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Jesús Ildefonso Díaz, David Gómez-Castro, Tatiana A. Shaposhnikova

NONLINEAR REACTION-DIFFUSION PROCESSES FOR NANOCOMPOSITES

ANOMALOUS IMPROVED HOMOGENIZATION

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to Eryk and Oliver
to Cheri
to Stanislav and Alexander

Preface

The main goal of this book is to study the critical case of the homogenization of reaction-diffusion equations on periodical domains (or particles) where the nature of the reaction term changes in the homogenized effective equation. This fact has been linked by some authors to the surprising properties of nanocomposites (some of the so-called meta-materials). We give more details of this connection in the Introduction. This explains the title of the book. However, beyond the Introduction we take a fundamentally mathematical approach, and we do not provide too many specifics on the applications since our main goal is to present here the rigorous proof of the associate convergence results and the characterization of the new reaction terms.

This book essentially collects several researches by the authors (sometimes jointly with other colleagues) on the homogenization of nonlinear reaction-diffusion problems (mainly of elliptic or parabolic type) in the so-called "critical scale" in which an "anomalous" (or strange) term arises in the homogenized problem. In some sense, this is a *research-level book* written after the papers are published. So, this book gives the authors the opportunity (and duty) to explain the reasoning behind the arguments, but removes the burden to do all the details (since they can be found in the papers which are mentioned in the list of references). We include also some new results not present in the literature (we provide a list in Section 1.7).

Certainly, this book is not any introduction to homogenization: there is a long list of very good texts written with this purpose, such as we will mention in the Introduction. Some common facts of our exposition are (i) to go beyond the important restriction about the shape of the "particles" G_ε, (ii) to extend the results for a nonlinear diffusion operator (such as the p-Laplacian operator) and (iii) to offer a *common root* to different types of boundary conditions on ∂G_ε instead of presenting different proofs for the cases of conditions known under the names of Dirichlet, Neumann, Robin, Signorini, etc., conditions. This is done in the context of *maximal monotone graphs σ* of $\mathbb{R}^2$.

Parts of this book (which grew from the Doctoral Thesis, in 2017, of the second author at the Complutense University of Madrid, UCM) have been the subject of various courses by the authors: a 10-hour doctoral course (by the third author) at the UCM, in November 2015; a 20-hour mini-course, developed by the second author, at the "Modeling Week" congress at the UCM in June 2017; and a 20-hour Master course at the UCM, by the third author in 2019 and 2020.

We thank many people for their maintained collaboration: first of all our coauthors Carlos Conca, Delfina Gómez, Willi Jäger, Amable Liñán, Miguel Lobo, Maria Neuss-Radu, Maria Eugenia Pérez, Alexander V. Podol'skii, Evariste Sánchez-Palencia, Claudia Timofte and Maria N. Zubova (we never forget the important influence of Haïm Brezis, Jacques-Louis Lions and Olga A. Oleinik on the authors). Special thanks are given to Alexander V. Podol'skii, Claudia Timofte and Maria Zubova for their very

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Notation

Asymptotic comparison

Let a_ε and b_ε be two positive sequences. We denote:

$a_\varepsilon \sim b_\varepsilon$ There exists $\lim_{\varepsilon\to 0} a_\varepsilon/b_\varepsilon \in (0, +\infty)$.
$a_\varepsilon \simeq b_\varepsilon$ There exists $\lim_{\varepsilon\to 0} a_\varepsilon/b_\varepsilon = 1$.
$a_\varepsilon \ll b_\varepsilon$ There exists $\lim_{\varepsilon\to 0} a_\varepsilon/b_\varepsilon = 0$.
$a_\varepsilon \lesssim b_\varepsilon$ Either $a_\varepsilon \ll b_\varepsilon$ or $a_\varepsilon \sim b_\varepsilon$.

In the case $p = n$ we sometimes need to force this notation slightly, but we will make careful note of this (see Remarks 4.1 and 5.1 and Section 6.1).

Problem parameters

n Dimension of the ambient space. Usually $n \geq 3$ unless otherwise specified.
a_ε Scaling parameter of the particles. We always assume that $a_\varepsilon \leq \varepsilon$ and that there exists a limit of $a_\varepsilon/\varepsilon$. In the literature this value is usually $a_\varepsilon = C_0\varepsilon^\alpha$ but we take here a more general approach.
$a_\varepsilon^\star$ Critical value of a_ε. Its value depends on each concrete problem under consideration.
$\beta(\varepsilon)$ Coefficient for the Neumann boundary condition. Some authors use the precise value $\beta(\varepsilon) = \varepsilon^{-\gamma}$.
$\beta^\star(\varepsilon)$ Critical value of $\beta(\varepsilon)$. Coincides with $|S_\varepsilon|^{-1}$ with the notation below.
$\mathcal{A}_0, \mathcal{B}_0$ Parameters describing the strange term in the homogenized problem for balls. Their values depend on the relation between a_ε, $\beta(\varepsilon)$ and their critical values. They also depend on the geometric setting. For the case of particles over the whole domain see Section 4.7.1.

Geometric sets

Ω Domain for the PDE. Smooth open bounded set of $\mathbb{R}^n$.
Ω_ε Exterior part to the set of particles (chemical reactor). It may represent also a perforated domain.
$\partial\Omega$ Boundary of Ω.
$\overline{\Omega}$ Closure of the set Ω. The overline notation will be dropped in integration domains and measures when the n-dimensional Lebesgue measure of $\partial\Omega$ vanishes.
Y Open unit cube $(-\frac{1}{2}, \frac{1}{2})^n$.
A^+ For a generic set $A \subset \mathbb{R}^n$, $A^+ = \{(x_1, \cdots, x_n) \in A : x_n > 0\}$. Similarly for A^-.
A^0 For a generic set $A \subset \mathbb{R}^n$, $A^0 = \{(x_1, \cdots, x_n) \in A : x_n = 0\}$.
G_0 Shape of the model particle. Typically $0 \in G_0$ and $\overline{G}_0 \subset Y$ (for the cases of particles over the whole domain or on an interior manifold). In the case of particles on a part of the boundary $G_0 \subset Y \cap \mathbb{R}^{n-1} \times \{0\}$.

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Υ_ε The set of indexes $j \in \mathbb{Z}^n$ where we place particles. When G_0 is n-dimensional we request that $\varepsilon j + \varepsilon\overline{Y} \subset \Omega$ (case of particles over the whole domain), while when G_0 is $(n-1)$-dimensional (case of particles on a part of the boundary) we request $\varepsilon j + \varepsilon\overline{Y}^+ \subset \Omega$.

S_ε Boundary of the set of n-dimensional particles over which the nonlinear boundary condition is set. In the case of particles over a part of the boundary S_ε denotes the own set of $(n-1)$-dimensional particles.

Functional spaces

$C_c^\infty(\Omega)$ Set of functions $\varphi : \Omega \to \mathbb{R}$ with infinitely many derivatives and compact support inside Ω.

L^p The Lebesgue space of functions such that the p-th power of their absolute value is integrable. When $p = \infty$ the functions are bounded a.e.

$W^{1,p}$ Usual Sobolev space of functions in L^p with gradient in L^p.

$W_0^{1,p}(\Omega)$ Closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ when Ω is bounded.

$W^{1,p}(\Omega,\Gamma)$ Closure in $W^{1,p}(\Omega)$ of the set of functions $C^\infty(\Omega)$ that vanish in a neighborhood of Γ when Ω is bounded and $\Gamma \subset \partial\Omega$.

Operators

$|\cdot|$ Throughout this text we will be quite loose with the notation $|\cdot|$. For numbers, it indicates absolute value; for vectors, norms; for sets the Hausdorff in k-dimensions which is finite ($k = n$ or $n-1$); for finite sets, its cardinality. This will not lead to confusion.

s_+ $= \max\{s, 0\}$ for a real number s.

s_- $= (-s)_+$ for a real number s.

P_ε Extension operator from $W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega)$. It will also operate from $W^{1,p}(\Omega_\varepsilon, \partial\Omega) \to W_0^{1,p}(\Omega)$.

ν Exterior unit vector to $\partial\Omega$. The corresponding set Ω will be clear from the context; when there can be doubts, we will note it by ν_Ω.

Δ_p p-Laplace operator: $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

∂_{ν_p} Given a function φ we define $\partial_{\nu_p}\varphi = |\nabla\varphi|^{p-2}\nabla\varphi \cdot \nu$.

$\mathrm{d}x$ Volume form for n-dimensional integrals in variable x. Same for $\mathrm{d}y$.

$\mathrm{d}S_x$ Surface form for $(n-1)$-dimensional integrals. Same for $\mathrm{d}S_y$.

p-cap Capacity of a set in terms of the operator Δ_p. See Remark 3.11.

λ_{G_0} The 2-cap(G_0).

$\mathcal{H}^s$ s-dimensional Hausdorff measure.

Functions

σ Nonlinear reaction function. It could be a multivalued maximal monotone graph of $\mathbb{R}^2$. Generally non-decreasing, unless otherwise specified.

$\mathcal{H}$ Reaction function appearing in the homogenized equation for the critical case. It is usually obtained from another function denoted by H.

Φ Convex function of which σ is its subdifferential, i. e., $\sigma = \partial\Phi$. Usually $\Phi(0) = 0$ and $\Phi \geq 0$.

$\widehat{\kappa}$ Auxiliary function used in the definition of the capacity. See Remark 3.11.

J_ε The energy function associated with the boundary value problem on Ω_ε. Defined in (2.4).

Contents

1 Introduction and modeling

1.1 Motivation

This book deals with the mathematical homogenization process applied to some reaction-diffusion models. More specifically, we will fix our attention to the delicate point of how a mathematical model in the "microscopic" scale may induce the justification of a quite different "macroscopic model." One first intuitive idea of this curious different modeling arises in the study of cases in which the reaction takes place only on the boundary of many "microscopic particles," as for instance

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & \text{in } \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon) g^\varepsilon(x) & \text{on } S_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

(and, in one of the cases, with a small modification on a part of the boundary condition on $\partial\Omega$; see problem (1.6) below), where the details on the domain G_ε, the internal boundary S_ε and the rest of data f, β, σ and g will be presented later. The diffusion is modeled here by the quasilinear operator $\Delta_p u_\varepsilon \equiv \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$ with $p > 1$. Note that $p = 2$ corresponds to the usual linear Laplacian diffusion operator. This kind of problems mainly arises in the study of chemical reactive flows through the exterior of a domain containing periodically distributed reactive solid grains (or reactive particles).

Particulate filters arise in many applications (as, for instance, in the exhaust purification systems of Diesel and gasoline vehicles). The same model also applies when the chemical fluid reacts on walls of a porous medium (which we assume periodically distributed) so that the flows take place on the holes of the solid porous medium. It is the so-called *adsorption phenomenon*: the adhesion of atoms, ions or molecules from a gas, liquid or dissolved solid to a surface. This process differs from *absorption*, in which a fluid is dissolved by or permeates a liquid or solid, respectively. For some presentations of the chemical aspects involved in the model (and also for some mathematical and historical backgrounds) we refer to a series of works which we collect here in alphabetical order: [12, 14, 23, 104, 102, 150, 165, 166, 167, 179, 203, 205, 211, 171] and [263], among others. We point out that the case $p \neq 2$ corresponds to a simplification of the modeling when the flow is turbulent and also when the fluid is non-Newtonian (see, e. g., [102]). Moreover, as is well known, this nonlinear diffusion operator appears also in many other contexts and is one of the best examples of quasilinear operators leading to a formulation in terms of nonlinear monotone operators (see, e. g., [192, 47, 92]). Here, the "normal derivative" must be understood as $\partial_{\nu_p} u_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nu$, where ν is an outward unit normal vector on the boundary of the particles $S_\varepsilon \subset \partial\Omega_\varepsilon$.

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The function σ in (1.1) is assumed to be given: mainly σ is a monotone continuous function such that $\sigma(0) = 0$ as it corresponds to the so-called Freundlich kinetics

$$\sigma(v) = [v_+]^r, \quad \text{where } v_+ = \max\{v, 0\},\ 0 < r \leq 1$$

(in this framework v represents a concentration and thus $v \geq 0$). The exponent r is called *the order of the reaction*. In some applications the limit case ($r = 0$) is of great relevance and its mathematical treatment is carried out in terms of the maximal monotone graph of $\mathbb{R}^2$ (see [48]) given by $\sigma(s) = 0$ if $s < 0$ and $\sigma(s) = 1$ if $s > 0$, $\sigma(0) = [0, 1]$.

To exemplify, let us assume that the particles are spread over the whole domain. A first "surprise" arises when it is shown (first *formally* by the "two-scale asymptotic method" and then rigorously in suitable functional spaces and using, as a fundamental tool, the notion of weak convergence) that we can take a limit in some rigorous sense such that

$$u_\varepsilon \text{ “} \longrightarrow \text{” } u$$

as $\varepsilon \to 0$ and this "limit" $u(x)$ satisfies a global reaction-diffusion in which the reaction takes place on the whole domain Ω.

In the first cases studied in the literature, the particles were "not too small" with respect to their repetition. In that setting, the diffusion operator could be modified but the same kind of chemical kinetics σ modeled the "macroscopic reaction term"

$$\begin{cases} -\operatorname{div}(A^{\text{eff}} \nabla u) + \beta_1^{\text{eff}} \sigma(u) = f + \beta_2^{\text{eff}} g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Besides the occurrence of the global reaction term (from *adsorption* to *absorption* nonlinear terms), the different *macroscopic (or effective) diffusion operator*, A^{eff}, allows to justify some *non-isotropic* propagation phenomena. When the scale of the particles is "too small" with respect to the repetition, then they are too small to be meaningful in the limit and we have

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

There exists a *critical scale* of the size of the particles with respect to their distance that separates the two behaviors above. If the particles have this precise scaling that separates the behaviors above, we have a "new surprise." The constitutive law of the homogenized virtual reaction term does not coincide with the one of the adsorption constitutive law σ and it is possible to show that the global equation satisfied by "the limit" $u(x)$ now involves the presence of an *anomalous* or *strange term* $\mathcal{H}(u)$

$$\begin{cases} -\Delta_p u + \mathcal{H}(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This effect of the relation between size and distance giving rise to a different reaction behavior is typical of many processes in *nanotechnology*.

New materials, in particular the so-called “mechanical meta-materials,” are built as artificial structures which have mechanical properties defined by their geometric structure rather than their chemical composition. They can be seen as a counterpart to the rather well-known family of “optical meta-materials.” These materials can be designed to have properties outside the scope found in nature. In the context of this book, the choice of such a critical size scale can be identified as an improved homogenization.

Note that in this *critical scale* with respect to the repetition the diffusion does not suffer any important modification (in contrast to the abovementioned case) since the particles are “too small” to affect the diffusion. The *critical scale* in which such new behavior arises and the correct identification of the *strange term* $\mathcal{H}(u)$, and its connection with the microscopic law given by σ, are the main subjects which are object of study in this book.

Moreover, as an indication of a potential important success of the nanoscale approach to building new materials with better structural properties than the materials existing in nature, we will show in this book that the presence of this new *strange reaction term* $\mathcal{H}(u)$ “improves” the process, leading, for instance, to a better *chemical effectiveness* and preventing the formation of the so-called *dead cores*. We will discuss this below in Sections 4.9.3, 4.9.4, 5.7 an 6.6 and Appendices A and C.

In the rest of this long Introduction we will make clear the notations of the book, we will precise the data and assumptions required for the occurrence of the *strange terms* and we will give the keys to the proofs of the main results. We also provide some historical and bibliographic notes (see Sections 1.4 and 1.6).

The case of σ being non-monotone also arises in the applications. This is the case, for instance, for the so-called Langmuir–Hinshelwood kinetics in which

$$\sigma(v) = \lambda[v_+]^m \frac{\delta + 1}{\delta + [v_+]^{m+k}}, \quad \text{for some } \lambda, \delta, k, m > 0,\ \delta \text{ small, and for any } v \geq 0$$

(see [14]), or the case, arising in enzyme kinetics, in which

$$\sigma(v) = \lambda \frac{[v_+]^m}{\delta + [v_+]^{m+k}}, \quad \text{for some } \lambda, \delta, k, m > 0,\ \delta \text{ small, and for any } v \geq 0$$

(see [21]). The case in which σ is non-monotone and singular,

$$\sigma(v) = [v_+]^{-k}, \quad k \in (0,1),$$

considered in [96], is also interesting. The case in which σ is non-monotone and possibly discontinuous was treated in [189] and [7].

1.2 Open domain with solid particles

The aim of this text is the study of a nonlinear reaction-diffusion problem on the exterior of a set of periodically placed particles over a domain $\Omega \subset \mathbb{R}^n$ (bounded and regular, for simplicity) where the reaction takes place over a periodical part of the boundary and the diffusion is of p-Laplace type, introduced above (see (1.1)). For a small parameter $\varepsilon > 0$, the particles (or holes) will be the translation at distance $\varepsilon > 0$ of a characteristic shape G_0 scaled by a parameter $a_\varepsilon \leq \varepsilon$. This particle will be either n-dimensional and placed periodically in the interior of Ω or $(n-1)$-dimensional and placed over an internal manifold or on the boundary.

In the appendices at the end of this volume we give some insights into different related problems.

1.2.1 The cases with *n*-dimensional particles G_0

Let the shape of an elementary inclusion (in our setting a particle, but it applies also to the case of a hole) represented by a domain G_0 be an open set such that $\overline{G_0} \subset Y = (-\frac{1}{2}, \frac{1}{2})^n$. In most of the main cases we will assume G_0 is homeomorphic to a ball (i. e., there exists an invertible continuous map $\Psi : U \to V$ between open sets of $\mathbb{R}^n$, U and V, where $G_0 \subset U$ and V contains the open ball of radius one, $\Psi(G_0)$ is the ball and Ψ^{-1} is continuous).

In this setting, we define

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (\varepsilon j + a_\varepsilon G_0), \quad S_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (\varepsilon j + a_\varepsilon \partial G_0),$$

where $\Upsilon_\varepsilon \subset \mathbb{Z}^n$ indexes the set of points where we will place particles. We furthermore request that

$$\Upsilon_\varepsilon \subset \{ j \in \mathbb{Z}^n : \varepsilon j + \varepsilon \overline{Y} \subset \Omega \}.$$

With this choice we guarantee that $\overline{G_\varepsilon} \subset \Omega$ and $S_\varepsilon \cap \partial\Omega = \emptyset$. Thus, the Dirichlet boundary condition in (1.1) is taken on all $\partial\Omega$. We will sometimes consider that

$$a_\varepsilon = C_0 \varepsilon^\alpha.$$

The difference in scale can be appreciated in Figure 1.1.

Remark 1.1. In the case of small particles $a_\varepsilon \ll \varepsilon$ we will sometimes take $G_0 = B_1$. This is a small abuse of notation since $G_0 \not\subset Y$. This is not a problem since in this setting $a_\varepsilon \ll \varepsilon$, and hence $a_\varepsilon G_0 \subset \varepsilon Y$ for ε small.

Two families of boundary data g^ε are usually considered:

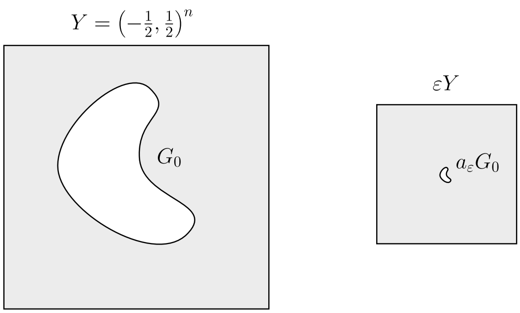

Figure 1.1: The reference cell Y and the scalings by ε and $a_\varepsilon = \varepsilon^\alpha$, for $\alpha > 1$. Note that, for $\alpha > 1$, $\varepsilon^\alpha G_0$ (for a general particle shaped as G_0) becomes smaller relative to εY, which scales as the repetition. In some first examples G_0 will be a ball $B_1(0)$ (see Remark 1.1).

1. The first case is the one in which the external source in the boundary reaction depends on the macroscopic scale

$$g^\varepsilon(x) = g(x),$$

   for $g : \Omega \to \mathbb{R}$ being in an adequate Sobolev space.
2. The second case considers the reaction with the same periodicity as the particle

$$g^\varepsilon(x) = g\left(\frac{x - \varepsilon j}{a_\varepsilon}\right), \quad x \in \partial(\varepsilon j + a_\varepsilon G_0), j \in \Upsilon_\varepsilon, \tag{1.2}$$

   where g needs only be defined on ∂G_0 and be integrable. We will usually assume that $g \in L^{p'}(\partial G_0)$ with $p' = p/(p-1)$.

A way to write both these behaviors in only one expression is

$$g^\varepsilon(x) = g_{\mathrm{st}}(x) + g_{\mathrm{per}}\left(\frac{x - \varepsilon j}{a_\varepsilon}\right). \tag{1.3}$$

Remark 1.2. Even though we take G_0 as a single connected particle, much of the work could be extended to G_0 with a finite number of connected components (each one diffeomorphic to a ball). This case is very relevant in applications to, for example, chemical engineering. An interesting problem we will not discuss here corresponds to the situation in which G_0 is composed of several types of components (see [174]). The case where the particle at each εj is picked from a finite set of particle shapes A^j was studied in [206] for the case of two different big particles with different regular kinetics functions σ and in [225] for the case of critical particles and $p = n = 2$ under the assumption that all $|\partial A_j|$ coincide, i. e., the particles are isoperimetric (see also Remark 4.23).

Remark 1.3. In some occasions we will also consider the Signorini problem, also known as the boundary obstacle problem,

$$\begin{cases} -\Delta_p u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ (\frac{\partial u_\varepsilon}{\partial \nu_p} + \beta(\varepsilon)\sigma(u_\varepsilon) - \beta(\varepsilon)g^\varepsilon)u_\varepsilon = 0 & \text{on } S_\varepsilon, \\ u_\varepsilon \ge 0 & \text{on } S_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_p} + \beta(\varepsilon)\sigma(u_\varepsilon) \ge \beta(\varepsilon)g^\varepsilon & \text{on } S_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us now properly introduce the three types of geometries which are usually considered, depending on Υ_ε (i. e., where the particles are placed and thus the internal boundary conditions on S_ε).

1.2.1.1 Particles over the whole domain

This is the setting which originally attracted most attention and interest from the mathematical community. This can be easily seen by the amount of work over the years. In fact, once this case is mastered, the remaining cases can be attacked very much in a similar fashion. The general strategy of the book will consist in giving the precise results and techniques for this case and proving the equivalent results for the other two cases.

In this case, we consider

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : \varepsilon j + \varepsilon \overline{Y} \subset \Omega\}.$$

Then Ω_ε appears as in Figure 1.2. Note that in this setting the number of particles scales like

$$|\Upsilon_\varepsilon| \simeq \varepsilon^{-n}|\Omega|.$$

The proof of this fact is simple. Due to the choice of Υ_ε

$$|\Omega| - \varepsilon^n|\Upsilon_\varepsilon| = |\Omega| - \left|\bigcup_{j\in\Upsilon_\varepsilon}(\varepsilon j + \varepsilon Y)\right| = \left|\Omega \setminus \bigcup_{j\in\Upsilon_\varepsilon}(\varepsilon j + \varepsilon Y)\right|.$$

Note that the last set is contained in the set

$$\Omega \setminus \bigcup_{j\in\Upsilon_\varepsilon}(\varepsilon j + \varepsilon Y) \subset \bigcup_{\substack{j\in\mathbb{Z}^n \\ (\varepsilon j+\varepsilon Y)\cap\partial\Omega\neq\emptyset}} (\varepsilon j + \varepsilon Y).$$

Since $\partial\Omega$ is a smooth manifold, the latter set has n-dimensional measure converging to zero (take, for example, a tubular neighborhood).

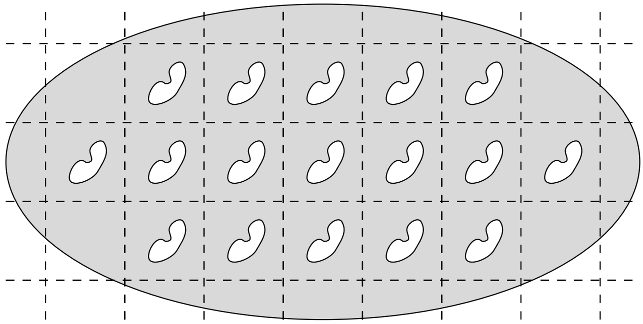

Figure 1.2: The set Ω_ε in the case of solid particles over the whole domain. The adjacent particles are at distance of the order of ε.

Remark 1.4. As indicated in the list of notations, we will use the notation $|\cdot|$ for many purposes. Here, $|\Upsilon_\varepsilon|$ denotes the cardinality of the finite set Υ_ε, whereas $|\Omega|$ denotes the Lebesgue measure of Ω.

Remark 1.5. Many variants of problem (1.1) are relevant in the applications and present interesting mathematical results in their treatment. This is the case, for instance, when the particles (over the whole domain) are assumed to be permeable. In that case, we must assume that there is an internal reaction inside the particles, instead just on their boundaries. In fact, the modeling leads now to a transmission problem with an unknown flux on the boundary of each particle:

$$\begin{cases} -D_f \Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ -D_p \Delta v_\varepsilon + \alpha\sigma(v_\varepsilon) = 0 & \text{in } G_\varepsilon, \\ u_\varepsilon = v_\varepsilon, \text{ and } D_f \frac{\partial u_\varepsilon}{\partial \nu} = D_p \frac{\partial v_\varepsilon}{\partial \nu} & \text{on } S_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where G_ε is the set of all particles, and with the diffusion coefficients D_f and D_p usually quite different. The homogenization of this problem was already considered in [86] and [84]. A dynamic boundary transmission condition was considered in [85]. The treatment of the critical size for some related problems was made in [32] and [136]. The techniques presented in our book can be adapted to this framework.

1.2.1.2 Particles over a manifold splitting the domain

Quite often in the applications (for instance in adsorption processes in chemical engineering) the reactant medium is located merely on some kind of grill (or perforated surface); see, e. g., the nice presentation on the modeling made in [150].

As in the previous problem, there is a useful duality and the same formulation applies to a set of isolated particles which are periodically located over an internal surface in the chemical reactor. It corresponds to the so-called *fluidized bed reactor* used for many industrial applications. In this type of reactor, a fluid (gas or liquid) is passed through a solid granular material (usually a catalyst possibly shaped as tiny spheres).

Other problems of a radically different nature also lead (after some simplifications) to quite similar formulations. It is the case, for instance, of some problems in elasticity associated to *lattice type structures* such as honeycombs and reinforced structures (see, e. g., [214, 83] and the references therein).

We will place the particles only over a manifold, which for simplicity we assume to be $\Omega \cap \{x_n = 0\}$ (see Figure 1.3). Let us introduce some notation. For any arbitrary set $\omega \subset \mathbb{R}^n$ we denote

$$\omega^+ = \{x \in \omega : x_n > 0\}, \quad \omega^- = \{x \in \omega : x_n < 0\} \quad \text{and} \quad \omega^0 = \{x \in \omega : x_n = 0\}. \tag{1.4}$$

In this case, we assume that Ω^+ and Ω^- are both non-empty and take

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}^{n-1} \times \{0\} : \overline{\varepsilon j + \varepsilon Y} \subset \Omega\}.$$

Unlike in the previous case, in this setting the number of particles scales like

$$|\Upsilon_\varepsilon| \sim \varepsilon^{n-1}\mathcal{H}^{n-1}(\Omega^0),$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-Hausdorff measure. Here, the equivalent to (1.3) is

$$g^\varepsilon(x) = g_{\mathrm{st}}(x) + g_{\mathrm{per}}\left(\frac{x - \varepsilon j}{a_\varepsilon}\right), \quad x \in \varepsilon j + a_\varepsilon \partial G_0 \text{ for some } j \in \Upsilon_\varepsilon. \tag{1.5}$$

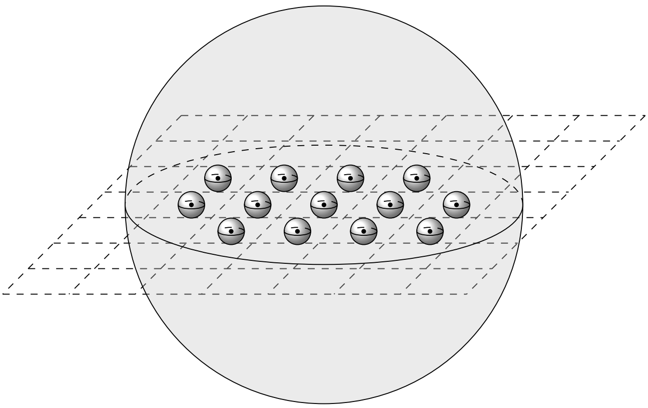

Figure 1.3: The set Ω_ε in the case of solid particles over a manifold. The adjacent particles are at distance of the order of ε.

1.2.2 The case of $(n-1)$-dimensional particles contained in $\partial\Omega$

The third type of *model problem* we will consider in this book is, in some sense, related with the above model problem with particles over a manifold but with the important difference that this manifold is located on a part of the boundary of the domain Ω. In this context, the particles are contained in $\partial\Omega$ and hence are $(n-1)$-dimensional. Problems of this nature arise in many different contexts, as for instance in chemical engineering (see [179]), elasticity (see, e. g., [214]), nanocomposites (see, e. g., [264]) and reverse osmosis (see, e. g., [115] and the many references therein).

In this last setting, which we present for the sake of completion, the particles will be contained in the boundary, and hence they are $(n-1)$-dimensional (see Figure 1.4). For simplicity, we consider that the part of the boundary with particles is

$$(\partial\Omega)^0 = \{x \in \partial\Omega : x_n = 0\}.$$

We therefore assume that $G_0 \subset Y \cap \mathbb{R}^{n-1} \times \{0\}$. In this case we assume that $(\Omega_\varepsilon)^- = \emptyset$ and define

$$G_\varepsilon = \emptyset, \quad S_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (\varepsilon j + a_\varepsilon G_0),$$

where

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}^{n-1} \times \{0\} : \overline{(\varepsilon j + \varepsilon Y)^0} \subset (\partial\Omega)^0 \text{ and } (\varepsilon j + \varepsilon Y)^+ \subset \Omega\}.$$

Note that, in contrast to the two precedent cases, S_ε is not the boundary of the particles but the own set of $(n-1)$-dimensional particles. We have

$$\partial\Omega_\varepsilon = (\partial\Omega)^+ \cup [(\partial\Omega)^0 \setminus S_\varepsilon] \cup S_\varepsilon.$$

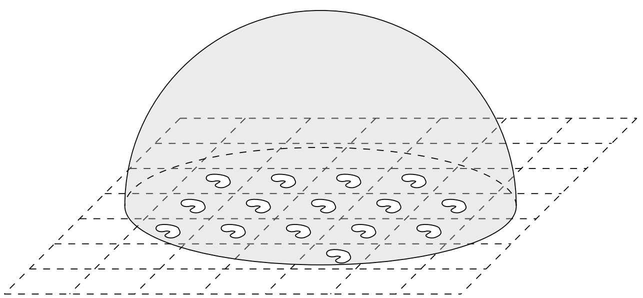

Figure 1.4: The set Ω_ε in the case of solid particles over the boundary. The adjacent particles are $(n-1)$-dimensional and are at distance of the order of ε.

The problem we will consider is the following:

$$\begin{cases} -\Delta_p u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } (\partial\Omega)^+, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = g^\varepsilon & \text{on } S_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon = 0 & \text{on } (\partial\Omega)^0 \setminus S_\varepsilon. \end{cases} \tag{1.6}$$

Note that it is different from the problem in which on $(\partial\Omega)^0 \setminus S_\varepsilon$ we ask for the Dirichlet boundary condition

$$u_\varepsilon = 0 \quad \text{on } (\partial\Omega)^0 \setminus S_\varepsilon.$$

1.3 Homogenized problem: effective reaction-diffusion behavior

The aim of the homogenization process is to study the function u such that the solutions of (1.1), u_ε, converge, $u_\varepsilon \to u$, in some sense as $\varepsilon \to 0$. The idea is that there exists an effective reaction-diffusion behavior given by a modification of the parameters and nonlinearity, such that u_0 is a solution of a limit problem, which depends on the geometry. The different nature of these three problems gives rise to three different behaviors of the limit.

1.3.1 Solid particles over the whole domain

The complete discussion of this case can be found in Chapter 4. We summarize the result here. First, let us indicate that it is easy to compute that

$$|S_\varepsilon| = |\Upsilon_\varepsilon| a_\varepsilon^{n-1} |\partial G_0| \simeq \varepsilon^{-n} a_\varepsilon^{n-1} |\partial G_0||\Omega|.$$

To illustrate the different behavior, let g^ε be given by (1.3). For some reasons that we will explain below, we assume that $\beta(\varepsilon) \lesssim 1/|S_\varepsilon|$. In this case the effective problem is given by

$$\begin{cases} -\operatorname{div} a^{\mathrm{eff}}(\nabla u) + \beta_1^{\mathrm{eff}} \sigma^{\mathrm{eff}}(u) = f + \beta_2^{\mathrm{eff}} g^{\mathrm{eff}} & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

In this setting, the so-called effective diffusion coefficient $a^{\mathrm{eff}} : \mathbb{R}^n \to \mathbb{R}$ has a simple form in some cases given by

$$a^{\mathrm{eff}}(\xi) = \begin{cases} A^{\mathrm{eff}} \xi & a_\varepsilon \sim \varepsilon \text{ and } p = 2, \\ |\xi|^{p-2}\xi & a_\varepsilon \ll \varepsilon \text{ and } p \in (1, \infty). \end{cases}$$

When $a_\varepsilon \ll \varepsilon$, the diffusion operator $-\Delta_p$ is not altered. For $a_\varepsilon \sim \varepsilon$, a^{eff} is obtained by solving an auxiliary problem, known as *cell problem*, which will be discussed in Section 4.4. When $p = 2$, the linearity is preserved, so we only need to recover the form of the matrix A^{eff}. This allows to classify two types of cases:

1. Big particles: $a_\varepsilon \sim \varepsilon$. There is an effective-diffusion matrix depending on the shape of the particles, and the nonlinearity stays the same.
2. Small particles: $a_\varepsilon \ll \varepsilon$. This different effective diffusion is not present.

In this book we are mostly interested in the possible change of the nature of the nonlinear reaction term. We will show that there exists a *critical size* $a_\varepsilon^\star$ that differentiates three regimes:

$$\sigma^{\text{eff}}(x,s) = \begin{cases} \sigma(s) & a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon \text{ and } a_\varepsilon \leq \varepsilon, \\ \mathcal{H}(x,s) & a_\varepsilon \sim a_\varepsilon^\star, \\ 0 & a_\varepsilon \ll a_\varepsilon^\star. \end{cases} \tag{1.7}$$

This function $\mathcal{H}(x,s)$ is, in general, different from the original reaction σ and depends on the scaling of $\beta(\varepsilon)$ (as we will explain below). It is the so-called "strange term" of critical-scale homogenization (a terminology popularized by [80, 81] and preserved by many authors).

We give some historical notes in Section 1.6. This is the reason why this critical case is "anomalous," as point out in the title of the book. The determination of this function $\mathcal{H}$ is one the main difficulties we will face in this book. When G_0 is a ball it is given by (4.16)–(4.17), where for G_0 general it is given by (4.32) and the notations in Section 3.1.5.3.

In this setting, we will show that

$$a_\varepsilon^\star = \begin{cases} \varepsilon^{\frac{n}{n-p}} & p \in (1,n), \\ \varepsilon e^{-\alpha\varepsilon^{-\frac{n}{n-1}}} & p = n \text{ for any } \alpha > 0, \\ 0 & p > n. \end{cases}$$

The last case is a short-hand notation to indicate that for $p > n$ there is no critical scale. The case $p = n$ needs to be understood in the sense of Remark 4.1. In Chapter 4 we will show how this critical scale is deduced.

Finally, let us look at the influence of the rest of the terms on the homogenized equation. We define

$$\beta^0 = \lim_{\varepsilon\to 0} \frac{\beta(\varepsilon)}{1/|S_\varepsilon|} \in [0,+\infty).$$

We will show in Chapter 4 that then we have

$$\beta_1^{\mathrm{eff}} = \begin{cases} \frac{\beta^0}{|\Omega||Y\setminus G_0|} & a_\varepsilon \sim \varepsilon, \\ \frac{\beta^0}{|\Omega|} & a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon, \\ 1 & a_\varepsilon \lesssim a_\varepsilon^\star, \end{cases} \quad \beta_2^{\mathrm{eff}} = \begin{cases} \beta_1^{\mathrm{eff}} & a_\varepsilon \gg a_\varepsilon^\star, \\ 0 & a_\varepsilon \sim a_\varepsilon^\star \end{cases} \tag{1.8}$$

and

$$g^{\mathrm{eff}}(x) = g_{\mathrm{st}}(x) + \frac{1}{|\partial G_0|} \int_{\partial G_0} g_{\mathrm{per}}(y)\, \mathrm{d}y. \tag{1.9}$$

The fact that $\beta_2^{\mathrm{eff}} = 0$ when $a_\varepsilon \sim a_\varepsilon^\star$ happens (as we will see below) because g^ε is "written into" the strange term $\mathcal{H}$. In the supercritical range $a_\varepsilon \ll a_\varepsilon^\star$, we always assume that $g^\varepsilon = 0$, so we give no information of β_2^{eff}. A classification of the different values of the function H(u) when $\beta(\varepsilon) = \varepsilon^{-\gamma}$ and $a_\varepsilon = C_0 \varepsilon^\alpha$ is presented in Section 4.8.

As mentioned before, the critical scale for β is

$$\beta^\star(\varepsilon) = |S_\varepsilon|^{-1},$$

and then the resulting homogenized equation can be classified according to Table 1.1.

Table 1.1: Homogenized equation in the different ranges when $\sigma^{-1}(0) = 0$ and $g^\varepsilon = 0$. When $a_\varepsilon \sim a_\varepsilon^\star$, the case $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$ behaves like the one with homogeneous Neumann conditions (i. e., $\sigma = 0$) and $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ behaves like the case of homogeneous Dirichlet conditions.

	$a_\varepsilon \sim \varepsilon$	$a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$	$a_\varepsilon \sim a_\varepsilon^\star$	$a_\varepsilon \ll a_\varepsilon^\star$
$\beta(\varepsilon) \ll \beta^\star(\varepsilon)$	$-\operatorname{div}(a^{\mathrm{eff}}(\nabla u)) = f$	$-\Delta_p u = f$	$-\Delta_p u = f$	$-\Delta_p u = f$
$\beta(\varepsilon) \sim \beta^\star(\varepsilon)$	$-\operatorname{div}(a^{\mathrm{eff}}(\nabla u)) + \beta^{\mathrm{eff}}\sigma(u) = f$	$-\Delta_p u + \beta^{\mathrm{eff}}\sigma(u) = f$	$-\Delta_p u + \mathcal{H}(u) = f$	$-\Delta_p u = f$
$\beta(\varepsilon) \gg \beta^\star(\varepsilon)$	$u = 0$	$u = 0$	$-\Delta_p u + \mathcal{A}_0 \lvert u\rvert^{p-2} u = f$	$-\Delta_p u = f$

1.3.2 Particles over a manifold

In this case, the limit satisfies the homogenized equation

$$\begin{cases} -\Delta_p u = f & \Omega^+ \cup \Omega^-, \\ u = 0 & \partial\Omega, \\ [u]_{\Omega^0} = 0, & \\ [|\nabla u|^{p-2} \frac{\partial u}{\partial x_n}]_{\Omega^0} = \beta_1^{\mathrm{eff}} \sigma^{\mathrm{eff}}(x,u) - \beta_2^{\mathrm{eff}} g^{\mathrm{eff}} & \Omega \cap \{x_n = 0\}, \end{cases}$$

where

$$[f]_{\Omega^0}(x) = \lim_{h\to 0}\big(f(x+he_n) - f(x-he_n)\big).$$

The description of the effective values of $\mathcal{H}$, (1.8) and (1.9) are more or less preserved. The values of $a_\varepsilon^\star$ and $\beta(\varepsilon)^\star$ have to be adapted to this case. We have different constants

$$\beta_1^{\text{eff}} = \begin{cases} \frac{\beta^0}{|\Omega^0|} & a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon, \\ 1 & a_\varepsilon \lesssim a_\varepsilon^\star, \end{cases} \qquad \beta_2^{\text{eff}} = \begin{cases} \beta_1^{\text{eff}} & a_\varepsilon \gg a_\varepsilon^\star, \\ 0 & a_\varepsilon \sim a_\varepsilon^\star. \end{cases}$$

We will discuss the precise values of the effective parameters below. A table similar to Table 1.1 can be drafted, written for the conditions on $\Omega^0 = \Omega \cap \{x_n = 0\}$. The detailed results can be found in Chapter 5. The results are summarized in Table 1.2.

Table 1.2: Homogenized boundary condition on the interior manifold Ω^0 in the different ranges when $\sigma^{-1}(0) = 0$ and $g^\varepsilon = 0$. When $[|\nabla u|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = 0$, then simply $-\Delta_p u = f$ in the whole domain Ω.

	$a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon$	$a_\varepsilon \sim a_\varepsilon^\star$	$a_\varepsilon \ll a_\varepsilon^\star$
$\beta(\varepsilon) \ll \beta^\star(\varepsilon)$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = 0$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = 0$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = 0$
$\beta(\varepsilon) \sim \beta^\star(\varepsilon)$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = \beta^{\text{eff}}\sigma(u)$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = \mathcal{H}(u)$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = 0$
$\beta(\varepsilon) \gg \beta^\star(\varepsilon)$	$u = 0$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = \mathcal{A}_0\|u\|^{p-2}u$	$[\|\nabla u\|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = 0$

1.3.3 Particles over the boundary

In this case we recover

$$\begin{cases} -\Delta_p u = f & \Omega \subset \{x_n > 0\}, \\ u = 0 & \partial\Omega \setminus \{x_n = 0\}, \\ \partial_{\nu_p} u + \beta_1^{\text{eff}}\sigma^{\text{eff}}(u) = \beta_2^{\text{eff}} g^{\text{eff}} & \partial\Omega \cap \{x_n = 0\}. \end{cases}$$

We will also discuss the explicit values of these effective parameters below, with

$$g^{\text{eff}}(x) = g_{\text{st}}(x) + \frac{1}{|G_0|}\int_{G_0} g_{\text{per}}(y)\,\mathrm{d}y \tag{1.10}$$

and

$$\beta_1^{\text{eff}} = \begin{cases} \frac{\beta^0}{|(\partial\Omega)^0|} & a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon, \\ 1 & a_\varepsilon \lesssim a_\varepsilon^\star, \end{cases} \qquad \beta_2^{\text{eff}} = \begin{cases} \beta_1^{\text{eff}} & a_\varepsilon \gg a_\varepsilon^\star, \\ 0 & a_\varepsilon \sim a_\varepsilon^\star. \end{cases}$$

We refer the reader to Chapter 6 for the results. The results are summarized in Table 1.3.

Table 1.3: Homogenized boundary condition on $\Omega^0 \subset \partial\Omega$ when in the different ranges when $\sigma^{-1}(0) = 0$ and $g^\varepsilon = 0$. When $a_\varepsilon \sim a_\varepsilon^\star$, the case $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$ behaves like the one with homogeneous Neumann conditions (i. e., $\sigma = 0$) and $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ behaves like if we start with homogeneous Dirichlet conditions on $\partial\Omega \cap \{x_n = 0\}$.

	$a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon$	$a_\varepsilon \sim a_\varepsilon^\star$	$a_\varepsilon \ll a_\varepsilon^\star$
$\beta(\varepsilon) \ll \beta^\star(\varepsilon)$	$\partial_{\nu_p} u = 0$	$\partial_{\nu_p} u = 0$	$\partial_{\nu_p} u = 0$
$\beta(\varepsilon) \sim \beta^\star(\varepsilon)$	$\partial_{\nu_p} u + \beta^{\mathrm{eff}}\sigma(u) = 0$	$\partial_{\nu_p} u + \mathcal{H}(u) = 0$	$\partial_{\nu_p} u = 0$
$\beta(\varepsilon) \gg \beta^\star(\varepsilon)$	$u = 0$	$\partial_{\nu_p} u + \mathcal{A}_0 \lvert u\rvert^{p-2} u$	$\partial_{\nu_p} u = 0$

1.4 Different homogenization techniques

Here we will briefly present some of the most relevant methodologies applied in homogenization for the types of problems mentioned above. Most of them have been applied to our problem, as we will see later.

1.4.1 The multiple-scales method

One of the possibilities (and, in fact, a pioneering method, see [242]) in dealing with identifying the limit consists of considering an expansion known as *asymptotic expansion* of the solutions. In the case $a_\varepsilon = \varepsilon$ we can formally imagine that our solution is of the form

$$u_\varepsilon(x) = u(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots \tag{1.11}$$

and derive the behavior from there. This method, which is now known as the *multiple-scales method*, is still very much in use (see, e. g., [104, 126, 57] among many others). In this direction we recommend the famous books [31, 239, 77, 214] (a more detailed list of references can be found in Section 1.6).

This kind of argument works in two steps. Take for example the simple case of

$$\begin{cases} -\operatorname{div}(A(\frac{x}{\varepsilon})\nabla u_\varepsilon) = f & \Omega, \\ u_\varepsilon = 0 & \partial\Omega, \end{cases}$$

where A is Y-periodic. First, a formal deduction of the good approximation can be made and a later rigorous proof can be given. In particular, we can use repeatedly the computation that if $v = v(x, y)$, then

$$\frac{\partial}{\partial x_i}\left[v\left(x, \frac{x}{\varepsilon}\right)\right] = \frac{\partial v}{\partial x_i}\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\frac{\partial v}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right).$$

Substituting (1.11) into $-\operatorname{div}(A(\frac{x}{\varepsilon})u_\varepsilon) = f$ and gathering terms, one can recover that there is a natural choice

$$u_1\left(x, \frac{x}{\varepsilon}\right) = \hat{\xi}\left(\frac{x}{\varepsilon}\right) \cdot \nabla u, \quad u_2\left(x, \frac{x}{\varepsilon}\right) = \hat{\theta} : D^2 u, \quad \cdots,$$

where the equations for u_0, $\hat{\xi}$ and $\hat{\theta}$ can be found explicitly and the remaining terms are formally of higher order (as in the usual Taylor expansion). The second part of this kind of arguments is to estimate the convergence, as a typical result for this problem is that

$$\left\| u_\varepsilon(x) - \left(u(x) + \varepsilon\hat{\xi}\left(\frac{x}{\varepsilon}\right) \cdot \nabla u + \varepsilon^2 \hat{\theta} : D^2 u\right)\right\|_{H^1(\Omega)} \to 0, \quad \text{as } a_\varepsilon \to 0.$$

The details of similar examples can be found, e. g., in [239, Chapter 5], [76, Chapter 7]. Furthermore, one can find some rates of this convergence.

Naturally, the situation becomes more complicated when $a_\varepsilon \ll \varepsilon$ (for the application of this formal expansion, see, e. g., [150]). We will not apply this technique in this book.

1.4.2 The Γ-convergence method

This method was introduced by De Giorgi [98] and later developed in [97, 92, 266] (among many other authors). The essential idea behind the Γ-convergence method is to study the problem in its energy variational formulation and the conditions under which convergence of the energies implies convergence of their minimizers, i. e., of the solutions of the elliptic problems. Here we present some results extracted from [92].

Definition 1.6. Let X be a topological space. The Γ-lower limit and Γ-upper limit of a sequence (F_n) of functions $X \to [-\infty, \infty]$ are defined as follows:

$$\left(\Gamma - \liminf_{n\to+\infty} F_n\right)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{n\to+\infty} \inf_{y\in U} F_n(y),$$

$$\left(\Gamma - \limsup_{n\to+\infty} F_n\right)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{n\to+\infty} \inf_{y\in U} F_n(y),$$

where $\mathcal{N}(x) = \{U \subset X,\ U \text{ open} : x \in U\}$. If there exists $F : X \to [-\infty, +\infty]$ such that $F = \Gamma - \liminf_{n\to+\infty} F_n = \Gamma - \limsup_{n\to+\infty} F_n$, then we say that F_n Γ-converges to F, and we denote it as

$$F = \Gamma - \lim_{n\to+\infty} F_n.$$

For the sake of convenience, in this section we will denote

$$F' = \Gamma - \liminf_{n\to+\infty} F_n,$$

$$F'' = \Gamma - \limsup_{n\to+\infty} F_n.$$

The results that make this technique interesting for us are the following.

Theorem 1.7. *Suppose that* (F_n) *are equi-coercive in* X. *Then* F' *and* F'' *are coercive and*

$$\inf_{x\in X} F'(x) = \lim_{n\to+\infty} \inf_{x\in X} F_n(x).$$

Proposition 1.8. *Let* x_n *be a minimizer of* F_n *in* X *and assume that* $x_n \to x$ *in* X. *Then*

$$F'(x) = \liminf_{n\to\infty} F_n(x_n), \quad F''(x) = \limsup_{n\to\infty} F_n(x_n).$$

In the context of homogenization we are mainly interested in the behavior of functionals

$$F_\varepsilon(u, A) = \begin{cases} \int_A f(\frac{x}{\varepsilon}, u(x), Du(x))\,\mathrm{d}x & u \in W^{1,p}(A), \\ +\infty & \text{otherwise}, \end{cases}$$

where $p > 1$. The main result of this method is the following. Let

$$f_0(\xi) = \inf_{v\in W^{1,p}_{\mathrm{per}}(Y)} \int_Y f\big(y, v(y), \xi + Dv(y)\big)\,\mathrm{d}y.$$

Then, under some mild assumptions on f, for every sequence $\varepsilon_n \to 0$, we have that F_{ε_n} Γ-converges to F_0, the functional defined by

$$F_0(u, A) = \begin{cases} \int_A f_0(Du)\,\mathrm{d}x & u \in W^{1,p}(A), \\ +\infty & \text{otherwise}. \end{cases}$$

The characterization of this function f_0 allows to recover the homogenized limit of problems. We refer the reader to [92] for complete details in this direction.

This method was applied to our cases of interest presented above, with some modification, by Kaizu [178] and Goncharenko [162].

1.4.3 The two-scale convergence method

The two-scale method was introduced by Nguetseng [210] and later developed by some authors, amongst which we highlight the work of Allaire [5, 6] (see also the survey [267] and its many references). The central definition of this method is the following.

Definition 1.9. Let (v_ε) be a sequence in $L^2(\Omega)$. We say that the sequence v_ε *two-scale converges* to a function $v_0 \in L^2(\Omega \times Y)$ if, for any function $\psi = \psi(x, y) \in \mathcal{D}(\Omega; \mathcal{C}^\infty_{\mathrm{per}}(Y))$ (i. e., functions which are smooth in x, y, have compact domain in Ω and are periodic in y) one has

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y v_0(x, y)\psi(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

By taking $\psi = \psi(x)$ in the previous definition it is immediate to see that

$$v_\varepsilon \rightharpoonup V^0 = \frac{1}{|Y|} \int_Y v_0(\cdot, y)\,dy,$$

weakly in $L^2(\Omega)$. The key point of this theory is to study the convergence of functions of the type $\psi(x, \frac{x}{\varepsilon})$ and then apply them suitably to the weak formulation. This method has also been applied for the homogenization of general Hamiltonians (see, e. g., [29] and the references therein).

1.4.4 Tartar's method of oscillating test functions

This method (initially called "energy method") is due to Tartar (see [254, 255, 256, 209]). The general idea behind it is to consider the appropriate weak formulation and select suitable test functions φ^ε with properties that, in the limit, reveal a weak formulation of the homogeneous problem.

Unfortunately, there is not any specific rule to choose the oscillating test functions and thus it must be built for each particular problem under consideration. Many references will be indicated in Section 1.6.

This is the general method applied to obtain the results of this book. As we shall see, it is not a straightforward recipe, and the choice of test function and their analysis can become a very hard task. Many detailed examples will be given in the following chapters. Perhaps the simplest presentation corresponds to the case considered in Section 4.6, but, without any doubt, the more interesting (and harder) application corresponds to the critical cases studied in Section 4.7.

A difficulty that arises with this method in domains with particles or holes is the need of a common functional space, since $u_\varepsilon \in L^p(\Omega_\varepsilon)$. This leads to the construction of extension operators $P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega)$, which will be discussed in Section 3.1.1.

1.4.5 The periodical unfolding method

The periodical unfolding method was introduced by Cioranescu, Damlamian and Griso in [73, 75] (see the monograph [74]). It consists on transforming the solution to a fixed domain $\Omega \times Y$. The case of particles (or holes) was considered in [72, 71, 59]. See also [78].

Let us present the reasoning in domains with particles (or holes). The idea is to decompose every point in Ω as a sum

$$x = [x]_Y + \{x\}_Y,$$

where $[x]_Y$ is the unique element in $\mathbb{Z}^n$ such that $x - [x]_Y \in [0,1)^n$. That is, we have that $[\cdot]_Y$ is constant over Y_ε^j. We define the operator

$$\mathcal{T}_{\varepsilon,\delta} : \varphi \in L^2(\Omega) \mapsto \mathcal{T}_{\varepsilon,\delta}(\varphi) \in L^p(\Omega \times \mathbb{R}^n)$$

as

$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z) = \begin{cases} \varphi(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon\delta z) & (x,z) \in \hat{\Omega}_\varepsilon \times \frac{1}{\delta} Y, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\hat{\Omega}_\varepsilon = \operatorname{interior}\Big(\bigcup_{\substack{\xi \in \mathbb{Z}^n: \\ \varepsilon(\xi + Y) \subset \Omega}} \varepsilon(\xi + \overline{Y}) \Big).$$

Note that $\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z)$ is piecewise constant in x. The boundary of G_ε^j corresponds to $\hat{\Omega}_\varepsilon \times \partial G_0$.

The great advantage of this approach is that it removes the need to construct extension operators. Therefore, it allows to consider non-smooth shapes of G_0. This method has shown very good results, and the properties of $\mathcal{T}_{\varepsilon,\delta}(u_\varepsilon)$ are well understood, at least in the non-critical cases.

1.5 Structure of the proofs and main ideas: oscillating test functions

The structure of the proofs below will follow a general scheme which we have found to be a winning strategy. In order to fix ideas in this introduction, let us focus on the case $p = 2$.

1.5.1 Showing the solutions u_ε have a limit

Since u_ε are functions defined over the sets Ω_ε it is immediate to find how to formalize the intuition of why $u_\varepsilon \to u$.

Uniform boundedness

The natural energy spaces for existence and uniqueness of solutions are the Sobolev spaces:

$$W^{1,p}(\Omega_\varepsilon, \partial\Omega) = \overline{\{u \in \mathcal{C}^\infty(\Omega_\varepsilon) : u \text{ vanishes on a neighborhood of } \partial\Omega\}}^{W^{1,p}(\Omega_\varepsilon)}.$$

Checking uniform boundedness in these spaces

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C,$$

where C does not depend on ε, is relatively standard in most cases.

Finding a common space: extension to Ω

Since we want the convergence to occur in some functional space, we need to find a *common ground*. When G_0 is $(n-1)$-dimensional this is not needed. The classical approach to solve this problem is to construct extension operators

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon, \partial\Omega) \to W_0^{1,p}(\Omega),$$

which are uniformly continuous with the norms above:

$$\int_\Omega |\nabla P_\varepsilon u|^p \, dx \le C \int_{\Omega_\varepsilon} |\nabla u|^p \, dx, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega),$$

for some C not depending on ε.

Compactness

Using both facts above, it is immediate that there exists uniform boundedness in $W_0^{1,p}(\Omega)$:

$$\|P_\varepsilon u_\varepsilon\|_{W_0^{1,p}(\Omega)} \le C.$$

Therefore, by well-known weak compactness results (since $1 < p < +\infty$), there exists a limit

$$P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega).$$

1.5.2 Characterizing an effective equation

We pass to the limit in the weak formulation of the problem to detect the weak formulation for the effective (homogenized) problem. For simplicity, let us assume $p = 2$.

Effective diffusion

In order to have “convergence” of the gradient in some space, we can either study $\nabla P_\varepsilon u_\varepsilon$ or introduce

$$\widetilde{\nabla u_\varepsilon} = \begin{cases} \nabla u_\varepsilon & \Omega_\varepsilon, \\ 0 & \Omega \setminus \Omega_\varepsilon. \end{cases} \tag{1.12}$$

The latter has the advantage that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi \, dx = \int_{\Omega} \widetilde{\nabla u_\varepsilon} \nabla \varphi \, dx,$$

so it is sufficient to take weak limits. Note that it is unlikely that $\widetilde{\nabla u_\varepsilon}$ coincides with $\nabla P_\varepsilon u_\varepsilon$. Since

$$\|\widetilde{\nabla u_\varepsilon}\|_{L^2(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C,$$

it has a limit

$$\widetilde{\nabla u_\varepsilon} \rightharpoonup \xi_0 \quad \text{in } L^2(\Omega).$$

Then, the diffusion term can be rewritten

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi \, dx \to \int_{\Omega} \xi_0 \nabla \varphi \, dx.$$

Therefore, if we are able to deal adequately with the other terms, the diffusion term in the effective problem is $-\operatorname{div}(\xi_0)$.

Since

$$|\Omega_\varepsilon| \to \begin{cases} \theta|\Omega| & a_\varepsilon \sim \varepsilon, \ a_\varepsilon \leq \varepsilon \\ & \text{and the particles are over the whole domain,} \\ |\Omega| & a_\varepsilon \ll \varepsilon, \end{cases}$$

for some $\theta < 1$, the extension by 0 given by (1.12) only produces an effect if the holes are large: $a_\varepsilon \sim \varepsilon$. We will show

$$\xi_0 = \begin{cases} a^{\text{eff}}(\nabla u) & a_\varepsilon \sim \varepsilon, \ a_\varepsilon \leq \varepsilon \\ & \text{and the particles are over the whole domain,} \\ \nabla u & \text{otherwise.} \end{cases}$$

This characterization is the main difficulty studied in Section 4.4.

Detecting the critical cases

The idea is to first study the integral $\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon}$. A good approach is to study the trace inequality. This is one of the main focuses of Chapter 3. Assume that g is a smooth function. Typically a value $a_\varepsilon^\star$ appears such that the behavior is as follows: for a smooth function g

$$\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} g \, dS = \rho_\varepsilon + C_\varepsilon \int_{T_\varepsilon} g \, dx,$$

where T_ε is made up of ε-scaled balls and $\rho_\varepsilon \to 0$ for $a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon$. Some auxiliary functions m_ε are usually used in this task, allowing to pass the integral from S_ε to T_ε. This allows us to determine the critical value $a_\varepsilon^\star$.

Thus, we will recover

$$\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} g\,\mathrm{d}S \to \begin{cases} \frac{1}{|\Omega|}\int_\Omega g\,\mathrm{d}x & \text{if the particles are over the whole domain,} \\ \frac{1}{|\Omega^0|}\int_{\Omega^0} g\,\mathrm{d}x & \text{if the particles are centered in } \Omega^0, \\ \frac{1}{|(\partial\Omega)^0|}\int_{(\partial\Omega)^0} g\,\mathrm{d}S & \text{if the particles are in } (\partial\Omega)^0, \end{cases}$$
$$\text{when } a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon. \tag{1.13}$$

We recall the notation $\cdot^0$ is introduced in (1.4). This drives the effective reaction term. In the case $a_\varepsilon \ll a_\varepsilon^\star$ we will be able to remove the reaction term with smart test functions that vanish on S_ε (see, for example, Section 4.6 below).

1.5.3 Study of the critical case: the appearance of the strange term

This is the trickiest case. We apply Tartar's method of oscillating test functions, again for the case $p = 2$. The choice of these functions will be rather involved.

Weak formulation

For simplicity, let us study the case (1.1). First, we write our problem in a weak formulation (which will be justified later), of the form (for any good test function v)

$$\int_{\Omega_\varepsilon} \nabla v \nabla(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} (\sigma(v) - g^\varepsilon)(v - u_\varepsilon)\,\mathrm{d}S \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon)\,\mathrm{d}x. \tag{1.14}$$

If σ is a maximal monotone graph this requires some refinement which we will discuss in Section 2.2.

Choice of oscillating test functions

Then, we select as test function $v_\varepsilon = v - W_\varepsilon(x; v)$, where v is a generic test function for the homogenized problem and W_ε is a sequence of functions converging weakly to 0 with special properties. The aim of this term is to control the singular term $\beta(\varepsilon)\int_{S_\varepsilon}$. Note that we can write

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla v_\varepsilon \nabla(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x = \int_{\Omega_\varepsilon} \nabla v \nabla(v - u_\varepsilon)\,\mathrm{d}x - \int_{\Omega_\varepsilon} \nabla v \nabla W_\varepsilon\,\mathrm{d}x \\ - \int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla(v - W_\varepsilon - u_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

In the case $p \neq 2$, we will have a similar result with some additional error terms (see Lemma 4.38). The first term has a clear limit, and it yields the diffusion term of the effective equation. The second integral usually vanishes in the limit due to the construction of W_ε. The function W_ε is chosen with the properties that

$$\begin{cases} \Delta W_\varepsilon = 0 & x \in \bigcup_{j\in\Upsilon_\varepsilon}(\varepsilon j + \frac{\varepsilon}{4}B_1 \setminus a_\varepsilon \overline{G_0}), \\ W_\varepsilon = 0 & x \notin \bigcup_{j\in\Upsilon_\varepsilon}(\varepsilon j + \frac{\varepsilon}{4}B_1). \end{cases}$$

We will add more requirements below. With this choice, the last integral becomes

$$\int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (v - W_\varepsilon - u_\varepsilon)\,\mathrm{d}x = \int_{S_\varepsilon} (\partial_\nu W_\varepsilon)(v - W_\varepsilon - u_\varepsilon)\,\mathrm{d}S_x + \sum_{j\in\Upsilon_\varepsilon} \int_{\varepsilon j + \frac{\varepsilon}{4}\partial B_1} (\partial_\nu W_\varepsilon)(v - u_\varepsilon)\,\mathrm{d}S_x.$$

The normal derivative on S_ε, $(\partial_\nu W_\varepsilon)|_{S_\varepsilon}$ is chosen so that we have a cancelation of the integral in S_ε of (1.14). The other normal derivative, $(\partial_\nu W_\varepsilon)|_{\varepsilon j + \frac{\varepsilon}{4}\partial B_1}$, with the averaging limit (1.13), yields the so-called *strange term*, which we have denoted $\mathcal{H}$.

Let us focus on the simpler case in which $g^\varepsilon = 0$ and $G_0 = B_1$ (even though this is not contained in Y, see Remark 1.1). On each ball $\varepsilon j + \frac{\varepsilon}{4}B_1$ we can pick

$$W_\varepsilon(x;v) = H\big(v(x)\big)w_\varepsilon(x - \varepsilon j)$$

(for a suitable w_ε as explained below) and we get

$$\int_{S_\varepsilon} (\partial_\nu W_\varepsilon)(v - W_\varepsilon - u_\varepsilon)\,\mathrm{d}S_x = \sum_{j\in\Upsilon_\varepsilon} \int_{\varepsilon j + a_\varepsilon \partial G_0} \big(\partial_\nu w_\varepsilon(x - \varepsilon j)\big)H(v)\big(v - H(v)w_\varepsilon(x-\varepsilon j) - u_\varepsilon\big)\,\mathrm{d}S_x.$$

The proof is made simpler by assuming that $w_\varepsilon = 1$ on ∂G_0. With the conditions we already set on W_ε we arrive at the capacity type problem, which plays a very important role to this purpose,

$$\begin{cases} \Delta w_\varepsilon = 0 & x \in \frac{\varepsilon}{4}B_1 \setminus a_\varepsilon \overline{G_0}, \\ w_\varepsilon = 1 & a_\varepsilon \partial G_0, \\ w_\varepsilon = 0 & \frac{\varepsilon}{4}\partial B_1. \end{cases}$$

Since we are assuming that G_0 is a ball this solution is radially symmetric and explicit (see Section 3.1.5). Hence, $W_\varepsilon = 1$ on S_ε and we have

$$\int_{S_\varepsilon} (\partial_\nu W_\varepsilon)(v - W_\varepsilon - u_\varepsilon)\,\mathrm{d}S_x = B_\varepsilon \int_{S_\varepsilon} H(v)\big(v - H(v) - u_\varepsilon\big)\,\mathrm{d}S_x,$$

where $B_\varepsilon = \partial_\nu w_\varepsilon|_{a_\varepsilon \partial G_0}$. To get the cancelation of integrals in S_ε we want that

$$\begin{aligned} B_\varepsilon \int_{S_\varepsilon} H(v)\big(v - H(v) - u_\varepsilon\big)\,\mathrm{d}S_x &\simeq \beta(\varepsilon) \int_{S_\varepsilon} \sigma(v_\varepsilon)(v_\varepsilon - u_\varepsilon)\,\mathrm{d}S_x \\ &= \beta(\varepsilon) \int_{S_\varepsilon} \sigma\big(v - H(v)\big)\big(v - H(v) - u_\varepsilon\big)\,\mathrm{d}S_x. \end{aligned}$$

This cancelation is obtained if H is taken such that for every $s \in \mathbb{R}$

$$\left(\lim_{\varepsilon\to 0} \frac{B_\varepsilon}{\beta(\varepsilon)}\right) H(s) = \sigma\big(s - H(s)\big). \tag{1.15}$$

We point out that the above limit is related with another constant $\mathcal{B}_0$, which will be introduced later (see Remark 4.31). In Section 4.7.1.1, we will show that this functional equation has a single solution H. On the other hand,

$$-\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} (\partial_\nu W_\varepsilon)(v - u_\varepsilon)\,\mathrm{d}S_x = A_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} H(v)(v - u_\varepsilon)\,\mathrm{d}S_x,$$

where $A_\varepsilon = -\partial_\nu w_\varepsilon|_{\frac{\varepsilon}{4}\partial B_1}$. Since these integrals are now over non-critical balls, we prove through the averaging result described above that

$$A_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} H(v)(v - u_\varepsilon)\,\mathrm{d}S_x \to \mathcal{A}_0 \int\limits_\Omega H(v)(v-u)\,\mathrm{d}x.$$

The constant $\mathcal{A}_0$ is related with the capacity, as we will explain later (see Remark 4.31). Joining this information, as $\varepsilon \to 0$, we will show that (for any good test function v)

$$\int\limits_\Omega \nabla v \nabla(v-u)\,\mathrm{d}x + \mathcal{A}_0 \int\limits_\Omega H(v)(v-u)\,\mathrm{d}x \geq \int\limits_\Omega f(v-u)\,\mathrm{d}x.$$

This is how we have that the new reaction term is given by $\mathcal{H}(u) = \mathcal{A}_0 H(u)$, with H satisfying (1.15). It is, in general, different from σ. Moreover, there are some subcases which arise according to the different values of $\beta(\varepsilon)$ (see Table 1.1 below). We will obtain many properties of H later. As a first property, note that taking a derivative in s in (1.15) we recover the estimate

$$H'(s) = \frac{\sigma'(s - H(s))}{(\lim_{\varepsilon\to 0}\frac{B_\varepsilon}{\beta(\varepsilon)}) + \sigma'(s - H(s))} \in [0,1].$$

This function H is always non-decreasing and Lipschitz continuous, and there exists a universal bound of H'.

When G_0 is not a ball, then the choice of W_ε is more involved. We will get back to this in Section 3.1.5.3.

1.6 A literature review

First, we want to point out the classical references [31, 239] and some more modern presentations in [76] and [256]. Most of the classical papers in homogenization refer to lecture notes by Luc Tartar [254], but they can be difficult to access. Those notes apparently led to [209].

Before indicating, in a detailed form, many of the papers in the literature dealing with homogenization processes giving rise to some strange terms, perhaps, it is a good place to make mention of a general (long but surely far to be complete) list of books, by chronological order, dealing with homogenization methods (we will not collect here any of the many books dealing with proceedings of international conferences on homogenization). Very few fields of mathematics have exhibited such an intensive development in so few years. Among the pioneering books we could mention [39, 201] (which already presents the phenomenon of the appearance of strange terms), [184, 31, 239, 194, 18, 15, 135, 214, 238, 175, 212, 248] (already considering the occurrence of strange terms), [213, 165, 222, 82, 76, 4, 207, 63, 201] (enlarged English version of [202]) and [67, 204, 256, 264, 179, 74, 33]. See also [44].

We point out that the occurrence of a strange term for a critical scale is discussed in detail at least in the books [201, 248, 67, 74] and [33].

In the rest of this section we will refer to some specific references dealing with the three types of problems mentioned in Section 1.3 giving unity to this book.

1.6.1 Particles over the whole domain

Big particles $a_\varepsilon \sim \varepsilon$

In the case $p = 2$ the presence of an effective diffusion has been known since the 1970s (see [242, 241, 19, 17, 31]). It is not difficult to recover via asymptotic expansion. A very nice presentation can be found in [76]. The addition of boundary conditions on the holes (which is independent of the effective diffusion, as we will see below) appears in the linear setting [83] (Dirichlet boundary condition) and [77] (linear σ), later for the obstacle problem in [89] and in the semilinear setting [84]. In [206] the case of two different big particles with different regular kinetics functions σ was considered. The case $a_\varepsilon \sim \varepsilon$, $p \neq 2$ and $\sigma = 0$ was studied in [128] (see also the references therein).

Subcritical particles $a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$

In this setting the work has been long but incremental:

– $p = 2$. First the homogeneous Neumann case was studied [87], followed by the linear reaction in [218]. The case of σ nonlinear was first studied by Goncharenko in [162] for dimension 3. This paper is quite singular since it is the first known appearance of a functional equation for the effective reaction H for the critical scale. Later this work was extended by [270, 268, 173]. The reader may find the obstacle problem in [89]. We also refer the reader to previous work by Kaizu [178, 177].
– $p \in (2, n)$. This range was covered in [226] and [245], and later for Signorini type problems in [106].
– $p \in (1, 2)$. This range was developed in [228].

- $p \in (1, n)$. A unified approach for the whole range was presented in [110].
- $p = n$. This case was studied in [158].
- $p > n$. The range without critical scale was developed in [114].

Critical particles $a_\varepsilon \sim a_\varepsilon^\star$

That is the more interesting case.

- $p = 2$. As a first result in the literature we find [169], for the case of Dirichlet boundary conditions (see also [168] and [170]). The case of the obstacle problem was treated in [16]. Hruslov's work on the Dirichlet boundary conditions was later improved in the famous papers [80, 81] (see the English translation in [79]), which introduced the notion of "strange term." At the same time [218] studied σ linear, and [162] studied σ nonlinear and $N = 3$. This is the first case where a functional equation for the strange term appears. This equation, which only holds true for balls, reads
$$H(s) = C\sigma\big(s - H(s)\big), \quad \forall s \in \mathbb{R}. \tag{1.16}$$
We will discuss later the value of the constant. Further improvement of the dimension and the nature of the boundary conditions when G_0 is a ball can be found in [270, 172, 268]. The obstacle problem was studied in [89].
Again, we point out parallel work of Kaizu [178, 177] in this direction. He gives no characterization of the term H.
The case of G_0 not a ball was studied in [116]. The surprising result is that there is no equation (1.16), but rather H is recovered from a capacity type problem. It was later generalized in [269] to include dependence on x of σ and g^ε as (1.2).
- The case $p \in (2, n)$ was discussed in [245].
- The case $p \in (1, n)$ was discussed in [112, 111]. A result for Dirichlet boundary conditions for systems can be found in [11], with a proof based on Γ-convergence.
- The case $p = n$ can be found in [229].

Supercritical case $a_\varepsilon \ll a_\varepsilon^\star$

As we will see, it is not difficult to show that in this case the reaction term vanishes (see (1.13)).

The results for this case appear usually alongside one of the previous cases, but it is not worth showing the original proof. The case $p = 2$ can be found in [268], and here we extend the same philosophy for $p \in (1, n)$.

1.6.2 Particles over a manifold

The case of particles along a manifold is possibly the least studied case. In this direction we refer to [54, 156, 151, 161, 159, 160, 195, 196, 273, 157] for further references.

[273] studies variational inequalities for the biharmonic operator; [54, 159, 160, 195, 196] consider the Laplace operator and linear problems; [54] contains an extra advection term related with the flow velocity; [156, 151] consider variational inequalities for the Laplace operator for certain $\beta(\varepsilon)$ and a_ε; [196] considers a boundary value problem for the p-Laplacian for the another particular case. In [157] the authors deal with the p-Laplace case with the Signorini boundary condition and all a_ε and $\beta(\varepsilon)$.

1.6.3 Particles over the boundary

This kind of problem was first studied in [240, 95, 215]. In the subcritical setting we have the work of Chechkin [64] that deals with linear σ. In [143, 213] the authors deal with Dirichlet boundary conditions, in the critical and non-critical settings.

The reader will find results in the subcritical setting and $p = 2$ in [65] (where there is a complete asymptotic expansion, see also [25]), [141] (for the parabolic case and σ linear), [66] (where $\sigma(x, u) = a(x)u$) and [223] (Signorini boundary condition), [132] (general elliptic operator in a finite planar strip perforated by small holes along a curve).

In the critical setting, some relevant works are [272] (Signorini problem for $p = 2$ and $n = 3$), [106, 115] (where $p = 2 \le n$ and G_0 is a ball), [231] ($p = 2 < n$ and G_0 a general shape) and [230] (the case $p = n$ for G_0 a ball). The case of dynamical boundary conditions can be found in [107].

The eigenvalue problem in $n = 2$ for Dirichlet and Neumann boundary conditions was studied in [144, 40, 43]. In [246] the case of transport terms is considered. Similar results with the Steklov boundary condition can be found in [145, 68, 70]. The elasticity equation was studied in [197, 55, 198, 198]. For the case where the particles are replaced by strips on a cylinder we refer to [41, 42].

1.7 Novelties

As already mentioned, besides providing a unified approach to this subject, which as pointed out above is spread across a vast literature, we will provide in this book some new results which are not (to the best of our knowledge) available elsewhere. We list now some of the main novelties of our book:
– We write a_ε and $\beta(\varepsilon)$ in all the cases, removing the usual structural restrictions.
– We give an intuition of the appearance of the critical scale, when $p < n$.
– In Chapter 1 we give a very detailed but clear overview of the approaches to the different cases and provide the most complete literature review of the subject available in any of the papers.
– In Chapter 2 we give a very detailed introduction to the inequality formulations that were used in the papers by some authors.

- In Chapter 3 we detect the critical value τ_ε of the estimate in Lemma 3.6 in the different cases $p < n$, $p = n$, $p = n$ and each geometrical setting, which allows us to precisely write the scales a_ε such that the averaging lemmas from S_ε to the corresponding set hold. When this lemma fails, we find the critical scale. We provide new estimates and properties on the strange term and on the auxiliary function $\widehat{w}$ when G_0 is not a ball, showing that there is a uniform Lipschitz continuity constant linked to the capacity on G_0. We show, for the first time in the literature, that as the nonlinearity approaches the maximal monotone graph for the Dirichlet boundary condition, the respective strange term converges. Finally, we explain the connection between the case of n-dimensional particles and $(n-1)$-particles. In each of three settings, we provide details on the uniform trace theorems and averaging lemmas which are difficult to find in the papers.
- In Chapter 4 we give a very detailed explanation of how the strange term appears and why it has its particular form. We provide a uniform presentation of the averaging in Theorem 4.5, where the right-hand is given in all situations according to the relation between p and n. Usually, in the previous literature, only one case is presented isolated from other possibilities. We also point out that the smilingly surprising coefficients in the equation have a very natural explanation related to the p-capacity. We also prove a new convergence result when the data are in L^1.
- In Chapter 5 we provide a detailed explanation of the relation between the weak and strong formulations of the term of “jump across a manifold.”
- In Chapter 6 we point out the behavior in the case $a_\varepsilon \sim a_\varepsilon^\star$ and $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ we recover in the limit of the behavior of the Dirichlet case, as in the rest of the cases for σ. We also point out that a Dirichlet boundary condition, in the critical case, passes to be (after the homogenization limit) a Robin type boundary condition.
- In Appendix A we show how the elliptic results translate directly to the parabolic case. This represents, as it is common in parabolic problems, a decoupling between the time and spatial variables.
- In Appendix B we provide many new details on the problem with dynamic boundary condition. We provide new estimates on the strange term for critical-size homogenization.
- In Appendix C, on random particles, we provide a new application of the main result for the Signorini type boundary condition at the critical scale, showing that for negative functions f the homogenized solution becomes negative in some suitable regions of the domain.

2 Preliminary results and comments

The internal core philosophy of this text is that a unified approach can be given to the treatment of different types of conditions on the boundaries of the particles applying maximal monotone graphs (which are described below). However, the treatment of such operators is difficult in the framework of the homogenization procedure. Thus, we will only provide a complete proof of the homogenization in the most general setting with respect to σ for the simplest case with respect to the shape of the particles. For the rest of the cases we will show that the homogenization result is true when σ is smooth (or even Hölder continuous) and that the rigorous passage to the limit and the *strange term* are well defined and *nice* for the general shape case.

In order to give the most general setting, let us recall some classical results from the literature. As general references for elliptic partial differential equations (PDEs), we refer the reader to the textbooks [45, 138, 148].

2.1 Maximal monotone graphs. *A common roof*

In some contexts (when, for instance, $p = 2$), it is desirable to formulate the nonlinear Robin type condition

$$\frac{\partial u_\varepsilon}{\partial \nu} + \beta(\varepsilon)\sigma(u_\varepsilon) = 0 \quad \text{on } S_\varepsilon, \tag{2.1a}$$

where ν is the unitary outward vector to S_ε, in a general framework which also includes, as particular cases, other types of boundary conditions as, for instance, the Dirichlet boundary condition

$$u_\varepsilon = 0 \quad \text{on } S_\varepsilon, \tag{2.1b}$$

or even the case of Signorini type boundary condition (also known as boundary obstacle problem) with a given non-decreasing function σ_0,

$$\begin{cases} u_\varepsilon \geq 0 & \text{on } S_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma_0(u_\varepsilon) \geq 0 & \text{on } S_\varepsilon, \\ u_\varepsilon(\partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma_0(u_\varepsilon)) = 0 & \text{on } S_\varepsilon. \end{cases} \tag{2.1c}$$

There is a unified presentation of such a goal leading to the respective weak formulations (even for the general case $p > 1$). The idea is to use the framework of *maximal monotone operators* (see, for instance, [47, 48, 192, 56] and the more recent exposition made in [22]).

Remark 2.1. Later we will be able to add a term g^ε to the right-hand side of these conditions (see, e. g., Theorem 2.13).

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Definition 2.2. Let X be a Banach space and $A : X \to \mathcal{P}(X')$ (as usual, $\mathcal{P}(X')$ denotes the set of all the subsets of X'). We say that A is a monotone operator if, for all $x, \hat{x} \in X$,

$$\langle x - \hat{x}, \xi - \hat{\xi} \rangle_{X \times X'} \geq 0 \quad \forall \xi \in A(x), \hat{\xi} \in A(\hat{x}).$$

We define the domain of A as $\mathrm{dom}(A) = \{x \in X : A(x) \neq \emptyset\}$. Here $\emptyset$ is the empty set. We say that A is a maximal monotone operator if there is no other monotone operator $\widetilde{A}$ such that $\mathrm{dom}(A) \subset \mathrm{dom}(\widetilde{A})$ and $A(x) \subset \widetilde{A}(x)$ for all $x \in X$.

Examples 2.3. Some examples of monotone operators $A = \sigma$, when $X = \mathbb{R}$, are:
1. Any continuous non-decreasing functions $\sigma : \mathbb{R} \to \mathbb{R}$.
2. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be discontinuous and let $(x_n)_n$ be its set of discontinuity points. Then, the function

$$\widetilde{\sigma}(x) = \begin{cases} \sigma(x) & x \in \mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}, \\ [\sigma(x_n^-), \sigma(x_n^+)] & x = x_n \text{ for some } n \in \mathbb{N} \end{cases}$$

   is a maximal monotone operator. In this framework, maximal monotone operators in $\mathbb{R}$ can be seen as maximal monotone graphs of $\mathbb{R}^2$, and vice versa.
3. The Dirichlet boundary condition (2.1b) can be written in terms of maximal monotone operators as (2.1a) with

$$\sigma_{\mathrm{D}}(x) = \begin{cases} \emptyset & x < 0, \\ \mathbb{R} & x = 0, \\ \emptyset & x > 0. \end{cases} \tag{2.2}$$

   We call this graph the maximal monotone graph associated with the homogeneous Dirichlet boundary condition.
4. The Signorini boundary condition (2.1c) can be written formally as (2.1a) with

$$\sigma(x) = \begin{cases} \emptyset & x < 0, \\ (-\infty, 0] & x = 0, \\ \sigma_0(x) & x > 0. \end{cases} \tag{2.3}$$

Note also that if $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function and sublinear $|\sigma(u)| \leq C(1 + |u|)$, then its associated Nemytskii operator, which maps $u \in L^2(\Omega)$ to $\sigma(u) \in L^2(\Omega)$, is a maximal monotone operator in $X = L^2(\Omega)$. To be absolutely correct, when σ is multivalued we should write the above nonlinear Robin type boundary condition as

$$-\frac{\partial u_\varepsilon}{\partial \nu} \in \beta(\varepsilon)\sigma(u_\varepsilon) \quad \text{on } S_\varepsilon.$$

Nevertheless, for the sake of simplicity (and as an abuse of the notation) we will avoid such unusual expression. Of course, when σ fails to be continuous, the use of maximal monotone operators escapes the usual framework of classical solutions of PDEs.

Another advantage of maximal monotone operators is the simplicity to define their inverses. For $\sigma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$, we define its inverse in the sense of maximal monotone operators as the map $\sigma^{-1} : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$\sigma^{-1}(s) = \{x \in \mathbb{R} : s \in \sigma(x)\}.$$

It is a trivial exercise to show that σ^{-1} is also a maximal monotone operator.

Subdifferentials

Using maximal monotone graphs introduces some degree of delicateness in the treatment of the equations. The good thing is that the variational theory can still be applied. The main idea of the variational formulation is that the energy functional is typically convex, and thus it admits a minimizer. This philosophy is preserved also when the boundary condition involves a maximal monotone graph. This is already a well-known theory which was developed in many articles and books (see, e. g., [47, 48, 133, 134, 22]).

Let us see the connection. The main idea is that a smooth function is nondecreasing if and only if its primitive is convex. Hence, an easy way to construct a maximal monotone graph is by using a convex function. Instead of the usual derivative we need to generalize it by introducing a possibly *multivalued* concept of generalized derivative.

Definition 2.4. Let $\Psi : X \to (-\infty, +\infty]$ be convex and lower semicontinuous. Let $\mathrm{Dom}(\Psi) = \{u \in X : \Psi(u) < +\infty\}$. We define its subdifferential, $\partial\Psi$, at $x \in X$ as

$$\partial\Psi(x) = \{\xi \in X' : \langle \xi, y - x\rangle \le \Psi(y) - \Psi(x), \forall y \in X\}.$$

It is well known (see, e. g., [134, 22]) that if $\Psi : \mathbb{R} \to (-\infty, +\infty]$ is convex, then $\partial\Psi$ is a maximal monotone graph, and vice versa, if $\sigma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a maximal monotone operator, then there exists a convex function $\Psi : \mathbb{R} \to (-\infty, +\infty]$ such that $\sigma = \partial\Psi$. Since Ψ is convex it is lower continuous and bounded below, hence up to translation we can always assume that $\Psi \ge 0$. We will keep this assumption for the remainder of the book (except in some few cases dealing with non-monotone functions σ).

Example 2.5.

1. For a reaction of order r, $\sigma(v) = [v_+]^r$, we have $\Psi(v) = \frac{[v_+]^{r+1}}{r+1}$.
2. For a reaction of zero order $\Psi(v) = [v_+]$.
3. For the graph of Dirichlet boundary conditions $\Psi(0) = 0$ and $\Psi(v) = +\infty$ if $v \neq 0$.
4. For the Signorini boundary conditions $\Psi(v) = +\infty$ if $v \le 0$ and $\Psi(v) = \int_0^v \sigma_0(s)\,ds$ if $v > 0$.

Remark 2.6. The term g_ε can be introduced as an x-dependence in σ by considering $\widetilde{\sigma}(x, u) = \sigma(u) - g(x)$. In this setting $\widetilde{\sigma}(x, \cdot)$ is an m. m. g. for every x fixed, and we can construct $\widetilde{\Psi}(x, u) = \Phi(u) - g(x)u$ convex for each x fixed (see, e. g., [134, 22]).

Remark 2.7. Since we always assume that $\sigma(0) \ni 0$, its primitive Ψ will always be such that $\Psi(0) = 0$ and $\Psi \geq 0$.

2.2 Variational formulation of the problems

Thus, if $\sigma = \partial\Psi$, all the different boundary conditions listed above allow a common variational formulation given by the minimization of the energy functional

$$J_\varepsilon(u_\varepsilon) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p \, dx + \beta(\varepsilon)\int_{S_\varepsilon} \Psi(u_\varepsilon)\, dS - \int_{\Omega_\varepsilon} f u_\varepsilon \, dx - \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon u_\varepsilon \, dS. \tag{2.4}$$

More details on the regularity on the external data will be given later (see Theorem 2.13 below). Actually, to be more correct, we have to work with

$$\widetilde{J}_\varepsilon(u_\varepsilon) = \begin{cases} J_\varepsilon(u_\varepsilon) & \text{if } u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega) \text{ and } u_\varepsilon(x) \in \operatorname{dom}(\Psi) \text{ for a. e. } x \in S_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

This energy functional is convex. When the particles are contained on $\partial\Omega$ see Remark 2.8.

Remark 2.8. In the case of particles in the interior of Ω (either on the whole space or on a manifold), we minimize over the energy space $X = W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. However in the case of particles on the boundary we work on the energy space $X = W^{1,p}(\Omega, (\partial\Omega)^+)$ and $\widetilde{J}(u_\varepsilon)$ is suitably modified. Note that there is a trace operator $T : X \to L^p(S_\varepsilon)$ in the cases of particles on the whole domain and on an interior manifold (see, e. g., [45]). By a usual abuse of the notation, we are identifying $T(u)$ with u. For the case of particles on the boundary the good trace operator is operating from X into $L^p(G_\varepsilon)$.

The equivalence between the weak and the variational formulation will be recalled later.

Lemma 2.9 ([134]). *Let X be a reflexive Banach space, let $J : X \to (-\infty, +\infty]$ be a convex functional and let $A = \partial J : X \to \mathcal{P}(X')$ be its subdifferential. Then the following conditions are equivalent:*
(a) *u is a minimizer of J;*
(b) *$u \in \operatorname{dom}(A)$ and $0 \in Au$.*

If either holds, then:
(c) *For every $v \in \operatorname{dom}(A)$ and $\xi \in Av$*

$$\langle \xi, v - u \rangle \geq 0. \tag{2.5}$$

Furthermore, assume that J is Gâteaux-differentiable on X and A is continuous on X. Then condition (c) *is also equivalent to condition* (a).

Remark 2.10. Naturally, if there is uniqueness of u satisfying (c), then conditions (a)–(c) are also equivalent.

Remark 2.11. When $p = 2$, one should not confuse condition (c) with the – very similar – formulation in Stampacchia's theorem (see, e. g., [45, Theorem 5.6]). For a bilinear form a and a linear function G the Stampacchia formulation is

$$a(u, v - u) \geq G(v - u),$$

for all v in the correspondent space, whereas in formulation (c) we have $a(v, v - u)$. In the nonlinear setting we point also to the work by Brézis and Sibony [51].

2.2.1 Formulation as variational inequalities

From Lemma 2.9 we find some equivalent expressions of the weak solution of our problems (see expression (a) below) which are called variational inequalities. Some other equivalent expressions, which will be very useful later, are given in the next proposition. These formulations are particularly useful in the treatment of quasilinear equations and also when σ is a multivalued maximal monotone graph.

Proposition 2.12. *Let $p > 1$, $\sigma = \partial\Psi$, and let u_ε be a minimizer of J_ε over the energy space:*
- $X = W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ *for the case of particles in the interior of* Ω*;*
- $X = W^{1,p}(\Omega, (\partial\Omega)^+)$ *for the case of particles on the boundary.*

Then, u_ε satisfies the following three characterizations:

(a) *For all $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ (respectively $v \in W^{1,p}(\Omega, (\partial\Omega)^+)$)*

$$\begin{aligned}\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int\limits_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon))\,\mathrm{d}S \\ \geq \int\limits_{\Omega_\varepsilon} f(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int\limits_{S_\varepsilon} g^\varepsilon(v - u_\varepsilon)\,\mathrm{d}S. \end{aligned} \tag{2.6}$$

(b) *For $v \in W^{1,\infty}(\Omega_\varepsilon, \partial\Omega)$ (respectively $v \in W^{1,\infty}(\Omega, (\partial\Omega)^+)$) and any $\xi \in L^1(S_\varepsilon)$ such that $\xi(x) \in \sigma(v(x))$ for a. e. $x \in S_\varepsilon$*

$$\begin{aligned}\int\limits_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int\limits_{S_\varepsilon} \xi(v - u_\varepsilon)\,\mathrm{d}S \\ \geq \int\limits_{\Omega_\varepsilon} f(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int\limits_{S_\varepsilon} g^\varepsilon(v - u_\varepsilon)\,\mathrm{d}S. \end{aligned} \tag{2.7}$$

(c) *For all* $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ *(respectively* $v \in W^{1,p}(\Omega, (\partial\Omega)^+)$*)*

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \big(\Psi(v) - \Psi(u_\varepsilon)\big)\,\mathrm{d}S$$
$$\geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon(v - u_\varepsilon)\,\mathrm{d}S. \tag{2.8}$$

Proof. Let us prove first (2.8). Consider the map $x \in \mathbb{R}^n \mapsto |x|^p \in \mathbb{R}$. It is a convex map with derivative $D|x|^p = p|x|^{p-2}x$. Hence, for $a, b \in \mathbb{R}^n$ we have

$$|a|^p - |b|^p \geq p|b|^{p-2}b \cdot (a - b).$$

Hence

$$|b|^p - |a|^p \leq p|b|^{p-2}b \cdot (b - a).$$

Considering $b = \nabla v$ and $a = \nabla u_\varepsilon$ we have

$$|\nabla v|^p - |\nabla u_\varepsilon|^p \leq p|\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon).$$

Taking into account this fact and that u_ε is a minimizer of J_ε we have

$$\begin{aligned}
0 &\leq J_\varepsilon(v) - J_\varepsilon(u_\varepsilon)\\
&= \frac{1}{p}\int_{\Omega_\varepsilon} \big(|\nabla v|^p - |\nabla u_\varepsilon|^p\big)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \big(\Psi(v) - \Psi(u_\varepsilon)\big)\,\mathrm{d}S\\
&\quad - \int_{\Omega_\varepsilon} f(v - u_\varepsilon)\,\mathrm{d}x - \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon(v - u_\varepsilon)\,\mathrm{d}S\\
&\leq \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \big(\Psi(v) - \Psi(u_\varepsilon)\big)\,\mathrm{d}S\\
&\quad - \int_{\Omega_\varepsilon} f(v - u_\varepsilon)\,\mathrm{d}x - \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon(v - u_\varepsilon)\,\mathrm{d}S.
\end{aligned}$$

Thus, we have obtained (2.8).

Let us assume that u_ε is a minimizer of J_ε. Considering characterization Lemma 2.9(c) we have

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla(v-u_\varepsilon)\cdot\nabla(v-u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \xi(v-u_\varepsilon)\,\mathrm{d}S \geq \int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon(v-u_\varepsilon)\,\mathrm{d}S,$$

for some ξ such that $\xi(x) \in \sigma(v(x))$ a. e. in S_ε. Since Ψ is convex and $\sigma = \partial\Psi$ we have

$$\Psi(v) - \Psi(u_\varepsilon) \geq \xi(v - u_\varepsilon).$$

Hence, (2.7) is proved. In order to prove (2.6), we can repeat the same argument from the usual weak formulation. Equation (2.7) can be obtained by considering the Brézis–Sibony characterization of the weak formulation of (1.1) (see Lemma 1.1 of [51] or Theorem 2.2 of Chapter 2 in [192]). □

2.2.2 Existence and uniqueness of solutions

The aim of this section is to prove the following.

Theorem 2.13. *Let $\varepsilon > 0, p > 1, f \in L^{p'}(\Omega_\varepsilon)$ and $g^\varepsilon \in L^{p'}(S_\varepsilon)$. Then, there exists a unique $u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ (respectively in $W^{1,p}(\Omega, (\partial\Omega)^+)$) satisfying* (2.8).

To prove the existence of solutions we can use convex analysis to show the existence of minimizers of J_ε (see, e. g., [133, 134, 22]), or by applying an abstract result in a very general framework. To state in its broadest generality we introduce (as in [46]) the following definition.

Definition 2.14. Let V be a reflexive Banach space. We say that $A : V \to V'$ is a *pseudo-monotone operator* if it is bounded and it has the following property: if $u_j \rightharpoonup u$ in V and

$$\limsup_{j\to+\infty} \langle A(u_j), u_j - u\rangle \le 0,$$

then, for all $v \in X$,

$$\liminf_{j\to+\infty} \langle A(u_j), u_j - v\rangle \ge \langle A(u), u - v\rangle.$$

We can now recall the following well-known result.

Theorem ([46], also Theorem 8.5 in [192]). *Let $A : V \to V'$ be a pseudo-monotone operator and let φ be a proper convex function lower semicontinuous such that*

$$\begin{cases} \text{there exist } v_0 \text{ such that } \varphi(v_0) < \infty \text{ and} \\ \frac{(Au, u - v_0) + \varphi(u)}{\|u\|} \to \infty, \text{ as } \|u\| \to \infty. \end{cases}$$

Then, for $f \in V'$, there exists a solution of the problem

$$\big(A(u) - f, v - u\big) + \varphi(v) - \varphi(u) \ge 0, \quad \forall v \in V.$$

In this setting A is simply given by the p-Laplacian, and the hypotheses are easily checked. The uniqueness of solutions is a consequence of the strict monotonicity of the operator $A + \partial\Psi$ (see, e. g., [192, 58]). For the case of smooth nonlinear terms we refer the reader to the monographs [186, 185]. We point out that some other comments and references on the existence and uniqueness of solutions will be presented later when dealing with non-monotone functions $\sigma(s)$ (Section 2.6.2), when considering L^1 data (Section 4.9.1) and when dealing with the spectral problem (Section 4.9.6).

2.3 The critical scaling of the reaction constant β

The scaling constant $\beta(\varepsilon)$ is relevant to determine the reaction term in the effective problem. Although the different problems under consideration are rather different, they all include in their energy a term coming from the reaction of the form

$$\beta(\varepsilon) \int_{S_\varepsilon} \Phi(u_\varepsilon)\, \mathrm{d}S,$$

for different choices of the function Φ. The more favorable case in the study of this term arises when the integrand is constant, and then

$$\beta(\varepsilon) \int_{S_\varepsilon} \mathrm{d}S = \beta(\varepsilon)|S_\varepsilon|.$$

In order for this kind of term to scale properly, we introduce the definition of the critical scaling

$$\beta^\star(\varepsilon) = |S_\varepsilon|^{-1}.$$

We have

$$\beta^\star(\varepsilon) \int_{S_\varepsilon} \mathrm{d}S = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \mathrm{d}S,$$

so this is the usual average operator. For functions $g \in \mathcal{C}(\overline{\Omega})$, it is clear that

$$\left| \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} g\, \mathrm{d}S \right| \leq \|g\|_\infty.$$

Hence, up a to a subsequence, there is a limit of $\beta^\star(\varepsilon) \int_{S_\varepsilon} g\, \mathrm{d}S$. Thus

$$\beta(\varepsilon) \int_{S_\varepsilon} g\, \mathrm{d}S = \frac{\beta(\varepsilon)}{\beta^\star(\varepsilon)} \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} g\, \mathrm{d}S \to \lim_{\varepsilon\to 0} \frac{\beta(\varepsilon)}{\beta^\star(\varepsilon)} \lim_{\varepsilon\to 0} \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} g\, \mathrm{d}S.$$

This is why the scaling constant

$$\beta^0 = \lim_{\varepsilon\to 0} \frac{\beta(\varepsilon)}{\beta^\star(\varepsilon)}$$

is relevant. If $\beta^0 = 0$, the only possibility in the limit (as $\varepsilon \to 0$) is to lose the reaction term in the homogenized equation.

We will see that if $\beta^\star(\varepsilon) \ll \beta(\varepsilon)$ (i. e., $\beta^0 = +\infty$) and $\sigma' > 0$, then the reaction term is dominant, and in the limit we have no diffusion. This last case spoils the narrative, and we will only discuss it in Section 4.8. The relevant case we will be interested in is $\beta \sim \beta^\star$ (i. e., $\beta^0 \in (0, +\infty)$).

Remark 2.15. Note that with this choice, we have

$$\beta^\star(\varepsilon) = |S_\varepsilon|^{-1} = |\Upsilon_\varepsilon|^{-1}|a_\varepsilon \partial G_0|^{-1}.$$

The last measure will scale like the dimension of $|\partial G_0|$.

This choice of $\beta^\star(\varepsilon)$ has a significant advantage. Since we are now averaging, the embeddings of $L^r(S_\varepsilon)$ in $L^s(S_\varepsilon)$ are uniform in ε.

Lemma 2.16. *Let* $1 < r < s$. *Then,*

$$\Big(\beta^\star(\varepsilon)\int_{S_\varepsilon} |u|^r \,\mathrm{d}S\Big)^{\frac{1}{r}} \le \Big(\beta^\star(\varepsilon)\int_{S_\varepsilon} |u|^s \,\mathrm{d}S\Big)^{\frac{1}{s}}.$$

Proof. The proof is a simple application of Hölder's theorem for $q = \frac{s}{r}$. We have

$$\int_{S_\varepsilon} |u|^r \,\mathrm{d}S \le \Big(\int_{S_\varepsilon} |u|^s \,\mathrm{d}S\Big)^{\frac{r}{s}} \Big(\int_{S_\varepsilon} 1^{\frac{s}{s-r}} \,\mathrm{d}S\Big)^{\frac{s-r}{s}} = |S_\varepsilon|^{\frac{s-r}{s}} \Big(\int_{S_\varepsilon} |u|^s \,\mathrm{d}S\Big)^{\frac{r}{s}} = \beta^\star(\varepsilon)^{-\frac{s-r}{s}} \Big(\int_{S_\varepsilon} |u|^s \,\mathrm{d}S\Big)^{\frac{r}{s}}.$$

□

2.4 Uniform approximation results

The case of $\sigma \in \mathcal{C}(\mathbb{R})$, non-decreasing and $\sigma(0) = 0$ and the case $\beta(\varepsilon)\beta^*(\varepsilon)^{-1} \to 0$ can be treated thanks to some uniform approximation arguments. Assume for the moment (we will prove it later) that there exists $C > 0$ independent of ε such that

$$\beta^\star(\varepsilon)\int_{S_\varepsilon} |u|^p \,\mathrm{d}x \le C \int_{\Omega_\varepsilon} |\nabla u|^p \,\mathrm{d}x, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega). \tag{2.9}$$

Let u_ε and $\bar{u}_\varepsilon$ be, respectively, the solution of our problem with kinetic functions σ and $\bar{\sigma}$. Using them as test functions in the weak formulation of our problem we get

$$\int_{\Omega_\varepsilon} \big(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla \bar{u}_\varepsilon|^{p-2}\nabla \bar{u}_\varepsilon\big) \cdot \nabla(u_\varepsilon - \bar{u}_\varepsilon) \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \big(\sigma(u_\varepsilon) - \bar{\sigma}(\bar{u}_\varepsilon)\big)(u_\varepsilon - \bar{u}_\varepsilon) \,\mathrm{d}S = 0.$$

Adding, subtracting and using the monotonicity of σ we recover

$$\begin{aligned}&\int_{\Omega_\varepsilon} \big(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla \bar{u}_\varepsilon|^{p-2}\nabla \bar{u}_\varepsilon\big) \cdot \nabla(u_\varepsilon - \bar{u}_\varepsilon) \,\mathrm{d}x \\ &\quad = -\beta(\varepsilon)\int_{S_\varepsilon} \big(\sigma(u_\varepsilon) - \bar{\sigma}(\bar{u}_\varepsilon)\big)(u_\varepsilon - \bar{u}_\varepsilon) \,\mathrm{d}S\end{aligned}$$

$$\begin{aligned}
&= -\beta(\varepsilon) \int_{S_\varepsilon} (\sigma(u_\varepsilon) - \sigma(\bar{u}_\varepsilon))(u_\varepsilon - \bar{u}_\varepsilon)\, \mathrm{d}S \\
&\quad + \beta(\varepsilon) \int_{S_\varepsilon} (\sigma(\bar{u}_\varepsilon) - \bar{\sigma}(\bar{u}_\varepsilon))(u_\varepsilon - \bar{u}_\varepsilon)\, \mathrm{d}S \\
&\le \beta(\varepsilon) \int_{S_\varepsilon} (\sigma(\bar{u}_\varepsilon) - \bar{\sigma}(\bar{u}_\varepsilon))(u_\varepsilon - \bar{u}_\varepsilon)\, \mathrm{d}S.
\end{aligned}$$

Thus

$$\int_{\Omega_\varepsilon} (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla \bar{u}_\varepsilon|^{p-2} \nabla \bar{u}_\varepsilon) \cdot \nabla(u_\varepsilon - \bar{u}_\varepsilon)\, \mathrm{d}x \le \beta(\varepsilon) \|\sigma - \bar{\sigma}\|_\infty \int_{S_\varepsilon} |u_\varepsilon - \bar{u}_\varepsilon|\, \mathrm{d}S.$$

Using Lemma 2.16 we have

$$\int_{\Omega} (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla \bar{u}_\varepsilon|^{p-2} \nabla \bar{u}_\varepsilon) \cdot \nabla(u_\varepsilon - \bar{u}_\varepsilon)\, \mathrm{d}x \le \beta(\varepsilon) \beta^\star(\varepsilon)^{-1} \|\sigma - \bar{\sigma}\|_\infty \left(\beta^\star(\varepsilon) \int_{S_\varepsilon} |u_\varepsilon - \bar{u}_\varepsilon|^p\, \mathrm{d}S \right)^{\frac{1}{p}}.$$

Case $p \ge 2$

When $p \ge 2$ due to $(|b|^{p-2}b - |a|^{p-2}a) \cdot (b - a) \ge 2^{2-p}|b - a|^p$ (see [102, Lemma 4.10] or [190, Formula (I) in Chapter 10]) and (2.9) we recover

$$\|u_\varepsilon - \bar{u}_\varepsilon\|^{p-1}_{W^{1,p}(\Omega_\varepsilon)} \le C\beta(\varepsilon)\beta^\star(\varepsilon)^{-1} \|\sigma - \bar{\sigma}\|_\infty. \tag{2.10}$$

Case $1 < p < 2$

In this setting we only have (see the abovementioned references) the weaker inequality

$$(|b|^{p-2}b - |a|^{p-2}a) \cdot (b - a) \ge (p - 1)|b - a|^2 (1 + |a|^2 + |b|^2)^{\frac{p-2}{2}}.$$

We recover

$$\begin{aligned}
&(p - 1) \int_{\Omega_\varepsilon} |\nabla u_\varepsilon - \nabla \bar{u}_\varepsilon|^2 (1 + |\nabla u_\varepsilon|^2 + |\nabla \bar{u}_\varepsilon|^2)^{\frac{p-2}{2}}\, \mathrm{d}x \\
&\quad \le C\beta(\varepsilon)\beta^\star(\varepsilon)^{-1} \|\sigma - \bar{\sigma}\|_\infty (\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \|\nabla \bar{u}_\varepsilon\|_{L^p(\Omega_\varepsilon)}).
\end{aligned} \tag{2.11}$$

2.5 The range $\beta(\varepsilon) \not\sim \beta^\star(\varepsilon)$

We present an intuitive argument of what happens in this case. The rigorous details must be analyzed for each case (see, e. g., Section 4.8). The philosophy is that when $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$, the reaction vanishes in the limit. This is shown by taking $\bar{\sigma} = 0$. Since

$\beta(\varepsilon)\beta^\star(\varepsilon)^{-1} \to 0$, we show that u_ε and $\bar{u}_\varepsilon$ share a limit. This is in contrast to the above-mentioned case; when $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ and $a_\varepsilon \gg a_\varepsilon^\star$, we will show that the reaction term is dominant and we simply end with $\sigma(u) = 0$. The critical case, as usual, is special.

2.6 Comments

The aim of the general theory is usually to cover all types of boundary conditions, and hence we aim to use maximal monotone operators. However, when we do not work in the most general setting in order to simplify the presentation, or due to technical difficulties, we usually assume that σ is Lipschitz continuous. We provide below some tricks that allow a direct extension of results from smooth σ to broader classes of functions.

2.6.1 Uniformly continuous σ

Consider a sequence of well-behaved functions σ_δ that converges to σ. The main idea is to show that the solutions associated to σ_δ, say $u_{\varepsilon,\delta}$, approximate uniformly the one with σ, i. e.,

$$\|u_\varepsilon - u_{\varepsilon,\delta}\|_X \le C\|\sigma - \sigma_\delta\|_\infty, \tag{2.12}$$

in some functional space X. As $\varepsilon \to 0$, $P_\varepsilon u_{\varepsilon,\delta} \rightharpoonup u_\delta$ in $W^{1,p}(\Omega)$, where u_δ is the solution of the corresponding limit problem with σ_δ as nonlinear kinetics. Furthermore, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$, and the uniform continuous dependence holds in the limit

$$\|u - u_\delta\|_X \le C\|\sigma - \sigma_\delta\|_\infty. \tag{2.13}$$

It is easy to show that as $\delta \to 0$, we have $u_\delta \rightharpoonup \widehat{u}$ in $W^{1,p}(\Omega)$, the solution of the corresponding limit equation. Taking limits as $\delta \to 0$ in (2.13), we deduce that $u = \widehat{u}$.

In order to make a selection of the right-hand side of (2.12) we first provide a lemma which is quite similar to the Yosida approximation of a given maximal monotone operator (see [48]).

Lemma 2.17. *Let σ be non-decreasing, uniformly continuous such that $\sigma(0) = 0$. Then, there exists σ_δ Lipschitz (furthermore, piecewise linear), non-decreasing and such that $\sigma_\delta(0) = 0$ and*

$$\sup_{x\in\mathbb{R}} |\sigma(x) - \sigma_\delta(x)| < \delta.$$

Proof. Since σ is uniformly continuous, for $\delta > 0$ fixed, let γ be small enough so that if $|x - y| < \gamma$, then $|\sigma(x) - \sigma(y)| < \delta$. Let σ_δ be the piecewise linear interpolation of the values of σ in $k\gamma/2$ for $k \in \mathbb{Z}$. The conclusion holds. □

Recovering (2.12) usually passes by sharp embeddings of the spaces $L^p(S_\varepsilon)$ in $L^q(S_\varepsilon)$, and applying the equation. For an example in terms of the interior problem we refer the reader to [110, Lemma 9].

2.6.2 Non-monotone σ

There are some relevant cases in the applications in which σ is non-monotone. This is the case, for instance, of the Langmuir–Hinshelwood kinetics and other examples mentioned in the Introduction. Let us show that there exists a small value $k_1 > 0$ such that if $\sigma' \geq -k_1$, the theory still works. The existence of solutions when σ is non-monotone was already shown in [52] (and the papers cited in Section 1.1). Here we will make only some considerations concerning the convergence of suitable approximations and consider only the case $p = 2$. First, for $a_\varepsilon \gtrsim a_\varepsilon^\star$, as we will show later, there exists a finite trace constant (see Lemma 4.2 for the case of particles over the whole domain)

$$C_0 = \sup_{0<\varepsilon<1} \sup_{\substack{v \in H^1(\Omega_\varepsilon, \partial\Omega) \\ v \neq 0}} \frac{\beta^*(\varepsilon) \int_{S_\varepsilon} |v|^2 \,\mathrm{d}S}{\int_{\Omega_\varepsilon} |\nabla v|^2 \,\mathrm{d}x}.$$

If $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$, then we take

$$k_1 < \frac{1}{C_0} \inf_{\varepsilon<1} \frac{\beta^*(\varepsilon)}{\beta(\varepsilon)}.$$

Therefore, if $a_\varepsilon \gtrsim a_\varepsilon^\star$, $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$ and $\sigma' \geq -k_1$, the following statements hold:

- $\int_{\Omega_\varepsilon} |\nabla v|^2 + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(v) v \geq C \int_{\Omega_\varepsilon} |\nabla v|^2$ for some $C > 0$.
- The operator with the boundary condition is monotone. In particular, for each $f \in L^2(\Omega)$ there is a unique $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ that satisfies

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi \,\mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(u_\varepsilon) \varphi \,\mathrm{d}S = \int_{\Omega} f \varphi \,\mathrm{d}x, \tag{2.14}$$

  for all $\varphi \in H^1(\Omega_\varepsilon, \partial\Omega)$. Furthermore, by using the mean value theorem for σ, it holds that

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla v \nabla (v - u_\varepsilon) \,\mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(v)(v - u_\varepsilon) \,\mathrm{d}S - \int_{\Omega_\varepsilon} f(v - u_\varepsilon) \,\mathrm{d}x \\
&\quad = \int_{\Omega_\varepsilon} |\nabla (v - u_\varepsilon)|^2 \,\mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \big(\sigma(v) - \sigma(u_\varepsilon)\big)(v - u_\varepsilon) \,\mathrm{d}S \\
&\quad = \int_{\Omega_\varepsilon} |\nabla (v - u_\varepsilon)|^2 \,\mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \sigma'\big(\eta_\varepsilon(x)\big) |v - u_\varepsilon|^2 \,\mathrm{d}S
\end{aligned}$$

$$\geq \int_{\Omega_\varepsilon} |\nabla(v - u_\varepsilon)|^2 \, dx - k_1 \frac{\beta(\varepsilon)}{\beta^\star(\varepsilon)} \beta^\star(\varepsilon) \int_{S_\varepsilon} |v - u_\varepsilon|^2 \, dS$$
$$\geq 0,$$

where $\eta_\varepsilon(x)$ is a function coming from the mean value theorem. Thus, the convergence result to the solution of the corresponding homogenized problem in this setting can be recovered uniformly from the theory for the case $\sigma' \geq 0$.

3 Estimates over one periodicity cell

As pointed out in Section 1.5.2, we need to characterize the limits as $\varepsilon \to 0$ of the type $\int_{S_\varepsilon} g$. Since S_ε is made up of disconnected elements, we can work with the integral in each one of them. Up to translation, S_ε is given as the repetition of $a_\varepsilon \partial G_0$. The particles we study can be n-dimensional (Sections 1.2.1.1 and 1.2.1.2) or $(n-1)$-dimensional (Section 1.2.2), and these situations need to be treated separately. This chapter is of a very technical nature but, as we will see later, it supplies very fine and useful results which will be crucial for the delicate proofs presented in Chapter 4 to 6.

3.1 Case of *n*-dimensional particles

Let us consider the cell $\varepsilon Y \setminus a_\varepsilon \overline{G_0}$ as seen in Figure 1.1. The aim of this section is to show:

1. There exists an extension operator P_ε that is able to "fill in" the particles (or holes) with suitable information, in a way that does not increase significantly the $W^{1,p}$ norm.
2. There exist some uniform Poincaré inequalities for the spaces $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.
3. There are suitable estimates on the trace operator over $a_\varepsilon \partial G_0$, of the type

$$\int_{a_\varepsilon \partial G_0} |u|^p \, dS \le C_1(\varepsilon) \int_{\varepsilon Y \setminus a_\varepsilon G_0} |u|^p \, dx + C_2(\varepsilon) \int_{\varepsilon Y \setminus a_\varepsilon G_0} |\nabla u|^p \, dx. \tag{3.1}$$

4. In order to obtain the limit of the reaction term, when $a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon$, we want to be able to write

$$\int_{a_\varepsilon \partial G_0} g \, dS = \mu_\varepsilon \int_{\varepsilon Y \setminus a_\varepsilon G_0} g \, dx + \rho_\varepsilon \tag{3.2}$$

   for suitable values of μ_ε (which should converge to a constant) and of ρ_ε (which should converge to 0). Later we will need a proper scaling, $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$, as mentioned in Section 2.3.
5. The situation for the critical case $a_\varepsilon \sim a_\varepsilon^\star$ is more difficult and we need to introduce some special auxiliary functions which later will allow the interplay between diffusion and reaction terms in the weak formulation. We provide some estimates on them in this chapter.

Remark 3.1. Note that in inequality (3.1) we have written $\varepsilon Y \setminus a_\varepsilon G_0$ instead of $\varepsilon Y \setminus a_\varepsilon \overline{G}_0$, for simplicity. We will do this in integrals and Lebesgue measures, since the n-dimensional measure of ∂G_0 is 0.

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3.1.1 Extension operators

For this section we follow the approach in [217]. Let $A \subset B$. We say that P is an extension operator if $P : F(A) = \{f : A \to \mathbb{R}\} \to F(B)$ and has the property that $P(f)|_A = f$. Let $p > 1$. We will say that a family of linear extension operators

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega) \tag{3.3}$$

is uniformly bounded if there exists a constant $C > 0$, independent of ε, such that

$$\|P_\varepsilon u\|_{W^{1,p}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega_\varepsilon)} \quad \forall u \in W^{1,p}(\Omega_\varepsilon). \tag{3.4}$$

Note that, since there are no particles tangent to the boundary, we also have

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon, \partial\Omega) \to W_0^{1,p}(\Omega).$$

We will prove that this mapping has also a uniform constant in the gradient norm, i. e.,

$$\big\|\nabla(P_\varepsilon u)\big\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega_\varepsilon)}, \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega), \tag{3.5}$$

where $C > 0$ also does not depend on ε, and this will yield a uniform Poincaré inequality (see Theorem 3.4 below). A family of operators with this property, for $1 \le p < +\infty$, was constructed in [228]. The aim of this section is to prove the following lemma.

Lemma 3.2. *Let $G_0 \in \mathcal{C}^{0,1}$ such that $\overline{G}_0 \subset Y$. Then in either of the settings of Section* 1.2.1 *there exists a uniformly bounded family of linear extension operators* (3.3) *such that* (3.4) *and* (3.5) *hold.*

The idea is to apply the following theorem.

Theorem (Theorem 7.25 in [148]). *Let Ω be a $C^{k-1,1}$ domain in $\mathbb{R}^n$, $k \ge 1$. Then (i) $\mathcal{C}^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$, $1 \le p < +\infty$, and (ii) for any open set $\Omega' \supset\supset \Omega$ there exists a linear extension operator $E : W^{k,p}(\Omega) \to W_0^{k,p}(\Omega')$ such that $Eu = u$ in Ω and*

$$\|Eu\|_{W^{k,p}(\Omega')} \le C\|u\|_{W^{k,p}(\Omega)},$$

where $C = C(k, \Omega, \Omega')$.

Remark 3.3. Going back to the construction in [148], we can check that the extension of the constant function 1 is the function 1, i. e., $E(1) = 1$.

Proof of Lemma 3.2. We consider a large ball B such that $Y \Subset B$ and the linear extension operator

$$E : W^{1,p}(Y \setminus \overline{G_0}) \to W^{1,p}(B)$$

such that

$$\|Eu\|_{W^{1,p}(B)} \le C_0 \|u\|_{W^{1,p}(Y \setminus \overline{G_0})}.$$

Let us scale it down by a_ε as

$$E_\varepsilon u(x) = E\big[u(a_\varepsilon \cdot)\big]\left(\frac{x}{a_\varepsilon}\right).$$

In other words, we construct

$$E_\varepsilon : W^{1,p}(a_\varepsilon Y \setminus a_\varepsilon \overline{G_0}) \to W^{1,p}(Y \setminus \overline{G_0}) \xrightarrow{E} W^{1,p}(B) \to W^{1,p}(a_\varepsilon B).$$

Note that rather than $\varepsilon Y \setminus a_\varepsilon \overline{G_0}$ we are considering the a_ε-rescale of Y. By a simple change in variable we observe that

$$\|E_\varepsilon u\|_{W^{1,p}(a_\varepsilon B)} \le C_1 \|u\|_{W^{1,p}(a_\varepsilon Y \setminus a_\varepsilon \overline{G_0})}.$$

Thus we can define

$$P_\varepsilon u(x) = \begin{cases} E_\varepsilon[u(\cdot - \varepsilon j)](x + \varepsilon j) & x \in \varepsilon j + a_\varepsilon Y \text{ for some } j \in \Upsilon_\varepsilon, \\ u(x) & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \|\nabla P_\varepsilon u\|^p_{W^{1,p}(\Omega)} &= \|u\|^p_{W^{1,p}(\Omega \setminus \bigcup_{j\in\Upsilon_\varepsilon}(\varepsilon j + a_\varepsilon Y))} + \sum_{j\in\Upsilon_\varepsilon} \big\|E_\varepsilon[u(\cdot - \varepsilon j)]\big\|^p_{W^{1,p}(a_\varepsilon Y)} \\ &\le \|u\|^p_{W^{1,p}(\Omega_\varepsilon)} + C_1 \sum_{j\in\Upsilon_\varepsilon} \big\|u(\cdot - \varepsilon j)\big\|^p_{W^{1,p}(a_\varepsilon Y \setminus a_\varepsilon \overline{G_0})} \\ &= \|u\|^p_{W^{1,p}(\Omega_\varepsilon)} + C_1 \sum_{j\in\Upsilon_\varepsilon} \|u\|^p_{W^{1,p}(\varepsilon j + a_\varepsilon Y \setminus a_\varepsilon \overline{G_0})} \\ &\le (1 + C_1)\|u\|^p_{W^{1,p}(\Omega_\varepsilon)}. \end{aligned}$$

Since the extension operator is such that $E(1) = 1$, one can recover (3.5). For a given function u, let $C_0 = \int_{Y\setminus\overline{G_0}} u(y)dy$. Then, $E(u - C_0) = E(u) - C_0$. Using the Poincaré–Wirtinger inequality for $W^{1,p}(Y \setminus \overline{G_0})$ (see, e. g., [138]) we have

$$\begin{aligned} \big\|\nabla E(u)\big\|_{L^p(Y)} = \big\|\nabla E(u - C_0)\big\|_{L^p(Y)} &\le \big\|E(u - C_0)\big\|_{W^{1,p}(Y)} \le C\|u - C_0\|_{W^{1,p}(Y\setminus\overline{G_0})} \\ &\le C\big(\|u - C_0\|_{L^p(Y\setminus\overline{G_0})} + \big\|\nabla(u - C_0)\big\|_{L^p(Y\setminus\overline{G_0})}\big) \\ &\le C\|\nabla u\|_{L^p(Y\setminus\overline{G_0})}. \end{aligned}$$

As above, through scaling one recovers the result. This completes the proof. □

3.1.2 Uniform Poincaré inequality on Ω_ε

Given a bounded domain Ω, the existence of a positive Poincaré constant $C_{p,\Omega}$ such that

$$\|v\|_{L^p(\Omega)} \le C_{p,\Omega}\|\nabla v\|_{L^p(\Omega)}, \quad \forall v \in W_0^{1,p}(\Omega)$$

is well known. However, it is not trivial to show that all domains Ω_ε have a common constant for $\varepsilon > 0$ small. The following result is very often used in the literature but it is seldom stated.

Theorem 3.4. *Let $p > 1$. Under the assumptions of Section 1.2.1, we have*

$$\|u\|_{L^p(\Omega_\varepsilon)} \le C\|\nabla u\|_{L^p(\Omega_\varepsilon)} \quad \forall u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega) \text{ and } \varepsilon > 0,$$

where C does not depend on ε.

Proof. We apply only Lemma 3.2 and the Poincaré inequality in Ω. First, we use the fact that $P_\varepsilon u = u$ in Ω_ε. Thus

$$\|u\|_{L^p(\Omega_\varepsilon)} = \|P_\varepsilon u\|_{L^p(\Omega_\varepsilon)} \le \|P_\varepsilon u\|_{L^p(\Omega)} \le C_{p,\Omega}\|\nabla P_\varepsilon u\|_{L^p(\Omega)}.$$

We complete the proof by using (3.5), where the constant is also uniform in ε. □

Remark 3.5. We can use a similar argument to prove that we have a uniform Poincaré–Wirtinger inequality in a cell, i. e.,

$$\|v - \overline{v}\|_{L^p(\varepsilon Y \setminus a_\varepsilon \overline{G_0})} \le C\varepsilon\|\nabla v\|_{L^p(\varepsilon Y \setminus a_\varepsilon \overline{G_0})},$$

where $\overline{v} = \frac{1}{|\varepsilon Y \setminus a_\varepsilon G_0|}\int_{\varepsilon Y \setminus a_\varepsilon G_0} v(x)\,\mathrm{d}x$ and C does not depend on ε.

3.1.3 Sharp trace estimates on $a_\varepsilon \partial G_0$ in $\varepsilon Y \setminus a_\varepsilon \overline{G_0}$

As result of well-known trace inequalities (see, e. g., [138]), we know that

$$\int\limits_{a_\varepsilon \partial G_0} |u|^p\,\mathrm{d}S \le C_\varepsilon\Big(\int\limits_{\varepsilon Y \setminus a_\varepsilon G_0} |u|^p\,\mathrm{d}x + \int\limits_{\varepsilon Y \setminus a_\varepsilon G_0} |\nabla u|^p\,\mathrm{d}x\Big),$$

for some positive constant C_ε. In the following pages we present a sharp trace estimate with constants depending explicitly on ε. This analysis unifies some different cases with respect to p and n, and is quite similar to that of [87, Lemma 2.1], but includes also the cases $p \ne 2$.

Lemma 3.6. *Let G_0 be smooth, $u \in W^{1,p}(\varepsilon Y \setminus a_\varepsilon \overline{G_0})$, $p > 1$, and assume that $a_\varepsilon \leq \varepsilon$. Then*

$$\int_{a_\varepsilon \partial G_0} |u|^p \, \mathrm{d}S \leq C a_\varepsilon^{n-1} \Big(\varepsilon^{-n} \int_{\varepsilon Y \setminus a_\varepsilon G_0} |u|^p \, \mathrm{d}x + \tau_\varepsilon \int_{\varepsilon Y \setminus a_\varepsilon G_0} |\nabla u|^p \, \mathrm{d}x \Big),$$

where

$$\tau_\varepsilon \sim \begin{cases} a_\varepsilon^{p-n} & p < n, \\ (\ln \frac{2\varepsilon}{a_\varepsilon})^{p-1} & p = n, \\ \varepsilon^{p-n} & p > n, \end{cases} \tag{3.6}$$

and C is a constant independent of ε and u.

Proof. For simplicity, we take an extension of u outside of εY such that

$$\int_{\mathbb{R}^n \setminus a_\varepsilon G_0} |u|^p \, \mathrm{d}x \leq C \int_{\varepsilon Y \setminus a_\varepsilon G_0} |u|^p \, \mathrm{d}x, \qquad \int_{\mathbb{R}^n \setminus a_\varepsilon G_0} |\nabla u|^p \, \mathrm{d}x \leq C \int_{\varepsilon Y \setminus a_\varepsilon G_0} |\nabla u|^p \, \mathrm{d}x.$$

As for the extension, this can be done with a constant that is uniform in ε, as follows. Since Y and G_0 are good enough, there exists an extension operator from $T : W^{1,p}(Y \setminus \overline{G}_0) \to W^{1,p}(\mathbb{R}^n \setminus \overline{G}_0)$. We define this extension by

$$u(x) = \begin{cases} T[u|_{\varepsilon Y \setminus a_\varepsilon G_0}(\varepsilon \cdot)](x/\varepsilon) & x \in \mathbb{R}^n \setminus \varepsilon \overline{G}_0, \\ u(x) & x \in \varepsilon G_0 \setminus a_\varepsilon G_0. \end{cases}$$

First, let us take $R > 0$ such that $\overline{G}_0 \subset B_R$. We check that

$$|||v|||^p = \int_{\partial B_R} |v|^p \, \mathrm{d}S_y + \int_{B_R \setminus G_0} |\nabla v|^p \, \mathrm{d}y$$

is an equivalent norm of $W^{1,p}(B_R \setminus G_0)$ (this is a version of Friedrich's inequality). Let us prove that $C_1 \|v\|_{W^{1,p}(B_R \setminus \overline{G_0})} \leq |||v||| \leq C_2 \|v\|_{W^{1,p}(B_R \setminus \overline{G_0})}$ with $C_1, C_2 > 0$ and finite. The existence of C_2 follows from the continuity of the trace. Let $C_1 = \inf |||v|||/\|v\|_{W^{1,p}(B_R \setminus \overline{G_0})}$ and take a minimizing sequence v_n such that $\|v_n\|_{W^{1,p}(B_R \setminus \overline{G_0})} = 1$. Up to a subsequence v_n converges to some v weakly in $W^{1,p}(B_R \setminus \overline{G_0}), L^p(\partial G_0)$ and strongly in $L^p(B_R \setminus G_0)$. If $C_1 = 0$, then the gradients converge to zero and v is constant $1/|B_R \setminus G_0|$ but, due the weak lower semicontinuity, also $\int_{\partial G_0} |v|^p \, \mathrm{d}S \leq 0$. Hence $C_1 > 0$.

Hence, the continuity of the trace is stated as

$$\int_{\partial G_0} |v|^p \, \mathrm{d}S_y \leq C \Big(\int_{\partial B_R} |v|^p \, \mathrm{d}S_y + \int_{B_R \setminus G_0} |\nabla v|^p \, \mathrm{d}y \Big). \tag{3.7}$$

Taking a function $u \in W^{1,p}(a_\varepsilon B_R \setminus a_\varepsilon G_0)$ and $v(y) = u(a_\varepsilon y)$ we scale these integrals to deduce

$$\int_{a_\varepsilon \partial G_0} |u|^p \, \mathrm{d}S_x \le C\Big(\int_{a_\varepsilon \partial B_R} |u|^p \, \mathrm{d}S_x + a_\varepsilon^{p-1} \int_{a_\varepsilon B_R \setminus a_\varepsilon G_0} |\nabla u|^p \, \mathrm{d}x \Big).$$

Thus, it suffices to prove the estimate for $a_\varepsilon B_R$. Through the extension to $\mathbb{R}^n \setminus a_\varepsilon \overline{G_0}$ we can use εB_R instead of εY. We will work in spherical coordinates with a radius ρ and letting $\theta \in \mathbb{R}^{n-1}$ denote a parametrization of the ∂B_1. We will denote by Θ the set of these parameters. The Jacobian can be written as $\rho^{n-1} J(\theta)$. Let us write u in polar coordinates as $\chi(\rho, \theta) = u(x)$. Then, as in [87],

$$\int_{a_\varepsilon \partial B_R} |u|^p \, \mathrm{d}S = a_\varepsilon^{n-1} R^{n-1} \int_{\partial B_1} \big|\chi(a_\varepsilon R, \theta)\big|^p J(\theta) \, \mathrm{d}\theta. \tag{3.8}$$

We write, for any $\rho > a_\varepsilon R$ and $\theta \in \Theta$,

$$\chi(a_\varepsilon R, \theta) = \chi(\rho, \theta) - \int_{a_\varepsilon R}^{\rho} \frac{\partial \chi}{\partial t}(t, \theta) \, \mathrm{d}t.$$

For $p > 1$, due to the convexity of the p-power

$$\big|\chi(a_\varepsilon R, \theta)\big|^p \le 2^{p-1} \big|\chi(\rho, \theta)\big|^p + 2^{p-1} \left| \int_{a_\varepsilon R}^{\rho} \frac{\partial \chi}{\partial t}(t, \theta) \, \mathrm{d}t \right|^p.$$

On the other hand,

$$\begin{aligned} \left| \int_{a_\varepsilon R}^{\rho} \frac{\partial \chi}{\partial t}(t, \theta) \, \mathrm{d}t \right|^p &\le \left| \int_{a_\varepsilon R}^{\rho} \frac{\partial \chi}{\partial t}(t, \theta) t^{\frac{n-1}{p}} t^{-\frac{n-1}{p}} \, \mathrm{d}t \right|^p \\ &\le \Big(\int_{a_\varepsilon R}^{\rho} t^{-\frac{n-1}{p-1}} \, \mathrm{d}t \Big)^{p-1} \Big(\int_{a_\varepsilon R}^{\rho} \left| \frac{\partial \chi}{\partial t}(t, \theta) \right|^p t^{n-1} \, \mathrm{d}t \Big). \end{aligned}$$

We get

$$\big|\chi(a_\varepsilon R, \theta)\big|^p \le 2^{p-1} \big|\chi(\rho, \theta)\big|^p + 2^{p-1} \tau_\varepsilon \Big(\int_{a_\varepsilon R}^{\varepsilon R} \left| \frac{\partial \chi}{\partial t}(t, \theta) \right|^p t^{n-1} \, \mathrm{d}t \Big), \tag{3.9}$$

where we define

$$\tau_\varepsilon = \Big(\int_{a_\varepsilon R}^{\varepsilon} t^{-\frac{n-1}{p-1}} \, \mathrm{d}t \Big)^{p-1}.$$

A simple integration shows that τ_ε satisfies (3.6). Multiplying (3.9) by $\rho^{n-1}J$ and integrating over $B_{\varepsilon R} \setminus B_{a_\varepsilon R}$ yields

$$\begin{aligned}
&\int_{\partial B_1}\int_{a_\varepsilon R}^{\varepsilon R} |\chi(a_\varepsilon R,\theta)|^p \rho^{n-1} J(\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta \\
&\quad\le 2^{p-1}\int_{\partial B_1}\int_{a_\varepsilon R}^{\varepsilon R} |\chi(\rho,\theta)|^p\rho^{n-1}J\,\mathrm{d}\rho\,\mathrm{d}\theta + 2^{p-1}\int_{\partial B_1}\int_{a_\varepsilon R}^{\varepsilon R}\tau_\varepsilon\Big(\int_{a_\varepsilon R}^{\rho}\Big|\frac{\partial\chi}{\partial t}(t,\theta)\Big|^p t^{n-1}\,\mathrm{d}t\Big)\rho^{n-1}J\,\mathrm{d}\rho\,\mathrm{d}\theta \\
&\quad\le 2^{p-1}\int_{\partial B_1}\int_{a_\varepsilon R}^{\varepsilon R} |\chi(\rho,\theta)|^p\rho^{n-1}J\,\mathrm{d}\rho\,\mathrm{d}\theta + 2^{p-1}\tau_\varepsilon\tau_{2,\varepsilon}\int_{\partial B_1}\Big(\int_{a_\varepsilon R}^{\varepsilon R}\Big|\frac{\partial\chi}{\partial t}(t,\theta)\Big|^p t^{n-1}\,\mathrm{d}t\Big)J\,\mathrm{d}\rho\,\mathrm{d}\theta,
\end{aligned}$$

where

$$\tau_{2,\varepsilon} = \int_{a_\varepsilon R}^{\varepsilon R}\rho^{n-1}\,\mathrm{d}\rho \le C\varepsilon^n.$$

Note that $|\partial\chi/\partial t| \le |\nabla u|$. On the other hand,

$$\int_{\partial B_1}\int_{a_\varepsilon R}^{\varepsilon R} |\chi(a_\varepsilon R,\theta)|^p\rho^{n-1}J(\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta = \Big(\int_{\partial B_1}|\chi(a_\varepsilon R,\theta)|^p J(\theta)\,\mathrm{d}\theta\Big)\tau_{2,\varepsilon}.$$

Going back to (3.8) we have

$$\int_{a_\varepsilon \partial B_R}|u|^p\,\mathrm{d}S \le a_\varepsilon^{n-1}2^{p-1}\Big(\tau_{2,\varepsilon}^{-1}\int_{\varepsilon B_R\setminus a_\varepsilon B_R}|u|^p\,\mathrm{d}x + \tau_\varepsilon\int_{\varepsilon B_R\setminus a_\varepsilon B_R}|\nabla u|^p\,\mathrm{d}x\Big).$$

Since it is easy to check that $a_\varepsilon^{n-1}\tau_\varepsilon \lesssim a_\varepsilon^{p-1}$ in each of the cases we recover the result. □

Remark 3.7. It is not surprising that $W^{1,n}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ behaves differently. For example, the family of radial solutions of $\Delta_n u = 0$ in $\mathbb{R}^n$ includes $\ln|x|$, whereas for any other values of p radial solutions are of power type.

Remark 3.8. The denominator 2 in the case $p = n$ is included so that we can safely pick $a_\varepsilon = \varepsilon$.

Remark 3.9. In the literature, it is generally assumed that G_0 is star-shaped. With the addition inequality (3.7) it is sufficient that it holds for balls. This simplifies the assumptions on G_0 and the computations.

3.1.4 Auxiliary oscillating functions in the subcritical range $a_\varepsilon^\star \ll a_\varepsilon \lesssim \varepsilon$

Our control on the integral on S_ε is based first on the scaled trace lemma and on passing to volumetric integrals as $\varepsilon \to 0$. Our hope is to show something of the form (3.2).

This will play a crucial role in the proof of the "from surface to volume averaging convergence theorems" (see Theorem 4.5 and Theorem 4.11 below). This conversion can be made by studying a specific function $m_\varepsilon \in W^{1,p}(Y_\varepsilon)$ such that, for any test function $\varphi \in W^{1,p}(Y_\varepsilon)$, where $Y_\varepsilon = \varepsilon Y \setminus \overline{a_\varepsilon G_0}$,

$$\int_{\partial(a_\varepsilon G_0)} \varphi \, \mathrm{d}S = \mu_\varepsilon \int_{\varepsilon Y \setminus a_\varepsilon G_0} \varphi \, \mathrm{d}x + \int_{\varepsilon Y \setminus a_\varepsilon \overline{G_0}} |\nabla m_\varepsilon|^{p-2} \nabla m_\varepsilon \nabla \varphi \, \mathrm{d}x. \tag{3.10}$$

This is actually the weak formulation of a PDE with suitable boundary conditions that defines a unique m_ε. Let us define the function $m_\varepsilon(x)$ as the unique Y_ε-periodic function built through the solution of the boundary value problem (see, e. g., [82])

$$\begin{cases} \Delta_p m_\varepsilon = \mu_\varepsilon & x \in \varepsilon Y \setminus a_\varepsilon \overline{G_0}, \\ \partial_{\nu_p} m_\varepsilon = 1 & x \in a_\varepsilon \partial G_0, \\ \partial_{\nu_p} m_\varepsilon = 0 & x \in \varepsilon \partial Y, \end{cases} \qquad \int_{\varepsilon Y \setminus a_\varepsilon G_0} m_\varepsilon(x) \, \mathrm{d}x = 0,$$

where μ_ε is a positive constant defined so as to satisfy the compatibility condition

$$\mu_\varepsilon = \frac{\varepsilon^{-n} a_\varepsilon^{n-1} |\partial G_0|}{1 - (a_\varepsilon \varepsilon^{-1})^n |G_0|}, \tag{3.11}$$

which is obtained by taking $\varphi = 1$ as test function. Note that necessarily $m_\varepsilon(x)$ must change sign in Y_ε. The reason for the periodicity condition is that we will later construct an Y_ε-periodic function over Ω_ε by translating these functions, i. e.,

$$M_\varepsilon(x) = m_\varepsilon(x - \varepsilon j), \quad x \in \varepsilon j + \varepsilon Y \text{ for some } j \in \Upsilon_\varepsilon. \tag{3.12}$$

Note that the different repetition in the case of particles over the whole domain and only the boundary yield different M_ε, whereas m_ε is the same.

When we want to study the limit of oscillating functions, for example

$$\int_{S_\varepsilon} g^\varepsilon(x) \varphi(x) \, \mathrm{d}S_x, \quad \text{where } g^\varepsilon(x) = g\left(\frac{x - \varepsilon j}{a_\varepsilon}\right) \text{ if } x \in \varepsilon j + \partial(a_\varepsilon G_0),$$

for a given $g \in L^1(\partial G_0)$, we can define

$$\begin{cases} \Delta_p m_{g,\varepsilon} = \mu_{g,\varepsilon} & x \in \varepsilon Y \setminus a_\varepsilon \overline{G_0}, \\ \partial_{\nu_p} m_{g,\varepsilon} = g(\cdot / a_\varepsilon) & x \in a_\varepsilon \partial G_0, \\ \partial_{\nu_p} m_{g,\varepsilon} = 0 & x \in \varepsilon \partial Y, \end{cases} \qquad \int_{\varepsilon Y \setminus a_\varepsilon G_0} m_{g,\varepsilon}(x) dx = 0,$$

where $\mu_{g,\varepsilon}$ is a constant defined so as to satisfy the compatibility condition

$$\mu_{g,\varepsilon} = \frac{\varepsilon^{-n} a_\varepsilon^{n-1}}{1 - (a_\varepsilon \varepsilon^{-1})^n |G_0|} \int_{\partial G_0} g(s) \, \mathrm{d}S.$$

3.1.4.1 Estimates for m_ε

We will use the following fact.

Proposition 3.10. *Let $p > 1$. Then*

$$\|\nabla m_\varepsilon\|_{L^p(\varepsilon Y\setminus a_\varepsilon G_0)} \leq C \begin{cases} a_\varepsilon^{\frac{n}{p}} & p < n, \\ a_\varepsilon(\ln \frac{2\varepsilon}{a_\varepsilon})^{\frac{1}{n}} & p = n, \\ a_\varepsilon^{\frac{n-1}{p-1}} \varepsilon^{\frac{p-n}{p(p-1)}} & p > n. \end{cases}$$

Proof. Let, as before, $Y_\varepsilon = \varepsilon Y \setminus \overline{a_\varepsilon G_0}$. Setting in (3.10) $\varphi = m_\varepsilon$, from the definition of $m_\varepsilon(x)$, applying Lemma 3.6 and the Poincaré–Wirtinger inequality (see Remark 3.5), we obtain

$$\begin{aligned}
\|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^{p^2} &\leq \left(\left|\int_{a_\varepsilon \partial G_0} m_\varepsilon \, \mathrm{d}S\right| + |\mu_\varepsilon| \left|\int_{Y_\varepsilon} m_\varepsilon \, \mathrm{d}x\right|\right)^p \\
&\leq \left(\int_{a_\varepsilon \partial G_0} |m_\varepsilon| \, \mathrm{d}S + \mu_\varepsilon \times 0\right)^p \\
&\leq \left(\left(\int_{a_\varepsilon \partial G_0} 1^{p'} \, \mathrm{d}S\right)^{\frac{1}{p'}} \left(\int_{a_\varepsilon \partial G_0} |m_\varepsilon|^p \, \mathrm{d}S\right)^{\frac{1}{p}}\right)^p \\
&\leq \left(\int_{a_\varepsilon \partial G_0} 1 \, \mathrm{d}S\right)^{p-1} \|m_\varepsilon\|_{L^p(a_\varepsilon \partial G_0)}^p \\
&\leq C_1 a_\varepsilon^{(n-1)(p-1)} \|m_\varepsilon\|_{L^p(a_\varepsilon \partial G_0)}^p \leq \\
&\leq C_2 a_\varepsilon^{(n-1)(p-1)} a_\varepsilon^{n-1} (\varepsilon^{-n} \|m_\varepsilon\|_{L^p(Y_\varepsilon)}^p + \tau_\varepsilon \|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^p) \\
&\leq C_3 a_\varepsilon^{p(n-1)} (\varepsilon^{-n+p} + \tau_\varepsilon) \|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^p,
\end{aligned} \tag{3.13}$$

where τ_ε is given by (3.6). Therefore

$$\|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^{p-1} \leq C a_\varepsilon^{n-1} (\varepsilon^{-n+p} + \tau_\varepsilon)^{\frac{1}{p}}.$$

Now we can use the definition of τ_ε and the conclusion follows. □

3.1.4.2 Estimates for $m_{g,\varepsilon}$

Similarly to the arguments used in (3.13) we have

$$\|\nabla m_{g,\varepsilon}\|_{L^p(\varepsilon Y\setminus a_\varepsilon \overline{G_0})} \leq \begin{cases} C a_\varepsilon^{\frac{n}{p}} & p < n, \\ C a_\varepsilon(\ln \frac{2\varepsilon}{a_\varepsilon})^{\frac{1}{n}} & p = n, \\ C a_\varepsilon^{\frac{n-1}{p-1}} \varepsilon^{\frac{p-n}{p(p-1)}} & p > n, \end{cases}$$

where C depends only on $\|g\|_{L^{p'}(\partial G_0)}$, p and n.

3.1.5 Auxiliary functions in the critical case $a_\varepsilon \sim a_\varepsilon^\star$: capacity problems

As we discussed in Section 1.5.3, in the critical case the situation is more involved: we cannot simply work on the integral of S_ε, but go deeper into the complete expression of the notion of weak solution of the problem. To this aim we need to understand sharply the behavior of the solutions of some capacity type problems. The fine estimates on the respective solutions which we will obtain now will play a crucial role in the proof of the main homogenization convergence results in the next chapter (see Theorem 4.36 and Theorem 4.44).

3.1.5.1 When G_0 is a ball and $1 < p < n$

Let us fix $G_0 = B_1$ (we refer the reader to Remark 1.1). The simplest example of this kind of problems is

$$\begin{cases} \Delta_p w_\varepsilon = 0 & \frac{\varepsilon}{4}B_1 \setminus (a_\varepsilon \overline{G_0}), \\ w_\varepsilon = 1 & \partial(a_\varepsilon G_0), \\ w_\varepsilon = 0 & \partial(\frac{\varepsilon}{4}B_1). \end{cases} \tag{3.14}$$

The idea behind this auxiliary problem is to allow to trade the integral over $a_\varepsilon G_0$ for one over $\frac{\varepsilon}{4}B_1$, for which we can apply the subcritical theory. In particular, for w_ε we have

$$\int\limits_{\frac{\varepsilon}{4}B_1 \setminus a_\varepsilon G_0} |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla \varphi \, \mathrm{d}x = \int\limits_{\partial(\frac{\varepsilon}{4}B_1)} \varphi \partial_{\nu_p} w_\varepsilon \, \mathrm{d}S + \int\limits_{\partial a_\varepsilon G_0} \varphi \partial_{\nu_p} w_\varepsilon \, \mathrm{d}S.$$

The explicit solution for problem (3.14) is known:

$$w_\varepsilon(x) = \frac{|x|^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}{a_\varepsilon^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}, \quad a_\varepsilon \le |x| \le \frac{\varepsilon}{4}. \tag{3.15}$$

We can simply compute its gradient as

$$\begin{aligned} \frac{d}{dr} w_\varepsilon(r) &= -\frac{n-p}{p-1} \frac{r^{-\frac{n-p}{p-1}-1}}{a_\varepsilon^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}} = -\frac{n-p}{p-1} \frac{a_\varepsilon^{\frac{n-p}{p-1}} r^{-\frac{n-p}{p-1}-1}}{1 - (\frac{\varepsilon}{4a_\varepsilon})^{-\frac{n-p}{p-1}}} \\ &\simeq -\frac{n-p}{p-1} a_\varepsilon^{\frac{n-p}{p-1}} r^{\frac{1-n}{p-1}}. \end{aligned}$$

We have the precise integrability exponent, and if $q > \frac{n(p-1)}{n-1}$, then

$$\int\limits_{\frac{\varepsilon}{4}B_1 \setminus a_\varepsilon G_0} \left|\nabla w_\varepsilon(x)\right|^q \mathrm{d}x \sim \int_{a_\varepsilon}^{\frac{\varepsilon}{4}} \left|\frac{dw_\varepsilon}{dr}\right|^q r^{n-1} dr \sim a_\varepsilon^{n-q}. \tag{3.16}$$

Since this is a radially symmetric function, $\partial_{\nu_p} w_\varepsilon$ is a constant in either boundary with values

$$\begin{aligned} \partial_{\nu_p} w_\varepsilon|_{\frac{\varepsilon}{4}\partial B_1} &\simeq -\left(\frac{n-p}{p-1}\right)^{p-1} a_\varepsilon^{n-p} (\varepsilon/4)^{1-n}, \\ \partial_{\nu_p} w_\varepsilon|_{a_\varepsilon \partial G_0} &\simeq \left(\frac{n-p}{p-1}\right)^{p-1} a_\varepsilon^{1-p}. \end{aligned} \tag{3.17}$$

These estimates will play a crucial role in the proof of the important Lemma 4.38.

Note that through a change of scale $\widehat{w}_\varepsilon(y) = w_\varepsilon(a_\varepsilon y)$ this new function is the solution of

$$\begin{cases} \Delta_p \widehat{w}_\varepsilon = 0 & \frac{\varepsilon}{4a_\varepsilon} B_1 \setminus \overline{G}_0, \\ \widehat{w}_\varepsilon = 1 & \partial G_0, \\ \widehat{w}_\varepsilon = 0 & \partial(\frac{\varepsilon}{4a_\varepsilon} B_1). \end{cases}$$

Asymptotically, $\varepsilon/a_\varepsilon \to +\infty$, so these functions converge to the solution of

$$\begin{cases} \Delta_p \widehat{w} = 0 & \mathbb{R}^n \setminus \overline{G}_0, \\ \widehat{w} = 1 & \partial G_0, \\ \widehat{w} \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{3.18}$$

when $G_0 = B_1$, and we have

$$\widehat{w}(y) = |y|^{-\frac{n-p}{p-1}}. \tag{3.19}$$

Note that for $p = 2$ we get the usual Newtonian potential of fundamental relevance in mechanics and electrostatic studies. The values of the normal derivative are linked directly to scaling properties of $\widehat{w}$.

Remark 3.11. When $p \in (1, n)$, this function $\widehat{w}$ is usually called $\widehat{\kappa}$, and we will use this notation when G_0 is not a ball. This p-potential is systematically studied in [200]. It will often appear in our computations, so we state the PDE

$$\begin{cases} \Delta_p \widehat{\kappa} = 0 & \mathbb{R}^n \setminus \overline{G_0}, \\ \widehat{\kappa} = 1 & \partial G_0, \\ \widehat{\kappa} \to 0 & \text{as } |y| \to +\infty. \end{cases} \tag{3.20}$$

The function $\widehat{\kappa}$ is often used to compute the p-capacity of G_0, even when G_0 is not a ball. One defines

$$p\text{-cap}(G_0) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla w|^p \, dx : w \in C_c^\infty(\mathbb{R}^n), w \geq 1 \text{ in } G_0 \right\}.$$

The Euler–Lagrange equation for this functional is precisely (3.18) and $\widehat{\kappa}$ is the function for which the infimum is attained. Hence, integrating by parts

$$p\text{-cap}(G_0) = \int_{\mathbb{R}^n\setminus\overline{G_0}} |\nabla\widehat{\kappa}|^p \,\mathrm{d}x = -\int_{\mathbb{R}^n\setminus\overline{G_0}} \operatorname{div}(|\nabla\widehat{\kappa}|^{p-2}\nabla\widehat{\kappa})\widehat{\kappa}\,\mathrm{d}x + \int_{\partial G_0} \partial_{\nu_p}\widehat{\kappa}\,\mathrm{d}S = \int_{\partial G_0} \partial_{\nu_p}\widehat{\kappa}\,\mathrm{d}S. \tag{3.21}$$

We will use the notation

$$\lambda_{G_0} = 2\text{-cap}(G_0). \tag{3.22}$$

Remark 3.12. When $G_0 = B_R$ and $p \in (1,n)$, due to the explicit solution (3.19) (scaled if needed) we recover that

$$p\text{-cap}(B_R) = \left(\frac{n-p}{p-1}\right)^{p-1} |\partial B_1| R^{n-p}.$$

Alternatively, $\widehat{w}_\varepsilon$ is used to compute the relative p-capacity of G_0 in $\frac{\varepsilon}{4a_\varepsilon}B_1$. The limit as $\varepsilon \to 0$ is the above value.

3.1.5.2 When G_0 is a ball and $p = n$

In this setting, we still want to study problem (3.14). It was shown in [229] that the explicit solution is given by

$$w_\varepsilon(x) = \frac{\ln \frac{4|x|}{\varepsilon}}{\ln \frac{4a_\varepsilon}{\varepsilon}}. \tag{3.23}$$

Due to the properties of ln, this solution is quite similar to the solution (3.15) but replacing the power by a logarithm. We have

$$\partial_{\nu_n} w_\varepsilon|_{\partial\frac{\varepsilon}{4}B_1} \simeq -\left(\frac{4}{\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}}\right)^{n-1},$$

$$\partial_{\nu_n} w_\varepsilon|_{\partial a_\varepsilon G_0} \simeq \left(\frac{1}{a_\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}}\right)^{n-1}.$$

Since $a_\varepsilon \ll \varepsilon$ we have $\ln \frac{\varepsilon}{4a_\varepsilon} > 0$ for ε small.

Remark 3.13. When $p = n$, the solution of the n-harmonic problem is of logarithmic type and the behavior of w_ε becomes logarithmic. We see in Remark 3.12 that $n = p$ is a special case. In fact, for $p \in (1,n]$ we can define the relative p-capacity for two open sets such that $A \subset B$ as

$$p\text{-cap}_B(A) = \inf\left\{\int_B |\nabla w|^p \,\mathrm{d}x : w \in W_0^{1,p}(B) \text{ and } w \geq 1 \text{ in } A\right\}.$$

The Euler–Lagrange equation for this problem is

$$\begin{cases} -\Delta_p w = 0 & B \setminus \overline{A}, \\ w = 1 & \partial A, \\ w = 0 & \partial B. \end{cases} \tag{3.24}$$

The relative p-capacity for two concentric balls is well known (see [142]). First, note that as in (3.14) and (3.23) the solution of problem (3.24) is given by

$$w(x) = \frac{K(|x|) - K(R)}{K(r) - K(R)}, \quad K(r) = \begin{cases} \frac{p-1}{n-p} |\partial B_1|^{-\frac{1}{p-1}} r^{\frac{n-p}{p-1}} & p < n, \\ -|\partial B_1|^{-\frac{1}{p-1}} \ln r & p = n. \end{cases}$$

Thus

$$p\text{-cap}_{B_R}(B_r) = \big(K(r) - K(R)\big)^{1-p}.$$

When $p < n$ we can let $R \to +\infty$ and we recover the usual p-cap. However, when $p = n$ we have

$$p\text{-cap}_{B_R}(B_r) = |\partial B_1| \left(\ln \frac{R}{r} \right)^{1-n},$$

so the behavior is more delicate (if we simply let $R \to \infty$ we arrive at 0). This is the reason why when $p < n$ we recover the p-capacity directly, and hypothesis on a_ε will be simply stated in terms of $a_\varepsilon / a_\varepsilon^\star$, and when $p = n$ the $\ln a_\varepsilon / a_\varepsilon^\star$ is involved.

3.1.5.3 When G_0 is not a ball and $p = 2$

There have been many attempts to extend the above theory for G_0 a ball to the general case. However, since the solution of (3.14) could not be found explicitly, the problem remained open for a long time. The novel approach in [116] came from looking for $\widehat{w}$ instead of w_ε. In fact, the replacement for $\widehat{w}$ must now be linked to the corresponding nonlinearity. In the case of G_0 a ball, if we want to use a test function v for the homogenized problem, as shown in detail in Section 4.7.1, we will need to correct it by taking the associated oscillating test functions

$$v_\varepsilon(x) = v(x) - H\big(v(x)\big) w_\varepsilon(x - \varepsilon j)$$

on each cell. When G_0 is not a ball we would like to replace $H(v(x)) w_\varepsilon(x - \varepsilon j)$ by a single function $w_{\sigma,\varepsilon}(x - \varepsilon j; v(x))$ (see the proof of Theorem 4.44).

Assume that $g^\varepsilon = 0$ and $p = 2$. As we will show, the good way to go is by constructing a function $\widehat{w}_\sigma = \widehat{w}_\sigma(y, s)$ for $s \in \mathbb{R}$ and $y \in \mathbb{R}^n$ where, for each $s \in \mathbb{R}$, $\widehat{w}_\sigma(\cdot, s)$ solves

$$\begin{cases} \Delta \widehat{w}_\sigma = 0, & \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_\nu \widehat{w}_\sigma = C_0 \sigma(s - \widehat{w}_\sigma) & \partial G_0, \\ \widehat{w}_\sigma \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{3.25}$$

where

$$C_0 = \lim_{\varepsilon \to 0} a_\varepsilon \beta(\varepsilon). \tag{3.26}$$

Since $p = 2$ and we work with critical-size particles over the whole domain $a_\varepsilon \beta(\varepsilon) \sim a_\varepsilon \varepsilon^n a_\varepsilon^{1-n} = a_\varepsilon^{2-n} \varepsilon^n \sim 1$. As we will see below (see Theorem 4.36 and Theorem 4.44), this function $\widehat{w}_\sigma$ is chosen so that we have the correct cancelations of integrals in S_ε such as will be indicated in Remark 4.40.

With this choice, we will show that the effective reactive term $\mathcal{H}$ (to be defined later) is retrieved from the function $\widehat{H}_\sigma$ defined, for $s \in \mathbb{R}$, in the following way:

$$\widehat{H}_\sigma(s) = \int_{\partial G_0} \partial_\nu \widehat{w}_\sigma(y, s)\, \mathrm{d}S_y. \tag{3.27}$$

Remark 3.14. When $G_0 = B_1$, if we go back to $\widehat{w}(y) = |y|^{2-n}$, the solution of (3.18) for $p = 2$, it is easy to see that $\widehat{w}_\sigma(y, s) = H(s)\widehat{w}(y)$, if we consider $H(s)$ that solves (1.16) written with precise constants as

$$(n-2)H(s) = C_0 \sigma\big(s - H(s)\big).$$

This is none other than the equation for the "strange term" that appears when G_0 is a ball (see Section 4.7.1). In particular,

$$\widehat{H}_\sigma(s) = \int_{\partial G_0} (n-2)H(s)\, \mathrm{d}S_y = (n-2)|\partial B_1| H(s).$$

This constant is later assimilated to recover the effective reaction $\mathcal{H}$.

Remark 3.15. When $\sigma = \sigma(x, s)$ and $g^\varepsilon(x)$ is given by (1.3), then we should take $\widehat{w}_\sigma = \widehat{w}_\sigma(x, y, s)$, for $x \in \Omega$ and $s \in \mathbb{R}$ fixed, as the solution of

$$\begin{cases} \Delta \widehat{w}_\sigma = 0 & y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_\nu \widehat{w}_\sigma = C_0 \sigma(x, s - \widehat{w}_\sigma) - C_0 g_{\mathrm{st}}(x) - C_0 g_{\mathrm{per}}(y) & y \in \partial G_0, \\ \widehat{w}_\sigma \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{3.28}$$

and then we define

$$\widehat{H}_\sigma(x, s) = \int_{\partial G_0} \partial_\nu \widehat{w}_\sigma(x, y, s)\, \mathrm{d}S_y.$$

The estimates for this new function $\widehat{w}_\sigma$, and its rescaled counterparts, are notoriously more difficult to obtain.

Remark 3.16. In [108] (see also [271]) the authors study boundary conditions of type $\partial_\nu u_\varepsilon + b_\varepsilon(x) u_\varepsilon = 0$. This requires an additional modification in the boundary value for $\widehat{w}$, but we will not discuss it here.

Remark 3.17. When $p \neq 2$, and when g^ε is of type (1.3), we would look at the auxiliary problem

$$\begin{cases} \Delta_p \widehat{w}_\sigma = 0 & y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_{\nu_p} \widehat{w}_\sigma = C_0 \sigma(s - \widehat{w}_\sigma) - C_0 g_{\mathrm{st}}(x) - C_0 g_{\mathrm{per}}(y) & y \in \partial G_0, \\ \widehat{w}_\sigma \to 0 & \text{as } |y| \to +\infty. \end{cases}$$

Due to the scaling of ∂_{ν_p}, we would take $C_0 = \lim_{\varepsilon\to 0} a_\varepsilon^{p-1}\beta(\varepsilon)$ (see Remark 3.22 below) and then we can define

$$\widehat{H}_\sigma(x,s) = \int_{\partial G_0} \partial_{\nu_p} \widehat{w}_\sigma(x,y,s)\,\mathrm{d}S_y.$$

We expect this function to have similar nice properties to the case $p = 2$. However, obtaining approximation estimates like (3.38) below for the corresponding problem is possible, but it is a difficult task. We leave this as an open problem.

Some useful properties

Let us get some estimates of $\widehat{w}_\sigma$ and $\widehat{H}_\sigma$ when $g^\varepsilon = 0$. To have some pointwise estimate, let us look at

$$\widehat{\kappa}_\sigma(y,s) = \frac{\partial \widehat{w}_\sigma}{\partial s}(y,s).$$

It is a solution of

$$\begin{cases} \Delta \widehat{\kappa}_\sigma = 0 & \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_\nu \widehat{\kappa}_\sigma = C_0 \sigma'(s - \widehat{w}_\sigma)(1 - \widehat{\kappa}_\sigma) & \partial G_0, \\ \widehat{\kappa}_\sigma \to 0 & \text{as } |y| \to +\infty. \end{cases} \tag{3.29}$$

To get a bound, define $\widehat{\kappa}$ as the unique solution of (3.20) with $p = 2$. Note that we have already seen this boundary value problem when G_0 is a ball in (3.18). Then

$$\begin{cases} \Delta(\widehat{\kappa} - \widehat{\kappa}_\sigma) = 0 & \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_\nu(\widehat{\kappa} - \widehat{\kappa}_\sigma) + C_0\sigma'(s - \widehat{w}_\sigma)(\widehat{\kappa} - \widehat{\kappa}_\sigma) = \partial_\nu \widehat{\kappa} & \partial G_0, \\ \widehat{\kappa} - \widehat{\kappa}_\sigma \to 0 & \text{as } |y| \to +\infty. \end{cases} \tag{3.30}$$

Since $\sigma' \geq 0$ and $\partial_\nu \widehat{\kappa} \geq 0$ at ∂G_0, it is not hard to see that

$$0 \leq \widehat{\kappa}_\sigma = \frac{\partial \widehat{w}_\sigma}{\partial s} \leq \widehat{\kappa}.$$

Hence,

$$0 \leq \widehat{w}_\sigma(y,s) \leq \widehat{\kappa}(y)s, \quad \forall s \geq 0,$$

the converse for $s < 0$ and, in general

$$\left|\widehat{w}_\sigma(y,s)\right| \le \widehat{\kappa}(y)|s|, \quad \forall s \in \mathbb{R}. \tag{3.31}$$

Note that, since $\widehat{\kappa}$ is a harmonic function, it cannot achieve interior extrema, so

$$0 \le \widehat{\kappa} \le 1.$$

On the other hand, by the comparison principle

$$\widehat{\kappa}(y) \le \frac{K_0}{|y|^{n-2}},$$

where $K_0 = \max_{\partial G_0} |y|^{n-2}$.

Since $\widehat{w}_\sigma$ is harmonic, so is its partial derivative, and we have $\partial\widehat{w}_\sigma/\partial y_i = \nabla\widehat{w}_\sigma \cdot e_i = \operatorname{div}(\widehat{w}_\sigma e_i)$. Thus, by the mean value property (see, e. g., [148]), letting $R < \operatorname{dist}(y, G_0)$ we get

$$\frac{\partial\widehat{w}_\sigma}{\partial y_i}(x) = \frac{1}{|B_R|}\int_{B_R(x)} \frac{\partial\widehat{w}_\sigma}{\partial y_i}(y')\,\mathrm{d}y' = \frac{1}{|B_R|}\int_{\partial B_R(x)} \widehat{w}_\sigma(y')e_i \cdot \nu\,\mathrm{d}S_{y'}.$$

Hence

$$\left|\frac{\partial\widehat{w}_\sigma}{\partial y_i}(y)\right| \le CR^{-1}\big(|x| - R\big)^{2-n}, \quad \forall R < \operatorname{dist}(y, G_0). \tag{3.32}$$

We also have

$$\begin{aligned} 0 \le \widehat{H}'_\sigma(s) &= \int_{\partial G_0} \partial_\nu \widehat{\kappa}_\sigma\,\mathrm{d}S_y = \int_{\partial G_0} \widehat{\kappa}\partial_\nu\widehat{\kappa}_\sigma\,\mathrm{d}S_y \\ &= \int_{\mathbb{R}^n\setminus G_0} \nabla\widehat{\kappa}\nabla\widehat{\kappa}_\sigma dy = \int_{\partial G_0} \widehat{\kappa}_\sigma\partial_\nu\widehat{\kappa}\,\mathrm{d}S_y \\ &\le \int_{\partial G_0} \partial_\nu\widehat{\kappa}\,\mathrm{d}S_y. \end{aligned}$$

This gives us a universal bound of $\widehat{H}'_\sigma$ depending only on G_0, but not on σ:

$$0 \le \widehat{H}'_\sigma(s) \le \lambda_{G_0} = \int_{\partial G_0} \partial_\nu\widehat{\kappa}\,\mathrm{d}S_y. \tag{3.33}$$

The value of λ_{G_0} is precisely the so-called *2-capacity of* G_0 (see Remark 3.11) and it is quite relevant in many applications.

Remark 3.18. Even if we consider $g^\varepsilon \neq 0$ and $\widehat{w}_\sigma(x,y,s)$ the solution of (3.28), when we take the derivative in s we still recover (3.29). So (3.33) is universal also in g^ε (where the derivative is taken with respect to s). We can guarantee $\widehat{H}(x,s) \le H(x,0) + s\lambda_{G_0}$.

Lastly, we will get an energy estimate. Assume that $s \ge 0$. By the maximum principle, we have that $\widehat{w}_\sigma, \partial_\nu \widehat{w}_\sigma \ge 0$ in ∂G_0. Taking $\widehat{w}_\sigma$ as a test function in the weak formulation of (3.25) we have

$$\int_{\mathbb{R}^n \setminus G_0} |\nabla \widehat{w}_\sigma|^2 \, dx = \int_{\partial G_0} \widehat{w}_\sigma \partial_\nu \widehat{w}_\sigma \, dS \le s\widehat{H}_\sigma(s). \tag{3.34}$$

This holds true also for $s < 0$, since $\widehat{w}_\sigma, \partial_\nu \widehat{w}_\sigma < 0$ in ∂G_0.

Remark 3.19. Recalling Remark 3.11 we have that λ_{G_0} comes from the 2-capacity. In particular

$$\lambda_{B_1} = (n-2)|\partial B_1|.$$

Remark 3.20. Let us compare $\widehat{w}_{\sigma_1}$ and $\widehat{w}_{\sigma_2}$ for two different continuous functions σ. The difference solves

$$\begin{cases} \Delta(\widehat{w}_{\sigma_1} - \widehat{w}_{\sigma_2}) = 0 & \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_\nu(\widehat{w}_{\sigma_1} - \widehat{w}_{\sigma_2}) - C_0(\sigma_1(s - \widehat{w}_{\sigma_1}) - C_0\sigma_1(s - \widehat{w}_{\sigma_2})) & \\ \quad = C_0(\sigma_1(s - \widehat{w}_{\sigma_2}) - \sigma_2(s - \widehat{w}_{\sigma_2})) & \\ \quad \le C_0 \|\sigma_1 - \sigma_2\|_{L^\infty(-s,s)} & \partial G_0, \\ \widehat{w}_{\sigma_1} - \widehat{w}_{\sigma_2} \to 0 & |y| \to \infty. \end{cases}$$

Hence, if $\sigma_m \to \sigma$ uniformly over compacts, then $\widehat{w}_{\sigma_m}(s,\cdot) \to \widehat{w}_\sigma(s,\cdot)$ a. e. in x, for each $s > 0$.

Let us see what happens in the extreme case in which σ approaches a multivalued graph.

Remark 3.21. Consider the maximal monotone graph associated to the homogeneous Dirichlet boundary condition, σ_D, given by (2.2). Let us construct a sequence of functions σ such that (at least intuitively) $\sigma_m \to \sigma_D$. Going back to (3.30) we have the estimate

$$\int_{\mathbb{R}^n \setminus G_0} \left|\nabla(\widehat{\kappa} - \widehat{\kappa}_\sigma)\right|^2 dx + \int_{\partial G_0} C_0 \sigma_m'(s - \widehat{w}_{\sigma_m})(\widehat{\kappa} - \widehat{\kappa}_\sigma)^2 \, dS \le C \int_{\partial G_0} |\partial_\nu \widehat{\kappa}|^2 \, dS.$$

Take, for $m \in \mathbb{N}$, $\sigma_m(t) = mt$. Then,

$$m \int_{\partial G_0} \left(\widehat{\kappa} - \widehat{\kappa}_{\sigma_m}(s,y)\right)^2 dS_y \le C.$$

Then, as $m \to +\infty$, we have $\widehat{\kappa}_{\sigma_m}(s,\cdot) \to \widehat{\kappa}$ in $L^2(\partial G_0)$ (uniformly in s). Since they are harmonic functions such that they coincide at infinity, we have that $\widehat{\kappa}_{\sigma_m}(s,\cdot) \to \widehat{\kappa}$ in $\mathbb{R}^n \setminus$

G_0 and so $\widehat{w}_\sigma(s,\cdot) \to s\widehat{\kappa}(\cdot)$. Let $R > 0$ be such that $\overline{G}_0 \subset B_R$. The pointwise convergence together with (3.31) and (3.34) ensures that $\widehat{w}_\sigma(s,\cdot) \rightharpoonup s\widehat{\kappa}(\cdot)$ weakly in $H^1(B_R \setminus \overline{G_0})$. Thus, taking a smooth function η such that $\eta = 1$ in G_0 and $\eta = 0$ in $\mathbb{R}^n \setminus \overline{B_R}$ we have

$$\begin{aligned} \widehat{H}_{\sigma_m}(s) &= \int_{\partial G_0} \eta(y)\partial_\nu \widehat{w}_{\sigma_m}(s,y)\,\mathrm{d}S_y = \int_{B_R\setminus G_0} \nabla\eta(y)\nabla\widehat{w}_{\sigma_m}(s,y)\,\mathrm{d}y \\ &\to s\int_{B_R\setminus G_0} \nabla\eta(y)\nabla\widehat{\kappa}(y)\,\mathrm{d}y = s\int_{\partial G_0} \eta\partial_\nu\widehat{\kappa}(y)\,\mathrm{d}S_y = \lambda_{G_0}s. \end{aligned}$$

Hence, we recover that

$$\widehat{H}_{\sigma_\mathrm{D}}(s) = \lambda_{G_0}s,$$

i. e., the strange term associated to the Dirichlet boundary condition is a linear function of the unknown. The fact that the Dirichlet condition on the particles leads to a linear effective diffusion (related to the capacity) was one of the main results in [79], where the terminology *strange term* originated. Different authors put as coefficient a measure μ but we know now that at least under the abovementioned conditions $\mu = \lambda_{G_0}$. Note that $\widehat{H}_{\sigma_D}$ becomes an extremal case of the universal bound (3.33), so the bound is sharp.

A similar argument can be applied to recover the intuition that, when we deal with the Signorini boundary condition, we have

$$\widehat{H}_\sigma(s) \to \begin{cases} \widehat{H}_{\sigma_0}(s) & s \geq 0, \\ \lambda_{G_0}s & s < 0, \end{cases} \quad \text{as } \sigma \to \text{graph of Signorini condition} = \begin{cases} \sigma_0(s) & s \geq 0, \\ (-\infty, 0] & s = 0, \\ \emptyset & s < 0. \end{cases}$$

This behavior has been proved in many papers (see, for instance, the references given in Appendix C).

Approximation

Since we have now started from the function $\widehat{w}$, but wish to apply rescaled and cut-off functions of type w_ε, we need to introduce several approximation results. They were developed in [116]. We move on to defining

$$\widehat{w}_{\sigma,\varepsilon}(x) = \widehat{w}_\sigma\left(\frac{x}{a_\varepsilon}\right).$$

Remark 3.22. Note that due to the scaling we get $\partial_\nu \widehat{w}_{\sigma,\varepsilon} = C_0 a_\varepsilon^{-1}\sigma(s - \widehat{w}_{\sigma,\varepsilon})$. This is the reason for the scaling $C_0 \simeq a_\varepsilon\beta(\varepsilon)$.

Finally, we take $w_{\sigma,\varepsilon}$ as the solution of

$$\begin{cases} \Delta w_{\sigma,\varepsilon} = 0 & \frac{\varepsilon}{4}B_1 \setminus a_\varepsilon \overline{G_0}, \\ \partial_\nu w_{\sigma,\varepsilon} = C_0 a_\varepsilon^{-1} \sigma(s - w_{\sigma,\varepsilon}) & a_\varepsilon \partial G_0, \\ w_{\sigma,\varepsilon} = 0 & \frac{\varepsilon}{4}\partial B_1. \end{cases} \tag{3.35}$$

The following result improves some estimates given in [116] and it uses the independence with respect to σ'.

Lemma 3.23. *We have the following properties:*

(a) *If $s \geq 0$, then $w_{\sigma,\varepsilon} \leq \widehat{w}_{\sigma,\varepsilon}$ (and conversely if $s \leq 0$).*

(b) *We have*

$$|w_{\sigma,\varepsilon} - \widehat{w}_{\sigma,\varepsilon}| \leq \max_{|y| = \frac{\varepsilon}{4a_\varepsilon}} \left|\widehat{w}_\sigma(y, s)\right| \leq C|s|\varepsilon^2. \tag{3.36}$$

(c) *If $s \geq 0$, then*

$$0 \leq -\partial_\nu(\widehat{w}_{\sigma,\varepsilon} - w_{\sigma,\varepsilon}) \leq C_0 a_\varepsilon^{-1} \sigma(s), \quad \text{on } \partial G_0, \tag{3.37}$$

and the converse inequality holds for $s \leq 0$.

(d) *We have the estimate*

$$\int_{\frac{\varepsilon}{4}B_1 \setminus (a_\varepsilon G_0)} \left|\nabla w_{\sigma,\varepsilon}(x, s) - \nabla \widehat{w}_{\sigma,\varepsilon}(x, s)\right|^2 \mathrm{d}x \leq C|s|^2 \varepsilon^{n+2}, \tag{3.38}$$

where C does not depend on σ in any of the previous estimates.

Proof. Let $s \geq 0$ and $\varepsilon > 0$ be fixed. We have that $v = \widehat{w}_{\sigma,\varepsilon} - w_{\sigma,\varepsilon}$ solves

$$\begin{cases} \Delta v = 0 & \frac{\varepsilon}{4}B_1 \setminus a_\varepsilon \overline{G_0}, \\ \partial_\nu v = C_0 a_\varepsilon^{-1} \sigma(s - \widehat{w}_{\sigma,\varepsilon}) - C_0 a_\varepsilon^{-1} \sigma(s - w_{\sigma,\varepsilon}) & a_\varepsilon \partial G_0, \\ v = \widehat{w}_{\sigma,\varepsilon} & \frac{\varepsilon}{4}\partial B_1. \end{cases}$$

Thus, the weak formulation is

$$\int_{\frac{\varepsilon}{4}B_1 \setminus a_\varepsilon G_0} \nabla v \nabla \varphi \,\mathrm{d}x + C_0 a_\varepsilon^{-1} \int_{a_\varepsilon \partial G_0} (\sigma(s - w_{\sigma,\varepsilon}) - \sigma(s - \widehat{w}_{\sigma,\varepsilon}))\varphi \,\mathrm{d}S_x = \int_{\frac{\varepsilon}{4}\partial B_1} \varphi \partial_\nu v \,\mathrm{d}S_x,$$

for any test function $\varphi \in H^1(\frac{\varepsilon}{4}B_1 \setminus a_\varepsilon G_0)$. Take $\varphi = v_-$. On the one hand $(\sigma(s - w_{\sigma,\varepsilon}) - \sigma(s - \widehat{w}_{\sigma,\varepsilon}))v_- \geq 0$ by the monotonicity of σ. On the other hand, on $\frac{\varepsilon}{4}\partial B_1$, we have $\varphi = (\widehat{w}_{\sigma,\varepsilon})_- = 0$. Thus

$$\int_{\frac{\varepsilon}{4}B_1 \setminus a_\varepsilon G_0} |\nabla v_-|^2 \,\mathrm{d}x \leq 0.$$

That means $v \geq 0$ or, equivalently, $w_{\sigma,\varepsilon} \leq \widehat{w}_{\sigma,\varepsilon}$.

For the pointwise estimates, let us replace σ by $\sigma_m(s) = \sigma(s) + \frac{s}{m}$ for $m \in \mathbb{N}$, which is strictly increasing. Let $v_m = \widehat{w}_{\varepsilon,\sigma_m} - w_{\varepsilon,\sigma_m}$. Thus, it follows that on $a_\varepsilon \partial G_0$,

$$\partial_\nu v_m = C_0 a_\varepsilon^{-1} \sigma_m(s - \widehat{w}_{\varepsilon,\sigma_m}) - C_0 a_\varepsilon^{-1} \sigma_m(s - w_{\varepsilon,\sigma_m}) \le 0.$$

Furthermore, assume that $\partial_\nu v_m(x) = 0$ for some $x \in a_\varepsilon \partial G_0$. Then $\sigma_m(s - \widehat{w}_{\varepsilon,\sigma_m}(x,s)) = \sigma_m(s - w_{\varepsilon,\sigma_m}(x,s))$ and, since σ_m is strictly increasing, we recover $v_m = 0$. Hence, at each point of the boundary $\partial_\nu v_m(x) > 0$ or $v_m = 0$. Hence, the global maximum of v_m cannot happen in $a_\varepsilon \partial G_0$. Since v_m is harmonic it achieves its maximum on $\frac{\varepsilon}{4}\partial B_1$. Since the value there is explicit, we recover (3.36). Hence $0 \le w_{\varepsilon,\sigma_m} \le \widehat{w}_{\varepsilon,\sigma_m} \le s$, so we recover (3.37). We can pass to the limit as $m \to \infty$ applying Remark 3.20 and the equivalent argument for $w_{\sigma,m}$.

Finally, by Green's formula

$$\begin{aligned}
\int_{\frac{\varepsilon}{4}B_1 \setminus a_\varepsilon G_0} |\nabla v|^2 \,\mathrm{d}x &\le \left| \int_{\frac{\varepsilon}{4}\partial B_1} \widehat{w}_{\sigma,\varepsilon} \partial_\nu v \,\mathrm{d}S_x \right| \\
&\le \left| \int_{\frac{\varepsilon}{4}B_1 \setminus \frac{\varepsilon}{8}B_1} \nabla \widehat{w}_{\sigma,\varepsilon} \nabla v \,\mathrm{d}x \right| + \left| \int_{\frac{\varepsilon}{8}\partial B_1} \widehat{w}_{\sigma,\varepsilon} \partial_\nu v \,\mathrm{d}S_x \right| \\
&\le \left| \int_{\frac{\varepsilon}{4}\partial B_1} (\partial_\nu \widehat{w}_{\sigma,\varepsilon}) v \,\mathrm{d}S_x \right| + \left| \int_{\frac{\varepsilon}{8}\partial B_1} (\partial_\nu \widehat{w}_{\sigma,\varepsilon}) v \,\mathrm{d}S_x \right| + \left| \int_{\frac{\varepsilon}{8}\partial B_1} \widehat{w}_{\sigma,\varepsilon} \partial_\nu v \,\mathrm{d}S_x \right|.
\end{aligned}$$

The first two terms can be controlled by (3.32) and (3.36). First, since $|x| = \frac{\varepsilon}{4}$, $|y| = \frac{\varepsilon}{4a_\varepsilon}$ for ε small enough, $R = |y|/2 < \operatorname{dist}(y, G_0)$. Then (3.32) becomes

$$\left|\partial_\nu \widehat{w}_{\sigma,\varepsilon}(x)\right| \le a_\varepsilon^{-1} |\nabla_y \widehat{w}_\sigma| \le C a_\varepsilon^{-1} |s| |y|^{1-n} = C|s| \frac{a_\varepsilon^{n-2}}{\varepsilon^{n-1}} \le C|s|\varepsilon.$$

We have

$$\left| \int_{\frac{\varepsilon}{4}\partial B_1} (\partial_\nu \widehat{w}_{\sigma,\varepsilon}) v \,\mathrm{d}S_x \right| \le C|s|^2 \varepsilon^{n+2},$$

and the same in $\frac{\varepsilon}{8}\partial B_1$.

For the last integral, we repeat the proof of (3.32). Take x such that $|x| = \frac{\varepsilon}{8}$. For $\varepsilon > 0$ we have that taking $R = \frac{\varepsilon}{16} \le \frac{\varepsilon}{4} - |x| < \operatorname{dist}(x, G_0)$,

$$\left| \frac{\partial v}{\partial x_i}(x) \right| \le C R^{-1} \|v\|_\infty \le C|s|\varepsilon.$$

Using the explicit bound of $\widehat{w}_{\sigma,\varepsilon}$ and (3.36) we get

$$\left| \int_{\frac{\varepsilon}{8}\partial B_1} \widehat{w}_{\sigma,\varepsilon} \partial_\nu v \,\mathrm{d}S_x \right| \le C|s|^2 \varepsilon^{n+2}. \qquad \square$$

3.2 Case of $(n-1)$-dimensional particles

For $(n-1)$-dimensional particles we do not need an extension operator, since u_ε is also defined on S_ε. Thus, some of the computations in the n-dimensional case are not needed.

3.2.1 Trace estimates on $a_\varepsilon G_0$ in εY^+

Let $0 \in G_0 \subset Y^0$ be an $(n-1)$-manifold, $u \in W^{1,p}(\varepsilon Y^+)$, $p > 1$, and assume that $a_\varepsilon \le \varepsilon$. Then

$$\int_{a_\varepsilon G_0} |u|^p \,\mathrm{d}x \le C a_\varepsilon^{n-1} \Big(\varepsilon^{-n} \int_{\varepsilon Y^+} |u|^p \,\mathrm{d}x + \tau_\varepsilon \int_{\varepsilon Y^+} |\nabla u|^p \,\mathrm{d}x \Big), \tag{3.39}$$

where again τ_ε is given by (3.6). To prove this result, we proceed similarly to Section 3.1. Using the trace theorem in $W^{1,p}(B_R^+)$ and equivalence of norms, we know that

$$\int_{G_0} |v|^p \,\mathrm{d}S_y \le C \Big(\int_{(\partial B_R)^+} |v|^p \,\mathrm{d}S_y + \int_{B_R^+} |\nabla v|^p \,\mathrm{d}y \Big), \quad \forall v \in W^{1,p}(B_R^+).$$

Scaling this, we recover that

$$\int_{a_\varepsilon G_0} |u|^p \,\mathrm{d}S_x \le C \Big(\int_{a_\varepsilon (\partial B_R)^+} |u|^p \,\mathrm{d}S_x + a_\varepsilon^{p-1} \int_{a_\varepsilon (B_R)^+} |\nabla u|^p \,\mathrm{d}x \Big).$$

Now we can apply the same argument as in Lemma 3.6.

Remark 3.24. This last estimate says that since G_0 is an $(n-1)$-manifold (even though it is not the boundary of an open set), the trace estimates are the same as for those which actually are boundaries. This should not be surprising, if one thinks on continuous deformations.

3.2.2 Auxiliary functions in the subcritical case

We need to introduce some auxiliary functions similar to the ones defined in Section 3.1.4. Here, we do only the computations for $m_{g,\varepsilon}$:

$$\begin{cases} \Delta_p m_{g,\varepsilon} = \mu_\varepsilon & x \in \varepsilon Y^+, \\ \partial_{\nu_p} m_{g,\varepsilon} = g(x/a_\varepsilon) & x \in a_\varepsilon G_0, \\ \partial_{\nu_p} m_{g,\varepsilon} = 0 & x \in \partial(\varepsilon Y^+) \setminus a_\varepsilon G_0, \end{cases} \qquad \int_{\varepsilon Y^+} m_{g,\varepsilon}(x) \,\mathrm{d}x = 0. \tag{3.40}$$

In this setting

$$\mu_\varepsilon = \left|\varepsilon Y^+\right|^{-1} \int\limits_{a_\varepsilon G_0} g(x/a_\varepsilon)\, \mathrm{d}S_x = \frac{a_\varepsilon^{n-1}}{\varepsilon^n/2} \int\limits_{G_0} g(y)\, \mathrm{d}S_y.$$

As above, we have the following.

Lemma 3.25. *Let $p > 1$ and $a_\varepsilon \le \varepsilon$. Then, if $m_{g,\varepsilon}$ is the solution of* (3.40), *we have*

$$\|\nabla m_{g,\varepsilon}\|_{L^p(\varepsilon Y^+)} \le C \begin{cases} a_\varepsilon^{\frac{n}{p}} & p < n, \\ a_\varepsilon (\ln \frac{2\varepsilon}{a_\varepsilon})^{\frac{1}{n}} & p = n, \\ a_\varepsilon^{\frac{n-1}{p-1}} \varepsilon^{\frac{p-n}{p(p-1)}} & p > n, \end{cases}$$

where C depends only on $\|g\|_{L^{p'}(G_0)}$.

Proof. From the definition of $m_{g,\varepsilon}(x)$, applying Lemma 3.6 we obtain

$$\begin{aligned}
\|\nabla m_{g,\varepsilon}\|^{p^2}_{L^p(\varepsilon Y^+)} &\le \left(\left| \int\limits_{a_\varepsilon G_0} m_{g,\varepsilon} \partial_{\nu_p} m_{g,\varepsilon}\, \mathrm{d}S \right| + |\mu_\varepsilon| \left| \int\limits_{\varepsilon Y^+} m_{g,\varepsilon}\, \mathrm{d}x \right| \right)^p \\
&\le \left(\int\limits_{a_\varepsilon G_0} |m_{g,\varepsilon} \partial_{\nu_p} m_{g,\varepsilon}|\, \mathrm{d}S + |\mu_\varepsilon| \times 0 \right)^p \\
&\le \left(\left(\int\limits_{a_\varepsilon G_0} \left|g(x/a_\varepsilon)\right|^{p'} \mathrm{d}S \right)^{\frac{1}{p'}} \left(\int\limits_{a_\varepsilon G_0} |m_{g,\varepsilon}|^p\, \mathrm{d}S \right)^{\frac{1}{p}} \right)^p \\
&\le a_\varepsilon^{(n-1)(p-1)} \|g\|^p_{L^{p'}(G_0)} \|m_{g,\varepsilon}\|^p_{L^p(a_\varepsilon G_0)} \\
&\le C_1 a_\varepsilon^{(n-1)(p-1)} \|g\|^p_{L^{p'}(G_0)} \|m_{g,\varepsilon}\|^p_{L^p(a_\varepsilon G_0)} \le \\
&\le C_2 a_\varepsilon^{(n-1)(p-1)} a_\varepsilon^{n-1} \|g\|^p_{L^{p'}(G_0)} \left(\varepsilon^{-n} \|m_{g,\varepsilon}\|^p_{L^p(\varepsilon Y^+)} + \tau_\varepsilon \|\nabla m_{g,\varepsilon}\|^p_{L^p(\varepsilon Y^+)}\right) \\
&\le C_3 a_\varepsilon^{p(n-1)} \|g\|^p_{L^{p'}(G_0)} (\varepsilon^{-n+p} + \tau_\varepsilon) \|\nabla m_{g,\varepsilon}\|^p_{L^p(\varepsilon Y^+)},
\end{aligned}$$

where τ_ε is given by (3.6). Therefore

$$\|\nabla m_{g,\varepsilon}\|^{p-1}_{L^p(\varepsilon Y^+)} \le C a_\varepsilon^{n-1} (\varepsilon^{-n+p} + \tau_\varepsilon)^{\frac{1}{p}} \|g\|_{L^{p'}(G_0)}.$$

Now we can use the estimates on τ_ε and the conclusion holds. □

3.2.3 Auxiliary functions in the critical case

3.2.3.1 When G_0 is not a ball and $p = 2 < n$

Similarly to the case of n-dimensional particles, we define $\widehat{w}_\sigma(y,s)$ by

$$\begin{cases} \Delta \widehat{w}_\sigma = 0 & y \in (\mathbb{R}^n)^+, \\ \partial_\nu \widehat{w}_\sigma = C_0 \sigma(s - \widehat{w}_\sigma) & y \in G_0, \\ \partial_\nu \widehat{w}_\sigma = 0 & y \in (\mathbb{R}^n)^0 \setminus \overline{G_0}, \\ \widehat{w}_\sigma \to 0 & \text{as } |y| \to \infty. \end{cases}$$

Recall that in this setting $\nu = -e_n$. The effective reaction is related to the function

$$\widehat{H}_\sigma(s) = \int_{G_0} \partial_\nu \widehat{w}(y,s)\,\mathrm{d}S_y.$$

Exactly as we did in Section 3.1.5.3 we can introduce the auxiliary problem

$$\begin{cases} \Delta \widehat{\kappa} = 0 & y \in (\mathbb{R}^n)^+, \\ \widehat{\kappa} = 1 & y \in G_0, \\ \partial_\nu \widehat{\kappa} = 0 & y \in (\mathbb{R}^n)^0 \setminus \overline{G_0}, \\ \widehat{\kappa} \to 0 & \text{as } |y| \to \infty, \end{cases}$$

and define

$$\lambda_{G_0} = \int_{G_0} \partial_\nu \widehat{\kappa}(y)\,\mathrm{d}S_y.$$

We have

$$0 \le \frac{\partial \widehat{w}}{\partial s}(y,s) \le \widehat{\kappa}(y)s, \quad 0 \le \widehat{H}'_\sigma(y) \le \lambda_{G_0}.$$

3.2.3.2 When G_0 is a ball and $p = n$

In this setting $G_0 = B_1^0$ (again, recall Remark 1.1). The good choice in this setting, to operate as before, is to take q_ε as the solution of

$$\begin{cases} -\Delta_p q_\varepsilon = 0 & \frac{\varepsilon}{4} B_1^+, \\ q_\varepsilon = 0 & \frac{\varepsilon}{4} (\partial B_1)^+, \\ q_\varepsilon = 1 & a_\varepsilon G_0, \\ \partial_{\nu_p} q_\varepsilon = 0 & \frac{\varepsilon}{4} B_1^0 \setminus a_\varepsilon G_0. \end{cases}$$

There is no explicit expression for q_ε. Hence, we change it slightly by the new problem

$$\begin{cases} -\Delta_p w_\varepsilon = 0 & \frac{\varepsilon}{4}B_1^+ \setminus a_\varepsilon \overline{B_1^+}, \\ w_\varepsilon = 0 & \frac{\varepsilon}{4}(\partial B_1)^+, \\ w_\varepsilon = 1 & a_\varepsilon(\partial B_1)^+, \\ \partial_{\nu_p} w_\varepsilon = 0 & \frac{\varepsilon}{4}B_1^0 \setminus a_\varepsilon B_1^0. \end{cases}$$

By using the symmetrical extension $w(x_1, \dots, x_{n-1}, x_n) = w(x_1, \dots, x_{n-1}, -x_n)$, it is easy to see that it is enough to solve

$$\begin{cases} -\Delta_p w_\varepsilon = 0 & \frac{\varepsilon}{4}B_1 \setminus a_\varepsilon \overline{B_1}, \\ w_\varepsilon = 0 & \frac{\varepsilon}{4}\partial B_1, \\ w_\varepsilon = 1 & a_\varepsilon \partial B_1. \end{cases}$$

This is the same auxiliary function as in the previous setting: w_ε is given by (3.23), which we extend into $a_\varepsilon B_1$ by 1 (due to the explicit expression, it is easy to see that the extension lies in $W^{1,p}(\frac{\varepsilon}{4}B_1^+)$). It is not hard to check the following lemma.

Lemma 3.26. *Let $p = n$. Then*

$$\int_{\frac{\varepsilon}{4}B_1^+} \left|\nabla(w_\varepsilon - q_\varepsilon)\right|^n \mathrm{d}x \le C\left(\ln \frac{\varepsilon}{4a_\varepsilon}\right)^{-n}. \tag{3.41}$$

We proceed similarly to [106], where the result can be found for $p = 2$.

Proof. First, since $w_\varepsilon - q_\varepsilon = 0$ on $\frac{\varepsilon}{4}(\partial B_1)^+$ and $a_\varepsilon G_0$, using it as a test function in both equations we have

$$\int_{\frac{\varepsilon}{4}B_1^+} |\nabla q_\varepsilon|^{p-2}\nabla q_\varepsilon \nabla(w_\varepsilon - q_\varepsilon)\,\mathrm{d}x = 0,$$

$$\int_{\frac{\varepsilon}{4}B_1^+ \setminus a_\varepsilon B_1^+} |\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon \nabla(w_\varepsilon - q_\varepsilon)\,\mathrm{d}x = \int_{a_\varepsilon(\partial B_1)^+} \partial_{\nu_p} w_\varepsilon (w_\varepsilon - q_\varepsilon)\,\mathrm{d}S_x$$

(note that in $a_\varepsilon B_1^+$ we have $w_\varepsilon = 1$, so its gradient vanishes). Therefore,

$$\int_{\frac{\varepsilon}{4}B_1^+} (|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon - |\nabla q_\varepsilon|^{p-2}\nabla q_\varepsilon)\nabla(w_\varepsilon - q_\varepsilon)\,\mathrm{d}x = \int_{a_\varepsilon(\partial B_1)^+} \partial_{\nu_p} w_\varepsilon (w_\varepsilon - q_\varepsilon)\,\mathrm{d}S_x.$$

On the other hand,

$$\int_{a_\varepsilon(\partial B_1)^+} \partial_{\nu_p} w_\varepsilon (w_\varepsilon - q_\varepsilon)\,\mathrm{d}S_x \le |\partial_{\nu_p} w_\varepsilon| \int_{a_\varepsilon(\partial B_1)^+} |w_\varepsilon - q_\varepsilon|\,\mathrm{d}S_x.$$

Since w_ε is radial, we can find the explicit estimate

$$|\partial_{\nu_p} w_\varepsilon|\Big|_{a_\varepsilon \partial B_1} \sim \left(a_\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}\right)^{1-n}.$$

Take $v(y) = w_\varepsilon(a_\varepsilon y) - q_\varepsilon(a_\varepsilon y)$ defined in $\frac{\varepsilon}{4a_\varepsilon}B_1^+$ and $v = 0$ on $(\frac{\varepsilon}{4a_\varepsilon}\partial B_1)^+$. For ε small enough we can use the trace theorem, $W^{1,r}(B_2^+ \setminus B_1^+) \to L^1(\partial\Omega)$, and we have

$$\int\limits_{(\partial B_1)^+} |v(y)|\, dS_y \le C\Bigg(\int\limits_{B_2^+\setminus B_1^+} |\nabla v(y)|^p\, dy\Bigg)^{\frac{1}{p}} \le C\Bigg(\int\limits_{(\frac{\varepsilon}{4a_\varepsilon}B_1^+)} |\nabla v(y)|^p\, dy\Bigg)^{\frac{1}{p}}.$$

Scaling this down,

$$a_\varepsilon^{1-n} \int\limits_{a_\varepsilon(\partial B_1)^+} |w_\varepsilon - q_\varepsilon|\, dS_x \le C\Bigg(\int\limits_{\frac{\varepsilon}{4}B_1^+} |\nabla(w_\varepsilon - q_\varepsilon)|^p\, dx\Bigg)^{\frac{1}{p}}.$$

Hence

$$\begin{aligned}\int\limits_{\frac{\varepsilon}{4}B_1^+} |\nabla(w_\varepsilon - q_\varepsilon)|^p\, dx &\le C \int\limits_{\frac{\varepsilon}{4}B_1^+} \left(|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon - |\nabla q_\varepsilon|^{p-2}\nabla q_\varepsilon\right)\nabla(w_\varepsilon - q_\varepsilon)\, dx \\ &\le C\left(\ln \frac{\varepsilon}{4a_\varepsilon}\right)^{1-n}\Bigg(\int\limits_{\frac{\varepsilon}{4}B_1^+} |\nabla(w_\varepsilon - q_\varepsilon)|^p\, dx\Bigg)^{\frac{1}{p}},\end{aligned}$$

and we recover the result. □

4 Particles over the whole domain

In this chapter we study

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & \text{in } \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon) g^\varepsilon(x) & \text{on } S_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in the geometry presented in Section 1.2.1.1. To avoid repetition, we will not specify the setting in each result but we recall the equivalent formulations to the definition of weak solutions presented in Section 2.2. Going back to Remark 2.15 we have

$$\beta^\star(\varepsilon) \simeq \varepsilon^n a_\varepsilon^{1-n} |\Omega|^{-1} |\partial G_0|^{-1}. \tag{4.1}$$

Throughout this entire chapter the critical scaling will be

$$a_\varepsilon^\star = \begin{cases} \varepsilon^{\frac{n}{n-p}} & \text{if } 1 < p < n, \\ \varepsilon e^{-M\varepsilon^{-\frac{n}{n-1}}} & \text{for any } M > 0 \text{ if } p = n, \\ 0 & p > n, \end{cases}$$

where p is, as usual, the exponent of the diffusion operator in (1.1). Below we will explain why this is the correct value.

Remark 4.1. Note that the special case $p = n$ does not only have a critical scale, but a whole family of them indexed by a parameter M. Thus, we must update slightly the notation used in Section 1.3 for this case:

$$a_\varepsilon \sim a_\varepsilon^\star \iff \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(\ln \frac{\varepsilon}{4a_\varepsilon} \right)^{\frac{1-n}{n}} \in (0, \infty),$$

$$a_\varepsilon \ll a_\varepsilon^\star \iff \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(\ln \frac{\varepsilon}{4a_\varepsilon} \right)^{\frac{1-n}{n}} = 0,$$

and the corresponding for $\gg$, i. e., the criterion is not on the limit of a_ε directly, but on the one of $\varepsilon^{-1}(\ln \frac{\varepsilon}{4a_\varepsilon})^{\frac{1-n}{n}}$.

4.1 On the existence of a critical scale

Throughout this text, there exist four ranges of a_ε, as mentioned in the Introduction (see, e. g., Table 1.1). The first big distinction, $a_\varepsilon \sim \varepsilon$ or $a_\varepsilon \ll \varepsilon$, is natural since

$$|\Omega \setminus G_\varepsilon| = |\Upsilon_\varepsilon| |\varepsilon Y \setminus a_\varepsilon G_0| \simeq |\Omega| \varepsilon^{-n} |\varepsilon Y \setminus a_\varepsilon G_0|.$$

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Here we have used the fact that $|\Upsilon_\varepsilon| \simeq |\Omega|\varepsilon^{-n}$ (see Section 1.2.1.1). If $a_\varepsilon = \varepsilon$, then $|\varepsilon Y \setminus a_\varepsilon G_0| = \varepsilon^n |Y \setminus G_0|$, while if $a_\varepsilon \ll \varepsilon$, then $|\varepsilon Y \setminus a_\varepsilon G_0| \simeq \varepsilon^n |Y|$. This clearly affects most of the estimates. Furthermore, take a function v defined in $\mathbb{R}^n$ (later it will be chosen for suitable purposes in order to get good test functions) and let v_ε be defined periodically by

$$v_\varepsilon(x) = \begin{cases} v(\frac{x-\varepsilon j}{a_\varepsilon}) & x \in \varepsilon(j+Y) \text{ for some } j \in \Upsilon_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\int_\Omega v_\varepsilon(x)\,\mathrm{d}x = |\Upsilon_\varepsilon| \int_{\varepsilon Y} v(x/a_\varepsilon)\,\mathrm{d}x = |\Upsilon_\varepsilon| a_\varepsilon^n \int_{\frac{\varepsilon}{a_\varepsilon} Y} v(y)\,\mathrm{d}y.$$

If $a_\varepsilon = \varepsilon$, then

$$\int_\Omega v_\varepsilon(x)\,\mathrm{d}x \to |\Omega| \int_Y v(y)\,\mathrm{d}y.$$

However, if $a_\varepsilon \ll \varepsilon$ and $v \in L^1(\mathbb{R}^n)$, we have

$$\int_\Omega v_\varepsilon(x)\,\mathrm{d}x \sim \varepsilon^{-n} a_\varepsilon^n \int_{\mathbb{R}^n} v(y)\,\mathrm{d}y \to 0.$$

This will be very useful, especially for the supercritical case $a_\varepsilon \ll a_\varepsilon^\star$.

The second distinction in the ranges of a_ε is related to the existence of a critical value $a_\varepsilon^\star$, coming from the scaling of the p-Laplace operator and its energy. For $p \in (1, n)$, assume that v is compactly supported and $\nabla v \in L^p(\mathbb{R}^n)$. If we compute the p-Laplace energy when $a_\varepsilon \ll \varepsilon$ we have

$$\int_\Omega |\nabla v_\varepsilon|^p\,\mathrm{d}x = |\Upsilon_\varepsilon| a_\varepsilon^{n-p} \int_{\mathbb{R}^n} |\nabla v(y)|^p\,\mathrm{d}y \sim \varepsilon^{-n} a_\varepsilon^{n-p}. \tag{4.2}$$

The critical value, $a_\varepsilon^\star$, is taken such that the asymptotic expansion of the corresponding p-Laplacian satisfies $\varepsilon^{-n}(a_\varepsilon^\star)^{n-p} \sim 1$. When $p > n$, the classical Sobolev embedding $W^{1,p}(\Omega) \to \mathcal{C}(\overline{\Omega})$ suggests there is no critical scale (see [114]), and we can always compute Riemann style sums. The different cases are presented in Table 1.1 and their mathematical justification is the main goal of this chapter. As expected, the case $a_\varepsilon \sim a_\varepsilon^\star$ is by far the more difficult case. The case $p = n$ will be treated separately.

Below we give a rigorous justification of these intuitive estimates.

4.2 Integrals over S_ε

4.2.1 Improved global trace inequality

By applying Lemma 3.6 in each particle we deduce the following.

Lemma 4.2. *We have*

$$\beta^\star(\varepsilon)\int_{S_\varepsilon}|u|^p\,\mathrm{d}S\le C\bigg(\int_{\Omega_\varepsilon}|u|^p\,\mathrm{d}x+\varepsilon^n\tau_\varepsilon\int_{\Omega_\varepsilon}|\nabla u|^p\,\mathrm{d}x\bigg),$$

where τ_ε is given by (3.6).

Proof. From a direct computation, using Lemma 3.6 it follows that

$$\begin{aligned}\beta^\star(\varepsilon)\int_{S_\varepsilon}|u|^p\,\mathrm{d}S_x&=\frac{1}{|S_\varepsilon|}\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+a_\varepsilon\partial G_0}|u|^p\,\mathrm{d}S_x\\&\le\frac{C}{|\Upsilon_\varepsilon||a_\varepsilon\partial G_0|}a_\varepsilon^{n-1}\sum_{j\in\Upsilon_\varepsilon}\bigg(\varepsilon^{-n}\int_{\varepsilon j+\varepsilon Y\setminus a_\varepsilon G_0}|u|^p\,\mathrm{d}x+\tau_\varepsilon\int_{\varepsilon j+\varepsilon Y\setminus a_\varepsilon G_0}|\nabla u|^p\,\mathrm{d}x\bigg)\\&\le C\bigg(\int_{\Omega_\varepsilon}|u|^p+\varepsilon^n\tau_\varepsilon\int_{\Omega_\varepsilon}|\nabla u|^p\bigg),\end{aligned}$$

since $|\Upsilon_\varepsilon|\ge c\varepsilon^{-n}$ and $|a_\varepsilon\partial G_0|=a_\varepsilon^{n-1}|\partial G_0|$. □

Note that if τ_ε is given by (3.6), then

$$\varepsilon^n\tau_\varepsilon\sim\begin{cases}\varepsilon^n a_\varepsilon^{p-n} & p<n,\\ \varepsilon^n\ln(\frac{2\varepsilon}{a_\varepsilon})^{p-1} & p=n,\\ \varepsilon^p & p>n.\end{cases}$$

Thus, this coefficient is bounded (or tends to 0) if $a_\varepsilon\sim a_\varepsilon^\star$ (respectively $a_\varepsilon^\star\ll a_\varepsilon\le\varepsilon$). This is the first time that we are able to detect the critical scale.

Corollary 4.3. *Let $p>1$ and $a_\varepsilon\gtrsim a_\varepsilon^\star$. Then, there exists C independent of ε such that*

$$\beta^\star(\varepsilon)\int_{S_\varepsilon}|u|^p\,\mathrm{d}S\le C\bigg(\int_{\Omega_\varepsilon}|u|^p\,\mathrm{d}x+\int_{\Omega_\varepsilon}|\nabla u|^p\,\mathrm{d}x\bigg).$$

Remark 4.4. Due to Lemma 2.16, Corollary 4.3 and Theorem 3.4, if $a_\varepsilon\gtrsim a_\varepsilon^\star$, we have

$$\beta^\star(\varepsilon)\int_{S_\varepsilon}|u_\varepsilon|\,\mathrm{d}S\le\bigg(\beta^\star(\varepsilon)\int_{S_\varepsilon}|u_\varepsilon|^p\,\mathrm{d}S\bigg)^{\frac1p}\le C\bigg(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^p\bigg)^{\frac1p}\,\mathrm{d}x.$$

This is precisely (2.9). This is very useful to pass to the limit after approximating function σ.

4.2.2 Limit of the integral over S_ε of convergent sequences in the subcritical case

The aim of this section is to show the important "from surface to volume averaging convergence principle." It says that if $a_\varepsilon \gg a_\varepsilon^\star$, then the limit of the average over the set of boundaries of all the particles gives, in the limit, an average over the whole space. This gives a mathematical justification to the first "surprise" mentioned in Section 1.1.

Theorem 4.5. *Assume that $a_\varepsilon^\star \ll a_\varepsilon \le \varepsilon$ and that there exists the limit of $a_\varepsilon/\varepsilon$. Then, for any sequence $v_\varepsilon \in W^{1,p}(\Omega)$ with $\|\nabla v_\varepsilon\|_{L^p(\Omega)}$ bounded and such that $v_\varepsilon \to v$ in $L^p(\Omega)$, we have*

$$\left|\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} v_\varepsilon \, \mathrm{d}S - \frac{1}{|\Omega|}\int_\Omega v \, \mathrm{d}x\right| \le R(\varepsilon) + \begin{cases} C a_\varepsilon^{\frac{p-n}{p}} \varepsilon^{\frac{n}{p}} & p < n, \\ C\varepsilon (\ln \frac{2\varepsilon}{a_\varepsilon})^{\frac{n-1}{n}} & p = n, \\ C\varepsilon & p > n, \end{cases} \tag{4.3}$$

where $R(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Recall that $\beta^\star(\varepsilon) = |S_\varepsilon|^{-1}$. The second term in (4.3) is the reason why critical scales appear in the homogenization process for $p \le n$ and none can appear when $p > n$. Note that the critical scale is precisely the one in which this term becomes a constant. Beyond that value, the second term in (4.3) does not tend to zero. This result is sharp, in the sense that we will show that in the complementary range the behavior of the homogenization problem changes.

Remark 4.6. The assumption that there exists a limit of $a_\varepsilon/\varepsilon$ is always made throughout the text, and will not be repeated.

Remark 4.7. This kind of averaging lemmas are usually written in the literature with some dangling constants, which can make the reading rather tricky (see, e. g., [213, 113, 269]). As a mater of fact, later, in the proof of Theorem 4.36 below, we will apply Theorem 4.5 in the case of particles at the critical scale but on an artificial bigger surface set covering the set of particles, so that the corresponding passing to the limit is well guaranteed even in this case.

First, let us give an auxiliary result which probably is well known in the literature but we were unable to find a published proof of it. The most similar result we found was Theorem 2.6 of [76] but the statement is not entirely equivalent.

Lemma 4.8. *Let $a_\varepsilon \le \varepsilon$. Then,*

$$\chi_{\Omega_\varepsilon} \rightharpoonup \lim_{\varepsilon\to 0} \left| Y \setminus (a_\varepsilon \varepsilon^{-1} G_0) \right|$$

weakly-$\star$ in $L^\infty(\Omega)$.

Note that when the particles are small ($a_\varepsilon \ll \varepsilon$), then the convergence is strong in L^p, as we will prove in Lemma 4.26.

Proof. Let us first characterize the candidate to be a limit. If $v \in C^1(\overline{\Omega})$ we can check easily that

$$\int_\Omega v\chi_{\Omega_\varepsilon}\,\mathrm{d}x = \int_{\Omega_\varepsilon} v(x)\,\mathrm{d}x = \int_\Omega v(x)\,\mathrm{d}x - \sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+a_\varepsilon G_0} v(x)\,\mathrm{d}x.$$

Now we compute, taking a Taylor expansion,

$$\begin{aligned}\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+a_\varepsilon G_0} v(x)\,\mathrm{d}x &= \sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+a_\varepsilon G_0}\big(v(\varepsilon j)+\nabla v\big(\eta_\varepsilon(x)\big)\cdot(x-\varepsilon j)\big)\,\mathrm{d}x\\ &= a_\varepsilon^n\varepsilon^{-n}|G_0|\sum_{j\in\Upsilon_\varepsilon}\varepsilon^n v(\varepsilon j)+\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+a_\varepsilon G_0}\nabla v\big(\eta_\varepsilon(x)\big)\cdot(x-\varepsilon j)\,\mathrm{d}x.\end{aligned}$$

Since we are assuming that v is smooth, ∇v is bounded and the second integral tends to 0. The first integral converges as a Riemann sum to

$$\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+a_\varepsilon G_0} v(x)\,\mathrm{d}x = \Big(\lim_{\varepsilon\to 0} a_\varepsilon^n\varepsilon^{-n}|G_0|\Big)\int_\Omega v(x)\,\mathrm{d}x.$$

We point out that for any $a_0 \in (0,1)$, we have $1-a_0^n|G_0| = |Y\setminus a_0G_0|$. Hence,

$$\lim_{\varepsilon\to 0}\int_\Omega v\chi_{\Omega_\varepsilon}\,\mathrm{d}x = \int_\Omega v(x)\,\mathrm{d}x - \lim_{\varepsilon\to 0} a_\varepsilon^n\varepsilon^{-n}|G_0|\int_\Omega v(x)\,\mathrm{d}x = \Big(\lim_{\varepsilon\to 0}\big|Y\setminus(a_\varepsilon\varepsilon^{-1}G_0)\big|\Big)\int_\Omega v(x)\,\mathrm{d}x.$$

Let us consider the constant function $F_1 = \lim_{\varepsilon\to 0}|Y\setminus(a_\varepsilon\varepsilon^{-1}G_0)|$, and show it is the weak-$\star$ limit in $L^\infty(\Omega)$. Since $\chi_{\Omega_\varepsilon}$ is bounded in $L^\infty(\Omega)$, there exists a weak-$\star$ convergent subsequence. Let $F_2 \in L^\infty(\Omega)$ be its limit. This means precisely that

$$\lim_{\varepsilon\to 0}\int_\Omega v\chi_{\Omega_\varepsilon}\,\mathrm{d}x = \int_\Omega vF_2\,\mathrm{d}x, \quad \forall v \in L^1(\Omega).$$

Due to the inclusion $C^1(\overline{\Omega}) \subset L^1(\Omega)$, joining this information with the first step

$$\int_\Omega (F_1-F_2)v\,\mathrm{d}x = 0, \quad \forall v \in C^1(\overline{\Omega}).$$

Since $F_1 - F_2 \in L^\infty(\Omega)$, we deduce by density that $F_1 = F_2$. Since every weak-$\star$ convergent subsequence has the same limit, the whole sequence converges. □

Proof of Theorem 4.5. We recall the definitions of μ_ε and M_ε given by (3.11) and (3.12) from Section 3.1.4. Note that, by (3.11),

$$\mu_\varepsilon = \frac{\varepsilon^{-n}a_\varepsilon^{n-1}|\partial G_0|}{1-(a_\varepsilon\varepsilon^{-1})^n|G_0|} \simeq \frac{|S_\varepsilon|}{|\Omega|}\frac{1}{|Y\setminus(a_\varepsilon\varepsilon^{-1}G_0)|}.$$

For the sake of simplicity, in this proof we use the notation $Y_\varepsilon^j = \varepsilon j + \varepsilon Y \setminus a_\varepsilon G_0$. With this notation, taking M_ε given by (3.12) and applying (3.10),

$$\int_{S_\varepsilon} v_\varepsilon \, dS = \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} |\nabla M_\varepsilon|^{p-2} \nabla M_\varepsilon \nabla v_\varepsilon \, dx + \sum_{j \in \Upsilon_\varepsilon} \mu_\varepsilon \int_{Y_\varepsilon^j} v_\varepsilon \, dx.$$

Therefore

$$\begin{aligned} \left| \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} v_\varepsilon \, dS - \frac{1}{|\Omega|} \int_\Omega v \, dx \right| &\leq \left| \frac{1}{|S_\varepsilon|} \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} |\nabla M_\varepsilon|^{p-2} \nabla M_\varepsilon \nabla v_\varepsilon \, dx \right| \\ &\quad + \left| \frac{\mu_\varepsilon}{|S_\varepsilon|} \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} v_\varepsilon \, dx - \frac{1}{|\Omega|} \int_\Omega v \, dx \right|, \end{aligned}$$

where we used the product of weak and strong convergences. Using Hölder's inequality

$$\sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} |\nabla M_\varepsilon|^{p-1} |\nabla v_\varepsilon| \, dx \leq \|\nabla M_\varepsilon\|^{p-1}_{L^p(\bigcup_{j \in \Upsilon_\varepsilon} Y_\varepsilon^j)} \|\nabla v_\varepsilon\|_{L^p(\Omega_\varepsilon)}.$$

Due to Proposition 3.10 and the estimate on the number of particles we recover that

$$|S_\varepsilon|^{-1} \|\nabla M_\varepsilon\|^{p-1}_{L^p(\bigcup_{j \in \Upsilon_\varepsilon} Y_\varepsilon^j)} = |S_\varepsilon|^{-1} |\Upsilon_\varepsilon|^{\frac{p-1}{p}} \|\nabla m_\varepsilon\|^{p-1}_{L^p(\varepsilon Y \setminus a_\varepsilon G_0)} \leq \begin{cases} C a_\varepsilon^{\frac{p-n}{p}} \varepsilon^{\frac{n}{p}} & p < n, \\ C\varepsilon \ln(\frac{2\varepsilon}{a_\varepsilon})^{\frac{n-1}{n}} & p = n, \\ C\varepsilon & p > n. \end{cases}$$

For the last term, which we can denote as $R(\varepsilon)$, we write

$$\begin{aligned} \left| \frac{\mu_\varepsilon}{|S_\varepsilon|} \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} v_\varepsilon \, dx - \frac{1}{|\Omega|} \int_\Omega v \, dx \right| &\leq \frac{\mu_\varepsilon}{|S_\varepsilon|} \left| \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} v_\varepsilon \, dx - \int_{\Omega_\varepsilon} v_\varepsilon \, dx \right| \\ &\quad + \left| \frac{\mu_\varepsilon}{|S_\varepsilon|} \int_{\Omega_\varepsilon} v_\varepsilon \, dx - \frac{1}{|\Omega|} \int_\Omega v \, dx \right|, \end{aligned}$$

where

$$\left| \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} v_\varepsilon \, dx - \int_{\Omega_\varepsilon} v_\varepsilon \, dx \right| \leq \|v_\varepsilon\|_{L^p(\Omega)} \left| \Omega \setminus \bigcup_{j \in \Upsilon_\varepsilon} (\varepsilon j + \varepsilon Y) \right|^{\frac{p-1}{p}}$$

and

$$\begin{aligned} \left| \frac{\mu_\varepsilon}{|S_\varepsilon|} \int_{\Omega_\varepsilon} v_\varepsilon \, dx - \frac{1}{|\Omega|} \int_\Omega v \, dx \right| &\leq \left(\frac{\mu_\varepsilon}{|S_\varepsilon|} - \frac{1}{|\Omega|} \frac{1}{|Y \setminus (a_\varepsilon \varepsilon^{-1} G_0)|} \right) \left| \int_{\Omega_\varepsilon} v_\varepsilon \, dx \right| \\ &\quad + \frac{1}{|\Omega|} \left| \int_\Omega v_\varepsilon \frac{\chi_{\Omega_\varepsilon}}{|Y \setminus (a_\varepsilon \varepsilon^{-1} G_0)|} \, dx - \int_\Omega v \, dx \right|. \end{aligned}$$

Due to the estimate on μ_ε, Lemma 4.8 and the strong L^p-convergence of v_ε, we have $R(\varepsilon) \to 0$ and the result is proved. □

Remark 4.9. The right-hand side of (4.3) is rather significant. As will see immediately, the fact that this right hand goes to 0 as $\varepsilon \to 0$ is a sufficient condition for the integrals to behave nicely in the limit, and thus we will see later that we are in the subcritical case. A priori, these estimates need not be sharp. It is only in combination with the analysis of the case $a_\varepsilon \sim a_\varepsilon^\star$ that we know it is.

Remark 4.10. If $a_\varepsilon = C_0\varepsilon^\alpha$, then the result implies that

$$\varepsilon^{-(\alpha(n-1)-n)} \int_{S_\varepsilon} v_\varepsilon \,\mathrm{d}S \to C_0^{n-1}|\partial G_0| \int_\Omega v \,\mathrm{d}x,$$

as $\varepsilon \to 0$ if $\alpha < \frac{n}{n-p}$. This is how the result appears in most of the previous literature (see also Remark 4.23).

4.2.3 Limit of the integral over S_ε of oscillating sequences

When we have an oscillating function g^ε constructed through a function g, a naive analysis suggests

$$\begin{aligned}\sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} g\Big(\frac{x-\varepsilon j}{a_\varepsilon}\Big)\varphi(x)\,\mathrm{d}S &= \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} g\Big(\frac{x-\varepsilon j}{a_\varepsilon}\Big)\big(\varphi(\varepsilon j) + \nabla\varphi(\eta_\varepsilon(x))\cdot(x-\varepsilon j)\big)\,\mathrm{d}S \\ &\simeq \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} g\Big(\frac{x-\varepsilon j}{a_\varepsilon}\Big)\varphi(\varepsilon j)\,\mathrm{d}S = \sum_{j\in\Upsilon_\varepsilon} a_\varepsilon^{n-1}\varphi(\varepsilon j)\int_{\partial G_0} g(y)\,\mathrm{d}S,\end{aligned}$$

and hence, since $|S_\varepsilon|^{-1} \simeq \frac{\varepsilon^n a_\varepsilon^{1-n}}{|\Omega||\partial G_0|}$, we have

$$\begin{aligned}\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} g^\varepsilon\varphi\,\mathrm{d}S &\simeq \frac{1}{|\partial G_0|}\int_{\partial G_0} g(y)\frac{1}{|\Omega|}\sum_{j\in\Upsilon_\varepsilon}\varepsilon^n\varphi(\varepsilon j)\,\mathrm{d}S \\ &\to \frac{1}{|\partial G_0|}\int_{\partial G_0} g(y)\,\mathrm{d}S\,\frac{1}{|\Omega|}\int_\Omega \varphi\,\mathrm{d}x.\end{aligned}$$

The rigorous proof of this result passes by applying the functions $m_{g,\varepsilon}$ introduced in Section 3.1.4.

Theorem 4.11. *Assume that $a_\varepsilon^\star \ll a_\varepsilon \le \varepsilon$ and $g \in L^{p'}(\partial G_0)$. Then, for any sequence $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ with $\|\nabla v_\varepsilon\|_{L^p(\Omega_\varepsilon)}$ bounded and such that $v_\varepsilon \to v$ in $L^p(\Omega)$ we have*

$$\beta^\star(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j + a_\varepsilon\partial G_0} g\Big(\frac{x-\varepsilon j}{a_\varepsilon}\Big)v_\varepsilon(x)\,\mathrm{d}S \to \frac{1}{|\partial G_0|}\int_{\partial G_0} g(y)\,\mathrm{d}S\frac{1}{|\Omega|}\int_\Omega v(x)\,\mathrm{d}x. \tag{4.4}$$

This result was first proved in [77] when $a_\varepsilon \sim \varepsilon$ and for v_ε a constant sequence.

Remark 4.12. We always write integrals in their averaged form, so that there are no confusing constants.

4.3 A priori estimates for u_ε

The aim of this section is to prove the following proposition.

Proposition 4.13. *Let $p > 1$ and let u_ε be the minimizer of J_ε given by* (2.4) *and Theorem* 2.13. *Then:*

1. *If $g^\varepsilon = 0$, then*
$$\|\nabla u_\varepsilon\|^{p-1}_{L^p(\Omega_\varepsilon)} \leq C\|f\|_{L^{p'}(\Omega_\varepsilon)}.$$
2. *If $g^\varepsilon \neq 0$ and $a_\varepsilon \gtrsim a^\star_\varepsilon$, we have*
$$\|\nabla u_\varepsilon\|^{p-1}_{L^p(\Omega_\varepsilon)} \leq C\left(\|f\|_{L^{p'}(\Omega_\varepsilon)} + \beta(\varepsilon)\beta^\star(\varepsilon)^{-1}\left(\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} \left|g^\varepsilon\right|^{p'} \mathrm{d}S_x\right)^{\frac{1}{p'}}\right).$$

Proof. For the first estimate we use u_ε as a test function in the Euler–Lagrange weak formulation. Applying that σ is monotone we deduce
$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p \,\mathrm{d}x \leq \int_{\Omega_\varepsilon} f u_\varepsilon \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon u_\varepsilon \,\mathrm{d}S.$$
Due to Hölder's inequality we have
$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p \,\mathrm{d}x \leq \|f\|_{L^{p'}(\Omega_\varepsilon)}\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \beta(\varepsilon)\beta^\star(\varepsilon)^{-1}\left(\beta^\star(\varepsilon)\int_{S_\varepsilon}\left|g^\varepsilon\right|^{p'} \mathrm{d}S_x\right)^{\frac{1}{p'}}\left(\beta^\star(\varepsilon)\int_{S_\varepsilon}|u|^p \,\mathrm{d}S_x\right)^{\frac{1}{p}}.$$
When $g^\varepsilon = 0$ the result follows directly from Theorem 3.4. When $g^\varepsilon \neq 0$ we use Remark 4.4. □

We can also prove a result in terms of the energy. Here we follow the proof in [111].

Proposition 4.14. *Let $p > 1$ and let u_ε be the minimizer of J_ε given by* (2.4) *and thus satisfying the variational inequality (*2.6*). Then:*

1. *If $g^\varepsilon = 0$, then*
$$\beta(\varepsilon)\left\|\Psi(u_\varepsilon)\right\|_{L^1(S_\varepsilon)} \leq C\|f\|^{p'}_{L^{p'}(\Omega_\varepsilon)}.$$
2. *If $g^\varepsilon \neq 0$ and $a_\varepsilon \gtrsim a^\star_\varepsilon$, we have*
$$\beta(\varepsilon)\left\|\Psi(u_\varepsilon)\right\|_{L^1(S_\varepsilon)} \leq C\left(\|f\|_{L^{p'}(\Omega_\varepsilon)} + \beta(\varepsilon)\beta^\star(\varepsilon)^{-1}\left(\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon}\left|g^\varepsilon\right|^{p'} \mathrm{d}S_x\right)^{\frac{1}{p'}}\right)^{p'}.$$

Recall in the previous result that Ψ denotes a convex function such that $\sigma = \partial\Psi$.

Proof. Since u_ε is the minimizer, we always have that $J_\varepsilon(u_\varepsilon) \le J_\varepsilon(0) = 0$ (see Remark 2.7). We have

$$\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \Psi(u_\varepsilon)\,\mathrm{d}S \le \int_{\Omega_\varepsilon} f u_\varepsilon \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon u_\varepsilon \,\mathrm{d}S.$$

Going back to the proof of Proposition 4.13 we recover

$$\int_{\Omega_\varepsilon} f u_\varepsilon \,\mathrm{d}x \le C\|f\|_{L^{p'}(\Omega_\varepsilon)}\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C\|f\|_{L^{p'}(\Omega_\varepsilon)}\|f\|^{\frac{1}{p-1}}_{L^{p'}(\Omega_\varepsilon)} = C\|f\|^{p'}_{L^{p'}(\Omega_\varepsilon)}.$$

We can repeat the same argument when $g^\varepsilon \neq 0$ and $a_\varepsilon \gtrsim a_\varepsilon^\star$. □

4.4 Big particles $a_\varepsilon = \varepsilon$

In this setting, with the correct scaling of β, we get an effective problem of the form

$$\begin{cases} -\operatorname{div} a^{\mathrm{eff}}(\nabla u) + \beta^{\mathrm{eff}}\sigma(u) = f + \beta^{\mathrm{eff}} g^{\mathrm{eff}} & \Omega, \\ u = 0 & \partial\Omega, \end{cases}$$

where

$$\beta^{\mathrm{eff}} = \frac{\beta^0}{|\Omega||Y \setminus G_0|} \tag{4.5}$$

and g^{eff} is given by (1.9). We present the complete details in the case $p = 2$, and we only make some comments on the generalization when $p \neq 2$. The precise nature of a^{eff} is involved and is described below.

4.4.1 The linear case $p = 2$

To illustrate, in a very simple example, how some of the ideas work, let us go back to one of the earliest results in homogenization. The idea behind the following example is a G-convergence argument (owed to Spagnolo [249]). A modern presentation of further details of the proof can be found in [76].

Example 4.15. Let $a : \mathbb{R} \to \mathbb{R}$ be a $[0,1]$-periodic function such that $0 < \alpha \le a \le \beta$, $f \in L^2(0,1)$ and $a^\varepsilon(x) = a(\frac{x}{\varepsilon})$. We consider the one-dimensional problem

$$\begin{cases} -\frac{d}{dx}(a^\varepsilon \frac{du_\varepsilon}{dx}) = f & x \in (0,1), \\ u_\varepsilon(0) = u_\varepsilon(1) = 0. \end{cases}$$

By multiplying by u_ε and integrating, the sequence u_ε is bounded in $H_0^1(0,1)$, and therefore

$$u_\varepsilon \rightharpoonup u$$

in $H_0^1(0,1)$, and by the same argument

$$a^\varepsilon \frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} = \xi_\varepsilon \rightharpoonup \xi_0$$

is convergent in $H^1(0,1)$ (since $f \in L^2$) and in the limit

$$\begin{cases} -\frac{d}{dx}(\xi^0) = f & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

holds. It is easy to show (see Lemma 4.18 below) that if $h \in L^2(0,1)$, then $h(\frac{\cdot}{\varepsilon}) \rightharpoonup \int_0^1 h\,\mathrm{d}x$ in $L^2(0,1)$. Hence, up to a subsequence,

$$\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}x} = \frac{1}{a^\varepsilon}\xi^\varepsilon \rightharpoonup \int_0^1 \frac{1}{a(x)}\,\mathrm{d}x \cdot \xi^0$$

in $L^2(0,1)$. Hence $\xi^0 = \frac{1}{\int_0^1 \frac{1}{a(x)}\,\mathrm{d}x}\frac{du}{dx}$ and thus u satisfies

$$\begin{cases} -\frac{d}{dx}\big(\frac{1}{\int_0^1 \frac{1}{a(x)}\,\mathrm{d}x}\frac{du}{dx}\big) = f & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

The term $a_0 = \frac{1}{\int_0^1 \frac{1}{a(x)}\,\mathrm{d}x}$ is sometimes known as the *effective diffusion coefficient*. This concludes this example. Applying the Γ-convergence method described in Section 3.1.1 we can obtain the same result as in Example 4.15 in a way that can be generalized to higher dimensions.

The focus of this work is the problem of oscillating coefficients

$$L^\varepsilon u_\varepsilon = f, \quad L^\varepsilon v = \operatorname{div}\left(B\left(\frac{x}{\varepsilon}\right)\nabla v\right),$$

where $B = (b_{ij})$ is a matrix, $b_{ij} = b_{ji} \in L^\infty([0,1]^n)$ and they are extended by periodicity. This models the behavior of a periodical two-phase composite (a material formed by the inclusion of two materials with different properties). This work is, no doubt, based on previous results, for example by Spagnolo (see, e. g., [249]) on the limit problems of $-\operatorname{div}(B_k u_k)$ as $B_k \to B_\infty$.

In this setting $\beta^\star(\varepsilon) \sim \varepsilon$. Let us introduce the cell problem for $i = 1, \dots, n$

$$\begin{cases} -\Delta\chi_i = 0 & Y \setminus \overline{G_0}, \\ \nabla\chi_i \cdot \nu = -e_i \cdot \nu & \partial G_0, \\ \chi_i \text{ is } Y-\text{periodic}, \end{cases} \tag{4.6}$$

where e_i is the i-th vector of the standard Euclidean basis. It is easy to see that χ_i are bounded. We define the effective diffusion coefficients as

$$a_{ij}^{\text{eff}} = \delta_{ij} + \frac{1}{|Y \setminus G_0|} \int\limits_{Y \setminus G_0} \frac{\partial \chi_j}{\partial y_i} \, \mathrm{d}y. \tag{4.7}$$

It is a well-known fact (see, e. g., [30]) that the matrix $A^{\text{eff}} = (a_{i,j}^{\text{eff}})$ is symmetric and positive-definite. This special problem works best if $a_\varepsilon = \varepsilon$.

Theorem 4.16. *Let $f \in L^2(\Omega)$, $\sigma \in C(\mathbb{R})$ monotone non-decreasing with $\sigma(0) = 0$ such that one of the following holds:*
1. *The growth of the derivative of σ is controlled by*

$$|\sigma'(s)| \leq C(1 + |s|^q), \quad \text{for some } 0 \leq q < \frac{n}{n-2}. \tag{4.8}$$

2. *The growth of σ is controlled by*

$$|\sigma(s)| \leq C(1 + |s|^q), \quad \text{for some } 0 \leq q < \frac{n}{n-2}. \tag{4.9}$$

Let u_ε be the minimizer of J_ε in this setting, assume $a_\varepsilon = \varepsilon$ and let

$$\beta^0 = \lim_{\varepsilon \to 0} \frac{\beta(\varepsilon)}{\beta^\star(\varepsilon)}.$$

We distinguish two cases:
(a) *If $\beta(\varepsilon) \lesssim \beta^\star(\varepsilon)$ (i. e., $\beta^0 \in [0, \infty)$) and g^ε is given by (1.3), with $g_{st} \in H^1(\Omega), g_{per} \in L^2(\partial G_0)$, then $P_\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$, where u is the energy solution of*

$$\begin{cases} -\operatorname{div}(A^{\text{eff}} \nabla u) + \beta^{\text{eff}} \sigma(u) = f + \beta^{\text{eff}} g^{\text{eff}} & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

$\beta^{\text{eff}} = \frac{\beta^0}{|\Omega||Y \setminus G_0|}$ *and g^{eff} is given by (1.9).*
(b) *If $\beta^0 = +\infty$ and $g^\varepsilon = 0$, then up to a subsequence $P_\varepsilon u_\varepsilon \rightharpoonup u$ where $\sigma(u) = 0$.*

Remark 4.17. In [84] the authors consider $\beta(\varepsilon) = a\varepsilon$. Since in this setting $\beta^\star(\varepsilon) \simeq \frac{\varepsilon}{|\Omega||\partial G_0|}$ we recover $\beta^0 = a|\Omega||\partial G_0|$ and $\beta^{\text{eff}} = a\frac{|\partial G_0|}{|Y \setminus G_0|}$ as the authors do.

We begin with a lemma that is only relevant for big particles. The proof can be found in [76, Theorem 2.6] (we recall that $|Y| = 1$ in our setting).

Lemma 4.18 (Limit of ε periodic functions). *Let g be Y-periodic and $L^p(Y)$ for $p \in (1, \infty)$. Then*

$$g\left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup \int\limits_Y g(y) \, \mathrm{d}y \quad \text{weakly in } L^p(\Omega).$$

Intuition of the result. Let $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$. First we check that the sequence is bounded. We compute

$$\int_\Omega |g_\varepsilon(x)|^p \,\mathrm{d}x = \sum_{j\in\mathbb{Z}^n} \int_{(\varepsilon j+\varepsilon Y)\cap\Omega} \left| g\left(\frac{x-\varepsilon j}{\varepsilon}\right)\right|^p \mathrm{d}x$$
$$\leq \left|\{j \in \mathbb{Z}^n : (\varepsilon j + \varepsilon Y)\cap\Omega \neq \emptyset\}\right| \varepsilon^n \int_Y |g(y)|^p \,\mathrm{d}y.$$

Since Ω is contained in a large ball, $|\{j \in \mathbb{Z}^n : (\varepsilon j + \varepsilon Y)\cap\Omega \neq \emptyset\}| \leq C\varepsilon^{-n}$ and hence this quantity is uniformly bounded. Thus, up to a subsequence g_ε has a weak limit in $L^p(\Omega)$. Let us call the limit g_0. Let $\varphi \in C_c^\infty(\Omega)$. First, we split

$$\int_\Omega g_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \sum_{j\in\Upsilon_\varepsilon} \int_{\varepsilon j+\varepsilon Y} g_\varepsilon(x)\varphi(x)\,\mathrm{d}x + \int_{\Omega\setminus\bigcup_{j\in\Upsilon_\varepsilon}\varepsilon j+\varepsilon Y} g_\varepsilon(x)\varphi(x)\,\mathrm{d}x.$$

The second term tends to zero, since the measure of the integration domain tends to zero. For the first term, we expand φ on every εj,

$$\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+\varepsilon Y} g_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \sum_{j\in\Upsilon_\varepsilon} \varepsilon^n\varphi(\varepsilon j)\int_Y g(y)\,\mathrm{d}y + \sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+\varepsilon Y} g(x/\varepsilon)\nabla\varphi(\eta_\varepsilon(x))\cdot(x-\varepsilon j)\,\mathrm{d}x.$$

Again, the second term tends to zero, and we recover the result from the Riemann sum. □

Remark 4.19. A similar argument can be made in $L^1(\Omega)$ using the uniform integrability.

Remark 4.20. We will only prove the case of (4.8). The proof in the other case is very similar, passing to the inequality formulation (2.6). See [84] for the details. Actually, since the hypothesis is applied for Ψ, we could assume that σ is multivalued but satisfying the growth assumption. The result for Signorini can also be obtained by modifications of the argument [89].

Proof of Theorem 4.16 (b). First, due to the a priori estimates and properties of the extension operator, we know that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$. Pick $v \in C_c^\infty(\overline{\Omega})$. We write

$$\beta(\varepsilon)^{-1}\beta^\star(\varepsilon)\int_{\Omega_\varepsilon}\nabla u_\varepsilon\cdot\nabla v\,\mathrm{d}x + \beta^\star(\varepsilon)\int_{S_\varepsilon}\sigma(u_\varepsilon)v\,\mathrm{d}S = \beta(\varepsilon)^{-1}\beta^\star(\varepsilon)\int_{\Omega_\varepsilon} fv\,\mathrm{d}x.$$

Since, from the assumption on σ, we deduce that $\sigma(Pu_\varepsilon)$ is bounded in $W^{1,r}(\Omega)$ with $r = \frac{2n}{q(n-2)+n}$ and thus $\sigma(Pu_\varepsilon) \to \sigma(u)$ strongly in $L^r(\Omega)$ (thanks to the Lebesgue dominated theorem since $Pu_\varepsilon \to u$ strongly in $L^2(\Omega)$) we can apply Theorem 4.5 and we have

$$\beta^\star(\varepsilon)\int_{S_\varepsilon}\sigma(u_\varepsilon)v\,\mathrm{d}S \to \frac{1}{|\Omega|}\int_\Omega \sigma(u)v\,\mathrm{d}x.$$

The other integrals are bounded, and as we pass to the limit we recover

$$\int_\Omega \sigma(u)v\,\mathrm{d}x = 0.$$

Thus $\sigma(u) = 0$. □

Proof of Theorem 4.16 (a). We follow the scheme of the proof in [84] considering the case in which σ is so smooth that (4.8). We have (for any good test function v)

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \sigma(u_\varepsilon)v\,\mathrm{d}S = \int_{\Omega_\varepsilon} fv\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} gv\,\mathrm{d}S. \tag{4.10}$$

Again, due to the a priori estimates and properties of the extension operator, we know that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$. The sequence

$$\xi_\varepsilon(x) = \widetilde{\nabla u_\varepsilon}(x) = \begin{cases} \nabla u_\varepsilon(x) & x \in \Omega_\varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

is bounded in $L^2(\Omega)$, and hence it has a weak limit ξ^{eff}. Thus, for any good test function v

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\,\mathrm{d}x = \int_\Omega \xi_\varepsilon \cdot \nabla v\,\mathrm{d}x \to \int_\Omega \xi^{\mathrm{eff}} \nabla v\,\mathrm{d}x.$$

Since $\sigma(u_\varepsilon)$ is uniformly bounded in $W^{1,r}(\Omega)$ with $r = \frac{2n}{q(n-2)+n}$, for any $v \in C_c^\infty(\Omega)$ we have by Theorem 4.5

$$\beta(\varepsilon)\int_{S_\varepsilon} \sigma(u_\varepsilon)v\,\mathrm{d}S \longrightarrow \frac{\beta^0}{|\Omega|}\int_\Omega \sigma(u)v\,\mathrm{d}x, \quad \beta(\varepsilon)\int_{S_\varepsilon} g_{\mathrm{st}}(x)v\,\mathrm{d}S \longrightarrow \frac{\beta^0}{|\Omega|}\int_\Omega g_{\mathrm{st}}v\,\mathrm{d}x.$$

Due to Theorem 4.11 we have

$$\beta(\varepsilon)\int_{S_\varepsilon} g_{\mathrm{per}}\left(\frac{x-\varepsilon j}{a_\varepsilon}\right)v(x)\,\mathrm{d}S \longrightarrow \frac{\beta^0}{|\Omega||\partial G_0|}\int_{\partial G_0} g_{\mathrm{per}}(y)\,\mathrm{d}S \int_\Omega v(x)\,\mathrm{d}x.$$

Lastly, due to Lemma 4.8 we have

$$\int_{\Omega_\varepsilon} fv\,\mathrm{d}x = \int_\Omega \chi_{\Omega_\varepsilon} fv\,\mathrm{d}x \longrightarrow |Y \setminus G_0| \int_\Omega fv\,\mathrm{d}x.$$

Thus u is a solution of

$$\int_\Omega \xi^{\mathrm{eff}} \cdot \nabla v\,\mathrm{d}x + \frac{\beta^0}{|\Omega|}\int_\Omega \sigma(u)v\,\mathrm{d}x = |Y \setminus G_0| \int_\Omega fv\,\mathrm{d}x + \frac{\beta^0}{|\Omega|}\int_\Omega g^{\mathrm{eff}} v\,\mathrm{d}x.$$

This is to say

$$-\operatorname{div}\xi^{\text{eff}} + \frac{\beta^0}{|\Omega|}\sigma(u) = |Y \setminus G_0|f + \frac{\beta^0}{|\Omega|}g^{\text{eff}}.$$

As a last step, we need to characterize ξ^{eff}. This is the peculiarity of the case of big particles. Let us recover one component of ξ_i^{eff} for $i = 1, \dots, n$. Consider the test function

$$v_{i,\varepsilon}(x) = \varepsilon\left(\chi_i\left(\frac{x}{\varepsilon}\right) + \frac{x_i}{\varepsilon}\right),$$

where χ_i is given by (4.6). By extending χ_i into G_0 to produce an $H^1(Y) \cap L^\infty(Y)$ function $\overline{\chi}_i$, we can construct

$$\overline{v}_{i,\varepsilon}(x) = \varepsilon\overline{\chi}_i\left(\frac{x}{\varepsilon}\right) + x_i,$$

which are uniformly bounded in $H^1(\Omega)$. Furthermore, the first term tends to zero in $L^\infty(\Omega)$ so

$$\overline{v}_{i,\varepsilon} \to x_i \quad \text{weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega).$$

Consider the extension by zero of its gradient inside the particles

$$\widetilde{\frac{\partial v_{i,\varepsilon}}{\partial x_j}}(x) = \widetilde{\frac{\partial \chi_i}{\partial y_j}}\left(\frac{x}{\varepsilon}\right) + \delta_{ij}\chi_{Y\setminus G_0}\left(\frac{x}{\varepsilon}\right), \quad \forall x \in \bigcup_{k\in\Upsilon_\varepsilon} Y_\varepsilon^k.$$

The measure of the remainder set of points of Ω tends to zero. Due to Lemma 4.18

$$\widetilde{\frac{\partial v_{i,\varepsilon}}{\partial x_j}} \rightharpoonup \int_Y \left(\widetilde{\frac{\partial \chi_i}{\partial y_j}}(y) + \delta_{ij}\chi_{Y\setminus G_0}(y)\right) \mathrm{d}y = \int_{Y\setminus G_0} \frac{\partial \chi_i}{\partial y_j}(y)\,\mathrm{d}y + \delta_{ij}|Y \setminus G_0| = |Y \setminus G_0|\, a_{ij}^{\text{eff}},$$

weakly in $L^2(\Omega)$, where a_{ij}^{eff} is given by (4.7). Hence

$$\widetilde{\nabla v_{i,\varepsilon}} \rightharpoonup |Y \setminus G_0|(a_{i1}^{\text{eff}}, \dots, a_{in}^{\text{eff}}) = |Y \setminus G_0| a_i^{\text{eff}},$$

weakly in $L^2(\Omega)^n$. Due to the definition of the cell problem

$$\begin{cases} -\Delta v_{i,\varepsilon} = 0 & \text{in } \Omega_\varepsilon, \\ \nabla v_{i,\varepsilon} \cdot \nu = 0 & \text{on } S_\varepsilon, \end{cases}$$

and we do not care what happens on $\partial\Omega$. Fix $\varphi \in C_c^\infty(\Omega)$. Using φu_ε as a test function in the weak formulation of the problem for $v_{i,\varepsilon}$ we recover

$$\int_{\Omega_\varepsilon} \nabla v_{i,\varepsilon} \cdot \nabla(\varphi u_\varepsilon)\,\mathrm{d}x = 0. \tag{4.11}$$

Using $\varphi v_{i,\varepsilon}$ as a test function in (4.10)

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla(v_{i,\varepsilon}\varphi)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \sigma(u_\varepsilon)v_{i,\varepsilon}\varphi\,\mathrm{d}S = \int_{\Omega_\varepsilon} f v_{i,\varepsilon}\varphi\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon v_{i,\varepsilon}\varphi\,\mathrm{d}S.$$

Note that, similarly to above,

$$\begin{aligned}\beta(\varepsilon)\int_{S_\varepsilon}(\sigma(u_\varepsilon) - g^\varepsilon)v_{i,\varepsilon}\varphi\,\mathrm{d}S &= \varepsilon\beta(\varepsilon)\int_{S_\varepsilon}\big(\sigma\big(u_\varepsilon(x)\big) - g^\varepsilon(x)\big)\chi_i\Big(\frac{x}{\varepsilon}\Big)\varphi(x)\,\mathrm{d}S \\ &\quad + \beta(\varepsilon)\int_{S_\varepsilon}\big(\sigma\big(u_\varepsilon(x)\big) - g^\varepsilon(x)\big)x_i\varphi(x)\,\mathrm{d}S \\ &\to \frac{\beta^0}{|\Omega|}\int_\Omega\big(\sigma\big(u(x)\big) - g(x)\big)x_i\varphi(x)\,\mathrm{d}x,\end{aligned}$$

$$\int_{\Omega_\varepsilon} f v_{i,\varepsilon}\varphi\,\mathrm{d}x \to |Y \setminus G_0|\int_\Omega f x_i\varphi\,\mathrm{d}x.$$

To study the gradient we have, applying (4.11),

$$\begin{aligned}\int_{\Omega_\varepsilon}\nabla u_\varepsilon\cdot\nabla(v_{i,\varepsilon}\varphi)\,\mathrm{d}x &= \int_{\Omega_\varepsilon} v_{i,\varepsilon}\nabla u_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x + \int_{\Omega_\varepsilon}\varphi\nabla u_\varepsilon\cdot\nabla v_{i,\varepsilon}\,\mathrm{d}x \\ &= \int_{\Omega_\varepsilon} v_{i,\varepsilon}\nabla u_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x - \int_{\Omega_\varepsilon} u_\varepsilon\nabla\varphi\cdot\nabla v_{i,\varepsilon}\,\mathrm{d}x \\ &= \int_\Omega \bar v_{i,\varepsilon}\widetilde{\xi_\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega u_\varepsilon\nabla\varphi\cdot\widetilde{\nabla v_{i,\varepsilon}}\,\mathrm{d}x \\ &\to \int_\Omega x_i\xi^{\mathrm{eff}}\cdot\nabla\varphi\,\mathrm{d}x - |Y\setminus G_0|\int_\Omega u\nabla\varphi\cdot a_i^{\mathrm{eff}}\,\mathrm{d}x.\end{aligned}$$

We recover that

$$\operatorname{div}(x_i\xi^{\mathrm{eff}} - |Y\setminus G_0|u a_i^{\mathrm{eff}}) + \frac{\beta^0}{|\Omega|}\sigma(u)x_i = |Y\setminus G_0| f x_i + \frac{\beta^0}{|\Omega|}g^{\mathrm{eff}}x_i.$$

Since a_i^{eff} is constant $\operatorname{div}(u a_i^{\mathrm{eff}}) = a_i^{\mathrm{eff}}\cdot\nabla u$. On the other hand, $\operatorname{div}(x_i\xi^{\mathrm{eff}}) = x_i\operatorname{div}\xi^{\mathrm{eff}} + \xi^{\mathrm{eff}}\cdot\nabla x_i = x_i\operatorname{div}\xi^{\mathrm{eff}} + \xi_i^{\mathrm{eff}}$. Thus

$$\xi_i^{\mathrm{eff}} = |Y\setminus G_0|a_i^{\mathrm{eff}}\cdot\nabla u.$$

Joining this information

$$\xi^{\mathrm{eff}} = |Y\setminus G_0|A^{\mathrm{eff}}\nabla u. \qquad \square$$

4.4.2 The nonlinear case $p \neq 2$

In this setting it was shown in [128], for the case $\sigma = 0$ and $g^\varepsilon(x) = g_{\mathrm{per}}(\frac{x-\varepsilon j}{a_\varepsilon})$, that an effective nonlinear diffusion still exists. Repeating the argument above for ξ^{eff}, in this setting we take the limit of $|\nabla\xi_\varepsilon|^{p-2}\nabla\xi_\varepsilon$ and this converges to some vector ξ^{eff}. Using the techniques in [128], one can characterize $\xi^{\mathrm{eff}} = a^{\mathrm{eff}}(\nabla u)$ by means of the effective diffusion $a^{\mathrm{eff}} : \mathbb{R}^n \to \mathbb{R}^n$. For $\xi \in \mathbb{R}^n$, we take v as the solution of

$$\begin{cases} \int_{Y\setminus G_0} |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi \, \mathrm{d}y = 0 \text{ for all } \varphi \in W^{1,p}(Y \setminus \overline{G_0}), Y\text{-periodic}, \\ v - \xi \cdot y \text{ is } Y\text{-periodic} \end{cases}$$

and define

$$a^{\mathrm{eff}}(\xi) = \frac{1}{|Y \setminus G_0|} \int\limits_{Y\setminus G_0} |\nabla v|^{p-2}\nabla v \, \mathrm{d}y.$$

In [128] the authors prove that the homogenized nonlinear diffusion $a^{\mathrm{eff}}(\xi)$ is again a $(p-1)$-homogeneous function of ξ. Note that the involved formulation matches the linear setting. It seems an easy task to extend the results of [128] to the case $\sigma \neq 0$, as presented above.

4.5 Subcritical cases $a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$ and $p > 1$

This is possibly the simplest case of relevance due to the combination of Theorem 4.5 and the fact that the set of all the particles vanishes in measure as $\varepsilon \to 0$

$$|\Omega \setminus \Omega_\varepsilon| = |\Upsilon_\varepsilon||a_\varepsilon G_0| \sim \varepsilon^{-n} a_\varepsilon^n |\Omega||G_0| \to 0. \tag{4.12}$$

This fact will allow us to skip now the difficulty concerning the computation of the effective diffusion. The aim of this subsection will be to prove the following result concerning the case of a regular function σ (we refer the reader to Section 1.6.1 for some references concerning other choices of σ).

Theorem 4.21. *Let $p \in (1,\infty)$, $f \in L^{p'}(\Omega)$, $g^\varepsilon \in W^{1,\infty}(\Omega)$, $a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$, $\sigma' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ non-decreasing such that $\sigma(0) = 0$ and let*

$$\beta^0 = \lim_{\varepsilon\to 0} \beta(\varepsilon)\beta^\star(\varepsilon)^{-1}.$$

We distinguish two cases, depending on the value of β^0:

(a) *If $\beta^0 < +\infty$ and g^ε is given by (1.3) with $g_{st} \in W^{1,\infty}(\Omega)$, $g_{per} \in L^{p'}(\partial\Omega)$, then $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, where u is the unique solution of*

$$\begin{cases} -\Delta_p u + \beta^{\mathrm{eff}}\sigma(u) = f + \beta^{\mathrm{eff}} g^{\mathrm{eff}} & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.13}$$

$\beta^{\mathrm{eff}} = \beta^0/|\Omega|$ and g^{eff} given by (1.9).

(b) *If $\beta^0 = +\infty$, $g^\varepsilon = 0$ and $|\sigma(t)| \le C(1 + |t|^{\frac{p}{p-1}})$, then, up to a subsequence, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ with u such that*

$$\sigma(u) = 0$$

a. e. in Ω. In particular, if σ is strictly increasing, then $u = 0$.

Remark 4.22. Note that when $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$ we have $\beta^{\text{eff}} = 0$ and the reaction terms coming from the Robin boundary condition vanish. However, if $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$, then they dominate the rest (in this subcritical case and also in the case of big particles).

Remark 4.23. When $a_\varepsilon = C_0\varepsilon^\alpha$ with $\alpha \in (1, \frac{n}{n-p})$ and $\beta(\varepsilon) = \varepsilon^{n-\alpha(n-1)} \sim \beta^\star(\varepsilon)$, then applying (4.1) we have

$$\beta^{\text{eff}} = |\partial G_0| C_0^{n-1}.$$

This is the result that one typically finds in the literature (see, e. g., [177]). Note that this fact explains the existence of several homogenization results in the literature concerning the case of several isoperimetric shapes for G_0 mentioned in Remark 1.2.

Remark 4.24. It is not difficult to check that the conclusion of Theorem 4.21 remains valid for the case in which σ is not autonomous, i. e., when replacing $\sigma(u)$ by $\sigma(x, u)$ with $\sigma(x, 0) = 0$. A special case which was considered in the previous literature concerns the case $p = 2$ and $\sigma(x, u) = V(x)u$ (the existence and uniqueness of solutions is guaranteed if, for instance, $V \in H^1(\Omega)$ and $V \ge 0$, but much more general "potentials" can be also considered). In that case, the homogenized equation for the subcritical case becomes the linear equation

$$-\Delta u + \beta^{\text{eff}} V(x)u = f,$$

and thus several authors say that we get the (stationary) Schrödinger equation.

Remark 4.25. In [270] the authors prove the strong convergence in the case $p = 2$ and $\beta^0 \in (0, \infty)$, i. e., $\|\nabla(u_\varepsilon - u)\|_{L^2(\Omega_\varepsilon)} \to 0$. Due to the properties of P_ε, this implies that $P_\varepsilon u_\varepsilon \to u$ strongly in $H_0^1(\Omega)$. The proof is long and involved, and we will not reproduce it here. As a matter of fact, we will prove later a related result (see Section 4.7.1.4) for the case of the critical scale.

We state a lemma that is a direct consequence of (4.12) and that improves the corresponding result given in Lemma 4.8 concerning this special case.

Lemma 4.26. *Let $a_\varepsilon \ll \varepsilon$. Then $\chi_{\Omega_\varepsilon} \to 1$ strongly in $L^q(\Omega)$ for any $q \in [1, \infty)$.*

Proof. We have

$$\|\chi_{\Omega_\varepsilon} - 1\|_{L^q(\Omega)}^q = \int_\Omega |\chi_{\Omega_\varepsilon} - 1|^q \, dx = |\Omega \setminus \Omega_\varepsilon| = |\Upsilon_\varepsilon||a_\varepsilon G_0| \to 0. \qquad \square$$

Proof of Theorem 4.21 (a). The a priori estimate from Proposition 4.13 guarantees that there is a limit of $P_\varepsilon u_\varepsilon$. Our aim is to pass to the limit in formulation (2.7). Consider a test function $v \in W_0^{1,\infty}(\Omega)$. Then $\sigma(v) \in W_0^{1,\infty}(\Omega)$. Thus $\sigma(v)(v - P^\varepsilon u_\varepsilon) \rightharpoonup \sigma(v)(v-u)$ weakly in $W_0^{1,p}(\Omega)$.

Let $\beta(\varepsilon) \lesssim \beta^\star(\varepsilon)$. We study the different terms. First

$$\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x = \int_\Omega f(v-u_\varepsilon)\chi_{\Omega_\varepsilon}\,\mathrm{d}x.$$

Since $f \in L^{p'}(\Omega)$, $v - P_\varepsilon u_\varepsilon \rightharpoonup v-u$ in $L^q(\Omega)$ (for some $q > p$ due to the Sobolev embedding theorem) and $\chi_{\Omega_\varepsilon} \to 1$ strongly in $L^{\frac{qp'}{qp'-1}}(\Omega)$, we have

$$\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x \to \int_\Omega f(v-u)\,\mathrm{d}x.$$

Now, concerning the diffusion term we have

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\cdot\nabla(v-u_\varepsilon)\,\mathrm{d}x = \int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla(v-P_\varepsilon u_\varepsilon)\chi_{\Omega_\varepsilon}\,\mathrm{d}x.$$

Since $|\nabla v|^{p-2}\nabla v \in L^\infty(\Omega)$, $\nabla(v-P_\varepsilon u_\varepsilon) \rightharpoonup \nabla(v-u)$ in $L^p(\Omega)$ and $\chi_{\Omega_\varepsilon} \to 1$, strongly in $L^{\frac{p}{p-1}}(\Omega)$ we can pass to the limit

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\cdot(v-u_\varepsilon)\,\mathrm{d}x \to \int_\Omega |\nabla v|^{p-2}\nabla v\cdot(v-u)\,\mathrm{d}x.$$

Finally, the passing to the limit in the more relevant term, associated to the reaction term, is now quite easy through Theorems 4.5 and 4.11 since

$$\beta(\varepsilon)\int_{S_\varepsilon}(\sigma(v)-g^\varepsilon)(v-u_\varepsilon)\,\mathrm{d}S \to \frac{\beta^0}{|\Omega|}\int_\Omega(\sigma(v)-g^{\mathrm{eff}})(v-u)\,\mathrm{d}x,$$

and we recover

$$\int_\Omega |\nabla v|^{p-2}\nabla v\cdot(v-u)\,\mathrm{d}x + \frac{\beta^0}{|\Omega|}\int_\Omega \sigma(v)(v-u)\,\mathrm{d}x \ge \int_\Omega f(v-u)\,\mathrm{d}x + \frac{\beta^0}{|\Omega|}\int_\Omega g^{\mathrm{eff}}(v-u)\,\mathrm{d}x,$$

which is equivalent to the weak definition of solution of the problem (4.13). □

Proof of Theorem 4.21 (b). The case $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ can be studied as above. In the weak formulation we get

$$\beta^\star(\varepsilon)\beta(\varepsilon)^{-1}\int_{\Omega_\varepsilon} |\nabla v|^p\nabla v\cdot\nabla(v-u_\varepsilon)\,\mathrm{d}x + \beta^\star(\varepsilon)\int_{S_\varepsilon}\sigma(v)(v-u_\varepsilon)\,\mathrm{d}S \ge \beta^\star(\varepsilon)\beta(\varepsilon)^{-1}\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x,$$

for any $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Arguing as above the integrals over Ω_ε are bounded, and since they are multiplied by a vanishing coefficient these terms vanish in the limit. Due to Theorem 4.5

$$\int_\Omega \sigma(v)(v - u)\,\mathrm{d}x \geq 0.$$

We now take $v_k = \rho_k * u + \lambda\varphi$, where ρ_k are convolution kernels and $0 \leq \varphi \in C_c^\infty(\Omega)$. First, $v_k \to u + \lambda\varphi$ strongly in $L^p(\Omega)$. Due to boundedness and a. e. convergence, $\sigma(v_k) \rightharpoonup \sigma(u + \lambda\varphi)$ weakly in $L^{p'}(\Omega)$. Thus

$$\int_\Omega \sigma(u + \lambda\varphi)\lambda\varphi\,\mathrm{d}x \geq 0.$$

As $\lambda \to 0^+$ we recover

$$\int_\Omega \sigma(u)\varphi\,\mathrm{d}x \geq 0.$$

As $\lambda \to 0^-$ we recover

$$\int_\Omega \sigma(u)\varphi\,\mathrm{d}x \leq 0.$$

Thus $\sigma(u) = 0$. □

Remark 4.27. Note that we only need σ to be locally Lipschitz, since we use $\sigma(v)$ and v is assumed to be bounded. Finally a natural question is whether the nonlinearity commutes with the extension so as to use work intuitively with $P^\varepsilon\sigma(u_\varepsilon)$. Assume $a_\varepsilon \ll \varepsilon$. Let σ be uniformly Lipschitz and let u_ε be a bounded sequence in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Then, up to a subsequence,

$$P^\varepsilon\sigma(u_\varepsilon) - \sigma(P^\varepsilon u_\varepsilon) \to 0 \quad \text{strongly in } L^p(\Omega).$$

Note that, up to a subsequence, $P^\varepsilon\sigma(u_\varepsilon) - \sigma(P^\varepsilon u_\varepsilon)$ is strongly convergent in $L^p(\Omega)$, and they are only different in $\Omega \setminus \Omega_\varepsilon$, a set whose measure tends to zero. Hence, the limit is characterized, and the whole sequence converges.

4.6 Supercritical case $a_\varepsilon \ll a_\varepsilon^\star$ and $p \in (1, n)$

As mentioned before this case is not very relevant. The proof is very simple. Here we present briefly a proof similar to that in [268], there for the case $p = 2$. We do not cover the case $p = n$, since it is a little more delicate.

Theorem 4.28. *Let* $1 < p < n$, $a_\varepsilon \ll a_\varepsilon^\star$, *any sequence* $\beta(\varepsilon) > 0$, σ *continuous non-decreasing such that* $\sigma(0) = 0$ *and let us consider* u_ε *the solution of* (1.1), *where* $f \in L^{p'}(\Omega)$ *and* $g^\varepsilon = 0$. *Then,* $P_\varepsilon u_\varepsilon \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$, *where* u *is the unique solution of*

$$\begin{cases} -\Delta_p u = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{4.14}$$

Proof. Due to the a priori estimate in Proposition 4.13 we have, up to a subsequence, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Let

$$K_0 = \max_{y \in G_0} |y|.$$

Consider a radial function $\bar{\psi} : \mathbb{R}^n \to [0,1]$ such that

$$\bar{\psi}(y) = \begin{cases} 0 & \text{if } |y| \geq 2K_0, \\ 1 & \text{if } |y| \leq K_0, \end{cases} \qquad |\nabla \bar{\psi}| \leq K,$$

and let

$$\psi_\varepsilon(x) = \sum_{j \in \Upsilon_\varepsilon} \bar{\psi}\left(\frac{x - \varepsilon j}{a_\varepsilon} \right).$$

It is clear that $\psi_\varepsilon = 1$ in $\varepsilon j + a_\varepsilon G_0$ and 0 in $\partial\Omega$. As mentioned in Section 4.1, we can apply the scaling (4.2) and thus

$$\int_\Omega \left|\nabla \psi_\varepsilon(x)\right|^p \mathrm{d}x \leq |\Upsilon_\varepsilon| a_\varepsilon^{n-p} \int_Y \left|\nabla \overline{\psi}(y)\right|^p \mathrm{d}y \sim \varepsilon^{-n} a_\varepsilon^{n-p} = \left(\frac{a_\varepsilon}{a_\varepsilon^\star} \right)^{n-p} \varepsilon^{-n} (a_\varepsilon^\star)^{n-p} \to 0.$$

Note that this is not true for $p = n$. Let $\varphi \in W_0^{1,p}(\Omega)$. Taking $\varphi_\varepsilon = \varphi(1 - \psi_\varepsilon)$ as a test function we have that $\varphi_\varepsilon \to \varphi$ in $W_0^{1,p}(\Omega)$ and moreover

$$\int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^{p-2} \nabla \varphi_\varepsilon \nabla(\varphi_\varepsilon - u_\varepsilon) \,\mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(\varphi_\varepsilon)(\varphi_\varepsilon - u) \,\mathrm{d}S \geq \int_{\Omega_\varepsilon} f(\varphi_\varepsilon - u_\varepsilon) \,\mathrm{d}x.$$

Since $\varphi_\varepsilon = 0$ in S_ε and $\sigma(0) = 0$, the integral on S_ε is always zero. Due to the strong convergence, Lemma 4.26 and the considerations on the gradient made in Section 4.1, we have

$$\int_\Omega |\nabla \varphi|^{p-2} \nabla \varphi \cdot (\varphi - u) \,\mathrm{d}x \geq \int_\Omega f(\varphi - u) \,\mathrm{d}x.$$

This limit is the weak formulation of (4.14), which admits a unique solution, so the whole sequence $P_\varepsilon u_\varepsilon$ converges to the desired function. □

Remark 4.29. As it is clearly seen in the above proof, the information of the homogenized weak formulation is revealed by the choice of a specific sequence of test functions. The auxiliary function ψ_ε oscillates, by construction, with the repetition of the particles. This is precisely why this method is known as *oscillating test functions*. This oscillating character of the test functions is also present in the arguments used for other scales of a_ε but in a more implicit way.

4.7 Critical case $a_\varepsilon \sim a_\varepsilon^\star$: the anomalous (or strange) nonlinear term

In the following sections we will present the results obtained by the authors in the critical cases which form part of the main core of this book. As a matter of fact, we have structured this book by putting firstly a series of basic (and technical) results just to get now a clearer proof of the main results, which, as previously said, are exceptional among the possible scales of the particles.

The aim of this section is to show that the limit of $P_\varepsilon u_\varepsilon$ is u, the unique solution of

$$\begin{cases} -\Delta_p u + \mathcal{H}(x,u) = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{4.15}$$

The function $\mathcal{H}$ is typically different from σ in this setting.

4.7.1 Case of G_0 a ball, $p \in (1, n)$ and $g^\varepsilon = g$

Again we assume that $G_0 = B_1$ (we refer the reader, once again, to Remark 1.1). If $g^\varepsilon = g \in W^{1,\infty}(\Omega)$ for any ε, then

$$\mathcal{H}(x,s) = \mathcal{A}_0 |H(x,s)|^{p-2} H(x,s) \tag{4.16}$$

and H is at each point x the solution of the functional equation

$$\mathcal{B}_0 |H(x,s)|^{p-2} H(x,s) \in \sigma(s - H(x,s)) - g(x), \tag{4.17}$$

where

$$\mathcal{A}_0 = \left(\frac{n-p}{p-1}\right)^{p-1} |\partial B_1| \lim_{\varepsilon\to 0} (a_\varepsilon^{n-p} \varepsilon^{-n}), \quad \mathcal{B}_0 = \left(\frac{n-p}{p-1}\right)^{p-1} \lim_{\varepsilon\to 0} \frac{a_\varepsilon^{1-p}}{\beta(\varepsilon)}.$$

Remark 4.30. Note that $a_\varepsilon^{1-p}/\beta(\varepsilon) \sim a_\varepsilon^{1-p} a_\varepsilon^{n-1} \varepsilon^{-n} \sim a_\varepsilon^{n-p} \varepsilon^{-n}$. This tends to a constant if $a_\varepsilon \sim a_\varepsilon^\star$.

Remark 4.31. When $a_\varepsilon = C_0 \varepsilon^{\frac{n}{n-p}}$ we recover the well-known value $\mathcal{A}_0 = (\frac{n-p}{p-1})^{p-1} |\partial B_1| C_0^{n-p}$. If we also take $\beta(\varepsilon) = \varepsilon^{\frac{n}{n-p}(p-1)}$, we recover $\mathcal{B}_0 = (\frac{n-p}{p-1})^{p-1} C_0^{1-p}$ (see, e. g., [112]).

Remark 4.32. The appearance of the strange formula for $\mathcal{A}_0$ is not fortuitous. As we will see later (see Lemma 4.38) it comes from the normal derivative of the auxiliary function w_ε in $a_\varepsilon \partial G_0$, and therefore it is related to the p-cap of G_0 as mentioned in Remark 3.11. This will play a fundamental role in the stochastic framework presented in Appendix C.

4.7.1.1 Properties of the strange term

Proposition 4.33. *Let σ be a maximal monotone graph such that $\sigma(0) = 0$ and let g be pointwise defined. Then:*

(a) *For every x, there exists exactly one value $H(x,s) \in \mathbb{R}$ such that* (4.17) *holds. If $g(x) = 0$, then $H(x,s) = H(s)$ and $H(0) = 0$.*

(b) *We have that H is non-decreasing in s and*

$$\big|H(x,s) - H(x,t)\big| \le |s-t|.$$

(c) *If $g(x) = 0$ and σ is differentiable, then*

$$\frac{\partial H}{\partial s}(s) = \frac{\sigma'(s - H(s))}{\sigma'(s-H(s)) + \mathcal{B}_0 |H(s)|^{p-2}} \in [0,1]. \tag{4.18}$$

(d) *When σ is a maximal monotone graph of $\mathbb{R}^2$ and $g = 0$, then H is given through the inverse of the composition of two maximal monotone graphs:*

$$H = \big(I + \sigma^{-1} \circ \Theta_{n,p}\big)^{-1} \text{ where } \Theta_{n,p}(s) = \mathcal{B}_0 |s|^{p-2} s.$$

Sketch of proof. First, let us understand the case where σ is a continuous function defined for every $s \in \mathbb{R}$; $\Theta_{n,p}$ is a strictly increasing function and $\Theta_{n,p}(\pm\infty) = \pm\infty$. For every s fixed we define the function

$$\Phi_s(h) = \Theta_{n,p}(h) - \sigma(s-h),$$

which is also strictly increasing in h. Furthermore, since

$$\Phi_s(\pm\infty) = \Theta_{n,p}(\pm\infty) - \sigma(s \mp \infty) = \Theta_{n,p}(\pm\infty) - \sigma(\mp\infty) = \pm\infty,$$

we have a bijection $\Phi_s : \mathbb{R} \to \mathbb{R}$. Hence, for every x and s there exists a unique solution of (4.17) and it is given by

$$H(x,s) = \Phi_s^{-1}\big(-g(x)\big).$$

We have that $\Phi_s^{-1}(0) = 0$. Hence $H(x,0) = 0$ (and $\mathcal{H}(x,0) = 0$) if and only if $g(x) = 0$. This requires some care when dealing with $\mathcal{H}(x,s)$.

When σ is a maximal monotone graph, then Φ_s is a maximal monotone graph, but since it is strictly increasing, its inverse is a pointwise function and we have the explicit expression of the statement. We recall that the inverse of a maximal monotone graph of $\mathbb{R}^2$ is also a maximal monotone graph. Moreover, $\sigma^{-1} \circ \Theta_{n,p}$ is a maximal monotone graph and thus $(I + \sigma^{-1} \circ \Theta_{n,p})^{-1}$ is an injective real application.

If σ is a non-decreasing differentiable function, we can take a derivative in s in (4.17) to recover

$$\mathcal{B}_0 \big|H(s)\big|^{p-2} \frac{\partial H}{\partial s}(s) = \sigma'\big(s - H(s)\big)\bigg(1 - \frac{\partial H}{\partial s}(s)\bigg).$$

Hence, we recover (4.18). Hence, when σ is differentiable, H is non-decreasing and Lipschitz of constant at most 1. By approximation, we recover Proposition 4.33 (b). □

Naturally, if g is defined a. e., then so is H. But this is good enough for our theory. We conclude collecting different examples, which have been studied in separate scenarios.

Examples 4.34. Some explicit examples of H can be found if we assume that $g = 0$.

(a) **No reaction.** If $\sigma \equiv 0$, then $H \equiv 0$.

(b) **Dirichlet boundary conditions.** Since $\operatorname{Dom} \sigma_{\mathrm{D}} = \{0\}$, equation (4.17) is only defined if $s - H(s) = 0$ and so $H(s) = s$. Thus, the strange term results:

$$\mathcal{H}(s) = \mathcal{A}_0 |s|^{p-2} s.$$

This is the unusual behavior proved for $p = 2$ by Hruslov [169] and Cioranescu and Murat [80, 81, 79].

(c) **Signorini boundary conditions.** A relevant case in the applications corresponds to the Signorini type boundary condition (2.1c), which can be written with the maximal monotone operator (2.3), given by

$$\mathcal{H}(s) = \begin{cases} \mathcal{A}_0 |H_0(s)|^{p-2} H_0(s) & s \geq 0, \\ \mathcal{A}_0 |s|^{p-2} s & s < 0, \end{cases}$$

where

$$\mathcal{B}_0 \big|H_0(s)\big|^{p-2} H_0(s) = \sigma_0\big(s - H_0(s)\big), \quad s > 0.$$

Hence, $\mathcal{H}$ coincides with the one from the Dirichlet condition if $s < 0$ and with the one from σ_0 if $s \geq 0$. This result was obtained previously in [89] (if $p = 2$) and [173] (if $p \neq 2$) by ad hoc techniques. Once the theory for maximal monotone operators is applied, it can be found as a corollary.

A comment of the behavior of u when f is very negative in a small region can be found in Proposition C.2.

Remark 4.35. As in Remark 4.24, we can also consider the case in which we replace $\sigma(u)$ by $\sigma(x,u)$ with $\sigma(x,0) = 0$. For instance, for the special case of $p = 2$ and $\sigma(x,u) = V(x)u$ (with $V \in H^1(\Omega)$ and $V \geq 0$) in the critical case we also get a (stationary) Schrödinger linear equation

$$-\Delta u + \mathcal{A}_0 \widetilde{V}(x)u = f,$$

but now with an effective potential which does not coincide with the original potential $V(x)$ since it is given by the bounded function

$$\widetilde{V}(x) = \frac{V(x)}{V(x) + \mathcal{B}_0}$$

(see, e. g., [271]).

4.7.1.2 Proof of the homogenization result

Theorem 4.36. *Let $p \in (1,n)$, $G_0 = B_1$, $a_\varepsilon \sim a^\star_\varepsilon$, $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$, $g^\varepsilon = g \in W^{1,\infty}(\Omega)$ for any ε, σ a maximal monotone graph. Then $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, where u is the unique solution of* (4.15) *with the reaction term $\mathcal{H}$ given by* (4.16)–(4.17).

Coming back to the function w_ε constructed in Section 3.1.5.1 we seek to apply oscillating test functions of the form $v_\varepsilon = v - hW_\varepsilon$, where v is a test function of the limit problem, h is a function correcting the amplitude of the oscillating function which will be chosen later and

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon(x - \varepsilon j) & a_\varepsilon < |x - \varepsilon j| < \frac{\varepsilon}{4} \text{ for some } j \in \Upsilon_\varepsilon, \\ 1 & |x - \varepsilon j| \leq a_\varepsilon \text{ for some } j \in \Upsilon_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is a well-defined construction, since for each x there is at most one $j \in \Upsilon_\varepsilon$ such that $x \in \varepsilon j + \varepsilon Y$ (i. e., $|x - \varepsilon j| < \varepsilon$).

The function h will be, in fact, of the form $h(x) = H(x, v(x))$, where the function $H(x,s)$ will be chosen through equation (4.17). Since v_ε is a valid test function for the weak formulation (2.8) we start by writing that

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \cdot \nabla (v_\varepsilon - u_\varepsilon)\, dx + \beta(\varepsilon) \int_{S_\varepsilon} \big(\Psi(v_\varepsilon) - \Psi(u_\varepsilon) - g(v_\varepsilon - u_\varepsilon)\big)\, dS - \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\, dx \geq 0. \tag{4.19}$$

Remark 4.37. In the critical case, the integral on S_ε has to be controlled by an interplay with the diffusion term. This is the second surprising fact indicated in the Introduction of the book (Section 1.1) and it does not happen in the sub- or supercritical regimes. It is a special phenomenon arising only in the critical case.

A useful decomposition of the diffusion balance term was given in [111] in terms of the following lemma.

Lemma 4.38. *Let* $1 < p < n$ *and let* $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ *be a sequence of functions with uniform bounded norm, let* $v \in \mathcal{C}_c^\infty(\Omega)$, $h \in W^{1,\infty}(\Omega)$ *and let* $v_\varepsilon = v - hW_\varepsilon$ *be the oscillating test function correcting the test function* v *of the homogenized problem. Then*

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x = I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + R(\varepsilon), \tag{4.20}$$

where $R(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *and*

$$I_{1,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x, \tag{4.21}$$

$$I_{2,\varepsilon} = -B_\varepsilon \int_{S_\varepsilon} |h|^{p-2}h(v - h - u_\varepsilon)\,\mathrm{d}S, \tag{4.22}$$

$$I_{3,\varepsilon} = A_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\varepsilon j + \frac{\varepsilon}{4}\partial B_1} |h|^{p-2}h(v - u_\varepsilon)\,\mathrm{d}S, \tag{4.23}$$

with the constants given by

$$A_\varepsilon \simeq \left(\frac{n-p}{p-1}\right)^{p-1} a_\varepsilon^{n-p}(\varepsilon/4)^{1-n}, \quad B_\varepsilon \simeq \left(\frac{n-p}{p-1}\right)^{p-1} a_\varepsilon^{1-p}.$$

We will give the proof of this lemma below, after a set of important considerations.

Remark 4.39. Note that since we are in the critical range $a_\varepsilon \sim \varepsilon^{\frac{n}{n-p}}$, $A_\varepsilon \sim a_\varepsilon^{n-p}\varepsilon^{1-n} \sim \varepsilon$, whereas

$$\beta^\star(\varepsilon) \sim |S_\varepsilon|^{-1} \sim \varepsilon^n (a_\varepsilon^\star)^{1-n} \sim (a_\varepsilon^*)^{n-p}(a_\varepsilon^\star)^{1-n} \sim (a_\varepsilon^\star)^{1-p}.$$

Hence $B_\varepsilon \sim \beta^\star(\varepsilon)$.

Remark 4.40. This is, perhaps, the most important remark of this book. There is a nice way to have an intuition for how (4.17) (the equation that characterizes H) appears. It is a formal synthesis which, in our opinion, is absent in most of the papers dealing with homogenization to the critical scale and the occurrence of strange terms. Assume for a second that σ is smooth and we look at the variational inequality (2.8), which is equivalent to the weak formulation of the problem (1.1)

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} (\sigma(v_\varepsilon) - g)(v_\varepsilon - u_\varepsilon)\,\mathrm{d}S - \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x \geq 0,$$

for any oscillating test function of the form $v_\varepsilon = v - hW_\varepsilon$. We have two integrals over S_ε, namely $I_{2,\varepsilon}$ and the one with the reaction term. Hence, if we manage to have

$$B_\varepsilon \int_{S_\varepsilon} |h|^{p-2}h(v - h - u_\varepsilon)\,\mathrm{d}S \simeq \beta(\varepsilon)\int_{S_\varepsilon} (\sigma(v_\varepsilon) - g)(v_\varepsilon - u_\varepsilon)\,\mathrm{d}S,$$

then we cancel out the terms related to the critically scaled S_ε. On the other hand, we can compute the limit of $I_{3,\varepsilon}$ by applying the "from surface to volume averaging convergence principle" given in Theorem 4.5 to the surface

$$\widehat{S}_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}\left(\varepsilon j + \frac{\varepsilon}{4}\partial B_1\right).$$

Thus, if we force the equality to hold pointwise, since $W_\varepsilon = 1$ on S_ε, we would like to have

$$\frac{B_\varepsilon}{\beta(\varepsilon)}|h|^{p-2}h \simeq \sigma(v_\varepsilon) - g = \sigma(v-h) - g.$$

This is the reason why we take $h = H(v)$, where H solves pointwise equation (4.17). Our proofs for general σ come from developing this idea in terms of subdifferentials.

Proof of Theorem 4.36. As usual, we know that u_ε is bounded in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, and hence up to a subsequence, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. The term $I_{1,\varepsilon}$ tends to the term corresponding to $-\Delta_p$ in the effective variational inequality formulation. Note that since $a_\varepsilon \ll \varepsilon$ and $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ we can compute

$$\lim_{\varepsilon\to 0} I_{1,\varepsilon} = \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u)\,\mathrm{d}x.$$

As mentioned in Remark 4.40, the term (4.22) will interplay with the term on S_ε on the weak formulation, where $I_{3,\varepsilon}$ yields the effective reaction term in the homogenized equation. Since the weak formulation is given, equivalently, by the variational inequality, we could kill the terms on S_ε if we show that

$$\limsup_{\varepsilon\to 0}\left(I_{2,\varepsilon} + \beta(\varepsilon)\int_{S_\varepsilon}\big(\Psi(v_\varepsilon) - \Psi(u_\varepsilon) - g(v_\varepsilon - u_\varepsilon)\big)\right)\mathrm{d}S \le 0.$$

First, note that, since the integral is controlled and due to the definition of $\mathcal{B}_0$, we have

$$I_{2,\varepsilon} = -\mathcal{B}_0\beta(\varepsilon)\int_{S_\varepsilon}|h|^{p-2}h(v-h-u_\varepsilon)dS + R(\varepsilon),$$

where $R(\varepsilon) \to 0$. Due to the choice of v_ε and g_ε we have, since $W_\varepsilon = 1$ on S_ε,

$$\Psi(v_\varepsilon) - \Psi(u_\varepsilon) - g(v_\varepsilon - u_\varepsilon) = \Psi(v-h) - \Psi(u_\varepsilon) - g(x)(v-h-u_\varepsilon).$$

Hence, we would want to have

$$\Psi(v-h) - \Psi(u_\varepsilon) - g(x)(v-h-u_\varepsilon) \le \mathcal{B}_0|h|^{p-2}h(v-h-u_\varepsilon). \tag{4.24}$$

Here is where the magic of convex analysis comes into play. This kind of inequality is reminiscent of the properties of the subdifferential of a convex function

$$\overline{\Psi}(x,s) - \overline{\Psi}(x,t) \le \xi(s-t), \quad \forall \xi \in \partial\overline{\Psi}(x,s) \tag{4.25}$$

for a general function $\overline{\Psi}(x,\cdot)$ assumed to be convex for every x, and where $\partial\overline{\Psi}$ denotes its subdifferential. This is what we get when we assume that σ is a maximal monotone graph. Taking

$$\overline{\Psi}(x,s) = \Psi(s) - g(x)s,$$

in terms of its subdifferential we have $\partial_s\overline{\Psi}(x,s) = \sigma(s) - g(x)$. If $h = H(x,v)$, where H is the pointwise solution of (4.17), we recover (4.24) by taking in (4.25)

$$s = v(x) - h(x), \quad t = u_\varepsilon(x), \quad \xi = \mathcal{B}_0|h|^{p-2}h = \mathcal{B}_0\big|H(.,v)\big|^{p-2}H(.,v) \in \partial\Psi(t,s).$$

To compute the limit of $I_{3,\varepsilon}$ we apply the important "from surface to volume averaging convergence principle" given in Theorem 4.5 to the surface

$$\widehat{S}_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}\left(\varepsilon j + \frac{\varepsilon}{4}\partial B_1\right).$$

We have that $|\widehat{S}_\varepsilon| \simeq \frac{|\Omega||\partial B_1|}{4^{n-1}}\varepsilon^{-1}$, so by Theorem 4.5 and the definition of $\mathcal{A}_0$ we get the effective reaction balance term

$$\lim_{\varepsilon\to 0} I_{3,\varepsilon} = \mathcal{A}_0\int_\Omega \big|H(x,v)\big|^{p-2}H(x,v)(v-u)\,\mathrm{d}x.$$

We compute now the limit of the last term of the variational inequality associated to f. Due to (3.16), for $q > \frac{n(p-1)}{n-1}$ it is a direct computation to check that

$$\int_\Omega |\nabla W_\varepsilon|^q\,\mathrm{d}x \sim |\Upsilon_\varepsilon| \int_{\frac{\varepsilon}{4}B_1} \big|\nabla w_\varepsilon(x)\big|^q\,\mathrm{d}x \sim \varepsilon^{-n}a_\varepsilon^{n-q} \sim a_\varepsilon^{p-q} \tag{4.26}$$

(since $a_\varepsilon \sim a_\varepsilon^\star$). Hence, due to the Sobolev embedding inequalities we get

$$W_\varepsilon \to 0 \quad \begin{cases} \text{strongly in } W_0^{1,q}(\Omega) \text{ if } 1 \le q < p, \\ \text{weakly in } W_0^{1,p}(\Omega). \end{cases} \tag{4.27}$$

The second statement may not seem obvious. However, in $W^{1,p}$ the norm is bounded, and hence there must exist a weak limit. This limit must coincide with the $W^{1,q}$ limits for $q < p$, and therefore it must be 0. Hence, by compactness, $v_\varepsilon \to v$ in $L^p(\Omega)$ and since $a_\varepsilon \ll \varepsilon$, by Lemma 4.8 we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x = \int_\Omega f(v-u)\,\mathrm{d}x.$$

Joining all the information above,

$$\begin{aligned}
0 \le\ & \liminf_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot (v_\varepsilon - u_\varepsilon)\,\mathrm{d}x \\
&+ \beta(\varepsilon)\int_{S_\varepsilon} \big(\Psi(v_\varepsilon) - \Psi(u_\varepsilon) - g(v_\varepsilon - u_\varepsilon)\big)\,\mathrm{d}S - \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x \Big) \\
\le\ & \liminf_{\varepsilon\to 0}\Big(I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + \beta(\varepsilon)\int_{S_\varepsilon} \big(\Psi(v_\varepsilon) - \Psi(u_\varepsilon) - g(v_\varepsilon - u_\varepsilon)\big)\,\mathrm{d}S - \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x \Big) \\
\le\ & \liminf_{\varepsilon\to 0}\Big(I_{1,\varepsilon} + I_{3,\varepsilon} - \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x \Big) \\
=\ & \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u)\,\mathrm{d}x + \mathcal{A}_0 \int_\Omega \big|H(x,v)\big|^{p-2} H(x,v)(v-u)\,\mathrm{d}x - \int_\Omega f(v-u)\,\mathrm{d}x.
\end{aligned} \tag{4.28}$$

This concludes the proof. □

4.7.1.3 Proof of Lemma 4.38

The proof of Lemma 4.38 is fairly technical and depends on the case of whether $p < 2$ or $p \ge 2$. A first result in which it is followed by a presentation similar to the one in this book was the paper [268] relative to the case $p = 2$. The case $p \in (2, n)$ was analyzed in [245]. The case $p \in (1, 2)$ was analyzed in [113]. We refer the reader to [190] for some techniques associated to the p-Laplace operator.

Let us present first the case $p = 2$, which is quite simple, and then the general setting $p > 1$. We have

$$\int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x = J_{1,\varepsilon} - J_{2,\varepsilon},$$

where

$$\begin{aligned}
J_{1,\varepsilon} &= \int_{\Omega_\varepsilon} \nabla v \cdot \nabla(v - hW_\varepsilon - u_\varepsilon)\,\mathrm{d}x, \\
J_{2,\varepsilon} &= \int_{\Omega_\varepsilon} \nabla(hW_\varepsilon) \cdot \nabla(v - hW_\varepsilon - u_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

Moreover, due to the estimate (4.26)

$$J_{1,\varepsilon} = I_{1,\varepsilon} - \int_{\Omega_\varepsilon} \nabla v \cdot \nabla(hW_\varepsilon)\,\mathrm{d}x = I_{1,\varepsilon} + R(\varepsilon).$$

On the other hand, using Green's formula, the fact that W_ε satisfies problem (3.14) and the estimates on W_ε given in Section 3.1.5.1 we have

$$\begin{aligned}
J_{2,\varepsilon} &= \int_{\Omega_\varepsilon} W_\varepsilon \nabla h \cdot \nabla(v - hW_\varepsilon - u_\varepsilon)\,\mathrm{d}x + \int_{\Omega_\varepsilon} h\nabla W_\varepsilon \cdot \nabla(v - hW_\varepsilon - u_\varepsilon)\,\mathrm{d}x \\
&= R(\varepsilon) - \int_{\Omega_\varepsilon} (\nabla W_\varepsilon \cdot \nabla h)(v - hW_\varepsilon - u_\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{j\in\Upsilon_\varepsilon} \int_{\partial(\varepsilon j+\frac{\varepsilon}{4}B_1)} \partial_\nu w_\varepsilon^j h(v - u_\varepsilon)\,\mathrm{d}S \\
&\quad + \sum_{j\in\Upsilon_\varepsilon} \int_{\partial(\varepsilon j+a_\varepsilon G_0)} \partial_\nu w_\varepsilon^j h(v - h - u_\varepsilon)\,\mathrm{d}S,
\end{aligned}$$

where ∂_ν is the usual normal derivative. Using the estimates on the gradient and normal derivatives of W_ε given in Section 3.1.5.1 (see (3.17)) we have

$$J_{2,\varepsilon} = R(\varepsilon) + A_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\partial(\varepsilon j+\frac{\varepsilon}{4}B_1)} h(v - u_\varepsilon)\,\mathrm{d}S - B_\varepsilon \int_{S_\varepsilon} h(v - h - u_\varepsilon)\,\mathrm{d}S.$$

This completes the proof for $p = 2$.

To consider the case of $p \neq 2$ we need the following auxiliary result.

Lemma 4.41. *Let $p > 1$, $a_\varepsilon \sim a_\varepsilon^\star$, $n \geq 3$, $v \in W_0^{1,\infty}(\Omega)$ and let $\varphi_\varepsilon \in W_0^{1,p}(\Omega)$ be uniformly bounded. Let also $\eta_\varepsilon \in W^{1,p}(\Omega)$ be such that $\|\nabla\eta_\varepsilon\|_{L^q(\Omega)} \to 0$, for all $q \in [1, p)$, as $\varepsilon \to 0$. Then*

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \big|\nabla(v - \eta_\varepsilon)\big|^{p-2} \nabla(v - \eta_\varepsilon) \cdot \nabla\varphi_\varepsilon\,\mathrm{d}x \\
&\quad = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi_\varepsilon\,\mathrm{d}x - \int_{\Omega_\varepsilon} |\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon \cdot \nabla\varphi_\varepsilon\,\mathrm{d}x + R(\varepsilon),
\end{aligned}$$

where $R(\varepsilon) \to 0$.

We split the proof in the cases $p \in (1, 2)$, $(2, 3]$ and $(3, +\infty)$. We recall two not too difficult technical lemmas dealing with the different values of p [113] (see also [190]).

Lemma 4.42. *Let $p \in (1, 2)$. Then there exists a positive constant $C = C(p)$ such that the inequality*

$$\big||\mathbf{a} - \mathbf{b}|^{p-2}(\mathbf{a} - \mathbf{b}) - (|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b})\big| \leq C(|\mathbf{a}||\mathbf{b}|)^{\frac{p-1}{2}} \tag{4.29}$$

is valid for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

Lemma 4.43. *Let $p > 3$. Then, for all $a, b \geq 0$ we have*

$$(a + b)^{p-2} - b^{p-2} \leq (p - 2)a(a + b)^{p-3}.$$

Proof of Lemma 4.41. **Case** $p \in (1,2)$**.** By Lemma 4.42, by applying Hölder's inequality, we have

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon} \big(\big|\nabla(v-\eta_\varepsilon)\big|^{p-2}\nabla(v-\eta_\varepsilon)\cdot\nabla\varphi_\varepsilon - \big(|\nabla v|^{p-2}\nabla v - |\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon\big)\cdot\nabla\varphi_\varepsilon\big)\,\mathrm{d}x\Big|\\ \le C\int_{\Omega_\varepsilon} |\nabla v|^{\frac{p-1}{2}}|\nabla\eta_\varepsilon|^{\frac{p-1}{2}}|\nabla\varphi_\varepsilon|\,\mathrm{d}x\\ \le K\|\nabla v\|_\infty^{\frac{p-1}{2}}\|\nabla\eta_\varepsilon\|_{L^{\frac{p+1}{2}}(\Omega_\varepsilon)}^{\frac{p-1}{2}}\|\nabla\varphi_\varepsilon\|_{L^{\frac{p+1}{2}}(\Omega_\varepsilon)},\end{aligned}$$

since $1 < (p+1)/2 < p$. This proves the result.
Case $p \in [2,3]$. When $p \ge 2$ we write

$$\begin{aligned}\int_{\Omega_\varepsilon} \big|\nabla(v-\eta_\varepsilon)\big|^{p-2}\nabla(v-\eta_\varepsilon)\cdot\nabla\varphi_\varepsilon\,\mathrm{d}x\\ = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi_\varepsilon\,\mathrm{d}x - \int_{\Omega_\varepsilon} |\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon\cdot\nabla\varphi_\varepsilon\,\mathrm{d}x\\ + \int_{\Omega_\varepsilon}\big(\big|\nabla(v-\eta_\varepsilon)\big|^{p-2} - |\nabla v|^{p-2}\big)\nabla v\cdot\nabla\varphi_\varepsilon\,\mathrm{d}x\\ - \int_{\Omega_\varepsilon}\big(\big|\nabla(v-\eta_\varepsilon)\big|^{p-2} - |\nabla\eta_\varepsilon|^{p-2}\big)\nabla\eta_\varepsilon\cdot\nabla\varphi_\varepsilon\,\mathrm{d}x.\end{aligned} \tag{4.30}$$

Hence, $R(\varepsilon)$ is given by the last two terms. For the first term, in the case $p \in [2,3]$, through Minkowski's inequality, we have

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon}\big(\big|\nabla(v-\eta_\varepsilon)\big|^{p-2} - |\nabla v|^{p-2}\big)\nabla v\cdot\nabla\varphi_\varepsilon\,\mathrm{d}x\Big|\\ \le \int_{\Omega_\varepsilon}\big(\big(|\nabla v| + |\nabla\eta_\varepsilon|\big)^{p-2} - |\nabla v|^{p-2}\big)|\nabla v||\nabla\varphi_\varepsilon|\,\mathrm{d}x\\ \le \int_{\Omega_\varepsilon} |\nabla\eta_\varepsilon|^{p-2}|\nabla v||\nabla\varphi_\varepsilon|\,\mathrm{d}x\\ \le \|\nabla v\|_{L^\infty}\Big(\int_{\Omega_\varepsilon}|\nabla\eta_\varepsilon|^{(p-2)q}\,\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{\Omega_\varepsilon}|\nabla\varphi_\varepsilon|^{\frac{q}{q-1}}\,\mathrm{d}x\Big)^{\frac{q-1}{q}}.\end{aligned}$$

It suffices to pick any q such that $(p-2)q < p$ and $\frac{q}{q-1} \le p$. Thus, any $q \in [\frac{p}{p-1}, \frac{p}{p-2})$ is valid. A similar argument works for the last term in (4.30).
Case $p > 3$**.** We look again at the remainder terms in (4.30) and use Lemma 4.43. Then

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon}\big(\big|\nabla(v-\eta_\varepsilon)\big|^{p-2} - |\nabla v|^{p-2}\big)\nabla v\cdot\nabla\varphi_\varepsilon\,\mathrm{d}x\Big|\\ \le \int_{\Omega_\varepsilon}\big(\big(|\nabla v| + |\nabla\eta_\varepsilon|\big)^{p-2} - |\nabla v|^{p-2}\big)|\nabla v||\nabla\varphi_\varepsilon|\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}
&\le C \int_{\Omega_\varepsilon} |\nabla \eta_\varepsilon| \big(|\nabla v| + |\nabla \eta_\varepsilon|\big)^{p-3} |\nabla \varphi_\varepsilon| \, \mathrm{d}x \\
&\le C \int_{\Omega_\varepsilon} (|\nabla \eta_\varepsilon| + |\nabla \eta_\varepsilon|^{p-2}) |\nabla \varphi_\varepsilon| \, \mathrm{d}x \\
&\le C \Big(\int_{\Omega_\varepsilon} |\nabla \eta_\varepsilon|^{q_1} \, \mathrm{d}x \Big)^{\frac{1}{q_1}} \Big(\int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^{\frac{q_1}{q_1-1}} \, \mathrm{d}x \Big)^{\frac{q_1-1}{q_1}} + \\
&\quad + C \Big(\int_{\Omega_\varepsilon} |\nabla \eta_\varepsilon|^{(p-2)q_2} \, \mathrm{d}x \Big)^{\frac{1}{q_2}} \Big(\int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^{\frac{q_2}{q_2-1}} \, \mathrm{d}x \Big)^{\frac{q_2-1}{q_2}} .
\end{aligned}$$

Again, we get a large set of possible choices for q_1, q_2. The last term in (4.30) behaves similarly. □

The splitting Lemma 4.38 (for $p \neq 2$) follows as a direct consequence of this lemma.

Proof of Lemma 4.38 *(for $p \neq 2$).* Since $v_\varepsilon = v - hW_\varepsilon$, by Lemma 4.41, we have

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \cdot \nabla (v_\varepsilon - u_\varepsilon) \, \mathrm{d}x = J_{1,\varepsilon} - J_{2,\varepsilon} + R(\varepsilon),$$

where

$$\begin{aligned}
J_{1,\varepsilon} &= \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla (v - hW_\varepsilon - u_\varepsilon) \, \mathrm{d}x, \\
J_{2,\varepsilon} &= \int_{\Omega_\varepsilon} \big|\nabla (hW_\varepsilon)\big|^{p-2} \nabla (hW_\varepsilon) \cdot \nabla (v - hW_\varepsilon - u_\varepsilon) \, \mathrm{d}x.
\end{aligned}$$

Moreover, due to (4.26)

$$J_{1,\varepsilon} = I_{1,\varepsilon} - \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla (hW_\varepsilon) \, \mathrm{d}x + R(\varepsilon) = I_{1,\varepsilon} + R(\varepsilon).$$

On the other hand, similarly to the case $p = 2$,

$$\begin{aligned}
J_{2,\varepsilon} &= \int_{\Omega_\varepsilon} |h \nabla W_\varepsilon|^{p-2} h \nabla W_\varepsilon \cdot \nabla (v - hW_\varepsilon - u_\varepsilon) \, \mathrm{d}x + R(\varepsilon) \\
&= R(\varepsilon) + \sum_{j \in \Upsilon_\varepsilon} \int_{\partial(\varepsilon j + \frac{\varepsilon}{4} B_1)} \big|\nabla w_\varepsilon^j\big|^{p-2} \partial_\nu w_\varepsilon^j |h|^{p-2} h (v - u_\varepsilon) \, \mathrm{d}S \\
&\quad + \sum_{j \in \Upsilon_\varepsilon} \int_{\partial(\varepsilon j + a_\varepsilon G_0)} \big|\nabla w_\varepsilon^j\big|^{p-2} \partial_\nu w_\varepsilon^j |h|^{p-2} h (v - h - u_\varepsilon) \, \mathrm{d}S.
\end{aligned}$$

From the explicit estimates given in Section 3.1.5.1 we have

$$J_{2,\varepsilon} = R(\varepsilon) + A_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial(\varepsilon j+\frac{\varepsilon}{4}B_1)} |h|^{p-2}h(v-u_\varepsilon)\,\mathrm{d}S - B_\varepsilon \int\limits_{S_\varepsilon} |h|^{p-2}h(v-h-u_\varepsilon)\,\mathrm{d}S.$$

This completes the proof. □

4.7.1.4 Strong convergence with correctors when $g^\varepsilon = 0$

By using similar arguments to the ones of the above section, it seems possible to prove the strong convergence with a corrector when we assume some additional regularity on u as, for instance, $u \in W^{1,\infty}(\Omega)$. At least for the case $p = 2$ we can prove that

$$\|u_\varepsilon + H(u)W_\varepsilon - u\|_{W^{1,p}(\Omega_\varepsilon,\partial\Omega)} \to 0 \tag{4.31}$$

as $\varepsilon \to 0$. Using that W_ε converges strongly to 0 in $W^{1,q}$ for $q < p$ we also deduce immediately that

$$\|u - u_\varepsilon\|_{W^{1,q}(\Omega_\varepsilon,\partial\Omega)} \to 0, \quad \text{for } q < p.$$

This result can be found for $p = 2$ in [270] and σ Lipschitz continuous, and the Signorini problem in [173]. Let us give an idea of the proof (for $p = 2$). The proof in the case $p \neq 2$ is not present in the literature, to the best of our knowledge. However, we expect it should follow from not too different ideas from the case $p = 2$.

The weak formulation for u_ε is given by

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\varphi\,\mathrm{d}x + \beta(\varepsilon)\int\limits_{S_\varepsilon} \sigma(u_\varepsilon)\varphi\,\mathrm{d}S = \int\limits_{\Omega_\varepsilon} f\varphi\,\mathrm{d}x.$$

Let us define $U_\varepsilon = u - H(u)W_\varepsilon$. Plugging it back on the weak formulation of the problem for u_ε we have

$$\begin{aligned}\int\limits_{\Omega_\varepsilon} \nabla U_\varepsilon \nabla\varphi\,\mathrm{d}x + \beta(\varepsilon)\int\limits_{S_\varepsilon} \sigma(U_\varepsilon)\varphi\,\mathrm{d}S &= \int\limits_{\Omega_\varepsilon} \nabla u\nabla\varphi\,\mathrm{d}x - \int\limits_{\Omega_\varepsilon} \nabla\big(H(u)W_\varepsilon\big)\nabla\varphi\,\mathrm{d}x \\ &\quad + \beta(\varepsilon)\int\limits_{S_\varepsilon} \sigma\big(u - H(u)\big)\varphi\,\mathrm{d}S.\end{aligned}$$

We must take into account that

$$\int\limits_{\Omega_\varepsilon} \nabla u\nabla\varphi\,\mathrm{d}x = \int\limits_{\Omega} f\varphi\,\mathrm{d}x - \mathcal{A}_0\int\limits_{\Omega} H(u)\varphi\,\mathrm{d}x - \int\limits_{\Omega\setminus\Omega_\varepsilon} \nabla u\nabla\varphi\,\mathrm{d}x.$$

Hence, if we assume the additional regularity $u \in W^{1,\infty}(\Omega)$, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla(U_\varepsilon - u_\varepsilon)\nabla\varphi \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon}(\sigma(U_\varepsilon) - \sigma(u_\varepsilon))\varphi \,\mathrm{d}S \\ = \int_{\Omega\setminus\Omega_\varepsilon} f\varphi \,\mathrm{d}x - \mathcal{A}_0\int_\Omega H(u)\varphi \,\mathrm{d}x - \int_{\Omega\setminus\Omega_\varepsilon} \nabla u\nabla\varphi \,\mathrm{d}x \\ - \int_{\Omega_\varepsilon} \nabla\big(H(u)W_\varepsilon\big)\nabla\varphi \,\mathrm{d}x + \beta(\varepsilon)\mathcal{B}_0\int_{S_\varepsilon} H(u)\varphi \,\mathrm{d}S.\end{aligned}$$

Now the aim is to take $\varphi = U_\varepsilon - u_\varepsilon$ and show that the right-hand side tends to zero. One looks for similar cancelations as in the above result and with the advantage that $P_\varepsilon u_\varepsilon - u$ and W_ε tend weakly to 0 in $H_0^1(\Omega)$. As before, the case $p \neq 2$ is more technical, but follows the same philosophy. We refer the reader to [270, 113] for the details. We point out that the regularity $u_\varepsilon \in W^{1,\infty}(\Omega_\varepsilon)$ was shown in [50], for $p = 2$, when σ is a maximal monotone graph and $f \in L^\infty(\Omega)$. The regularity $u \in W^{1,\infty}(\Omega)$, when $f \in L^\infty(\Omega)$ and $p = 2$, is well known since function H is Lipschitz continuous (see, e. g., [186]) and even under weaker assumptions on f (see [49]). Then it makes sense to ask if the above convergence, in fact, takes place also in $W^{1,\infty}(\Omega)$ when $u \in W^{1,\infty}(\Omega)$: as far a we know, this is an open problem.

4.7.2 Case of G_0 a ball, $p = n$ and $g^\varepsilon = g$

This setting was first considered in [229]. The main arguments are as above, but replacing now w_ε by the value given in Section 3.1.5.2. Again, we assume that $G_0 = B_1$ and $g^\varepsilon = g \in W^{1,\infty}(\Omega)$. We recover the homogenized problem (4.15), where

$$\mathcal{H}(x,s) = \mathcal{A}_0\big|H(x,s)\big|^{p-2}H(x,s)$$

and H is at each point (x,s) the solution of the functional equation

$$\mathcal{B}_0\big|H(x,s)\big|^{n-2}H(x,s) \in \sigma\big(s - H(x,s)\big) - g(x),$$

where

$$\mathcal{A}_0 = |\partial B_1| \lim_{\varepsilon\to 0} \varepsilon^{-n}\left(\ln\frac{\varepsilon}{4a_\varepsilon}\right)^{1-n}, \quad \mathcal{B}_0 = \lim_{\varepsilon\to 0}\left(\beta(\varepsilon)a_\varepsilon \ln\frac{\varepsilon}{4a_\varepsilon}\right)^{1-n}.$$

Going back to Remark 4.1, we observe that the parameter α of the family of critical scales appears when substituting the assumption on a_ε in $\mathcal{A}_0$ and $\mathcal{B}_0$.

4.7.3 Case of G_0 not a ball, $p = 2 < n$

First, let us look at the case $g^\varepsilon = 0$. The first paper in this setting is [116] for σ Hölder continuous.

Theorem 4.44. *Assume $a_\varepsilon \sim a_\varepsilon^\star$, $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$, $f \in L^2(\Omega)$, $g^\varepsilon = 0$ and σ Hölder continuous for some $\alpha \le 1$. Then, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the unique solution of the homogenized problem* (4.15) *with*

$$\mathcal{H}(x,s) = \mathcal{A}_0 \widehat{H}_\sigma(s), \quad \textit{where } \mathcal{A}_0 = \lim_{\varepsilon\to 0} \frac{a_\varepsilon^{n-2}}{\varepsilon^n} \tag{4.32}$$

and $\widehat{H}_\sigma$ is given by (3.27).

We will use the auxiliary functions and notation given in Section 3.1.5.3. We recall that as proved in Section 3.1.5.3, $0 \le \widehat{H}_\sigma'(s) \le \lambda_{G_0}$. This constant λ_{G_0} depends only on G_0 and, even though we will assume that σ is a smooth function, there is a natural way to extend the results to maximal monotone graphs, as pointed out in Remarks 3.20 and 3.21. This intuition matches nicely with the explicit results when $G_0 = B_1$ given in Section 4.7.1.1.

We can proceed similarly in the case of a ball, with some additional constructions. For $v \in C^\infty(\overline{\Omega})$ we define, for $x \in \Omega_\varepsilon$,

$$W_{\sigma,\varepsilon}(x; v) = \begin{cases} w_{\sigma,\varepsilon}(x - \varepsilon j, v(\varepsilon j)) & x - \varepsilon j \in \frac{\varepsilon}{4} B_1 \setminus (a_\varepsilon \overline{G}_0) \text{ for some } j \in \Upsilon_\varepsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{4.33}$$

where $w_{\sigma,\varepsilon}$ is the solution of (3.35). As above, this is well defined since there is at most one $j \in \Upsilon_\varepsilon$ so that $x \in \varepsilon j + \varepsilon Y$.

As with the previous case, it is not hard to prove that $W_{\varepsilon,\sigma}$ is bounded in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Furthermore, by extending $w_{\sigma,\varepsilon}$ into the particle by preserving the gradient, we can see that there is an extension $\widetilde{W}_{\sigma,\varepsilon}$ bounded in $W_0^{1,p}(\Omega)$. Arguing as in Section 4.1, it is not difficult to see that

$$\widetilde{W}_{\sigma,\varepsilon}(\cdot; v) \rightharpoonup 0 \quad \text{in } W_0^{1,p}(\Omega).$$

Note that, unlike before, we cannot define the function inside the particles. The oscillating test function is now of the form

$$v_\varepsilon(x) = v(x) - W_{\sigma,\varepsilon}(x; v).$$

In this setting, we need the following adapted version of Lemma 4.38 (the proof is left as an exercise for the reader).

Lemma 4.45. *Let $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ be a sequence with uniformly bounded norm, $v \in C_c^\infty(\Omega)$, $v_\varepsilon(x) = v(x) - W_{\sigma,\varepsilon}(x; v)$. Then*

$$\int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x = I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + R(\varepsilon),$$

where $R(\varepsilon) \to 0$ and

$$\begin{aligned}
I_{1,\varepsilon} &= \int_{\Omega_\varepsilon} \nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x,\\
I_{2,\varepsilon} &= -\sum_{j\in\Upsilon_\varepsilon} \int_{\partial(\varepsilon j + a_\varepsilon G_0)} (v_\varepsilon(x) - u_\varepsilon(x))\partial_\nu w_{\sigma,\varepsilon}(x - \varepsilon j; v(\varepsilon j))\,\mathrm{d}S_x,\\
I_{3,\varepsilon} &= -\sum_{j\in\Upsilon_\varepsilon} \int_{\partial(\varepsilon j + \frac{\varepsilon}{4}B_1)} (v(x) - u_\varepsilon(x))\partial_\nu w_{\sigma,\varepsilon}(x - \varepsilon j; v(\varepsilon j))\,\mathrm{d}S.
\end{aligned}$$

Going back to (4.40), this splitting shows how we arrive at a formulation of problem (3.25).

Let us study $I_{2,\varepsilon}$. From our choice (3.35) we have

$$I_{2,\varepsilon} = -C_0 a_\varepsilon^{-1} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial(\varepsilon j + a_\varepsilon G_0)} (v(x) - W_{\sigma,\varepsilon}(x) - u_\varepsilon(x))\sigma(v(\varepsilon j) - W_{\sigma,\varepsilon}(x))\,\mathrm{d}S_x.$$

Therefore, we recover

$$\begin{aligned}
&I_{2,\varepsilon} + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(v - W_{\sigma,\varepsilon})(v_\varepsilon - u_\varepsilon)\,\mathrm{d}S_x\\
&= \sum_{j\in\Upsilon_\varepsilon} \int_{\partial(\varepsilon j + a_\varepsilon G_0)} (v_\varepsilon(x) - u_\varepsilon(x))\big(\beta(\varepsilon)\sigma(v(x) - W_{\sigma,\varepsilon}(x)) - C_0 a_\varepsilon^{-1}\sigma(v(\varepsilon j) - W_{\sigma,\varepsilon}(x))\big)\,\mathrm{d}S_x.
\end{aligned}$$

When σ is smooth, it is easy to show that it tends to zero due to our choice of C_0 and by taking Taylor expansions.

With this lemma, we can almost repeat the computations made in (4.28) except for the cancelation of $I_{2,\varepsilon}$. The term $I_{1,\varepsilon}$ is as for the case of balls. For the term $I_{3,\varepsilon}$ we need an adapted version of Theorem 4.11. By taking into account (3.38) and an argument similar to Theorem 4.11 we recover that if $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $H_0^1(\Omega)$ and $v \in C^\infty(\overline{\Omega})$, then

$$I_{3,\varepsilon} = -\sum_{j\in\Upsilon_\varepsilon} \int_{\varepsilon j + \frac{\varepsilon}{4}\partial B_1} \varphi_\varepsilon(x)\partial_\nu w_{\sigma,\varepsilon}(x - \varepsilon j; v(\varepsilon j))\,\mathrm{d}S_x \longrightarrow \left(\lim_{\varepsilon\to 0} \frac{a_\varepsilon}{a_\varepsilon^\star}\right)^{n-2} \int_\Omega \varphi(x)\widehat{H}_\sigma(v(x))\,\mathrm{d}x. \tag{4.34}$$

Let us explain this limit. For a fixed v smooth, we construct the operator on $H^1(\Omega)$

$$\mu_\varepsilon(\varphi) = -\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} \varphi(x)\partial_\nu w_{\sigma,\varepsilon}(x-\varepsilon j; v(\varepsilon j))\,\mathrm{d}S_x.$$

For a function $\varphi \in C^1(\bar{\Omega})$

$$\int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} \varphi(x)\partial_\nu w_{\sigma,\varepsilon}(x-\varepsilon j; v(\varepsilon j))\,\mathrm{d}S_x \simeq -\varphi(\varepsilon j)\int\limits_{\frac{\varepsilon}{4}\partial B_1} \partial_\nu w_{\sigma,\varepsilon}(x; v(\varepsilon j))\,\mathrm{d}S_x.$$

Since $\Delta w_{\sigma,\varepsilon} = 0$, we have

$$0 = \int\limits_{\frac{\varepsilon}{4}\partial B_1\setminus a_\varepsilon G_0} \Delta w_{\sigma,\varepsilon}\,\mathrm{d}x = \int\limits_{\frac{\varepsilon}{4}\partial B_1} \partial_\nu w_{\sigma,\varepsilon}\,\mathrm{d}S_x + \int\limits_{a_\varepsilon\partial G_0} \partial_\nu w_{\sigma,\varepsilon}\,\mathrm{d}S_x. \tag{4.35}$$

Hence

$$-\int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} \varphi(x)\partial_\nu w_{\sigma,\varepsilon}(x-\varepsilon j; \varphi(\varepsilon j))\,\mathrm{d}S_x \simeq \varphi(\varepsilon j)\int\limits_{a_\varepsilon\partial G_0} \partial_\nu w_{\sigma,\varepsilon}\,\mathrm{d}S_x.$$

Now, we note that due to (3.37)

$$\begin{aligned}\int\limits_{a_\varepsilon\partial G_0} \partial_\nu w_{\sigma,\varepsilon}(x; v(\varepsilon j))\,\mathrm{d}S_x &\simeq \int\limits_{a_\varepsilon\partial G_0} \partial_\nu \widehat{w}_{\sigma,\varepsilon}(x; v(\varepsilon j))\,\mathrm{d}S_x \\ &= a_\varepsilon^{n-2}\int\limits_{\partial G_0} \partial_\nu \widehat{w}_{\sigma}(y; v(\varepsilon j))\,\mathrm{d}S_y \\ &= a_\varepsilon^{n-2}\widehat{H}_\sigma(v(\varepsilon j)).\end{aligned}$$

Thus

$$\mu_\varepsilon(\varphi) \simeq \frac{a_\varepsilon^{n-2}}{\varepsilon^n}\sum_{j\in\Upsilon_\varepsilon} \varepsilon^n\varphi(\varepsilon j)\widehat{H}_\sigma(v(\varepsilon j)).$$

We recover the limit by Riemann sums. Hence, if the operator has a limit, this limit is

$$\mu(\varphi) = \lim_{\varepsilon\to 0}\frac{a_\varepsilon^{n-2}}{\varepsilon^n}\int\limits_\Omega \varphi(x)\widehat{H}_\sigma(v(x))\,\mathrm{d}x.$$

The proof that μ_ε is strongly convergent (which is needed to show the convergence of $\mu_\varepsilon(\varphi_\varepsilon)$ where φ_ε converges weakly), and the rigorous proof of (4.34), is as follows:

1. One proves that

$$-\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} \varphi_\varepsilon(x)\partial_\nu \widehat{w}_{\sigma,\varepsilon}(x-\varepsilon j; v(\varepsilon j))\,\mathrm{d}S_x \longrightarrow \left(\lim_{\varepsilon\to 0}\frac{a_\varepsilon^{n-2}}{\varepsilon^n}\right)\int\limits_\Omega \varphi(x)\widehat{H}_\sigma(v(x))\,\mathrm{d}x.$$

   Note the difference with (4.34). This is done by defining functions similar to $m_{g,\varepsilon}$ and noting that, since $\Delta\widehat{w}_{\sigma,\varepsilon} = 0$, similarly to (4.35), we have

$$-\int\limits_{\frac{\varepsilon}{4}\partial B_1} \partial_\nu \widehat{w}_{\sigma,\varepsilon}(y;s)\,\mathrm{d}S_y = a_\varepsilon^{n-2}\widehat{H}_\sigma(s).$$

2. We prove that

$$\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} \varphi_\varepsilon(x)\big(\partial_\nu w_{\sigma,\varepsilon}(x-\varepsilon j; v(\varepsilon j)) - \partial_\nu \widehat{w}_{\sigma,\varepsilon}(x-\varepsilon j; v(\varepsilon j))\big)\,\mathrm{d}S_x \longrightarrow 0.$$

   This is done by integrating by parts and applying the estimates in Lemma 3.23. We have

$$\begin{aligned}
&\int\limits_{\varepsilon j+\frac{\varepsilon}{4}\partial B_1} \varphi_\varepsilon(x)\big(\partial_\nu w_{\sigma,\varepsilon}(x-\varepsilon j;s) - \partial_\nu \widehat{w}_{\sigma,\varepsilon}(x-\varepsilon j;s)\big)\,\mathrm{d}S_x \\
&\quad = \int\limits_{\frac{\varepsilon}{4}B_1\setminus a_\varepsilon G_0} \nabla\varphi_\varepsilon(x+\varepsilon j)\nabla\big(w_{\sigma,\varepsilon}(x;s) - \widehat{w}_{\sigma,\varepsilon}(x;s)\big)\,\mathrm{d}x \\
&\qquad - \int\limits_{a_\varepsilon\partial G_0} \varphi_\varepsilon(x+\varepsilon j)\big(\partial_\nu w_{\sigma,\varepsilon}(x;s) - \partial_\nu \widehat{w}_{\sigma,\varepsilon}(x;s)\big)\,\mathrm{d}S_x.
\end{aligned}$$

The first term is controlled via the H^1 estimate. For the second term we use that

$$\begin{aligned}
&\int\limits_{a_\varepsilon\partial G_0} \varphi_\varepsilon(x+\varepsilon j)\big(\partial_\nu w_{\sigma,\varepsilon}(x;s) - \partial_\nu \widehat{w}_{\sigma,\varepsilon}(x;s)\big)\,\mathrm{d}S_x \\
&\quad = C_0 \int\limits_{a_\varepsilon\partial G_0} \varphi_\varepsilon(x+\varepsilon j)\big(\sigma(s - w_{\sigma,\varepsilon}(x;s)) - \sigma(s - \widehat{w}_{\sigma,\varepsilon}(x;s))\big)\,\mathrm{d}S_x.
\end{aligned}$$

Now, we can use the regularity of σ and the estimate in $L^2(a_\varepsilon\partial G_0)$ for the difference. It seems that this in the only point in the proof where the regularity of σ is applied, since all relevant estimates on Lemma 3.23 are independent of σ. We point the reader to [116] for the full details and finish the section with some comments.

When g^ε is as in (1.2) and σ depends on x, the authors in [269] use the test functions mentioned in Remark 3.15 to recover similar results. Likewise, the case $g^\varepsilon(x) = g(x)$ can be treated where in (4.32) we recover $\mathcal{H}$ depending also on x.

Remark 4.46. The study of the case when σ is a general monotone graph is possible although involved. It would require revising the estimates presented for $w_{\varepsilon,\sigma}$ and a sharp analysis of the term $I_{2,\varepsilon}$.

Remark 4.47. The study of this problem when $p \in (1, n)$ should follow similarly to the case $G_0 = B_1$ with the additions above. However, it has not been done yet due to the difficulty of obtaining estimates for $\widehat{w}_{\varepsilon,\sigma}$ equivalent to (3.38).

4.7.4 Case of G_0 not a ball, $1 < p \leq n$ and $g^\varepsilon \neq 0$

To be best of our knowledge, the results for this problem do not exist in the literature. However, we expect the effective homogenized equation to be

$$\begin{cases} -\Delta_p u + \mathcal{H}(x, u) = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases}$$

where, with the notations of Remark 3.17 (which allow g^ε to be of the general class (1.3)), we would have

$$\mathcal{H}(x, s) = \mathcal{A}_0 \widehat{H}_\sigma(x, s),$$

for some $\mathcal{A}_0$. As already mentioned in Remark 3.17, there are important severe technical difficulties to overcome due to the quasilinear behavior of the p-Laplacian (specially if $p < 2$), but we expect the program to be similar to the remaining cases.

4.8 The case $\beta \not\sim \beta^\star$

Now let us justify the different homogenized results in this framework mentioned in Table 1.1.

The case $a_\varepsilon \ll a_\varepsilon^\star$ is not relevant, since the reaction term never appears as proved in Theorem 4.28 for $g^\varepsilon = 0$, and the behavior is independent of $\beta(\varepsilon)$. We simply point out that if $\beta(\varepsilon) \lesssim \beta^\star(\varepsilon)$, the solution has the same limit as $\sigma = 0$.

For subcritical scales we already showed in Theorem 4.16 and Theorem 4.21 that when $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$, then the problem has the same limit as the homogeneous Neumann case (i. e., corresponding to $\sigma = 0$), and when $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ and $g^\varepsilon = 0$, we have $\sigma(u) = 0$ in the limit, i. e., the reaction term becomes dominant.

We can give some additional information on the case $a_\varepsilon^\star \ll a_\varepsilon \leq \varepsilon$, by recalling Section 2.4. Due to Remark 4.4, we can compare the solutions for two kinetics σ and $\bar{\sigma}$. The computation in Section 2.4 is rigorous in this case if $a_\varepsilon \gtrsim a_\varepsilon^\star$, and hence we have (2.10) if $p \geq 2$ and (2.11) if $1 < p \leq 2$. Hence, the case $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$ behaves like $\sigma = 0$, i. e., the homogeneous Neumann boundary condition on the particles.

The behavior of $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ and $a_\varepsilon \sim a_\varepsilon^\star$ is interesting. When G_0 is a ball, let us look at (4.17). We note that if $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$, then $\mathcal{B}_0$ becomes 0 (while the rest of the argument works) and we recover the equation

$$-\sigma\big(s - H(x,s)\big) = g(x).$$

For σ strictly increasing

$$H(s) = s - \sigma^{-1}\big(-g(x)\big).$$

If $g(x) = 0$ simply $H(s) = s$. This is the same behavior as when $\sigma = \sigma_D$ corresponding the Dirichlet boundary (given by (2.2)). We have that $\mathcal{H}$ is, as for the Dirichlet boundary condition,

$$\mathcal{H}(s) = \mathcal{A}_0|s|^{p-2}s.$$

If, on the other hand, we look at the auxiliary problem (3.25) (corresponding to $g^\varepsilon = 0$), with C_0 given by (3.26), so it would be $C_0 = +\infty$. Changing it to the left-hand side, the auxiliary problem becomes

$$\begin{cases} \Delta\widehat{w}_\sigma = 0 & \mathbb{R}^n \setminus \overline{G_0}, \\ 0 = \sigma(s - \widehat{w}_\sigma) & \partial G_0, \\ \widehat{w}_\sigma \to 0 & \text{as } |y| \to +\infty. \end{cases}$$

In other words, if σ is strictly increasing,

$$\begin{cases} \Delta\widehat{w}_\sigma = 0 & \mathbb{R}^n \setminus \overline{G_0}, \\ \widehat{w}_\sigma = s & \partial G_0, \\ \widehat{w}_\sigma \to 0 & \text{as } |y| \to +\infty, \end{cases}$$

so $\widehat{w}_\sigma(s,y) = s\widehat{\kappa}(y)$ and hence

$$H(s) = s\lambda_{G_0}.$$

Again, we have recovered the same reaction homogenized term as the one associated to the case of Dirichlet boundary conditions.

Although for different problems, the similar cases in Tables 1.2 and 1.3 are recovered by similar procedures.

In the previous literature for $p < n$, many authors considered the special choices

$$a_\varepsilon = C_0\varepsilon^\alpha, \quad \beta(\varepsilon) = \varepsilon^{-\gamma}. \tag{4.36}$$

Under this framework, the different resulting homogenized equations can be classified according to the different cases indicated in Figure 4.1

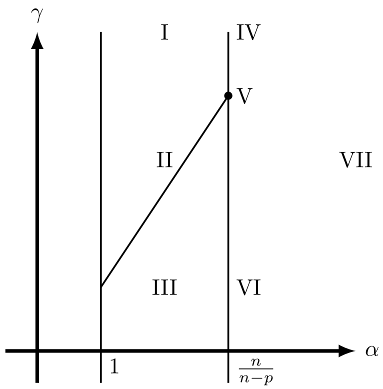

Figure 4.1: Representation of Table 1.1 with the choices (4.36). Note that $\alpha = 1$ corresponds to big particles, whereas $\alpha = \frac{n}{n-p}$ corresponds to $a_\varepsilon^\star$. Regions I and IV correspond to $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$, II and V to $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$ and III and VI to $\beta(\varepsilon) \ll \beta^\star(\varepsilon)$. Region VII is the supercritical case of small particles. The line for $\alpha = 1$ is also split in three regions by line II, but we do not number them for the sake of simplicity. Note also that for the critical case, $\alpha = \frac{n}{n-p}$, there are three different cases of homogenized equations according to the values of $\beta(\varepsilon) = \varepsilon^{-\gamma}$, as in Table 1.1.

4.9 Further comments

4.9.1 L^1 data

In many applications the data f and g^ε are less regular than usual and it is impossible to get the solution u_ε of the problem (1.1) in the energy space $W^{1,p}(\Omega^\varepsilon, \partial\Omega)$. Here we only will consider the case of particles of critical scale over the whole domain and $p = 2$, but the results can be extended to many other formulations. So, in this subsection, we consider the problem

$$\begin{cases} -\Delta u_\varepsilon = f(x) & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon) g^\varepsilon(x) & x \in S_\varepsilon, \\ u_\varepsilon = 0 & \partial\Omega, \end{cases} \tag{4.37}$$

under the assumptions

$$\sigma \text{ is a maximal monotone graph of } \mathbb{R}^2,\ 0 \in \sigma(0), \tag{4.38}$$

$$f \in L^1(\Omega), \tag{4.39}$$

$$g^\epsilon \in L^1(S_\varepsilon). \tag{4.40}$$

We start by considering the basic theory of the existence and uniqueness of a weak solution as a first step to the homogenization process. We will need now some further results beyond the exposition made in Chapter 2 for the usual case considered in the book in which $f \in L^2(\Omega)$ and $g^\epsilon \in L^2(S_\varepsilon)$.

One possibility to carry out the homogenization process is in the framework of the so-called *renormalized solutions*; see [38, 208, 53, 147] and, especially, [127], in which the case of big particles (with a more complex linear diffusion operator) was considered under the assumption that σ is a continuous increasing function such that $\sigma(0) = 0$. Nevertheless, in this section we will follow a different approach in order to obtain the convergence of the direct sequence of solutions u_ε and not only on the sequence of its truncations. It is well known that, at least under some additional assumptions, there is some equivalence between renormalized solutions, entropy solutions (see [27, 247]) and L^1-weak solutions ([131]), but here we will make a direct presentation for weak solutions.

Following the pioneering and fundamental work [52] and similar to Chapter 2, given a maximal monotone graph of $\mathbb{R}^2$, σ, we define the following notion of solution.

Definition 4.48. We say that $u_\varepsilon \in W^{1,1}(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of problem (4.37) if there exists $b_\varepsilon \in L^1(S_\varepsilon)$, with $b_\varepsilon(x) \in \sigma(u_\varepsilon(x))$ on S_ε such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\varphi \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} b_\varepsilon \varphi \,\mathrm{d}S = \int_{\Omega_\varepsilon} f\varphi \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon \varphi \,\mathrm{d}S, \quad \forall\varphi \in W^{1,\infty}(\Omega_\varepsilon, \partial\Omega). \quad (4.41)$$

Since we have *mixed* type boundary conditions in problem (4.37) on $\partial\Omega_\varepsilon$ (of Dirichlet type on $\partial\Omega$ and of Robin type on S_ε), the corresponding result given in [48, Theorem 22] does not apply directly to our framework. In fact, in this paper it is assumed that the diffusion equation contains a positive absorption term (i. e., the equation is of the type $-\Delta u_\varepsilon + a u_\varepsilon = f(x)$, with $a > 0$). This is crucial in some parts of their arguments. Nevertheless, instead of this fact, we will use here the property that there is a part of the boundary of Ω_ε in which we have a Dirichlet condition (since otherwise it must be assumed some additional conditions on f and g^ε; see [28]).

We point out that the paper [52] was the object of many generalizations and that some of them are valid also for the quasilinear case $p \neq 2$ (see, e. g., [26, 188]). For homogenization purposes we will need some uniform estimates which are given in the following result and which look new in the literature.

Theorem 4.49. *Assume* (4.38) *on σ and let f and g^ε satisfy* (4.39) *and* (4.40), *respectively. Then, there exists a unique weak solution $u_\varepsilon \in W^{1,q}(\Omega_\varepsilon, \partial\Omega)$, with $1 \le q < \frac{n}{n-1}$, of problem* (4.37). *Moreover, if $\widehat{u}_\varepsilon \in W^{1,q}(\Omega_\varepsilon, \partial\Omega)$ is a solution* (4.37) *corresponding solution to different data $\widehat{f}$ and $\widehat{g}^\varepsilon$ that satisfy* (4.39) *and* (4.40), *and $\widehat{b}_\varepsilon \in L^1(S_\varepsilon)$ with $\widehat{b}_\varepsilon(x) \in \sigma(\widehat{u}_\varepsilon(x))$ on S_ε, we have*

$$\beta(\varepsilon)\|b_\varepsilon - \widehat{b}_\varepsilon\|_{L^1(S_\varepsilon)} \le \|f - \widehat{f}\|_{L^1(\Omega_\varepsilon)} + \beta(\varepsilon)\|g_\varepsilon - \widehat{g}_\varepsilon\|_{L^1(S_\varepsilon)} \quad (4.42)$$

and for any $1 \le q < \frac{n}{n-1}$, there exists a constant C (depending only on Ω and q) such that

$$\|\nabla(u_\varepsilon - \widehat{u}_\varepsilon)\|_{L^q(\Omega_\varepsilon)} \le C\big(\|f - \widehat{f}\|_{L^1(\Omega_\varepsilon)} + \beta(\varepsilon)\|g_\varepsilon - \widehat{g}_\varepsilon\|_{L^1(S_\varepsilon)}\big). \quad (4.43)$$

Since we will adapt to our setting some of the results of [52] and its generalization made in [28] we will give only a sketch of the proof, except when proving that the constant $C > 0$, appearing in (4.43), is independent of ε. The proof is a consequence of several ingredients. A basic fact is the L^1-treatment of the linear Laplace problem with mixed boundary conditions

$$\begin{cases} -\Delta v = f & x \in \Omega_\varepsilon, \\ \partial_\nu v = g & x \in S_\varepsilon, \\ v = 0 & \partial\Omega, \end{cases} \tag{4.44}$$

for $f \in L^1(\Omega_\varepsilon)$ and $g \in L^1(S_\varepsilon)$. We say that $v \in W^{1,1}(\Omega_\varepsilon, \partial\Omega)$ is an L^1-weak solution of (4.44) if

$$\int_{\Omega_\varepsilon} \nabla v \nabla \varphi \, \mathrm{d}x = \int_{\Omega_\varepsilon} f\varphi \, \mathrm{d}x + \int_{S_\varepsilon} g\varphi \, \mathrm{d}S, \quad \forall \varphi \in W^{1,\infty}(\Omega_\varepsilon, \partial\Omega). \tag{4.45}$$

When f, g are smooth, existence is classical as presented in Chapter 2. Existence and uniqueness with L^1 are a consequence of the following continuous dependence estimate.

Lemma 4.50. *Assume $f \in L^1(\Omega)$ and $g \in L^1(S_\varepsilon)$ and let v be a weak solution of* (4.44). *Then, for any $1 \le q < \frac{n}{n-1}$, there exists a constant C (depending only on Ω and q) such that*

$$\|v\|_{W^{1,q}(\Omega_\varepsilon)} \le C\big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(S_\varepsilon)}\big). \tag{4.46}$$

Proof. We study the cases $q \in (1, \frac{n}{n-1})$. The case $q = 1$ follows any of these cases by applying Hölder's inequality

$$\|\nabla v\|_{L^1(\Omega_\varepsilon)} \le |\Omega_\varepsilon|^{\frac{q-1}{q}} \|\nabla v\|_{L^q(\Omega_\varepsilon)} \le |\Omega|^{\frac{q-1}{q}} \|\nabla v\|_{L^q(\Omega_\varepsilon)}.$$

Let $r = q' = \frac{q}{q-1} \in (n, \infty)$. We will prove that, for $h_0, \dots, h_n \in L^r(\Omega_\varepsilon)$, we have

$$\int_{\Omega_\varepsilon} \left(h_0 v + \sum_{i=1}^n h_i \frac{\partial v}{\partial x_i} \right) \mathrm{d}x \le C\big(\|f\|_{L^1(\Omega_\varepsilon)} + \|g\|_{L^1(S_\varepsilon)}\big) \sum_{i=0}^n \|h_i\|_{L^r(\Omega_\varepsilon)}, \tag{4.47}$$

for all $r > n$, with $C > 0$ depending only on Ω and q. We can use this estimate; we take $h_0 = |v|^{q-2} v$, $h_i = |\nabla v|^{q-2} \frac{\partial v}{\partial x_i}$ and we recover

$$\begin{aligned} &\int_{\Omega_\varepsilon} (|v|^q + |\nabla v|^q) \, \mathrm{d}x \\ &\quad \le C\big(\|f\|_{L^1(\Omega_\varepsilon)} + \|g\|_{L^1(S_\varepsilon)}\big) \left(\left(\int_{\Omega_\varepsilon} |v|^q \, \mathrm{d}x \right)^{\frac{q-1}{q}} + \sum_{i=1}^n \left(\int_{\Omega_\varepsilon} |\nabla v|^{\frac{q-2}{q-1} q} \left| \frac{\partial v}{\partial x_i} \right|^{\frac{q}{q-1}} \mathrm{d}x \right)^{\frac{q-1}{q}} \right) \\ &\quad \le C\big(\|f\|_{L^1(\Omega_\varepsilon)} + \|g\|_{L^1(S_\varepsilon)}\big) \left(\left(\int_{\Omega_\varepsilon} |v|^q \, \mathrm{d}x \right)^{\frac{q-1}{q}} + \left(\int_{\Omega_\varepsilon} |\nabla v|^q \, \mathrm{d}x \right)^{\frac{q-1}{q}} \right). \end{aligned}$$

Hence, by simple manipulation (4.47) implies (4.46) without adding any dependence on ε.

To prove (4.47) we will construct some auxiliary functions. By density, it suffices to prove (4.47) for $h_0, \dots, h_n \in C_c^\infty(\Omega_\varepsilon)$. Fix one of these $(n+1)$-tuples and define $h = h_0 - \sum_{i=1}^n \frac{\partial h_i}{\partial x_i}$. Let φ_ε be the solution of the associate linear problem

$$\begin{cases} -\Delta \varphi_\varepsilon = h(x) & x \in \Omega_\varepsilon, \\ \partial_\nu \varphi_\varepsilon = 0 & x \in S_\varepsilon, \\ \varphi_\varepsilon = 0 & \partial\Omega. \end{cases}$$

Following the proof of [164, Lemma 7.3] we can deduce the estimate

$$\|\varphi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le C(\Omega_\varepsilon) \sum_{i=0}^n \|h_i\|_{L^r(\Omega_\varepsilon)}. \tag{4.48}$$

The argument consists of taking $\varphi(x) = \max\{v(x) - k, 0\}$, for $k \ge 0$, as test function and iterating the inequality obtained through the weak formulation. A careful revision of the proof of [164, Lemma 7.3] allows to see that

$$C(\Omega_\varepsilon) = \widehat{C}_n \Psi_n(|\Omega_\varepsilon|),$$

where $\widehat{C}_n$ is a universal constant (only depending on r and n) and $\Psi_n(s)$ is a strictly increasing function of s (depending also on r and n). Indeed, the estimate of the proof of Lemma 7.3 of [164] only depends on the Sobolev inequality constant and the Lebesgue measure of the domain. According to [252] we know that such a constant depends increasingly on the measure of the set where it is applied. Thus, in our special case we have

$$C(\Omega_\varepsilon) \le C = \widehat{C}_n \Psi_n(|\Omega|). \tag{4.49}$$

Let us use this estimate to prove (4.47). Using φ_ε as a test function in the equation of v,

$$\int_{\Omega_\varepsilon} \nabla v \nabla \varphi_\varepsilon \, \mathrm{d}x = \int_{\Omega_\varepsilon} f \varphi_\varepsilon \, \mathrm{d}x + \int_{S_\varepsilon} g \varphi_\varepsilon \, \mathrm{d}S.$$

On the other hand, using v as a test function in the problem for φ_ε we have

$$\int_{\Omega_\varepsilon} \nabla v \nabla \varphi_\varepsilon \, \mathrm{d}x = \int_{\Omega_\varepsilon} v \left(h_0 - \sum_{i=1}^n \frac{\partial h_i}{\partial x_i} \right) \mathrm{d}x = \int_{\Omega_\varepsilon} h_0 v \, \mathrm{d}x + \sum_{i=1}^n \int_{\Omega_\varepsilon} h_i \frac{\partial v}{\partial x_i} \, \mathrm{d}x,$$

since $h_0, \dots, h_n$ are compactly supported. Joining the last two equations,

$$\int_{\Omega_\varepsilon} h_0 v + \sum_{i=1}^n \int_{\Omega_\varepsilon} h_i \frac{\partial v}{\partial x_i} \, \mathrm{d}x = \int_{\Omega_\varepsilon} f \varphi_\varepsilon \, \mathrm{d}x + \int_{S_\varepsilon} g \varphi_\varepsilon \, \mathrm{d}S.$$

Using (4.48) we deduce (4.47) and hence obtain the result. □

Remark 4.51. Let τv denote the trace of $v \in W^{1,1}(\Omega_\varepsilon, \partial\Omega)$ in $L^1(S_\varepsilon)$. Let M be the operator in $L^1(\Omega_\varepsilon) \times L^1(S_\varepsilon)$ with the graph

$$\{[(v,w),(f,g)] : (v,f,g) \in W^{1,1}(\Omega_\varepsilon,\partial\Omega) \times L^1(\Omega_\varepsilon) \times L^1(S_\varepsilon),$$
$$w = \tau v \text{ and } v \text{ is solution of (4.44)}\}.$$

Then, M is a linear, single-valued and densely defined operator in $L^1(\Omega_\varepsilon) \times L^1(S_\varepsilon)$. This is proved by an argument which consists in regularizing (f,g) and passing to the limit (see [28]). In fact, the above arguments prove that operator M is m-accretive in $L^1(\Omega) \times L^1(S_\varepsilon)$ (i. e., $(I+\lambda M)^{-1}$ is an everywhere defined non-expansive self-mapping of $L^1(\Omega) \times L^1(S_\varepsilon)$ for $\lambda > 0$).

Remark 4.52. If P is the projection of $L^1(\Omega_\varepsilon) \times L^1(S_\varepsilon)$ onto $L^1(\Omega_\varepsilon)$ the estimate (4.46) proves that

$$P(I+M)^{-1} : L^1(\Omega_\varepsilon) \times L^1(S_\varepsilon) \to W^{1,q}(\Omega_\varepsilon, \partial\Omega),$$

boundedly for $1 \le q < \frac{n}{n-1}$. Thus M has a compact resolvent. In particular $(f,g) \in R(M)$ if and only if (f,g) is orthogonal to the null space of the adjoint M^* of M. Note that if (u,z) is in the null space of the adjoint M^*, we have $z = \tau u$ and

$$\begin{cases} -\Delta u = 0 & x \in \Omega_\varepsilon, \\ \partial_\nu u = 0 & x \in S_\varepsilon, \\ u = 0 & \partial\Omega. \end{cases} \tag{4.50}$$

Then, by the Friedrich inequality we get that $u = 0$ and then $z = 0$. Note that for pure Neumann boundary conditions this null space is much bigger (it contains to $(a,0)$ for any constant a) and this is the reason to ask for some supplementary conditions to the data (f,g) in that case (see [28]).

Proof of Theorem 4.49. The strategy consists in several steps.
(i) We approximate f and g^ε by a sequence f_λ and g^ε_λ of bounded data $f_\lambda \in L^\infty(\Omega)$ and $g^\varepsilon_\lambda \in L^\infty(S_\varepsilon)$ converging to f in $L^1(\Omega)$ and to $g^\varepsilon \in L^1(S_\varepsilon)$, respectively, and we solve the corresponding problem (4.37) with σ replaced by its Yosida approximation $\sigma_\lambda = (I - (I+\lambda\sigma)^{-1})/\lambda$, or even better, we solve the approximate problem

$$\begin{cases} -\Delta u = f_\lambda(x) & x \in \Omega_\varepsilon, \\ \partial_\nu u + \beta(\varepsilon)\sigma_\lambda(u) = \beta(\varepsilon) g^\varepsilon_\lambda(x) & x \in S_\varepsilon, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.51}$$

with $\lambda > 0$. Thus we obtain a unique solution $u_{\varepsilon,\lambda} \in H^1(\Omega_\varepsilon, \partial\Omega) \cap L^\infty(\Omega)$ as indicated in Chapter 2.
Step (ii) consists in proving that we can pass to the limit in L^1 as $\lambda \downarrow 0$. This is an easy variation of Theorem B' of [28]. It was shown there that as $\lambda \downarrow 0$, the sequences $\{u_{\varepsilon,\lambda}\}$

and $\{\sigma_\lambda(u_{\varepsilon,\lambda})\}$ converge to limits $\{u_\varepsilon\}$ and $\{b_\varepsilon\}$ in $L^1(\Omega_\varepsilon)$ and $L^1(S_\varepsilon)$, respectively, with $b_\varepsilon(x) \in \sigma(u_\varepsilon(x))$ on S_ε, and that u_ε is a solution of (4.37).

Estimate (4.42) was essentially proved in [28, Proposition E]. A different alternative proof consists in substituting the expressions as weak solutions for $u_{\varepsilon,\lambda}$ and $\widehat{u}_{\varepsilon,\lambda}$ and taking $\varphi = \phi(u_{\varepsilon,\lambda} - \widehat{u}_{\varepsilon,\lambda})$ as test function, where $\phi \in C^1(\mathbb{R})$, ϕ non-decreasing, $\phi(0) = 0$. Then

$$\begin{aligned}\int_{\Omega_\varepsilon} \phi'(u_{\varepsilon,\lambda} - \widehat{u}_{\varepsilon,\lambda})\left|\nabla(u_{\varepsilon,\lambda} - \widehat{u}_{\varepsilon,\lambda})\right|^2 \mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon}(\sigma_\lambda(u_{\varepsilon,\lambda}) - \sigma_\lambda(\widehat{u}_{\varepsilon,\lambda}))\phi(u_{\varepsilon,\lambda} - \widehat{u}_{\varepsilon,\lambda})\,\mathrm{d}S \\ \leq \int_{\Omega_\varepsilon}(f_\lambda - \widehat{f}_\lambda)\phi(u_{\varepsilon,\lambda} - \widehat{u}_{\varepsilon,\lambda})\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon}(g_\lambda^\varepsilon - \widehat{g}_\lambda^\varepsilon)\phi(u_{\varepsilon,\lambda} - \widehat{u}_{\varepsilon,\lambda})\,\mathrm{d}S.\end{aligned}$$

The first term is positive, so it can be dropped. As ϕ approximates the sign function, we recover

$$\beta(\varepsilon)\int_{S_\varepsilon}\left|\sigma_\lambda(u_{\varepsilon,\lambda}) - \sigma_\lambda(\widehat{u}_{\varepsilon,\lambda})\right|\mathrm{d}S \leq \int_{\Omega_\varepsilon}|f_\lambda - \widehat{f}_\lambda|\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon}\left|g_\lambda^\varepsilon - \widehat{g}_\lambda^\varepsilon\right|\mathrm{d}S.$$

Here we have used that σ_λ is non-decreasing and $\sigma(0) = 0$. Then, as $\lambda \searrow 0$, we get the estimate (4.42). In order to prove (4.43) (which was absent in [28]) we point out that $v_\varepsilon = u_\varepsilon - \widehat{u}_\varepsilon$ satisfies

$$\begin{cases} -\Delta v_\varepsilon = f - \widehat{f} & x \in \Omega_\varepsilon, \\ \partial_\nu v_\varepsilon = \beta(\varepsilon)[(g^\varepsilon - \widehat{g}^\varepsilon) - (b_\varepsilon - \widehat{b}_\varepsilon)] & x \in S_\varepsilon, \\ w_\varepsilon = 0 & \partial\Omega. \end{cases} \tag{4.52}$$

Then, it suffices to apply estimate (4.46), the triangular inequality, and then estimate (4.42) to arrive at (4.43) with the constant $(C+1)$, C given by (4.49) and thus independent of ε. □

Now we are in conditions to state and prove a convergence result, as $\varepsilon \to 0$. Although many different possibilities are allowed, here we will consider only the case of balls at the critical space making arise a strange term, as an alternative to Theorem 4.36.

Theorem 4.53. *Let $a_\varepsilon \sim a_\varepsilon^*$, G_0 a ball, $\beta(\varepsilon) \sim \beta^*(\varepsilon)$, $f \in L^1(\Omega)$, $g^\varepsilon \in W^{1,\infty}(\Omega)$ and let σ be a maximal monotone graph such that $0 \in \sigma(0)$. Let $u_\varepsilon \in W^{1,q}(\Omega_\varepsilon, \partial\Omega)$, with $1 \leq q < \frac{n}{n-1}$, be the unique weak solution of problem (4.37). Then*

$$P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,q}(\Omega), \quad \text{with } 1 \leq q < \frac{n}{n-1}, \tag{4.53}$$

as $\varepsilon \to 0$, where u is the unique weak solution of the homogenized problem

$$\begin{cases} -\Delta u + \mathcal{H}(x,u) = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.54}$$

with $\mathcal{H}(x,u)$ given by (4.16) and (4.17).

Proof. By Theorem 4.49 and the existence of the extension operators presented in Chapter 2 we have

$$P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,q}(\Omega),$$

as $\varepsilon \to 0$, for some $u \in W_0^{1,q}(\Omega)$, for any $1 \leq q < \frac{n}{n-1}$. Moreover, as in the proof of Theorem 4.49, we approximate f by a sequence of functions $f_k \in L^\infty(\Omega)$ such that

$$f_k \to f \quad \text{in } L^1(\Omega),$$

as $k \to +\infty$. We construct the auxiliary solutions $u_{\varepsilon,k}$ of problem (4.37) with (f_k, g) as external data. Due to Theorem 4.36, we have

$$P_\varepsilon u_{\varepsilon,k} \rightharpoonup u_k \quad \text{in } H_0^1(\Omega),$$

as $\varepsilon \to 0$, where u_k is the unique solution of (4.54) with f_k as external datum. Then, by the estimate (4.43) we get

$$\|\nabla(u_{\varepsilon,k} - u_\varepsilon)\|_{L^q(\Omega_\varepsilon)} \leq C\|f_k - f\|_{L^1(\Omega)},$$

for all $k \in \mathbb{N}$, $\varepsilon > 0$ and $1 \leq q < \frac{n}{n-1}$. This estimate is preserved in the limit due to weak convergence and, hence, $u_k \to u$ in $W_0^{1,q}(\Omega)$, and thus $\mathcal{H}(x, u_k) \to \mathcal{H}(x, u)$ in $L^1(\Omega)$, as $k \to +\infty$, since H is uniformly Lipschitz. From the equivalent very weak formulation (see, e. g., [124]) of problem (4.54) with f_k as external datum, we have

$$\int_\Omega (-u_k \Delta\varphi + \mathcal{H}(x, u_k)\varphi)\,\mathrm{d}x = \int_\Omega f_k \varphi\,\mathrm{d}x \quad \forall \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega).$$

Passing to the limit, we deduce that

$$\int_\Omega (-u\Delta\varphi + \mathcal{H}(x, u)\varphi)\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x \quad \forall \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega),$$

and hence u is a very weak solution of problem (4.54) with f as external datum. Since $u \in W_0^{1,q}(\Omega)$, u is also a weak solution of (4.54). □

Remark 4.54. Many generalizations of the above theorem seem possible but they will be the object of a separated work.

4.9.2 Additional properties of the strange term

It is useful to have some uniform approximation of function H being characterized in terms of the primitive maximal monotone graph σ (in the spirit of Lemma 2.17 it can always be applied since H is also a maximal monotone graph). For a maximal monotone graph σ we will denote

$$\sigma^0(r) = s_r, \text{ such that } s_r \in \sigma(r) \text{ and } \|s_r\| = \inf_{s\in\sigma(r)} \|s\|.$$

This element s_r is unique since $\sigma(r)$ is a closed convex set (see [48]).

Proposition 4.55. *Let G_0 be a ball and let $g \in W^{1,\infty}(\Omega)$. Let σ_m and σ be maximal monotone graphs such that $D(\sigma^{-1}) \subset D(\sigma_m^{-1}) \subset \overline{D(\sigma^{-1})}$ for any $m = 0, 1, 2, \ldots$ Assume that*

$$\textit{for any } r \in D(\sigma^{-1}) \textit{ the exists } s_m \in \sigma_m^{-1}(r) \textit{ such that } s_m \to (\sigma^{-1})^0(r). \tag{4.55}$$

Then, for any $r \in \overline{D(\sigma^{-1})}$ and for any fixed $x \in \overline{\Omega}$, the sequence $H_m(x, r)$ (associated to σ_m) given by (4.17) *converges uniformly to $H(x, r)$ (associated to σ) as $m \to \infty$.*

We shall need the following auxiliary result.

Lemma 4.56 (Proposition 2.8 of [48]). *Let A_m and A be maximal monotone operators on a Hilbert space X such that $D(A) \subset D(A_m) \subset \overline{D(A)}$ for any $m = 0, 1, 2, \ldots$. Assume that*

$$\textit{for any } r \in D(A)\ \exists s_m \in A_m(r) \textit{ such that } s_m \to A^0(r). \tag{4.56}$$

Then, for any $r \in \overline{D(A)}$ the function $(I + cA_m)^{-1}(r) \to (I + cA_m)^{-1}(r)$ uniformly for any c in a bounded set of $[0, +\infty)$.

Proof. From formula (4.17) we have

$$H(x, s) + \sigma^{-1}\big(\Theta_{n,p}(H(x, s)) + g(x)\big) \ni s,$$

and thus we get the characterization

$$H(x, \cdot) = (I + \sigma^{-1}(\Theta_{n,p}(\cdot) + g(x))^{-1}, \text{ where } \Theta_{n,p}(r) = \mathcal{B}_0|r|^{p-2}r. \tag{4.57}$$

The result will be a direct application of the above lemma once we prove that, for any fixed $x \in \overline{\Omega}$ (note that g is continuous on Ω and can be extended to $\overline{\Omega}$), the operator $r \mapsto \sigma^{-1}(\Theta_{n,p}(r) + g(x))$ is a maximal monotone operator A_x (depending of the parameter $x \in \overline{\Omega}$) and that the approximation of σ_m implies a good approximation of A_x. Obviously, for any fixed $x \in \overline{\Omega}$, the operator

$$r \mapsto \Phi(x, r) = \Theta_{n,p}(r) + g(x) \tag{4.58}$$

is also a maximal monotone operator in $\mathbb{R}$ and its inverse is given by

$$\Phi^{-1}(x, s) = \left|\frac{s - g(x)}{\mathcal{B}_0}\right|^{\frac{1}{(p-1)}} \operatorname{sign}\big(s - g(x)\big).$$

Then we define the operator $A_x = (\Phi^{-1}(x,.)\circ\sigma)^{-1} = \sigma^{-1}\circ\Phi(x,.)$ (with $D(A_x) = R(\Phi^{-1}(x,.)\circ\sigma)$), which is the inverse of the maximal monotone operator

$$A_x^{-1} = \Phi^{-1}(x,.) \circ \sigma, \text{ with } D(A_x^{-1}) = D(\sigma).$$

Moreover, since $\Phi^{-1}(x,.)$ is single-valued, the principal section of A_x is given by

$$A_x^0 = (\sigma^{-1})^0 \circ \Phi(x,.)$$

(recall that the value of the principal section of a maximal operator $A(s)$, for any $s \in D(A)$, is given by the element of minimum norm in the closed and convex set $A(s)$; see [48]). Finally, if $r \in D(A_x)$, defining $A_{x,m} = \sigma_m^{-1} \circ \Phi(x,.)$, with σ_m satisfying (4.55), then for any $r \in D(A_x)$ we have that the element $\widehat{s}_{x,m} = \Phi(x, s_m)$ satisfies $\widehat{s}_{x,m} \in A_{x,m}(r)$ and $\widehat{s}_{m,x} \to A_x^0(s)$, so that the assumptions of the above lemma are fulfilled and we get, for any fixed $x \in \overline{\Omega}$,

$$H_m(x,r) = (I + \sigma_m^{-1} \circ \Phi(x,.))^{-1}(r) \to H(x,r) = (I + \sigma^{-1} \circ \Phi(x,.))^{-1}(r), \text{ uniformly.} \quad \square$$

An important consequence of the above result arises when we approximate both maximal monotone graphs σ and σ^{-1} by their (Lipschitz) Yosida approximation. We recall that if $\sigma = \partial\psi$, the Yosida approximation associated to the convex function ψ is given by

$$\sigma_m = \partial\psi_m, \text{ where } \psi_m(s) = \min_{r\in\mathbb{R}} \left\{ \frac{1}{2m}|r-s|^2 + \psi(r) \right\}.$$

We know (see Propositions 2.6 and 2.11 of [48]) that $\psi_m \in W^{2,\infty}(\mathbb{R})$ and $|\psi_m'(s)| \uparrow |\sigma^0(s)|$ for any $s \in \mathbb{R}$. Moreover, it is also well known that

$$\sigma^{-1} = \partial\psi^*,$$

where ψ^* is the Fenchel–Moreau convex conjugate of ψ defined by

$$\psi^*(r) = \sup_{s\in\mathbb{R}}\{rs - \psi(s)\}$$

(see [48] page 41). In addition, it is known that at least if ψ is “weakly coercive” in the sense that

$$\forall r_n \text{ such that } \sup_n \psi(r_n) < +\infty \text{ then } \sup_n |r_n| < +\infty, \tag{4.59}$$

then the Yosida approximations of σ^{-1} are given by the conjugate $(\psi_m)^*$ of the Yosida approximations ψ_m of σ (see, e. g., [15]). As a matter of fact, since $(\psi)^{**} = \psi$ and $(\psi_m)^{**} = \psi_m$ we have that if ψ^* is “weakly coercive,” i. e.,

$$\forall r_n \text{ such that } \sup_n \psi^*(r_n) < +\infty \text{ then } \sup_n |r_n| < +\infty, \tag{4.60}$$

then the Yosida approximations of σ are given by the conjugates of the Yosida approximations of σ^{-1}. Then we arrive at the following result.

Corollary 4.57. *Let $\sigma = \partial\psi$ be a maximal monotone graph of $\mathbb{R}^2$ and let σ_m be the Yosida approximations of σ. Assume that either* (4.59) *or* (4.60) *holds. Let G_0 be a ball and $g \in W^{1,\infty}(\Omega)$. Define $H_m(x,s) = (I + [(\psi_m)^*]'(\Phi(x,.)))^{-1}(s)$ with $\Phi(x,.)$ given by* (4.58). *Then, for any $r \in R(\sigma)$ and for any fixed $x \in \overline{\Omega}$ we have that $H_m(x,r) \to H(x,r)$ uniformly as $m \to 0$.*

Remark 4.58. It is very easy to check that if σ is the maximal monotone graph associated to reactions of order $k \geq 0$, mentioned in Section 1.1, then at least one of the assumptions, (4.59) or (4.60), is satisfied. In fact it also holds for any maximal monotone graph which is given by a Hölder continuous non-decreasing function and for any graph with a bounded range (as, for instance, the Heaviside function). We also mention that this is not the case of the special maximal monotone graph associated to the Signorini boundary conditions, but there are other ad hoc methods to study the approximations of the conjugate convex functions in this case (see, e. g., the last chapter of the monograph [16]).

4.9.3 Homogenization of the effectiveness factor

A relevant parameter in chemical engineering is the so-called *effectiveness factor*, which indicates the fraction of a chemical reaction taking place. This is given by integrating the reaction term, in either the heterogeneous media or the homogenized setting

$$\frac{1}{|S_\varepsilon|}\int\limits_{S_\varepsilon} \sigma(u_\varepsilon)\,\mathrm{d}S, \qquad \frac{1}{|\Omega|}\int\limits_{\Omega} \sigma(u)\,\mathrm{d}x.$$

Since the main interest of the effective equation is to give approximate information of the real-life problem in the heterogeneous setting, it is relevant to ask whether

$$\frac{1}{|S_\varepsilon|}\int\limits_{S_\varepsilon} \sigma(u_\varepsilon)\,\mathrm{d}S \to \frac{1}{|\Omega|}\int\limits_{\Omega} \sigma(u)\,\mathrm{d}x \quad \text{as } \varepsilon \to 0. \tag{4.61}$$

As we know, we can only expect this kind of behavior in the subcritical regime. When σ is smooth, this result is just a form of our averaging lemmas. In more general settings, this result have also been proved. We refer the reader to [84, 110] for results in this direction.

In [119, 118] the authors study the effectiveness of the homogenized problem with respect to the shape of G_0. For $a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$ the reaction term is proportional to $|\partial G_0|\sigma(u)$ and this is the only effect of G_0 on the equation. The higher is $|\partial G_0|$, the faster is the reaction, and the lower is u, the lower is the effectiveness. When $a_\varepsilon \sim \varepsilon$ the effective diffusion appears. Amongst the convex shapes of G_0 of fixed volume, they found there are maximizers of the effectiveness.

4.9.4 Pointwise comparison of critical and non-critical problems

Since we do not have a natural definition of effectiveness in the critical case, the claim that this critical scale is “more effective” than the non-critical ones – a claim that is often made in the nanotechnology community – is difficult to test. However, we know that for the non-critical cases the effectiveness is increasing with the value of the respective solutions (since the function σ is assumed to be increasing). Thus, it is interesting to study whether we can find a pointwise comparison of the solutions corresponding to critical and non-critical homogenized problems.

Let us give a concise example, which is a modification of those presented in [109]. Assume we are placing particles over the whole domain of radius $a_\varepsilon = \varepsilon^\alpha$ and we would like to pick α such that the concentration is higher (or lower). Let us assume we are also able to pick the scaling coefficient $\beta(\varepsilon) = \varepsilon^\gamma$ with adequate γ (in this setting $\gamma = n - \alpha(n-1)$). For simplicity we assume that $p = 2$ and $G_0 = B_1$.

First, let us consider the case of $g^\varepsilon = 0$
If $\alpha \in (1, \frac{n}{n-2})$ we recover the homogenized PDE

$$-\Delta u_{\mathrm{nc}} + |\partial B_1| \sigma(u_{\mathrm{nc}}) = f.$$

Note that this is independent of α. If $\alpha = \frac{n}{n-2}$, then the homogenized equation becomes

$$-\Delta u_{\mathrm{c}} + (n-2)|\partial B_1| H(u_{\mathrm{c}}) = f,$$

where

$$(n-2)H(s) = \sigma\big(s - H(s)\big).$$

As we have already shown, $H(0) = 0$ and $H' \geq 0$. Both problems are completed with the boundary condition $u = 0$ on $\partial\Omega$ (although this could be generalized). Since we are dealing with chemical reactions we assume that the functions are non-negative $u_{\mathrm{nc}}, u_{\mathrm{c}} \geq 0$ (this is natural if they represent concentrations). Then, by applying that $0 \leq H(u_{\mathrm{c}}) \leq u_{\mathrm{c}}$ we have

$$\begin{aligned} -\Delta(u_{\mathrm{c}} - u_{\mathrm{nc}}) + |\partial B_1|\big(\sigma(u_{\mathrm{c}}) - \sigma(u_{\mathrm{nc}})\big) &= |\partial B_1|\big(\sigma(u_{\mathrm{c}}) - (n-2)H(u_{\mathrm{c}})\big) \\ &= |\partial B_1|\big(\sigma(u_{\mathrm{c}}) - \sigma\big(u_{\mathrm{c}} - H(u_{\mathrm{c}})\big)\big) \\ &\geq 0, \end{aligned}$$

due to the monotonicity of σ. Hence, we get

$$u_{\mathrm{c}} \geq u_{\mathrm{nc}} \quad \text{in } \Omega.$$

Non-homogeneous exterior Dirichlet boundary condition

In the setting of chemical engineering it is common to assume that on the "walls" of Ω we have a constant high concentration and one may pose the problem

$$\begin{cases} \Delta U_\varepsilon = f & \Omega_\varepsilon, \\ \partial_\nu U_\varepsilon + \beta(\varepsilon)\sigma(U_\varepsilon) = 0 & S_\varepsilon, \\ U_\varepsilon = 1 & \partial\Omega, \end{cases}$$

where again we assume $f \geq 0$. By the change in variable $u_\varepsilon = 1 - U_\varepsilon$ we get

$$\begin{cases} \Delta u_\varepsilon = -f & \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \beta(\varepsilon)(\sigma(1) - \sigma(1-u_\varepsilon)) = \beta(\varepsilon)\sigma(1) & S_\varepsilon, \\ u_\varepsilon = 0 & \partial\Omega. \end{cases}$$

Letting $\sigma(s) = \bar{\sigma}(1) - \bar{\sigma}(1-s)$ we recover a problem in our usual formulation in this book. Note that $\sigma(0) = \bar{\sigma}(0) = 0$ and $\sigma(1) = \bar{\sigma}(1)$. By the comparison principle $0 \leq u_\varepsilon \leq 1$, when $\alpha \in (1, \frac{n}{n-2})$ we recover the homogenized PDE

$$-\Delta u_{\mathrm{nc}} + |\partial B_1|\big(\sigma(1) - \sigma(1 - u_{\mathrm{nc}})\big) = -f + |\partial B_1|\sigma(1),$$

or, equivalently, for $U_{\mathrm{nc}} = 1 - u_{\mathrm{nc}}$,

$$-\Delta U_{\mathrm{nc}} + |\partial B_1|\sigma(U_{\mathrm{nc}}) = f.$$

If $\alpha = \frac{n}{n-2}$, then the homogenized equation is

$$-\Delta u_{\mathrm{c}} + (n-2)|\partial B_1|\overline{H}(u_{\mathrm{c}}) = -f,$$

where

$$(n-2)\overline{H}(s) = \big(\sigma(1) - \sigma\big(1 - \big(s - \overline{H}(s)\big)\big)\big) - \sigma(1) = -\sigma\big(1 - \big(s - \overline{H}(s)\big)\big).$$

Thus

$$-\Delta U_{\mathrm{c}} + (n-2)|\partial B_1|H(U_{\mathrm{c}}) = f,$$

where $(n-2)H = \sigma(s-H)$ is exactly the same characterization as for the case of a homogeneous Dirichlet boundary condition. It is easy to see, by contradiction, that $H \leq 0$ in $[0,1]$. Repeating the above argument, we arrive at the comparison

$$U_{\mathrm{c}} \geq U_{\mathrm{nc}} \quad \text{in } \Omega.$$

In the chemical engineering framework, this is read in terms of the concentration as the important conclusion that the critical case leads to a "better" reaction than the non-critical cases.

Remark 4.59. This kind of behavior has been shown in some experimental works in heterogeneous nanocatalysis; see references to the use of gold nanoparticles, for instance, in [244] and in the survey [220].

Remark 4.60. It was shown in [109] (see also [102]) that, in the case of chemical reactions of order less than one ($\sigma(s) = cs^k$ with $k \in (0,1)$) and big particles, the homogenized problem (in terms of U_{nc}) may generate a "dead core region" where $U_{\mathrm{nc}} = 0$. This is very negative from the point of view of the chemical reaction. In contrast to that, for this same kinetics (reaction of order $k < 1$) the use of critical-size particles leads to an improved result since it can be proved that for the equation with the strange term no dead core region can be formed (see [102]).

4.9.5 Homogenization and optimal control

Many different formulations involving an optimal control associated to problem (1.1) can be considered. For instance we can assume a distributed control $v(x)$

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) + v(x) & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(x, u_\varepsilon) = \beta(\varepsilon) g^\varepsilon(x) & x \in S_\varepsilon, \\ u_\varepsilon = 0 & \partial\Omega \end{cases} \tag{4.62}$$

and search for the minimization of the energy

$$J_\varepsilon(v) = \frac{1}{p}\|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \frac{\Lambda}{2}\|v\|^2_{L^2(\Omega_\varepsilon)},$$

for some weight constant $\Lambda > 0$. The assumption most often taken in the literature on this optimal control problem was $p = 2$. The existence of an optimal control, for any $\varepsilon > 0$, is a well-known result since a long time ago [191]. The associated homogenization problem was first considered in [181, 182, 237, 234] for Dirichlet boundary conditions on the boundary of the particles, for Neumann boundary conditions in [182] and for Signorini boundary conditions in [250]. The passing to the limit, as $\varepsilon \to 0$, in the case of a critical scale was analyzed in [234, 250]. Note that this requires some stronger convergence results since we must pass to the limit in the energies and not only in the own solutions u_ε.

The extension to nonlinear Robin conditions is the main goal of the paper [123] (where the case of particles on a manifold is also considered). A remarkable corollary in the framework of the applications in catalysis (in chemical engineering), when dealing with particles of critical size, is that in order to control the homogenized chemical reaction (for instance, trying to maximize the effectiveness when we write $w = 1 - u$, i. e., with $w = 1$ on the boundary of the chemical reactor Ω, as mentioned in Section 4.9.3) we can do that by controlling the microproblems (1.1) with a cost functional

J_ε which is independent of the own concentration w_ε and only depends on its gradient ∇v_ε. We point out that other connections between homogenization and control theory will be mentioned in Appendix A, in the framework of parabolic problems and the so-called controllability property for the homogenized equation.

4.9.6 Convergence of spectra

A very interesting problem which has received much attention in the literature concerns the *homogenization of linear and semilinear eigenvalue problems*. So, we assume $p = 2$ and for a given maximal graph $\sigma(x, u)$, we consider the eigenvalue problem

$$\begin{cases} -\Delta u_\varepsilon = \lambda^\varepsilon u_\varepsilon & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta(\varepsilon)\sigma(x, u_\varepsilon) \ni 0 & \text{for } x \in S_\varepsilon. \end{cases} \tag{4.63}$$

A special case which was intensively considered concerns the *linear non-autonomous eigenvalue problem* in which

$$\sigma(x, u) = a(x)u,$$

where $a \equiv a(x)$ is a strictly positive continuously differentiable function of the variable $x \in \overline{\Omega}$.

In the linear case, for each $\varepsilon > 0$, problem (4.63) is a standard spectral problem in the couple of spaces $H^1(\Omega_\varepsilon, \partial\Omega) \subset L^2(\Omega_\varepsilon)$, with a discrete spectrum. Let us denote

$$\lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_k^\varepsilon \leq \cdots < \infty,$$

the sequence of eigenvalues repeated according to their multiplicities, and let $\{u_k^\varepsilon\}_{k=1}^\infty$ be the set of associated eigenfunctions (which we know form an orthonormal basis in $L^2(\Omega_\varepsilon)$). Some pioneering convergence results were given in [199, 235, 261] (see also [221] including also a stochastic formulation). A very general result (containing an abstract formulation) from the spectral perturbation theory proving the convergence for eigenvalues and the corresponding eigenfunctions of spectral problems was given in Lemma 1.6, Chapter III, of [214]. Of course, the formulation of the final homogenized eigenvalue problem depends on $\beta(\varepsilon)$, a_ε and n, but, in general, it corresponds to a linear eigenvalue problem of the type

$$\begin{cases} -\operatorname{div}(A^{\text{eff}}\nabla u) + \beta^{\text{eff}}(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see, e. g., [236]). For some other spectral convergence results concerning particles on a manifold or on part of the boundary (problems which will be considered in Chapters 5 and 6), see [160, 155]. See also [219] for the non-periodic case, [216] for the case of the bi-Laplacian operator and [154] for the case of cylinders instead of particles.

When $\sigma(x,u)$ is nonlinear, problem (4.63) becomes a bifurcating problem and the existence of a family of solutions u_ε, for each $\varepsilon > 0$, requires different tools (see references in [176, 122]). Once again, the final homogenized eigenvalue problem depends on $\beta(\varepsilon)$, a_ε, n and $\sigma(x,u)$, but it can always be formulated in terms of a nonlinear eigenvalue problem of the type

$$\begin{cases} -\operatorname{div}(A^{\mathrm{eff}}\nabla u)+\mathcal{H}(x,u) \ni \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.64}$$

for some suitable maximal monotone graph $\mathcal{H}(x,u)$ (which in the critical case becomes a non-decreasing Lipschitz function). The convergence theorem for the special case of big particles and $\sigma(x,u) = |u|^{k-1}u$ with $k \in [0, n/(n-2))$ was given in [176]. It seems possible to extend such results to the case of particles of critical size, arriving then to a eigenvalue problem with a strange term $\mathcal{H}(x,u)$, different from $\sigma(x,u)$. This subject is object of study by some of the authors. For instance, it is well known (see, e. g., [122, 100, 121]) that if $\mathcal{H}(x,u) = c\sigma(x,u) = c|u|^{k-1}u$, with $k \in (0,1)$, the nodal solutions u_λ of (4.64) are unstable for any value of the parameter $\lambda > 0$ leading to the existence of solution. This is in contrast to the case corresponding to the critical scale since then $\mathcal{H}(x,u)$ becomes a Lipschitz continuous increasing function and the *linearization principle* ([121]) allows to show the existence of stable stationary solutions u_λ, for some $\lambda > 0$.

Finally, we mention that there is also active research on the homogenization convergence for the case in which the expression of the eigenvalue is placed in the boundary condition on S_ε (and not in the partial differential equation). This corresponds to the so-called *Steklov problem*; see, e. g., the works (on some linear problems) [224, 69], and the references therein.

5 Particles over an interior manifold

In this chapter we will prove how to get the homogenized problem for the case in which the particles are placed on an interior manifold of Ω. We will tackle the geometry presented in Section 1.2.1.2. We point out that in this setting

$$|S_\varepsilon| = |\Upsilon_\varepsilon||a_\varepsilon \partial G_0| \simeq \varepsilon^{1-n}|\Omega^0|a_\varepsilon^{n-1}|\partial G_0|,$$

where we recall, going back to (1.4), that $\Omega^0 = \{x_n = 0\} \cap \Omega$ and hence it is an $(n-1)$-dimensional manifold. In this setting the critical scale is given by

$$a_\varepsilon^\star = \begin{cases} \varepsilon^{\frac{n-1}{n-p}} & \text{if } p < n, \\ \varepsilon e^{-\frac{\alpha^2}{\varepsilon}} & \text{for any } \alpha > 0 \text{ if } p = n, \\ 0 & p > n. \end{cases}$$

Remark 5.1. When $p = n$ we point out that the notion $a_\varepsilon \sim a_\varepsilon^\star$ must be taken through ln as we mentioned in Remark 4.1, i. e.,

$$a_\varepsilon \sim a_\varepsilon^\star \iff \lim_{\varepsilon \to 0} \left(\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon} \right)^{-1} = C,$$

$$a_\varepsilon \ll a_\varepsilon^\star \iff \lim_{\varepsilon \to 0} \left(\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon} \right)^{-1} = 0,$$

and the corresponding criterion for $\gg$.

In order to obtain the homogenized equation, first, we need to understand the limit of the integrals over S_ε. For a very smooth function g we have

$$\begin{aligned} \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} g(x)\, \mathrm{d}S &= \frac{\varepsilon^{n-1} a_\varepsilon^{1-n}}{|\Omega^0||\partial G_0|} \sum_{j \in \Upsilon_\varepsilon} \int_{\varepsilon j + a_\varepsilon \partial G_0} g(x)\, \mathrm{d}S \\ &= \frac{\varepsilon^{n-1} a_\varepsilon^{1-n}}{|\Omega^0||\partial G_0|} \sum_{j \in \Upsilon_\varepsilon} \int_{a_\varepsilon G_0} g(x + \varepsilon j)\, \mathrm{d}S \\ &\simeq \frac{\varepsilon^{n-1}}{|\Omega^0|} \sum_{\substack{j = (j_1, \dots, j_{n-1}, 0) \\ \varepsilon j + \varepsilon \overline{Y}^+ \subset \Omega}} g(\varepsilon j) \\ &\to \frac{1}{|\Omega^0|} \int_{\Omega^0} g(x)\, \mathrm{d}S. \end{aligned}$$

Hence, similarly to the case of particles over the whole domain, we can expect to arrive at the weak formulation, for any given good test function v,

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla(v - u)\, \mathrm{d}x + \beta_1^{\mathrm{eff}} \int_{\Omega^0} \sigma^{\mathrm{eff}}(v)(v - u)\, \mathrm{d}S \ge \int_\Omega f(v - u)\, \mathrm{d}x + \beta_2^{\mathrm{eff}} \int_{\Omega^0} g(v - u)\, \mathrm{d}S.$$

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This formulation with an integral over an interior manifold is quite uncommon, although not completely surprising. It represents, in some sense, a variation (or discontinuity) of the flux. Before we consider the homogenization results, let us give a sense to this new term.

5.1 Jumps over an interface

In order to characterize the weak formulation associated with this new term in Ω^0 we look at what this means in pointwise terms. Let us assume that u is smooth in Ω^+ and Ω^- (defined in (1.4)) and that for some function $G : \Omega^0 \to \mathbb{R}$ we have (for any good test function v)

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla (v-u)\, \mathrm{d}x + \int_{\Omega^0} G(v-u)\, \mathrm{d}S \geq \int_{\Omega} f(v-u)\, \mathrm{d}x.$$

Using $v = u + \lambda\varphi$, where $\varphi \in C_c^\infty(\Omega)$, and letting $\lambda \to 0^\pm$ we have

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, \mathrm{d}x + \int_{\Omega^0} G\varphi\, \mathrm{d}S = \int_{\Omega} f\varphi\, \mathrm{d}x.$$

Applying the divergence theorem in Ω^+ we have

$$\int_{\Omega^+} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, \mathrm{d}x - \int_{\Omega^0} \varphi |\nabla u|^{p-2} \nabla u \cdot \nu_{\Omega^+}\, \mathrm{d}S = -\int_{\Omega^+} \varphi \Delta_p u\, \mathrm{d}x.$$

The exterior normal vector is $\nu_{\Omega^+} = -e_n$ on Ω^0, and we have

$$-\int_{\Omega^0} |\nabla u|^{p-2} \nabla u \cdot \nu_{\Omega^+} \varphi\, \mathrm{d}S = \int_{\Omega^0} |\nabla u|^{p-2} \frac{\partial u}{\partial x_n} \varphi\, \mathrm{d}S.$$

Since there will be a jump in the gradient, let us denote

$$|\nabla_+ u(x)|^{p-2} \frac{\partial u}{\partial x_n^+}(x) = \lim_{h \to 0^+} |\nabla u(x + he_n)|^{p-2} \frac{\partial u}{\partial x_n}(x + he_n).$$

Repeating the process in Ω^-, where now $\nu_{\Omega^-} = e_n$, we have

$$\int_{\Omega^-} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, \mathrm{d}x - \int_{\Omega^0} |\nabla_- u|^{p-2} \frac{\partial u}{\partial x_n^-} \varphi\, \mathrm{d}S = -\int_{\Omega^-} \varphi \Delta_p u\, \mathrm{d}x.$$

Joining the terms from Ω^+ and Ω^- we recover

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, \mathrm{d}x + \int_{\Omega^0} \left(|\nabla_+ u|^{p-2} \frac{\partial u}{\partial x_n^+} - |\nabla_- u|^{p-2} \frac{\partial u}{\partial x_n^-} \right) \varphi\, \mathrm{d}S = -\int_{\Omega} \varphi \Delta_p u\, \mathrm{d}x.$$

We introduce the notation for the jump of a general function v across Ω^0 by

$$[v]_{\Omega^0}(x) = \lim_{h\to 0^+}\big(v(x+he_n) - v(x-he_n)\big).$$

Matching the boundary and interior terms we deduce, since φ is arbitrary, that

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega^+ \cup \Omega^-, \\ [|\nabla u|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = G & \text{on } \Omega^0. \end{cases}$$

Since we look for solutions $u \in W^{1,p}(\Omega)$, in particular they are in $W^{1,q}(\Omega)$ for some $q \in (1,n)$ (see [139, Theorem 4.19]). Hence, there is a representative of u (i. e., a choice of the a. e. class) that is continuous except for a set ω of q-cap$(\omega) = 0$. In particular, $\mathcal{H}^s(\omega) = 0$ for $s > n-q$. Hence jump $[u]_{\Omega^0} = 0$ except in $\omega \cap \Omega^0$, and $\mathcal{H}^{n-1}(\omega\cap\Omega^0) = 0$. The exterior boundary condition on $\partial\Omega$ comes from the functional dependence $u \in W_0^{1,p}(\Omega)$.

5.2 Existence of a critical scale

As we did in the previous case, we can discover the critical scale through the scaling. In this setting, the perturbations are only included along the interior manifold. Let v be defined on $\mathbb{R}^n$ compactly supported, and take

$$v_\varepsilon(x) = \begin{cases} v(\frac{x-\varepsilon j}{a_\varepsilon}) & x \in \varepsilon j + \varepsilon Y \text{ for some } j \in \Upsilon_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Note that due to the construction of Υ_ε, we have that $v_\varepsilon(x) = 0$ if $|x_n| > \varepsilon$. If $a_\varepsilon \ll \varepsilon$, we have

$$\int_\Omega |\nabla v_\varepsilon(x)|^p \,\mathrm{d}x = |\Upsilon_\varepsilon| a_\varepsilon^{n-p} \int_{\frac{\varepsilon}{a_\varepsilon}Y} |\nabla v(y)|^p \,\mathrm{d}y \sim \varepsilon^{1-n} a_\varepsilon^{n-p} \int_{\frac{\varepsilon}{a_\varepsilon}Y} |\nabla v(y)|^p \,\mathrm{d}y. \tag{5.1}$$

Hence, this integral changes its limit behavior when $\varepsilon^{1-n}a_\varepsilon^{n-p} \sim 1$. The case $p = n$ is always special.

Unlike when particles are spread over the whole domain, in this setting we do not have that the scale of big particles, $a_\varepsilon \sim \varepsilon$, has any significant additional behavior.

5.3 Integrals over S_ε

5.3.1 Trace theorem

By applying Lemma 3.6 in each particle and with some additional arguments with respect to the ones presented in Section 4.2.1 we deduce the following.

Lemma 5.2. *For all $u \in W^{1,p}(\Omega_\varepsilon)$ we have*

$$\beta^\star(\varepsilon) \int_{S_\varepsilon} |u|^p \, dS \le C\Big(\int_{\Omega_\varepsilon} |u|^p \, dx + \varepsilon^{n-1}\tau_\varepsilon \int_{\Omega_\varepsilon} |\nabla u|^p \, dx \Big).$$

Sketch of proof. Repeating the argument in Lemma 4.2 we obtain

$$\beta^\star(\varepsilon) \int_{S_\varepsilon} |u|^p \, dS \le C\Big(\varepsilon^{-1} \int_{\Omega_\varepsilon \cap \{|x_n|<\varepsilon\}} |u|^p \, dx + \varepsilon^{n-1}\tau_\varepsilon \int_{\Omega_\varepsilon \cap \{|x_n|<\varepsilon\}} |\nabla u|^p \, dx \Big),$$

due to the fact that $|\Upsilon_\varepsilon| \sim \varepsilon^{1-n}$ instead of ε^{-n}, and the special location of the particles. Since we do not want to use higher order derivatives, we estimate

$$\int_{\Omega_\varepsilon \cap \{|x_n|<\varepsilon\}} |\nabla u|^p \, dx \le \int_{\Omega_\varepsilon} |\nabla u|^p \, dx.$$

For the first term, we work a little roughly through the extension $v = P_\varepsilon u$ (although more direct arguments are possible). We have

$$\int_{\Omega_\varepsilon \cap \{|x_n|<\varepsilon\}} |u|^p \, dx \le \int_{\Omega \cap \{|x_n|<\varepsilon\}} |v|^p \, dx.$$

Formally, since

$$v(x + he_n) = v(x) + \int_0^h \frac{\partial v}{\partial x_n}(x + se_n) \, ds,$$

it is not hard to compute

$$\begin{aligned}
\int_{\Omega \cap \{|x_n|<\varepsilon\}} |v|^p \, dx &\le C\varepsilon\Big(\int_{\Omega^0} |v(x)|^p \, dS + \int_{\Omega \cap \{|x_n|<\varepsilon\}} |\nabla v(x)|^p \, dx \Big) \\
&\le C\varepsilon\Big(\int_{\Omega} |v(x)|^p \, dx + \int_{\Omega \cap \{|x_n|<\varepsilon\}} |\nabla v(x)|^p \, dx \Big),
\end{aligned}$$

where we used for the last step the trace theorem $W^{1,p}(\Omega^+) \to L^p(\Omega^0)$. We conclude the proof by using the continuity of the extension operator. □

Note that in this setting

$$\varepsilon^{n-1}\tau_\varepsilon \sim \begin{cases} \varepsilon^{n-1} a_\varepsilon^{p-n} & p < n, \\ \varepsilon^{n-1} \ln(\frac{2\varepsilon}{a_\varepsilon})^{p-1} & p = n, \\ \varepsilon^{p-1} & p > n. \end{cases}$$

This is bounded (or tends to 0) if $a_\varepsilon^\star \lesssim a_\varepsilon \le \varepsilon$. Like in Section 4.2.1 we see the presence of the critical scale.

Remark 5.3. Note that in the above expression we have $\varepsilon^{n-1}\tau_\varepsilon$ instead of $\varepsilon^n\tau_\varepsilon$ given in Section 4.2.1, due to the different scaling of $|\Upsilon_\varepsilon|$.

Lemma 5.4. *Let $p > 1$ and $a_\varepsilon \gtrsim a_\varepsilon^\star$. Then, there exists C independent of ε such that, for any $u \in W^{1,p}(\Omega_\varepsilon)$,*

$$\beta^\star(\varepsilon)\int_{S_\varepsilon}|u|^p\,\mathrm{d}S \le C\Big(\int_{\Omega_\varepsilon}|u|^p\,\mathrm{d}x + \int_{\Omega_\varepsilon}|\nabla u|^p\,\mathrm{d}x\Big).$$

5.3.2 Limit of integrals over S_ε

Now we work with M_ε and $M_{g,\varepsilon}$ defined with the new Υ_ε but the same m_ε and $m_{g,\varepsilon}$ defined in Chapter 3. Working exactly as in the previous chapter, we deduce

$$\chi_{\Omega_\varepsilon} \rightharpoonup 1 \quad \text{weak} - \star \text{ in } L^\infty(\Omega), \tag{5.2}$$

even when $a_\varepsilon = \varepsilon$. The volumetric integrals do not detect the perturbations on a manifold.

Theorem 5.5. *Assume that $a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$ or $a_\varepsilon = \varepsilon$. Let $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ be a sequence with $\|\nabla v_\varepsilon\|_{L^p(\Omega_\varepsilon)}$ bounded and such that $v_\varepsilon \to v$ in $L^p(\Omega^0)$. Then we have*

$$\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} v_\varepsilon\,\mathrm{d}S \to \frac{1}{|\Omega^0|}\int_{\Omega^0} v\,\mathrm{d}x. \tag{5.3}$$

The only improvement needed in the proof is to remind that the trace operator $W^{1,p}(\Omega) \to L^p(\Omega^0)$ is compact (see, e. g., [35]). Similarly we have the following theorem.

Theorem 5.6. *Assume that $a_\varepsilon^\star \ll a_\varepsilon \le \varepsilon$ and $g \in L^{p'}(\partial G_0)$. Then, for any sequence $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ with $\|\nabla v_\varepsilon\|_{L^p(\Omega_\varepsilon)}$ bounded and such that $P_\varepsilon v_\varepsilon \to v$ in $L^p(\Omega^0)$, we have*

$$\beta^\star(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j + a_\varepsilon\partial G_0} g\Big(\frac{x-\varepsilon j}{a_\varepsilon}\Big)v_\varepsilon(x)\,\mathrm{d}S \to \frac{1}{|\partial G_0|}\int_{\partial G_0} g(y)\,\mathrm{d}S\frac{1}{|\Omega^0|}\int_{\Omega^0} v(x)\,\mathrm{d}x. \tag{5.4}$$

5.4 A priori estimates for u_ε

Following exactly the same proof as in Section 4.3 (but applying Lemma 5.4) we have the following proposition.

Proposition 5.7. *Let $p > 1$ and let u_ε be the minimizer of J_ε defined in Section 2.2. Then:*
1. *If $g^\varepsilon = 0$, then*

$$\|\nabla u_\varepsilon\|^{p-1}_{L^p(\Omega_\varepsilon)} \le C\|f^\varepsilon\|_{L^{p'}(\Omega_\varepsilon)}. \tag{5.5}$$

2. *If* $g^\varepsilon \neq 0$ *and* $a_\varepsilon \geq a_\varepsilon^\star$, *we have*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{p-1} \leq C\left(\|f^\varepsilon\|_{L^{p'}(\Omega_\varepsilon)} + \beta(\varepsilon)\beta^\star(\varepsilon)^{-1}\left(\frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} |g^\varepsilon|^{p'}\,\mathrm{d}S \right)^{\frac{1}{p'}} \right). \tag{5.6}$$

5.5 Subcritical particles $a_\varepsilon^\star \ll a_\varepsilon \leq \varepsilon$

Using similar test functions, we write the usual weak formulation (for any good test function v)

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon) \int_{\Omega_\varepsilon} (\sigma(v) - g^\varepsilon)(v - u_\varepsilon)\,\mathrm{d}S \geq \int_\Omega f(v - u_\varepsilon)\,\mathrm{d}x.$$

Assume that $g^\varepsilon(x)$ is given by (1.3), where $g_{\mathrm{st}} \in W^{1,\infty}(\Omega)$ and $g_{\mathrm{per}} \in L^{p'}(\partial G_0)$. Taking into account the a priori estimates, we know that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Let

$$\beta^0 = \lim_{\varepsilon \to 0} \beta(\varepsilon)|S_\varepsilon|.$$

By taking into account Section 5.3.2 we can pass to the limit, at least when σ is smooth (and by approximation for other regularities as in Chapter 4), and deduce

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u)\,\mathrm{d}x + \frac{\beta^0}{|\Omega^0|} \int_{\Omega^0} \sigma(v)(v - u)\,\mathrm{d}S \geq \int_\Omega f(v - u)\,\mathrm{d}x + \frac{\beta^0}{|\Omega^0|} \int_{\Omega^0} g^{\mathrm{eff}}(v - u)\,\mathrm{d}x,$$

where

$$g^{\mathrm{eff}}(x) = g_{\mathrm{per}}(x) + \frac{1}{|\partial G_0|} \int_{\partial G_0} g_{\mathrm{per}}(y)\,\mathrm{d}S.$$

As we have shown above, this is the weak formulation of

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega^+ \cup \Omega^-, \\ [|\nabla u|^{p-2} \frac{\partial u}{\partial x_n}]_{\Omega^0} = \beta^{\mathrm{eff}}(\sigma(u) - g^{\mathrm{eff}}) & \text{on } \Omega^0, \\ u = 0 & \partial\Omega, \end{cases}$$

where $\beta^{\mathrm{eff}} = \beta^0/|\Omega^0|$.

5.6 Supercritical particles $a_\varepsilon \ll a_\varepsilon^\star$

In this setting, consider again a radial function $\bar\psi : \mathbb{R}^n \to [0,1]$ such that

$$\bar\psi(y) = \begin{cases} 0 & \text{if } |y| \geq 2K_0, \\ 1 & \text{if } |y| \leq K_0, \end{cases} \qquad |\nabla \bar\psi| \leq K,$$

and let

$$\psi_\varepsilon(x) = \sum_{j\in\Upsilon_\varepsilon} \bar{\psi}\Big(\frac{x-\varepsilon j}{a_\varepsilon}\Big).$$

It is clear that $\psi_\varepsilon = 1$ in $\varepsilon j + a_\varepsilon G_0$ and 0 in $\partial\Omega$. Moreover, due to (5.1)

$$\int_\Omega |\nabla\psi_\varepsilon(x)|^p \,\mathrm{d}x \lesssim \Big(\frac{a_\varepsilon}{a_\varepsilon^\star}\Big)^{n-p} \varepsilon^{1-n}(a_\varepsilon^\star)^{n-p} \to 0.$$

Taking $\varphi_\varepsilon = \varphi(1-\psi_\varepsilon) \to \varphi$ in $W_0^{1,p}(\Omega)$ as a test function we again recover that if $g^\varepsilon = 0$, then the limit is

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ [|\nabla u|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = 0 & \text{on } \Omega^0, \\ u = 0 & \partial\Omega. \end{cases}$$

5.7 Critical-size particles $a_\varepsilon \sim a_\varepsilon^\star$

As in the previous setting, we can use either the adequate corrector functions W_ε (and deal with $G_0 = B_1$ and with σ a maximal monotone graph) or $W_{\sigma,\varepsilon}$ (and deal with a general G_0 but for σ regular). For the sake of simplicity, we take the first case: $G_0 = B_1$, $1 < p < n$ and $g_{\mathrm{per}} = 0$, which is sufficiently illustrative. We leave to the reader to make the general extension. We can repeat the argument in (4.28) to recover

$$\begin{cases} -\Delta_p u = f & \Omega^+ \cup \Omega^-, \\ [|\nabla u|^{p-2}\frac{\partial u}{\partial x_n}]_{\Omega^0} = \mathcal{H}(x,u) & \Omega^0, \\ u = 0 & \partial\Omega, \end{cases} \tag{5.7}$$

where

$$\mathcal{H}(x,s) = \mathcal{A}_0|H(x,s)|^{p-2}H(x,s) \tag{5.8}$$

at each point $x \in \Omega$, $H(x,\cdot)$ is the solution of the functional equation

$$\mathcal{B}_0|H(x,s)|^{p-2}H(x,s) \in \sigma\big(s - H(x,s)\big) - g_{\mathrm{st}}(x) \tag{5.9}$$

and

$$\mathcal{A}_0 = \Big(\frac{n-p}{p-1}\Big)^{p-1}|\partial B_1| \lim_{\varepsilon\to 0}(a_\varepsilon^{n-p}\varepsilon^{1-n}),$$
$$\mathcal{B}_0 = \Big(\frac{n-p}{p-1}\Big)^{p-1} \lim_{\varepsilon\to 0} \frac{a_\varepsilon^{1-p}}{\beta(\varepsilon)}.$$

Let us give the intuition of the proof in this setting. Taking W_ε corresponding to the new Υ_ε (but with the same w_ε as before), we still obtain the equivalent result to Lemma 4.38. Letting $v_\varepsilon = v - hW_\varepsilon$ we get

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x = I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + R(\varepsilon),$$

where

$$I_{1,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x, \tag{5.10}$$

$$I_{2,\varepsilon} = -B_\varepsilon \int_{S_\varepsilon} |h|^{p-2}h(v - h - u_\varepsilon)\,\mathrm{d}S, \tag{5.11}$$

$$I_{3,\varepsilon} = A_\varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\varepsilon j + \frac{\varepsilon}{4}\partial B_1} |h|^{p-2}h(v - u_\varepsilon)\,\mathrm{d}S, \tag{5.12}$$

and the constants are given by

$$A_\varepsilon \simeq \left(\frac{n-p}{p-1}\right)^{p-1} a_\varepsilon^{n-p}(\varepsilon/4)^{1-n}, \quad B_\varepsilon \simeq \left(\frac{n-p}{p-1}\right)^{p-1} a_\varepsilon^{1-p}.$$

The values of A_ε and B_ε are precisely as before when written in this way, but scaling with the changes of a_ε. In this setting, letting

$$\widehat{S}_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}\left(\varepsilon j + \frac{\varepsilon}{4}\partial B_1\right),$$

we recover

$$|\widehat{S}_\varepsilon| = |\Upsilon_\varepsilon|\left(\frac{\varepsilon}{4}\right)^{n-1}|\partial B_1| \simeq \frac{|\Omega^0||\partial B_1|}{4^{n-1}}.$$

Due to the critical scale we also get $A_\varepsilon \sim 1$. On the other hand,

$$|S_\varepsilon| = |\Upsilon_\varepsilon| a_\varepsilon^{n-1}|\partial G_0| \sim \varepsilon^{1-n}a_\varepsilon^{n-1} \sim a_\varepsilon^{p-1}.$$

Thus

$$\begin{aligned}\mathcal{A}_0 &= \lim_{\varepsilon\to0}\frac{A_\varepsilon|\widehat{S}_\varepsilon|}{|\Omega^0|} = \lim_{\varepsilon\to0}\left(\frac{n-p}{p-1}\right)^{p-1} a_\varepsilon^{n-p}(\varepsilon/4)^{1-n}\varepsilon^{1-n}(\varepsilon/4)^{n-1}|\partial B_1| \\ &= \left(\frac{n-p}{p-1}\right)^{p-1}|\partial B_1|\lim_{\varepsilon\to0}(a_\varepsilon^{n-p}\varepsilon^{1-n})\end{aligned}$$

and

$$\mathcal{B}_0 = \lim_{\varepsilon\to0}\frac{B_\varepsilon}{\beta(\varepsilon)} = \left(\frac{n-p}{p-1}\right)^{p-1}\lim_{\varepsilon\to0}\frac{a_\varepsilon^{1-p}}{\beta(\varepsilon)}.$$

Remark 5.8. When $g^\varepsilon \neq 0$, G_0 is not a ball or $p = n$, we make the adequate modifications to the test function w_ε as in the previous setting. Restrictions of σ will make some estimates more manageable, but are philosophically not needed in order to recover the estimates.

5.8 Some remarks

Remark 5.9. In the special case of Signorini boundary conditions with $g^\varepsilon = 0$ (which corresponds to the case of the maximal monotone graph σ given by (2.3)), the solution of the variational inequality converges at the critical scale to the solution of problem (5.7) with the corresponding Lipschitz increasing function $\mathcal{H}$ given by Example 4.34 (c) that was shown in [157] by a refined version of the technique of proof presented in this chapter.

Remark 5.10. There are several cases mentioned in Table 1.2 which merit some comments. For instance, we can see there that in some of the homogenized limits the solution must vanish on the manifold. In some others the manifold does not play any significant role since there is no jump on the gradients over such manifold. Moreover, in the critical case there is a subcase in which the strange term is clearly different from σ, and another subcase in which the jump of the gradients is proportional to the value of u if $p = 2$ independently of the value of function σ.

Remark 5.11. The case in which G_0 is not a ball was considered in [233]. The technique of proof has some common points with the proof presented in Section 4.7.3 but some important adaptations are needed.

In terms of the homogenized problem, in the subcritical or critical case (problem (5.7)), it seems complicated to prove directly a comparison principle for two different limit kinetics $\mathcal{H}$ and $\widehat{\mathcal{H}}$ (which we assume for simplicity to be non-increasing continuous functions such that $\mathcal{H}(0) = \widehat{\mathcal{H}}(0) = 0$), which are well ordered in the sense that

$$\mathcal{H}(r) \leq \widehat{\mathcal{H}}(\widehat{r}) \quad \text{for any } r \leq \widehat{r}. \tag{5.13}$$

Some comparison results of this nature are well known for the case of nonlinear Robin type boundary conditions (see, e. g., [49] and [103]). The main difficulty in our case comes from the fact that the transmission condition, on the manifold, depends of the unknown value of the corresponding solution (let us say u and $\widehat{u}$). Nevertheless, the homogenization process supplies an argument to get such type of comparison results, which, in particular, allows to conclude that the "change of velocity" across the manifold is smaller in the case of a critical scale.

Corollary 5.12. *Let u and $\widehat{u}$ be the solutions of the problems* (5.7) *corresponding to increasing functions $\mathcal{H}$ and $\widehat{\mathcal{H}}$ satisfying* (5.13) *with the same rest of the data. Then $u \leq \widehat{u}$ on Ω.*

Proof. By using the fact that $\mathcal{H}(r) = (I + \sigma^{-1} \circ \Theta_{n,p})^{-1}(r)$ (see Proposition 4.33 d), where we assumed some constants equal to one) we assume that

$$\sigma(r) \leq \hat{\sigma}(\hat{r}) \quad \text{for any } r \leq \hat{r}. \tag{5.14}$$

Then, for any given $\varepsilon > 0$, let u_ε and $\hat{u}_\varepsilon$ be the solutions of the problems corresponding to the respective problems with particles on the manifold and increasing functions σ and $\hat{\sigma}$. Then, by the results of [49] we conclude that $u_\varepsilon(x) \leq \hat{u}_\varepsilon(x)$ on Ω_ε and by the convergence in $L^p(\Omega)$ of their extensions as $\varepsilon \to 0$ (given in Section 5.7) we get that $u \leq \hat{u}$ on Ω. □

Remark 5.13. There are some special three-dimensional problems ($n = 3$) in which the periodicity of the reactant objects is bidimensional and thus the critical scale (for semilinear problems $p = 2$, for instance) is the one which curiously corresponds in this chapter to $p = n$ (although this balance is not true in this framework). This happens, for instance, in the homogenization of reactive thin tubes. We refer the reader to [153, 154, 152].

6 Particles over a part of the boundary

Let us study now the geometrical setting presented in Section 1.2.2. We recall that, in contrast to the two precedent chapters, S_ε is not the boundary of the particles but the own set of $(n-1)$-dimensional particles. Going back to Remark 2.15 we have

$$|S_\varepsilon| = |\Upsilon_\varepsilon||a_\varepsilon G_0| \simeq \left|(\partial\Omega)^0\right||G_0|\varepsilon^{1-n}a_\varepsilon^{n-1}.$$

Since the particles are now contained in the boundary of $\Omega_\varepsilon = \Omega$ we do not need any extension operator. The critical scale $a_\varepsilon^\star$ is as in Chapter 5.

6.1 Existence of a critical scale

This case works very similarly to Sections 4.1 and 5.2 with only few modifications. Let v defined on $\mathbb{R}^n$ compactly supported and take

$$v_\varepsilon(x) = \begin{cases} v(\frac{x-\varepsilon j}{a_\varepsilon}) & x \in \varepsilon j + \varepsilon Y^+ \text{ for some } j \in \Upsilon_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Note that due to the construction of Υ_ε, we have $v_\varepsilon(x) = 0$ if $x_n > \varepsilon$. If $a_\varepsilon \ll \varepsilon$ we have

$$\int_\Omega \left|\nabla v_\varepsilon(x)\right|^p \mathrm{d}x = |\Upsilon_\varepsilon| a_\varepsilon^{n-p} \int_{\varepsilon Y^+} \left|\nabla v(x)\right|^p \mathrm{d}x \sim \varepsilon^{1-n} a_\varepsilon^{n-p} \int_{\frac{\varepsilon}{a_\varepsilon} Y^+} \left|\nabla v(y)\right|^p \mathrm{d}y. \tag{6.1}$$

Hence, this integral changes its limit behavior when $\varepsilon^{1-n}a_\varepsilon^{n-p} \sim 1$. The case $p = n$ is always special.

6.2 Integrals over S_ε

First, we can prove a trace estimate: if $a_\varepsilon \le \varepsilon$, then

$$\beta^\star(\varepsilon) \int_{S_\varepsilon} |u|^p \,\mathrm{d}S \le C\left(\int_\Omega |u|^p \,\mathrm{d}x + \varepsilon^{n-1}\tau_\varepsilon \int_\Omega |\nabla u|^p \,\mathrm{d}x \right), \tag{6.2}$$

where τ_ε is given by (3.6). This follows directly from (3.39) by applying a similar argument to that of Lemma 5.2. Again, $\varepsilon^{n-1}\tau_\varepsilon$ is bounded if $a_\varepsilon^\star \lesssim a_\varepsilon \le \varepsilon$.

Similarly to the previous cases, taking the functions $m_{g,\varepsilon}$ given by Section 3.2.2 one recovers the following result.

Theorem 6.1. *Assume that $a_\varepsilon^\star \ll a_\varepsilon \le \varepsilon$ and let $g \in L^{p'}(G_0)$. Then, for any sequence $v_\varepsilon \in W^{1,p}(\Omega)$ with $\|\nabla v_\varepsilon\|_{L^p(\Omega)}$ bounded and such that $v_\varepsilon \to v$ in $L^p(\Omega)$ we have*

$$\frac{1}{|S_\varepsilon|} \sum_{j\in\Upsilon_\varepsilon} \int_{\varepsilon j + a_\varepsilon G_0} g\left(\frac{x-\varepsilon j}{a_\varepsilon}\right) v_\varepsilon(x)\,\mathrm{d}S \longrightarrow \frac{1}{|G_0|}\int_{G_0} g(y)\,\mathrm{d}S \frac{1}{|(\partial\Omega)^0|} \int_{(\partial\Omega)^0} v(x)\,\mathrm{d}S. \tag{6.3}$$

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From this, we obtain the following corollary.

Corollary 6.2. *Assume that $a_\varepsilon^\star \ll a_\varepsilon \le \varepsilon$. Then, for any sequence $v_\varepsilon \in W^{1,p}(\Omega)$ with $\|\nabla v_\varepsilon\|_{L^p(\Omega)}$ bounded and such that $v_\varepsilon \to v$ in $L^p(\Omega)$ we have*

$$\frac{1}{|S_\varepsilon|}\int\limits_{S_\varepsilon} v_\varepsilon \,\mathrm{d}S \longrightarrow \frac{1}{|(\partial\Omega)^0|}\int\limits_{(\partial\Omega)^0} v(x)\,\mathrm{d}S. \tag{6.4}$$

6.3 A priori estimates

As in the previous cases, using (6.2) we can prove the following proposition.

Proposition 6.3. *Let $p > 1$ and let u_ε be the minimizer of J_ε given in Section 2.2 (see Remark 2.8). Then:*

1. *If $g^\varepsilon = 0$, we have*
$$\|\nabla u_\varepsilon\|_{L^p(\Omega)}^{p-1} \le C\|f\|_{L^{p'}(\Omega)}. \tag{6.5}$$
2. *If $g^\varepsilon \ne 0$ and $a_\varepsilon \gtrsim a_\varepsilon^\star$, we have*
$$\|\nabla u_\varepsilon\|_{L^p(\Omega)}^{p-1} \le C\left(\|f\|_{L^{p'}(\Omega)} + \beta(\varepsilon)\beta^\star(\varepsilon)^{-1}\left(\frac{1}{|S_\varepsilon|}\int\limits_{S_\varepsilon}|g^\varepsilon|^{p'}\,\mathrm{d}S\right)^{\frac{1}{p'}}\right). \tag{6.6}$$

6.4 Subcritical particles $a_\varepsilon^\star \ll a_\varepsilon \le \varepsilon$

In this setting, it can be proved that the homogenized problem is

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega,\\ u = 0 & \text{on } (\partial\Omega)^+,\\ \partial_{\nu_p} u + \beta^{\mathrm{eff}}\sigma(u) = \beta^{\mathrm{eff}} g^{\mathrm{eff}}(x) & \text{on } (\partial\Omega)^0, \end{cases} \tag{6.7}$$

where $\beta^{\mathrm{eff}} = \beta^0/|(\partial\Omega)^0|$ and, if g^ε is given by (1.5), where $g_{\mathrm{st}} \in W^{1,\infty}(\Omega)$ and $g_{\mathrm{per}} \in L^{p'}(G_0)$, then we have that the effective term g^{eff} is given by (1.10). As in Chapter 4, we assume that σ is smooth but the results can be extended to more general classes of σ.

We start by writing the weak formulation of the problem: for suitable test functions v we have

$$\int\limits_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla(v-u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int\limits_{S_\varepsilon}(\sigma(v)-g^\varepsilon(x))(v-u_\varepsilon)\,\mathrm{d}S \ge \int\limits_\Omega f(v-u_\varepsilon)\,\mathrm{d}x.$$

Using Theorem 6.1 and Corollary 6.2 we have

$$\beta^\star(\varepsilon)\int\limits_{S_\varepsilon}(\sigma(v)-g^\varepsilon(x))(v-u_\varepsilon)\,\mathrm{d}S \longrightarrow \frac{1}{|(\partial\Omega)^0|}\int\limits_{(\partial\Omega)^0}\left(\sigma(v(x)) - g_{\mathrm{st}}(x) - \frac{1}{|G_0|}\int\limits_{G_0} g_{\mathrm{per}}(y)\,\mathrm{d}S\right)\mathrm{d}S.$$

Since u_ε has a limit u in $L^p(\Omega)$, we recover (for any good test function v)

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u)\,\mathrm{d}x + \frac{\beta^0}{|(\partial\Omega)^0|}\int_{(\partial\Omega)^0}\left(\sigma\big(v(x)\big) - g_{\mathrm{st}}(x) - \frac{1}{|G_0|}\int_{G_0} g_{\mathrm{per}}(y)\,\mathrm{d}S\right)\mathrm{d}S$$
$$\geq \int_\Omega f(v-u)\,\mathrm{d}x.$$

This is the weak formulation of the proposed homogenized problem.

6.5 Supercritical particles with $1 < p < n$

We take again $K_0 = \max_{y\in G_0}|y|$ and a radial function $\bar\psi : \mathbb{R}^n \to [0,1]$ such that

$$\bar\psi(y) = \begin{cases} 0 & \text{if } |y| \geq 2K_0, \\ 1 & \text{if } |y| \leq K_0, \end{cases} \qquad |\nabla\bar\psi| \leq K, \tag{6.8}$$

and we let

$$\psi_\varepsilon(x) = \sum_{j\in\Upsilon_\varepsilon} \bar\psi\left(\frac{x-\varepsilon j}{a_\varepsilon}\right). \tag{6.9}$$

Thus, due to (6.1)

$$\int_\Omega \big|\nabla\psi_\varepsilon(x)\big|^p\,\mathrm{d}x \lesssim \varepsilon^{1-n}a_\varepsilon^{n-p} \to 0.$$

Using $v_\varepsilon = v(1-\psi_\varepsilon)$ as a test function, we have that $v_\varepsilon \to v$ strongly in $W^{1,p}(\Omega)$ and $v_\varepsilon = 0$ in S_ε. Thus, if we assume that $g^\varepsilon = 0$, the homogenized problem is

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } (\partial\Omega)^+, \\ \partial_{\nu_p} u = 0 & \text{on } (\partial\Omega)^0. \end{cases} \tag{6.10}$$

6.6 Critical-size particles

6.6.1 Case of $p = 2 < n$ when G_0 is not a ball

The procedure is very much like in the case of particles over a manifold, and we have

$$\begin{cases} -\Delta u = f & \Omega, \\ \partial_\nu u + \mathcal{H}(s,u) = 0 & (\partial\Omega)^0, \\ u = 0 & (\partial\Omega)^+, \end{cases}$$

where

$$\mathcal{H}(x,s) = C_0^{n-2}\widehat{H}_\sigma(x,s),$$

for

$$C_0 = \lim_{\varepsilon\to 0} a_\varepsilon \beta(\varepsilon),$$

and $\widehat{H}_\sigma$ is defined in Section 3.2.3.1. We recall again that we have a universal bound $0 \le \widehat{H}_\sigma' \le \lambda_{G_0}$ depending only on G_0. The details, which are very similar to the ones already presented, can be found in [115] for the special case of the Signorini boundary condition.

Remark 6.4. The case of critical-size particles $a_\varepsilon \sim a_\varepsilon^\star$ and $\beta(\varepsilon) \gg \beta^\star(\varepsilon)$ is specially illustrative of the strange boundary condition satisfied by the homogenized solution. As indicated in Table 1.3, if we assume, for instance, that in the starting problem we have Dirichlet conditions on S_ε, then after the homogenization process, we find that the limit function satisfies a linear Robin boundary condition on the part of the boundary given by $(\partial\Omega)^0$, if for instance $p = 2$.

6.6.2 Case of $p = n$ when $G_0 = B_1^0$

In this setting we work only in the case $g^\varepsilon = g$, as usual for G_0 a ball, and we can prove that

$$\begin{cases} -\Delta_n u = f & \Omega, \\ u = 0 & (\partial\Omega)^+, \\ \partial_{\nu_n} u + \mathcal{H}(x,u) = 0 & \Omega^0, \end{cases}$$

where

$$\mathcal{H}(x,s) = \mathcal{A}_0 \big|H(x,s)\big|^{n-2} H(x,s), \quad \mathcal{B}_0 \big|H(x,s)\big|^{n-2} H(x,s) = \sigma\big(s - H(x,s)\big) - g(x),$$

and, with $\beta^0 = \lim_{\varepsilon\to 0} \beta(\varepsilon)|S_\varepsilon| < \infty$,

$$\mathcal{A}_0 = \big|(\partial B_1)^+\big| \lim_{\varepsilon\to 0}\left(\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}\right)^{1-n}, \quad \mathcal{B}_0 = \frac{|(\partial B_1)^+|}{|G_0|} \lim_{\varepsilon\to 0} \beta(\varepsilon)^{-1}\left(a_\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}\right)^{1-n}.$$

The argument is slightly different from the previous case. We present here a generalization of the argument in [106], where the case $n = 2$ was considered. The details can be found in [230].

Remark 6.5. In [106], $G_0 = (-\ell_0, \ell_0)$ (for us $\ell_0 = 1$) and $a_\varepsilon = C_0 \varepsilon e^{\frac{-\alpha^2}{\varepsilon}}$, so $\mathcal{A}_0 = \pi/\alpha^2$ and $\beta(\varepsilon) = e^{\frac{\alpha^2}{\varepsilon}}$, so $\mathcal{B}_0 = \frac{\pi}{2C_0\alpha^2}$.

The correct oscillating test function is

$$v_\varepsilon = v - hQ_\varepsilon,$$

where Q_ε is given by

$$Q_\varepsilon(x) = \begin{cases} q_\varepsilon(x - \varepsilon j) & x \in (\varepsilon j + \varepsilon B_1^+) \text{ for some } j \in \Upsilon_\varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

and q_ε is as defined in Section 3.2.3.2. Since we want to apply an argument similar to (4.38), we need to show that Q_ε tends weakly to zero. We could probably do this directly. However, as we mentioned in Section 3.2.3.2, we estimate this function through the auxiliary function

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon(x - \varepsilon j) & x \in (\varepsilon j + \frac{\varepsilon}{4} B_1^+ \setminus a_\varepsilon B_1^+) \text{ for some } j \in \Upsilon_\varepsilon, \\ 1 & x \in (\varepsilon j + a_\varepsilon B_1^+) \text{ for some } j \in \Upsilon_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

From (3.41) we deduce that

$$\int_\Omega \left|\nabla(Q_\varepsilon - W_\varepsilon)\right|^n \mathrm{d}x = |\Upsilon_\varepsilon| \int_{\frac{\varepsilon}{4}B_1^+} \left|\nabla(w_\varepsilon - q_\varepsilon)\right|^n \mathrm{d}x \le C\varepsilon\left(\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}\right)^{-n} \to 0,$$

due to the critical scale. Since W_ε converges weakly to 0 in $W^{1,n}(\Omega)$, so does Q_ε.

Remark 6.6. This methodology fails when $p < n$ because this convergence fails to be strong.

We would like to repeat an argument as in Theorem 4.36. Using v_ε as a test function we would like to recover something similar to Lemma 4.38:

$$\int_\Omega |\nabla v_\varepsilon|^{n-2} \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\,\mathrm{d}x = I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + R(\varepsilon), \tag{6.11}$$

where $R(\varepsilon) \to 0$ and

$$I_{1,\varepsilon} = \int_\Omega |\nabla v|^{n-2} \nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x, \tag{6.12}$$

$$I_{2,\varepsilon} = -\int_{S_\varepsilon} \partial_{\nu_n} Q_\varepsilon |h|^{p-2} h(v - h - u_\varepsilon)\,\mathrm{d}S, \tag{6.13}$$

$$I_{3,\varepsilon} = -\sum_{j \in \Upsilon_\varepsilon} \int_{\varepsilon j + \frac{\varepsilon}{4}(\partial B_1)^+} \partial_{\nu_p} Q_\varepsilon |h|^{n-2} h(v - u_\varepsilon)\,\mathrm{d}S. \tag{6.14}$$

The only problem arising when applying such a program of proof is that we do not know the explicit value of $\partial_{\nu_p} q_\varepsilon$ as we do with w_ε. For convenience, let us define

$$\widehat{S}_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}\Big(\varepsilon j + \frac{\varepsilon}{4}(\partial B_1)^+\Big).$$

Note that $|\widehat{S}_\varepsilon| = |\Upsilon_\varepsilon||\frac{\varepsilon}{4}(\partial B_1)^+| \simeq 4^{1-n}|(\partial\Omega)^0||(\partial B_1)^+|$.

First, one needs to show that we can find some balance between $I_{2,\varepsilon}$ and $I_{3,\varepsilon}$ by the corresponding term for W_ε and this new term can cancel out in the limit of the reaction term. Due to the strong convergence of $Q_\varepsilon - W_\varepsilon$ we can write

$$\begin{aligned}
-(I_{2,\varepsilon} + I_{3,\varepsilon}) &= \sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+\frac{\varepsilon}{4}B_1^+} |\nabla Q_\varepsilon|^{n-2}\nabla Q_\varepsilon \nabla\big(|h|^{n-2}h(v - hQ_\varepsilon - u_\varepsilon)\big)\,\mathrm{d}x\\
&= R(\varepsilon) + \sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+\frac{\varepsilon}{4}B_1^+} |\nabla W_\varepsilon|^{n-2}\nabla W_\varepsilon \nabla\big(|h|^{n-2}h(v - hW_\varepsilon - u_\varepsilon)\big)\,\mathrm{d}x\\
&= R(\varepsilon) + \sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j+\frac{\varepsilon}{4}B_1^+\setminus a_\varepsilon B_1^+} |\nabla W_\varepsilon|^{n-2}\nabla W_\varepsilon \nabla\big(|h|^{n-2}h(v - hW_\varepsilon - u_\varepsilon)\big)\,\mathrm{d}x\\
&= R(\varepsilon) + \int_{\widetilde{S}_\varepsilon} \partial_{\nu_n} W_\varepsilon |h|^{n-2}h(v - h - u_\varepsilon)\,\mathrm{d}S\\
&\quad + \int_{\widehat{S}_\varepsilon} \partial_{\nu_n} W_\varepsilon |h|^{n-2}h(v - u_\varepsilon)\,\mathrm{d}S\\
&= R(\varepsilon) - (J_{2,\varepsilon} + J_{3,\varepsilon}).
\end{aligned}$$

Since we have the explicit value of $\partial_{\nu_p} W_\varepsilon$, the rest of the work is easier. Recovering the explicit values from Section 3.1.5.2 we have

$$\begin{aligned}
J_{3,\varepsilon} &= \left(\frac{4}{\varepsilon \ln\frac{\varepsilon}{4a_\varepsilon}}\right)^{n-1}\int_{\widehat{S}_\varepsilon} |h|^{n-2}h(v - u_\varepsilon)\,\mathrm{d}S\\
&\longrightarrow \lim_{\varepsilon\to 0}\left(\left(\frac{4}{\varepsilon \ln\frac{\varepsilon}{4a_\varepsilon}}\right)^{n-1}|\widehat{S}_\varepsilon|\right)\frac{1}{|(\partial\Omega)^0|}\int_{(\partial\Omega)^0} |h|^{n-2}h(v - h - u)\,\mathrm{d}S\\
&= \big|(\partial B_1)^+\big|\lim_{\varepsilon\to 0}\left(\varepsilon\ln\frac{\varepsilon}{4a_\varepsilon}\right)^{1-n}\int_{(\partial\Omega)^0} |h|^{n-2}h(v - h - u)\,\mathrm{d}S,
\end{aligned}$$

where we recover $\mathcal{A}_0$. On the other hand,

$$J_{2,\varepsilon} = \left(\frac{1}{a_\varepsilon \ln\frac{\varepsilon}{4a_\varepsilon}}\right)^{n-1}\sum_{j\in\Upsilon_\varepsilon}\int_{\varepsilon j + a_\varepsilon(\partial B_1)^+} |h|^{p-2}h(v - h - u_\varepsilon)\,\mathrm{d}S.$$

With this construction we can show that if h_ε is a bounded sequence in $W^{1,n}(\Omega)$, we have the following result (which improves a similar lemma in [230]).

Lemma 6.7. *Let $h_\varepsilon \in W^{1,n}(\Omega, (\partial\Omega)^+)$ be a bounded sequence. Then*

$$\left| \frac{1}{|\Upsilon_\varepsilon||a_\varepsilon(\partial B_1)^+|} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j + a_\varepsilon(\partial B_1)^+} h_\varepsilon \, \mathrm{d}S - \frac{1}{|S_\varepsilon|} \int\limits_{S_\varepsilon} h_\varepsilon \, \mathrm{d}S \right| \le C\big(a_\varepsilon \varepsilon^{-1}\big)^{n-1} \varepsilon^{1/n} \big(1 + \|\nabla h_\varepsilon\|^n_{L^n(\Omega)}\big) \to 0.$$

Proof. To be able to apply a balance of this integral with the one on S_ε, we construct a function θ:

$$\begin{cases} \Delta_n \theta = 0 & \text{in } B_1^+, \\ \partial_{\nu_n} \theta = \frac{1}{|(\partial B_1)^+|} & (\text{on } \partial B_1)^+, \\ \partial_{\nu_n} \theta = -\frac{1}{|G_0|} & \text{on } G_0 = B_1^0, \end{cases} \qquad \theta_\varepsilon(x) = \varepsilon \theta\left(\frac{x - \varepsilon j}{a_\varepsilon}\right) \text{ when } x \in \varepsilon j + a_\varepsilon B_1^+.$$

We have the scaling $\partial_{\nu_n} \theta_\varepsilon = (\varepsilon a_\varepsilon^{-1})^{n-1} \partial_{\nu_n} \theta$. Thus

$$\begin{aligned} &\frac{1}{|\Upsilon_\varepsilon||a_\varepsilon(\partial B_1)^+|} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j + a_\varepsilon(\partial B_1)^+} h_\varepsilon \, \mathrm{d}S - \frac{1}{|S_\varepsilon|} \int\limits_{S_\varepsilon} h_\varepsilon \, \mathrm{d}S \\ &\quad = \frac{\varepsilon^{n-1}}{|\Upsilon_\varepsilon|} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j + a_\varepsilon B_1^+} |\nabla \theta_\varepsilon|^{n-2} \nabla \theta_\varepsilon \nabla h_\varepsilon \, \mathrm{d}x. \end{aligned}$$

In each of these balls, we have two estimates. On the one hand,

$$\int\limits_{\varepsilon j + a_\varepsilon B_1^+} |\nabla_x \theta_\varepsilon|^n \, \mathrm{d}x \le C\varepsilon^n.$$

On the other hand, applying Young's inequality

$$\int\limits_{\varepsilon j + a_\varepsilon B_1^+} |\nabla \theta_\varepsilon|^{n-2} \nabla \theta_\varepsilon \nabla h_\varepsilon \, \mathrm{d}x \le C\left(\delta_1^{-n} \int\limits_{\varepsilon j + a_\varepsilon B_1^+} |\nabla_x \theta_\varepsilon|^n \, \mathrm{d}x + \delta_1^{n/(n-1)} \int\limits_{\varepsilon j + a_\varepsilon B_1^+} |\nabla_x h_\varepsilon|^n \, \mathrm{d}x \right),$$

where δ_1 is an arbitrary positive number. Going back to the sum, we get the bound

$$\sum_{j\in\Upsilon_\varepsilon} \int\limits_{\varepsilon j + a_\varepsilon B_1^+} |\nabla \theta_\varepsilon|^{n-2} \nabla \theta_\varepsilon \nabla h_\varepsilon \, \mathrm{d}x \le C\left(\delta_1^{-n} \varepsilon^n |\Upsilon_\varepsilon| + \delta_1^{n/(n-1)} \int\limits_{B_1^+} |\nabla h_\varepsilon|^n \, \mathrm{d}x \right).$$

We now take $\delta_1 = \varepsilon^{(n-1)/n^2}$ to recover the desired result. □

Thus, we only need to identify the right constant. To compute the constant $\mathcal{B}_0$ we write

$$-J_{2,\varepsilon} \simeq \left(\frac{1}{a_\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}} \right)^{n-1} \frac{|\Upsilon_\varepsilon||a_\varepsilon(\partial B_1)^+|}{|S_\varepsilon|\beta(\varepsilon)} \beta(\varepsilon) \int\limits_{S_\varepsilon} h_\varepsilon \, \mathrm{d}S.$$

Hence, we take $h = H(x, v)$, where H solves

$$\mathcal{B}_0 |H(s)|^{n-2} H(s) = \sigma(s - H(s)) - g(x)$$

and

$$\mathcal{B}_0 = \lim_{\varepsilon \to 0} \left(\frac{1}{a_\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}} \right)^{n-1} \frac{|\Upsilon_\varepsilon||a_\varepsilon(\partial B_1)^+|}{|S_\varepsilon|\beta(\varepsilon)}.$$

6.7 Further comments

Remark 6.8. Many variants are possible: we refer the reader to the list of papers mentioned in Section 1.6.3.

Remark 6.9. The case of σ a general maximal monotone graph, when G_0 is not a ball, remains for us an open problem. The results of [115] in which σ is the Signorini maximal monotone graph seem to indicate that a positive answer could be obtained in a more general setting but some new argument are needed to pass to the limit after regularizing σ.

Remark 6.10. For a possible connection between the results of this chapter and the homogenization for fractional operators we refer the reader to Section 6.3 of [115].

A Comments on the parabolic case

All the results of this book concerning the three elliptic problems presented in Section 1.3 admit a corresponding version in the framework of parabolic equations. As a matter of fact, the application of homogenization techniques to evolution problems were present already in the pioneering papers and books on the subject (see, e. g., among many others [242, 31, 239, 175, 256]) and for the case of the occurrence of strange terms [201, 202] and especially [162], the first paper in which the occurrence of a strange term was proved for nonlinear Robin type boundary conditions (see also [34, 67, 163, 74]). Concerning the parabolic problem associated to the p-Laplacian operator we mention the paper [227].

The aim of this appendix is merely to present some few comments on the adaptation of the results of previous chapters, in particular Chapter 4, to the following parabolic problem dealing with particles over the whole spatial domain Ω:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta_p u_\varepsilon = f & (0,T)\times\Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_p} + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon)g^\varepsilon & (0,T)\times S_\varepsilon, \\ u_\varepsilon = 0 & (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \Omega_\varepsilon. \end{cases} \tag{A.1}$$

This problem can be treated very similarly to the elliptic one, at least under good integrability of f, g^ε and u_ε. For strong solutions, the weak formulation of (A.1) leads to the formulation

$$\int_0^T\int_{\Omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial t}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$+\beta(\varepsilon)\int_0^T\int_{S_\varepsilon}(\sigma(u_\varepsilon)-g^\varepsilon)\varphi\,\mathrm{d}S\,\mathrm{d}t = \int_0^T\int_{\Omega_\varepsilon} f\varphi\,\mathrm{d}x\,\mathrm{d}t, \tag{A.2}$$

for all φ smooth, where

$$u_\varepsilon \in L^\infty(0,T;W^{1,p}(\Omega_\varepsilon,\partial\Omega)) \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial t} \in L^2((0,T)\times\Omega_\varepsilon). \tag{A.3}$$

We will only develop here the variational theory, but an extension to L^1 data is possible as in Chapter 4.

A priori estimates

Gradient estimates on parabolic problems are usually harder to prove than their elliptic counterparts. From some variation of the regularity results for subdifferential operators (see, e. g., Theorem 3.6 of [48]), or through Galerkin approximation we can

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prove that

$$\|\partial_t u_\varepsilon\|_{L^2((0,T)\times\Omega_\varepsilon)} + \sup_{t\in[0,T]} \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C,$$

where C depends only on $\|u_0\|_{W_0^{1,p}(\Omega)}$, $\|f\|_{L^2((0,T)\times\Omega_\varepsilon)}$ and $\beta(\varepsilon)\beta(\varepsilon)^{-1}\|g^\varepsilon\|_{L^\infty(0,T;L^{p'}(S_\varepsilon)\cap L^2(S_\varepsilon))}$.

Extension operator

This can be easily generalized from the theory for stationary functions since the extension is spatial and has no relation with time. We construct

$$P_\varepsilon : L^p(0,T;W^{1,p}(\Omega_\varepsilon,\partial\Omega)) \longrightarrow L^p(0,T;W_0^{1,p}(\Omega))$$

by extending for almost any t fixed. Since this operator is linear and purely spatial and does not interact with the time variable, it preserves the estimates of the time derivatives. Therefore, there is a limit of the extension $P_\varepsilon u_\varepsilon$ up to a subsequence such that

$$\begin{aligned} P_\varepsilon u_\varepsilon &\rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\\ P_\varepsilon u_\varepsilon &\to u \quad \text{strongly in } L^p((0,T)\times\Omega),\\ \frac{\partial}{\partial t}(P_\varepsilon u_\varepsilon) &\rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in } L^2((0,T)\times\Omega). \end{aligned} \tag{A.4}$$

A.1 A weak formulation in terms of a variational inequality

As in the elliptic case, passing to the limit in the weak formulation directly is not possible due to the weak convergences. It is much better to find a suitable weaker formulation.

Lemma A.1. *Let u_ε satisfy* (A.2), (A.3). *Then it satisfies*

$$\begin{aligned} &\int_0^T\int_{\Omega_\varepsilon} \frac{\partial v}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\nabla(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \beta(\varepsilon)\int_0^T\int_{S_\varepsilon}(\sigma(v)-g^\varepsilon)(v-u_\varepsilon)\,\mathrm{d}S\,\mathrm{d}t\\ &\quad\ge \int_0^T\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{\Omega_\varepsilon}\big(u_\varepsilon(0,x)-v(0,x)\big)^2\,\mathrm{d}x, \end{aligned} \tag{A.5}$$

for all $v \in C_c^1([0,T]\times\Omega)$.

Proof. Assume that u_ε is a weak solution and as usual take $\varphi = v - u_\varepsilon$. We have

$$\begin{aligned} &\int_0^T\int_{\Omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \beta(\varepsilon)\int_0^T\int_{S_\varepsilon}(\sigma(u_\varepsilon)-g^\varepsilon)(v-u_\varepsilon)\,\mathrm{d}S\,\mathrm{d}t = \int_0^T\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

By monotonicity arguments that we used already in the elliptic setting, we know that

$$\int_0^T\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \beta(\varepsilon)\int_0^T\int_{S_\varepsilon}(\sigma(u_\varepsilon)-g^\varepsilon)(v-u_\varepsilon)\,\mathrm{d}S\,\mathrm{d}t$$
$$\le \int_0^T\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \beta(\varepsilon)\int_0^T\int_{S_\varepsilon}(\sigma(v)-g^\varepsilon)(v-u_\varepsilon)\,\mathrm{d}S\,\mathrm{d}t.$$

Let us look at the "new" term coming from the time derivative. We write

$$\begin{aligned}
\int_0^T\int_{\Omega_\varepsilon}\frac{\partial u_\varepsilon}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\Omega_\varepsilon}\frac{\partial(u_\varepsilon - v)}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon}\frac{\partial v}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{2}\int_0^T\frac{d}{dt}\int_{\Omega_\varepsilon}(u_\varepsilon - v)^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon}\frac{\partial v}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{2}\Big(\int_{\Omega_\varepsilon}\big(u_\varepsilon(T,x)-v(T,x)\big)^2\,\mathrm{d}x - \int_{\Omega_\varepsilon}\big(u_\varepsilon(0,x)-v(0,x)\big)^2\Big)\,\mathrm{d}x\\
&\quad + \int_0^T\int_{\Omega_\varepsilon}\frac{\partial v}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{1}{2}\int_{\Omega_\varepsilon}\big(u_\varepsilon(0,x)-v(0,x)\big)^2\,\mathrm{d}x + \int_0^T\int_{\Omega_\varepsilon}\frac{\partial v}{\partial t}(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Joining these computations, we recover (A.5).

Conversely, assume that u_ε is time differentiable and that it satisfies (A.5). Take φ smooth, $\lambda \in \mathbb{R}$ and $v = u_\varepsilon + \lambda\varphi$. We get

$$\lambda\int_0^T\int_{\Omega_\varepsilon}\frac{\partial(u_\varepsilon+\lambda\varphi)}{\partial t}\varphi\,\mathrm{d}x\,\mathrm{d}t + \lambda\int_0^T\int_{\Omega_\varepsilon}|\nabla(u_\varepsilon+\lambda\varphi)|^{p-2}\nabla(u_\varepsilon+\lambda\varphi)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$+\beta(\varepsilon)\lambda\int_0^T\int_{S_\varepsilon}(\sigma(u_\varepsilon+\lambda\varphi)-g^\varepsilon)\varphi\,\mathrm{d}S\,\mathrm{d}t \ge \lambda\int_0^T\int_{\Omega_\varepsilon}f\varphi\,\mathrm{d}x\,\mathrm{d}t - \frac{\lambda^2}{2}\int_{\Omega_\varepsilon}\varphi(0,x)^2\,\mathrm{d}x.$$

Assuming that $\lambda > 0$, dividing by λ and passing to the limit as $\lambda \to 0^+$ we get

$$\int_0^T\int_{\Omega_\varepsilon}\frac{\partial u_\varepsilon}{\partial t}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$+\beta(\varepsilon)\int_0^T\int_{S_\varepsilon}(\sigma(u_\varepsilon)-g^\varepsilon)\varphi\,\mathrm{d}S\,\mathrm{d}t \ge \int_0^T\int_{\Omega_\varepsilon}f\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

When $\lambda \to 0^-$ we recover the converse inequality and the proof is complete. □

A.2 Small subcritical particles $a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$

When the particles are subcritical we go back to our averaging results Theorems 4.5 and 4.11. Due to (A.4), we recover that, when σ is smooth and $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$,

$$\int_0^T \int_\Omega \frac{\partial v}{\partial t}(v-u)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega |\nabla v|^{p-2}\nabla v \nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t + \beta^{\mathrm{eff}} \int_0^T \int_\Omega (\sigma(v) - g^{\mathrm{eff}})(v-u)\,\mathrm{d}S\,\mathrm{d}t$$
$$\geq \int_0^T \int_\Omega f(v-u)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_\Omega \big(u(0,x) - v(0,x)\big)^2\,\mathrm{d}x.$$

This is the weak formulation of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + \beta^{\mathrm{eff}} \sigma(u_\varepsilon) = f + \beta^{\mathrm{eff}} g^{\mathrm{eff}} & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \\ u = u_0 & t = 0, \end{cases}$$

where the effective elements are those in Section 4.5. Of course, in the case of big particles on the whole domain the diffusion operator must be modified in the homogenized problem (see, e. g., [84] for the case $p = 2$ and σ a non-decreasing function as in Section 4.4).

A.3 Supercritical particles $a_\varepsilon \ll a_\varepsilon^\star$ and $p \in (1, n)$

Just like in Section 4.6 we can take the same spatial test function $v_\varepsilon(t,x) = v(t,x)(1 - \psi_\varepsilon(x))$ to “remove” the boundary term and get, if $g^\varepsilon = 0$ and $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$, in the limit,

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = f & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \\ u = u_0 & t = 0. \end{cases}$$

A.4 Critical-size particles $a_\varepsilon \sim a_\varepsilon^\star$

We can still take same oscillating test functions $v_\varepsilon(t,x) = v(t,x) - W_{\sigma,\varepsilon}(x; v(t,\cdot))$ and we get in all the assumptions in Chapter 4 that u is the unique solution of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + \mathcal{H}(x,u) = f & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \\ u = u_0 & t = 0, \end{cases}$$

where $\mathcal{H}(x,u)$ is given by (4.16)–(4.17) when G_0 is a ball and $p < n$ (see [227]), and when G_0 is not a ball through the auxiliary function w_σ.

Remark A.2. We point out that the elliptic problem (1.1) can be regarded also as the stationary problem associated to doubly nonlinear parabolic problems in which the equation is of the form

$$\frac{\partial}{\partial t}\gamma(u_\varepsilon) - \Delta_p u_\varepsilon = f,$$

where γ is a non-decreasing continuous function (or, more in general, a maximal monotone graph of $\mathbb{R}^2$) considered intensively by many authors in the last 50 years (see references, e. g., in [105] and [262]). The homogenization of the case of the one-phase Stefan problem, and $p = 2$, was carried out in [94]. For the case $\gamma(s) = |s|^{1/m}\operatorname{sign}(s)$ and $p = 2$ see [104]. The homogenization of this problem can be also treated with the abstract results of Section 3.9.2 of [16] on the convergence of the associated semi-groups. The study of the free boundary of the obtained homogenization problem, at the critical scale, can be compared with the different behavior of solutions of the problems before homogenizing, leading to some improvements, as in Chapter 4. Note that now the comparison techniques are more delicate (see, e. g., [262]) but some energy methods can be also applied (see [13]).

A.5 A remark on controllability

Once we know the convergence of parabolic problems of the type (A.1) to its respective homogenized parabolic problem (as, e. g., the problem in Section A.4), according to the size of the particles a_ε, many different questions related to the controllability of both problems were investigated in the literature. For instance, in terms of problem (A.1), the so-called "approximate controllability" property of the parabolic problem assumes that one of the data v is variable in a subset of the data (the set of admissible controls, for instance the boundary data on S_ε),

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta_p u_\varepsilon = f & (0,T)\times\Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_p} + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon)v_\varepsilon\chi_{(0,T)\times S_\varepsilon^\omega} & (0,T)\times S_\varepsilon, \\ u_\varepsilon = 0 & (0,T)\times\partial\Omega, \\ u_\varepsilon(x,0) = u_0(x) & \Omega_\varepsilon, \end{cases} \tag{A.6}$$

and the question is to search if, given a "target state" $u_{\varepsilon,T}$, which represents a possible value of the solution (the "observation of the state," for instance the value on Ω of u_ε at $t = T$), and an arbitrary $\delta > 0$, we are able to find a control v_ε such that

$$\|u_\varepsilon(T; v_\varepsilon) - u_{\varepsilon,T}\|_{L^2(\Omega_\varepsilon)} \le \delta. \tag{A.7}$$

When $\delta = 0$ this property is called "exact controllability" and requires to assume the "target state" $u_{\varepsilon,T}$ in a small subset of $L^2(\Omega_\varepsilon)$ due to the regularizing effects appearing in parabolic problems. On the other hand, in practice it can be difficult to act on the whole domain S_ε and so it is convenient to assume that the support of the control v_ε can be reduced to a "small" subset S_ε^ω of S_ε. Most of the available results on approximate controllability are reduced to the case $p = 2$ and deal with semilinear parabolic equations and Dirichlet (or Neumann) boundary conditions (see [193, 140, 125]). The controllability of problem (A.6), i. e., for the case of nonlinear Robin boundary conditions, was proved in Section 3.2 of [120] when σ is assumed to be *sublinear at infinity*, i. e.,

$$\left|\sigma(s)\right| \le C\big(1 + |s|\big) \quad \text{for } |s| > M, \text{ for some } M > 0.$$

A real-life application of the above controllability problem is the following: consider a polluted sand filter occupying some domain Ω (with a fixed flow rate of pollutant). We add a suitable chemical reactant with concentration v_ε (a control) on parts of the surface of the particles. Let $u_\varepsilon(T; v_\varepsilon)$ be the resulting concentration of the pollutant at time $T > 0$. The problem is to find the concentration of reagent v_ε to control the contaminant in a desired way throughout the whole region Ω_ε at this time.

A natural question is to know if the sequence of good controls $\{v_\varepsilon\}$ converges, by an homogenization process, to some global control function $v \in L^2(\Omega)$ allowing to prove the approximate controllability for the homogenized problem. For instance, in the case of a critical scale, the control problem would be of the type

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u_\varepsilon + \mathcal{H}(x,u) = f + v\chi_{(0,T)\times\omega} & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \\ u(x,0) = u_0(x) & \Omega, \end{cases} \tag{A.8}$$

for some subregion $\omega \subset \Omega$ and with a desired state $u_T = \lim_{\varepsilon\to 0} u_{\varepsilon,T}$. A favorable answer to the above question was obtained in [1] for big particles $a_\varepsilon = \varepsilon$ (and thus, essentially, with $\mathcal{H}(x,u) = C\sigma(u)$ and σ sublinear at infinity) and $p = 2$. Several different authors produced previously some related results (always for $p = 2$) for some variants of the above setting (see [130, 129, 90, 91, 180, 88]). We point out that for semilinear parabolic problems as (A.8) it was shown in [99] that if $\mathcal{H}(x,u)$ is superlinear at infinity (i. e., $|\mathcal{H}(x,u)| \ge C(1 + |s|^r)$ for $|s| > M$, for some $r > 1$ and $M > 0$), then an *obstruction phenomenon* arises and the parabolic problem does not satisfy, in general, the approximate controllability property. In consequence, this shows that the result of [1] is optimal in the sense that if we consider big particles $a_\varepsilon = \varepsilon$ and σ superlinear at infinity, then the sequence of controls $\{v_\varepsilon\}$ cannot be convergent to a useful control for the limit problem (since in this case $\mathcal{H}(x,u) = C\sigma(u)$, with σ superlinear at infinity). Nevertheless, the answer may be entirely different if a_ε corresponds to the critical scale since now $\mathcal{H}(x,u)$ is globally Lipschitz continuous and no obstruction phenomenon occurs. This could be of interest in the framework of application to climate models (see, e. g., [99] and [101]). The extension of the above situation to the case $p \ne 2$ remains an open problem.

B Dynamic boundary condition

In contrast to the previous appendix, a much less studied variant of our problem, although very relevant in the applications (see the references below), is the case of the so-called dynamic boundary conditions

$$\begin{cases} -\Delta_p u_\varepsilon = f & (0,T)\times\Omega_\varepsilon, \\ \beta(\varepsilon)\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon}{\partial \nu_p} + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon)g^\varepsilon & (0,T)\times S_\varepsilon, \\ u_\varepsilon = 0 & (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & S_\varepsilon. \end{cases} \tag{B.1}$$

A quite complete list of references dealing with nonlinear problems with dynamic boundary conditions, starting already in 1901, can be found, e. g., in the survey articles [24] and [20]. The PDE is sometimes an elliptic equation (and thus there is a great contrast between a stationary interior law and a dynamic boundary condition). Nevertheless, the dynamic boundary condition may coexist with a parabolic equation. In the context of reaction-diffusion equations, dynamical boundary conditions have been rigorously derived in [149] and [146].

The main goal of this section is to present only some comments on the homogenization arguments to this peculiar type of problems and, more specifically, how to identify the homogenized problem, containing some strange term, in the case of critical-size particles. For more details we refer the reader to the papers on this subject which will be indicated below for each one of the cases which can be presented according the different size of the particles.

In this setting, it can be shown that, under sufficient regularity of the data, we have

$$u_\varepsilon \in L^\infty(0,T;W^{1,p}(\Omega_\varepsilon,\partial\Omega)) \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial t} \in L^2((0,T)\times S_\varepsilon). \tag{B.2}$$

We will only deal here with the case of particles over the whole domain. We leave to the reader to adapt the details in the other geometrical settings. We repeat, briefly, the basic preliminaries. For some homogenization results when the dynamic boundary condition holds on the boundary of particles placed on an interior manifold see [274].

A priori estimates

In this setting the estimates are even a little bit more difficult than in the parabolic setting. There is still a unique weak solution and if $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$ it satisfies

$$\beta^\star(\varepsilon)\|\partial_t u_\varepsilon\|_{L^2((0,T)\times S_\varepsilon)} + \sup_{t\in[0,T]} \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C.$$

The argument passes, as for the parabolic problem, by the abstract theory of subdifferential operators or by Galerkin approximation. We refer the reader to [117] for $p = 2$ and [232] and [10] for the case $p \neq 2$.

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Extension operator

The extension operator we have constructed for the parabolic problem is also valid in this setting and we still recover

$$\begin{aligned} P_\varepsilon u_\varepsilon &\rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\\ P_\varepsilon u_\varepsilon &\to u \quad \text{strongly in } L^p((0,T)\times\Omega). \end{aligned} \tag{B.3}$$

Formulation as a variational inequality

Repeating the arguments in Appendix A for the parabolic problem, the inequality formulation is

$$\begin{aligned} &\int_0^T\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\nabla(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \beta(\varepsilon)\int_0^T\int_{S_\varepsilon}\left(\frac{\partial v}{\partial t}+\sigma(v)-g^\varepsilon\right)(v-u_\varepsilon)\,\mathrm{d}S\,\mathrm{d}t\\ &\quad \ge \int_0^T\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \frac{\beta(\varepsilon)}{2}\int_{S_\varepsilon}\big(u_0(x)-v(0,x)\big)^2\,\mathrm{d}x. \end{aligned} \tag{B.4}$$

B.1 Small subcritical particles $a_\varepsilon^\star \ll a_\varepsilon \ll \varepsilon$

Since the weak formulation contains no troublesome terms when σ is smooth, we can pass to the limit in each term to recover

$$\begin{cases} \beta^{\mathrm{eff}}\frac{\partial u}{\partial t} - \Delta_p u + \beta^{\mathrm{eff}}\sigma(u) = f + \beta^{\mathrm{eff}} g^{\mathrm{eff}} & (0,T)\times\Omega,\\ u = 0 & (0,T)\times\partial\Omega,\\ u = u_0 & t = 0, \end{cases}$$

where $u_0 \in L^2(\Omega)$ and the effective elements are those in Chapter 4.

Of course, as in the case of small particles, the case of big particles ($a_\varepsilon = \varepsilon$) requires to modify the diffusion operator in a similar way as indicated in Section 4.4. That was done in [258, 259, 8, 10, 9] (see also the case of random particles in [265]).

B.2 Supercritical particles $a_\varepsilon \ll a_\varepsilon^\star$

This situation is more delicate. We present only the case where σ is smooth, $g^\varepsilon = 0$, $\beta(\varepsilon) \sim \beta^\star(\varepsilon)$ and $u_0 = 0$. Then, we take $v_\varepsilon(t,x) = v(t,x)(1-\psi_\varepsilon(x))$, which still vanishes in S_ε, and we show the homogenized problem is

$$\begin{cases} -\Delta_p u = f(t,x) & (0,T)\times\Omega,\\ u = 0 & (0,T)\times\partial\Omega. \end{cases}$$

There is no time derivative in the limit problem, and the solution has the regularity in time dictated by f. The initial data are lost.

B.3 Critical particles $a_\varepsilon^\star \sim a_\varepsilon$ and $G_0 = B_1$

This is, as usual, the most surprising and interesting case. The surprise for dynamical boundary conditions is the appearance of a memory term. This was first noted in the linear setting in [117]. For a result on the convergence in the nonlinear setting see [232], but we only want to give here the structure of the proof. We will do only the case $g^\varepsilon = g \in W^{1,\infty}(\Omega)$, but the details can be adapted to the general setting.

As in Section 4.7 take $v_\varepsilon(t,x) = v(t,x) - h(t,x)W_\varepsilon(x)$. We still want to have $h = H(v)$ being the solution of a certain equation. Going back to Remark 4.40, in order to get the suitable cancelation of the integrals over $(0,T) \times S_\varepsilon$, we now get

$$\frac{B_\varepsilon}{\beta(\varepsilon)}|h|^{p-2}h \simeq \frac{\partial v_\varepsilon}{\partial t} + \sigma(v_\varepsilon) - g = \frac{\partial v}{\partial t} - \frac{\partial h}{\partial t} + \sigma(v-h) - g$$

(where B_ε is given in Lemma 4.38). This shows that the functional equation that appeared in the elliptic setting is now replaced by a pointwise ordinary differential equation (ODE). This ODE needs an initial condition. We now go the integral in S_ε on the right-hand side of (B.4). If we want v_ε to cancel the integral given for $x \in S_\varepsilon$ at $t = 0$ we need to request

$$0 = u_0(x) - v_\varepsilon(0,x) = u_0(x) - v(0,x) + h(0,x).$$

But this is a valid choice for h. For a given v smooth enough and every x fixed we take $h(t,x) = H_v(t,x)$ as the unique solution of

$$\begin{cases} \frac{\partial H_v}{\partial t} = \frac{\partial v}{\partial t} - \mathcal{B}_0|H_v|^{p-2}H_v + \sigma(v - H_v) - g & t \in (0,T), \\ H_v(0,x) = v(0,x) - u_0(x). \end{cases} \tag{B.5}$$

Note that the existence and uniqueness of solution of (B.5) hold even if σ is a maximal monotone graph. With this choice of H_v, which does not depend on ε and is regular if v and g are smooth, we can pass to the limit as in Section 4.7.1 to recover

$$\int_0^T\int_\Omega |\nabla v|^{p-2}\nabla v\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{A}_0\int_0^T\int_\Omega |H_v|^{p-2}H_v(v-u)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_\Omega f(v-u)\,\mathrm{d}x\,\mathrm{d}t.$$

To recover the equation again we must take $v = u + \lambda\varphi$. We will deal in Remark B.4 with the fact that the equation for $H_{u+\lambda\varphi}$ requires $\frac{\partial u}{\partial t}$. We will show that by a change in variable, we can avoid this difficulty. The continuous dependence of (B.5) (which we prove in the general setting for $p = 2$ in Lemma B.10) and the usual trick of passing $\lambda \to 0^\pm$ show that this is the weak formulation of

$$\begin{cases} -\Delta_p u + \mathcal{A}_0|H_u|^{p-2}H_u = f & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \end{cases} \tag{B.6a}$$

coupled with the pointwise ODE that, for every $x \in \Omega$, is given as the solution of

$$\begin{cases} \frac{\partial H_u}{\partial t} + \mathcal{B}_0 |H_u|^{p-2} H_u = \frac{\partial u}{\partial t} + \sigma(u - H_u) - g & (0,T) \times \Omega, \\ H_u(0,x) = u(0,x) - u_0(x) & t = 0. \end{cases} \tag{B.6b}$$

Remark B.1. This term H_u behaves as a memory term for the equation of u. Note that for $p = 2$ and σ linear this problem becomes a linear ODE that can be solved explicitly (see [117]).

Remark B.2. It is very interesting to point out that $u(0,x)$ is not necessarily $u_0(x)$, but rather the solution of the elliptic PDE

$$\begin{cases} -\Delta_p u(0,x) + \mathcal{A}_0 |u(0,x) - u_0(x)|^{p-2} (u(0,x) - u_0(x)) = f(0,x) & \Omega, \\ u = 0 & \partial\Omega. \end{cases}$$

Thus, $u(0,x)$ is a *strange initial datum*. Even if $u_0(x) = 0$, the solution of the homogenized problem is not $u(0,x) = 0$ unless $f(0,x) = 0$.

Existence of a solution of (B.6a), (B.6b) comes from our proof on the convergence of the limit. When $p = 2$ and σ is linear the ODE can be explicitly solved (see [117]). In order to show that (B.6) has a unique solution, we show that the *strange term*

$$\mathcal{H}[u] = \mathcal{A}_0 |H_u|^{p-2} H_u$$

is monotone in the following sense.

Lemma B.3. *Let $u, \overline{u} : [0,\infty) \to \mathbb{R}$ be smooth functions. Let H_u and $H_{\overline{u}}$ be the solutions of* (B.6a), (B.6b) *corresponding to u and $\overline{u}$ and, finally, $\mathcal{H}[u] = \mathcal{A}_0 |H_u|^{p-2} H_u$ and $\mathcal{H}[\overline{u}] = \mathcal{A}_0 |H_{\overline{u}}|^{p-2} H_{\overline{u}}$. Then*

$$\int_0^T (\mathcal{H}[u] - \mathcal{H}[\overline{u}])(u - \overline{u})\,\mathrm{d}t \geq 0. \tag{B.7}$$

Proof. This property comes from a direct computation. We write

$$\begin{aligned} &\int_0^T (\mathcal{H}[u] - \mathcal{H}[\overline{u}])(u - \overline{u})\,\mathrm{d}t \\ &\quad = \mathcal{A}_0 \int_0^T (|H_u|^{p-2} H_u - |H_{\overline{u}}|^{p-2} H_{\overline{u}})(u - H_u - (\overline{u} - H_{\overline{u}}))\,\mathrm{d}t \\ &\quad\quad + \mathcal{A}_0 \int_0^T (|H_u|^{p-2} H_u - |H_{\overline{u}}|^{p-2} H_{\overline{u}})(H_u - H_{\overline{u}})\,\mathrm{d}t \end{aligned}$$

$$\geq \int_0^T \left(\frac{\partial(u - H_u - (\overline{u} - H_{\overline{u}}))}{\partial t} + \sigma(u - H_u) - \sigma(\overline{u} - H_{\overline{u}}) \right) (u - H_u - (\overline{u} - H_{\overline{u}}))\, \mathrm{d}t$$
$$\geq \int_0^T \left(\frac{\partial(u - H_u - (\overline{u} - H_{\overline{u}}))}{\partial t} \right) (u - H_u - (\overline{u} - H_{\overline{u}}))\, \mathrm{d}t$$
$$= \frac{1}{2} (u - H_u - (\overline{u} - H_{\overline{u}}))^2 \Big|_0^T$$
$$\geq -\frac{1}{2} (u(0) - H_u(0) - \overline{u}(0) - H_{\overline{u}}(0))^2 = 0.$$

This completes the proof. □

With this property, given two solutions u and $\overline{u}$ of (B.6a), (B.6b), we use $u - \overline{u}$ as a test function in the weak formulation of (B.6a) and recover

$$\int_0^T \int_\Omega (|\nabla u|^{p-2} \nabla u - |\nabla \overline{u}|^{p-2} \nabla \overline{u}) \cdot \nabla(u - \overline{u})\, \mathrm{d}x\, \mathrm{d}t$$
$$\leq \int_0^T \int_\Omega (|\nabla u|^{p-2} \nabla u - |\nabla \overline{u}|^{p-2} \nabla \overline{u}) \cdot \nabla(u - \overline{u})\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega (\mathcal{H}[u] - \mathcal{H}[\overline{u}])(u - \overline{u})\, \mathrm{d}x\, \mathrm{d}t = 0.$$

This guarantees that $\nabla u = \nabla \overline{u}$ in $(0, T) \times \Omega$. Since $u, \overline{u} = 0$ on the boundary, we recover that $u = \overline{u}$.

In Lemma B.10 below, we show for $p = 2$ the continuous dependence of H with respect to u, even when G_0 is not a ball. We do not present here the proof for $p \neq 2$, which can be found in [232].

Remark B.4. Note that in (B.6b) the equation contains a $\frac{\partial u}{\partial t}$ that could be hard to locate in a functional space. One could take $\overline{H}_u = u - H_u$ which has as equation

$$\begin{cases} \frac{\partial \overline{H}_u}{\partial t} + \sigma(\overline{H}_u) = \mathcal{B}_0 |u - \overline{H}_u|^{p-2}(u - \overline{H}_u) + g & (0, T) \times \Omega, \\ \overline{H}_u = u_0 & t = 0. \end{cases}$$

This problem no longer depends on $\frac{\partial u}{\partial t}$. The continuous dependence can be proved in this setting.

Remark B.5. If we consider mixed time derivatives in both Ω_ε and S_ε, we can write in general the parabolic problem

$$\begin{cases} a \frac{\partial u_\varepsilon}{\partial t} - \Delta_p u_\varepsilon = f & (0, T) \times \Omega_\varepsilon, \\ b\beta(\varepsilon) \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon}{\partial \nu_p} + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon) g^\varepsilon & (0, T) \times S_\varepsilon, \\ u_\varepsilon = 0 & (0, T) \times \partial\Omega, \\ a\, u(0, \cdot) = a\, u_0 & \Omega_\varepsilon, \\ b\, u(0, \cdot) = b\, u_0 & S_\varepsilon, \end{cases}$$

for $a, b \geq 0$. Then the limit problem becomes

$$\begin{cases} a\frac{\partial u}{\partial t} - \Delta_p u + \mathcal{A}_0 |H_u|^{p-2} H_u = f & (0,T)\times\Omega, \\ b\frac{\partial H_u}{\partial t} + \mathcal{B}_0 |H_u|^{p-2} H_u = b\frac{\partial u}{\partial t} + \sigma(u - H_u) - g & (0,T)\times\Omega, \\ u = 0 & (0,T)\times\partial\Omega, \\ a\, u(0,x) = a\, u_0(x) & \Omega, \\ b\, H_u(0,x) = b(u(0,x) - u_0(x)) & \Omega. \end{cases}$$

The values a and b are added to the initial condition so that they become trivial when a or b vanishes. If $a > 0$, then the last condition is just $bH_u = 0$. However, if $a = 0$ we can recover our *strange initial datum*, pointed out above. Due to a misprint, in [117, equation (1.5)], the initial condition for H_u is written exclusively $bH_u = 0$. However, in that paper $\sigma(u) = \lambda u$ and the correct explicit value of H_u is provided [117, equation (2.10)] as well as the equation for u when $a = 0$ is written at the mentioned paper.

B.4 Critical particles $a_\varepsilon^\star \sim a_\varepsilon$ when $p = 2$ and general G_0

Repeating the argument in Section 4.7.3 we must pick our function $\widehat{w}_\sigma$ so that we get a cancelation of the already famous term $I_{2,\varepsilon}$ from Lemma 4.45. Now, instead of a single value $s \in \mathbb{R}$, we must be able to input a time-dependent function $\phi : [0,\infty) \to \mathbb{R}$. Using a similar argument as before, we recover that the auxiliary function $\widehat{w}(t,y;x,\phi)$ should solve, for each $x \in \Omega$,

$$\begin{cases} \Delta \widehat{w}_\sigma = 0 & (0,\infty)\times\mathbb{R}^n \setminus \overline{G_0}, \\ C_0 \frac{\partial \widehat{w}_\sigma}{\partial t} + \frac{\partial \widehat{w}_\sigma}{\partial \nu} = C_0 \frac{\mathrm{d}\phi}{\mathrm{d}t} + C_0 \sigma(\phi - \widehat{w}_\sigma) - C_0 g(x) & (0,\infty)\times\partial G_0, \\ \widehat{w}_\sigma \to 0 & \text{as } |y| \to +\infty \text{ and } t > 0, \\ \widehat{w}_\sigma(t,0) = \phi(0) - u_0 & \text{at } t = 0, \end{cases} \tag{B.8}$$

and

$$H[x,\phi](t) = \int_{\partial G_0} \partial_\nu \widehat{w}_\sigma(t,y;x,\phi)\,\mathrm{d}S_y.$$

In the homogenized equation we have

$$\begin{cases} -\Delta u(t,x) + A_0 H[x, u(\cdot,x)](t) = f & (0,\infty)\times\Omega, \\ u = 0 & (0,\infty)\times\partial\Omega. \end{cases}$$

The initial data are, as above, encoded in H.

Remark B.6. Note that when $G_0 = B_1$, then $\widehat{w}_\sigma(t,y;x,\phi) = H_\phi(t,x)\widehat{w}(y) = H_\phi(t,x)\widehat{\kappa}(y)$, where $\widehat{\kappa}$ is given by (3.20).

Again, to prove uniqueness of solution of the homogenized problem, it is sufficient to prove H is a monotone operator. We state the following lemma.

Lemma B.7. *Let $\phi, \overline{\phi} \in C^1([0,+\infty))$. Then, for each x fixed*

$$\int_0^T \big(H[x,\phi] - H[x,\overline{\phi}]\big)(\phi - \overline{\phi})\,\mathrm{d}t \geq 0.$$

Proof. For the sake of convenience with respect to the length of the proof, let us denote $w = \widehat{w}_\sigma[\phi]$ and $\overline{w} = \widehat{w}_\sigma[\overline{\phi}]$. We have

$$\begin{aligned}
\int_0^T \big(H[\phi] - H[\overline{\phi}]\big)(\phi - \overline{\phi})\,\mathrm{d}t &= \int_0^T \int_{\partial G_0} \partial_\nu (w - \overline{w})(\phi - \overline{\phi})\,\mathrm{d}S\,\mathrm{d}t \\
&= \int_0^T \int_{\partial G_0} \partial_\nu (w - \overline{w})\big(\phi - w - (\overline{\phi} - \overline{w})\big)\,\mathrm{d}S\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\partial G_0} \partial_\nu (w - \overline{w})(w - \overline{w})\,\mathrm{d}S\,\mathrm{d}t \\
&= C_0 \int_0^T \int_{\partial G_0} \left(\frac{\partial(\phi - w - (\overline{\phi} - \overline{w}))}{\partial t}\right)\big(\phi - w - (\overline{\phi} - \overline{w})\big)\,\mathrm{d}S\,\mathrm{d}t \\
&\quad + C_0 \int_0^T \int_{\partial G_0} \big(\sigma(\phi - w) - \sigma(\overline{\phi} - \overline{w})\big)\big(\phi - w - (\overline{\phi} - \overline{w})\big)\,\mathrm{d}S\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^n \setminus \partial G_0} |\nabla (w - \overline{w})|^2\,\mathrm{d}S\,\mathrm{d}t \\
&\geq 0,
\end{aligned}$$

as we did in Lemma B.3. □

The continuity of H is proved similarly to previous cases.

Remark B.8. As above, if one does not wish to use time derivatives so that the operator can be applied to functions which are just $L^2(0,T;L^2(\partial G_0))$, then the change in variable $\overline{w}_\sigma = \phi\widehat{\kappa} - \widehat{w}_\sigma$ is the solution of

$$\begin{cases}
\Delta \overline{w}_\sigma = 0 & (0,\infty) \times \mathbb{R}^n \setminus \overline{G_0}, \\
C_0 \frac{\partial \overline{w}_\sigma}{\partial t} + \frac{\partial \overline{w}_\sigma}{\partial \nu} + C_0 \sigma(\overline{w}_\sigma) = \phi \frac{\partial \widehat{\kappa}}{\partial \nu} + C_0 g & (0,\infty) \times \partial G_0, \\
\overline{w}_\sigma \to 0 & \text{as } |y| \to +\infty \text{ and } t > 0, \\
\overline{w}_\sigma = u_0 & \text{at } t = 0.
\end{cases}$$

Note that

$$H[x,\phi](t) = \int_{\partial G_0} (\phi \partial_\nu \hat{\kappa} - \partial_\nu \overline{w}_\sigma)\, \mathrm{d}S = \phi(t)\lambda_{G_0} - \int_{\partial G_0} \partial_\nu \overline{w}_\sigma(t,y;x,\phi)\, \mathrm{d}S_y. \tag{B.9}$$

Similarly as we did to H, it is not too difficult to show the continuous dependence. First, we do this in terms of w_σ.

Lemma B.9. *We have*

$$\int_0^T \int_{\mathbb{R}^n \setminus G_0} |\nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])|^2 \, \mathrm{d}S\, \mathrm{d}t \le \lambda_{G_0} \int_0^T |\phi - \bar{\phi}|^2 \, \mathrm{d}t.$$

Proof. We check that

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus G_0} |\nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])|^2 \, \mathrm{d}x &= \int_{\mathbb{R}^n \setminus G_0} \nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])\, \mathrm{d}x \\
&= \int_{\partial G_0} (\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \partial_\nu (\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])\, \mathrm{d}S \\
&= -\frac{C_0}{2} \int_{\partial G_0} \frac{\partial}{\partial t} (\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])^2 \, \mathrm{d}S \\
&\quad - C_0 \int_{\partial G_0} (\sigma(\overline{w}_\sigma[\phi]) - \sigma(\overline{w}_\sigma[\overline{\phi}]))(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])\, \mathrm{d}S \\
&\quad + \int_{\partial G_0} (\phi - \bar{\phi})(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \partial_\nu \hat{\kappa}\, \mathrm{d}S.
\end{aligned}$$

Integrating in $[0,T]$,

$$\int_0^T \int_{\mathbb{R}^n \setminus G_0} |\nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])|^2 \, \mathrm{d}x\, \mathrm{d}t \le \int_0^T \int_{\partial G_0} (\phi - \bar{\phi})(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \partial_\nu \hat{\kappa}\, \mathrm{d}S\, \mathrm{d}t.$$

Note that

$$\begin{aligned}
\int_{\partial G_0} (\phi - \bar{\phi})(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \partial_\nu \hat{\kappa}\, \mathrm{d}S &= (\phi - \overline{\phi}) \int_{\mathbb{R}^n \setminus G_0} \nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \nabla \hat{\kappa}\, \mathrm{d}x \\
&\le |\phi - \overline{\phi}| \Big(\int_{\mathbb{R}^n \setminus G_0} |\nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n \setminus G_0} |\nabla \hat{\kappa}|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \\
&\le \frac{1}{2} \int_{\mathbb{R}^n \setminus G_0} |\nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])|^2 \, \mathrm{d}x + \frac{1}{2} |\phi - \overline{\phi}|^2 \int_{\mathbb{R}^n \setminus G_0} |\nabla \hat{\kappa}|^2 \, \mathrm{d}x.
\end{aligned}$$

Therefore, we have

$$\int_0^T \int_{\mathbb{R}^n \setminus G_0} |\nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \int_{\mathbb{R}^n \setminus G_0} |\nabla \widetilde{\kappa}|^2 \, \mathrm{d}x \int_0^T |\phi - \bar{\phi}|^2 \, \mathrm{d}t.$$

Lastly, we point out that this constant is precisely the capacity (3.21). □

This allows us to recover the continuous dependence of H.

Lemma B.10. *We have*

$$\int_0^T |H[\phi] - H[\overline{\phi}]|^2 \, \mathrm{d}t \leq 2\lambda_{G_0}^2 \int_0^T |\phi - \overline{\phi}|^2 \, \mathrm{d}t.$$

Proof. We recall first that (B.9). Now

$$\begin{aligned}
\int_{\partial G_0} \partial_\nu (\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \, \mathrm{d}S &= \int_{\partial G_0} \widetilde{\kappa} \partial_\nu (\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \, \mathrm{d}S \\
&= \int_{\mathbb{R}^n \setminus G_0} \nabla \widetilde{\kappa} \nabla (\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}]) \, \mathrm{d}x \\
&\leq \lambda_{G_0}^{\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus G_0} |\nabla(\overline{w}_\sigma[\phi] - \overline{w}_\sigma[\overline{\phi}])|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned}$$

Taking power 2 and integrating we recover the result. □

Remark B.11. It is likely that the 2 in the previous estimate can be avoided through sharper analysis.

C Critical-size particles with a stochastic perturbation

This is a research subject which is currently being very active. Such as we have indicated before, there are many mathematical books on periodic homogenization, and some of them already consider a stochastic framework (see, e. g., [31, 175]), developing pioneering works by Papanicolaou, Varadhan, Kozlov, Yurinskij and others in the 1970s. Nevertheless new methods and many different applications are presently being proposed for a large array of models (see, e. g., [37, 67, 33, 137, 243, 257] and [260], among many other references).

In this appendix we will follow a concrete approach (the assumption of "stationary and ergodic" random media) which was initiated by Dal Maso and Modica [93] (in their paper they acknowledge a suggestion from L. Russo). More specifically we will illustrate the occurrence of a strange term, for the *critical size* of the particles, in the context of stochastic homogenization applied to obstacle type problems according mainly to the papers by Caffarelli and Mellet [60] (obstacle problem for $p = 2$) and by Tang [253] (obstacle problem with $p \neq 2$). See also [61].

We start by introducing a different notation with respect to the rest of the book: here the spatial domain is denoted by D and not Ω, since this symbol is traditionally used in the context of probability to denote a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ which we assume given in this appendix. Here, for each $\omega \in \Omega$ and $\varepsilon > 0$, the set of random particles is denoted by $G_\varepsilon(\omega)$, so that it defines its complementary set $D_\varepsilon(\omega) = D \setminus G_\varepsilon(\omega)$ where the diffusion-reaction process takes place.

The technique introduced by Caffarelli-Mellet for $p = 2$ allows us to work with

$$G_\varepsilon(\omega) = \left(\bigcup_{j \in \mathbb{Z}^n} G_{\varepsilon,j}(\omega) \right) \cap D. \tag{C.1}$$

In the theory introduced by Caffarelli–Mellet the particles are still essentially periodically placed, since they assume

$$G_{\varepsilon,j}(\omega) \subset B_{a_\varepsilon}(\varepsilon j),$$

where $a_\varepsilon \sim a_\varepsilon^\star$. For consistency with our previous notation we take

$$\Upsilon_\varepsilon = \{ j \in \mathbb{Z}^n : G_{\varepsilon,j}(\omega) \cap D \neq \emptyset \}.$$

The key assumption in passing to the limit is that the particles have similar capacities,

$$\operatorname{cap}\big(G_{\varepsilon,j}(\omega)\big) = \varepsilon^n \gamma(j, \omega).$$

Recall the definition of the important notion of capacity given in Remark 3.11.

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The process $\gamma : \mathbb{Z}^n \times \Omega \to [0, +\infty)$, a process that relates these capacities, needs to be bounded,

$$0 < \underline{\gamma} \leq \gamma(j, \omega) \leq \overline{\gamma}, \tag{C.2}$$

such that there exists a family of measure-preserving transformations $\tau_j : \Omega \to \Omega$ satisfying

$$\gamma(j + j', \omega) = \gamma(j, \tau_{j'}(\omega)), \quad \forall j, j' \in \mathbb{Z}^n \text{ and } \omega \in \Omega \tag{C.3}$$

and such that if $A \subset \Omega$ and

$$\tau_j(A) = A \quad \text{for all } j \in \mathbb{Z}^n, \text{then } P(A) \in \{0, 1\}. \tag{C.4}$$

The approach by Tang [253] for $p \in (1, n]$ is geometrically more modest and assumes (C.1) where only

$$G_{\varepsilon,j}(\omega) = B_{a_\varepsilon(j,\omega)}(\varepsilon j). \tag{C.5}$$

Hence, the position εj and shape (a ball) are prescribed, but the radius of this ball is stochastic. Still, the assumption is

$$p\text{-cap}(G_{\varepsilon,j}(\omega)) = \varepsilon^n \gamma(j, \omega), \tag{C.6}$$

such that conditions (C.2)–(C.4) hold. For $p = n$ we recall Remark 3.13, and by n-capacity we mean the relative n-capacity with respect to B_1. Note that the capacity of a ball can be explicitly computed and given by a monotone function F as

$$a_\varepsilon(j, \omega) = F(\varepsilon^n \gamma(j, \omega)). \tag{C.7}$$

Solving explicitly, this assumption means that $a_\varepsilon \sim a_\varepsilon^\star$.

We will describe the results for the following model problem (given $1 < p \leq n$ and $\omega \in \Omega$):

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & x \in D_\varepsilon(\omega), \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = 0 & x \in S_\varepsilon(\omega), \\ u_\varepsilon = 0 & \partial D_\varepsilon(\omega) \setminus S_\varepsilon(\omega), \end{cases} \tag{C.8}$$

where σ is the maximal monotone graph of $\mathbb{R}^2$ associated to the Signorini microscopic boundary conditions,

$$\sigma(r) = \begin{cases} 0 & \text{if } r > 0, \\ [0, +\infty) & \text{if } r = 0, \\ \emptyset & \text{if } r < 0. \end{cases}$$

Note that now, in this special case, the value of $\beta(\varepsilon)$ is irrelevant. Nevertheless we keep this formulation to maintain the coherence with the formulation maintained in previous chapters of this book. Of course this is not the case when σ is a continuous function (see, especially, [183], where the case of a Lipschitz function σ and $p = 2$ was considered by following a different technique).

C.1 Some comments on the ergodicity hypothesis

Let us stress the fact that the assumption (C.6) is not directly made on the shape of the particles but on their capacity. In this sense, the shape of the particles is left unspecified and may change with ε [60]. Note also that (C.6) implies that the diameters of the particles decrease faster than ε, which implies that the capacities of neighboring sets separate at the limit, and we can recover

$$p\text{-cap}\Big(\bigcup_{j\in\Upsilon_\varepsilon} G_{\varepsilon,j}(\omega)\Big) \sim \sum_{j\in\Upsilon_\varepsilon} p\text{-cap}(G_{\varepsilon,j}(\omega)) = \varepsilon^n \sum_{j\in\Upsilon_\varepsilon} \gamma(j,\omega). \tag{C.9}$$

Since the particles are spread over the whole domain we conclude that $|\Upsilon_\varepsilon| \sim \varepsilon^{-n}$.

Note that the process γ can be understood as a dynamical system over the set of indexes $\mathbb{Z}^n$ (instead of the real interval $[0,+\infty)$, as is usual in ODEs; see [175]). In this sense, the word "stationary" simply means that the random variable defined by $\gamma(j,.) : \Omega \to [0,+\infty)$ is independent of $j \in \mathbb{Z}^n$ (i. e., for all $a \in [0,+\infty)$, $P(\{\omega \in \Omega : \gamma(j,.) > a\}$ is independent of $j \in \mathbb{Z}^n$). This condition is the most general extension of the periodicity assumption made in the precedent chapters.

The ergodicity part (if $A \subset \Omega$ and $\tau_j(A) = A$ for all $j \in \mathbb{Z}^n$, then $P(A) = 1$ or $P(A) = 0$) means that the translation-invariant subsets of Ω have either full or zero measure [2].

We refer the reader to the presentations made in [257, 3] and [33] for some basic examples such as the *random checkerboard* and the *Poisson cloud*. Moreover, it is easy to see that a deterministic periodic location of particles of the same shape (as in previous chapters of this book) can be associated to a family of measure-preserving transformations satisfying the stationary and ergodic assumptions (see, e. g., Example 8.1 of [33]). It is also convenient to recall the Birkhoff theorem [36]. Define the *spatial average* of any given function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$\langle f\rangle_x = \lim_{\rho\to\infty} \frac{1}{\rho^n|K|} \int_{\rho K} f(x)\,\mathrm{d}x,$$

with K any arbitrary compact set of $\mathbb{R}^n$. Consider now $f(x,\omega) = \tilde{f}(\gamma(x)\omega)$, with $\tilde{f} \in L^1(\Omega,P)$, a stationary random field, with $\gamma(x)$ a group of measure-preserving transformations. Then if γ is ergodic, the spatial average is the same as the average over Ω (the expectation), i. e.,

$$\big\langle \tilde{f}(\gamma(x)\omega)\big\rangle_x = E[\tilde{f}],$$

for almost all $\omega \in \Omega$ and, in particular, it does not depend on the realization ω.

C.2 Convergence results

The result in [60] and [253] states the following. Let $u_\varepsilon(x,\omega)$ be the minimizer of the energy functional

$$J(v) = \frac{1}{p}\int_D |\nabla v|^{p-2}\,\mathrm{d}x - \int_D fv\,\mathrm{d}x, \tag{C.10}$$

in the convex set

$$K(\varepsilon,\omega) = \{v \in W_0^{1,p}(D) : v \geq 0 \text{ in } G_\varepsilon(\omega)\}.$$

Then,

$$u_\varepsilon(\cdot,\omega) \rightharpoonup u \quad \text{in } W_0^{1,p}(D) \text{ almost surely in } \omega \in \Omega,$$

where $u \in W_0^{1,p}(D)$ is the minimizer of the energy functional

$$J^{\text{eff}}(v) = \frac{1}{p}\int_D |\nabla v|^{p-2}\,\mathrm{d}x + \int_D \frac{\alpha_0}{p}(v_-)^p\,\mathrm{d}x - \int_D fv\,\mathrm{d}x,$$

for some $\alpha_0 > 0$. In other words, the effective equation for the obstacle problem includes a strange term of the form

$$\mathcal{H}(s) = \mathcal{A}_0(s_-)^{p-1}.$$

This reproduces the behavior of the periodical case (see Example 4.34 (c)) with $\mathcal{A}_0$ a constant related to the capacity (see Remark 4.31).

C.3 Auxiliary test function

The key point of the argument is based on the construction of an adequate auxiliary function like those in Sections 3.1.5 and 4.7. However, instead of taking one function w_ε and reproducing it by periodicity, they tackle W_ε directly.

Lemma C.1 ([60, 253]). *Assume that $G_\varepsilon(\omega)$ satisfies the hypothesis above. Then, there exist a positive real number α_0 and a function $W_\varepsilon(x,\omega)$ such that, for a. s. $\omega \in \Omega$,*

$$\begin{cases} -\Delta_p W_\varepsilon = \alpha_0 & \textit{in } D_\varepsilon(\omega), \\ W_\varepsilon(x,\omega) = 1 & \textit{in } G_\varepsilon(\omega), \\ W_\varepsilon(x,\omega) = 0 & \textit{on } \partial D_\varepsilon(\omega) \setminus G_\varepsilon(\omega), \end{cases}$$

with

$$W_\varepsilon(.,\omega) \rightharpoonup 0 \textit{ weakly in } W_0^{1,p}(D(\omega)), \textit{ as } \varepsilon \to 0.$$

Moreover, W_ε satisfies the following properties:

(a) *for any $\phi \in \mathcal{D}(D)$ and $0 < q < p$,*

$$\lim_{\varepsilon\to 0}\int_D |\nabla w_\varepsilon|^q \phi\,\mathrm{d}x = 0;$$

(b) *for any* $\phi \in \mathcal{D}(D)$,

$$\lim_{\varepsilon\to 0} \int_D |\nabla w_\varepsilon|^p \phi \,\mathrm{d}x = \alpha_0 \int_D \phi \,\mathrm{d}x;$$

(c) *for any sequence* $\{v_\varepsilon(., \omega)\}$ *in* $W_0^{1,p}(D)$ *with the property* $v_\varepsilon \rightharpoonup v$ *weakly in* $W_0^{1,p}(D)$, *as* $\varepsilon \to 0$ *and* $v_\varepsilon = 0$ *on* $G_\varepsilon(\omega)$, *and for any* $\phi \in \mathcal{D}(D)$, *we have*

$$\lim_{\varepsilon\to 0} \int_D |\nabla w_\varepsilon|^{p-2} \nabla W_\varepsilon \cdot \nabla v_\varepsilon \phi \,\mathrm{d}x = -\alpha_0 \int_D v\phi \,\mathrm{d}x.$$

The choice of the value α_0 is strongly related to the computation (C.9). We refer the reader to either text for the details.

C.4 Structure of the proof

The structure of the proof is based on some uniform estimates, for each ω. Thus, there must exist a weak limit (which could, in principle, depend on ω). The characterization of the limit is done by a Γ-convergence type argument similar to [80, 81, 79].

The first step is proving that letting u be the weak limit of u_ε, we have

$$J^{\mathrm{eff}}(u) \le \liminf_{\varepsilon\to 0} J(u_\varepsilon).$$

For $p = 2$ in [60] the authors use [81, Proposition 3.1], whereas for $p \neq 2$ the proof in [253] is direct. Using the corrector term the authors prove that for smooth test functions v, we have

$$\lim_{\varepsilon\to 0} J(v + v_- W_\varepsilon) = J^{\mathrm{eff}}(v).$$

The argument is completed by mixing these two limits:

$$J^{\mathrm{eff}}(u) \le \liminf_{\varepsilon\to 0} J(u_\varepsilon) \le \liminf_{\varepsilon\to 0} J(v + v_- W_\varepsilon) = J^{\mathrm{eff}}(v).$$

Thus, u is the minimizer of J^{eff}. Since this minimization problem does not depend on ω, neither does the limit ω.

C.5 Final comments

As shown above we have an *anomalous homogenization* with the presence of a "strange term" $(\sigma_0(u_-)^{p-1})$ under the critical size assumption (C.7). Note that what is happening here is that the limit u of functions $u_\varepsilon(., \omega)$, which are non-negative over big subregions of $D(\omega)$ (to be more precise, on the union $G_\varepsilon(\omega)$ of many small balls), may become negative on a region of D where $f(x)$ is very negative. More precisely, we have the following proposition.

Proposition C.2. *Let $f \in L^2(\Omega)$ such that the unique solution of* (C.8) *is such that $u \in L^\infty(\Omega)$. Let $\delta > 0$ and assume that the set $\Omega_{f,\Lambda} := \{x \in \Omega : f(x) \leq \Lambda < 0\}$ is not empty for some*

$$\Lambda < -\alpha_0 \delta^{p-1}. \tag{C.11}$$

Then $u(x) \leq -\delta$ for a. e. $x \in \Omega_{f,\Lambda}$ such that $d(x, \partial\Omega_{f,\Lambda}) \geq R$ with

$$R = \left(\frac{\|u\|_{L^\infty(\Omega)} + \delta}{C} \right)^{\frac{p-1}{p}}, \quad C = \frac{(p-1)(\alpha_0 \delta^{p-1} - \Lambda)^{\frac{1}{p-1}}}{pN^{\frac{1}{p-1}}}. \tag{C.12}$$

Proof. Given $x_0 \in \Omega_{f,\Lambda}$ we will use the local barrier function

$$\overline{u}(x; x_0) = C|x - x_0|^{\frac{p}{p-1}} - \delta,$$

with $C > 0$ to be chosen later. We have (see, e. g., Remark 2.7 of [102])

$$-\Delta_p \overline{u} = -C^{p-1} \frac{p^{(p-1)}N}{(p-1)^{(p-1)}}.$$

Thus

$$-\Delta_p \overline{u} + H(\overline{u}) = -\Delta_p \overline{u} - \sigma_0 |r|^{p-2} r_- \geq -C^{p-1} \frac{p^{(p-1)}N}{(p-1)^{(p-1)}} - \alpha_0 \delta^{p-1} \geq \Lambda \geq f(x) \quad \text{on } \Omega_{f,\Lambda},$$

if we assume C given by (C.12), thanks to the assumption (C.11). Then, if $B_R(x_0) \subset \Omega_{f,\Lambda}$ we get that $\overline{u}(x; x_0)$ will be a local supersolution assumed that

$$CR^{\frac{p}{p-1}} - \delta \geq \|u\|_{L^\infty(\Omega)}.$$

This is satisfied once we take R given by (C.12). Then, by the comparison principle we get

$$u(x) \leq C|x - x_0|^{\frac{p}{p-1}} - \delta \quad \text{a. e. on } B_R(x_0), \tag{C.13}$$

which implies the result (note that if $u \in C^0(B_R(x_0))$ we get from (C.13) that $u(x_0) \leq -\delta$). □

Remark C.3. The above proposition (which seems to have been unadvertised before in the literature) represents a mathematical rigorous proof of behaviors comparable to the experiments made for some new materials as the so-called "mechanical meta-materials": some artificial structures with mechanical properties defined by their structure rather than their composition. They can be seen as a counterpart to the rather well-known family of "optical meta-materials." Their mechanical properties can be designed to have values which cannot be found in nature (see, e. g., the survey paper [251]). For a rigorous mathematical approach to "optical meta-materials" see, e. g., [187].

Remark C.4. As mentioned in Chapter 6, the homogenization for particles on the boundary is related to the homogenization of equations given by suitable fractional operators. In the case of random particles it was considered by Caffarelli and Mellet in [62]. They prove that if the fractional operator is $(-\Delta)^s$, with $s \in (0,1]$, then the critical exponent is now $\frac{n}{n-2s}$.

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Index

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