime X_1 under the condition $X_2 = x_2$ can be expressed as

$$P(X_1 > x_1 | X_2 = x_2) = \int_{x_1}^{\infty} f(v|x_2)\,dv. \tag{3.6}$$

The density $f(v|x_2)$ corresponds to the conditional p.d.f. of lifetime X_1 given that $X_2 = x_2$ and can be expressed as

$$f(v|x_2) = \frac{f(v, x_2)}{f(x_2)},$$

where $f(v,x_2)$ is the joint p.d.f. of X_1,X_2, while $f(x_2)$ corresponds to the p.d.f. of lifetime X_2. Recalling expressions (3.2) and (3.3), the integral appeared in eq. (3.6) can be restated as

$$P(X_1>x_1|X_2=x_2)=\int_{x_1}^{\infty}\frac{a(a+\beta-a')\beta'\exp\big(-(a+\beta-\beta')(v-x_2)\big)}{(a+\beta)(a-a')+\beta a' exp\big((a+\beta-a')x_2\big)}dv,$$

and the proof is complete after some straightforward manipulations.

□

We then focus on the reliability attributes of series and parallel systems with two units under the bivariate Freund's distribution model. It is worth mentioning that an analogous effort has been already accomplished by several researchers for different bivariate cases [3, 12–14].

The following proposition provides explicit expressions for the computation of the survival functions of series and parallel reliability structures consisting of two Freund's distribution-based components.

Proposition 3. *We symbolize by $X_{(1)},X_{(2)}$ the random lifetimes of a series and a parallel structure consisting of two components X_1,X_2. If (X_1,X_2) follow the bivariate Freund's distribution (defined in eq. (3.1)), then the following ensue:*

(i) the reliability function of the parallel structure $X_{(2)}$ is expressed as

$$R_{(2)}(x)=1-\exp(-(a+\beta)x), \tag{3.7}$$

(ii) the reliability function of the series structure $X_{(1)}$ is given by

$$R_{(1)}(x)=\frac{\exp\big(-(a+\beta+a')x\big)\big(2\beta\exp(a+\beta)x+(3a+\beta-3a')\exp(a'x)\big)+a'-a-\beta}{a+\beta-a'}. \tag{3.8}$$

Proof. (i) The reliability function of the parallel structure with two units can be written in terms of the joint c.d.f. $H(x_1,x_2)$ as follows:

$$R_{(2)}(x)=P(X_{(2)}>x)=P(X_1>x \text{ or } X_2>x)=1-H(x,x).$$

By substituting expression (3.2) into this equality, the desired result is effortlessly derived.

(ii) In order to determine the reliability function of the series system consisting of two components under the bivariate Freund's distribution, we first recall that

$$R_{(1)}(t)=2R_i(t)-R_{(2)}(t), \quad i=1,2 \tag{3.9}$$

(see [15]) where $R_i(x)$ corresponds to the survival function of lifetimes X_i, $i=1,2$. Based on eq. (3.3), the c.d.f. of the lifetimes X_i, $i=1,2$ can be written as

$$F_i(x_i)=\frac{\beta-\beta\exp(x_i-a')+(a-a')(1-\exp(-(a+\beta)x_i)}{a+\beta-a'} \tag{3.10}$$

We then substitute formulae (3.7) and (3.10) into eq. (3.9), and the proof is complete. □

When a reliability system operates, it is important to estimate whether its failure may occur with low, medium, or high probability during the next time period. Generally speaking, if we denote by X an absolutely continuous random variable with the reliability function and p.d.f. $R(x)$ and $f(x)$, respectively, then the so-called failure rate is computed via the following expression:

$$r(x)=\frac{f(x)}{R(x)} \tag{3.11}$$

for all x so that $R(x)>0$ [16, 17]. The following proposition offers explicit expressions for the computation of the failure rate of reliability structures under the bivariate Freund's distribution model.

Proposition 4. *We symbolize by $X_{(1)},X_{(2)}$ the random lifetimes of a series and a parallel structure consisting of two components X_1,X_2. If (X_1,X_2) follow the bivariate Freund's distribution (defined in (3.1)), then the following ensue*:
(i) The failure rate of the parallel structure $X_{(2)}$ is given by

$$r_{(2)}(x)=\frac{a+\beta}{\exp((a+\beta)x)-1}, \tag{3.12}$$

(ii) the failure rate of the series structure $X_{(1)}$ is stated as

$$r_{(1)}(t)=-\frac{2\beta a'\exp((a+\beta)x)+(a+\beta)(3a+\beta-3a')\exp(a'x)}{2\beta\exp((a+\beta)x)+(3a+\beta-3a')\exp(a'x)-(a+\beta-a')\exp((a+\beta+a')x)}. \tag{3.13}$$

Proof. (i) We first derive an explicit expression for the p.d.f. of random variable $X_{(2)}$ as follows:

$$f_{(2)}(x)=-(a+\beta)\exp(-(a+\beta)x). \tag{3.14}$$

The outcome we are looking for is readily deduced by substituting expressions (3.7) and (3.14) into eq. (3.11).

(ii) Employing analogous arguments, we express the p.d.f. of random variable $X_{(1)}$ as

$$f_{(1)}(t) = \frac{\left(-2a'\beta\exp((a+\beta)x) - (a+\beta)(3a+\beta-3a')\exp(a'x)\right)\exp\left(-(a+\beta+a')x\right)}{a+\beta-a'} \tag{3.15}$$

and the desired outcome is reached by substituting expressions (3.8) and (3.15) into eq. (3.11).

□

When a reliability system is under investigation, it is of great importance to estimate its remaining operating time. The most common way of checking it over is based on its average time till the system fails, namely, the so-called mean residual lifetime (*MRL*, hereafter) of the underlying reliability structure. For more details, one may refer to the detailed monograph of Kuo and Zuo [18] or the survey of Ram [19]. The following proposition provides formulae for computing the *MRL* function of reliability systems under the bivariate Freund's distribution model.

Proposition 5. *We symbolize by $X_{(1)}, X_{(2)}$ the lifetimes of a series and a parallel structure with two units X_1, X_2. If (X_1, X_2) follows the Freund's distribution (defined in eq. (3.1)), then the following ensue:*
(i) the MRL of the parallel structure $X_{(2)}$ is given by

$$m_{(2)}(x) = -\frac{a+\beta}{1-\exp(-(a+\beta)x)}B(\exp(-(a+\beta)x;2,2) \tag{3.16}$$

(ii) the MRL of the series structure $X_{(1)}$ is stated as

$$m_{(1)}(x) = \frac{(2\beta B(\exp(-x);-\beta'+1;1)+(3(a-a')+\beta)B(\exp(-x);}{\exp\left(-(a+\beta+a')x\right)(2\beta\exp(a+\beta)x} \frac{-(a+\beta)+1;1)-(a+\beta-a')\exp(-x))}{+(3a+\beta-3a')\exp(a'x))+a'-a-\beta}, \tag{3.17}$$

where $B(y;v,w) = \int_0^y q^{v-1}(1-q)^{w-1}dq$ is the incomplete Beta function.

Proof. (i) The *MRL* of the parallel structure $X_{(2)}$ consisting of two components under the bivariate Freund's distribution model is defined as

$$m_{(2)}(x) = E(X_{(2)} - x \mid X_{(2)} > x) = \frac{1}{R_{(2)}(x)}\int_x^\infty R_{(2)}(t)dt$$

$$= -\frac{a+\beta}{1-\exp(-(a+\beta)x)}\int_x^\infty (1-\exp(-(a+\beta)t))dt, \tag{3.18}$$

where the respective reliability function $R_{(2)}(t)$ can be determined by the aid of eq. (3.7). We then apply the transformation $u = \exp(-(a+\beta)t)$ in the integral appeared in eq. (3.18) and the following ensues:

$$m_{(2)}(x) = -\frac{a+\beta}{1-\exp(-(a+\beta)x)} \int_0^{\exp(-(a+\beta)x)} u(1-u)du.$$

The outcome is now straightforward.

(ii) The *MRL* of the series structure $X_{(1)}$ consisting of two components under the bivariate Freund's distribution model is defined as

$$m_{(1)}(x) = E(X_{(1)} - x \big| X_{(1)} > x) = \frac{1}{R_{(1)}(x)} \int_x^{\infty} R_{(1)}(t)dt, \tag{3.19}$$

where the respective reliability function $R_{(1)}(t)$ can be determined by the aid of eq. (3.8). We then apply the transformation $u = \exp(-x)$ and the integral appeared in eq. (3.19) takes on the following form:

$$\int_x^{\infty} R_{(1)}(t)dt = \frac{2\beta}{a+\beta-a'} \int_0^{\exp(-x)} u^{-a'}du + \frac{3a+\beta-3a'}{a+\beta-a'} \int_0^{\exp(-x)} u^{-a-\beta}du - \exp(-x).$$

The result is now straightforward.

□

It is worth mentioning that one should substitute $x = 0$ into the earlier formulae, and the mean time to failure of both series and parallel structures with two units under the bivariate Freund's distribution model can be readily determined [20, 21]. Figures 3.1–3.8 display the failure rates of a series and a parallel system under the bivariate Freund's distribution.

Based on Figures 3.1–3.8, we can readily deduce that, under the bivariate Freund's distribution model, the following remarks hold true:

- the failure rate of the parallel structure consisting of two units decreases as the parameter a increases;
- the failure rate of the parallel structure consisting of two units decreases as the parameter β increases;
- the failure rate of the series structure consisting of two units decreases as the parameter a increases;
- the failure rate of the series system consisting of two units increases as the parameter β increases.

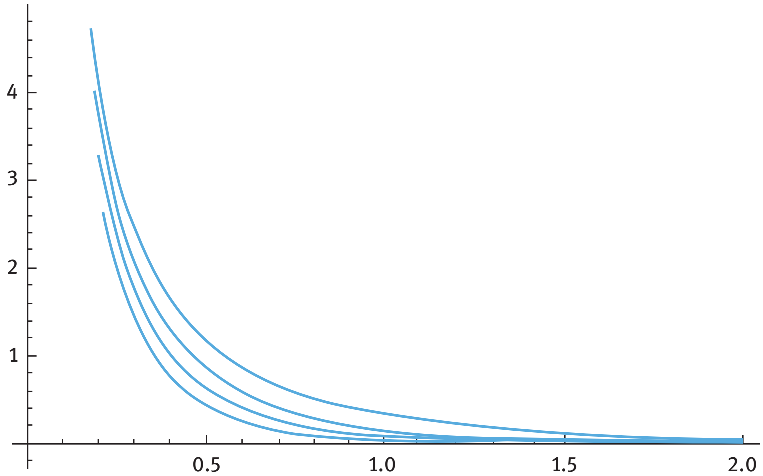

Figure 3.1: Failure rate of $X_{(2)}$ under the bivariate Freund's distribution for several designs ($a = 1$, $\beta = 1, 2, 3, 4$).

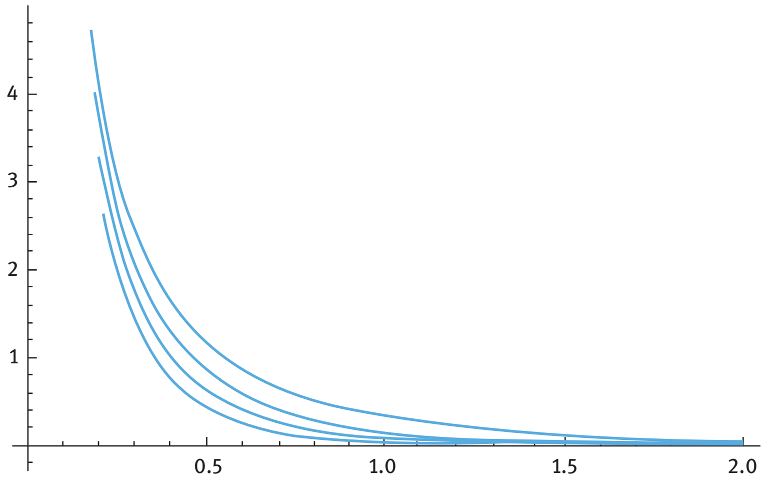

Figure 3.2: Failure rate of $X_{(2)}$ under the bivariate Freund's distribution for several designs ($a = 1, 2, 3, 4$, $\beta = 1$).

3.3 Gumbel's Type III bivariate reliability modeling

Let us next consider once again a reliability structure consisting of two (plausibly) dependent units with exchangeable lifetimes X_1, X_2. In this section, the random vector (X_1, X_2) is supposed to follow the Gumbel's Type III bivariate exponential model, which was first proposed by Gumbel [22] and since then has been studied by many

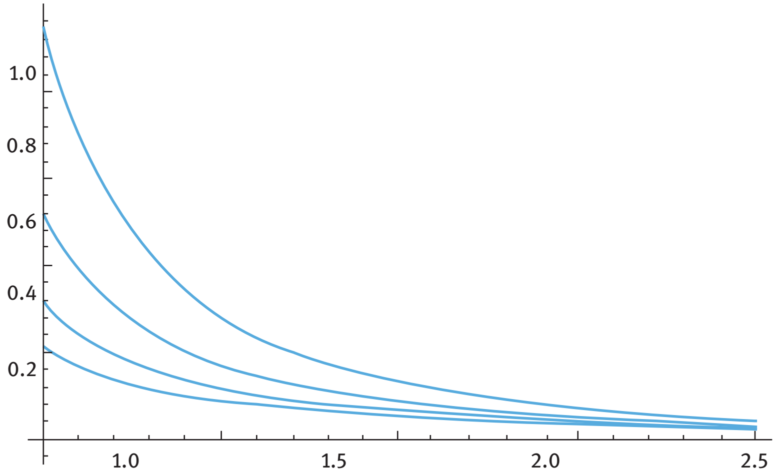

Figure 3.3: Failure rate of $X_{(1)}$ under the bivariate Freund's distribution for several designs ($a = 3, 4, 5, 6, \beta = a' = 1$).

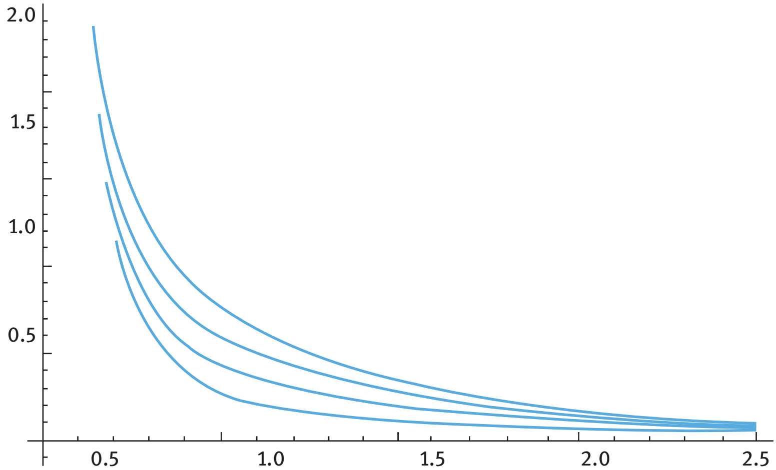

Figure 3.4: Failure rate of $X_{(1)}$ under the bivariate Freund's distribution for several designs ($a = 6, a' = 1, \beta = 1, 2, 3, 4$).

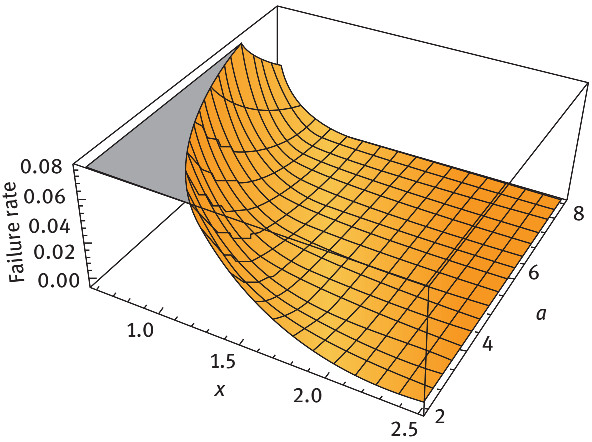

Figure 3.5: Failure rate of $X_{(2)}$ under the bivariate Freund's distribution for several designs (a, $\beta = 1$).

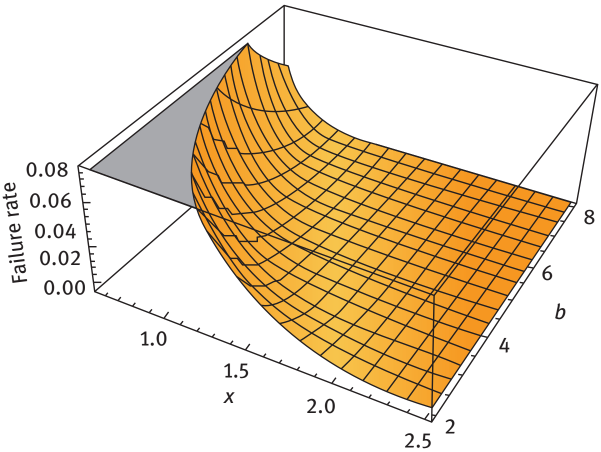

Figure 3.6: Failure rate of $X_{(2)}$ under the bivariate Freund's distribution for several designs ($a = 1$, β).

authors [15, 23, 24]. Under the Gumbel's Type III bivariate model, the joint p.d.f. of vector (X_1, X_2) is given by (see, e.g., [23])

$$f(x_1, x_2) = (\theta_1\theta_2)^{-1/\delta}(x_1x_2)^{1/\delta-1}\left[(x_1/\theta_1)^{1/\delta} + (x_2/\theta_2)^{1/\delta}\right]^{\delta-2} \times \left[\left\{(x_1/\theta_1)^{1/\delta} + (x_2/\theta_2)^{1/\delta}\right\}^{\delta} + 1/\delta - 1\right] \exp\left[-\left\{(x_1/\theta_1)^{1/\delta} + (x_2/\theta_2)^{1/\delta}\right\}^{\delta}\right], \tag{3.20}$$

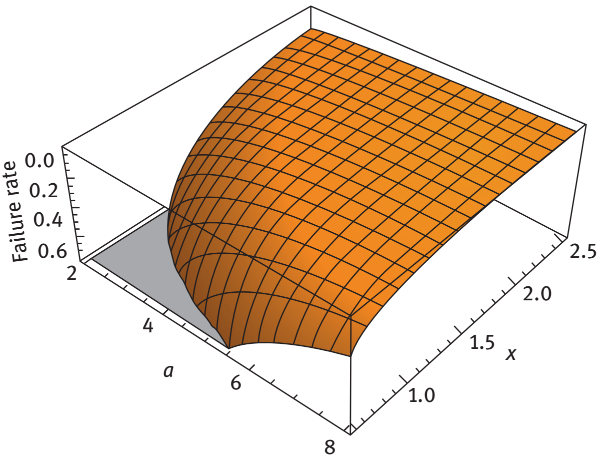

Figure 3.7: Failure rate of $X_{(1)}$ under the bivariate Freund's distribution for several designs (a, $a' = \beta = 1$).

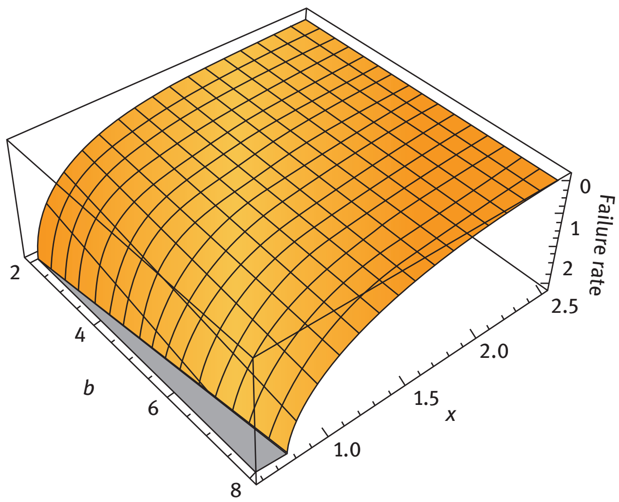

Figure 3.8: Failure rate of $X_{(1)}$ under the bivariate Freund's distribution for several designs ($a = 3$, $a' = 1$, β).

while the corresponding joint survival function is expressed as

$$\bar{F}(x_1, x_2) = \exp\left(-\left((x_1/\theta_1)^{1/\delta} + (x_2/\theta_2)^{1/\delta}\right)^{\delta}\right). \tag{3.21}$$

The Gumbel's Type III bivariate model involves two scale parameters θ_1 and θ_2 and a dependence parameter δ. Lu and Bhattacharyya [23] argued that because of the complicated form of its p.d.f., the aforementioned distribution model did not manage to attract further research attention. However, it is noteworthy that the marginals of the Gumbel's Type III bivariate model are exponential with means θ_1 and θ_2, respectively.

In the sequel, we shall investigate several properties of reliability structures consisting of two components under the Gumbel's Type III bivariate distribution model. We first consider the so-called *bivariate survival odds ratio* of such exchangeable lifetimes X_1, X_2.

Proposition 6. *Let X_1, X_2 denote the lifetimes of two units of a reliability structure, which follow the Gumbel's Type III bivariate model (defined in eq. (3.20)). The bivariate survival odds ratio of X_1, X_2 is given as*

$$\frac{P(X_1 \geq x_1 \text{ or } X_2 \geq x_2)}{P(X_1 \leq x_1,\ X_2 \leq x_2)}$$
$$= \frac{\exp(-x_1/\theta_1) + \exp(-x_2/\theta_2) - \exp\left(-\left((x_1/\theta_1)^{1/\delta} + (x_2/\theta_2)^{1/\delta}\right)^{\delta}\right)}{1 - \exp(-x_1/\theta_1) - \exp(-x_2/\theta_2) + \exp\left(-\left((x_1/\theta_1)^{1/\delta} + (x_2/\theta_2)^{1/\delta}\right)^{\delta}\right)}. \tag{3.22}$$

Proof. The *bivariate survival odds ratio* of X_1, X_2 can be expressed as $P(X_1 \geq x_1$ or $X_2 \geq x_2)/P(X_1 \leq x_1,\ X_2 \leq x_2)$. We then recall eq. (3.20), and the result that we are looking for is straightforward.

□

We then focus on reliability attributes of series and parallel structures with two units under the Gumbel's Type III bivariate distribution model. The following proposition offers closed formulae for the survival functions of well-known reliability structures with two Gumbel's Type III bivariate distribution-based components.

Proposition 7. *We symbolize by $X_{(1)}, X_{(2)}$ the lifetimes of a series and a parallel structure with two units X_1, X_2. If the random vector (X_1, X_2) follow the Gumbel's Type III bivariate distribution with scale parameters θ_1 and θ_2 such that $\theta_1 = \theta_2 = \theta$ (defined in (3.20)), then the following ensue:*
(i) the reliability function of the parallel structure $X_{(2)}$ is given by

$$R_{(2)}(x) = 2\exp(-x/\theta) - \exp\left(-2^{\delta}x/\theta\right), \tag{3.23}$$

(ii) the reliability function of the series structure $X_{(1)}$ is stated as

$$R_{(1)}(x) = \exp\left(-2^{\delta}x/\theta\right). \tag{3.24}$$

Proof. (i) The reliability function of the parallel structure with two units could be restated in terms of the joint survival function as follows:

$$R_{(2)}(x) = P(X_{(2)} > x) = P(X_1 > x \text{ or } X_2 > x).$$

Recalling expression (3.20), the result is straightforward.

(ii) Since the marginals of the bivariate Gumbel's Type III model are exponential with means θ_1 and θ_2, respectively, the proof is complete by combining eqs. (3.9) and (3.23).

□

The following proposition offers explicit expressions for the computation of the failure rate of reliability structures under the bivariate Gumbel's Type III distribution model.

Proposition 8. *We symbolize by $X_{(1)}, X_{(2)}$ the lifetimes of a series and a parallel structure with two units X_1, X_2. If the random vector (X_1, X_2) follow the bivariate Gumbel's Type III distribution with scale parameters θ_1 and θ_2 such that $\theta_1 = \theta_2 = \theta$ (defined in eq. (3.20)), then the following ensue:*

(i) *the failure rate of the parallel structure $X_{(2)}$ is given by*

$$r_{(2)}(x) = -\frac{2x\exp(2^\delta x/\theta) + 2^\delta x\exp(x/\theta)}{\theta x\exp(x/\theta) - 2\theta\exp(2^\delta x/\theta)}, \tag{3.25}$$

(ii) *the failure rate of the series structure $X_{(1)}$ is stated as*

$$r_{(1)}(t) = 2^\delta. \tag{3.26}$$

Proof. (i) The p.d.f. of the lifetime $X_{(2)}$ could be stated as

$$f_{(2)}(x) = \frac{2\exp(-x/\theta) + 2^\delta\exp\left(-2^\delta x/\theta\right)}{\theta}. \tag{3.27}$$

The outcome we are looking for is readily deduced by substituting expressions (3.23) and (3.27) into eq. (3.11).

(ii) Employing analogous arguments, we express the p.d.f. of $X_{(1)}$ as

$$f_{(1)}(t) = \frac{2^\delta\exp\left(-2^\delta x/\theta\right)}{\theta}, \tag{3.28}$$

and the desired outcome is reached by substituting expressions (3.24) and (3.28) into eq. (3.11).

□

Figures 3.9–3.12 display the failure rates of a series and a parallel system under the bivariate Gumbel's Type III model.

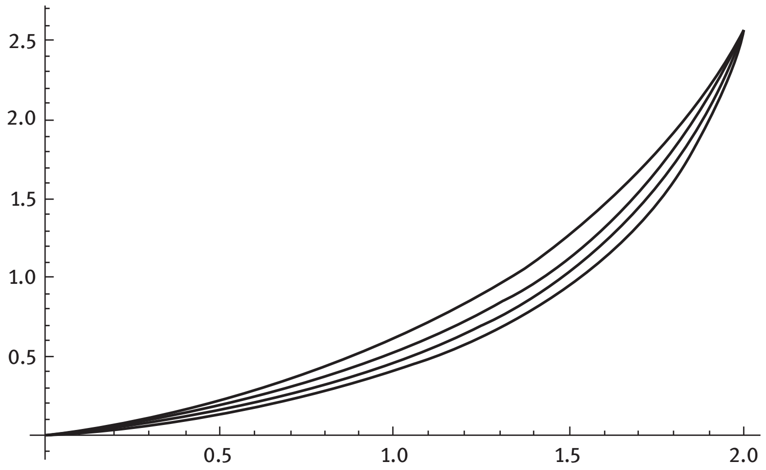

Figure 3.9: Failure rate of $X_{(2)}$ under the bivariate Gumbel Type III distribution for fixed δ.

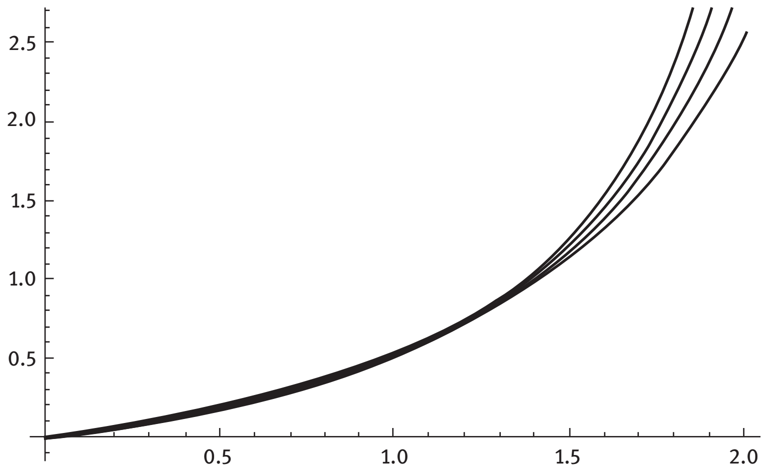

Figure 3.10: Failure rate of $X_{(2)}$ under the Gumbel Type III distribution for fixed θ.

Based on Figures 3.9–3.12, we can readily deduce that, under the bivariate Gumbel Type III distribution model, the following remarks hold true:

- the failure rate of the parallel structure consisting of two units decreases as the parameter θ increases and
- the failure rate of the parallel structure consisting of two units decreases as the parameter δ increases.

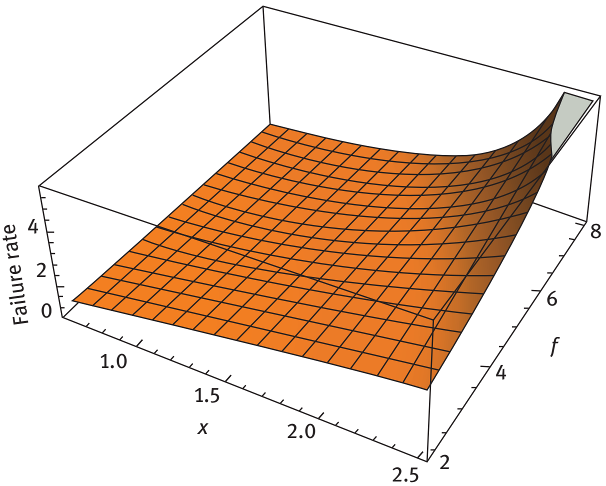

Figure 3.11: Failure rate of $X_{(2)}$ under the bivariate Gumbel Type III distribution for $\delta = 0.9$.

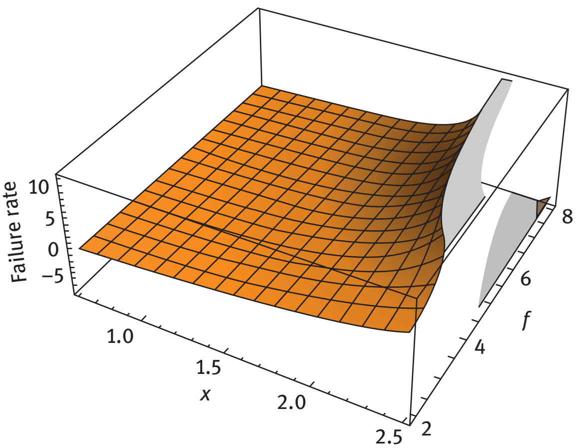

Figure 3.12: Failure rate of $X_{(2)}$ under the bivariate Gumbel Type III distribution for fixed $\delta = 0.5$.

3.4 Discussion

In this chapter, two specific bivariate distribution models have been taken into account in order to investigate several reliability characteristics of series and parallel structures consisting of two (possibly) dependent units. The performance of the underlying systems is evaluated by the aid of their failure rate or MRL. Several figures were constructed in order to depict the impact of the design parameters of each distribution model on the behavior of the structures. It is of some future research interest to consider alternative bivariate distribution models in order to shed light on the overall performance of the underlying reliability systems with two components. In addition, it would be interesting to examine the corresponding characteristics of reliability systems with more than two components by thinking over well-known multivariate distribution models.

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Alka Munjal and Jasvir Kaur

4 Reliability and redundancy in performance of equipment

Abstract: The chapter focuses primarily on the understanding of the reliability of equipment. It investigates the history, nature, and the importance of reliability of the system. It also includes the failure rate curve and the reliability theory for reliability engineering. This chapter describes the factors influencing the reliability of the equipment/system such as intrinsic and extrinsic factors. It includes the types of reliability and how we can make improvement in each case. Relation and difference between reliability and validity of the system, and how redundancy plays a crucial role in making the system reliable are discussed. The applications of reliability and methods that can be used for improvement of reliability of the system are described.

Keywords: Reliability, intrinsic and extrinsic factors, failure rate curve, redundancy in performance

4.1 Introduction

Reliability is the quality of being able to be trusted or believed because of working or behaving well. It is an important attribute of a product for the utilization and maintenance of any system. In other words, reliability is also termed as the ability of an item to perform functions that are required under specific conditions such as environmental and operational conditions for a long period of time.

In the area of statistics, reliability is nothing but the consistency of a measure. A measure is of high reliability only if similar results can be obtained from the system under consistent conditions. Reliability engineering progressed over the years, which led to more reliable products. The requirement of reliability is for enhancing the capacity of a system to work properly and prevent this from disaster. Reliability deals with the advancement and technique used in effectiveness of the system by reducing failure frequency. It involves cost minimization.

Reliability plays a vital role in every phase of the life cycle of the equipment/product, that is, from design to manufacture. Cost-effective systems can be developed, which will result in better performance. According to some mathematicians, reliability is the absence of failure. Reliability of the components of complex equipment depends

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on the number of random factors. There is no specific equipment to measure the reliability of the system. Reliability depends on many factors such as how the system is manufactured, the surrounding where equipment performs the functions, and others. Hence, we can say that reliability of the equipment depends on many uncertain factors. We cannot say anything about the failures and the timing of the failures.

4.2 Review of literature

Reliability becomes one of the most important issues in modern era. Every person who built new equipment wants to make that equipment as much reliable as possible. In our daily life, we deal with a number of complex systems, and their efficiency is closely related to the reliability of the system as reliability is the study of failures. Failure of any of the parts of the system is the main cause of the reduction of efficiency of whole systems. In this chapter, we describe the connection between reliability and validity. Hence, reliability is the main quality of the system. Usually, people use the redundancy design for improving the reliability of the system. k-Out-of-m models are very useful for improving the reliability of electrical systems.

Several studies on the analysis of reliability of the complex system have been done. Yusuf et al. [1] analyzed the stochastic modeling of a two-unit parallel system under two types of failures. Coit et al. [2] have studied the optimization of reliability of a system with k-out-of-n subsystems and also investigated k-out-of-n:G systems with dependent failures and imperfect coverage. Varma [3] has examined the stochastic behavior of a complex system with standby redundancy. Goel et al. [4] have examined the stochastic behavior of a two-unit parallel system with partial and catastrophic failures and analyzed prevention measures for maintenance. Bazovsky [5] has discussed the reliability theory and practice. Oliveira et al. [6] also studied the system with the help of a supplementary variable technique. Dhillon et al. [7] have studied the reliability of an identical unit parallel system with its failures. Chung [8] has estimated the analysis of reliability of a k-out-of-n redundant system. Zhang [9] dealt with a repairable system having $(n+1)$ units and a single repair facility, in which unit 1 has preemptive priority both in getting operated and repaired. Nailwal et al. [10] have analyzed the reliability of a complex system with three possibilities in repair with the application of copula. Nailwal et al. [11] have applied copula in reliability measures and sensitivity analysis of a complex matrix system including power failure. Goel et al. [12] analyzed a 1-out-of-3 system with a warm and a cold spare units and inspection. A lot of literature is available in the field of Markov repairable system. To cite a few, Zheng et al. [13] discussed a single-unit Markov repairable system with repair time omission, and Cui et al. [14] considered the several indexes including availability for aggregated Markov repairable system. Ram and

Singh [15] studied on the availability, mean time to failure, and cost analysis of a complex system with the help of Gumbel–Hougaard family copula.

4.3 Reliability theory

Reliability theory came into existence on the solid foundations of various equipments used in World War II with the origin of the reliability discipline. Before this, reliability was linked mostly to repeatability: when a test was considered "reliable," the same results would be obtained repeatedly. The discipline is defined as qualification and assessment of probability for performing the actions properly by a device for a long period of time intended under operating conditions. Reliability engineers work with systems having elements that show configurations, such as serial configuration, parallel configuration, mixed configuration, series–parallel configuration, and parallel–series configuration, and calculate their reliability. The study of causes of failures and their consequences is known as reliability. It is one of the prerequisite characteristics for the good quality of products.

Let "T" be the random variable denoting the lifetime or the time before the failure occurs. Now the probability that a system will work under the given environmental and operating conditions until time "t" reaches is given by

$$R(t) = P(T > t).$$

Clearly, reliability is a function of time. Since $R(t)$ is a probability, it must lie in the closed interval $[0, 1]$. But it is not possible to attain values 0 and 1 in real life. Thus, it always satisfies the following:

$$0 < R(t) < 1 \text{ and } R(t) = 0, \text{ if } t = \infty;\ R(t) = 1, \text{ if } t = 0.$$

4.4 Factors affecting the reliability

Factors influencing the reliability of test scores are classified into two categories.

4.4.1 Intrinsic factors

The intrinsic factors are those factors that occur when the test is conducted, and the principal intrinsic factors are given further.

4.4.1.1 Length of the test

Reliability is directly related to the length of the test. The reliability of the system has increased as the number of items contained in a test increases. In other words, more should be taken to get the more reliable system/equipment. However, it is not easy to attain the maximum length of the test to get the best reliable system because it became more difficult to handle that system. But tests containing small number of items lead to fatigue effects. Thus, it is better to use tests of large length rather than of small length. It is necessary that whenever a person measures the length of a test, he/she should note that the items added to increase the length of the test must satisfy the conditions such as equal range of difficulty, desired discrimination power, and comparability with other test items.

Let n be the random variable that denotes the number of times a test to be lengthened, and this is calculated as

$$n = \frac{r_{\mathrm{d}}(1 - r_{\mathrm{o}})}{r_{\mathrm{o}}(1 - r_{\mathrm{d}})}$$

where r_{d} is the desired reliability, r_{o} is the obtained reliability, and n is the number of times a test to be lengthened.

4.4.1.2 Homogeneity of items

Homogeneity of items includes two phases: one is item reliability and other is homogeneity of traits measured from one item to another. It is found that if different functions are measured by the items in a system and their intercorrelations are nearly zero, then the reliability of the system is very low.

4.4.1.3 Difficulty level of test

The level of test and clarification of terms that will be used in a test plays a vital role in reliability of test scores. If the test items are difficult, then it becomes more difficult for the group members to understand this. Hence, it will reduce the reliability of the system. It restricts the effectiveness of test scores.

4.4.1.4 Discriminative value

A discriminative value affects the reliability of the system to a large extent. If items can be recognized between superior and inferior, then their total correlation is high. Hence, the reliability of the system is high.

4.4.1.5 Test instructions

It is known that if the instructions of a test are concise and can be easily perceived, then the test can be conducted in a regular manner. Hence, it will increase the reliability. On the other hand, if the directions are complex and unclear, then these create the problem in understanding the procedure and performing the test, and may decrease the reliability.

4.4.1.6 Item selection

Item selection is a factor that we cannot ignore during the system's test scores. If there are large numbers of items involved in a test and they depend on each other, then the reliability of the system is reduced.

4.4.1.7 Maturity of the scorer

The maturity of the scorer also influences reliability of the test. Suppose the scorer is unpredictable and of fluctuating type, then the scores will vary from time to time according to his/her mood. Thus, it leads to low reliability.

4.4.2 Extrinsic factors

The extrinsic factors are those factors that do not lie in the test, and the important factors influencing the reliability are mentioned further.

4.4.2.1 Group variability

It is observed that when the group of pupils is tested and found to be homogeneous in ability, then the reliability of the test scores is less and vice versa.

4.4.2.2 Guessing and chance errors

Guessing in a test is one of the main factors influencing the reliability. It increased the chances of error, which reduces the reliability of the system. For example, if there are two alternatives and we have to guess the correct one, then there are only 50% chances of selection of the correct item.

4.4.2.3 Environmental conditions

We need uniform environmental conditions during the whole test, but it is not possible in real life. Hence, it may adversely affect the reliability of the test scores.

4.4.2.4 Momentary fluctuations

The next factor influencing the reliability of the equipment is momentary fluctuations. It may decrease the reliability of the system. For example, distraction by the sound of a train moving outside and a broken pencil while writing a work may lead to mistakes in the final answer and this creates a huge problem in further manufacturing of equipment.

4.5 Failure rate curve

The state of omission of an expected action from the device and a change in its properties that seriously affect or completely stop the functioning is known as failure.

The concept of failure helps in evaluating the reliability of a component, and the study of failure in detail is very helpful in making the system reliable. There are some components that have well-defined failures and some have not.

As we know, the component may fail with high frequency at commencement. These are termed as infant failures or early failures. The main reason for these failures is the defects in processing and manufacturing of the equipment.

The constant or stable failures happened because we use the component for a longer period. Such failures are termed as catastrophic failures/service failures. We can characterize these failures by the constant number of failures occurred in one unit of time.

With the passage of time, the component starts to deteriorate. Thus, the frequency of failures is rapidly increased. This happens because service life of various components have been exceeded. These failures are called as wear-out failures.

The pattern of failures of an item can be easily understood from a bathtub curve, which is just a hazard function. This curve is very useful in reliability engineering. It is not necessary that every item follows the bathtub curve. This depicts the relative failure rate of products over time. But there may be cases such that some components fail at an early stage, some components work until wear out then fail, and some components fail after their service life. In the bathtub curve (Figure 4.1), three periods are shown: first is infant mortality period (start-up commissioning), second is normal life period (useful life period), and third is normal wear-out period. First, the infant mortality period is delineated with a decreasing failure rate, and the main causes of these failures are defects and blunders. Normal life period is delineated with low and constant failure rate. It includes mostly those failures that occurred randomly. The last period is characterized with an increasing failure rate, and the main causes of these failures are fatigue and depletion of materials.

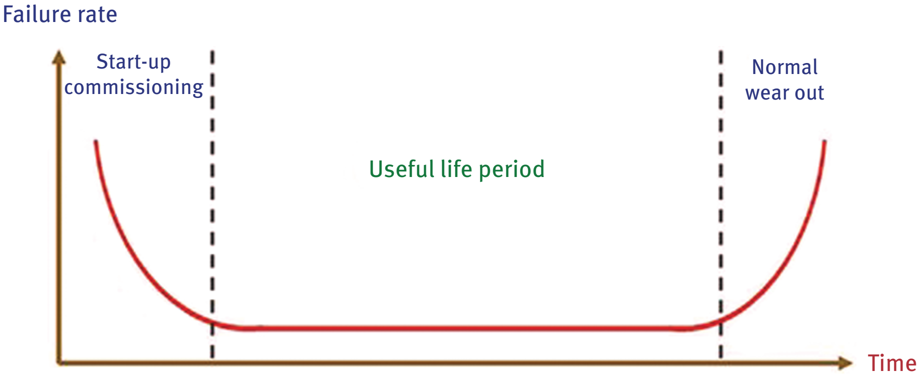

Figure 4.1: Bathtub curve.

4.6 Types of reliability

The following are several general classes of reliability:
(a) Inter-rater reliability
(b) Intermethod reliability
(c) Test–retest reliability
(d) International consistency reliability.

4.6.1 Inter-rater reliability

It is also known as inter-rater concordance or interobserver reliability. This reliability is the extent of annexure among two or more than two raters in the appraisals. Inter-rater reliability is the measure of equipollency or consensus occurs in the ratings stated by different judges. For example, a student of 10th class in DAV school made a project on some topic, but four examiners in the department submitted different results for the same student's project. This is the example of low inter-rater reliability.

4.6.1.1 Improving inter-rater reliability

– Criteria for rating, counting, and categorizing the variables that we will use in the system should be preplanned.
– Variables and methods that are used to measure inter-rater reliability should be defined properly.
– Same information and training should be given to all researchers who are involved.

4.6.2 Intermethod reliability

It is also known as parallel form of reliability when we deal with forms. This reliability is the extent of annexure the consistency has in the output of test score when there is a variation in the methods or instruments that are used in the test. This allows inter-rater reliability to be ruled out.

4.6.2.1 Improving intermethod reliability

– Tests or questionnaires and instruments involved each time should be based on the same theory.
– Make sure that all tests are formulated to measure the same thing.

4.6.3 Test–retest reliability

It is also known as repeatability test–retest reliability. This reliability is the extent of annexure the consistency has in the output of a test score from one test regime to the other. Output is collected from the same rater under same testing conditions. It weaves intra-rater reliability. For example, students of 10th class in DAV school complete a questionnaire for measuring personality traits. After days, weeks, or months apart, the same questionnaire was given to the same class and they give the same answers. This is the example of high test–retest reliability.

4.6.3.1 Improving test–retest reliability

- Tests or questionnaires should be designed and formulated in such a way that answers of the participants would not be influenced by their mood and concentration.
- During the collection of measurements, testing conditions should remain the same.

4.6.4 Internal consistency reliability

This reliability is the extent of annexure the consistency of output of a test scores across items within a test. Internal consistency is a degree that is basically based on the correlations among distinct items involved in a particular test.

Internal consistency is an informal way to compare the similarity of answers. In real life, it is not easy to check that whether the internal consistency is high or low because there is a wide variety of answers and changes with time also. Cronbach's alpha is a widely used statistical internal for consistency.

4.6.4.1 Improving internal consistency

- Tests or questionnaires and instruments involved and designed to deem the same concept should be based on the same theory.
- Surveys should be sent out at the same time because over different periods of time, confounding variables could be introduced.

4.7 The reliability coefficient

A reliability coefficient is an index for checking how well a particular test measures accomplishment (Table 4.1). It is defined as the ratio of variance in observed scores ascribable to actual scores. It measures the accuracy of the test or consistency of scrolling. There are several methods for calculating the reliability coefficient, including test–retest, inter-rater, and internal consistency reliability.

Table 4.1: Methods for calculating the reliability.

	Name of method	Types of reliability
1.	Cronbach's alpha	Internal consistency coefficient
2.	Spearman–Brown's formula	Reliability coefficient for split-half tests
3.	Pearson's correlation	Theoretical reliability coefficient between parallel tests
4.	Cohen's kappa	Inter-rater reliability

4.8 Connection between reliability and validity

Both validity and reliability are used for evaluating the quality of a product. They both deal with the method of measurement and technique used for it. Hence, they indicate the level of a product or a method used. No doubt, reliability and validity are related to each other but there are also some differences; reliability mentions about how consistent the measurement is and validity mentions about how accurate the measurement is. Hence, high reliability is one of the indicators of validity, that is, if a method is not reliable, then it implies that the method is not valid.

During the research, both reliability and validity are important to be considered when we are planning our quantitative research design, methods adopted during research, and preparing the final thesis. As validity implies the reliability but converse is not true. There may exist a method that is consistent in nature under same circumstances, but it is not valid.

It is observed that if our research has high validity, then it is highly reliable. The results obtained are valuable, and their properties and characteristics are real. Reliability can be assessed easily when compared with the validity of the method. We can estimate the reliability of a method by comparing the change in calculations of the same measurement in different versions. But we need some standard data for comparison of the obtained result to estimate the validity of the method.

The validity and reliability of our measurements depend on many parameters (Figure 4.2). It includes the research design, methods used during the research, and

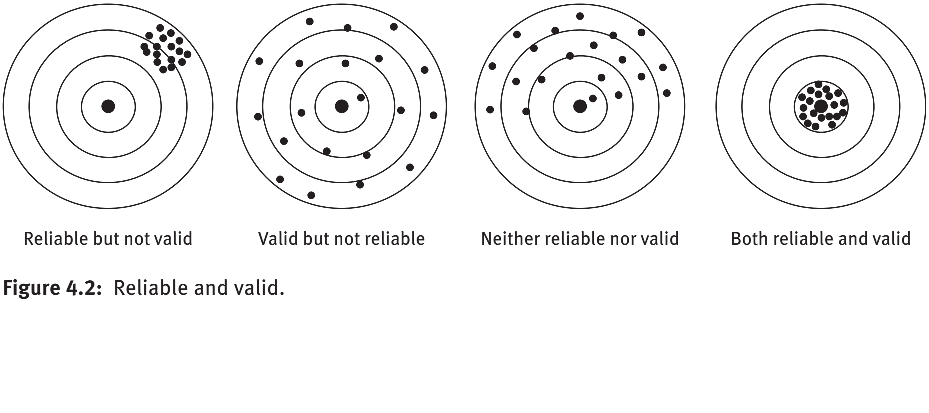

Figure 4.2: Reliable and valid.

how carefully we conduct our research. The scores or ratings used for measuring the variations in psychological traits and physical properties must give the real variations accurately. When we are going to collect our data at early stages of research, we should consider the validity of a system. The results should be as precise and stable as possible.

4.9 Reliability and redundancy in performance

Redundancy plays an important role to obtain a reliable system with less reliable components. It is based on the establishment of new parallel paths in a system. Actually, redundancy is the technique that makes use of more than one unit to achieve a reliable system. It probably increases the cost of a product, weight of a product, and complexity of a system to a significant extent. Thus, it is necessary to define the constraints, first, that must be subjected while optimizing reliability of the system. These constraints can be defined on the cost and weight of a product or on allocation of redundancy itself. As in series-type system having k stages, then it will work only if each stage out of k stages functions.

Redundancy makes use of more than one unit, so if an unit that is active fails, an alternate unit can be used. In this way, the alternate units may serve as spares. Hence, the system can be protected from failure, and addition of redundancies to the system is easy when compared with the improvement of reliability of the unreliable system. Thus, we need redundancy allocation, satisfying required constraints to maximize reliability.

4.10 Methods for improvement of reliability of the system

Following are the methods for improving the reliability of the system during the design phase of the system:
(a) Large number of components should not be used in the system.
(b) Components that we will use in the manufacturing of system must be of better quality.
(c) Highly derated components should be used.
(d) Environmental conditions should be suitable, for example, use of cooling fans.
(e) Fault tolerance should be used in the system.

Following are the methods for improving the reliability of the system during the production phase of the system:
(a) An increase in drift must be allowed in component values.
(b) The length of the test should be long.
(c) Methods should be carefully chosen for testing of components.

4.11 Conclusion

The main goal of reliability engineering is to solve all the problems concerned with the complexities and safety of the system. The domain of applications of reliability includes a wide range of engineering and technology. This also plays a role in aerospace, mechanical, and electrical engineering. Reliability analysis is a recent topic in the research area.

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5 Stochastic behavior of a system with degradation, fault detection, and replacement policy

Abstract: A system of two nonidentical units with the concept of degradation, fault detection, inspection, and replacement policy is considered and studied in this chapter. Initially, in the system, the first unit is operative and other is in cold standby condition. On failure of the first unit, it is first put for fault detection and then for its repair if required. The first unit becomes degraded after its first repair, and in the degraded mode if the unit fails again, then it will go for inspection to check if there is any feasibility for further repair or need to be replaced with a new unit. Two types of repairmen are available to perform the job of repair as well as for fault detection, inspection, and replacement: ordinary and expert. The expert repairman repairs and will perform all work in the system during its stay in the degraded unit. Several measures of reliability have been obtained and with the help of these measures, effectiveness of the system is analyzed.

Keywords: Reliability measures, cold standby, repairmen, Markov process

5.1 Introduction

The improvement in performance and reliability of systems has always been on priority in the industrial sector, and the use of redundancy is one of the most common practices, including additional number of units attached to the system such as a standby unit which increases the effectiveness of the system. Usually the redundant unit is used when the main unit goes for its repair. A large number of researchers have assumed that the unit becomes as perfect as a new one after repair facility. However, in many situations, it has been observed that the unit does not perform its work perfectly after repair and in fact the failure rate increases after its first repair. In such situations, the unit becomes degraded, that is, a unit is functional but with higher failure rate. In this direction, Bashir et. al. [1] worked on a model with repair, inspection, and post repair, while Bhatti et. al. [2] put forth the concept of inspection and two types of failure. Chander and Bansal [3] studied a single-unit reliability model with repair at different failure modes. Kumar et al. [4] used the

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concept of degradation after repair, and Kumar [5] worked on a redundant system with inspection and priority subject to degradation. Kumar and Kadyan [6] worked on a system with degradation and replacement. Malik et al. [7] studied a system with priority to repair and degradation. Further, Malik et al. [8] also used the concept of stochastic analysis with two types of inspection that is subject to degradation.

Continuing the work in this direction, a system that consists of two nonidentical units with the concept of degradation, fault detection, inspection, and replacement policy is considered and studied in this chapter. Initially, in the system, the first unit (A) is operative and other unit (B) is in cold standby condition. On failure of the first unit, it is first put for fault detection and then for its repair, if required. The first unit becomes degraded after its first repair and during the degraded mode if the unit fails again, then it will go for inspection to check if there is feasibility for further repair or need to be replaced with the new unit. Two types of repairmen are available to perform the job of repair as well as for fault detection, inspection, and replacement: ordinary and expert. The expert repairman repairs and performs all work in the system during its stay only for the degraded unit. Several measures of reliability have been obtained and with the help of these measures, effectiveness of the system is analyzed.

5.1.1 Assumptions

i) Priority for both operation and repair is given to the first unit (A).
ii) Failure time for both the units A and B associated with different events are independent random variable and follows the exponential distribution but with different parameters.
iii) Inspection time distribution of the first unit (A) is also exponential.
iv) Repair time distribution for both the units is taken to be general but with different parameters.
v) Replacement time distribution of the first unit A is also taken as general.

5.1.2 Notations

α	Failure rate of the operative unit (A)
α_1	Fault detection rate of the operative unit (A)
p_1	Probability that the fault is detected
q_1	Probability that no fault is detected $(p_1 + q_1 = 1)$
β	Failure rate of the cold standby unit (B)

(continued)

λ_1	Failure rate of the degraded unit (A)
λ_2	Inspection rate of the degraded unit (A)
p_2	Probability that further repair is feasible
q_2	Probability that further repair is not feasible $(p_2 + q_2 = 1)$
$H_1(\cdot)/H(\cdot)$	cdf of repair time distribution of unit A/B
$H_2(\cdot)$	cdf of replacement time distribution of the degraded unit (A)

5.1.3 Transition probabilities and mean sojourn time

Let X_n be the state visited at epoch T_{n^+} just after the transition at T_n, where $T_1, T_2, \ldots$ represents the regenerative epochs, then $\{X_n, T_n\}$ constitutes a Markov renewal process with state space E and

$$Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \le t | X_n = i].$$

Taking limits $t \to \infty$, the steady-state transition probabilities can be obtained:

$$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) = Q_{ij}(\infty),$$

$$p_{01} = p_{34} = p_{713} = p_{910} = p_{1210} = 1, \tag{5.1}$$

$$p_{89} = p_1, \quad p_{813} = q_1, \quad p_{117} = p_2, \quad p_{1112} = q_2, \tag{5.2}$$

$$p_{10} = \frac{q_1\alpha_1}{(\alpha_1 + \beta)}, \tag{5.3}$$

$$p_{12} = \frac{p_1\alpha_1}{(\alpha_1 + \beta)}, \tag{5.4}$$

$$p_{19}^{8} = \frac{\beta p_1}{(\alpha_1 + \beta)}, \tag{5.5}$$

$$p_{113}^{8} = \frac{\beta q_1}{(\alpha_1 + \beta)}, \tag{5.6}$$

$$p_{23} = h_1^*(\beta), \tag{5.7}$$

$$p_{210}^{9} = 1 - h_1^*(\beta), \tag{5.8}$$

$$p_{45} = \frac{p_2\lambda_2}{(\lambda_2 + \beta)}, \tag{5.9}$$

$$p_{46} = \frac{q_2\lambda_2}{(\lambda_2 + \beta)}, \tag{5.10}$$

$$p_{47}^{11} = \frac{\beta q_2}{(\lambda_2 + \beta)}, \tag{5.11}$$

$$p_{412}^{11} = \frac{\beta p_2}{(\lambda_2 + \beta)}, \tag{5.12}$$

$$p_{53} = g^*(\beta), \tag{5.13}$$

$$p_{510}^{12} = 1 - g^*(\beta), \tag{5.14}$$

$$p_{60} = h_2^*(\beta), \tag{5.15}$$

$$p_{613}^{7} = 1 - h_2^*(\beta), \tag{5.16}$$

$$p_{103} = h^*(\lambda_1), \tag{5.17}$$

$$p_{1011} = 1 - h^*(\lambda_1), \tag{5.18}$$

$$p_{130} = h^*(\alpha), \tag{5.19}$$

$$p_{138} = 1 - h^*(\alpha). \tag{5.20}$$

From the above steady-state probabilities, we can justify that

$$\sum_j p_{ij} = 1.$$

In order to analyze the behavior of the stochastic system model and to obtain various reliability measures, transition diagram along with all transitions for the proposed model is shown in Figure 5.1.

5.2 Mean sojourn time

If T_i denotes the sojourn time in state S_i, then the mean sojourn time is defined as the expected time of stay in S_i before transiting to any other state:

$$\mu_i = E(T_i) = \int_0^\infty \Pr[T_i > t]dt.$$

The expressions for mean sojourn time are obtained as follows:

$$\mu_0 = \frac{1}{\alpha}, \tag{5.21}$$

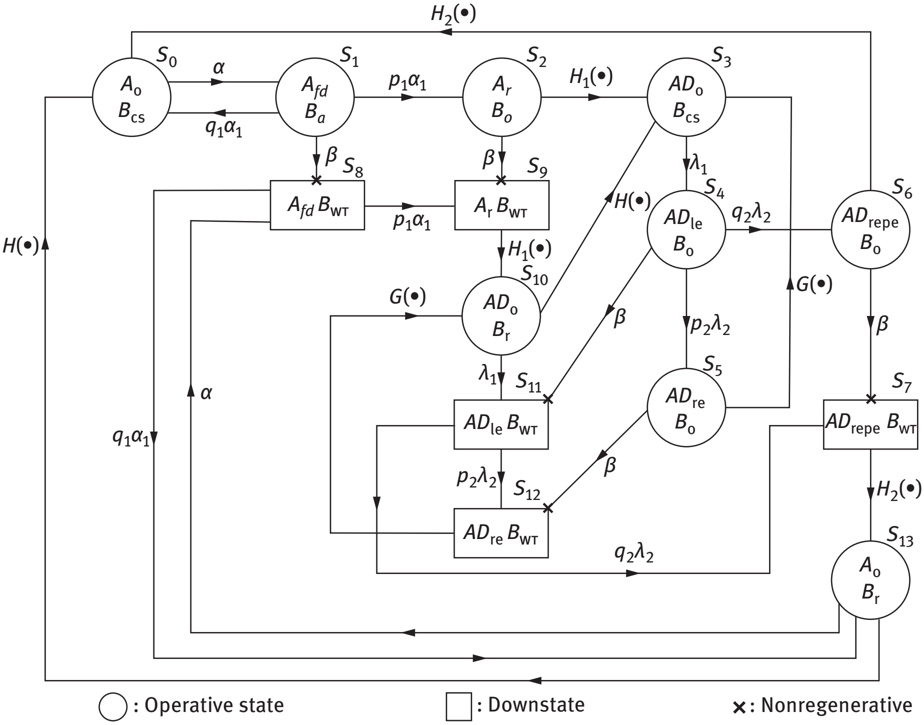

Figure 5.1: Transition diagram.

$$\mu_1 = \frac{1}{(\alpha_1 + \beta)}, \tag{5.22}$$

$$\mu_2 = \frac{1}{\beta}\left[1 - h_1^*(\beta)\right], \tag{5.23}$$

$$\mu_3 = \frac{1}{\lambda_1}, \tag{5.24}$$

$$\mu_4 = \frac{1}{(\lambda_2 + \beta)}, \tag{5.25}$$

$$\mu_5 = \frac{1}{\beta}\left[1 - g^*(\beta)\right], \tag{5.26}$$

$$\mu_6 = \frac{1}{\beta}\left[1 - h_2^*(\beta)\right], \tag{5.27}$$

$$\mu_7 = \int_0^\infty \bar{H}_2(u)\,du, \tag{5.28}$$

$$\mu_8 = \frac{1}{\alpha_1}, \tag{5.29}$$

$$\mu_9 = \int_0^\infty \bar{H}_1(u)du, \tag{5.30}$$

$$\mu_{10} = 1 - h^*(\lambda_1), \tag{5.31}$$

$$\mu_{11} = \frac{1}{\lambda_2}, \tag{5.32}$$

$$\mu_{12} = \int_0^\infty \bar{G}(u)du, \tag{5.33}$$

$$\mu_{13} = \frac{1}{\alpha}\left[1 - h^*(\alpha)\right]. \tag{5.34}$$

5.2.1 Mean time to system failure

It is defined as the expected time taken by the system to reach its failed state for the first time. Using simple probabilistic concepts, recurrence relations for cumulative density functions of the first passage time can be obtained and the expression for mean time to system failure (MTSF) is obtained and can be verified as follows:

$$\text{MTSF} = \frac{\alpha\beta\Big[(\lambda_2+\beta-p_2\lambda_2)g^*(\beta)\left[\beta+p_1\alpha_1\left(1-h_1^*(\beta)\right)\right]+p_1\alpha_1h_1^*(\beta)}{\left[(\alpha_1+\beta)+\alpha\beta+p_1\alpha_1\left(1-h_1^*(\beta)\right)\alpha\right]\left[\lambda_2+\beta-\lambda_2p_2g^*(\beta)\right]+p_1\alpha_1\left(h_1^*(\beta)\right)}$$

$$\frac{\left[p_2\lambda_2(1-g^*(\beta))+q_2\lambda_2\left(1-h_1^*(\beta)+\beta\right)\right]\Big]}{\left[\beta(\lambda_2+\beta)+\lambda_2(1-g^*(\beta))(\lambda_2+\beta+p_2\lambda_2)+\lambda_2q_2\left(1-h_2^*(\beta)\right)(\alpha_1+\beta)\right].} \tag{5.35}$$

5.2.2 Availability analysis

$A_i\ (t)$ is defined as the probability that a system will be in operational service during a scheduled operating period, that is, a system is available for use when required. By using simple probabilistic concepts, an expression for availability of the system is obtained as follows:

$$A_0 = \frac{N_2(0)}{D'_2(0)}, \tag{5.36}$$

where

$$N_2(0) = (1 - p_{138}p_{813})\left[\left(\mu_0 + \mu_1 + \mu_2 p_{12}\right)(1 - p_{45}p_{53}) + p_{12}p_{23}(\mu_3 + \mu_4 + p_{45}\mu_5 + p_{46}\mu_6) + \mu_{13}\left\{p_{113}^{8}(1 - p_{45}p_{53}) + p_{12}p_{23}\left(p_{46}p_{613}^{7} + p_{47}^{7}\right)\right\}\right]\left[(1 - p_{138}p_{813}) \left\{(1 - p_{45}p_{53})(1 - p_{1011}p_{1112}) - p_{103}\left(p_{45}p_{510}^{12} + p_{412}^{11}\right)\right\} - p_{138}p_{89}\left(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)\right)\right] + \left[\left\{\mu_0(1 - p_{45}p_{53}) + p_{103}(\mu_3 + \mu_4 + p_{45}\mu_5 + p_{46}\mu_6)\right\}(1 - p_{138}p_{813}) + \mu_{13}(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right))\right]\left[(1 - P_{138}p_{813})\left\{p_{12}p_{210}^{9}(1 - p_{45}p_{53}) + p_{12}p_{23}\left(p_{45}p_{510}^{12} + p_{412}^{11}\right) + p_{19}^{8}(1 - p_{45}p_{53})\right\} + p_{138}p_{89}\left\{p_{113}^{8}(1 - p_{45}p_{53}) + p_{12}p_{23}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)\right\}\right], \tag{5.37}$$

$$D_2(0) = \mu_1(1 - \mathrm{p}_{45}\mathrm{p}_{53})\left[(1 - p_{138}p_{813})p_{103}p_{46}p_{60} + p_{130}\left\{p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103} \left(p_{613}^{7}p_{46} + p_{413}^{11}\right)\right]\right] + \mu_2 p_{12}(1 - p_{45}p_{53})(1 - p_{138}p_{813})\left[(1 - p_{138}p_{813})p_{103}p_{46}p_{60} + p_{130}\left\{p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{413}^{11}\right)\right\}\right] + \mu_4\left[p_{12}(1 - p_{138}p_{813})p_{23} \left\{p_{103}p_{46}p_{60}(1 - p_{138}p_{813}) + p_{130}(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}(p_{46}p_{613}^{7} + p_{412}^{11})\right\}\right] + \mu_5 p_{12}p_{23}p_{45}\left[(1 - p_{138}p_{813})p_{103}p_{46}p_{60} + p_{130}(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{412}^{11}\right)\right] + \mu_6\left[p_{46}(1 - p_{138}p_{813})(1 - p_{45}p_{53})p_{103}p_{12} + p_{12}p_{23} p_{103}p_{130}p_{117} + p_{12}p_{23}p_{138}p_{89}p_{117} + p_{103}p_{19}^{8} + p_{138}p_{89}p_{103}p_{113}^{8}\right] + \mu_7\left[p_{12}p_{23}p_{47}^{11}(1 - p_{138}p_{813})\right]\left[(1 - p_{138}p_{813})p_{103}p_{34}p_{46}p_{60} + p_{130}\left\{p_{1011}p_{117}p_{713} (1 - p_{34}p_{45}p_{53}) + p_{103}\left(p_{34}p_{46}p_{613}^{7} + p_{34}p_{412}^{11}\right)\right\}\right] + \mu_8\left[\left\{p_{113}^{8}(1 - p_{45}p_{53}) - p_{12}p_{23}p_{138}p_{46}p_{613}^{7} - p_{12}p_{23}p_{138}p_{47}^{11}\right\} \left\{(p_{30} + p_{138}p_{89})p_{103}p_{46}p_{60} + p_{130}(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{412}^{11}\right)\right\}\right] + \mu_9\left[(1 - p_{138}p_{813})(1 - p_{45}p_{53})p_{12} - p_{12}p_{23}(1 - p_{138}p_{813})\left(p_{46} + p_{47}^{11}\right) \left\{p_{138}p_{89}(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)\right\}\right] + \mu_{11}\left[p_{1011}(1 - p_{138}p_{813}) (1 - p_{45}p_{53})\left\{(1 - p_{10})(1 - p_{138}p_{813})(1 - p_{45}p_{53}) - p_{12}p_{23}p_{46}p_{60}(1 - p_{138}p_{813}) - (1 - p_{45}p_{53})p_{103}p_{113}^{8} + p_{12}p_{23}p_{130}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)\right\}\right] + \mu_{12}\left[\left\{(1 - p_{138}p_{813})(1 - p_{45}p_{53})p_{1011}p_{1112} + p_{103}p_{412}^{11}(1 - p_{138}p_{813})\right\} \left\{(1 - p_{45}p_{53})(1 - p_{10}) - p_{12}p_{23}p_{46}p_{60}(1 - p_{138}p_{813}) - p_{130}(p_{113}^{8}(1 - p_{45}p_{53}) + p_{12}p_{23}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)) + p_{12}p_{23}p_{412}^{11}(1 - p_{138}p_{813})\left(p_{103}p_{46}p_{60}(1 - p_{138}p_{813}) p_{130}\left(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{412}^{11}\right)\right)\right)\right\}\right] + \mu_{13}\left[\left\{p_{12}p_{23}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)\right.\right.$$

$$+p_{113}^{8}(1-p_{45}p_{53})(1-p_{138}p_{813})p_{103}p_{46}p_{60}$$
$$+p_{130}(p_{1011}p_{117}(1-p_{45}p_{53}))+p_{103}(p_{46}p_{613}^{7}+p_{412}^{11})\}]. \tag{5.38}$$

5.2.3 Busy period analysis for ordinary and expert repairman

$B_i\ (t)$ and $B_i^{e}(t)$ denote the probabilities that the system is under repair by ordinary and expert repairmen, respectively, at time t, because of the failure of the units. Therefore, in steady state, these probabilities are obtained as follows:

$$B_0(s)=\frac{N_3(0)}{D'_2(0)}, \tag{5.39}$$

where

$$\begin{aligned}N_3(0)=&\left[(1-p_{138}p_{813})(\mu_1+\mu_2p_{12}+p_{19}^{8}\mu_9)(1-p_{45}p_{53})+(\mu_{13}+p_{138}\mu_2+p_{138}p_{89}\mu_9)\right.\\&\left.\{p_{113}^{8}(1-p_{45}p_{53})+p_{12}p_{23}(p_{46}p_{613}^{7}+p_{47}^{11})\}\right]\left[\{(1-p_{1011}p_{1112})(1-p_{45}p_{53})-\right.\\&\left.p_{103}(p_{45}p_{510}^{12}+p_{412}^{11})(1-p_{138}p_{813})\}-p_{138}p_{89}\{p_{1011}p_{117}+p_{103}(p_{46}p_{613}^{7}+p_{47}^{11})\}\right]\\&+\left[\{p_{19}^{8}(1-p_{45}p_{53})+p_{12}p_{23}(p_{45}p_{510}^{12}+p_{412}^{11})+p_{12}p_{210}^{9}(1-p_{45}p_{53})\}\right.\\&\left.(1-p_{138}p_{813})+p_{138}p_{89}\{p_{113}^{8}(1-p_{45}p_{53})+p_{12}p_{23}(p_{46}p_{613}^{7}+p_{47}^{11})\}\right]\left[(1-p_{45}p_{53})\right.\\&\left.(1-p_{138}p_{813})\mu_0+\{(1-p_{45}p_{53})p_{1011}p_{117}+p_{103}(p_{46}p_{613}^{7}+p_{47}^{11})\}\right]\end{aligned} \tag{5.40}$$

$D'_2(0)$ is the same as obtained in availability and is given by eq. (5.38):

$$B_0^{e}(s)=\frac{N_4(0)}{D'_2(0)}, \tag{5.41}$$

where

$$\begin{aligned}N_4(0)=&\left[p_{12}p_{23}(1-p_{138}p_{813})(\mu_4+\mu_5p_{45}+\mu_6p_{46}+p_{412}^{11}\mu_{12}+p_{47}^{11}\mu_7)\right]\left[\{(1-p_{45}p_{53})\right.\\&(1-p_{1011}p_{1112})-p_{103}(p_{45}p_{510}^{12}+p_{412}^{11})\}(1-p_{138}p_{813})-p_{138}p_{89}\{p_{1011}p_{117}\\&\left.(1-p_{45}p_{53})+p_{103}(p_{46}p_{613}^{7}+p_{47}^{11})\}\right]+\left[(1-p_{138}p_{813})\{p_{12}p_{210}^{9}(1-p_{45}p_{53})+p_{12}\right.\\&p_{23}(p_{45}p_{510}^{12}+p_{412}^{11})+p_{19}^{8}(1-p_{45}p_{53})+p_{138}p_{89}\{p_{12}p_{23}(p_{46}p_{613}^{7}+p_{47}^{11})+p_{113}^{8}\\&\left.(1-p_{45}p_{53})\}\right]\left[p_{103}(1-p_{138}p_{813})(\mu_4+p_{45}\mu_5+p_{46}\mu_6+p_{412}^{11}\mu_{12}+\mu_7p_{47}^{11})\right.\\&\left.+(1-p_{138}p_{813})(1-p_{45}p_{53})(p_{1011}\mu_{11}+p_{1011}p_{117}\mu_7+p_{1011}p_{1112}\mu_{12})\right].\end{aligned} \tag{5.42}$$

$D'_2(0)$ is the same as obtained in availability and is given by eq. (5.38).

5.2.4 Expected number of visits by ordinary and expert repairmen

$V_i(t)$ and $V_i^{\mathrm{e}}(t)$ denote the expected number of visits by the ordinary and expert repairmen, respectively, in (0,t], given that the system initially starts from regenerative state S_i. By probabilistic reasoning, the recurrence relations for $V_i(t)$ and $V_i^{\mathrm{e}}(t)$ are obtained, and the number of times the repairmen arrive in the system is given as follows:

$$V_0 = \frac{N_5(0)}{D'_2(0)}, \tag{5.43}$$

where

$$\begin{aligned} N_5(0) = {} & (1 - p_{138}p_{813})\left[1 + p_{12}p_{210}^{9}(1 - p_{45}p_{53}) + p_{12}p_{23}\left(p_{45}p_{510}^{12} + p_{412}^{11}\right) + p_{19}^{8}(1 - P_{45}p_{53})\right] \\ & \left[\left((1 - p_{45}p_{53})(1 - p_{1011}p_{1112}) - p_{103}\left(p_{45}p_{510}^{12} + p_{412}^{11}\right)\right)(1 - p_{138}p_{813})\right. \\ & \left. - p_{138}p_{89}\left(p_{1011}p_{117}(1 - p_{45}p_{53}) + p_{103}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)\right)\right]. \end{aligned} \tag{5.44}$$

$D'_2(0)$ is the same as obtained in availability and is given by eq. (5.38):

$$V_0^{\mathrm{e}} = \frac{N_6(0)}{D'_2(0)}, \tag{5.45}$$

where

$$\begin{aligned} N_6(0) = {} & p_{103}(1 - p_{138}p_{813})\left[p_{19}^{8}(1 - p_{45}p_{53}) + p_{12}p_{23}\left(p_{45}p_{510}^{12} + p_{412}^{11}\right) + p_{12}p_{210}^{9}\right. \\ & \left.(1 - p_{45}p_{53}) + p_{138}p_{89}\left\{p_{113}^{8}(1 - p_{45}p_{53}) + p_{12}p_{23}\left(p_{46}p_{613}^{7} + p_{47}^{11}\right)\right\}\right] \end{aligned} \tag{5.46}$$

and $D'_2(0)$ is the same as obtained in availability and is given by eq. (5.38).

5.2.5 Profit analysis

Let the expected profit incurred by the system during time (0, t] be denoted by (t).

$P(t)$ = expected total revenue in (0,t] – expected total service cost in (0,t] – expected cost of visit for repair facility in (0,t]:

$$P = Z_0A_0 - Z_1B_0 - Z_2B_0^{\mathrm{e}} - Z_3V_0 - Z_4V_0^{\mathrm{e}},$$

where

Z_0 is the revenue per unit time of the system, $Z_{1,}Z_2$ are costs per unit time for which ordinary and expert repairmen are busy, and $Z_{3,}Z_4$ are costs per visit by the ordinary and expert repairmen.

The expressions for A_0, B_0, B_0^e, V_0, and V_0^e are given in eqs. (5.36), (5.39), (5.41), (5.43), and (5.45), respectively.

5.2.6 Graphical study of the system model

The model proposed in Figure 5.1 can also be studied graphically by analyzing the behavior of characteristics such as MTSF, availability, and profit function.

First, graphs are plotted with respect to failure rate α for different values of repair rate θ_1 as 0.1, 0.7, 1.3, and 1.9 for fixing all other parameters as $\beta = 0.65$, $\alpha_1 = 0.49$, $Z_0 = 1050$, $Z_1 = 700$, $Z_2 = 450$, $Z_3 = 350$, $Z_4 = 150$, $\theta = 0.1$, $\theta_2 = 0.73$, $\lambda_1 = 0.20$, $\lambda_2 = 0.30$.

It can be easily verified from Figures 5.2–5.4, respectively, that MTSF, availability, and profit decrease with the increase in the rate of failure and it increases as the rate of repair is increased. Therefore, in order to make the system more efficient, the failure rate needs to be minimized and the repair rate should be maximized.

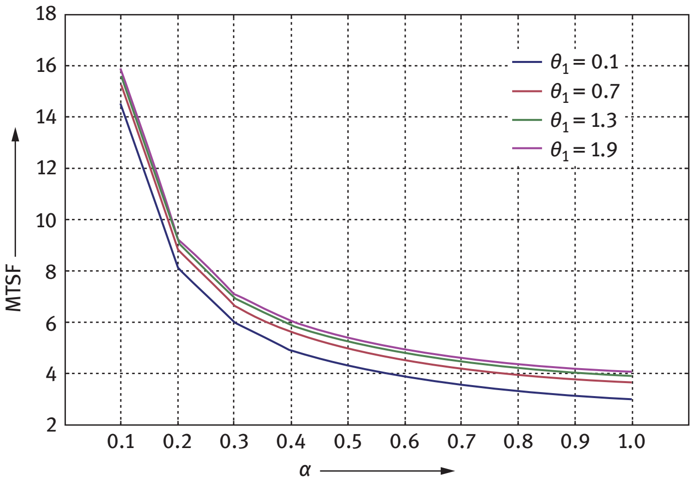

Figure 5.2: Behavior of MTSF with respect to failure rate α for different values of repair rate θ_1.

Similarly, graphs are also plotted with respect to failure rate β for different values of repair rate θ as 0.25, 0.5, 0.75, and 1.0, keeping all other parameters constant as $\beta = 0.65$, $\alpha_1 = 0.49$, $Z_0 = 1050$, $Z_1 = 700$, $Z_2 = 450$, $Z_3 = 350$, $Z_4 = 150$, $\theta_1 = 0.6$, $\theta_2 = 0.73$, $\lambda_1 = 0.20$, $\lambda_2 = 0.30$.

From Figures 5.5–5.7, respectively, we have observed that MTSF, availability, and profit function show a decline when the rate of failure increases, and these

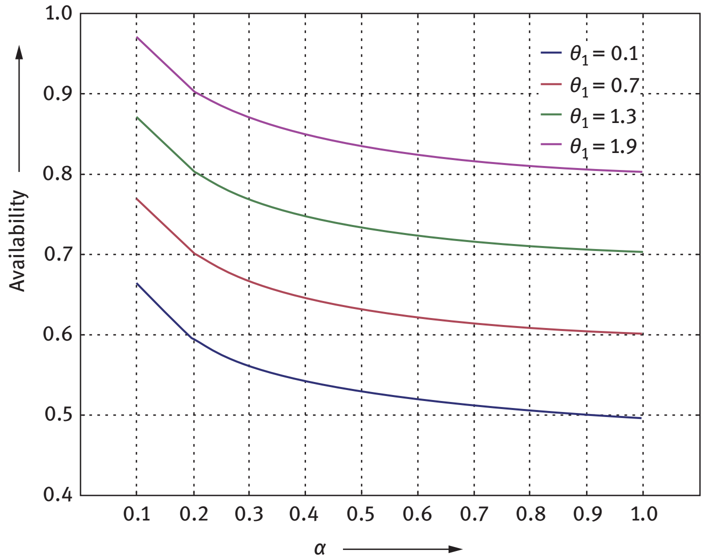

Figure 5.3: Behavior of availability with respect to failure rate α for different values of repair rate θ_1.

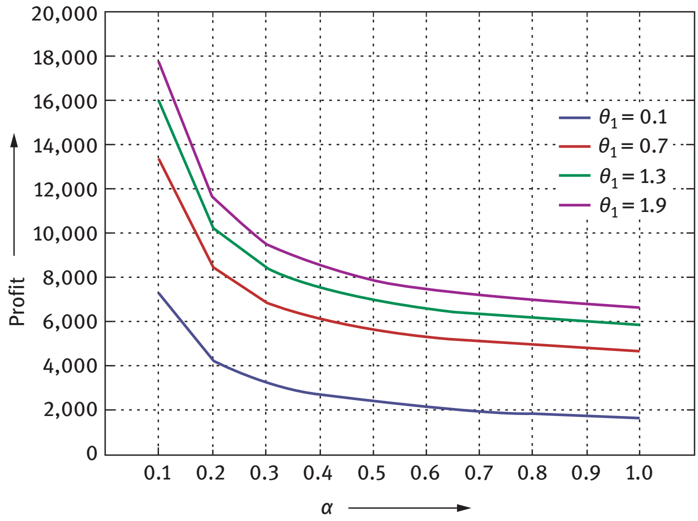

Figure 5.4: Behavior of profit with respect to failure rate α for different values of repair rate θ_1.

characteristics increase as the rate of repair increases. Therefore, we concluded that in order to make the system more efficient, we must increase the repair facility and minimize the failure rate.

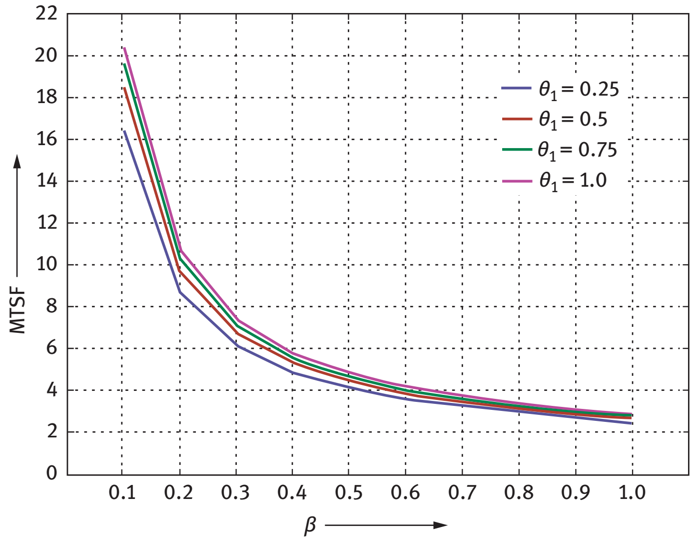

Figure 5.5: Behavior of MTSF with respect to failure rate β for different values of repair rate θ.

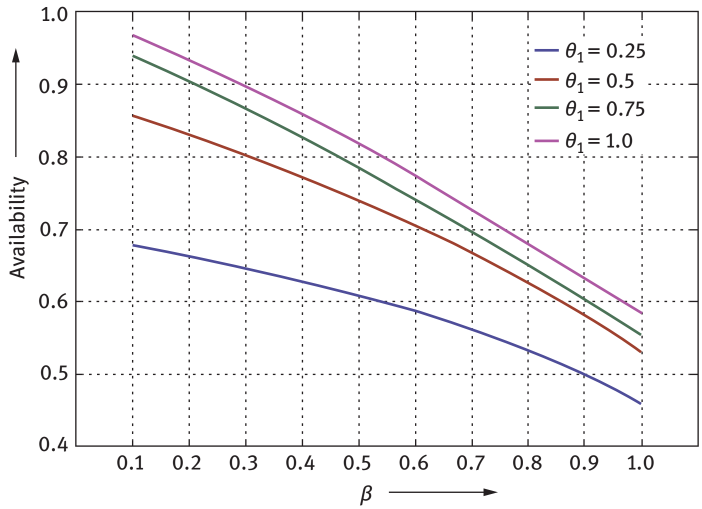

Figure 5.6: Behavior of availability with respect to failure rate β for different values of repair rate θ.

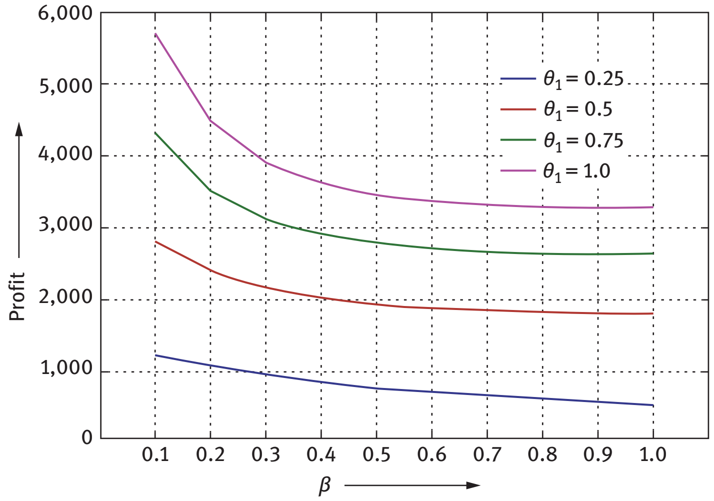

Figure 5.7: Behavior of profit with respect to failure rate β for different values of repair rate θ.

5.3 Conclusion

After graphically analyzing the behavior of the characteristics of reliability, it is concluded that MTSF decreases with the increase in the rate of failure and it increases as the rate of repair is increased. Hence, in order to make the system more efficient, the repair rate should be maximized. Further, it is analyzed that availability also decreases with the increase in the failure rate of the system and increases with the increase in the repair facility, and profit function also increases as the rate of repair increases, and it shows a decline when the failure rate increases. Therefore, the performance of the system can be amplified by providing appropriate repair facility to the system, as thorough repair of the units enhance the reliability of the system.

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Reetu Malhotra

6 Reliability evaluation with variation in demand

Abstract: This chapter is an attempt to strengthen the reliability models that consider unique conditions based on the variation in demand and production capacity of systems. A brief description of models with varying demands and comparison between the models taken two at a time for the system is given in this chapter. The author describes the foundation of reliability followed by different concepts introduced to improve reliability. She also discusses how varying demands affect the reliability in single-unit and two-unit redundant systems. Graphical analysis has been done to compare the systems with varying demands.

Keywords: Reliability measure, redundant system, price analysis, semi-Markov approach

Modern society is *technical* as well as technological. Being technical implies that it is being intricate in structures and substructures. Varied with instructions and categories and influenced by more than one social control capabilities. If there is any place at which it fails due to stress of the subjects then it is to be maintained and mentored. Being technological implies mechanics and techniques has a thorough potential in industrialization. Advent and advancements in science and technological know-how have expanded the opportunity and variety of *manufacturing* to a massive extent. With the upward jab in the quantum of demands, extra complicated and state-of-the-art systems are being added day by day to meet the needs of the people. A machine is expected to function flawlessly over a long period. If it features for the supposed time, it is said to be reliable. However, failure of the tool or the machinery may take place at any instance which may fall in the context beyond human control; nonetheless, the buyers assume best tiers of talent and perfection in the manufactured tools that are strongly implying “reliability.” The reliability of a device is the likelihood that it will perform the function of making merchandise on a nonstop groundwork and with suited carrier first rate and price effectiveness. Effective reliability analysis is a vital thing in the operation and planning of a system. This ever-increasing focus on reliability has resulted in more than one repercussion, which includes development in science to produce first-class products/machinery and to make the technique of manufacturing and *production* a costly affair.

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Early foundations of reliability may be determined in actuarial standards used in the insurance industry, especially in the study of human survival probabilities. In 1920, Bell Telephone Laboratories developed statistical strategies to clear up great manipulation problems. Later, societies such as American Society for Testing Materials and American Standards Associations have joined to popularize the first-class manipulated techniques. In the mid-1930s, Swedish engineer and mathematician Weibull in 1939 recommended a simple and handy mathematical model for the description, which is recognized as the Weibull distribution. It was once the first without a doubt giant result in the future development of reliability theory. The *probabilistic* thinking of reliability has grown out of the wishes of cutting-edge science and mainly out of the experiences in World War II with intricate military buildings as excessive failure prices have been determined especially in vacuum tubes.

A study about this was conducted, which shows the following facts:

1. The electronic equipment/systems used via the navy were operative for only 30% of its complete handy time due to the fact of common failures/maintenance problems.
2. Army equipment had been either under repair/breakdown or commissioning for nearly 60–75% of its complete time, which once more created issues for the profitable mission.
3. Air force findings for about 5 years showed that the maintenance expenses had been very high and even now and again increase many times from the unique price of the equipment/systems due to generic breakdowns.

The above data may be the cause that at some stage in hostilities duration, the availability of gear is of high importance barring its cost. With the above statistics, investigations had been similarly carried out, and a concept of reliability group/engineer came into existence. Following the war, Aeronautical Radio, Inc. was once installed by the commercial airlines to improve airborne electronic tools (referred to as avionics by using the military). In 1950, the U.S. Air Force shaped an ad hoc group to improve time-honored gear reliability, and in 1952, the defense department hooked up the Advisory Group on Reliability of Electronic Equipment. The requirement of reliability making an attempt out and demonstration of new structures emerged from this advisory group. Work posted at some stage in the 1950s located on the use of the exponential distribution to signify failure times. Davis [1] mentioned failure records and *goodness of fit* for competing failure distributions. Epstein [2–4] labored in the concern of a way of existence checking out with the useful resources of wondering about the exponential distribution. After these studies, the exponential failure time distribution acquired a distinct characteristic in reliability analysis.

Toward the supply in the 1950s and the establishment in the 1960s, the United States tried to put their whole effort in harmful missiles and space research. They used Mercury and Gemini Programs. It was essential for them to succeed in launching spacecraft than Russia, who were the first to touch the Moon. Shortly, an engineering association was established. In the 1960s, *IEEE Transactions on Reliability* journal and few textbooks on reliability have been introduced. To account for normal distribution, strategies like embedded *Markov chains* and section techniques have been considerably used to remedy queuing (modeling) problems. The basic queuing process can be described by the

1. input (arrival pattern),
2. queue (waiting line),
3. the service discipline (queue discipline), and
4. the service mechanism (service pattern).

Availability and several different measures of effectiveness of a machine [5–8] have been obtained from various researchers. Black and Proschan [9] regarded a problem of the most beneficial redundancy, in which a gadget is assumed to work successfully over a fixed period. Black and Proschan [10] considered a hassle of the most advisable redundancy in which a device is assumed to work correctly over a constant period. Barlow and Hunter [11] had referred to preventive protection policies. Gaver [12] considered "time to failure" and the availability of parallel machines with repair. Srinivasan [13] studied a standby redundant model with the noninstantaneous switchover. Kulshreshtha [14] determined the reliability of a reparable multicomponent computer with *redundancy* in parallel. Rasmussen [15] analyzed the so-called Rasmussen report, in which WASH-1400 was used to be a multimillion-dollar project. Branson and Shah [16] studied the reliability of structures consisting of devices with arbitrary repair time distributions. Srinivasan and Gopalan [17] studied two-unit standby gadgets with a single restore facility. Nakagawa [18] discussed an alternative problem of a cumulative injury model. Garg and Kumar [19] discussed a complicated gadget with two sorts of failure and repair. Subramanian and Ravichandran [20] studied a two-unit *standby* redundant system with an imperfect switchover. Mine and Kaiwal [21] enhanced the system reliability through assigning precedence to restore disciplines. Dhillon [22] analyzed a multistate computing device redundant gadget with familiar motive failures. Chiang and Niu [23] analyzed the reliability of the consecutive-k-out-of-n:F system. Murari and Goyal [24] cited the reliability device with two types of restore facilities. Naidu and Gopalan [25] analyzed a one-server two-unit computer project to arbitrary failure, random inspection, and failure modes. Murari received the reliability of a one-unit system project to random shocks and preventive maintenance. Tuteja et al. [26] stated a two-unit laptop with two sorts of repairmen and venture to random inspection. Kumar et al. [27] analyzed a two-unit redundant gadget with guidelines at need. Pan [28] predicted the reliability of imperfect switching structures' situation to more than one stress. Attahiru and Zhao [29] dealt with the comparison of a repairable system with three devices and two restore facilities. Singh et al. [30] analyzed a two-unit standby computing device with an accident and some kinds of repair. Yadavalli and Botha

[31] decided the asymptotic self-assurance limits. Gupta and Shivakar [32] analyzed two nonidentical parallel systems with equipped time distribution of repairmen. Xu et al. [33] decided the asymptotic balance of a repairable computing device with an imperfect switching mechanism. Gupta et al. [34] analyzed the reliability and availability of serial strategies of a plastic pipe manufacturing plant. Gupta and Tiwari [35] studied the simulation modeling of a complicated gadget of a thermal electricity plant. Damcese and Temraz [36] analyzed the availability and reliability of repairable parallel constructions with special failure rates. Gupta and Tiwari [37] examined a two-unit heat standby laptop with a two-phase repair and geometric distributions of the events. Godwin and Nsobundu [38] noted an impact on universal safety and overall performance in the cable manufacturing industry. Singh et al. [39] evaluated the reliability and availability of a database gadget with a standby unit provided with the resource of the machine provider. Hazra and Nanda [40] examined redundancy at the issue stage, which is last to that at the computing device stage regarding one of a variety of stochastic orders. Qingqing et al. [41] studied a *k*-out-of-*n* redundant machine problem to schedule backups, and the place of elements are online and operating, with the last factors being equipped in the unpowered, cold standby mode. Vrignat et al. [42] referred to a synthetic "hidden Markov model" of degradation to produce tournament sequences.

To attain monetary-appropriate factors from systems, there is a difficulty in procedures along with cost constraints which used to be imperative and after studying it, as a result, this component was once considered by a number of researchers from 1980s onward in the field of reliability. Goel et al. [43] gave the charge analysis of a two-unit cold standby gadget with two kinds of operation and repair. Murari et al. [44] labored out a fee evaluation of a two-unit heat standby machine with daily repairman and persistence time. Gupta et al. [45] labored out a cost–benefit evaluation of a single-server three-unit redundant device and multicomponent standby laptop with inspection and gradual switch. Gupta and Goel [46] gave the profits evaluation of the two-unit precedence standby device with administrative prolong in repair. Gopalan and Muralidhar [47] dealt with the cost comparison of a one-unit repairable machine subject to online *preventive* maintenance/repair. Tuteja and Taneja [48] evaluated the cost–benefit evaluation of a two-server two-unit redundant gadget with special kinds of failure. Gupta et al. [49] determined the profits evaluation of a two-unit priority standby system that is difficult to degradation and random shocks. Kumar and Saini [50] dealt with a price evaluation of a two-unit standby device beneath to have an impact on earthquakes. Pandey and Jacob [51] evaluated the cost, availability, and mean time to failure of a three-state standby intricate desktop beneath general reason and human failures. Taneja et al. [52] noted a reliability *mannequin* for two-unit cold standby devices with instructions and two types of repair. Rizwan and Taneja [53] evaluated the profits of a machine with an ideal restore at complete/partial failure. Siwach et al. [54] studied a two-unit redundant desktop with practice and accident. Tuteja et al. [55] stated a device whereby the standby unit working in the kingdom may additionally stop even except failure.

These researches, while making the contrast, took the assumed values of failure, repair, and different costs, that is, the actual facts on these costs were no longer taken into consideration. Taneja et al. [56] gathered the actual information on failure and fixed quotes of 232 programmable common sense controllers (programmable logic controllers (PLC)) and studied a single-unit PLC. Tandon et al. [57, 58] discussed subsystems of cable plants with maintenance categories and surplus produce.

Profit and other measures of system evaluation of different models are calculated, and positive steps are followed for improvement. The remaining goal is to have a sustainable machine in order to reap the most reliability. Although various researchers developed models and various authors wrote chapters/books on reliability fashions of industrial systems, yet none of these carried out reliability and *price analysis* of systems with the variation in demand. All the above-mentioned studies have considered the demand as fixed. However, there exist many practical situations where the demand for the units produced is not fixed. The author visited the following companies/plants and gathered information regarding work, failures, and repairs/replacements of the systems:

1. General Cable Energy India Private Ltd., Baddi, Himachal Pradesh, India
2. Promed Export Private Ltd., Baddi, Himachal Pradesh, India
3. Milk Food Ltd., Bahadurgarh, Patiala, Punjab, India

Based on the groundwork of the visits made and data gathered from these companies, the General Cable Energy was considered to be the most appropriate. The demand of the devices produced is not constant; therefore, the need for analyzing the reliability and *availability* of a gadget/redundant gadgets has been considered. In the text of reliability, significant studies [59–68] have been made on extraordinary types of one-unit or two-unit standby redundant systems with varying demand, owing to their familiar use in present-day enterprise and industrial systems.

Keeping the idea of variation in demand in view, the following reliability models have been developed and analyzed considering various situations. The simple idea in the evaluation of every developed mannequin/model is that of a state of the system. It includes the study of the system's conduct over time. The state of a machine can be categorized as transient state, steady state, failed state, and down state. When the running traits of a device are established on time, it is stated to be in a transient state. The probability distributions of arrivals, waiting time, and the service time of customers depend on time. In contrast to it, in steady state, waiting time, probability distribution of arrivals, and the service time of the customers are independent of the time, and the system is said to be in a steady state. In other words, the working traits of a device are impartial of time. This state takes place in the lengthy run of the system. In most of the stochastic models, regular state options exist unbiased of the preliminary state of the queue. In a failed state, the machine does no longer work. If the manufacturing of machines is greater than the demand acquired, it places into the downstate. Every device is analyzed using the *semi-Markov* approach (process) and the regenerative factor technique. In the Markov process, the state at nth time (suppose t_n) is only influenced by the state of the process at $(n - 1)$th time (t_{n-1}), that is, in the Markov process, the outcomes of any trial depend almost on the outcome of immediately preceding trial and not on any previous outcomes. Thus, a Markov process is uniquely determined by the initial distribution of probabilities and transition probabilities. The semi-Markov process is a stochastic method having an enumerable quantity of states (0, 1, 2, . . .). If t_1, t_2, . . . are the epochs at which the manner probabilistically restarts, that is, the procedure is unbiased of its previous and the stochastic conduct from epoch t_1 is equal as it had from t_0 = zero. Renewal process is a method in which transition from one state to any other is ruled by using the transition possibilities of a Markov procedure; however, the time spent in every state earlier than a transition happens is a random variable relying upon the closing transition made.

Malhotra and Taneja [58] developed a single-unit system followed by two devices/units that are frequently employed in the waiting time analysis. The first step is to generate the generating function and the second is a Laplace–Stieltjes transform. The generating function and *Laplace–Stieltjes* transform provide the principal tools for deriving many of the important stationary operating characteristics. Let a random variable take the positive values with the associated probabilities. The formulas for steady-state probabilities are specified, and the system's behavior is described by a Markov chain. Consider only those instants in time when a service had just been completed, and these are termed as regeneration points. Whenever a unit/system breaks down, it will result in an exceptional loss to the company. Hence, the machine restores issues that are very necessary in modeling. When a unit breaks down, the repairman starts its work. If at some stage in this time any other unit spoils, then it will be attended after the repair of the first unit. Thus, the damaged or failed devices structure a queue and wait for their repairs. There are quite a number of situations. There might also be a single repairman or more than that. If there is a single repairman, then it is known as the hassle of a single channel and if there is more than one repairman, it is referred to as a multichannel problem. The machines might also be repaired in a single phase or in multiphase.

Various models were developed on the groundwork of the statistics gathered by the author from the company/plant. Other various measures such as *cost–benefit analysis*, busy period, mean time to system failure (MTSF), and predicted a wide variety of visits with the aid of repairmen are estimated numerically by using the semi-Markov process and the regenerative factor technique. Computer programs using BASIC/C language/MATLAB were developed for plotting the graphs of MTSF, availability, and the profit for some numerical values of various rates, costs, revenues, and probabilities. This chapter discusses a brief introduction of the reliability and cost–benefit evaluation of author's models with variation in demand.

Profit analysis for a single-unit system with varying demand has additionally been examined collectively with the notion of introducing scheduled maintenance [69]. Revenue when demand ≥ manufacturing and when demand < production have also been taken into consideration while carrying out the profit analysis. A situation involving the thought of inspection [59] for detecting the kind of failure has additionally been examined together with the version in demand. A unit may additionally not be repairable on failure and is consequently been repaired or replaced accordingly. Three types of disasters have been determined while gathering statistics from the agency that is classified as *replaceable/repairable/reinstallation* failure. Graphical learning has additionally been made, which allows us to draw several conclusions regarding the profitability of the system.

Malhotra and Taneja [62, 66] developed two cold standby systems, wherein the first cold standby system both the gadgets/units can grow to be operative. Each of the gadgets may also become operative concurrently to complete the demand. In the second standby system, only one unit can be operated at a time. A graphical study has been done, which allows us to draw various conclusions in relation to the profitability of the systems. Thus, a range of models has been mentioned underneath different conditions. A system based on a specific model cannot be regarded as the best one. A model may be better in some situations and can also be worse in some other situations, and subsequently, the comparative study becomes significant. Keeping this in view, Malhotra and Taneja [63] elevated the comparative evaluation between the above-discussed models. Sunaina and Malhotra [70] studied many reliability models in a systematic manner. In this chapter, profits of single-unit and two-unit redundant units are compared when an additional unit is installed in a particular case.

6.1 Results and discussion

It has been observed that the profit gained by a single-unit system with varying demands is less than the profit gained by the same model with scheduled maintenance, which is further less than the profit gained by the model with inspection effect for any real value. Similarly, the interpretation as to which and when one model is extra advisable than the other with admire to the setup price of an extra unit is given in Table 6.1. (Data were taken from [59–68].) Graph of profits with respect to the additional cost due to scheduled maintenance/*inspection*/unit is shown in Figure 6.1: a single-unit system (model 1), single-unit system with scheduled maintenance (model 2), single-unit system with inspection (model 3), two-unit redundant system when only one unit is operative at a time (model 4), and two-unit redundant system when both units are operative at a time (model 5).

Table 6.1: Profits versus additional cost (scheduled maintenance/inspection/unit) in a system with varying demands.

Additional cost	Profit for the model	Greater than	Profit for the model
<2,250.1652	Model 4	>	Model 3
>2,250.1652	Model 3	>	Model 4
<2,924.3211	Model 5	>	Model 2
>2,924.3211	Model 2	>	Model 4
>2,924.3211	Model 1	>	Model 4
<6,317.8217	Model 5	>	Model 3
>6,317.8217	Model 3	>	Model 5
<6,915.368	Model 5	>	Model 1
>6,915.368	Model 1	>	Model 5
<6,915.368	Model 5	>	Model 2
>6,915.368	Model 2	>	Model 5
Taking any real value	Model 3	>	Model 2
Taking any real value	Model 2	>	Model 1
Taking any real value	Model 3	>	Model 1
Taking any real value	Model 5	>	Model 4

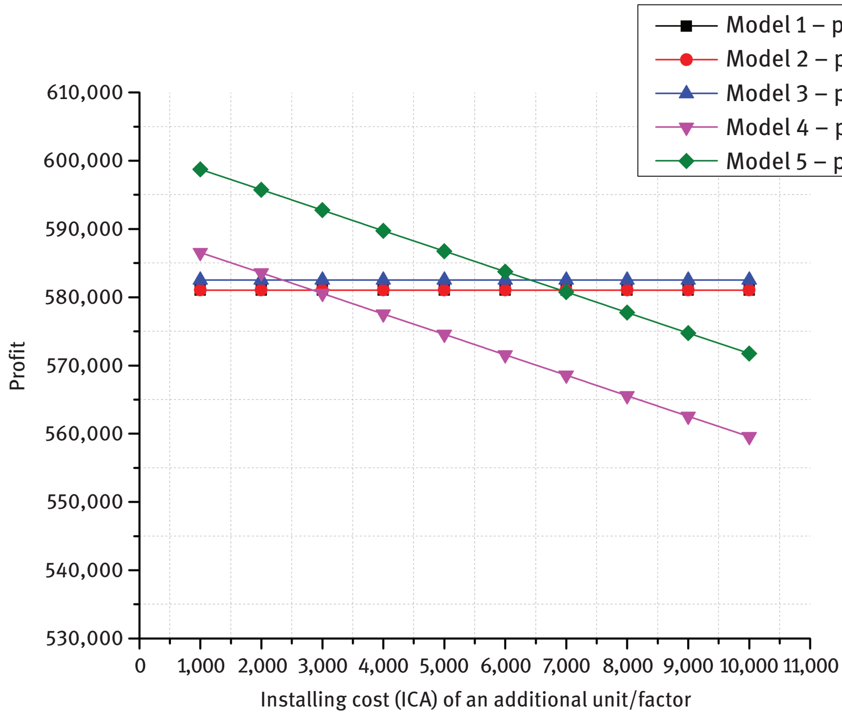

Figure 6.1: Graph of profits.

6.2 Conclusion

The chapter deals with models developed by introducing the concept of varying demands. These models can be used in various companies/organizations. The evaluation is achieved, considering the particular instances as already taken in the models wherein all the distributions for a variety of times have been taken as exponential. By considering a particular case, a graphical analysis has been performed to compare the results. One may, however, take that distribution, which will best fit with the actual data on time to failure, repair, scheduled maintenance, and inspection.

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Surabhi Sengar

7 Reliability analysis and cost optimization in manufacturing systems

Abstract: This chapter focuses on the cost optimization techniques used by manufacturing systems or industries and also various replacement policies that affect the cost of the system. A non-Markovian model is discussed to understand the behavior and reliability aspects of production industries; also the supplementary variable technique is used to convert it into a Markov model. Laplace transform is used to solve the mathematical model. Various reliability measures such as availability of the system, meantime failure for different failure rates, and steady-state nature of the system are evaluated.

Keywords: Cost optimization, Markov model, supplementary variable technique, steady-state nature

7.1 Introduction

In this optimistic scenario of competition, a rapid and cost-effective production is a key obligation for endurance. To attain this objective, the thought of the production team is to fetch people nowadays. In this sector, nonidentical equipment are arranged rationally to execute the desired procedure for conversion of unprocessed materials into processed materials. This helps to attain the accumulation production at an optimal cost with global quality principles. Furthermore, simultaneous and well-established systems are now an essential part of our life. Regular performance of these systems is a prime concern for their plenty of users who depend on them for their existence.

On the whole, a system is an arrangement of elements, that is, there is a practical correlation among its parts.

A system has a chain of commands of its parts that go by the different steps of operations, which can be operational or upstage, breakdown or failure, degraded or in repair. System breakdown is not always complete; it can be fractional as well.

In industries, most of the systems are repairable. As the revenue of any venture depends on the superiority of the manufactured goods, the cost of the production, and the service to the client, all these facts are influenced by the behavior of the system; hence, the behavior of the system or its performance is an important aspect for a venture or a manufacturer. Due to the rise in the convolution and mechanization

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of the industries, present repairable systems become very multifaceted. When we talk about manufacturing operations, apart from reliability, other aspects like steady-state accessibility or availability analysis, mean time between failure investigations are also necessary.

The aim of a production industry is basically to make a profit and also for industry. It is difficult to stay alive in the market for a long period without minimum monetary income for its investments. There is no uncertainty in saying that the expenditure associated with reliability measures raises the original cost of every machine, apparatus, or organization. However, because of the unsatisfactory performance of the product, an industry can lose its customers. It has no alternative other than to sustain with this expenditure. In a particular case, the significance of reliability cost depends on the expenditure of the system and breakdown free performance of the system. When the failure of a part of equipment results in a huge economical loss or human casualty, the importance of reliability and other aspects must be taken into consideration and evaluation. Once the name of a manufacturer is well known in the market for his/her reliability, his/her production volume and revenue will be substantially increased. If he/she wants to stay long in the field with the same image, then he/she has to optimize his/her costs and also income to serve maximum satisfaction to his/her client.

7.2 Review of literature

Due to factors such as convolution, cost, and design constraint, reliability assessment and availability of repairable systems are becoming important. The last decades have witnessed a budding concern in the growth and relevance of reliability/availability techniques in the area of different manufacturing sectors related to maintenance engineering and management. Many researchers have made their contribution to the development of different techniques to find the reliability characteristics of complex systems. They have focused their studies on the analysis of various repairable and nonrepairable systems. In this study, they assumed the repair time distributions exponential, Weibull, gamma, and general time distributions, and evaluated the reliability parameters for simple and complex systems.

Initially, Monte Carlo–embedded Markov chains and phase techniques were used to get solutions to the reliability issues. Later on, the technique of supplementary variables was introduced by Cox [1], and regenerative stochastic processes were employed by Smith [2]. Garg employed the supplementary variable technique in reliability analysis to solve the models.

An excellent account of the early development of the mathematical hypothesis of reliability was given by Barlow and Proschan [3]. Srinath [4] has compiled in his work the methods that were used to approximate and evaluate the reliabilities of machinery and systems. He also studied the means to develop reliability necessities of machinery and subsystem to meet the expected system reliability.

Brown and Proschan [5] proposed a survey on "imperfect repair." Goel and Gupta [6] analyzed the multistandby structure with repair and alternate strategy.

In history, various researchers worked on the reliability assessment of different manufacturing industries with different methods and techniques. Chakraborty and Dutta worked on the sugar production industry. Buzacott and Yao [7] have reviewed critical models for flexible industrialized systems. Singh and Kumar [8] worked on the reliability of the feeding system in the paper industry. Al-Jaar and Desrochers [9] discussed about the behavior assessment of automated production systems using general stochastic Petri nets. Singh and Mahajan [10] studied the reliability and behavior of an apparatus manufacturing industry using Laplace transforms. Altiok and Melamed [11] studied a case for modeling correlation in manufacturing systems. Tewari et al. [12] evaluated the accessibility of a crystallization unit of a sugar plant. Surabhi and Singh [13] found out the reliability measures of an engine assembly process of automobiles with an examination facility.

7.3 Cost optimization in the manufacturing industries

7.3.1 Reliability cost

Reliability cost may be categorized into five parts:

7.3.1.1 In-house or internal failure cost

Under those different types of costs like failure analysis, scrap value, testing, and corrective measures will be considered. Some more examples are the cost of crumb, cost of ruin, operating cost, replacement of revised products, clearance of faulty goods, system failure because of quality issues, investigation of the root of defects in the manufacturing, and retesting of revised goods.

7.3.1.2 Preventive cost

Instead of searching and eliminating defects from the product, it is better to prevent fault. The costs spend to shun or reduce the number of fault in first place are known as prevention costs. Few illustrations of prevention costs are enhancement of manufacturing growth, staff training, excellence engineering, statistical process control, quality circles, management of obstacle, and quality enhancement projects.

7.3.1.3 Cost of external failures

When imperfect products are served to consumers, external failure costs get increased, including service contract, alternate or replacement, lost in sales due to appalling reputation in the market, and expenses for damages because of the use of defective products. The delivery of faulty products can disappoint consumers, thereby reducing sales and income. Few more cases are expenses of field repair and handling grievance, product recollect repairs, and replacements further than the guarantee period.

7.3.1.4 Managerial cost

Managerial cost includes reviewing contracts, preparing proposals, performing data analysis, budget preparation, and management and forecasting.

7.3.1.5 Inspection and detection costs

Inspection or detection cost comes into sight to recognize faulty products before they are shipped to consumers. Entire costs that are related to the activities that are carried out during manufacturing processes to guarantee the quality standards are incorporated in this cost. For identifying defective or imperfect products, an inspection team needs to appoint by the manufacturing industry, which is sometimes very expensive for industries.

7.3.2 How the cost of a system affected by reliability

If a manufacturer wants to increase the reliability of his/her products, then it will result in the raised reliability design costs and in-house breakdown costs. The external costs like shipping cost do not depend on reliability but setup, commission, and preservation costs will show a turn down with an increase in reliability. Although as time passes, in-house breakdown costs or internal failure costs will start lessening:
- A design having the lowest cost will be always there among all possible designs.
- For the similar costs in two or more designs, the system may have different reliability levels.

– For a mixture of machinery or components that result in poor system cost, the reliability level can be high.
– Design has the maximum reliability among all the possible designs that will always be there.

7.3.3 How replacement policies affect the cost of the system

In the case of degraded items, a very common problem is to balance the cost of replacement of the old items with the new ones besides the cost of sustaining the old items efficiently.

Here are a few theories that will be pursued:

– Components are efficient unless or until they fail, and after that, they are finally useless.
– Wait-in-line troubles that arise when several things or items failed at the same time are disregarded as it is assumed that there is sufficient maintenance and repair facility and these activities are carried out continuously as per requirements.
– The items or machines that are failed or not working are replaced with the same item or machine, that is, both the replaced item and failed item have equal lifetime distribution.
– The time required for replacing the item is assumed to be minor or negligible.

The need for replacement is realized when the working elements like staff, apparatus, machines, and units are failed or degraded because of steady corrosion in their efficiency or unexpected failure or breakdown. If the replacement or alternate option is available in time, then maintenance and other operating costs can be reduced; on the other hand, these activities may increase the principal cost for replacement.

Illustrations are:

(a) Suppose that an automobile deteriorates due to its continuous usage, then we need to spend additional cost for its increased repairs and services. After a particular period, it will be too expensive to maintain the automobile, and it is better to be replaced with a new one. In this case, it is required to take replacement decisions to balance the rising maintenance cost with the falling cost value of the vehicle over time.

(b) If we consider an example of lights or bulbs on the road, they need to be replaced after their individual failure because in this case the time of failure cannot be predicted for a bulb; on the other hand, if the replacement is performed in a scheduled manner before their failure, then it will be cost-effective or economical. Hence, to make equilibrium among the useless life of a bulb before failure and cost occur during its service period, a replacement decision has to be taken here.

Here we discuss the following two types of replacement situations:
- Items whose effectiveness or efficiency deteriorates over time due to its regular use and requires extra repair and operating expenses.
- Items that fail suddenly or unexpectedly and become useless without giving any indication with time.

7.3.4 Types of replacement policies

Replacement policies can be divided into two types in regard to the timing of the replacement:
- Preventive replacement policy
- Failure replacement policy

7.3.4.1 Preventive replacement

- Steady interval or constant interval replacement policy
- Aged replacement policy
- Time-dependent replacement policy
- Inspection replacement policy
- In time replacement policy

7.3.4.2 Failure replacement

- Run to failure
- Accidental failure replacement

7.4 Model: operational behavior of gas production unit in a yarn plant with equipment renewal policy

7.4.1 Model description

This model discusses the behavior of gas production unit in a fiber plant for different cases, concerning a special policy called old age replacement policy or equipment renewal policy.

- The gas production unit consists of six subunits, namely, H_2SO_4 pit, charcoal unit, electric heater, H_2SO_4 separator, gas production unit, and freezing water unit, connected in series. H_2SO_4 separator has two units: H_2SO_4 collecting vessel and recovery pit.
- Equipment renewal policy has been applied for the freezing water unit as this unit has completed an age, say T, and still functional.
- Two cases have been discussed in this chapter. In the first case, freezing water unit is fully operational even after it attains the age T but the system is running at high risk, and in the second case, this unit has been replaced by a new one when the age T is attained.
- Here because of the failure of any subunit, the system can completely fail. It is also assumed that the system can be nonoperational due to a labor strike.
- Joint probability distribution has been applied in the repair of the H_2SO_4 separator since it can fail because of the failure of both of its subunits S_1 and $S_{2.}$

7.4.2 Assumptions

- Initially, at $t = 0$, the system is operating well.
- Failures are assumed to be statistically independent.
- The time required for repair of the subunits is assumed to be arbitrarily distributed.
- It is also assumed that after repair, the unit works like new.
- All failures follow exponential time distribution.
- Equipment renewal age replacement policy has been applied for the freezing water units.
- Two cases have been considered. In the first case, chilled water unit is working well even after it attains an age, say T, but the system is working with high risk, and in the second case old chilled water unit has been replaced with a new one.
- The whole system can also fail due to deliberate failures like workers strike.
- Joint probability distribution has been obtained in the repair of the H_2SO_4 separator (Figure 7.1 and Table 7.1).

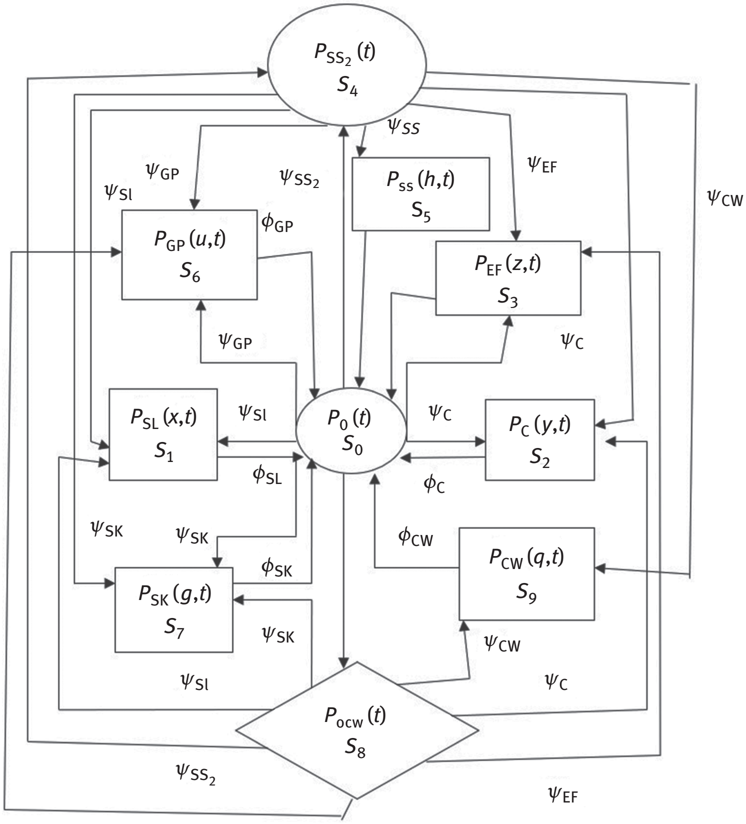

Figure 7.1: Transition state diagram.

Table 7.1: State specifications of the system.

States	Description	System state
S_0	Denotes that the system is in a fully operational condition	G
S_1	Denotes that the system is in a failed state because of the failure of the H_2SO_4 pit	F_R
S_2	Denotes that the system is in a failed state because of the failure of the charcoal unit	F_R
S_3	Denotes that the system is in a failed state because of the failure of the electric furnace	F_R
S_4	Denotes that the system is in the working condition when the recovery pit of the H_2SO_4 separator has been failed	G

Table 7.1 (continued)

States	Description	System state
S_5	When the system is in a failed state due to the failure of H_2SO_4 collecting vessel of H_2SO_4 separator when recovery pit has already failed	F_R
S_6	Denotes that the system is in a failed state because of the failure of the gas production unit	F_R
S_7	Denotes that the system is in a failed state because of worker's strike	F_R
S_8	Denotes that the system is in a wear-out condition but operational due to the old chilled water unit (at this state, the system is in a high risk of failure)	RS
S_9	Denotes the system is in a failed state from state S_8 due to the failure of the chilled water unit	F_R

Note: G, operational or working state; D_R, ruined state and under repair; F_R, breakdown or failed state and under repair.

7.4.3 Notations

Pr	Probability
$P_0(t)$	Pr (at time t, the system is in a good operational state S_0)
$P_i(k,t)$	Pr {of failed state because of failure of the ith subsystem at time t and elapsed repair time lies between k and $k+\Delta$}, where i = SL, C, EF, GP, CW, SK, and $k = x, y, z, u, q, g$.
SL/C/EF/	H_2SO_4 pit/charcoal unit/electric furnace/gas production unit/freezing
GP/CW/SK/OCW	Water unit/labor strike/old freezing water unit
k	Elapsed repair time, where $k = x, y, z, u, q, g$
ψ_{SL}/ψ_C	Failure rates of the H_2SO_4 pit/charcoal unit
ψ_{EF}/ψ_{GP}	Failure rates of electric furnace/gas production unit
ψ_{CW}/ψ_{SK}	Failure rates of chilled water unit/workers strike
ψ_{SS_1}	The failure rate of one unit (recovery pit) of H_2SO_4 separator
ψ_{SS}	The failure rate of other unit (H_2SO_4 collecting vessel) of H_2SO_4 separator
γ_{CW}	Wear-out failure rate or risk rate
$\phi i(k)$	General repair rate of the ith system in the time interval (k, $k+\Delta$), where i = SL, C, EF, GP, CW, SK, and $k = x, y, z, u, q, g$
$P_{SS_2}(t)$	Pr (of good operational state at time t when one unit (recovery pit) of H_2SO_4 separator failed)
$P_{OCW}(t)$	Pr (at time t, system is in the operational state but is at high risk due to old chilled water unit which is working after attaining an age, say T)
$P_{NCW}(t)$	Pr (at time t, system is in the good operational state when the old chilled water unit has been replaced by a new one)
K_1, K_2	Profit cost per unit time and repair cost per unit time, respectively

Let $u_1 = e^h$ and $u_2 = \phi_{SS}(h)$, then joint probability by using Gumbel–Hougaard family of the copula is given as $\phi_{SS} = \exp\left[h^\theta + \left[\log \phi_{SS}(h)\right]^\theta\right]^{1/\theta}$.

7.4.4 Mathematical model

The following differential equations have been obtained by considering limiting procedures and different probability constraints that satisfy the model:

$$\left[\frac{d}{dt} + \psi_{SL} + \psi_{C} + \psi_{EF} + \psi_{GP} + \psi_{SK} + \psi_{SS_2} + \gamma_{CW}\right] P_0(t)$$
$$= \int_0^\infty \phi_{SL}(x) P_{SL}(x,t)dx + \int_0^\infty \phi_{C}(y) P_{C}(y,t)dy + \int_0^\infty \phi_{EF}(z) P_{EF}(z,t)dz +$$
$$\int_0^\infty \phi_{GP}(u) P_{GP}(u,t)du + \int_0^\infty \phi_{SK}(g) P_{SK}(g,t)dg + \int_0^\infty \phi_{SS}(h) P_{SS}(h,t)dh +$$
$$\int_0^\infty \phi_{CW}(q) P_{CW}(q,t)dq, \tag{7.1}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \phi_{SL}(x)\right] P_{SL}(x,t) = 0, \tag{7.2}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \phi_{C}(y)\right] P_{C}(y,t) = 0, \tag{7.3}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \phi_{EF}(z)\right] P_{EF}(z,t) = 0, \tag{7.4}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial u} + \phi_{GP}(u)\right] P_{GP}(u,t) = 0, \tag{7.5}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial g} + \phi_{SK}(g)\right] P_{SK}(g,t) = 0, \tag{7.6}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial q} + \phi_{CW}(q)\right] P_{CW}(q,t) = 0, \tag{7.7}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial h} + \phi_{SS}(h)\right] P_{SS}(h,t) = 0, \tag{7.8}$$

$$\left[\frac{d}{dt} + \psi_{SL} + \psi_{C} + \psi_{EF} + \psi_{GP} + \psi_{SK} + \psi_{SS} + \psi_{CW}\right] P_{SS_2}(r,t) = \psi_{SS_2} P_0(t) + \psi_{SS_2} P_{OCW}(t), \tag{7.9}$$

$$\left[\frac{d}{dt}+\psi_{\mathrm{SL}}+\psi_{\mathrm{C}}+\psi_{\mathrm{EF}}+\psi_{\mathrm{GP}}+\psi_{\mathrm{SK}}+\psi_{\mathrm{SS}_2}+\psi_{\mathrm{CW}}\right]P_{\mathrm{OCW}}(t)\ =\ \gamma_{\mathrm{CW}}P_0(t), \tag{7.10}$$

where

$$\phi_{\mathrm{SS}} = \exp\left[h^{\theta}+\left[\log\phi_{\mathrm{SS}}(h)\right]^{\theta}\right]^{1/\theta}. \tag{7.11}$$

Boundary conditions:

$$P_{\mathrm{SL}}(0,t)=\psi_{\mathrm{SL}}\left[P_0(t)+P_{\mathrm{SS}_2}(t)+P_{\mathrm{OCW}}(t)\right], \tag{7.12}$$

$$P_{\mathrm{C}}(0,t)=\psi_{\mathrm{C}}\left[P_0(t)+P_{\mathrm{SS}_2}(t)+P_{\mathrm{OCW}}(t)\right], \tag{7.13}$$

$$P_{\mathrm{EF}}(0,t)=\psi_{\mathrm{EF}}\left[P_0(t)+P_{\mathrm{SS}_2}(t)+P_{\mathrm{OCW}}(t)\right], \tag{7.14}$$

$$P_{\mathrm{GP}}(0,t)=\psi_{\mathrm{GP}}\left[P_0(t)+P_{\mathrm{SS}_2}(t)+P_{\mathrm{OCW}}(t)\right], \tag{7.15}$$

$$P_{\mathrm{SK}}(0,t)=\psi_{\mathrm{SK}}\left[P_0(t)+P_{\mathrm{SS}_2}(t)+P_{\mathrm{OCW}}(t)\right], \tag{7.16}$$

$$P_{\mathrm{CW}}(0,t)=\psi_{\mathrm{CW}}\left[P_{\mathrm{SS}_2}(t)+P_{\mathrm{OCW}}(t)\right], \tag{7.17}$$

$$P_{\mathrm{SS}}(0,t)=\psi_{\mathrm{SS}}P_{\mathrm{SS}_2}(t). \tag{7.18}$$

Initial condition:

$$P_0(0)=1;\ \text{otherwise, zero.} \tag{7.19}$$

Solving eqs. (7.1)–(7.10) by taking Laplace transform and by using initial and boundary conditions, we obtained the following probabilities of the system is in upstate (i.e., good or partially failed or good with high risk) and downstate at time *t*:

$$\begin{aligned}\overline{P}_{\mathrm{up}}(s)\ &=\ \overline{P}_0(s)+\overline{P}_{\mathrm{SS}_2}(s)+\overline{P}_{\mathrm{OCW}}(s)\\ &=[1+B(s)+A(s)]\,\frac{1}{K(s)},\end{aligned} \tag{7.20}$$

$$\overline{P}_{\mathrm{down}}(s)=\overline{P}_{\mathrm{SL}}(s)+\overline{P}_{\mathrm{C}}(s)+\overline{P}_{\mathrm{EF}}(s)+\overline{P}_{\mathrm{GP}}(s)+\overline{P}_{\mathrm{SK}}(s)+\overline{P}_{\mathrm{CW}}(s)+\overline{P}_{\mathrm{SS}}(s). \tag{7.21}$$

Also,

$$\overline{P}_{\mathrm{up}}(s)+\overline{P}_{\mathrm{down}}(s)=1/s. \tag{7.22}$$

7.4.5 Steady-state behavior of the system

By Abel's lemma we have

$$\lim_{s\to 0}\{s\bar{F}(s)\} = \lim_{t\to\infty} F(t).$$

In eqs. (7.20) and (7.21), we get the following upstate and downstate time-independent probabilities:

$$P_{\mathrm{up}} = \frac{1}{K'(0)}\,[1+B(0)+A(0)], \tag{7.23}$$

$$P_{\mathrm{down}} = \frac{[1+B(0)+A(0)]}{K'(0)}[\psi_{\mathrm{SL}}M_{\mathrm{SL}}+\psi_{\mathrm{C}}M_{\mathrm{C}}+\psi_{\mathrm{EF}}M_{\mathrm{EF}}+\psi_{GP}M_{\mathrm{GP}}+\psi_{\mathrm{SK}}M_{\mathrm{SK}}]+$$

$$\psi_{\mathrm{CW}}M_{\mathrm{CW}}\frac{[B(0)+A(0)]}{K'(0)}+\psi_{\mathrm{SS}}M_{\mathrm{SS}}\frac{B(0)}{K'(0)}, \tag{7.24}$$

where

$$K'(0) = \left[\frac{d}{ds}K(s)\right]_{s=0}.$$

$M_i = -\bar{S}_i(0)$ = Mean time to repair the ith failure.

$$\mathrm{S}_{\phi_i}(s) = \frac{\phi_1}{s+\phi_1} \text{ where } i = \mathrm{SL, C, EF, GP, SK, CW, SS}.$$

When the system is not repairable, then probabilities are independent of x and then the reliability function is calculated as follows:

From eq. (7.20), we have

$$\bar{R}(s) = \frac{1}{s+\psi_{\mathrm{SL}}+\psi_{\mathrm{C}}+\psi_{\mathrm{EF}}+\psi_{\mathrm{GP}}+\psi_{\mathrm{SK}}+\psi_{\mathrm{SS}_2}+\gamma_{\mathrm{CW}}},$$

where $\bar{R}(s)$ denotes the Laplace transform of the reliability function.

So the reliability is

$$R(t) = exp\,[-(\psi_{\mathrm{SL}}+\psi_{\mathrm{C}}+\psi_{\mathrm{EF}}+\psi_{\mathrm{GP}}+\psi_{\mathrm{SK}}+\psi_{\mathrm{SS}_2}+\gamma_{\mathrm{CW}})t]. \tag{7.25}$$

7.4.6 Availability of the system

$$P_{\mathrm{up}}(t) = L^{-1}\{\bar{P}_{up}(s)\},$$

$$P_{\mathrm{up}}(s) = \frac{1+\dfrac{0.003\left(1+\frac{0.6}{s+0.681}\right)}{s+0.687}+\dfrac{0.01}{s+0.681}}{s+0.881}.$$

Taking inverse Laplace transforms, we have

$$P_{\mathrm{up}}(t) = 4.500000000e^{-(0.6810000000t)} - 1.530927835e^{-(0.6870000000t)} - 1.969072165e^{-(0.8810000000t)}. \quad (7.26)$$

7.4.7 MTTF of the system

The mean time to failure (MTTF) is

$$\mathrm{MTTF} = \int_0^\infty R(t)\,dt = \frac{1}{\psi_{\mathrm{SL}} + \psi_{\mathrm{C}} + \psi_{\mathrm{EF}} + \psi_{\mathrm{GP}} + \psi_{\mathrm{SK}} + \psi_{\mathrm{SS}_2} + \gamma_{\mathrm{CW}}}. \quad (7.27)$$

7.4.8 Cost-effectiveness of the system

The cost of the system is

$$G(t) = K_1 \int_0^t P_{\mathrm{up}}(t)dt - K_2 t,$$

where K_1 and K_2 are profit and repair costs per unit time, respectively.

Also,

$$G(t) = K_1 \int_0^t \Big[0.1666666666e^{-(0.0910000000\ 0t)} - 0.02380952381e^{-(0.0970000000\ 0t)} + 0.8571428571e^{-(0.1810000000t)}\Big] dt - K_2 t. \quad (7.28)$$

7.4.9 Case I: when the freezing water unit is operating well after completing its life

For a realistic study of the system's nature, we calculate reliability, availability, and cost function of the system about the time, keeping the other parameter fixed and MTTF of the system for different failure rates.

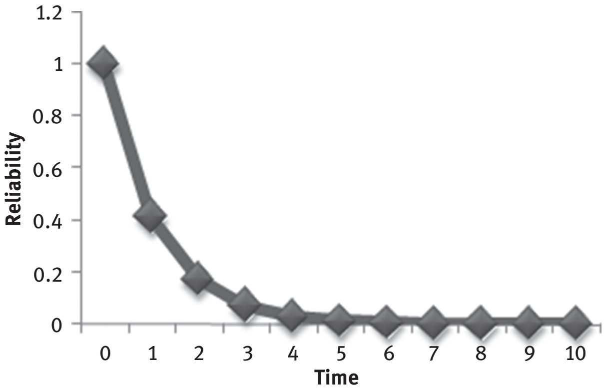

Figure 7.2: Time against reliability.

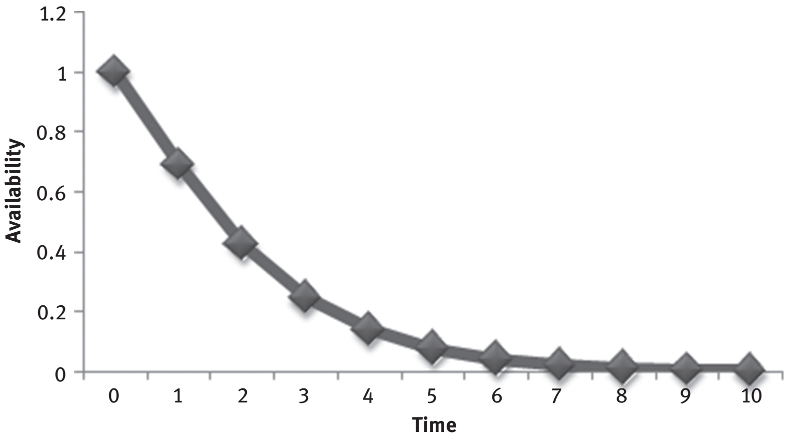

Figure 7.3: Time against availability.

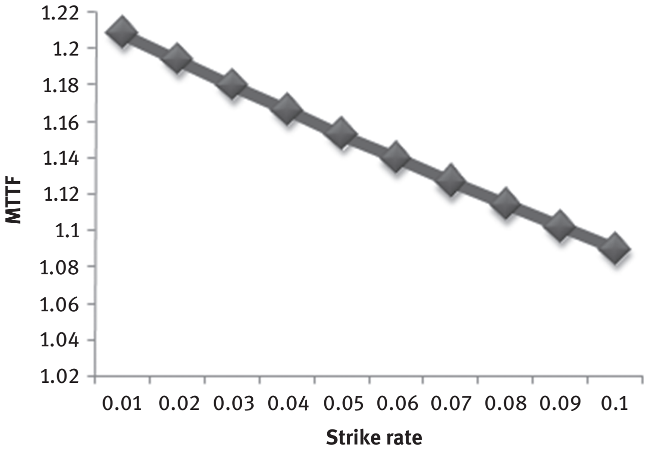

Figure 7.4: MTTF against labor strike rate.

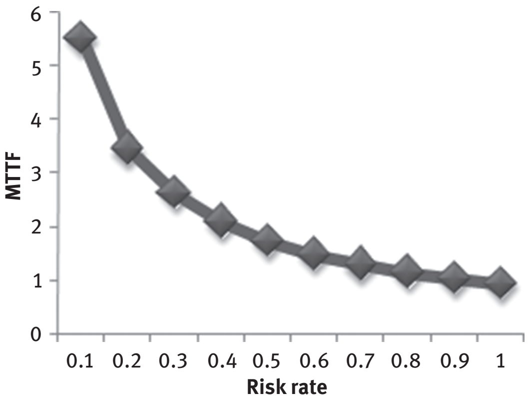

Figure 7.5: MTTF against risk rate.

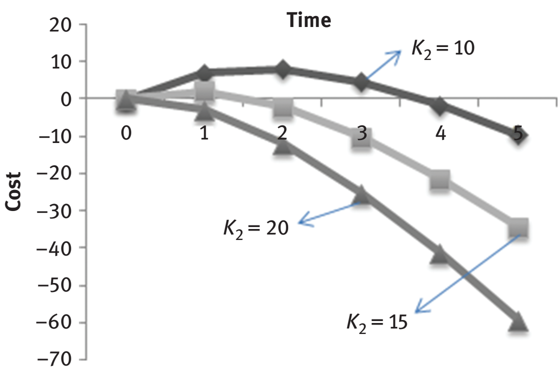

Figure 7.6: Profit function against time.

7.4.10 Case II: when the freezing water unit is replaced by a new one

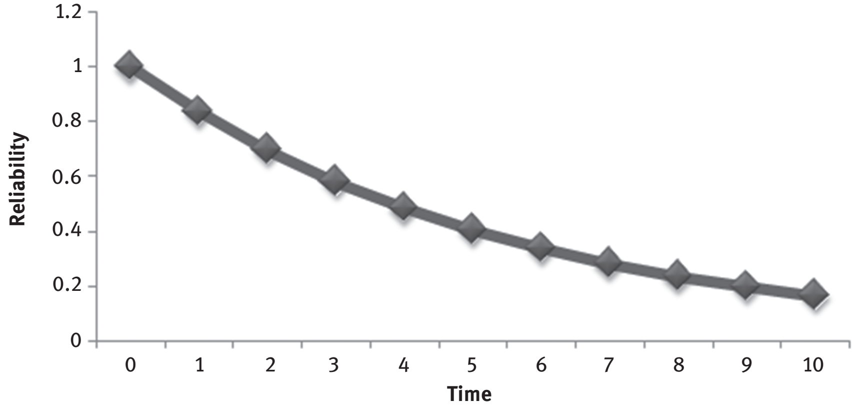

Figure 7.7: Time against reliability.

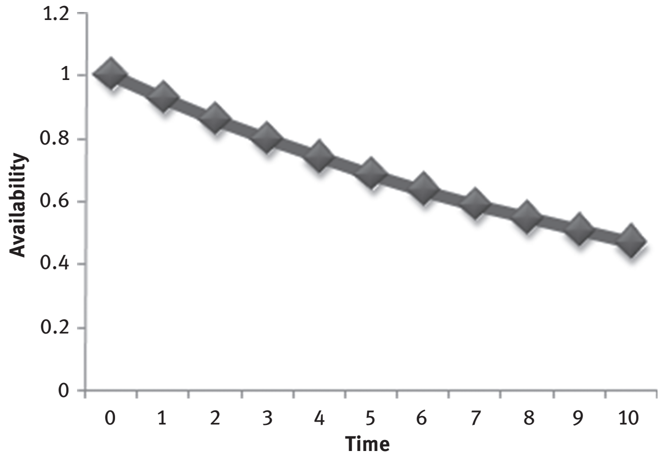

Figure 7.8: Time against availability.

Figure 7.9: MTTF against labor strike rate.

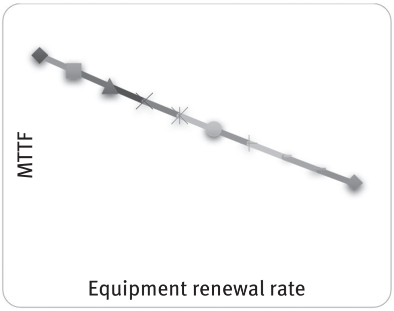

Figure 7.10: MTTF against equipment renewal rate.

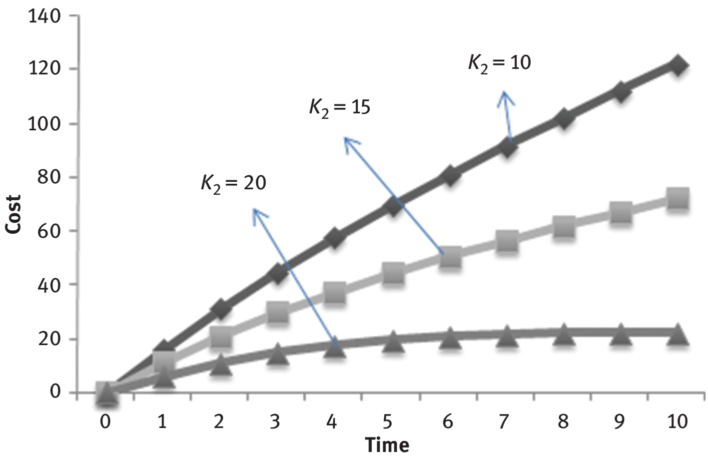

Figure 7.11: Profit function against time.

7.5 Results and discussion

In this study, two different cases are considered to find the reliability aspects of the considered system:

(a) When the freezing water unit is operating well after completing its life.
(b) The old freezing water unit has been replaced by a new one.

By critical examination of both the cases, it is found that in the first case, reliability of the system decreases more rapidly for the time than the second case, and later on becomes stable at a value of 0.0014 as shown in Figure 7.2. In the second case, it decreases approximately in a constant manner and becomes stable at 0.4 (shown in Figure 7.7).

The availability in the first case decreases rapidly as shown in Figure 7.3, but in the second case it decreases approximately in a constant manner as shown in Figure 7.8.

In the first case, the strike rate lies between 1.20772 and 1.10132, and in the second case it decreases hastily from 34.4827 to 8.40336 as shown in Figures 7.4 and 7.9, respectively. Thus, we can conclude that the mean operating time between successive failures is higher for every parameter in the second case when compared with the first case. In the first case, for the risk state, it decreases from 5.52486 to 0.925069 as shown in Figure 7.5, whereas in the second case, it decreases from 12.19512 to 11.1111 as depicted in Figure 7.10.

Fixing the profit cost K_1 at a value of Rs. 20 and varying service cost K_2 as 10, 15, and 20, the graphs reveal a significant conclusion that increased service cost leads to decrement in the expected profit. In the first case, the highest value of expected profit corresponding to the considered values of different parameters is 7.99223 for $K_2 = 10$ but when the service cost is equal to the revenue cost, then the profit function becomes negative. The industry will be facing a huge loss in this situation as shown in Figure 7.6. In the second case, the highest value of the expected profit is 121.8151 for $K_2 = 10$ as shown in Figure 7.11. The expected profit reduces with the increment in K_2.

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Amit Kumar and Arun Kumar Garov

8 Performance analysis of a paper mill plant by considering reworking/degradation of its components

Abstract: A paper mill plant (PMP) is investigated in this chapter by considering its different components such as digester, head box, dandy roll, press section, and dryer. The performance of PMP depends on the performance of its individual components along with a specific maintenance policy and on the interconnections of components inside the system. Individual components along with their various failures/repairs are critically analyzed, and a mathematical model is developed for their functioning through the Markov method to obtain reliability characteristics. For clear understanding, a graphical representation of outcomes has been depicted, and critical components of PMP have been identified.

Keywords: Paper mill plant, Markov process, mathematical modeling, multistate system, performance measures

8.1 Introduction

In this rapidly growing world, paper is widely used everywhere, for example, in education, offices, and packaging. Fulfilling this demand is not an easy task. A paper mill plant (PMP) is a place committed for production of paper by using wood pulp, old rags, and various items. A lot of work has been done in the past regarding the reliability of various industrial systems as well as PMP. Reliability is a popular concept that has been eminent for years as a creditable quality of a person or a product.

From the last few decades, an eminent work was done regarding the performability evaluation of a PMP by using different methods/techniques, for example, fuzzy technique, Markov method/processes, Petri nets, and fuzzy lambda–tau methodology [1–3]. Wastewater management is one of the most herculean tasks for the management of a PMP. The wastewater management of a PMP was analyzed in [4–6] and obtained an optimal solution for it. Kreetachat et al. [7] found out various components of wastewater in a PMP and obtained decolonization and organic removal efficiency and used a lignin-derived component and aliphatic compounds. Potential and cost reduction of CO_2 in a PMP was analyzed in [8]. Kopra et al. [9] discussed the

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refractive index measurements for determination of dissolved dry solids from a single washer filtrates and pulp filtrate fractions and also for the general control of washing loss levels in brown stock washing. Garg et al. [10] studied a hybridized technique, namely, ABCBLT to obtain the membership function for the performance measures of a complex repairable engineering system. Kumar et al. [11] calculated the sensitivity analysis and reliability characteristics for a thermal power plant by using the concept of multistate system. Ram et al. [12] determined the reliability of a system by the concept of k-out-of-n redundancy. Gupta and Tewari [13] discussed the development of a performance model of a power plant using the Markov process and a probabilistic approach. It has been done through two parts: first, the development of a predictive modal, and in the second phase, the performance evaluation of the developed model. Wang et al. [14] obtained the reliability optimization of a mixed configuration system by binary particle swarm optimization algorithm. The availability and maintenance cost of a PMP were investigated by Sachdeva et al. [15] to improve the performance of the same. Samrout et al. [16] found the maintenance cost of a mixed configuration system with the help of the ant colony optimization technique. Mon et al. [17] calculated the fuzzy system reliability analysis for different probability density functions. Sharma et al. [18] determined the availability with the help of KBARMs and examined the performance of a system with traditional analytical techniques. Huang et al. [19] used a predictive control system, which is based on an ANFIS, to the predictive model and controller use for performance of a paper-making wastewater treatment process on a PMP. Recalling the above-mentioned research works regarding PMP and also for several other industrial systems for reliability evaluation, here authors get an idea to use the mathematical modeling and reliability approach in a PMP.

8.2 System description

A paper mill consists of several units, whose collective functioning produced paper, in which digester, head box, dandy roll, dryer, and press section are main units. To begin the process, logs are passed through a debarker, where the bark is removed, and through chippers, where spinning blades cut the wood into one inch pieces. Those wood chips are then pressure-cooked with a mixture of water and chemicals in a digester. After the digester, it reaches the head box, dandy roll, press section, and then dryer for the final output, that is, paper. The various components of a PMP with their functioning are explained further.

– **Digester**

The wood chips are mixed into a large cylinder called a digester; here, they are soaked in a bath of chemicals mainly bisulfate of lime and cooked under pressure for about 8 h. All wood contains, along with the cellulose that makes a paper, a great deal of other materials that will decay slowly; the digesting process removes the other materials, leaving just the cellulose.

– **Head box**

After the digester, the next step is head box. In head box, the pulp passes through a wire belt and wire section.

– **Dandy roll**

As the pulp travels down the screen, water is drained away and recycled. The resulting crude paper sheet, or web, is squeezed between large rollers to remove most of the remaining water and ensure smoothness and uniform thickness.

– **Pressing**

After rolling, the product will reach the pressing section, where it will press according to size of paper.

– **Dryer**

After pressing, the product will reach the dryer section; here, the remaining water that is present after rolling and pressing sections will be completely removed from it

8.3 Assumptions

The following assumptions are considered in this chapter:
- A failure that occurs in the PMP will either reduce its performance or make it to fail.
- More than one component cannot be failed simultaneously.
- Repairmen will always be available to take care of failed components.
- For numerical computation, different failure and repairs are taken as constants.

8.4 Nomenclatures

For ease of reference, the major notations and states that are used in this paper are given in Tables 8.1 and 8.2, respectively.

Table 8.1: Nomenclature.

t/s	**Time unit/inverse Laplace variable**
$P_i(t);\ i = 0, 4, 5, 7$	The state probability of the paper mill plant is in state S_i; $i = 0, 4, 5, 7$
$P_j(x, t);\ j = 1, 2, 3, 6, 8$	The state probability of the paper mill plant is in state S_j; $j = 1, 2, 3, 6, 8$ with elapsed repair "x."
$\lambda_{DG}/\lambda_{HB}/\lambda_{DR}/\lambda_{PS}/\lambda_{DRO}/\lambda_{HO}$	Failure rates of the digester/head box/dryer/press section/dandy roll/human operator, respectively
$\mu_{DG}/\mu_{HB}/\mu_{DR}/\mu_{PS}/\mu_{DRO}/\mu_{HO}$	Repair rate of digester/head box/dryer/press section/dandy roll/human operator, respectively

Table 8.2: State description.

S_1	The state in which PMP is failed due to failure of digester
S_2	The state in which PMP is failed due to failure of head box
S_3	The state in which PMP is failed due to failure of dryer
S_4	The state in which PMP performance is degraded due to degradation of press section
S_5	The state in which PMP performance is degraded due to degradation of press section
S_6	The state in which PMP is failed due to press section failure
S_7	The state in which PMP performance is degraded due to failure of dandy roll
S_8	The state in which PMP is failed due to a human operator

8.5 Flow diagram and transition state diagram

By analyzing the various failures/repairs and functioning of the PMP, the flow diagram (Figure 8.1(a)) and state transition diagram (Figure 8.1(b)) have been devolved for PMP.

8.6 Mathematical formulation

Succeeding the rupture and repairs of the various components of PMP, throughout its functioning it takes different states, that is, good/degraded/failed states. These states are shown in the state transition diagram (Figure 8.1(b)). On the basis of these states, the subsequent introdifferential equations are generated as follows:

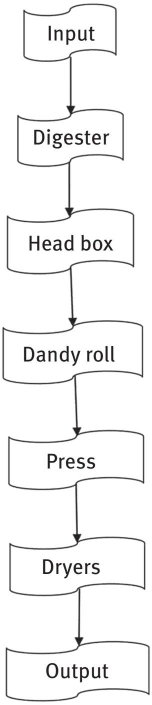

Figure 8.1a: Flow diagram.

For state $P_0(t)$

$$\left(\frac{\partial}{\partial t} + \lambda_{\mathrm{DG}} + \lambda_{\mathrm{HB}} + \lambda_{\mathrm{DR}} + 3\lambda_{\mathrm{PS}} + \lambda_{\mathrm{DRO}} + \lambda_{\mathrm{HO}}\right) P_0(t) = \mu_{\mathrm{DRO}} P_7(t) + \mu_{\mathrm{PS}} P_4(t) + \int_0^\infty \mu_{\mathrm{DG}} P_1(x,t)dx$$

$$+ \int_0^\infty \mu_{\mathrm{HB}} P_2(x,t)dx + \int_0^\infty \mu_{\mathrm{DR}} P_3(x,t)dx + \int_0^\infty \mu_{\mathrm{HO}} P_8(x,t)dx + \int_0^\infty \mu P_6(x,t)dx. \tag{8.1}$$

For state $P_1(x,t)$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \mu_{\mathrm{DG}}\right) P_1(x,t) = 0. \tag{8.2}$$

For state $P_2(x,t)$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \mu_{\mathrm{HB}}\right) P_2(x,t) = 0. \tag{8.3}$$

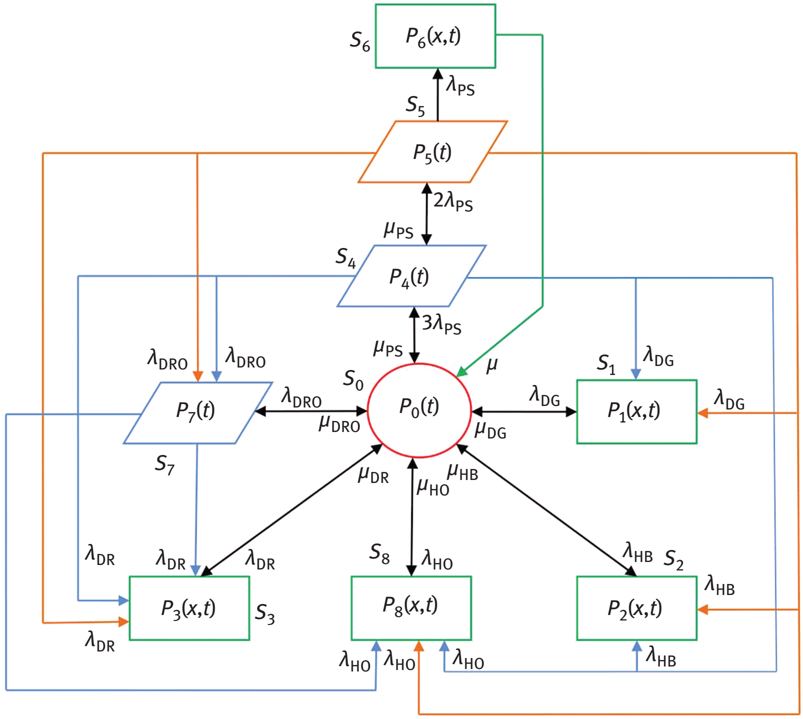

Figure 8.1b: State transition diagram.

For state $P_3(x,t)$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_{DR}\right)P_3(x,t)=0. \tag{8.4}$$

For state $P_4(x,t)$

$$\left(\frac{\partial}{\partial t}+\lambda_{DG}+\lambda_{HB}+\lambda_{HO}+\lambda_{DRO}+\lambda_{DR}+2\lambda_{PS}+\mu_{PS}\right)P_4(t)=3\lambda_{PS}P_0(t)+\mu_{PS}P_5(t). \tag{8.5}$$

For state $P_5(x,t)$

$$\left(\frac{\partial}{\partial t}+\lambda_{PS}+\lambda_{DG}+\lambda_{HB}+\lambda_{HO}+\lambda_{DR}+\lambda_{DRO}+\mu_{PS}\right)P_5(t)=2\lambda_{PS}P_4(t). \tag{8.6}$$

For state $P_6(x,t)$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu\right)P_6(x,t)=0. \tag{8.7}$$

For state $P_7(x,t)$

$$\left(\frac{\partial}{\partial t}+\lambda_{DR}+\lambda_{HO}+\lambda_{HO}+\mu_{DRO}\right)P_7(t)=\lambda_{DRO}P_0(t)+\lambda_{DRO}P_4(t)+\lambda_{DRO}P_5(t). \tag{8.8}$$

For state $P_8(x,t)$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_{HO}\right)P_8(x,t)=0. \tag{8.9}$$

Boundary conditions:

$$P_1(0,t)=\lambda_{DG}P_0(t)+\lambda_{DG}P_4(t)+\lambda_{DG}P_5(t), \tag{8.10}$$

$$P_2(0,t)=\lambda_{HB}P_0(t)+\lambda_{HB}P_4(t)+\lambda_{HB}P_5(t), \tag{8.11}$$

$$P_3(0,t)=\lambda_{DR}P_0(t)+\lambda_{DR}P_4(t)+\lambda_{DR}P_5(t)+\lambda_{DR}P_7(t), \tag{8.12}$$

$$P_6(0,t)=\lambda_{PS}P_5(t), \tag{8.13}$$

$$P_8(0,t)=\lambda_{HO}P_0(t)+\lambda_{HO}P_4(t)+\lambda_{HO}P_5(t)+\lambda_{HO}P_7(t). \tag{8.14}$$

Initial condition:

$$P_0(0)=1 \text{ and all other states are zero at } t=0. \tag{8.15}$$

The set of equations ((8.1)–(8.14) can be rewritten as follows (by using Laplace transformation):

$$(s+\lambda_{DG}+\lambda_{HB}+\lambda_{DR}+3\lambda_{PS}+\lambda_{DRO}+\lambda_{HO})\overline{P}_0(s)=1+\mu_{DRO}\overline{P}_7(s)+\mu_{PS}\overline{P}_4(s)+\int_0^\infty\mu_{DG}\overline{P}_1(x,s)dx$$

$$+\int_0^\infty\mu_{HB}\overline{P}_2(x,s)dx+\int_0^\infty\mu_{DR}\overline{P}_3(x,s)dx+\int_0^\infty\mu_{HO}\overline{P}_8(x,s)dx+\int_0^\infty\mu\overline{P}_6(x,s)dx, \tag{8.16}$$

$$\left(\frac{\partial}{\partial x}+s+\mu_{DG}\right)\overline{P}_1(x,s)=0, \tag{8.17}$$

$$\left(\frac{\partial}{\partial x}+s+\mu_{HB}\right)\overline{P}_2(x,s)=0, \tag{8.18}$$

$$\left(\frac{\partial}{\partial x}+s+\mu_{DR}\right)\overline{P}_3(x,s)=0, \tag{8.19}$$

$$(s+\lambda_{DG}+\lambda_{HB}+\lambda_{HO}+\lambda_{DRO}+\lambda_{DR}+2\lambda_{PS}+\mu_{PS})\overline{P}_4(s)=3\lambda_{PS}\overline{P}_0(s)+\mu_{PS}\overline{P}_5(s), \tag{8.20}$$

$$(s+\lambda_{PS}+\lambda_{DG}+\lambda_{HB}+\lambda_{HO}+\lambda_{DR}+\lambda_{DRO}+\mu_{PS})\overline{P}_5(s)=2\lambda_{PS}\overline{P}_4(s), \tag{8.21}$$

$$\left(\frac{\partial}{\partial x}+s+\mu\right)\overline{P}_6(x,s)=0, \tag{8.22}$$

$$\left(s+\lambda_{\mathrm{DR}}+\lambda_{\mathrm{HO}}+\mu_{\mathrm{DRO}}\right)\overline{P}_7(s)=\lambda_{\mathrm{DRO}}\overline{P}_0(s)+\lambda_{\mathrm{DRO}}\overline{P}_4(s)+\lambda_{\mathrm{DRO}}\overline{P}_5(s), \tag{8.23}$$

$$\left(\frac{\partial}{\partial x}+s+\mu_{\mathrm{HO}}\right)\overline{P}_8(x,s)=0. \tag{8.24}$$

Boundary conditions:

$$\overline{P}_1(0,s)=\lambda_{\mathrm{DG}}\overline{P}_0(s)+\lambda_{\mathrm{DG}}\overline{P}_4(s)+\lambda_{\mathrm{DG}}\overline{P}_5(s), \tag{8.25}$$

$$\overline{P}_1(0,s)=\lambda_{\mathrm{HB}}\overline{P}_0(s)+\lambda_{\mathrm{HB}}\overline{P}_4(s)+\lambda_{\mathrm{HB}}\overline{P}_5(s), \tag{8.26}$$

$$\overline{P}_3(0,s)=\lambda_{\mathrm{DR}}\overline{P}_0(s)+\lambda_{\mathrm{DR}}\overline{P}_4(s)+\lambda_{\mathrm{DR}}\overline{P}_5(s)+\lambda_{\mathrm{DR}}\overline{P}_7(s), \tag{8.27}$$

$$\overline{P}_6(0,s)=\lambda_{\mathrm{PS}}\overline{P}_5(s), \tag{8.28}$$

$$\overline{P}_8(0,s)=\lambda_{\mathrm{HO}}\overline{P}_0(s)+\lambda_{\mathrm{HO}}\overline{P}_4(s)+\lambda_{\mathrm{HO}}\overline{P}_5(s)+\lambda_{\mathrm{HO}}\overline{P}_7(s). \tag{8.29}$$

Solving eqs. (8.16)–(8.29) with the help of initial condition, we get

$$\overline{P}_0(s)=\frac{1}{(H_1-H_5-H_6-H_7-H_8)},$$

$$\overline{P}_4(s)=\frac{3\lambda_{\mathrm{PS}}H_3\overline{P}_0(s)}{(H_2H_3-2\lambda_{\mathrm{PS}}\mu_{\mathrm{PS}})},$$

$$\overline{P}_5(s)=\frac{2\lambda_{\mathrm{PS}}3\lambda_{\mathrm{PS}}\overline{P}_0(s)}{(H_2H_3-2\lambda_{\mathrm{PS}}\mu_{\mathrm{PS}})},$$

$$\overline{P}_7(s)=\left(\frac{\lambda_{\mathrm{DRO}}}{H_4}+\frac{\lambda_{\mathrm{DRO}}3\lambda_{\mathrm{PS}}H_3}{H_4(H_2H_3-2\lambda_{\mathrm{PS}}\mu_{\mathrm{PS}})}+\frac{\lambda_{\mathrm{DRO}}2\lambda_{\mathrm{PS}}3\lambda_{\mathrm{PS}}}{H_4(H_2H_3-2\lambda_{\mathrm{PS}}\mu_{\mathrm{PS}})}\right)\overline{P}_0(s),$$

where

$$H_1=(s+\lambda_{\mathrm{DG}}+\lambda_{\mathrm{HB}}+\lambda_{\mathrm{DR}}+3\lambda_{\mathrm{PS}}+\lambda_{\mathrm{DRO}}+\lambda_{\mathrm{HO}}),$$

$$H_2=(s+\lambda_{\mathrm{DG}}+\lambda_{\mathrm{HB}}+\lambda_{\mathrm{HO}}+\lambda_{\mathrm{DRO}}+\lambda_{\mathrm{DR}}+2\lambda_{\mathrm{PS}}+\mu_{\mathrm{PS}}),$$

$$H_3=(s+\lambda_{\mathrm{PS}}+\lambda_{\mathrm{DG}}+\lambda_{\mathrm{HB}}+\lambda_{\mathrm{HO}}+\lambda_{\mathrm{DR}}+\lambda_{\mathrm{DRO}}+\mu_{\mathrm{PS}}),$$

$$H_4=(s+\lambda_{\mathrm{DR}}+\lambda_{\mathrm{HO}}+\mu_{\mathrm{DRO}}),$$

$$H_5=\left(\frac{\lambda_{\mathrm{DRO}}}{H_4}+\frac{3\lambda_{\mathrm{PS}}H_3}{H_4(H_2H_3-\mu_{\mathrm{PS}}2\lambda_{\mathrm{PS}})}+\frac{\lambda_{\mathrm{DRO}}2\lambda_{\mathrm{PS}}3\lambda_{\mathrm{PS}}}{H_4(H_2H_3-\mu_{\mathrm{PS}}2\lambda_{\mathrm{PS}})}\right)\left(\mu_{\mathrm{DRO}}+\frac{\lambda_{\mathrm{DR}}\mu_{\mathrm{DR}}}{s+\mu_{\mathrm{DR}}}+\frac{\lambda_{\mathrm{HO}}\mu_{\mathrm{HO}}}{s+\mu_{\mathrm{HO}}}\right),$$

$$H_6=\frac{3\lambda_{\mathrm{PS}}H_3}{(H_2H_3-\mu_{\mathrm{PS}}2\lambda_{\mathrm{PS}})}\left(\mu_{\mathrm{PS}}+\frac{\lambda_{\mathrm{DG}}\mu_{\mathrm{DG}}}{s+\mu_{\mathrm{DG}}}+\frac{\lambda_{\mathrm{HO}}\mu_{\mathrm{HO}}}{s+\mu_{\mathrm{HO}}}+\frac{\lambda_{\mathrm{HB}}\mu_{\mathrm{HB}}}{s+\mu_{\mathrm{HB}}}+\frac{\lambda_{\mathrm{DR}}\mu_{\mathrm{DR}}}{s+\mu_{\mathrm{DR}}}\right),$$

$$H_7 = \frac{2\lambda_{PS}3\lambda_{PS}}{(H_2H_3 - \mu_{PS}2\lambda_{PS})}\left(\frac{\lambda_{DG}\mu_{DG}}{s+\mu_{DG}} + \frac{\lambda_{HO}\mu_{HO}}{s+\mu_{HO}} + \frac{\lambda_{HB}\mu_{HB}}{s+\mu_{HB}} + \frac{\lambda_{DR}\mu_{DR}}{s+\mu_{DR}} + \frac{\lambda_{PS}\mu_{PS}}{s+\mu_{PS}}\right),$$

$$H_8 = \left(\frac{\lambda_{DG}\mu_{DG}}{s+\mu_{DG}} + \frac{\lambda_{HO}\mu_{HO}}{s+\mu_{HO}} + \frac{\lambda_{HB}\mu_{HB}}{s+\mu_{HB}} + \frac{\lambda_{DR}\mu_{DR}}{s+\mu_{DR}}\right).$$

The probability of the system is in the form of upstate (good and degraded states) and downstate (failed state) are given as

$$\overline{P}_{up}(s) = \overline{P}_0(s) + \overline{P}_4(s) + \overline{P}_5(s) + \overline{P}_7(s), \tag{8.30}$$

$$\overline{P}_{down}(x,s) = \overline{P}_1(x,s) + \overline{P}_2(x,s) + \overline{P}_3(x,s) + \overline{P}_6(x,s) + \overline{P}_8(x,s). \tag{8.31}$$

8.7 Computation of various performance measures of PMP

8.7.1 Availability analysis

It is the measure of performance of maintained equipment. From the state transition diagram (Figure 8.1(b)), the availability of PMP is obtained as in eq. (8.30). Taking the value of different failure/repair as $\lambda_{DG} = 0.04$, $\lambda_{HB} = 0.01$, $\lambda_{DR} = 0.04$, $\lambda_{PS} = 0.03$, $\lambda_{DRO} = 0.02$, $\lambda_{HO} = 0.02$, $\mu_{DG} = 1$, $\mu_{DR} = 1$, $\mu_{HB} = 1$, $\mu_{PS} = 1$, $\mu = 1$, $\mu_{DRO} = 1$, $\mu_{HO} = 1$, in eq. (8.30) and then using inverse Laplace transformation, we get the availability of the PMP as follows:

$$P_{up}(t) = \begin{Bmatrix} 0.8689180707e^{-3.908891235t} + 0.03392638267e^{-2.018279532t} \\ + 0.005676112442e^{-1.218340496t} - 0.09147943424e^{-0.7444887366t} \end{Bmatrix} \tag{8.32}$$

The behavior of PMP availability with time is obtained as given in Figure 8.2 by varying time t in eq. (8.32).

8.7.2 Reliability

The reliability of PMP can be obtained by taking all repairs as zero and failure rates as $\lambda_{DG} = 0.04$, $\lambda_{HB} = 0.01$, $\lambda_{DR} = 0.04$, $\lambda_{PS} = 0.03$, $\lambda_{DRO} = 0.02$, $\lambda_{HO} = 0.02$ in eq. (8.30), and then taking inverse Laplace transformation:

$$R(t) = \begin{Bmatrix} 3.915254237e^{-0.81t} + 6e^{-0.7650000000t}\sinh(0.0450t) \\ -5.7e^{-0.72t} + 2.634146341e^{-0.6300000000t} + 0.1505994212e^{-0.22t} \end{Bmatrix}. \tag{8.33}$$

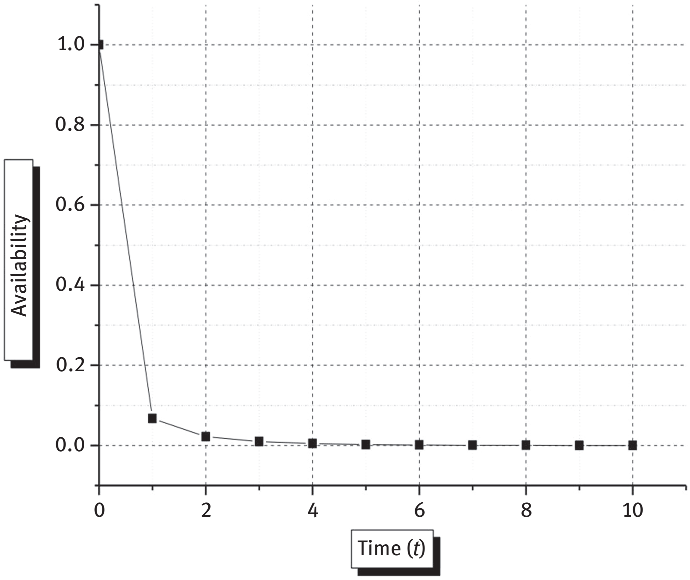

Figure 8.2: Availability versus time.

The behavior of PMP reliability with time is obtained as shown in Figure 8.3 by varying time t in eq. (8.33).

Table 8.3: Behavior of reliability versus time unit (t).

Time unit (t)	Reliability
0	1.0000000
1	0.6167107
2	0.3856004
3	0.2449810
4	0.1586895
5	0.1051808
6	0.0715327
7	0.0499807
8	0.0358584
9	0.0263585
10	0.0197860

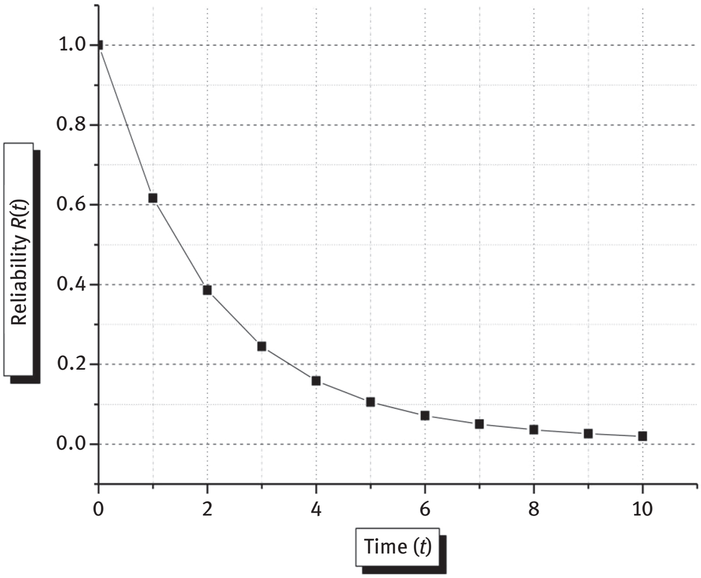

Figure 8.3: Reliability versus time.

8.7.3 Mean time to failure (MTTF) of PMP

MTTF (mean time to failure) is calculated as follows (from eq. (8.30)):

$$\mathrm{MTTF} = \lim_{s\to\infty}\overline{R}(s). \tag{8.34}$$

The behavior of MTTF with respect to various failure rates puts the value of failure rates as $\lambda_{\mathrm{DG}} = 0.04$, $\lambda_{\mathrm{HB}} = 0.01$, $\lambda_{\mathrm{DR}} = 0.04$, $\lambda_{\mathrm{PS}} = 0.03$, $\lambda_{\mathrm{DRO}} = 0.02$, $\lambda_{\mathrm{HO}} = 0.02$, and then varying each one by one from 0.01 to 0.10 with an interval of 0.01 in eq. (8.34). We obtain MTTF of PMP as given in Table 8.3 and corresponding Figure 8.4.

8.7.4 Sensitivity analysis

Sensitivity analysis helps us to identify critical components or portion of a system that is particularly sensitive to error. This chapter discusses further the sensitivity analysis of PMP's reliability with respect to MTTF.

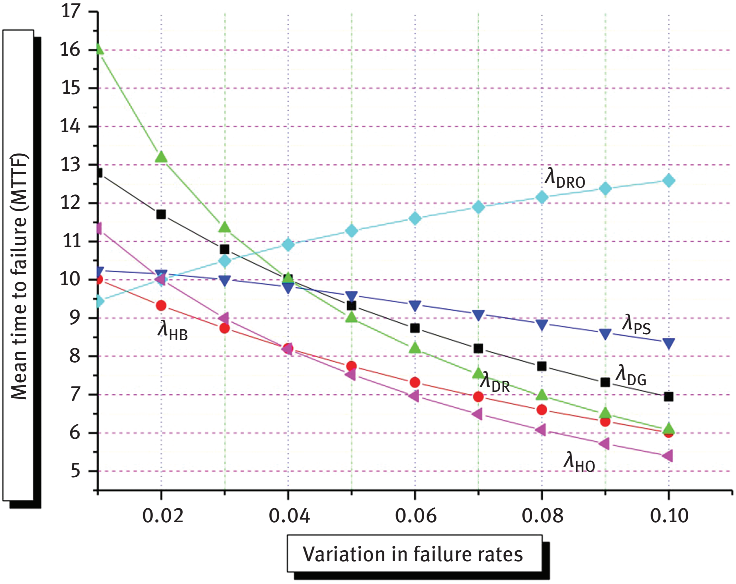

Figure 8.4: MTTF versus failure rates.

8.7.4.1 Sensitivity analysis of MTTF

Sensitivity analysis of PMP with respect to MTTF is performed by differentiating this expression with respect to various failure rates [20–22], and then place the value of various failure rates as $\lambda_{DG} = 0.04$, $\lambda_{HB} = 0.01$, $\lambda_{DR} = 0.04$, $\lambda_{PS} = 0.03$, $\lambda_{DRO} = 0.02$, $\lambda_{HO} = 0.02$ in these partial derivatives. Now varying the failure rates one by one, respectively, from 0.01 to 0.10 in these partial derivatives, we obtain Table 8.4 and corresponding Figure 8.5 for the sensitivity analysis of PMP for MTTF.

Table 8.4: Sensitivity of MTTF as a function of failure rates.

Variation in failure rates	$\frac{\partial(\mathrm{MTTF})}{\partial\lambda_{DG}}$	$\frac{\partial(\mathrm{MTTF})}{\partial\lambda_{HB}}$	$\frac{\partial(\mathrm{MTTF})}{\partial\lambda_{DR}}$	$\frac{\partial(\mathrm{MTTF})}{\partial\lambda_{PS}}$	$\frac{\partial(\mathrm{MTTF})}{\partial\lambda_{DRO}}$	$\frac{\partial(\mathrm{MTTF})}{\partial\lambda_{HO}}$
0.01	−117.3708	−72.99488	−359.8264	−4.776077	60.93474	−153.5449
0.02	−99.00666	−63.61309	−221.1514	−11.76449	52.10479	−114.6947
0.03	−84.55530	−55.90199	−153.5449	−17.28797	45.01668	−89.89174
0.04	−72.99488	−49.49202	−114.6947	−21.00240	39.24949	−72.86942
0.05	−63.61309	−44.10949	−89.89174	−23.24846	34.50029	−60.56055

Table 8.4 (continued)

Variation in failure rates	$\frac{\partial(\text{MTTF})}{\partial\lambda_{DG}}$	$\frac{\partial(\text{MTTF})}{\partial\lambda_{HB}}$	$\frac{\partial(\text{MTTF})}{\partial\lambda_{DR}}$	$\frac{\partial(\text{MTTF})}{\partial\lambda_{PS}}$	$\frac{\partial(\text{MTTF})}{\partial\lambda_{DRO}}$	$\frac{\partial(\text{MTTF})}{\partial\lambda_{HO}}$
0.06	−55.90199	−39.54839	−72.86942	−24.42663	30.54701	−51.30390
0.07	−49.49202	−35.65143	−60.56055	−24.86270	27.22415	−44.12768
0.08	−44.10949	−32.29691	−51.30390	−24.79650	24.40653	−38.42804
0.09	−39.54839	−29.38957	−44.12768	−24.39800	21.99816	−33.81113
0.10	−35.65143	−26.85402	−38.42804	−23.78579	19.92455	−30.00965

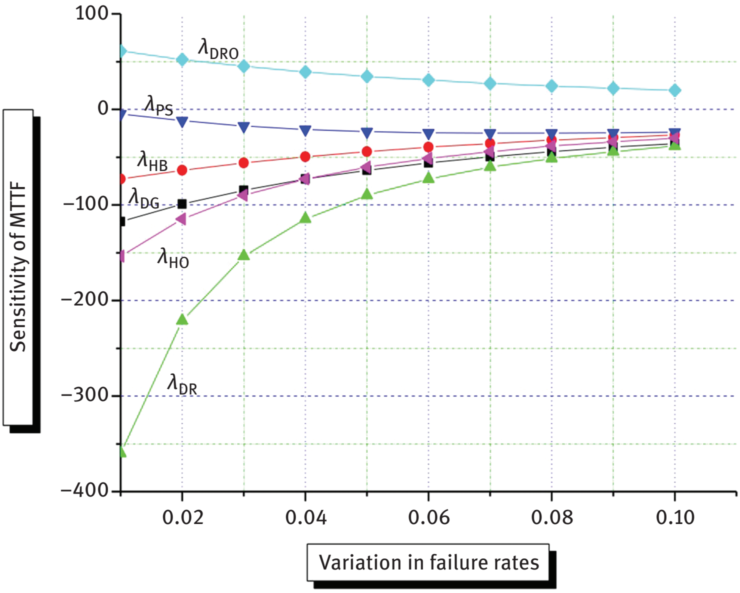

Figure 8.5: Sensitivity of MTTF as a function of failure rates.

8.7.4.2 Sensitivity analysis of reliability

Similarly, by performing the sensitivity analysis of PMP reliability for the same set of failure and repair rates, we obtain Table 8.5 and Figure 8.6.

Table 8.5: Sensitivity of reliability as a function of time unit (*t*).

Time (t)	$\frac{\partial R(t)}{\partial \lambda_{DG}}$	$\frac{\partial R(t)}{\partial \lambda_{HB}}$	$\frac{\partial R(t)}{\partial \lambda_{DR}}$	$\frac{\partial R(t)}{\partial \lambda_{PS}}$	$\frac{\partial R(t)}{\partial \lambda_{DRO}}$	$\frac{\partial R(t)}{\partial \lambda_{HO}}$
0	0	0	0	0	0	0
1	−0.887062	−0.813285	−0.896263	−0.0022443	0.02249	−0.896263
2	−1.574129	−1.331669	−1.608006	−0.0149316	0.081060	−1.608006
3	−2.095293	−1.646164	−2.165466	−0.0419248	0.164549	−2.165466
4	−2.479243	−1.820682	−2.594119	−0.0827112	0.264202	−2.594119
5	−2.750187	−1.900054	−2.915508	−0.1345124	0.373147	−2.915508
6	−2.928583	−1.915652	−3.147893	−0.1936298	0.486016	−3.147893
7	−3.031723	−1.889352	−3.306773	−0.2562620	0.598650	−3.306773
8	−3.074215	−1.836331	−3.405301	−0.3189660	0.707863	−3.405301
9	−3.068373	−1.767045	−3.454632	−0.3788860	0.811274	−3.454632
10	−3.024556	−1.688640	−.3.464209	−0.4338270	0.907149	−3.464209

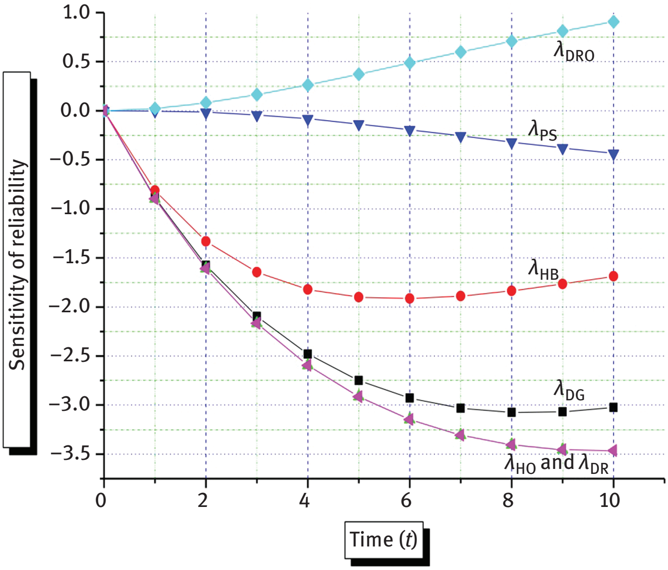

Figure 8.6: Sensitivity of reliability as a function of time unit (*t*).

8.8 Results and discussion

In this chapter, a PMP is analyzed for evaluating its various reliability characteristics using mathematical modeling and supplementary variable technique. By critically analyzing the various graphs of system performance, it is observed that the availability of PMP at ten units of time is given as 0.0000534, while the reliability of the same is given as 0.0263585. The difference between these two indices shows the importance of a good maintenance policy. The behavior of MTTF of PMP with respect to various failure rates is shown in Figure 8.4. By examining it critically, it is concluded that MTTF increases with respect to failure rate of dandy roll of PMP. Also MTTF is highest with respect to dryer failure and lowest with respect to dandy roll failure. Figure 8.5 reflects that the PMP MTTF is highly sensitive with respect to the failure of dandy roll and lowest with respect to failure of dryer. So observation of the graph points out that MTTF of the system is more sensitive with respect to dryer failure. Figure 8.6 shows the sensitivity analysis of PMP with respect to reliability. It reflects that the system's reliability is most sensitive with respect to the failure rate of dryer and human operator.

8.9 Conclusion

This chapter discussed about the application of reliability in performance analysis of a PMP that is subject to mathematical modeling and the Markov process. For PMP, various state probabilities are obtained in terms of different failure rates by using a supplementary variable technique, the Laplace transform, and the Markov process in order to evaluate its several performance measures. Finally, it has been demonstrated that how various reliability measures behave with respect to different failure/time. It has been found that performance of PMP could be improved overall by restricting its failure rates and make the plant less sensitive. With the obtained results, the authors conclude that PMP is most sensitive with respect to the failure rates of dryer and human operator. Also MTTF of PMP is more sensitive with respect to dryer. Finally, one can conclude that in order to make the optimal system designs more reliable for production, the maintenance team must give significant consideration on the failure rate of dryer, digester, and human operator along with other unit failures. It asserts that the results presented in this chapter are very helpful for the concerned management for the design and maintenance of PMP.

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9 Design and evaluation of coherent redundant system reliability

Abstract: In any real-life situation, the reliability of a component/design is determined by the natural phenomenon of cost consideration. But, in practice, there are many other extraneous factors that influence in designing and optimization of a given reliability system, apart from cost considerations like load, quantity, size, space, volume, and weight. This chapter provides a comprehensive study, design, analysis, and optimization of an integrated coherent redundant reliability design, which in the art of literature is not being reported. The system under investigation is designed and evaluated preliminarily by applying Lagrangean multiplier, which gives a genuinely accepted solution for the number of units, unit and phase reliabilities, and thus for the reliability of the design. For practical applicability of the system, an integer solution is derived for which the system is analyzed while optimizing the design reliability by applying integer and dynamic programming technique.

Keywords: Integrated reliability, Lagrangean multiplier, integer programming, dynamic programming

9.1 Introduction

The reliability design could be enhanced by retaining the excess values, or by using the high reliability unit, or simultaneously following one or any of the strategies [1, 2]. Both the methods devour extra assets or resources. Enhancing the design reliability, taken into account, depends on the availability of resources such as expense, load, and quantity. Reliability is generally seen as a component of expense; however, when it is applied on practical circumstances, the shrouded effect of limitations like load and quantity can have a direct influence in optimizing the design reliability. The innovative application of a multilimited redundant reliability model [3] is considered to enhance the expected design.

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The problem explores the unrevealed: that is, the number of units y_j, the reliability of the unit ru_j, and the reliability of phase RP_j in a given phase for a number of limitations to improve the reliability of the design, labeled as an integrated reliability model (IRM) [4]. IRM is enhanced by applying expense limitations, where connection among expense and reliability is formed. The unique feature of the proposed work is of load and quantity as supplementary limitations along with the design cost to enhance the redundant reliability design for coherent design configuration.

9.2 Coherent systems

The coherent designs are very well defined in literature under different practical application situations, and the basic definitions are presented in this chapter. The author has considered the two volatile situations of the coherent designs, that is, parallel–series [5] and the *k*-out-of-*n* configuration [6] designs in optimizing the design reliability with redundancy [7], particularly for the IRMs with multiple restriction case for this chapter.

9.2.1 The diversity of options for reliability improvement leads to the construction of several models

The variety of reliability improvement options leads to the development of several models as follows:

Model 1: y_j units at phase *j* are in parallel redundancy: $\mathrm{RP}_j\,(y_j) = 1 - ((1 - ru_j)^{y_j}$.

Model 2: y_j units at phase *j* form a *k*-out-of-*n* structure: $\mathrm{RP}_j\,(y_j) = \sum_{j=1}^{n} \binom{n}{v}.$ $y_j\,(1 - \mathrm{ru}_j)^{y_j - v}$.

Model 3: y_j units at phase *j* are in standby redundancy with a perfect switching:

$$\mathrm{RP}_j(y_j) = \text{ Probability of } \{y_{j,0} + y_{j,1} + y_{j,2} + \cdots + y_j,\, y_j > t\}.$$

Model 4: Unit y_j is chosen at phase *j* from a set of w_j units of different reliabilities:

$$\mathrm{RP}_j\,(y_j) = ru_j\,.\,y_j.$$

For all models, the design reliability can be written as a function $f(y_1, y_2, \ldots, y_n)$ of decision variables $y_1, y_2, \ldots, y_n$. Therefore, the problem of maximizing design reliability over units' choices and redundancy [8] options is represented as follows:

Maximize $f(y_1, y_2, \ldots, y_n)$

Subject to $\sum g_{ij}\,(y_j) \le b_i$, for $i = 1, 2, 3, \ldots, m$ $u_j \le y_j \le w_j$, for $j = 1, 2, 3, \ldots, n$.

(Since the design is coherent, $f(y_1, y_2, \ldots, y_n)$ is nondecreasing in each variable.)

9.2.2 Assumptions and notations

- Each unit in its respective phase is presumed to be indistinguishable, that is, every unit has similar reliability.
- Each unit is assumed to be indistinguishable in its respective phase, that is, each unit has similar reliability.
- The units are presumed to be statistically independent, that is, the failure of a unit does not influence the exhibition of other units in any design.
- It is assumed that the units are statistically independent, that is, the failure of a unit in any configuration does not influence the exhibition of other units.

R_d represents the system/design reliability; RP_j is the reliability of phase j, $0 < RP_j < 1$; ru_j is reliability of each unit in phase j, $0 < ru_j < 1$; y_j is the number of units in phase j; e_j is the expense coefficient of each unit in phase j; l_j is the load coefficient of each unit in phase j; q_j is the quantity coefficient of each unit in phase j; E_0 is the maximum allowable design expense; L_0 is the maximum allowable design load; Q_0 is the maximum allowable design quantity; and b_j, d_j, f_j, g_j, k_j, h_j are constants.

9.3 Optimization of parallel–series IRM with redundancy under multiple restrictions for the mathematical function

$$ru_j = \left[\frac{e_j}{b_j}\right]^{\frac{1}{d_j}} \text{ – Lagrangean approach}$$

9.3.1 Mathematical model

Design reliability for the given expense function

$$R_d = 1 - \prod_{i=1}^{n}\left(1 - \prod_{j=1}^{m} R_{ij}\right) \tag{9.1}$$

is subject to restrictions

$$\sum_{j=1}^{n} e_j.y_j \le E_0, \tag{9.2}$$

$$\sum_{j=1}^{n} l_j.y_j \le L_0, \tag{9.3}$$

$$\sum_{j=1}^{n} q_j.y_j \le Q_0. \tag{9.4}$$

Nonnegative restriction that y_j is an integer ru_j and RP_j, $\mathrm{RP}_j > 0$.

9.3.2 Mathematical function

To determine the mathematical model [9], the most frequently utilized function is considered for determination of reliability design and analysis [10, 11]. The mathematical function that is being proposed is as follows:

$$\mathrm{ru}_j = \left[\frac{e_j}{b_j}\right]^{\frac{1}{d_j}}, \tag{9.5}$$

where b_j and d_j are constants.

Reliability of the design for the given function

$$R_\mathrm{d} = 1 - \prod_{i=1}^{n}\left(1 - \prod_{j=1}^{m} R_{ij}\right).$$

The number of units at each phase y_j is given through the relation at each phase, and is given through the relation

$$y_j = \frac{\ln(\mathrm{RP}_j)}{\ln(\mathrm{ru}_j)}. \tag{9.6}$$

The problem under consideration is

$$\text{Maximize } Z = \sum_{j=1}^{n}\left[\left(b_j \,.\, \mathrm{ru}_j^{d_j}\right) y_j\right] \tag{9.7}$$

subject to restrictions

$$\sum_{j=1}^{n}\left[\left(b_j \,.\, \mathrm{ru}_j^{d_j}\right) . \, y_j\right] - E_0 \le 0, \tag{9.8}$$

$$\sum_{j=1}^{n}\left[\left(f_j \,.\, \mathrm{ru}_j^{g_j}\right) . \, y_j\right] - L_0 \le 0, \tag{9.9}$$

$$\sum_{j=1}^{n}\left[\left(k_j \,.\, \mathrm{ru}_j^{h_J}\right) . \, y_j\right] - Q_0 \le 0. \tag{9.10}$$

9.3.3 Lagrangean method for equality restrictions

The objective function is to [9]

$$\text{Minimize } z_j = f(y)$$
$$\text{subject to } g(y) = 0,$$

where $x = (y_1, y_2, \ldots, y_n)$ and $g = (g_1, g_2, \ldots, g_n)^T$.

Functions $f(y_i)$ and $g(y_i)$, $i = 1, 2, 3, \ldots, m$, are assumed twice continuously differentiable.

Let $L(Y, \lambda) = f(Y) - \lambda * g(Y)$. The function L is known as the Lagrangean function and the input parameter λ is called the Lagrangean multiplier, which is a constant.

Equations $(\delta L/\delta\lambda) = 0$ and $(\delta L/\delta Y) = 0$ yield the similar vital conditions stated earlier; hence, the Lagrangean function could be utilized directly to produce the vital conditions. This means that optimization [2] of $f(y)$ subject to $g(y) = 0$ is equivalent to optimization of the Lagrangean function $L(Y, \lambda)$, given a stationary point (Y_0, λ_0) for the Lagrangean function method $L(Y, \lambda)$ and the bordered Hessian matrix $\left(H^{\mathrm{B}}\right)$ evaluated at (Y_0, λ_0). Each of the real root $(n - m)$ of the polynomial $|\Delta| = 0$ must be negative if Y_0 is maximum; otherwise, if it is positive then Y_0 is minimum.

9.3.4 Lagrangean method for inequality restrictions

Suppose for the given problem

Maximize $z_j = f(y)$

Subject to $g_i = (y) \geq 0$, where $i = 1, 2, 3, \ldots, m$ and nonnegativity restriction $y \geq 0$.

The overall idea of expanding the Lagrangean strategy is that if the unconstrained optimum ideal of $f(y)$ does not fulfill every restriction, the restriction optimum must happen at a boundary point of the given solution space. This means that at least one or more than one of the m restrictions must be satisfied in a form that is accepted is known as the equation form. This procedural method involves the following steps:

Step 1: Work out the unconstrained problem Max $z_j = f(y)$. If the optimum result meets every restriction requirement, then take no further action, since all the restrictions are redundant. Otherwise, set $z = 1$ and proceed to Step 2.

Step 2: Enact any z restrictions, that is, first transform them into equalities and optimize $f(y)$ subject to "z" active restrictions by the Lagrangean method. If the desired outcome is appropriate with respect to the leftover restrictions, conclude the problem, and regard the outcome as a local optimum. On the contrary, enact a new set of "z" restrictions and repeat the step. If all sets of active restrictions taken as "z" at a time are favored without confronting a convenient working solution, then proceed to Step 3.

Step 3: If $z = m$, conclude the problem, as it demonstrates that there is no existence of working solutions. Else, set $z = z + 1$ and go to Step 2. A significant point frequently disregarded in introducing the technique depicted earlier is that, as expected, it does not ensure global optimum even when the issue is well behaved (possesses a unique optimum). An additional crucial point is the implicit misconception that, for $p < q$, the optimum of $f(y)$ subject to p equality restrictions is always better than its optimum subject to q equality restrictions. Unfortunately, this stands true, in general, only if the q restrictions form a subset of restrictions, p.

The Lagrangean multiplier method [12] considers the quantity of redundancies as real numbers. When the real number solution is achieved, the branch-and-bound technique or dynamic programming approach can be utilized to attain an integer solution. Procedures available for reliability evaluation and reliability optimization could be grouped through exact methods and iterative methods. The exact method acquires the solution analytically and produces more accurate solution. The Lagrangean multiplier method with Khun–Tucker conditions and dynamic programming are models for the exact methods. The iterative method acquires solution through redoing an algorithm or estimating the solutions. The branch-and-bound technique and the heuristic method [13] are models for the iterative method. In this chapter, the Lagrangean multiplier method with dynamic/integer programming for the proposed mathematical function is done by utilizing the MATLAB software [5].

9.3.5 The Lagrangean method

Solving the formulation that is proposed using the Lagrangean function [12] is formulated as follows:

$$\begin{aligned} F = R_{\mathrm{d}} + \lambda_1 \left[\sum_{j=1}^{n} \left\{ \left(b_j.\mathrm{ru}_j^{d_j} \right) . \frac{\ln(\mathrm{RP}_j)}{\ln(\mathrm{ru}_j)} \right\} - E_0 \right] + \lambda_2 \left[\sum_{j=1}^{n} \left\{ \left(f_j.\mathrm{ru}_j^{g_j} \right) . \frac{\ln(\mathrm{RP}_j)}{\ln(\mathrm{ru}_j)} \right\} - L_0 \right] \\ + \lambda_3 \left[\sum_{j=1}^{n} \left\{ \left(k_j.\mathrm{ru}_j^{h_j} \right) . \frac{\ln(\mathrm{RP}_j)}{\ln(\mathrm{ru}_j)} \right\} - Q_0 \right]. \end{aligned} \tag{9.11}$$

Then the stationary point can be obtained by differentiating the Lagrangean function with respect to RP_j, ru_j, λ_1, λ_2, and λ_3:

$$\begin{aligned}\frac{\partial F}{\partial \mathrm{ru}_j} &= \lambda_1\left[\left\{\sum_{j=1}^{n} b_j.\ln(\mathrm{RP}_j)\left(\ln(\mathrm{ru}_j.)d_j.ru_j^{d_j-1}-\frac{\mathrm{ru}_j^{d_j}}{(\mathrm{ru}_j)}\right)\right\}\right]\\ &+\lambda_2\left[\left\{\sum_{j=1}^{n} f_j.\ln(\mathrm{RP}_j)\left(\ln(\mathrm{ru}_j.)g_j.\mathrm{ru}_j^{g_j-1}-\frac{\mathrm{ru}_j^{g_j}}{(\mathrm{ru}_j)}\right)\right\}\right]\\ &+\lambda_3\left[\left\{\sum_{j=1}^{n} k_j.\ln(\mathrm{RP}_j)\left(\ln(\mathrm{ru}_j.)h_j.\mathrm{ru}_j^{h_j-1}-\frac{\mathrm{ru}_j^{h_j}}{(\mathrm{ru}_j)}\right)\right\}\right]=0,\end{aligned} \tag{9.12}$$

$$\frac{\partial F}{\partial \mathrm{RP}_j} = 1+\lambda_1\left[\sum_{j=1}^{n}\frac{b_j.\mathrm{ru}_j^{d_j}}{\ln(\mathrm{ru}_j)}.\frac{(1)}{(\mathrm{RP}_j)}\right]+\lambda_2\left[\sum_{j=1}^{n}\frac{f_j.\mathrm{ru}_j^{g_j}}{\ln(\mathrm{ru}_j)}.\frac{(1)}{(\mathrm{RP}_j)}\right]+\lambda_3\left[\sum_{j=1}^{n}\frac{k_j.\mathrm{ru}_j^{h_j}}{\ln(\mathrm{ru}_j)}.\frac{(1)}{(\mathrm{RP}_j)}\right]=0, \tag{9.13}$$

$$\frac{\partial F}{\partial \lambda_1} = \sum_{j=1}^{n}\left[\left(b_j.\mathrm{ru}_j^{d_j}\right).\frac{\ln(\mathrm{RP}_j)}{\ln(\mathrm{ru}_j)}\right]-E_0=0, \tag{9.14}$$

$$\frac{\partial F}{\partial \lambda_2} = \sum_{j=1}^{n}\left[\left(f_j.\mathrm{ru}_j^{g_j}\right).\frac{\ln(\mathrm{RP}_j)}{\ln(\mathrm{ru}_j)}\right]-L_0=0, \tag{9.15}$$

$$\frac{\partial F}{\partial \lambda_3} = \sum_{j=1}^{n}\left[\left(k_j.\mathrm{ru}_j^{h_j}\right).\frac{\ln(\mathrm{RP}_j)}{\ln(\mathrm{ru}_j)}\right]-Q_0=0. \tag{9.16}$$

9.4 Optimization of *k*-out-of-*n* IRM with redundancy under multiple restrictions for the mathematical function $\mathrm{ru}_j = \left[e_j/b_j\right]^{1/d_j}$ – Lagrangean approach

9.4.1 Mathematical model

Design reliability for the given expense function $R_d = \prod_{j=1}^{n} \mathrm{RP}_j$.

By considering the reliability function [12], the problem under consideration and the transformed equations becomes

Maximum $R_d = \prod_{j=1}^{n}\sum_{k=2}^{y_j}\binom{y_j}{k}\mathrm{ru}_j{}^k(1-\mathrm{ru}_j)^{y_j-k}$

Subject to

$$\sum_{j=1}^{n}\left[\left(b_j.\mathrm{ru}_j{}^{d_j}\right).y_j\right]-E_0 \le 0 \qquad \cdots \text{ expense restriction,}$$

$$\sum_{j=1}^{n}\left[\left(f_j.\,\mathrm{ru}_j^{g_j}\right).y_j\right]-L_0 \ \leq\ 0 \qquad \cdots \text{load restriction,}$$

$$\sum_{j=1}^{n}\left[\left(k_j.\,\mathrm{ru}_j^{h_j}\right).y_j\right]-\ Q_0\ \leq\ 0 \qquad \cdots \text{quantity restriction.}$$

A Lagrangean function is formulated as follows:

$$F=R_\mathrm{d}+\lambda_1\left[\sum_{j=1}^{n}\left\{b_j.\,\mathrm{ru}_j^{d_j}.y_j\right\}-e_0\right]+\lambda_2\left[\sum_{j=1}^{n}\left\{f_j.\,\mathrm{ru}_j^{g_j}.y_j\right\}-l_0\right]+\lambda_3\left[\sum_{j=1}^{n}\left\{k_j.\,\mathrm{ru}_j^{h_j}.y_j\right\}-q_0\right], \tag{9.17}$$

$$\begin{aligned}F=R_\mathrm{d}+\lambda_1&\left[\sum_{j=1}^{n}\left\{b_j.\,\mathrm{ru}_j^{d_j}.\frac{\log \mathrm{RP}_j}{\log \mathrm{ru}_j}\right\}-e_0\right]+\lambda_2\left[\sum_{j=1}^{n}\left\{f_j.\,\mathrm{ru}_j^{g_j}.\frac{\log \mathrm{RP}_j}{\log \mathrm{ru}_j}\right\}-l_0\right]\\&+\lambda_3\left[\sum_{j=1}^{n}\left\{k_j.\,\mathrm{ru}_j^{h_j}.\frac{\log \mathrm{RP}_j}{\log \mathrm{ru}_j}\right\}-q_0\right],\end{aligned} \tag{9.18}$$

$$\begin{aligned}\frac{\partial F}{\partial \mathrm{RP}_j}=1+\ \lambda_1&\left[\sum_{j=1}^{n}\left\{b_j.\mathrm{ru}_j^{d_j}.\frac{1}{\mathrm{RP}_j}.\frac{1}{\log \mathrm{ru}_j}\right\}\right]+\lambda_2\left[\sum_{j=1}^{n}\left\{f_j.\mathrm{ru}_j^{g_j}.\frac{1}{\mathrm{RP}_j}.\frac{1}{\log \mathrm{ru}_j}\right\}\right]\\&+\lambda_3\left[\sum_{j=1}^{n}\left\{k_j.\mathrm{ru}_j^{h_j}.\frac{1}{\mathrm{RP}_j}.\frac{1}{\log \mathrm{ru}_j}\right\}\right],\end{aligned} \tag{9.19}$$

$$\begin{aligned}\frac{\partial F}{\partial \mathrm{ru}_j}&=\lambda_1\left[\sum_{j=1}^{n}b_j.\ln\left(\mathrm{RP}_j\right)\left\{d_j.\,\mathrm{ru}_j^{d_j-1}.\frac{1}{\log \mathrm{ru}_j}+\mathrm{ru}_j^{d_j}\frac{-1}{(log\,\mathrm{ru}_j)^2}.\frac{1}{\mathrm{ru}_j}\right\}\right]+\\&\lambda_2\left[\sum_{j=1}^{n}f_j.\ln\left(\mathrm{RP}_j\right)\left\{g_j.\,\mathrm{ru}_j^{g_j-1}.\frac{1}{\log \mathrm{ru}_j}+\mathrm{ru}_j^{g_j}\frac{-1}{(\log \mathrm{ru}_j)^2}.\frac{1}{(\mathrm{ru}_j)}\right\}\right]+\\&\lambda_3\left[\sum_{j=1}^{n}k_j.\ln\left(\mathrm{RP}_j\right)\left\{h_j.\,\mathrm{ru}_j^{h_j-1}.\frac{1}{\log \mathrm{ru}_j}+\mathrm{ru}_j^{h_j}\frac{-1}{(log\,\mathrm{ru}_j)^2}.\frac{1}{(\mathrm{ru}_j)}\right\}\right],\end{aligned} \tag{9.20}$$

$$\begin{aligned}&\lambda_1\left[\sum_{j=1}^{n}b_j\ \mathrm{ru}_j^{d_j-1}.\frac{\log\ \mathrm{RP}_j}{\log\ \mathrm{ru}_j}\left[d_j-\frac{1}{\log\ \mathrm{ru}_j}\right]\right]+\\&\lambda_2\left[\sum_{j=1}^{n}f_j\ ru_j^{g_j-1}.\frac{\log \mathrm{RP}_j}{\log\ \mathrm{ru}_j}\left[g_j-\frac{1}{\log\ \mathrm{ru}_j}\right]\right]+\\&\lambda_3\left[\sum_{j=1}^{n}k_j\ ru_j^{h_j-1}.\frac{\log \mathrm{RP}_j}{\log\ \mathrm{ru}_j}\left[h_j-\frac{1}{\log\ \mathrm{ru}_j}\right]\right],\end{aligned} \tag{9.21}$$

$$\frac{\partial F}{\partial \lambda_1} = \sum_{j=1}^{n} \left[b_j . \mathrm{ru}_j^{d_j} . \frac{\log \mathrm{RP}_j}{\log \mathrm{ru}_j} \right] - e_0 = 0, \tag{9.22}$$

$$\frac{\partial F}{\partial \lambda_2} = \sum_{j=1}^{n} \left[f_j . \mathrm{ru}_j^{g_j} . \frac{\log \mathrm{RP}_j}{\log \mathrm{ru}_j} \right] - l_0 = 0, \tag{9.23}$$

$$\frac{\partial F}{\partial \lambda_3} = \sum_{j=1}^{n} \left[k_j . \mathrm{ru}_j^{h_j} \frac{\log \mathrm{RP}_j}{\log \mathrm{ru}_j} \right] - q_0 = 0, \tag{9.24}$$

where λ_1, λ_2, λ_3 are Lagrangean multipliers.

The quantity of units in each phase (y_j), the optimum unit [3] reliability (ru_j), the phase reliability (RP_j), and the design reliability (R_d) are obtained by utilizing the Lagrangean method. The method deduces a real (valued) solution with respect to expense, load, and quantity.

9.5 Case problem: parallel–series design and *k*-out-of-*n* design

Deriving the optimum unit reliability(ru_j), phase reliability (RP_j), number of units in each phase (y_j), and the design reliability (R_d) should not exceed the design expense – Rs. 300, load of the design – 400 kg, and quantity of the design – 600 cm^3.

9.5.1 Constants

The necessary data for constants in the case problem is presented in Table 9.1.

Table 9.1: The necessary data for constants in the case problem.

Phase	Parallel–series design						*k*-out-of-*n* design					
	Expense		Load		Quantity		Expense		Load		Quantity	
	b_j	d_j	f_j	g_j	k_j	h_j	b_j	d_j	f_j	g_j	k_j	h_j
01	100	2	100	2	250	2	50	2	70	2	100	2
02	75	3	100	3	120	3	40	3	35	3	80	3
03	75	4	105	4	130	4	35	4	72	4	115	4

9.5.2 Reliability design relating to expense, load, and quantity – without (y_j) rounding off

Table 9.2: Expense restriction details.

Phase	Parallel–series design					k-out-of-n design				
	ru_j	RP_j	y_j	e_j	$e_j.\,y_j$	ru_j	RP_j	y_j	e_j	$e_j\,.\,y_j$
01	0.9404	0.9343	1.10	88.44	98	0.9483	0.9352	1.26	44.96	56.65
02	0.9604	0.9311	1.77	66.44	117.60	0.9648	0.9448	1.52	36.43	54.60
03	0.9741	0.9874	0.48	67.52	32.40	0.9828	0.9569	2.54	32.65	82.93
				Total expense	**248.00**				**Total expense**	**194.18**

Table 9.3: Load restriction details.

Phase	Parallel–series design					k-out-of-n design				
	ru_j	RP_j	y_j	l_j	$l_j.\,y_j$	ru_j	RP_j	y_j	l_j	$l_j.\,y_j$
01	0.9404	0.9343	1.1	88.44	97.28	0.9483	0.9352	1.26	62.94	79.31
02	0.9604	0.9311	1.77	88.58	155.02	0.9648	0.9448	1.52	31.43	47.77
03	0.9741	0.9874	0.48	99.04	47	0.9828	0.9569	2.54	67.17	170.61
				Total load	**300.00**				**Total load**	**297.69**

Table 9.4: Quantity restriction details.

Phase	Parallel–series design					k-out-of-n design				
	ru_j	RP_j	y_j	q_j	$q_j.\,y_j$	ru_j	RP_j	y_j	q_j	$q_j.\,y_j$
01	0.9404	0.9343	1.1	88.44	97.28	0.9483	0.9352	1.26	62.94	79.31
02	0.9604	0.9311	1.77	88.58	155.02	0.9648	0.9448	1.52	31.43	47.77
03	0.9741	0.9874	0.48	99.04	47	0.9828	0.9569	2.54	67.17	170.61
				Total quantity	**491.85**				**Total quantity**	**490.03**
				Design reliability R_d	**0.8589**				**Design reliability R_d**	**0.8544**

The reliability design is restored by regarding the values of y_j to be integers (by rounding the value of y_j to the nearest possible integer) and the relevant outcomes relating to expense, load, and quantity are presented in Tables 9.5–9.10. Furthermore, these accompany the information by calculating the variation due to expense, load, quantity, and design reliability (before and after rounding off y_j).

9.5.3 Reliability design relating to expense, load, and quantity – with y_j rounding off

Table 9.5: Expense restriction details.

Phase	Parallel–series design					*k*-out-of-*n* design				
	ru_j	RP_j	y_j	e_j	$e_j.\ y_j$	ru_j	RP_j	y_j	e_j	$e_j.\ y_j$
01	0.9404	0.9404	1	88.44	88.44	0.9483	0.9435	1	50	50
02	0.9604	0.9224	2	66.44	132.88	0.9648	0.9613	2	36.43	72.86
03	0.9741	0.9741	1	67.52	67.52	0.9828	0.9685	3	35.43	106.29
				Total expense	**288.84**				**Total expense**	**229.15**
				Variation in total expense	**16.46%**				**Variation in total expense**	**18.00%**

Table 9.6: Load restriction details.

Phase	Parallel–series design					*k*-out-of-*n* design				
	ru_j	RP_j	y_j	l_j	$l_j.\ y_j$	ru_j	RP_j	y_j	l_j	$l_j.\ y_j$
01	0.9404	0.9404	1	88.44	88.44	0.9483	0.9435	1	63.49	63.49
02	0.9604	0.9224	2	88.58	177.16	0.9648	0.9613	2	38.75	77.50
03	0.9741	0.9741	1	99.04	99.04	0.9828	0.9685	3	66.92	200.76
				Total load	**364.64**				**Total load**	**341.75**
				Variation in total load	**20.89%**				**Variation in total load**	**14.8%**

Table 9.7: Quantity restriction details.

Phase	Parallel–series design					*k*-out-of-*n* design				
	ru_j	RP_j	y_j	q_j	$q_j . y_j$	ru_j	RP_j	y_j	q_j	$q_j . y_j$
01	0.9404	0.9404	1	221.08	221.08	0.9483	0.9435	1	89.93	89.93
02	0.9604	0.9224	2	106.30	212.60	0.9648	0.9613	2	89.80	179.60
03	0.9741	0.9741	1	99.04	99.04	0.9828	0.9685	3	105.42	316.26
				Total quantity	**532.72**				**Total quantity**	**585.79**
				Variation in total quantity	**11.24%**				**Variation in total quantity**	**18.33%**
				Design reliability	**0.8649**				**Design reliability**	**0.8784**
				Variation in design reliability	**1.16%**				**Variation in design reliability**	**2.8%**

9.6 Optimization of integrated redundant reliability coherent systems – integer/dynamic programming approach

Table 9.8: Reliability design relating to expense.

Phase	Integer programming					Dynamic programming				
	Parallel–series design					*k*-out-of-*n* design				
	ru_j	RP_j	y_j	e_j	$e_j . y_j$	ru_j	RP_j	y_j	e_j	$e_j . y_j$
01	0.9404	0.9454	1	88.44	88.44	0.9471	0.9701	1	54	54
02	0.9604	0.9341	1	66.84	66.84	0.9608	0.9666	2	44.5	89
03	0.9741	0.9638	1	69.52	69.52	0.9882	0.9776	3	48.6	194
				Total expense	**226.8**				**Total expense**	**286**
				Variation in total expense	**11.2%**				**Variation in total expense**	**4.66%**

Table 9.9: Reliability design relating to load.

Phase	Integer programming					Dynamic programming				
	Parallel–series design					*k*-out-of-*n* design				
	ru_j	RP_j	y_j	l_j	$l_j.\ y_j$	ru_j	RP_j	y_j	l_j	$l_j.\ y_j$
01	0.9404	0.9454	1	88.44	88.44	0.9471	0.9701	1	87.91	88
02	0.9604	0.9341	1	88.58	88.58	0.9608	0.9666	2	62.09	124
03	0.9741	0.9638	1	99.04	99.08	0.9882	0.9776	3	57.22	172
				Total load	**275.60**				**Total load**	**384**
				Variation in total load	**13%**				**Variation in total load**	**4%**

Table 9.10: Reliability design relating to quantity.

Phase	Integer programming					Dynamic programming				
	Parallel–series design					*k*-out-of-*n* design				
	ru_j	RP_j	y_j	q_j	$q_j.\ y_j$	ru_j	RP_j	y_j	q_j	$q_j.\ y_j$
01	0.9404	0.9454	1	221.08	221.08	0.9471	0.9701	1	89.70	90
02	0.9604	0.9341	1	106.30	106.30	0.9608	0.9666	2	88.70	177
03	0.9741	0.9638	1	99.04	99.04	0.9882	0.9776	3	101.09	303
				Total quantity	**426.42**	**Total quantity**				**570**
				Variation in total quantity	**14.4%**	**Variation in total quantity**				**3.83%**
				Design reliability	**0.8676**	**Design reliability**				**0.9167**
				Variation in design reliability	**2.79%**	**Variation in design reliability**				**7.58%**

9.7 Comparison of results in Lagrangean and integer programming approaches for the function $ru_j = \left[e_j / b_j\right]^{1/d_j}$

The entire study provides the information in optimizing the design reliability for the proposed *k*-out-of-*n* configuration-integrated redundant reliability design under multiple restrictions in Table 9.11.

Table 9.11: Comparison of results in Lagrangean and integer programming approaches for the functionru$_j = [e_j/b_j]^{1/d_j}$.

Phase	Lagrangean method									
	Without rounding off					With rounding off				
	y_j	R_d	Expense	Load	Quantity	y_j	R_d	Expense	Load	Quantity
01	1.1	0.8589	248	300	491.85	1	0.8649	288.84	364.64	532.72
02	1.77					2				
03	0.48					1				
Phase	Integer programming									
	y_j	R_d	Expense	Load	Quantity					
01	1	0.8676	226.8	275.6	426.62					
02	1									
03	1									

9.8 Comparison of results in Lagrangean and dynamic programming approaches for the function ru$_j = \left[e_j/b_j\right]^{1/d_j}$

In optimizing the design reliability for the proposed k-out-of-n configuration [6] integrated redundant reliability design [14] under multiple restrictions, and the details are provided in Table 9.12.

Table 9.12: Comparison of results in Lagrangean and dynamic programming approaches for the function ru$_j = [e_j/b_j]^{1/d_j}$.

Phase	Lagrangean method									
	Without rounding off					With rounding off				
	y_j	R_d	Expense	Load	Quantity	y_j	R_d	Expense	Load	Quantity
01	1.25	0.8521	300	400	600	2	0.8995	387	529	761
02	1.17					2				
03	3.79					4				
Phase	Dynamic programming									
	y_j	R_d	Expense	Load	Quantity					
1	1	0.9167	286	384	570					
2	2									
3	3									

9.9 Conclusions

The results of the proposed mathematical model clearly inform that though the Lagrangean method provides a solution in real values, the number of units required is being rounded off to the nearest integer for practical applicability to real-life problems. This crude method of rounding off y_j values will have a significant impact on the design parameters such as expense, load, and quantity, thereby affecting the reliability design. This method also infers that it may satisfy the phenomena that "higher the expense, higher the reliability," but never suggests a scientific solution.

To derive an integer solution applying the best of scientific technique like dynamic programming method, the study and application of dynamic programming method for the proposed mathematical model do suggest a scientific solution and for the defined models that the design reliability is being optimized without any hindrance nor violating the restrictions of the parameters, namely, expense, load, and quantity, thus achieving the desired results. This chapter emphasizes on authorizing a novel multiple restrictions methodology for redundant reliability design optimization.

This chapter delivers a solid advanced research for researchers wishing to study the reliability. Various mathematical tools and approaches are outlined, and these have been used effectively in research on system reliability evaluation and optimal design. Furthermore, a primer on complexity analysis is included. The background prerequisites required for comprehending this chapter includes only calculus, basic probability theory, and some knowledge based on computer programming.

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10 Mathematical modeling of a delayed innovation diffusion model with media coverage in adoption of an innovation

Abstract: This chapter proposes a time-delayed diffusion model incorporating three classes of populations to examine the innovation diffusion of a product in an open region. The basic results for justifying the model have been derived and shown that all the positive solutions of the model system are lying in a defined region. The positivity of all the equations has also been discussed in detail. The basic influence number (BIN) R_A of the model system has been inspected. It is observed that the adopter free point E_0 is locally asymptotically stable if $R_A < 1$ and becomes unstable for $R_A > 1$. The local stability analysis is also executed for the null adopter point E_0 and the nonnegative steady point E_*. In the model, the delay parameter τ has been taken into consideration as a bifurcation parameter, which will be able to find out the threshold value of τ causing periodic oscillations due to Hopf bifurcation. The impact of media coverage has also been observed. The sensitivity analysis of the variables of the model has been tested for the interior steady state. Finally, numerical computations have been performed in support of theoretical outcomes by considering an example.

Keywords: Nonlinear model, local stability, bifurcation, sensitivity analysis

Mathematics Subject Classification (2000) 34K18, 92D25

10.1 Introduction

There were some attempts after 1950s to explain innovation diffusion. A well-known researcher, Rogers M. Everett, was the first to write a book about the diffusion of innovation [1]. Rogers defined innovation as “an idea, practice, or object that is perceived as new by an individual or other unit of adoption” [2], and diffusion as “the process by which an innovation is communicated through channels with time among the people of a social system.” Rogers continues that diffusion is “a special type of communication, in that the messages are concerned with new ideas” and that “Communication is a process in which people create and share ideas with one another in order to arrive at a mutual consent” [3].

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The most common model applied in the category of consumer products is the Bass model [4]:

$$\frac{dA}{dt} = p[m - A] + \frac{q}{m}A[m - A].$$

This model incorporates by definition both types of communication channels: interpersonal and mass media. When the products are a mass media mean itself, the model is mostly based on interpersonal communication channels, which occurred clearly in the cases of television's diffusion. Researchers tried different parameter estimation methods in their approaches but none concentrated on a better balance between the interpersonal and mass media communication channels. After the first attempt to model diffusion of innovation of consumer products with the Bass models, it seems that the scholars realized the need to extend it. The outcome was the Bass extension model, which tried to fulfill more requirements and overcome practical problems. It looks like the problem of the nonapplicability of the simple Bass model in most of the cases, and the open window for improvement might have been generated from the nondirect consideration of the communication channels.

Tuli et al. [5] investigated a stage-structured innovation diffusion model with media effect for the diffusion of a new product, and they [6] explained the dynamics by instant and thinker buyers in the innovation diffusion system with an impact of media coverage and established that the basic influence number (BIN) R_A is not affected by the coefficient of media coverage. Dhar et al. [7] examined the dynamical behavior in the marketing model with thinker's population. They considered marketing of a product with a fixed amount of sales advertising and developed that the marketing managers can decide about the new product marketing strategy with respect to advertising intensity. Dhar et al. [8, 9] have modeled the survival of an innovation in a competitive environment with BINs and justify the influence of word of mouth to the spreading of information of a new product with the market dynamics with two competitive products in a closed region. The mathematicians developed a framework of an innovation dynamics in which multiple innovations are affecting each other during the process. They have explored the interaction and diffusion of three innovations simultaneously. The model was established to see that the existence of alternate product has an effect on the active colony of the product.

The effect of distinct demographic processes in a model of innovation diffusion with a dynamic market was studied in [10]. They have observed a model in a market with exponential growth, which completes some of the existing works in the field by introducing an explicit birth–death mechanism, whereas Shukla et al. [11] investigated that the process of diffusion is affected by distinct demographic processes and investigated the role of external factors to find out the adopters population at equilibrium with faster rate. Kumar et al. [12] examined the impact of evaluation period and observed that it caused periodic orbits in the system, and they [13–15] have explained that the innovation diffusion model exhibits Hopf bifurcation after passing through the threshold value of time delay and laid stress on the role of external influences which are helpful in attaining maturity stage in the system. The sustenance of two competing innovations under the influence of media involving adoption experiences, and analyzed the diffusion of two competitive innovations [16]. Tuli et al. [17, 18] formulated the model in two distinct patches with interactions and delays in adoption for two competitive products, and examined how people behave toward two products in these patches.

From the literature survey [30, 35, 40, 41], it can be easily seen that a remarkable work has been completed in the direction of the product expansion. However, more issues need to be resolved yet. Here, the focus of my work is on the impact of media as well as on the frustration rate of the adopters population in potential markets. An important point of aspect about product diffusion is the effect of media coverage, that is, how the media affect the diffusion of innovation. It is also expected that few of the frustrated adopters have a kind of frustration about the product for a time being, and temporarily they have been shifted to the frustrated class. Shortly, when again the interest about the product arises in their minds, they can readily be moved to the adopters class. Considering these scenarios, a delay differential equations model has been proposed and further analyzed by using the classical methods of innovation diffusion modeling, and the model includes the effect of information received through mutual word-of-mouth interactions between nonadopters and adopters population, and by the external advertisements (media) leaving the nonadopter population virtually prompt to adopt the innovation in the market. Hence, moving of nonadopter population to the adopter class will occur by the effect of word-of-mouth interactions and media coverage after a certain time lag. The effects of other demographic processes of the population such as emigration rate and death rate have also been included in the innovation diffusion.

This chapter is structured as follows: in Section 10.2, a nonlinear innovation diffusion model incorporating time delay as a control strategy, and external and internal influences are presented. In Section 10.3, the well-posedness of the model including the positivity and boundedness of the system is verified. Section 10.4 discusses the existence of steady states and concept of BINs. Section 10.5 deals with the dynamical analysis of the time-delayed model, and helped to find the local asymptotic stability conditions of adopter-free and interior steady states. The system is analyzed for the Hopf bifurcation by using bifurcation theory in Section 10.6. Section 10.7 deals with the sensitivity analysis at the positive steady state. In Section 10.8, the numerical simulations are executed in support of analytical outcomes. Finally, in the concluding section, the basic results of mathematical findings are presented with their significance to real-world scenarios.

10.2 Formulation of the proposed model

The ideas for the development of the model system are stated as follows:
1. Assume a nonlinear dynamical system consisting of three categories of population, namely, $N(t)$, $A(t)$, and $R(t)$ for potential adopters, adopters, and frustrated adopters, respectively.
2. Assume that Λ is the recruitment rate of potential adopters $N(t)$.
3. With the impact of media coverage (advertisements) and active command by word of mouth (mutual interactions), some nonadopters will purchase the new product and be shifted to adopters class instantaneously. This category of nonadopters is known to be instant buyers. Let m be the coefficient of media coverage applied on the potential adopters class $N(t)$. Here, p is taken as the instantaneous rate of transformation of the potential adopters $N(t)$ to the adopter class $A(t)$.
4. With the impact of media, $pe^{-mA}N(t)A(t)$ is the population who will eventually be the members of the adopters category.
5. Suppose that with the impact of word-of-mouth interactions, the nonadopter population takes evaluation time period τ. Let q be the variable internal factors (word-of-mouth interactions) applied on the potential adopters population to move to the adopters class.
6. The mutual interactions between nonadopters $N(t)$ and adopters $A(t)$ occur in a time interval $[t-\tau, t]$. Hence, $qN(t-\tau)A(t-\tau)$ are the individuals joining as adopters in the adopters class $A(t)$ by the effect of word of mouths.
7. Here, δ is taken as death or emigration rate for all categories of populations.
8. Let γ be the frustration rate of the adopters population, and ε be the rate of frustrated adopters joining $N(t)$, who may become the member of the adopter group $A(t)$ later on.

Thus, the governing equations are

$$\begin{cases} \dfrac{dN}{dt} = \Lambda - pe^{-mA(t)}N(t)A(t) - qN(t-\tau)A(t-\tau) - \delta N(t) + \varepsilon R(t), \\ \dfrac{dA}{dt} = pe^{-mA(t)}N(t)A(t) + qN(t-\tau)A(t-\tau) - (\delta+\gamma)A(t), \\ \dfrac{dR}{dt} = \gamma A(t) - (\delta+\varepsilon)R(t). \end{cases} \tag{10.1}$$

Further, eq. (10.1) is to be examined under the initial values

$$N(\xi) = \phi_1(\xi),\ A(\xi) = \phi_2(\xi),\ R(\xi) = \phi_2(\xi),\ \phi_1(0) > 0,\ \phi_2(0) > 0,\ \phi_3(0) > 0. \tag{10.2}$$

The initial values of the system are $N(\xi) = \phi_1(\xi)$, $A(\xi) = \phi_2(\xi)$, $R(\xi) = \phi_3(\xi)$, $\xi \in [-\tau, 0]$, $\phi_i(0) \geq 0$, $\phi_i(\xi) \in C\left([-\tau, 0], \mathbb{R}^3_+\right)$.

10.3 Basic preliminaries of the model

Before investigating the dynamical behavior of model (10.1), it must check the well-posedness of eq. (10.1) along with its boundedness. So, let us state the subsequent lemmas:

Lemma 1: The solutions of eq. (10.1) with values (10.2) are positive.

Proof. Consider $(N(t), A(t), R(t))$ is a solution of model (10.1).

The last equation of model (10.1) may be rewritten as

$$\frac{dR}{dt} \geq -(\delta+\varepsilon)R,$$

This gives $R(t) \geq R(0)\exp\left(-\int_0^t(\delta+\varepsilon)dv\right) > 0,\ t > 0.$

The second equation of the model for $t \in [0, \tau]$ may be rewritten as

$$\frac{dA}{dt} \geq -(\delta+\gamma)A(t),$$

which gives $A(t) \geq A(0)\exp\left(-\int_0^t(\delta+\gamma)du\right) > 0,$ for all $t > 0$.

Also,

$$\frac{dN}{dt} \geq -\left(pe^{-mA}A+\delta\right)N - qN(t-\tau)A(t-\tau)$$

or

$$N(t) \geq \exp\left(-\int_0^t\left(pe^{-mA}A+\delta\right)ds\right)$$
$$\left[N(0)+\int_0^t\{\Lambda - qA(s-\tau)N(s-\tau)\}\exp\left(\int_0^s\left(pe^{-mA}A+\delta\right)d\theta\right)ds\right] > 0,$$
$$\text{for } t \in [0, \tau].$$

Using the result from [19], taking $[\tau, 2\tau], \ldots, [n\tau, (n+1)\tau],\ n \in N$. By induction process, it can be proved that $N(t) > 0,\ A(t) > 0,$ and $R(t) > 0$ for $t \geq 0$.

Lemma 2. All solutions of eq. (10.1) with initial values (10.2) are bounded uniformly in $\Sigma = \left\{(N(t), A(t), R(t)) \in \mathbb{R}^3_+ : 0 \leq N(t)+A(t)+R(t) \leq \frac{\Lambda}{\delta}\right\}$.

Proof. Consider $(N(t), A(t), R(t))$ is a solution of eq. (10.1).

Let $\Omega(t) = N(t)+A(t)+R(t),\ t > 0$.

$$\frac{d\Omega}{dt} = \Lambda - \delta\Omega, \quad \text{for } \delta > 0.$$

such that

$$\frac{d\Omega}{dt} + \delta\Omega \le \Lambda.$$

Implementing the theorem of differential inequalities [20], we obtain

$$0 \le \Omega(N(t), A(t), R(t)) \le \frac{\Lambda}{\delta} + \frac{\Omega(N(0), A(0), R(0))}{e^{\delta t}}.$$

Taking $t \to \infty$,

$$0 \le \Omega(N(t), A(t), R(t)) \le \frac{\Lambda}{\delta}.$$

Thus, all solutions of eq. (10.1) lie in $\Sigma = \{(N(t), A(t), R(t)) \in \mathbb{R}^3_+ : 0 \le N + A + R \le (\Lambda/\delta)\}$.

10.4 Basic influence number and existence of steady states

BIN is the adoption generated by an adopter. For this, a matrix from the system equations is required to be created. The matrix will state that either the product will diffuse among the nonadopters or not. Consider f_1 and f_2 as vectors generated to give information about the new adopters by word of mouth of the adopters with potential adopters [18, 21]:

$$F_1 = \begin{pmatrix} pNA + qe^{-mA}NA \\ 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} (\delta + \gamma)A \\ (\delta + \varepsilon)R(t) - \gamma A \end{pmatrix}.$$

The variational matrices of F_1 and F_2 at null adopter steady-state $E_0(\Lambda/\delta, 0, 0)$ are

$$\hat{F}_1 = J(F_1) = \begin{pmatrix} \frac{\Lambda}{\delta}(p+q) & 0 \\ 0 & 0 \end{pmatrix}.$$

Also

$$\hat{F}_2 = J(F_2) = \begin{pmatrix} \gamma & 0 \\ -\gamma & \delta + \varepsilon \end{pmatrix},$$

where $\hat{F}_1$ is nonnegative, and $\hat{F}_2$, a nonsingular matrix. Thus, $\hat{F}_2\hat{F}_2^{-1}$ is nonnegative, and

$$\hat{F}_2\hat{F}_2^{-1} = \begin{pmatrix} \frac{\Lambda(p+q)}{\delta(\delta+\gamma)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the BIN is R_A, which is the largest eigenvalue and

$$R_A = \rho\left(\hat{F}_2\hat{F}_2^{-1}\right) = \frac{\Lambda(p+q)}{\delta(\delta+\gamma)}. \tag{10.3}$$

The BIN R_A is used for the average number of potential adopters converted to adopters with the impact of word-of-mouth communications occurred between the adopters and the potential adopters over the adoption period.

The proposed model (10.1) have two steady points, and their analysis is detailed as follows:

1. The null adopter point $E_0(\Lambda/\delta, 0, 0)$ exists. The adopter population for the product is zero; only potential adopter survives in this case.
2. The nonnegative steady point $E_*(N_*, A_*, R_*)$, and N_*, A_*, R_* are

$$N_* = \frac{(\delta+\gamma)}{(pe^{-mA_*}+q)}, \quad R_* = \frac{\gamma A_*}{\delta+\varepsilon}, \tag{10.4}$$

   and A_* are the roots of the transcendental equation

$$\Lambda - (\delta+\gamma)A_* + \frac{\varepsilon\gamma A_*}{\delta+\varepsilon} - \frac{\delta(\gamma+\delta)}{(pe^{-mA_*}+q)} = 0.$$

It is not convenient to get the exact interior steady state in parametric form, but considering the relative positions of nullclines, visualize the existence of E_* (Figure 10.1).

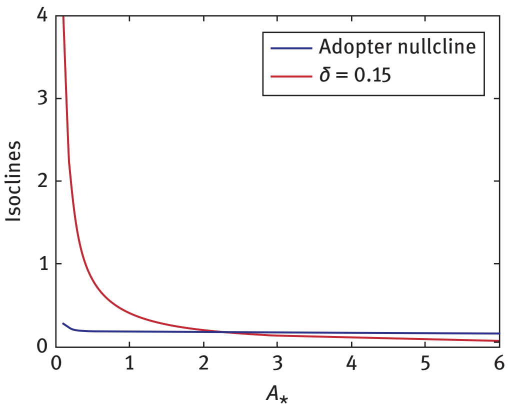

Figure 10.1: Existence of A_* for $\delta = 0.15$.

Hence, one can easily observe from eq. (10.4), the positive existence of $N^* > 0$ $A^* > 0$, $R^* > 0$ for further inspections for positive times.

10.5 Dynamical analysis of the model

10.5.1 Stability of the adopter-free steady-state E_0

The stability conditions for the steady-state E_0 are stated in the following result: The asymptotic stability conditions for E_0 are:

1. If $R_A < 1$, then the steady-state E_0 is asymptotically stable for all $0 \le \tau < \infty$.
2. If $R_A > 1$, then the steady-state E_0 becomes unstable, when $0 \le \tau < \infty$.

Proof. The determinant $\left| J^0 - \lambda I \right| = 0$ obtained with the help of variational matrix $[J^*]$ for the adopter-free steady-state $E_0(\Lambda/\delta, 0, 0)$ is

$$\begin{vmatrix} -\delta-\lambda & -\left(pe^{-\lambda\tau}+q\right)\frac{\Lambda}{\delta} & \varepsilon \\ 0 & -\left(pe^{-\lambda\tau}+q\right)\frac{\Lambda}{\delta}-(\delta+\gamma)-\lambda & 0 \\ 0 & \gamma & -(\delta+\varepsilon)-\lambda \end{vmatrix} = 0,$$

or

$$(\lambda+\delta)(\lambda+\delta+\varepsilon)\left[\left(pe^{-\lambda\tau}+q\right)\frac{\Lambda}{\delta}-(\delta+\gamma)-\lambda\right] = 0.$$

The eigenvalues are $\lambda = -\delta, \quad -(\delta+\varepsilon), \quad \left(pe^{-\lambda\tau}+q\right)(\Lambda/\delta)-(\delta+\gamma)$. For local asymptotic stability of E_0, all roots of λ must be negative. The earlier analysis provides us two negative roots of λ. The third root of λ is negative under $\left(pe^{-\lambda\tau}+q\right)(\Lambda/\delta)-(\delta+\gamma) < 0$. On simplification,

$$\left(pe^{-\lambda\tau}+q\right)\frac{\Lambda}{\delta} < (\delta+\gamma),$$

or

$$\left(pe^{-\lambda\tau}+q\right) < \frac{\delta}{\Lambda}(\delta+\gamma),$$

or

$$\left|\left(pe^{-\lambda\tau}+q\right)\right| < \left|\frac{\delta}{\Lambda}(\delta+\gamma)\right|,$$

or

$$(p+q) < \frac{\delta}{\Lambda}(\delta+\gamma), \ as \ |e^{-\lambda\tau}| = 1.$$

This will give $\Lambda(p+q)/\delta(\delta+\gamma)$ or will result in $R_A < 1$. Therefore, the steady state with null adopter is asymptotically stable when $R_A < 1$, and $0 \le \tau < \infty$. Also, the characteristic equation has a positive real solution when $R_A > 1$, and therefore $E_0(\Lambda/\delta, 0, 0)$ is unstable. The implication of $R_A < 1$ is that large outbreaks will not occur, and the diffusion process cannot take off. So, the innovation will diffuse in markets only if $R_A > 1$.

10.5.2 Stability of interior steady-state E_*

Let us discuss the stability of $E_*(N_*, A_*, R_*)$. The variational matrix at $E_*(N_*, A_*, R_*)$ has the form

$$J^* = \begin{bmatrix} -\left(pe^{-mA_*} + qe^{-\lambda\tau}\right)A_* - \delta & -\left(pe^{-mA_*} + qe^{-\lambda\tau}\right)N_* + pmN_*A_*e^{-mA_*} & \varepsilon \\ \left(pe^{-mA_*} + qe^{-\lambda\tau}\right)A_* & \left(pe^{-mA_*} + qe^{-\lambda\tau}\right)N_* - pmN_*A_*e^{-mA_*} - (\delta+\gamma) & 0 \\ 0 & \gamma & -(\delta+\varepsilon) \end{bmatrix},$$

The determinant $\left| J^* - \lambda I \right| = 0$ gives

$$\left(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3\right) + \left(q_1\lambda^2 + q_2\lambda + q_3\right)e^{-\lambda\tau} = 0. \tag{10.5}$$

Here

$$p_1(\tau) = \delta + pA_*e^{-mA_*} - pN_*e^{-mA_*} + pmN_*A_*e^{-mA_*} + (\delta+\gamma) + (\delta+\varepsilon),$$

$$p_2(\tau) = -\delta\left\{pN_*e^{-mA_*} - pmN_*A_*e^{-mA_*} - (\delta+\gamma)\right\} - p\gamma\varepsilon A_*e^{-mA_*},$$

$$p_3(\tau) = -\delta(\delta+\varepsilon)\left\{pN_*e^{-mA_*} - pmN_*A_*e^{-mA_*} - (\delta+\gamma)\right\},$$

$$q_1(\tau) = q(A_* - N_*), \ q_2(\tau) = -q\delta N_*, \ q_3(\tau) = -q\delta(\delta+\varepsilon)N_* - q\gamma\varepsilon A_*.$$

When $\tau = 0$, then eq. (10.5) will be the cubic equation

$$\lambda^3 + T_1\lambda^2 + T_2\lambda + T_3\lambda = 0, \tag{10.6}$$

with the coefficients

$$T_1 = p_1(0) + q_1(0), \ T_2 = p_2(0) + q_2(0), \ T_3 = p_3(0) + q_3(0).$$

If (H):$T_1 > 0$, $T_2 > 0$, $T_3 > 0$, and when $T_1T_2 - T_3 > 0$, then Routh–Hurwitz stability result [22] provides that values of eq. (10.6) are negative.

10.6 Hopf bifurcation

To examine the stability of model (10.1) with time delay τ around the nonnegative steady-state E_* for purely imaginary roots $\lambda = i\omega$ $(\omega > 0)$ of eq. (10.5). Substituting $\lambda = i\omega$ into eq. (10.5) and using the approach [23] for the separation of real and imaginary parts, such that

$$p_3 - p_1\omega^2 = -q_2\omega\sin(\omega\tau) - \left[q_3 - q_1\omega^2\right]\cos(\omega\tau), \tag{10.7}$$

$$p_2\omega - \omega^3 = \left[q_3 - q_1\omega^2\right]\sin(\omega\tau) - q_2\omega\cos(\omega\tau), \tag{10.8}$$

Taking squares of eqs. (10.7) and (10.8), and summing the corresponding sides, one can obtain the following equation:

$$F(\omega,\tau) = \omega^6 + s_1(\tau)\omega^4 + s_2(\tau)\omega^2 + s_3(\tau) = 0, \tag{10.9}$$

where

$$s_1 = p_1^2 - 2p_2 - q_1^2, \;\; s_2 = p_2^2 - 2p_1p_3 + 2q_1q_3 - q_2^2,$$
$$s_3 = p_3^2 - q_3^2.$$

Putting $\omega^2 = z$, then eq. (10.9) becomes

$$H(z,\tau) = z^3 + s_1(\tau)z^2 + s_2(\tau)z + s_3(\tau). \tag{10.10}$$

Differentiating the function H in z, the following is obtained:

$$\frac{dH}{dz} = 3z^2 + 2s_1z + s_2 = 0,$$

which has two roots

$$z_1^* = \frac{-s_1 + \sqrt{\Delta}}{3}, \;\; z_2^* = \frac{-s_1 - \sqrt{\Delta}}{3},$$

where

$$\Delta(\tau) = s_1^2(\tau) - 3s_2(\tau).$$

Let $s_1, s_2,$ and s_3 be as above. Then

Lemma 3. [Song et.al. [24]]

1. If $s_3 < 0$, then eq. (10.9) has minimum one nonnegative root.
2. If $s_3 \geq 0$, $\Delta = s_1^2 - 3s_2 \leq 0$, then eq. (10.9) has no positive roots.
3. If $s_3 \geq 0$, $\Delta = s_1^2 - 3s_2 > 0$, eq. (10.9) has nonnegative roots iff $z_1^* = \left(-s_1 + \sqrt{\Delta}\right)/3 > 0$, $H\left(z_1^*\right) \leq 0$.

For $\tau = 0$, for eq. (10.6), Lemma 3 can be rewritten as follows.

Lemma 4: For $\tau = 0$, one has [25]:

1. If $s_3 \geq 0$ and $\Delta = s_1^2 - 3s_2 \leq 0$, then for any $\tau \in [0, \tau_0)$, the sum of all roots of the eq. (10.5) with a positive real part is equal to the sum of such roots of eq. (10.6).
2. If $s_3 < 0$ or $s_3 \geq 0$, $\Delta = s_1^2 - 3s_2 > 0$, $z_1^* = \left(-s_1 + \sqrt{\Delta}\right)/3 > 0$ $H'(z^*) \leq 0$, then for any $\tau \geq 0$, the sum of all roots of eq. (10.5) with a positive real part is equal to the sum of such roots of eq. (10.6).

By using Lemma 3 and Lemma 4, the subsequent lemma can be stated:

Lemma 5 [26]

1. The equilibrium E_* of eq. (10.1) is absolutely stable iff E_* of the corresponding system is asymptotically stable, and for any $\tau > 0$, the characteristic equation (10.5) has no pure imaginary roots.
2. The equilibrium E_* of eq. (10.1) is conditionally stable iff all roots of eq. (10.5) have negative real parts for $\tau = 0$, and there exists a positive τ such that eq. (10.5) has a pair of pure complex values $\pm i\omega$.

Now to find out the impact of time delay τ on the stability of the interior steady state, suppose that τ is the bifurcation parameter. Assume that $\tau > 0$, and $i\omega$ is a purely complex root of eq. (10.5). Substituting $\lambda = i\omega$ in eq. (10.5), find a minimum value of τ_k so that eq. (10.5) has a complex conjugate pair of roots. Therefore, multiplying eq. (10.7) by $q_3 - q_1\omega^2$, and eq. (10.8) by $q_2\omega$ and adding

$$\left[\left(q_3 - q_1\omega^2\right)^2 - q_2^2\omega^2\right]\cos(\omega\tau) = \left(p_3 - p_1\omega^2\right)\left(q_3 - q_1\omega^2\right) + \left(p_2\omega - \omega^3\right)q_2\omega,$$

$$\cos(\omega\tau) = \frac{(p_3 - p_1\omega^2)(q_3 - q_1\omega^2) + (p_2\omega - \omega^3)q_2\omega}{\left(q_3 - q_1\omega^2\right)^2 - q_2^2\omega^2}.$$

At the threshold value of τ, the stability switch occurs, that is, the least value of τ in eq. (10.5) has purely complex conjugate pair of roots:

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\left[\frac{\left(p_3 - p_1\omega_k^2\right)\left(q_3 - q_1\omega_k^2\right) + \left(p_2\omega_k - \omega_k^3\right)q_2\omega_k}{\left(q_3 - q_1\omega_k^2\right)^2 - q_2^2\omega_k^2}\right] + \frac{2j\pi}{\omega_k},\ k = 0, 1, 2,\ j \in [0, \infty), \tag{10.11}$$

where $\omega_k = \pm i\sqrt{z_k}$, $z_k > 0$. The minimum value of τ_0 at which the purely imaginary eigenvalues of the form $\lambda_0 = \pm i\omega_0$ occur is as follows:

$$\tau_0 = \min_{0 \leq k \leq 2, j \geq 0} \tau_k^{(j)},\ \tau_k^{(j)} > 0. \tag{10.12}$$

If $s_3 < 0$, there must be a positive root $\omega = \omega_0$ of eq. (10.9). The transversality condition must be required for Hopf bifurcation, that is, it is required to prove that $(d/d\lambda)\mathrm{Re}(\lambda) > 0$ at $\tau = \tau_0^*$. Consider that $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ is a value of eq. (10.5), near $\tau = \tau_0$ and $\mu(\tau_0) = 0,\ \omega(\tau_0) = \omega_0$. Substituting λ into eq. (10.5), and differentiating with respect to parameter τ, we obtain:

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{\tau}{\lambda} + \frac{3\lambda^2 + 2p_1\lambda + p_2}{\left(q_1\lambda^2 + q_2\lambda + q_3\right)\lambda e^{-\lambda\tau}} + \frac{2q_1\lambda + q_2}{\left(q_1\lambda^2 + q_2\lambda + q_3\right)\lambda}$$

$$= -\frac{\tau}{\lambda} + \frac{\left(3\lambda^2 + 2p_1\lambda + p_2\right)e^{\lambda\tau} + \left(2q_1\lambda + q_2\right)}{\left(q_1\lambda^2 + q_2\lambda + q_3\right)\lambda}.$$

Simplification will provide

$$\left[\frac{d}{d\tau}\mathrm{Re}(\lambda)\right]^{-1}_{\lambda = i\omega_0} = \frac{3\omega_0^4 + 2\left(p_1^2 - 2p_2 - q_1^2\right)\omega_0^2 + \left(p_2^2 - 2p_1p_3 + 2q_1q_3 - q_2^2\right)}{q_2^2\omega_0^4 + \left(q_3 - q_1\omega_0^2\right)^2\omega_0^2}$$

or

$$\left[\frac{d}{d\tau}\mathrm{Re}(\lambda)\right]^{-1}_{\lambda = i\omega_0} = \frac{\left[H'(s)\right]_{\omega_0^2}}{q_2^2\omega_0^4 + \left(q_3 - q_1\omega_0^2\right)^2\omega_0^2} > 0 \quad \text{if } H'(s) > 0.$$

Therefore, the values of (10.5) crosses the vertical line as τ continuously increases through τ_0. Thus, the transversality condition is verified, and conditions for Hopf bifurcation [27] are justified at $\tau = \tau_0$, and the minimum value of $\tau_k^{(j)}$ is obtained by eq. (10.12).

10.7 Sensitivity analysis

Without any evaluation period, the normalized sensitivity analysis of eq. (10.1) is given in Table 10.1 at $E_*(N_*, A_*, R_*)$.

The normalized forward sensitivity index of a variable u that depends on a parameter p is defined as [28]

$$\Upsilon_p^u = \frac{\partial u}{\partial p} \cdot \frac{p}{u}.$$

It is noticeable that parameters Λ, p, and q, will have a positive effect on A^*, that is, the adopter population increases with a unit change in Λ, p, and q. But the rest of parameters have a negative effect.

Also, the coefficient of media coverage (m) is the most sensitive parameter for the adopter population, and has a big role for constructing A_*.

Table 10.1: Sensitivity indices $\Upsilon_{y_j}^{x_i} = (\partial x_i / \partial y_j) \cdot (y_j / x_i)$ of variables in eq. (10.1) with respect to parameters y_j.

Parameter (y_j)	Values	N_*	A_*	R_*
Λ	0.4	0.59991	1.479304	0.11927
p	0.34	2.1736401	1.481769	1.22816
q	0.3	−0.83599	0.618231	1.3826
m	0.32	0.326422	−2.0716	0.20734
γ	0.01	−1.629645	−0.465635	1.4394
δ	0.15	−0.592719	−1.37106	−1.48219
ε	0.1	0.189604	−1.140215	−0.19437

10.8 Numerical simulations

The analytic results so obtained are interpreted by using the parametric values stated in Table 10.2 of eq. (10.1). The numerical computations are executed by using MATLAB2016 to justify the completeness of the analytic findings.

Table 10.2: Description of parameters for simulation and their values.

Parameters	Description	Value
Λ	Recruitment rate	0.4
p	Rate of transformation from nonadopter to adopter with an effect of media	0.34
q	Cumulative density of variable internal influences	0.3
m	Coefficient of media coverage	0.32
γ	Frustration rate of the adopters	0.01
δ	Natural death rate or emigration rate	0.15
ε	Frustrated population joining $N(t)$	0.1

Taking distinct initial level of populations, it has been noticed that the population is converging to a globally stable steady-state $E_*(0.3468, 2.23, 0.09587)$ in the absence of any time delay parameter τ (see Figure 10.2). Also, the Routh–Hurwitz conditions, (H): $p_1 + q_1 = 0.86631 > 0, p_2 + q_2 = 3.2946 > 0, p_3 + q_3 = 1.37613 > 0, (p_1 + q_1)(p_2 + q_2) - (p_3 + q_3) = 1.47801 > 0$ are also justified. Further, when system (10.1) is integrated in the presence of τ, it was producing local asymptotic stability for $\tau = 8.1 < 8.2 = \tau_0$ (see Figure 10.3). When it crosses over the threshold value of delay parameter τ_0, the system started

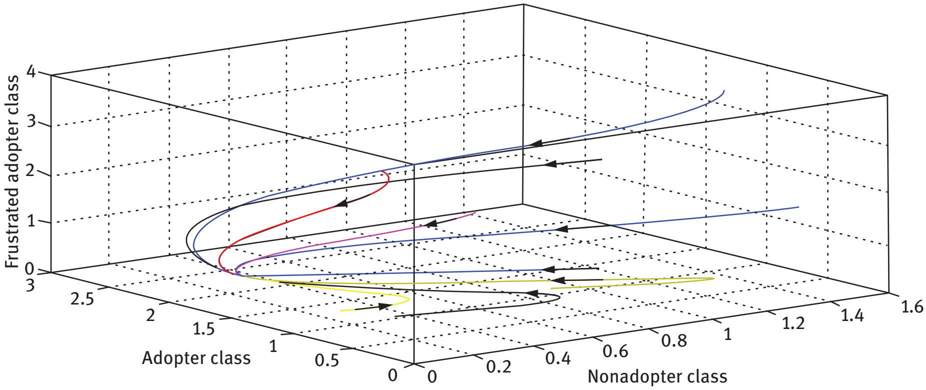

Figure 10.2: The phase portrait in the (N, A, R) phase space. $E_*(0.3468, 2.23, 0.09587)$ is globally asymptotically stable without any τ.

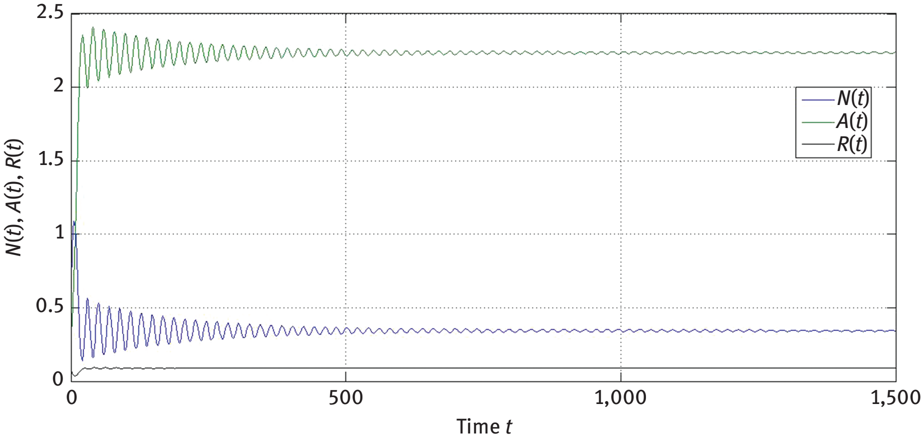

Figure 10.3: Convergence of solution trajectories for $\tau = 8.1 < 8.2 = \tau_0$ around the equilibrium point $E^*(0.3495, 2.207, 0.08859)$.

producing small amplitude periodic orbits around $E^*(0.3496, 2.208, 0.08861)$ (see Figure 10.4). The transversality condition $[(d/d\tau)\mathrm{Re}(\lambda)]^{-1}_{\lambda=i\omega_0} = 1.2853 > 0$ is verified. A more stable periodic orbit has been presented in Figure 10.5 when $\tau = 9.3$. As the media coverage m has been increased in system (10.1), it started producing oscillations, gradually moving toward generating solutions having double period, four period, and finally creating chaotic dynamics around E_* (Figure 10.6). Sensitivity analysis for the state variables at E_* has helped us to provide the most and least sensitive parameters for eq. (10.1).

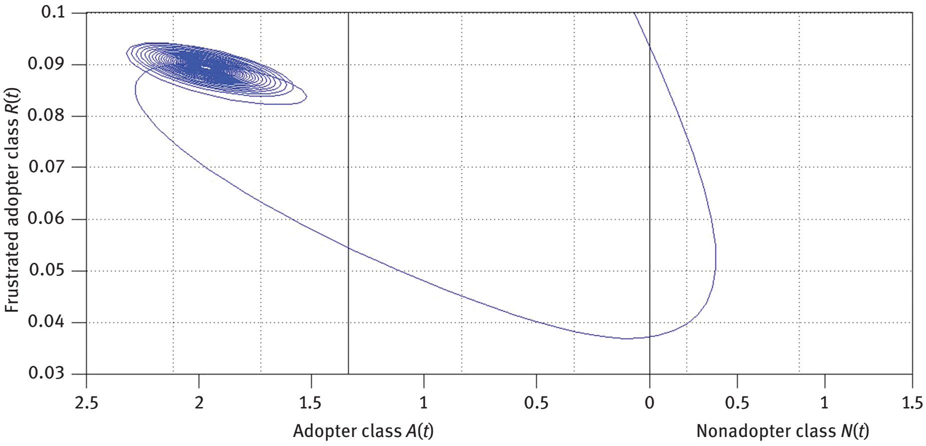

Figure 10.4: Periodic orbits of eq. (10.1) are obtained at a critical value $\tau_0 = 8.2$ around the equilibrium point $E^*(0.3496, 2.208, 0.08861)$.

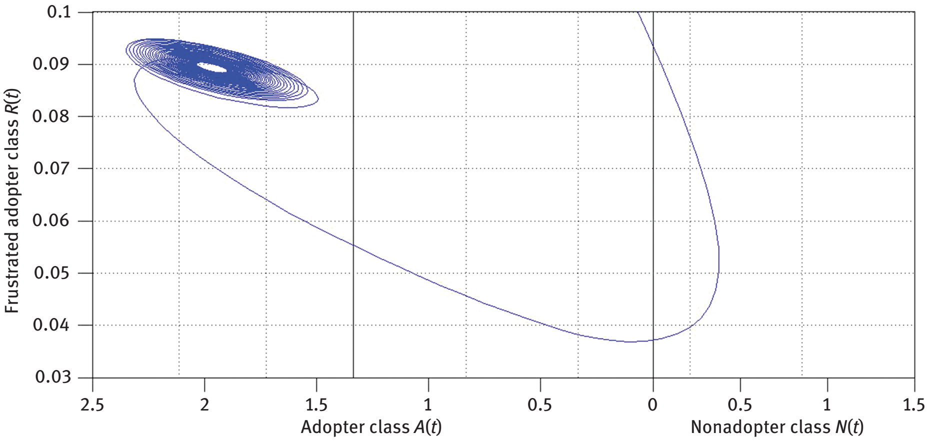

Figure 10.5: Stable limit cycle has been obtained for nonadopter, adopter, and frustrated adopter populations at a value $\tau = 9.3$ around $E^*(0.3465, 2.215, 0.08879)$.

10.9 Conclusion

The chapter is written by considering a mathematical system comprising the three categories of populations $N(t)$, $A(t)$, and $R(t)$, dealing with diffusion of a product. The dynamical analysis of system (10.1) has been performed, along with the basic results involving positivity of the solutions as well as all the solutions lying in the invariant region. The BIN for system (10.1) has also been computed. The stability

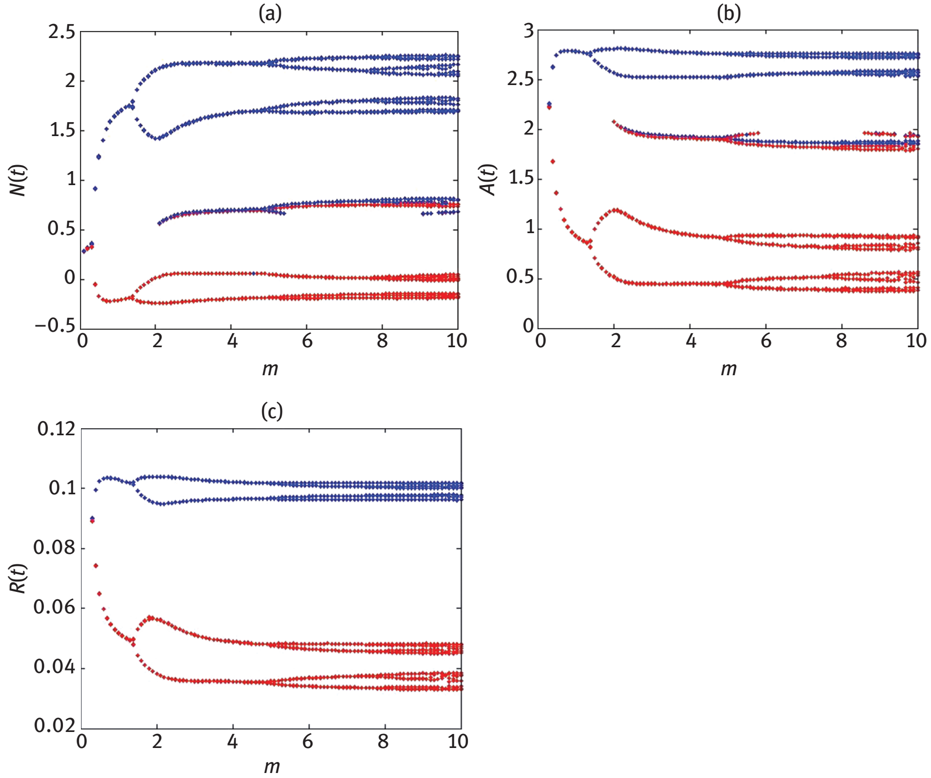

Figure 10.6: The bifurcation diagram of model (10.1) with respect to the parameter of media coverage *m*, where other parameters are kept the same as in Figure 10.2. Figure shows that model (10.1) becomes chaotic from stable dynamics with a small increase in the value of parameter *m*.

conditions of all the existing steady states of model (10.1) have been inspected. It has been noticed that the adopter free point E_0 is LAS (locally asymptotically stable) if $R_A < 1$, which implies that no demand for the innovation is lying in the region. System (10.1) without delay was producing globally stable behavior around the interior steady-state E_* (Figure 10.2). But the involvement of parameter τ in eq. (10.1) accelerated the adoption of the innovation, and driven all globally stable populations into Hopf bifurcation. For a particular set of parametric values mentioned in Table (10.1), it is seen that the interior steady-state E^* was locally stable for $\tau \in (0, 8.1)$ (Figure 10.3). But a local Hopf bifurcation caused in the system and a branch of periodic orbits of small amplitude bifurcated from the interior steady-state E_* when τ crosses over the threshold value $\tau_0 = 8.2$, and has been presented in Figure 10.4. Thus, it has been justified that the delay has forced the globally stable system toward the destabilization of the interior steady state, and the populations have coexisted in the form of limit cycles.

Also, the numerical computations predicted that the populations $N(t)$, $A(t)$, and $R(t)$ started observing small amplitude bifurcating oscillations with respect to the coefficient of media influence m. Also, as the value of media influence has been increased, it started producing two-period solution curves, then four-period solutions curves, and finally moved toward chaotic attractors (irregular solution curves; see Figure 10.6). Sensitivity analysis for the basic influence of the state variables at interior steady state has also been examined.

Key points

For the marketing managers, the analyzed points have been detailed as follows:
1. It will be able to find out a set of parameters under which adoption remains stable.
2. We obtained the conditions under which the system has oscillating character in the open markets.
3. It is helpful for making proper strategies before the launch of the product as the analysis has given controlling parameters to accelerate the adoption process.
4. The impact of media coverage will help managers to decide the extent they require to impose.

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Soni Bisht and S. B. Singh

11 Reliability and signature assessment of shuffle exchange networks using universal generating function

Abstract: This chapter deals with reliability and signature analysis of one of the most common multistage interconnection networks, namely, shuffle exchange network (SEN). This study investigates SEN by increasing the number of switching stages of SEN, with one (SEN + 1) and two (SEN + 2) additional stages. An effective method was implemented here to calculate the signature and mean time to failure of SEN using the Owen method, which incorporates an independent and identically distributed lifetime component. All perspectives of reliability, namely terminal, broadcast, and network reliability, have been analyzed using a universal generating function. Results corresponding to the reliability analysis of the considered interconnection networks are compared with the results obtained by the earlier studied methods.

Keywords: Multistage interconnection network, shuffle exchange networks, terminal reliability, broadcast reliability, network reliability, signature reliability, universal generating function

11.1 Introduction

With the rapid advancement in communication technologies, it is quite possible to form a network, through which hundreds or thousands of processors are connected. Many problems have been identified in the area of weather forecasting, air traffic control, and area of defense, whose solutions require a tremendous amount of computational power. Hence, the system performance in the future can considerably increase the concepts related to the parallel processing system. When numerous processors are interconnected with the memory modules through an interconnected network, then the network is called the interconnection network. Further, a computer network system plays an important role in the transmission of information/messages to the destinations [1, 2]. Moreover, for the transmission of information/messages to the destinations, networks depend on two types of switching topologies: circuit switching and packet switching. Gunawan [3] revealed that circuit switching depends on physical

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paths that exist between sources and sinks. Packet switching divides the data into small packets and routes the data through the interconnection network without using any physical path.

11.2 Preliminaries and review of the literature

11.2.1 Interconnection network

A processor is a parallel distributed system which spends a considerable amount of time to communicate among them under an efficient interconnection network. A parallel processing system is a system that requires subsystems to interconnect the control units, processors, memory modules, and an interconnection network [4–6]. The interconnection networks are classified into four categories: shared medium, direct, indirect, and hybrid networks. The shared medium networks are further classified into local area network and backplane bus networks. Direct interconnection networks like mesh, torus, and hypercube are the router-based topologies in which all nodes directly communicate through point-to-point links with each other. Indirect interconnection networks like crossbar networks and multistage interconnection networks (MINs) are the switch-based topologies in which communication takes place with nodes through different types of switching elements (SEs). Hybrid networks are the combination of direct and indirect topologies [4, 5, 7].

The performance of a multiprocessor system is basically based on the design of the interconnection network. To design interconnection networks, there are some prototype dimensions that are classified into three parts: the design of interconnection infrastructure, the design of interconnection management, and the design of interconnection performance. The design of the infrastructure depends on the way in which the design of interconnection network topology and switching techniques is carried out for the transmission of messages from the source to the sink nodes. The different types of switching techniques or elements that we encounter in real life are straight, exchange, upper broadcast, and lower broadcast. Moreover, in the interconnection management design, the routing strategies can manage the transmission of messages or information from source to destination. They are also dependent on the flow control strategies through which the information flows from the source to the sink to avoid the obstruction and poor performance of the network [2, 3]. In addition, the performance evaluation of the network relies on interconnection networks' reliability modeling and traffic configuration. The performance of interconnection networks can be measured by finding the reliability for a given period of time under certain specified conditions [3, 6].

11.2.2 Multistage interconnection networks (MINs)

MIN plays a key role in performance of the multiprocessor and the parallel computer systems. MINs are used in multiprocessing systems to provide communication between processors and memory modules in which a number of SEs are connected through the different interconnected wires. On the basis of accessibility of paths in the MIN, it is divided into two parts: blocking and nonblocking networks. In blocking networks, the simultaneous connection of more than one terminal reduces conflicts and increases the fault tolerance. While in nonblocking networks, any input port can be connected to any output port without affecting the existing connections. Hence, in this network, multiple paths are possible between every input and outputs, which leads to extra stages [8, 9].

In the context of switching, MINs can be divided into two classes: unidirectional and bidirectional. In the unidirectional MINs, switches are unidirectional while in the bidirectional MINs, the flow of messages can be transmitted through the switches simultaneously in opposite directions between the neighboring switches. Further, in the unidirectional MINs, X inputs are equal to the X outputs having $d \times d$ switches that require at least $\log_d X$ stages to allow a connection path between input port and output port. Also, addition of additional stages leads to the generation of more paths between an input port and an output port at the rate of hardware cost. Generally, in a single-path MIN, SEs are of the size of 2×2. Many single-path MINs are used in parallel processing systems such as shuffle exchange network (SEN), baseline network, generalized cube network, and extra stage cube network [3, 7]. If there is only one path between the source and the destination, and at the same time any one of the switches across the path is found to be faulty, then the entire network collapses. To improve the fault tolerance, we add the number of stages as SEs in the network. By doing so, the possibilities of alternate paths are raised [4, 10, 11].

11.2.3 Different methods for the reliability assessment of MINs

Using continuous-time Markov chains, Monte Carlo simulation, reliability limits, and reliability block diagram (RBD) methods, the reliability assessment of MINs has previously been carried out. Gunawan [3] and Bistouni and Jahanshahi [4] analyzed the reliability of MINs using RBD and reliability bounds. Trivedi [2] evaluated the reliability of MINs using a continuous-time Markov chain method. Rajkumar and Goyal [7] used the multivariable inversion algorithm to compute the reliability expressions using the path set-based method along with the comparison of results with that of previously obtained methods. Bistouni and Jahanshahi [29] introduced the concept of new fault-tolerant multistage interconnection network which named as improved extra group network (IEGN). Authors also proved that the IEGN attained much better result in terms of fault-tolerance, reliability, path-length and cost.

The basic idea about this method was primarily introduced by Ushakov [31]. Universal generating function (UGF) has been used for the reliability analysis of networks such as multistate node network, multistate flow networks, and acyclic transmission networks. The reliability of consecutively connected acyclic networks was assessed by Levitin [12] with the use of UGF method. In the literature, Levitin [13] developed algorithms for the reliability analysis of different types of systems using UGF method. Yeh [14] obtained all minimal paths and computed reliability of the considered network using UGF method. Yeh [15, 16] formed algorithms based on UGF for the reliability analysis of the acyclic transmission network and multistate node network.

Gertsbakh and Shpungin [1] evaluated the performance of any system component, which depends on the component's failure probabilities i.e. called signature. Samaneigo [17] was the first to develop the concepts of the signature reliability of systems having independent and identically distributed (i.i.d) components. The reliability signature can also be used in the network as a tool to examine the most significant switching components in the network. In the literature, Da et al. [18] evaluated the signature of k-out-of-n systems having different modules. Eryilmaz [19] provided the detailed description about the computations of signature vectors of series and parallel systems consisting of different modules and also derived the formulas for the signature and minimal signature of the series and parallel systems based on their modules. Franko and Tütüncü [20] examined the reliability based on the signature of weighted k-out-of-n:G system. Kumar and Singh [21, 22] evaluated the signature reliability of the coherent system and sliding window coherent systems. In this work, the authors determined the structure–function of the considered systems with the help of UGF and obtained the various reliability characteristics. Navarro and Rychlik [23] estimated the upper and lower bounds along with the expected lifetimes of different systems. The reliability of complex bridge networks was computed by Bisht and Singh [24], and the reliability of complex bridge networks was also compared on the basis of network flows. They also determined the signature reliability and mean time to failure (MTTF) of complex bridge networks. Navarro and Rychlik [23], Marichol and Mathonet [25], and Navarro and Rubio [26] computed the signatures of the systems having coherent and heterogeneous components based on the reliability functions. Navarro et al. [30] represent the reliability functions of coherent system based on signature. Researchers also extend this idea in the heterogeneous component. They also compare the performance of the systems with heterogeneous and homogeneous components. Negi and Singh [27] discussed the nonrepairable complex system that weighted k-out-of-n:G and weighted l-out-of-m:G configurations, respectively, with two subsystems A and B and also assessed reliability, MTTF, and sensitivity using the UGF method.

From the above discussion, it can easily be concluded that various methods such as decomposition method, RBD, continuous-time Markov chain, Monte Carlo simulation, and reliability bounds have analyzed the reliability assessment of MINs. But the reliability analysis of SEN, SEN + 1, and SEN + 2 with the help of RBD based on different flow strategies in the network and using UGF method is yet to be done. Keeping this fact in mind, in this chapter, we propose to study the same in the following sequence. First, we evaluate the reliability analysis of SEN, SEN + 1, and SEN + 2 with the help of UGF. Second, we calculate the signature from the structure functions of the considered MINs using Owen's method. Third, we compare the result obtained here in this study with the earlier results. Lastly, we conclude the paper.

11.3 Reliability of MINs

MINs are a network consisting of multiple layers of switching components arranged in a predefined topology in which various processors and memory modules interact with each other. These networks are used in multiprocessor systems, multicomputer, integrated circuits, and telecommunication switching. Because of wider applications of MINs, the study of their reliability and scope of improvement of reliability becomes an essential component. Reliability is the probability of the system to perform its required functions under some conditions for the specified period of time. Reliability of MINs can be classified into three categories: terminal reliability (TR), broadcast reliability (BR), and network reliability (NR).

11.3.1 Terminal reliability (TR) or two-terminal reliability

It is the probability of having at least one fault-free path from source to sink node in the network, which can either be directed or undirected. It is also known as one-to-one communication reliability [3, 4, 28].

11.3.2 Broadcast reliability (BR) or one-to-all terminal reliability

The probability of network is to flow data from the single-source terminal to multiple nodes of the sink terminal. If the data are not received by any of the sink nodes, this network will be in a failed state. The reliability of the broadcast is also known as source-to-multiple sink reliability or one-to-all reliability of communication [3, 4, 28].

11.3.3 Network reliability (NR) or all-terminal reliability

It is the probability of flow of the data from all sources to all sink nodes in the network. It is also known as all-to-all communication reliability [3, 4, 28].

11.4 Universal generating function

There are different methods to determine NR of the networks. Out of them, UGF is a useful method due to its time-reducing and straightforward nature. The UGF has also been shown to be an efficient tool for determining the reliability of binary/multistate systems.

UGF of an independent discrete random variable k is expressed in the polynomial form as follows:

$$U(z) = \sum_{m=1}^{M} r_m z^{k_m}, \tag{11.1}$$

where the variable k has m possible values and r_m is the probability of k which is equal to k_m.

A series system's UGF is expressed as follows:

$$\begin{aligned} U_a(z) \otimes U_b(z) &= \sum_{m=1}^{M} p_{am} z^{g_{am}} \underset{\text{ser}}{\otimes} \sum_{n=1}^{N} p_{bn} z^{g_{bn}} \\ &= \sum_{m=1}^{M} \sum_{n=1}^{N} p_{am} p_{bn} z^{\text{ser}(g_{am},\, g_{bn})}. \end{aligned} \tag{11.2}$$

A parallel system's UGF is obtained as

$$\begin{aligned} U_a(z) \otimes U_b(z) &= \sum_{m=1}^{M} p_{am} z^{g_{am}} \underset{\text{par}}{\otimes} \sum_{n=1}^{N} p_{bn} z^{g_{bn}} \\ &= \sum_{m=1}^{M} \sum_{n=1}^{N} p_{am} p_{bn} z^{\text{par}(g_{am},\, g_{bn})}, \end{aligned} \tag{11.3}$$

where $\underset{\text{ser}}{\otimes}$ and $\underset{\text{par}}{\otimes}$ are the composition operators when components are linked in series and parallel, respectively. Ser(·) and par(·) yield the complete subsystem performance rates consisting of elements a and b. p_{am} and p_{bn} are the probabilities of all considered components and g_{am} and g_{bn} are the performance rates of the system.

It is possible to express that the structure function for the series and parallel subsystem is given as $\phi_{\text{ser}}(G_1, \ldots, G_n) = \min\{G_1, \ldots, G_n\}$

$$\phi_{\text{par}}(G_1, \ldots, G_n) = \max\{G_1, \ldots, G_n\}$$

where $G_1, \ldots, G_n$ are the components present in the system.

11.5 Structure of mins

11.5.1 Shuffle exchange network

SEN is one of the most common MINs used in a parallel processing communication system. Classically, SEN is a single-path MIN. The SEN of size $X \times X$ is composed of $\log_2 X$ stages and each stage contains $X/2$ SEs. The SEN of size 8×8 consists of 12 SEs and each stage contains 4 SEs as shown in Figure 11.1 [4, 5, 7].

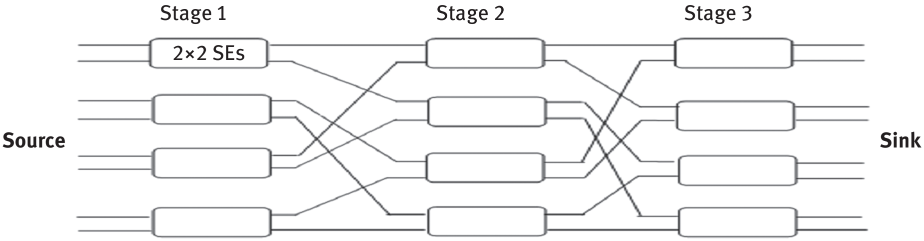

Figure 11.1: 8 × 8 shuffle exchange network (SEN).

11.5.2 Shuffle exchange network with one stage (SEN + 1)

SEN is a single-path MIN which becomes double path with the addition of one additional stage and designated as SEN + 1. The SEN + 1 of size $X \times X$ is composed of $\log_2 X + 1$ stages and each stage contains $X/2$ SEs. The SEN + 1 of size 8×8 consists of 16 SEs and each stage contains 4 SEs as shown in Figure 11.2.

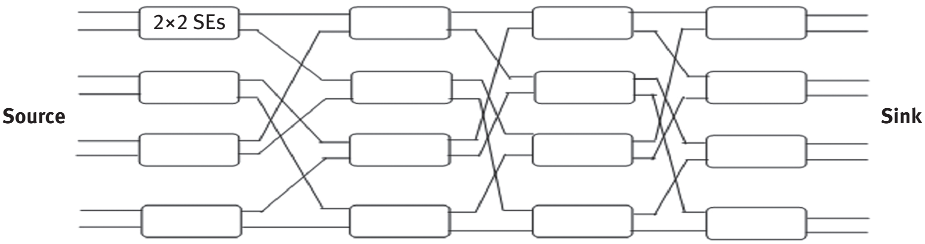

Figure 11.2: 8 × 8 shuffle exchange network with one additional stage (SEN + 1).

11.5.3 Shuffle exchange network with two additional stages (SEN + 2)

SEN becomes multipath with the addition of two additional stages and designated as SEN + 2. The SEN + 2 of size $X \times X$ is composed of $\log_2 X + 2$ stages and each stage contains $X/2$ SEs. Therefore, SEN + 2 of size 8 × 8 consists of 20 SEs and each stage contains 4 SEs as shown in Figure 11.3.

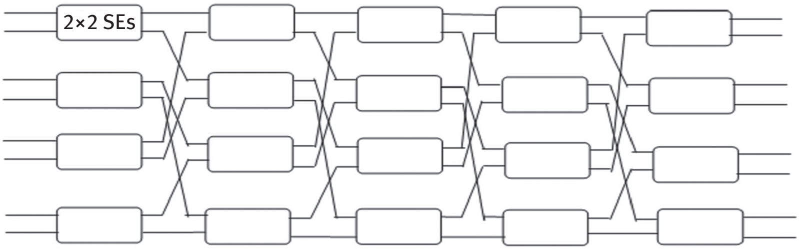

Figure 11.3: 8 × 8 shuffle exchange network with two additional stages (SEN + 2).

11.6 Reliability of SEN, SEN + 1, and SEN + 2

In the proposed work, reliability analysis of various SENs has been carried out on the basis of the following three reliabilities: TR, BR, and NR, unlike previously done, with the aid of UGF.

11.6.1 Terminal reliability of SEN, SEN + 1, and SEN + 2

TR is characterized as the probability of a successful path from the source to elements of sink switching. TR of SEN, SEN + 1, and SEN + 2 is discussed in this segment and compared to the reliability achieved by previously established methods.

11.6.1.1 Terminal reliability of SEN

The TR block diagram of SEN based on the available paths in the network is depicted in Figure 11.4.

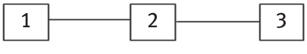

Figure 11.4: RBD for TR of 8 × 8 SEN.

The TR of a nonidentical component of SEN can be obtained by eq. (11.2) as follows:

$$R_{\mathrm{TR}}(\mathrm{SEN}) = \min(p_1, p_2, p_3),$$

where $p_i{}'$s are the probabilities of ith ($i = 1, 2, 3$) SEs present in the SEN.

If all components are equal, that is, all the switching component probabilities are the same, then the R_{TR}(SEN) structure function is expressed as

$$R_{TR}(\text{SEN}) = p^3. \tag{11.4}$$

TR of SEN with respect to different switching reliabilities with the help of the proposed method using eq. (11.4) along with the results obtained in the past has been listed in Table 11.1.

Table 11.1: Terminal reliability of 8 × 8 SEN.

Switching reliability	TR evaluation by UGF	TR [7]	TR [5]	TR [3]
0.9	0.72900	0.72900	0.72900	0.72900
0.95	0.85737	0.85737	0.85737	0.85737
0.96	0.88473	0.88473	0.88473	0.88473
0.98	0.94119	0.94119	0.94119	0.94119
0.99	0.97029	0.97029	0.97029	0.97029

11.6.1.2 Terminal reliability of SEN + 1

SEN + 1 is a double-path MIN. An RBD of SEN + 1 of size 8 × 8 network is depicted in Figure 11.5.

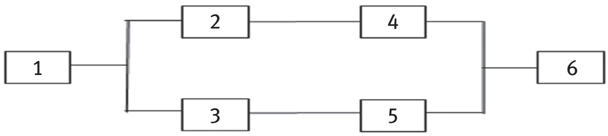

Figure 11.5: RBD for TR of SEN + 1.

The TR expression of SEN + 1 (R_{TR} (SEN + 1)) can be obtained with the help of eqs. (11.2) and (11.3) as follows:

$$R_{TR}(\text{SEN}+1) = \min((\max(\min(p_2, p_3)\min(p_4, p_5)))p_1 p_6),$$

where $p_i{}'$s are the probabilities of ith ($i = 1, 2, \ldots, 6$) SEs present in SEN + 1.

When all the components are identical $(p_i = p)$, then the structure–function of TR will be given by

$$R_{TR}(\text{SEN}+1) = 2p^4 - p^6. \tag{11.5}$$

TR corresponding to different switching reliabilities along with the previously obtained reliabilities by different methods are given in Table 11.2.

Table 11.2: Terminal reliability of 8 × 8 SEN + 1.

Switching reliability	TR evaluation by UGF	TR [7]	TR [5]	TR [3]
0.90	0.780759	0.780759	0.780759	0.780759
0.95	0.893920	0.893920	0.893920	0.893920
0.96	0.915935	0.915935	0.915935	0.915935
0.98	0.95889	0.95889	0.95889	0.95889
0.99	0.979712	0.979712	0.979712	0.979712

11.6.1.3 Terminal reliability of SEN + 2

In SEN + 2, the signal is transmitted through four different paths. An RBD of SEN + 2 of size 8 × 8 is shown in Figure 11.6.

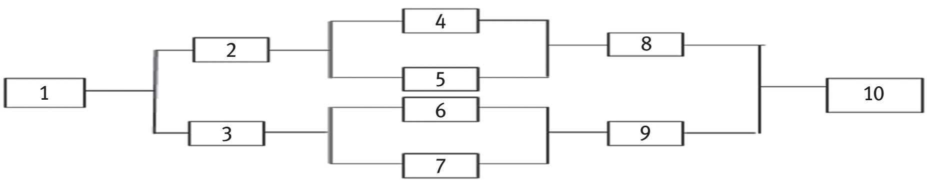

Figure 11.6: RBD for TR of 8 × 8 SEN + 2.

The TR expression of SEN + 2 (R_{TR} (SEN + 2)) is obtained with the help of eqs. (11.2) and (11.3) using UGF as

$$R_{\mathrm{TR}}(\mathrm{SEN}\ +\ 2) = \min((\min(\max(p_4, p_5)p_2p_8)(\min(\max(p_6, p_7)p_3p_9))p_1p_{10}),$$

where p_i′s are the probabilities of ith ($i = 1, 2,. . ., 10$) SEs present in SEN + 2.

If all the components are identical, then R_{TR} (SEN + 2) is given by

$$R_{\mathrm{TR}}(\mathrm{SEN}\ +\ 2) = -p^{10} + 6p^9 - 5p^8 - 3p^6 + 4p^5. \tag{11.6}$$

TR of SEN + 2 corresponding to different switching reliabilities along with the previously obtained reliabilities by different methods is listed in Table 11.3.

11.6.2 Broadcast reliability of SEN, SEN + 1, and SEN + 2

BR is expressed as the probability of transferring the signal from source to all destinations. In BR, all switches present in the last stage of the network play a vital role in reliability evaluation.

Table 11.3: Terminal reliability of 8 × 8 SEN + 2.

Switching reliability	TR evaluation by UGF	TR [7]	TR [5]	TR [4]
0.90	0.591145	0.7888415	0.778213	0.520995
0.95	0.755517	0.8971944	0.897194	0.694677
0.96	0.7966417	0.9182251	0.918225	0.742687
0.98	0.889761	0.9595733	0.958865	0.856787
0.99	0.942558	0.9798963	0.979708	0.924345

11.6.2.1 Broadcast reliability of SEN

The BR block diagram of 8 × 8 SEN is shown in Figure 11.7.

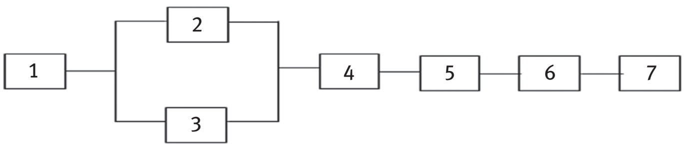

Figure 11.7: RBD for BR of 8 × 8 SEN.

Using eqs. (11.2) and (11.3), one can express BR as follows:

$$R_{\mathrm{BR}}(\mathrm{SEN}) = \min(\max(p_2 p_3) p_1 p_4 p_5 p_6 p_7),$$

where p_i's are the probabilities of ith ($i = 1, 2,\ldots, 7$) SEs in SEN.

If all components are identical, then the structure–function can be expressed as

$$R_{\mathrm{BR}}(\mathrm{SEN}) = 2p^6 - p^7. \tag{11.7}$$

BR of SEN with respect to different switching reliabilities with the help of the proposed method using eq. (11.7) along with the results obtained by the previously developed methods is listed in Table 11.4.

11.6.2.2 Broadcast reliability of SEN + 1

An RBD of SEN + 1 of size 8 × 8 is shown in Figure 11.8.

The BR of SEN + 1 can be computed with the help of eqs. (11.2) and (11.3) as follows:

$$R_{\mathrm{BR}}(\mathrm{SEN}\,+\,1) = (\min(\max(\min(\max(p_4, p_5)\max(p_6, p_7) p_2 p_3)) p_1 p_8 p_9 p_{10} p_{11}),$$

where p_i's are the probabilities of ith ($i = 1, 2,\ldots, 11$) SEs present in SEN + 1.

Table 11.4: Broadcast reliability of 8 × 8 SEN.

Switching reliability	BR evaluation by UGF	BR [7]	BR [5]	BR [3]
0.90	0.58458	0.478297	0.478297	0.478297
0.95	0.77184	0.698337	0.698337	0.698337
0.96	0.81406	0.751447	0.751447	0.751447
0.98	0.90359	0.868126	0.868126	0.868126
0.99	0.95089	0.932065	0.932065	0.932065

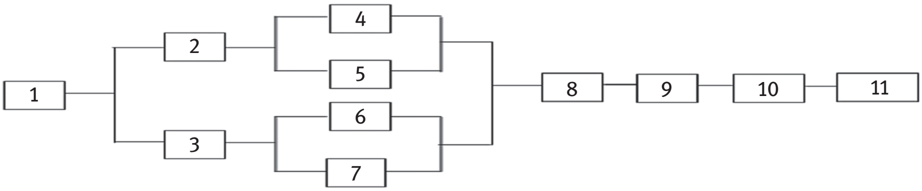

Figure 11.8: RBD for BR of 8 × 8 SEN + 1.

If all the components are identical $(p_i = p)$, then the structure–function of reliability will be given as follows:

$$R_{\mathrm{BR}}(\mathrm{SEN}\ +\ 1) = 2p^7 - p^9. \tag{11.8}$$

BR of SEN + 1 corresponding to different switching reliabilities along with the previously obtained reliabilities is listed in Table 11.5.

Table 11.5: Broadcast reliability of 8 × 8 SEN + 1.

Switching reliability	BR evaluation by UGF	BR [7]	BR [5]	BR [4]
0.90	0.56917	0.5548722	0.547124	0.554872
0.95	0.76642	0.7611920	0.758041	0.761192
0.96	0.81036	0.8067559	0.804540	0.806756
0.98	0.90250	0.9014617	0.900795	0.901462
0.99	0.95061	0.9503338	0.950151	0.950334

11.6.2.3 Broadcast reliability of SEN + 2

An RBD of SEN + 2 of size 8 × 8 is shown in Figure 11.9.

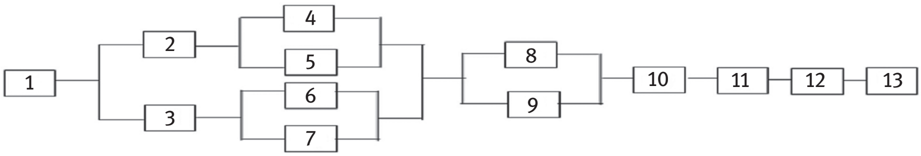

Figure 11.9: RBD for BR of 8 × 8 SEN + 2.

The BR expression of SEN + 2 can be obtained with the help of eqs. (11.2) and (11.3) using UGF as follows:

$$R_{\mathrm{BR}}(\mathrm{SEN}+2) = \min(\max(\max(\min(\max(p_4, p_5)p_2)\min(\max(p_6, p_7)p_3))p_8p_9)p_1 \\ p_{10}p_{11}p_{12}p_{13}),$$

where $p_i{}'$s are the probabilities of ith ($i = 1, 2, \ldots, 13$) SEs present in SEN + 2.

When all components are identical $(p_i = p)$, then the structure–function of reliability is given as follows:

$$R_{\mathrm{BR}}(\mathrm{SEN}+2) = p^{13} - 6p^{12} + 12p^{11} - 6p^{10} - 8p^9 + 8p^8. \tag{11.9}$$

BR corresponding to different switching reliabilities along with the previously computed reliabilities by different methods is given in Table 11.6.

Table 11.6: Broadcast reliability of 8 × 8 SEN + 2.

Switching reliability	BR evaluation by UGF	BR [7]	BR [5]	BR [4]
0.90	0.5776400	0.5669980	0.561638	0.408837
0.95	0.7697292	0.7668366	0.764166	0.610132
0.96	0.8126635	0.8108211	0.808875	0.667692
0.98	0.9042140	0.9027414	0.902115	0.809831
0.99	0.9508393	0.9506918	0.950515	0.897863

11.6.3 Network reliability of SEN, SEN + 1, and SEN + 2

NR can be expressed as the probabilities of successful communication taking place between all switching components present in the first and the last stages of the network.

11.6.3.1 Network reliability of SEN

RBD of SEN for computing NR is shown in Figure 11.10.

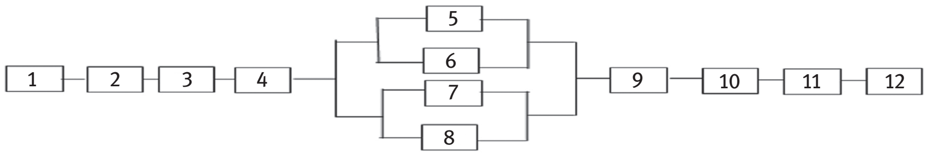

Figure 11.10: RBD for NR of 8 × 8 SEN.

NR of SEN can be computed with the help of eqs. (11.2) and (11.3) as follows:

$$R_{NR}(\text{SEN}) = \min(\max(\max(p_5, p_6), \max(p_7, p_8))p_1, p_2, p_3, p_4, p_9, p_{10}, p_{11}, p_{12}),$$

where p_i's are the probabilities of ith ($i = 1, 2, \ldots, 12$) SEs in SEN.

Here, considering all components to be identical, that is, the probabilities of all SEs are the same, then

$$R_{NR}(\text{SEN}) = p^{12} - 4p^{11} + 4p^{10}. \tag{11.10}$$

The reliabilities of SEN obtained by the present method and earlier developed methods are listed in Table 11.7.

Table 11.7: Network reliability of 8 × 8 SEN.

Switching reliability	NR evaluation by UGF	NR [7]	NR [5]	NR [4]
0.90	0.4219009	0.2824295	0.2824295	0.282430
0.95	0.6601074	0.540360	0.540360	0.540360
0.96	0.7190827	0.612709	0.612709	0.612710
0.98	0.8500825	0.7847147	0.7847147	0.784717
0.99	0.9225601	0.8863849	0.8863849	0.886385

11.6.3.2 Network reliability of SEN + 1

An RBD of SEN + 1 of size 8 × 8 is shown in Figure 11.11.

The NR expression of SEN + 1 (R_{NR} (SEN + 1)) can be obtained with the help of eqs. (11.2)–(11.5) using UGF as follows:

$R_{NR}(\text{SEN} + 1) = \min(\max(p_5, p_6) \max(p_7, p_8)(\max(\min(p_9, p_{10}) \min(p_{11}, p_{12})p_1 p_2 p_3 p_4 p_{13} p_{14} p_{15} p_{16}),$ where p_i's are probabilities of ith ($i = 1, 2, \ldots, 16$) SEs in SEN + 1.

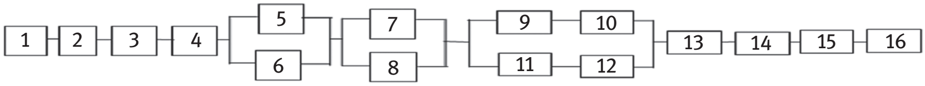

Figure 11.11: RBD for NR of 8 × 8 SEN + 1.

If all the components are identical ($p_i = p$), then the structure–function of reliability can be expressed as follows:

$$R_{\mathrm{NR}}(\mathrm{SEN}+1) = -p^{16} + 4p^{15} - 2p^{14} - 8p^{13} + 8p^{12}. \tag{11.11}$$

NR of SEN + 1 corresponding to different switching reliabilities along with the previously obtained reliabilities by different methods is listed in Table 11.8.

Table 11.8: Network reliability of 8 × 8 SEN + 1.

Switching reliability	NR evaluation by UGF	NR [7]	NR [5]	NR [4]
0.90	0.406669	0.388707	0.40667	0.388708
0.95	0.653831	0.645470	0.65383	0.645470
0.96	0.714663	0.708630	0.71466	0.708630
0.98	0.848744	0.8468415	0.84874	0.846842
0.99	0.922203	0.9216594	0.922194	0.921659

11.6.3.3 Network reliability of SEN + 2

NR of SEN + 2 of size 8 × 8 is shown in Figure 11.12.

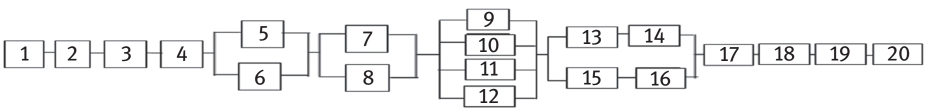

Figure 11.12: RBD for NR of 8 × 8 SEN + 2.

NR of size 8 × 8 can be computed as follows:

$$R_{\mathrm{NR}}(\mathrm{SEN}\ +\ 2) = \min(\max(p_5, p_6)\max(p_7, p_8)\max(p_9, p_{10}, p_{11}, p_{12})\max(\min(p_{13}, p_{14})$$
$$\min(p_{15}, p_{16}))p_1p_2p_3p_4p_{17}p_{18}p_{19}p_{20}),$$

where $p_i{}'$s are the probabilities of ith ($i = 1, 2, \ldots, 20$) SEs present in SEN + 2.

All components are identical $(p_i = p)$, then the structure–function of SEN + 2 can be expressed as follows:

$$R_{\text{NR}}(\text{SEN} + 2) = p^{20} - 8p^{19} + 24p^{18} - 28p^{17} - 12p^{16} + 72p^{15} - 80p^{14} + 32p^{13}. \qquad (11.12)$$

NR of SEN + 2 corresponding to different switching reliabilities along with the previously obtained reliabilities is shown in Table 11.9.

Table 11.9: Network reliability of 8 × 8 SEN + 2.

Switching reliability	NR evaluation by UGF	NR [7]	NR [5]	NR [4]
0.90	0.406628	0.406633	0.406629	0.262133
0.95	0.653830	0.655175	0.653828	0.480112
0.96	0.714668	0.715835	0.714661	0.549891
0.98	0.848751	0.849251	0.848749	0.733541
0.99	0.922294	0.922354	0.922194	0.854252

11.7 Signature reliability

As discussed earlier, Samaniego [17] introduced the signature to study the systems having continuous and i.i.d lifetimes of the components. Let $X_1, X_2, \ldots, X_m$ be the i.i.d. component and T be the lifetime of this network, then the reliability at a time $t \geq 0$ is given by

$$P(T > t) = \sum_{l=1}^{m} s_l P(X_{l:m} > t), \qquad (11.13)$$

where $X_{1:m} \leq X_{2:m} \leq \cdots \leq X_{m:m}$ are the ordered lifetimes of the components and s_l is the signature of the lth component given by $s_l = P(T = X_{l:m})$, $l = 1, 2, \ldots, m$.

The vector $s = (s_1, s_2, \ldots, s_m)$ is called the signature of the network.

11.7.1 Algorithm for evaluating the signature of SEN, SEN + 1, and SEN + 2 networks with the help of its structure–function

Step 1. By the structure–function, the signature is calculated as follows:

$$S_q = \frac{1}{\binom{p}{p-q+1}} \sum_{\substack{h \subseteq [p] \\ |h| = p-q+1}} \varphi(h) - \frac{1}{\binom{p}{p-q}} \sum_{\substack{h \subseteq [p] \\ |h| = p-q}} \varphi(h). \qquad (11.14)$$

A polynomial function of the considered MIN is computed as

$$h(r) = \sum_{i=1}^{a} C_i \binom{a}{i} p^i q^{n-i}, \tag{11.15}$$

where

$$C_i = \sum_{f=a-i+1}^{a} S_f, \quad i = 1, 2, \ldots, a.$$

Step 2. Tail signature of the system $\bar{S} = (\bar{S}_0, \ldots, \bar{S}_a)$ is assessed by using

$$\bar{S}_q = \sum_{f=q+1}^{a} s_f = \frac{1}{\binom{p}{p-q}} \sum_{|h|=p-q} \varphi(h). \tag{11.16}$$

Step 3. Compute the reliability function in the form of a polynomial by using the Taylor expansion with $y = 1$ by

$$p(y) = y^a h\left(\frac{1}{y}\right). \tag{11.17}$$

Step 4. With the use of eq. (11.16), the tail signature of the network can be estimated as

$$\bar{S}_q = \frac{(p-q)!}{p!} D^q p(1), \quad q = 0, 1, \ldots, a. \tag{11.18}$$

Step 5. Using eq. (11.18), the signature of SEN can be calculated as

$$s = \bar{S}_{q-1} - \bar{S}_q, \quad q = 1, \ldots, a. \tag{11.19}$$

11.7.2 Signature of SEN

11.7.2.1 Terminal signature

A terminal signature of SEN can be obtained with the help of two-terminal structure of SEN. Using structure–function, the signature of a two-terminal structure of SEN is computed.

Using eq. (11.15), we get the polynomial function of two SEN terminal structures from RBD as follows:

$$h(r) = r^3. \tag{11.20}$$

Using eqs. (11.18) and (11.19), respectively, to compute the tail signature ($\bar{S}_{\mathrm{TT}}$) and signature s_{TT} of SEN, we have

$$\bar{S}_{\mathrm{TT}} = (1, 0, 0, 0,),$$

$$s_{\mathrm{TT}} = (1, 0, 0).$$

11.7.2.2 Broadcast signature

Broadcast signature of SEN is obtained with the help of source-to-multiple terminal structure of SEN. It is computed with the help of structure–function of source-to-multiple terminal structure from eq. (11.7).

From eq. (11.15), we can get $h(r)$ function from the structure of source-to-multiple terminal as follows:

$$h(r) = 2r^6 - r^7. \tag{11.21}$$

Using eqs. (11.18) and (11.19), respectively, the tail signature ($\bar{S}_{\mathrm{SMT}}$) and signature (s_{SMT}) of the source-to-multiple terminal structure of SEN are evaluated:

$$\bar{S}_{\mathrm{SMT}} = \left(1, \frac{2}{7}, 0, 0, 0, 0, 0, 0\right),$$

$$s_{\mathrm{SMT}} = \left(\frac{5}{7}, \frac{2}{7}, 0, 0, 0, 0, 0\right).$$

11.7.2.3 Network signature

Network signature of SEN is computed with the help of all-terminal structure of SEN. Using eq. (11.10), we can compute function $h(r)$ from RBD of all-terminal structure of SEN by using eq. (11.15) as follows:

$$h(r) = -r^{12} + 4r^{11} - 6r^{10} + 4r^9. \tag{11.22}$$

Tail signature ($\bar{S}_{\mathrm{AT}}$) and signature (s_{AT}) of all-terminal structure of SEN with the help of eqs. (11.18) and (11.19), respectively, are computed as follows:

$$\bar{S}_{\mathrm{AT}} = \left(1, \frac{1}{3}, \frac{1}{11}, \frac{1}{55}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right),$$

$$s_{\mathrm{AT}} = \left(\frac{2}{3}, \frac{8}{33}, \frac{4}{55}, \frac{1}{55}, 0, 0, 0, 0, 0, 0, 0, 0\right).$$

11.7.3 Signature of SEN + 1

11.7.3.1 Terminal signature

Terminal signature of SEN + 1 can be computed with the help of two-terminal structure. To determine the polynomial function $h(r)$ of the two-terminal structure of SEN + 1, we can use eqs. (11.5) and (11.15), which yields

$$h(r) = 2r^4 - r^6. \tag{11.23}$$

Computing the tail signature $(\bar{S}_{\mathrm{TT}})$ and signature (s_{TT}) of SEN + 1 with the help of eqs. (11.18) and (11.19), respectively, we obtain

$$\bar{S}_{\mathrm{TT}} = \left(1, \frac{2}{3}, \frac{2}{15}, 0, 0, 0, 0\right),$$

$$s_{\mathrm{TT}} = \left(\frac{1}{3}, \frac{8}{15}, \frac{2}{15}, 0, 0, 0\right).$$

11.7.3.2 Broadcast signature

Broadcast signature of SEN + 1 is obtained from the source-to-multiple terminal structure. Using eqs. (11.8) and (11.15), one can compute the function $h(r)$ from RBD of source-to-multiple terminal structure of SEN + 2 as follows:

$$h(r) = 2r^7 - r^9. \tag{11.24}$$

The tail signature $(\bar{S}_{\mathrm{SMT}})$ and (s_{SMT}) of source-to-multiple terminal structure of SEN + 1 can be evaluated with the help of eqs. (11.18) and (11.19), respectively, as follows:

$$\bar{S}_{\mathrm{SMT}} = \left(1, \frac{2}{3}, \frac{2}{15}, 0, 0, 0, 0, 0, 0, 0\right),$$

$$s_{\mathrm{SMT}} = \left(\frac{1}{3}, \frac{8}{15}, \frac{2}{15}, 0, 0, 0, 0, 0, 0\right).$$

11.7.3.3 Network signature

Network signature of SEN + 1 can be evaluated from the structure–function of an all-terminal structure that is expressed in eq. (11.11). Using eq. (11.15), we get the polynomial function $h(r)$ from RBD of an all-terminal structure as follows:

$$h(r) = -p^{16} + 4p^{15} - 2p^{14} - 8p^{13} + 8p^{12}. \tag{11.25}$$

Computing the tail signature ($\bar{S}_{\mathrm{AT}}$) and signature (S_{AT}) of SEN + 1 with the help of eqs. (11.18) and (11.19), respectively, as follows:

$$\bar{S}_{\mathrm{AT}} = \left(1, \frac{1}{2}, \frac{11}{60}, \frac{3}{70}, \frac{2}{455}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right),$$

$$s_{\mathrm{AT}} = \left(\frac{1}{2}, \frac{19}{60}, \frac{59}{420}, \frac{49}{1274}, \frac{2}{455}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right).$$

11.7.4 Signature of SEN + 2

11.7.4.1 Terminal signature

It is calculated with the use of a two-terminal structure of SEN + 2. Substituting eq. (11.6) into eq. (11.15), we can compute the function $h(r)$ of a two-terminal structure of SEN + 2 as follows:

$$h(r) = -r^{10} + 6r^9 - 5r^8 + 4r^5 - 3r^6. \tag{11.26}$$

Tail signature ($\bar{S}_{\mathrm{TT}}$) and signature (s_{TT}) of SEN + 2 are computed with the help of eqs. (11.18) and (11.19), respectively, as follows:

$$\bar{S}_{\mathrm{TT}} = \left(1, \frac{2}{5}, \frac{17}{45}, \frac{7}{30}, \frac{17}{210}, \frac{1}{63}, 0, 0, 0, 0, 0\right),$$

$$s_{\mathrm{TT}} = \left(\frac{3}{5}, \frac{1}{45}, \frac{13}{90}, \frac{16}{105}, \frac{287}{4,410}, \frac{1}{63}, 0, 0, 0, 0\right).$$

11.7.4.2 Broadcast signature

Broadcast signature is computed with the help of source-to-multiple terminal structure of SEN + 2. Substituting eq. (11.9) into eq. (11.15), we obtain the polynomial function $h(r)$ of the source-to-multiple terminal structure of SEN + 2 as follows:

$$h(r) = r^{13} - 6r^{12} + 12r^{11} - 6r^{10} - 8r^9 + 8r^8. \tag{11.27}$$

The tail signature $(\bar{S}_{\mathrm{SMT}})$ and signature (s_{SMT}) of source-to-multiple terminal structure of SEN + 2 are computed with the help of eqs. (11.18) and (11.19), respectively, as follows:

$$\bar{S}_{\mathrm{SMT}} = \left(1, \frac{2}{3}, \frac{2}{15}, 0, 0, 0, 0, 0, 0, 0\right),$$

$$s_{\mathrm{SMT}} = \left(\frac{1}{3}, \frac{8}{15}, \frac{2}{15}, 0, 0, 0, 0, 0, 0\right).$$

11.7.4.3 Network signature

Network signature can be calculated with the help of an all-terminal structure of SEN + 2. Substituting eq. (11.12) into eq. (11.15), we obtain the polynomial function $h(r)$ of an all-terminal structure of SEN + 2 as follows:

$$h(r) = r^{20} - 8r^{19} + 24r^{18} - 28r^{17} - 12r^{16} + 72r^{15} - 80r^{14} + 32r^{13}. \quad (11.28)$$

The tail signature$(\bar{S}_{AT})$ and signature (s_{AT}) of an all-terminal structure of SEN + 2 can be evaluated with the help of eqs. (11.18) and (11.19), respectively, as follows:

$$\bar{S}_{\mathrm{AT}} = \left(1, \frac{3}{5}, \frac{6}{19}, \frac{41}{285}, \frac{268}{4,845}, \frac{11}{646}, \frac{6}{1,615}, \frac{4}{14,535}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right),$$

$$s_{\mathrm{AT}} = \left(\frac{2}{5}, \frac{27}{95}, \frac{931}{5,415}, \frac{429}{4,845}, \frac{119833}{3,129,870}, \frac{13889}{1,043,290}, \frac{50}{14,535}, \frac{4}{14,535}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right).$$

11.8 Mean time to failure (MTTF)

MTTF is described as the successive failures before disconnecting some sources from some destination. It is shown that MTTF and reliability are related to each other. The average time to failure of two terminals, source-to-multiple terminals, and all- terminal networks for SEN, SEN + 1, and SEN + 2 are described in this section.

11.8.1 Algorithm for finding the expected lifetime (MTTF) of SEN, SEN + 1, and SEN + 2 networks using minimal signature

Step 1. Determine the expected lifetime of the considered MINs, which have i.i.d components with mean μ.

Step 2. Use the structure–function to measure the minimum signature of the considered MINs:

$$H_T(u) = \sum_{f=1}^{n} C_f H_{1:f}(u), \quad (11.29)$$

where $H_{1:f}(u) = P_r(z_{1:f} > u)$ for $f = 1,2,\ldots, a$.

Step 3. Calculate the estimated lifetime of the network $E(T)$ with identical components by

$$E(T) = \mu \sum_{f=1}^{n} \frac{C_f}{f}, \tag{11.30}$$

where $C = (C_1, C_2, \ldots, C_a)$ is a coefficient of minimal signature.

11.8.2 Expected lifetime of SEN

11.8.2.1 MTTF of a two-terminal network of SEN

Using eq. (11.20), the estimated minimum signature lifetime is obtained as follows:

$$h(u) = u^3. \tag{11.31}$$

The minimum signature using eq. (11.31) of the two-terminal networks is given as follows:

Minimal signature $= (1, 0, 0)$.

It is possible to evaluate the expected lifetime using eq. (11.30) as follows:

$$E(T) = 0.3333.$$

11.8.2.2 MTTF of source-to-multiple sink network of SEN

The expected lifetime of SEN with the use of minimal signature using eq. (11.21) is expressed as

$$h(u) = 2u^6 - u^7. \tag{11.32}$$

The minimum signature with the help of eq. (11.32) of the source-to-multiple sink network is given as

Minimal signature $= (0, 0, 0, 0, 0, 2, 1)$.

Expected lifetime can be estimated by eq. (11.30) as follows:

$$E(T) = 0.190476.$$

11.8.2.3 MTTF of all-terminal network of SEN

Using eq. (11.22), the expected lifetime from minimal signature is obtained as follows:

$$h(u) = -u^{12} + 4u^{11} - 6u^{10} + 4u^9. \tag{11.33}$$

From eq. (11.33), the minimal signature is computed as $(0, 0, 0, 0, 0, 0, 0, 0, 4, -6, 4, -1)$.

Using eq. (11.30), the expected lifetime is evaluated as follows:

$$E(T) = 0.1247474.$$

11.8.3 Expected lifetime of SEN + 1

11.8.3.1 MTTF of a two-terminal network of SEN + 1

The expected lifetime of SEN with the use of minimal signature using eq. (11.23) is expressed as follows:

$$h(u) = 2u^4 - u^6. \tag{11.34}$$

Using eq. (11.34), the minimal signature is obtained as $(0, 0, 0, 2, 0, -1)$.

Expected lifetime can be assessed using eq. (11.30) as follows:

$$E(T) = 0.33333.$$

11.8.3.2 MTTF of source-to-multiple sink network of SEN + 1

From eq. (11.24), the expected lifetime from minimal signature is obtained as

$$h(u) = 2u^7 - u^9. \tag{11.35}$$

Minimal signature with the help of eq. (11.35) is given as $(0, 0, 0, 0, 0, 0, 2, 0, -1)$.

Expected lifetime can be estimated by eq. (11.30) as follows:

$$E(T) = 0.1746031.$$

11.8.3.3 MTTF of an all-terminal network of SEN + 1

From eq. (11.25), the expected lifetime from minimal signature is attained as follows:

$$h(u) = -u^{16} + 4u^{15} - 2u^{14} - 8u^{13} + 8u^{12}. \tag{11.36}$$

From eq. (11.36), the minimal signature is obtained as $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 8, -8, -2, 4, -1)$.

The expected lifetime can be estimated using eq. (11.30) as follows:

$$E(T) = 0.112591.$$

11.8.4 Expected lifetime of SEN + 2

11.8.4.1 MTTF of a two-terminal network of SEN + 2

The expected lifetime of SEN with the use of minimal signature from eq. (11.26) is expressed as

$$h(u) = -u^{10} + 6u^9 - 5u^8 + 4u^5 - 3u^6. \tag{11.37}$$

The minimum signature is defined using eq. (11.37) as follows:

Minimal signature $= (0, 0, 0, 0, 4, -3, 0, -5, 6, -1)$.

Expected lifetime can be calculated by eq. (11.30) as follows:

$$E(T) = 0.2416666.$$

11.8.4.2 MTTF of source-to-multiple sink network of SEN + 2

The estimated lifetime from minimal signature using eq. (11.27) is obtained as follows:

$$h(u) = u^{13} - 6u^{12} + 12u^{11} - 6u^{10} - 8u^9 + 8u^8. \tag{11.38}$$

The minimal signature can be computed with the help of eq. (11.38) as follows:

Minimal signature $= (0, 0, 0, 0, 0, 0, 0, 8, -8, -6, 12, -6)$.

Expected lifetime can be evaluated using eq. (11.30) as

$$E(T) = 0.178943278.$$

11.8.4.3 MTTF of all-terminal network of SEN + 2

The expected lifetime of SEN with the help of minimal signature using eq. (11.23) is expressed as

$$h(r) = r^{20} - 8r^{19} + 24r^{18} - 28r^{17} - 12r^{16} + 72r^{15} - 80r^{14} + 32r^{13}. \tag{11.39}$$

Minimal signature of all-terminal network of SEN + 2 from eq. (11.39) is determined as follows:

Minimal signature = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 32, − 80, 72, −12, −28, 24, −8, 1)

Finally, the expected lifetime of SEN + 2 can be evaluated by eq. (11.30) as follows:

$$E(T) = 0.112474625.$$

11.9 Conclusions

Reliability is one of the major concerns for most of the networks, especially in the field of communication network. This chapter shows how we can evaluate the reliability of SENs with the help of the proposed UGF. This study presents the reliability of SENs with an increasing number of stages in three different perspectives: TR, BR, and NR. Also, unlike done in the past, the signature reliability of multiprocessor networks with the aid of the Owens method has also been studied in this chapter. The expected lifetime of SENs with the aid of minimal signature is also studied for the first time.

A critical examination of the results of different reliabilities obtained from the proposed method and subsequent comparison with the results in [4, 5, 7] reveals the following:

(i) TR of SEN and SEN + 1 is obtained to be similar to that shown in [4, 5, 7], whereas in the case of SEN + 2, it is found to be higher than those obtained in [4] but lower than that in [5, 7].
(ii) BR of SEN, SEN + 1, and SEN + 2 is obtained to be greater than the methods in [4, 5, 7].
(iii) NR of SEN is obtained to be greater than that in [4, 5, 7]. Further, in the case of SEN + 1, the NR is observed to be higher than in [4, 7] and approximately similar to that in [5]. In the case of SEN + 2, it is observed to be greater, slightly greater, and lower than that in [4, 5, 7], respectively.
(iv) Finally, the study reveals that SEN + 1 architecture is more reliable than that of SEN and SEN + 2 architecture.

We also evaluated different failure probabilities corresponding to the different SEs in the considered networks. This has helped in enquiring the impact of different SEs on the reliability of the proposed architectures.

MTTF of a two-terminal structure of SEN and SEN + 1 is found to be highest, whereas it is lowest in case of the all-terminal structure of SEN + 2. We can conclude that the proposed algorithms can be an efficient way to evaluate reliability, signature, and MTTF of SENs.

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