A TEXTBOOK
ON

ENGINEERING MATHEMATICS

Ramakanta Meher

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Dedicated to My Parents

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Front Matter

I take great pleasure in presenting this book of engineering mathematics to the students of Engineering colleges. It is prepared in accordance with the syllabus of Bachelor's degrees in Engineering and polytechnic colleges. It has been prepared by keeping the modern method of education in mind as well as the aptitude and attitude of the students to participate in various competitive examinations. In this book, the concepts are explained in a lucid manner that makes the teaching and learning process more easy and effective. Each chapter has been prepared with strenuous efforts to present the principles of the subject in the easiest manner to understand and to work out the sum of each topic of the book. Similarly, each chapter has been started with an introduction, definitions, theorems, explanation and solved examples for the better understanding of concepts. I hope that this book serves the purpose of keeping in mind the changing needs of the society to make it lively and vibrating. The language used is very clear and simple, which is up to the level of comprehension of students.

This book has been divided in to seven chapters. Chapter-1 discusses the convergence of sequences and series and it also includes Monotonic increasing and decreasing

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sequence, Convergence of sequences and infinite series, different test for Convergence such as Comparison Test for Non-negative Series , Integral test, Ratio Test, Root Test, Rabbe’s Test, Logarithmic Test, Leibnitz’s test for Alternating series and Absolute and Conditional Convergence . Chapter-2 discusses the Fourier series expansion which includes Periodic function, Fourier Coefficients ,Validity of the Fourier series, Function with any period T and Half Range Expansions. Chapter-3 discusses the Fourier transform that includes Fourier Transform, Fourier Sine Transform, Fourier Cosine Transform, Properties of Fourier Transform and Solution of PDE Using Fourier transform. Chapter-4 discusses Matrices along with its properties that includes Operation with matrices, the law of matrix algebra, the inverse of a square matrix, procedure for finding an inverse of a matrix, and matrix multiplication in terms of columns. Chapter-5 discusses the system of linear equations that includes some special matrices i.e. Idempotent and Nilpotent Matrices and Elementary matrices , procedure for finding the inverse of a matrix, Non-Homogenous linear system, Elementary Transformation and Row Operations, Echelon Form and Reduced Row Echelon Form, Gaussian Elimination and Gaussian-Jordan Elimination, Criteria for

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Consistency and Uniqueness, Elementary row operations and row echolen form, Comparison of Gaussian and Gaussian – Jordan Elimination and Method of LU decomposition of a Matrix. Chapter-6 discusses the vector spaces and subspaces that includes Linear Combinations, Spanning a Vector space, Generating a vector space, Linear dependence and independence, properties of Bases and Basis and Dimensions. Chapter-7 discusses the Eigen values and Eigen vectors that includes Caylay-Hamilton theorem, Minimal Polynomials, Properties of Eigen values and Eigen vectors, Characteristic polynomial of Block Matrices, Minimal Polynomial and Block diagonalization, Triangular Form, Jordan Canonical Form and Rational Canonical Form.

I extend my deep sense of gratitude to my wife Premanjali and my cutest daughters Anushree and Aradhya and family members who took their sincere efforts in preparing this book.

R. Meher

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Chapter-7 1-32

Eigenvalues and Eigenvectors

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R.Meher

Chapter-1

Sequences and Infinite series

1.1. Sequences:

It is an arrangement of numbers that follows certain rule:

A sequence of numbers is a function with its domain as the set of positive integers. If the numbers are real, then we say, the sequence is a real sequence.

In set notation, one can write the sequence as $\left(n, u(n)\right)$, domain being $1,2,3,....,$, the set of positive integers N. As the domain is the same for every sequence, We often denote the sequence as $u_1, u_2, u_3, ..., u_n, ..$ or simply $\{u_n\}$.

Examples of real sequences are

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}, ..$ (Decreasing sequence)

$2, 0, 2, 0, 2, ..., 1+(-1)^{n+1}, ...$ (Oscillating sequence)

$1, 2, 4, 8, ..., 2^{n-1}, ...$ (Increasing sequence)

If we define an infinite series i.e. the sum of infinite number of sequences i.e. $u_1, u_2, u_3, ...u_n, ...$ i.e.

$$s_\infty = u_1 + u_2 + u_3 + ... + u_n + ...$$

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If we restrict the number of sequences up to nth term, then it becomes an nth partial sum i.e.

$$s_n = u_1 + u_2 + u_3 + ... + u_n$$

Bounded sequence:

Bounded Below

In bounded below, we know the lower term i.e. $1, 2, 3, 4, \ldots$,Where, $1 < 2, 1 < 3, 1 < 4, ..$

Here the lower term is 1 which is the bounded below term.

Bounded above

Consider a sequence $s_n = -n^2$ for all $n \in N$.

$$\begin{aligned} s_n &= \{-1^2, -2^2, -3^2, ...\} \\ &= \{-1, -4, -9, ...\} \end{aligned} \text{,Where } -4 < -1, -9 < -1, ...$$

Every other elements are less than 1,so -1 is the bounded above element.

Bounded:

A sequence which is both bounded above and bounded below is called a bounded sequence.

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Example: -1

$\langle s_n \rangle = \left\langle \frac{1}{n} \right\rangle$ for all $n \in N$.

$\langle s_n \rangle = \left\{ 1, \frac{1}{2}, \frac{1}{3}, ..., 0 \right\}$

It is a bounded sequence as 1 is bounded above and 0 is bounded below element.

Range $= \{0,1\}$

Example: -2

$\langle s_n \rangle = (-1)^n$

$\langle s_n \rangle = \{-1, 1, -1, 1, ...\}$

Range: $\{-1,1\}$

It is also an oscillatory sequence as it oscilates between -1 and 1.

1.2. Monotonic increasing and decreasing sequence:

A Sequence which is in increasing order is called a monotonic increasing sequence. Similarly, a sequence which is in decreasing order is called a monotonic decreasing sequence.

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For example:

$1,2,3,4.....$ monotonic increasing sequence

i.e. $u_n < u_{n+1}$.

$1,\frac{1}{2},\frac{1}{3},......$ monotonic decreasing sequence

i.e. $u_n > u_{n+1}$.

1.3. Convergence of sequences:

A sequence $u_1,u_2,u_3,....$ is said to converge to l, if for every positive ε (no matter how small, but not zero), we can find an integer M such that $|u_n - l| < \varepsilon$ for all $n > M$.

l is called the limit of the sequence and we write it as $\lim_{n\to\infty} u_n = l$ or simply $u_n \to l$. This definition of the limit of a sequence means that for $n > M$, each term u_n lies between $l-\varepsilon$ and $l+\varepsilon$.

At the most only, a finite number of terms of the sequence lie outside this interval.

A sequence, which is *not convergent* is said to be *divergent*.

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Example: -3

(i) $u_n = 1 + \frac{2}{n}$ .The sequence $\{u_n\}$ is convergent and has the limit 1.

Given any $\varepsilon > 0$,we need to find M such that $|u_n - l| < \varepsilon$ for all $n > M$ .That is, we want $\left|1 + \left(\frac{1}{n}\right) - 1\right| < \varepsilon$;that is $\left|\frac{1}{n}\right| < \varepsilon$, which is true if $n > \frac{1}{\varepsilon}$.

Thus ,we can take $M = \left[\frac{1}{\varepsilon}\right]$.

Since exists, the sequence is convergent.

(ii) The sequence $1, 2, 4, 8, \ldots.., 2^n, \ldots$ is divergent.

(iii) The sequence $2, 0, 2, 0, \ldots., 2^n, \ldots.$ is not convergent.

The limit oscillates between 0 and 2. Hence it is divergent; sometimes such a sequence is said to be an *oscillating sequence*, instead of generality saying that it is divergent.

Some useful limits:

(i) $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$,$(x > 0)$

(ii) $\lim_{n\to\infty} x^n = 0$, $(|x| < 1)$

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(iii) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(iv) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

(v) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ (any x)

1.4. Infinite series:

Consider the sequence $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ..., \frac{1}{2^n}, ..$

We can form a new sequence from this as follows:

$$s_1 = u_1 = 1$$

$$s_2 = u_1 + u_2 = \frac{3}{2}$$

$$s_3 = u_1 + u_2 + u_3 = \frac{7}{4}$$

In general,

$$s_{n+1} = s_n + u_{n+1},\ n = 1, 2, 3, ... ,$$

starting with

$$s_1 = u_1$$

$$\lim_{n\to\infty} s_n = u_1 + u_2 + u_3 + ... + \infty$$

If the above –mentioned limit exists and is equal to s , then we say that the infinite series $u_1 + u_2 + u_3 + ...$ converges and its sum is equal to s .if no such limit

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exists ,that is, if $\{s_n\}$ diverges, we say that the series is divergent.

Note:

$\sum u_n$ is understood to be $\sum_{n=1}^{\infty} u_n$ unless the range of summation is clearly indicated.

Similarly, $\lim u_n = l$ or $u_n \to l$ means the limit is l as $n \to \infty$.

1.5. Geometric Series:

The geometric series $1 + r + r^2 + ... + r^{n-1} + ...$ converges to $\frac{1}{(1-r)}$, If $|r| < 1$. If $|r| \geq 1$, the series diverges.

Proof:

Let the partial sum

$$s_n = 1 + r + r^2 + ... + r^{n-1} \qquad (1.5.1)$$

$$rs_n = r + r^2 + ... + r^{n-1} + r^n \qquad (1,5.2)$$

Subtracting (1.5.2) from (1.5.1), we get

$$s_n(1-r) = 1 - r^n$$

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If $r \neq 1, s_n = \frac{1-r^n}{1-r}$

$$\lim_{n\to\infty} s_n = \frac{1}{1-r} \quad \text{if } |r| < 1.$$

Thus the geometric series converges if its common ratio is less than unity in magnitude.

$$\lim_{n\to\infty} s_n = \infty \text{ if } \; r > 1 .$$

If $r < -1$, the limit does not exit. If $r = 1,\ s_n \to \infty$ and if $r = -1,\ s_n = \frac{1+(-1)^{n+1}}{2}$ and its limit does not exist. Thus a geometric series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Example: -4

Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Solution:

Here

$$u_r = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

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Then the nth partial sum

$$\begin{aligned} s_n &= \sum_{r=1}^{n} u_r \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \to 1 \end{aligned}$$

Thus the series is convergent and the sum is equal to 1.

Example: -5

Test the convergence of the series

$$1 + 2 + 3 + 4 + ...$$

Solution:

Here, $s_n = \frac{n(n+1)}{2} \to \infty$.

Therefore, the series is divergent.

It is often not possible to obtain the closed form expression for the nth partial sum of any given infinite series. Thus we are deprived of the means to test the convergence of the series by going through the limit of the nth partial sum. Various tests have been developed to apply them to the individual terms u_n of the series $\sum u_n$ to determine whether the series converges,

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without having to calculate the partial sum. Of course, in such cases we may not be able to find the actual sum of the series. We now study these tests one by one.

1.6. Test for Divergence:

Theorem: -1.1

If $lim u_n \neq 0$ or if $lim u_n$ fails to exist, then $\sum u_n$ diverges.

Proof:

We prove this assertion by showing that, if $\sum u_n$ converges, Then

$$lim u_n = 0.$$

Let $s_n = u_1 + u_2 + ... + u_n$ and suppose $s_n \to s$.

Further if

$$s_{n-1} = u_1 + u_2 + ... + u_{n-1}, \text{then } lim s_{n-1} = s.$$

Now,

$$\begin{aligned} \lim|s_n - s_{n-1}| &= \lim|u_n| \\ &= |s - s| = 0 \end{aligned}.$$

Example: -6

(1) $\sum n^2$ diverges, since $u_n = n^2 \to \infty$.

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(2) $\sum \frac{n+1}{n}$ diverges,since $u_n = 1+\frac{1}{n} \to 1(\neq 0)$.

(3) $\sum (-1)^{n+1}$ diverges, since $lim\, u_n = lim(-1)^{n+1}$ does not exist.

Note:

(i) If $\sum u_n$ converges, $\lim u_n$ is necessarily equal to 0. But $\lim u_n = 0$ is not enough to ensure convergence of the infinite series. $\lim u_n = 0$ is a necessary, but not sufficient condition for the series $\sum u_n$ to converge.

(ii) We can add or subtract two convergent series (term by term) and multiply a convergent series by a constant to give rise to new convergent series. That is, if $\sum u_n$ and $\sum v_n$ converge to s_1 and s_2 respectively,

$$\sum (\alpha u_n + \beta v_n) = \alpha \sum u_n + \beta \sum v_n$$

converges to $\alpha s_1 + \beta s_2$ for any finite scalars α and β.

(iii) Addition (or deletion) of a finite number of terms of an infinite series does not affect its convergence or divergence.

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1.7. Comparison Test for Non-negative series:

The series $\sum u_n$ of non-negative terms converges if there is a convergent series of non-negative terms $\sum v_n$ with $u_n \le v_n$ for all $n \ge n_0$ for some $n_0 \in N$.

Similarly, the series $\sum u_n$ of non-negative terms diverges if there is a divergent series of non-negative terms $\sum w_n$ with $u_n > w_n$ for $n \ge m_0$,for some $m_0 \in N$.

1.8. Integral test:

Let $f(x)$ be a function obtained by introducing the continuous variable x in place of the discrete variable n in the nth term of the series $\sum u_n$ of positive terms. Further, let $f(x)$ be a decreasing function of x for $x \ge 1$ .Then the series $\sum u_n$ and the integral $\int_1^\infty f(x)dx$ both converge or both diverge.

Note:

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The integral test enables us to test the convergence (or divergence) of an important series $\sum \frac{1}{n^p}$,which is extremely useful in the application of many other tests.

Example: -7

If p is a real number, the series

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots.. \qquad (1.7.1)$$

Converges if $p > 1$ and diverges if $p \leq 1$.

Proof:

Case-1

If $p \neq 1$.

$$u_n = \frac{1}{n^p}$$

Consider $f(x) = \frac{1}{x^p}$ for $n \geq 1$.

For any $m > 1$,

$$\int_1^m \frac{dx}{x^p} = \frac{1}{1-p}\left[m^{1-p} - 1\right] \rightarrow \frac{1}{p-1} \quad \text{if} \quad p > 1.$$

The limit is infinite if $p < 1$.

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Thus the series $\sum \frac{1}{n^p}$ is convergent for $p > 1$ and divergent if $p < 1$.

Case-2:

If $p = 1$

$$\int_1^m \frac{dx}{x} = \ln m \to \infty \text{ as } m \to \infty.$$

Thus the series (1) converges if $p > 1$ and diverges if $p \le 1$.

Example: -8

Test the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

Solution:

Applying integral test:

$$\int_2^{\infty} \frac{dx}{x \ln x} = ln(ln\, m) - ln(ln\, 2) \to \infty$$

as $m \to \infty$.

Therefore, the given series is divergent.

Example: -9

Test the convergence of

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$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}.$$

Solution:

Since $\ln n > 1$ for $n > 2$.

So $\frac{1}{\ln n} < 1$ and

hence

$$\frac{1}{n^p \ln n} < \frac{1}{n^p}.$$

We know that $\sum \frac{1}{n^p}$ converges for $p > 1$.

Hence the given series is convergent if $p > 1$(by the comparison test).

If $p < 1$, $n^p \leq n$ which implies that

$$\frac{1}{n^p} \geq \frac{1}{n} \text{ and}$$

hence $\frac{1}{n^p \ln n} > \frac{1}{n \ln n}$ $(n > 2)$.

From the previous example,

we have $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent.

Therefore $\sum \frac{1}{n^p \ln n}$, is also divergent.

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Thus from the above two examples (7) and (8),

$$\sum_{n=2}^{\infty}\frac{1}{n^{p}\ln n} \text{ is convergent}$$

if $p>1$ and divergent if $p\leq 1$.

1.9. Limit form of the comparison Test:

If two positive termed series $\sum u_n$ and $\sum v_n$ are such that $\underset{n\to\infty}{Lim}\frac{u_n}{v_n}=\lambda,(\neq 0)$,then both $\sum u_n$ and $\sum v_n$ converge or diverge together.

Example: -10

Test the convergence of the series

$$\sum\frac{\sqrt{n}}{n^{2}+1}.$$

Solution:

The exponents of n in the numerator and denominator are $\frac{1}{2}$ and 2 respectively. Their difference being $-\frac{3}{2}$.

We consider $\sum v_n$,

Where $v_n=\frac{1}{n^{\frac{3}{2}}}$ for applying the comparison test.

Now,

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$$\frac{u_n}{v_n} = \frac{\sqrt{n}}{n^2+1} n^{\frac{3}{2}}$$

$$= \frac{n^2}{n^2+1} \quad ,$$

$$= \frac{1}{1+\frac{1}{n^2}} \to 1$$

a finite non-zero number.

Hence both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But we know that $\sum \frac{1}{n^{\frac{3}{2}}}$ is convergent.

Therefore, the given series is convergent.

Example: -11

Test the convergence of

$$\sum \left[\left(n^3+1\right)^{1/3} - n \right].$$

Solution:

We have the identity

$$a-b = \frac{a^3-b^3}{a^2+b^2+ab} \text{ and}$$

hence

$$a^{1/3} - b^{1/3} = \frac{a-b}{a^{2/3}+b^{2/3}+a^{1/3}b^{1/3}}.$$

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Thus

$$u_n = \left(n^3+1\right)^{1/3} - \left(n^3\right)^{1/3}$$
$$= \frac{1}{\left(n^3+1\right)^{2/3} + n^2 + \left(n^3+1\right)^{1/3} n}$$

Take $v_n = \frac{1}{n^2}$,

Then

$$\frac{u_n}{v_n} = \frac{n^2}{\left(n^3+1\right)^{2/3} + n^2 + \left(n^3+1\right)^{1/3} n}$$
$$= \frac{1}{\left(1+\frac{1}{n^3}\right)^{2/3} + 1 + \left(1+\frac{1}{n^3}\right)^{1/3}}$$
$$\to \frac{1}{3}$$

Hence both $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum \frac{1}{n^2}$ converges.

Therefore, the given series is convergent.

1.10. Ratio Test:

Let $\sum u_n$ be a series of positive terms and

$$Lim \frac{u_{n+1}}{u_n} = \lambda \text{; then}$$

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(i) The series converges if $\lambda < 1$.

(ii) The series diverges if $\lambda > 1$. and

(iii) The test provides no conclusion if $\lambda = 1$.

Example: -12

Test the convergence of the series

$$\sum \frac{2^n n!}{n^n}.$$

Solution:

Here $u_n = \dfrac{2^n n!}{n^n}$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!}$$

$$= 2\left(\frac{n}{n+1}\right)^n \qquad \text{as } n \to \infty.$$

$$= 2\frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{2}{e} < 1$$

Therefore, the given series is convergent by the ratio test.

Example: -13

Test the convergence of the series

$$\frac{4}{1} + \frac{4.7}{1.2} + \frac{4.7.10}{1.2.3} + \ldots\ldots$$

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Solution:

$$u_n = \frac{4.7.10.......(3n+1)}{1.2.3......n}$$
$$= \frac{4.7.10.......(3n+1)}{n!}$$

$$u_{n+1} = \frac{4.7.10.......(3n+1)(3n+4)}{1.2.3......n.(n+1)}$$
$$= \frac{4.7.10.......(3n+1)(3n+4)}{(n+1)!}$$

Therefore

$$\frac{u_{n+1}}{u_n} = \frac{4.7.10.......(3n+1)(3n+4)}{(n+1)!} \cdot \frac{n!}{4.7.10.......(3n+1)}$$
$$= \frac{(3n+4)}{n+1}$$
$$= \frac{3+\frac{4}{n}}{1+\frac{1}{n}} \to 3 > 1$$

Therefore, the given series is divergent by the ratio test.

1.11. Root Test:

Let $\sum u_n$ be an infinite series of positive terms and $u_n^{1/n} \to \lambda$ Then

(i) The series converges if $\lambda < 1$.

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(ii) The series diverges if $\lambda > 1$ and

(iii) no conclusion can be drawn if $\lambda = 1$..

Example: -14

Test the convergence of the series

$$\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}.$$

Solution:

$$u_n^{1/n} = \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{1/n}$$

$$= \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} \to \frac{1}{\varepsilon} < 1$$

Therefore, the series converges by the root test.

Note: Two tests, Rabbe's test and the logarithmic test can provide a tool to test the convergence of an infinite series, when the ratio or root test fails to give a conclusion.

1.12. Rabbe's Test:

Let $\sum u_n$ be an infinite series and

$$\lim_{n} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lambda\,;$$

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then $\sum u_n$

(i) converges if $\lambda > 1$.

(ii) diverges if $\lambda < 1$ and

(iii) the test provides no conclusion if $\lambda = 1$.

1.13. Logarithmic Test:

The infinite series $\sum u_n$ of positive terms is convergent if $\lambda > 1$ and divergent if $\lambda < 1$, where

$$\lambda = \lim_{n\to\infty}\left[n.Ln\frac{u_n}{u_{n+1}}\right].$$

Example: -15

Discuss the convergence of the power series

$$\sum\frac{(n!)^2}{(2n!)}x^{2n}.$$

Solution:

$$\frac{u_{n+1}}{u_n} = \frac{((n+1)!)^2}{((2n+2)!)}x^{2n+2}\frac{(2n!)}{(n!)^2 x^{2n}}$$

$$= \frac{(n+1)^2}{(2n+2)(2n+1)}x^2$$

$$= \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)}x^2 \to \frac{x^2}{4}$$

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Therefore, by the ratio test,

The series is convergent if

$$x^2 < 4 ,\left[x \in (-2,2)\right]$$

and divergent if

$$x^2 > 4,\left[x \in (-\infty,2) \cup (2,\infty)\right].$$

The ratio test fails if

$$x^2 = 4 ,(x = \pm 2).$$

We apply Rabbe's test if $x^2 = 4$.

Then

$$\frac{u_n}{u_{n+1}} = \frac{2(2n+1)}{n+1}.\frac{1}{4} = \frac{(2n+1)}{2(n+1)}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{2n+1-2n-2}{2(n+1)}\right]$$

$$= \frac{-1}{2\left(1+\frac{1}{n}\right)} \to -\frac{1}{2} < 1$$

Therefore, the series is divergent.

Thus the given series is convergent if $x^2 < 4$ and divergent if $x^2 \geq 4$.

Example: -16

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Discuss the convergence of the power series $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+......$

Solution:

Here $u_n=\frac{n^nx^n}{n!}$

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{n^nx^n}$$

$$=\left(\frac{n+1}{n}\right)^n x$$

$$=\left(1+\frac{1}{n}\right)^n x\to ex$$

Thus the series is convergent,

if $x<\left(\frac{1}{e}\right)$ and divergent if $x>\left(\frac{1}{e}\right)$.

The test fails if $x=\left(\frac{1}{e}\right)$.

If $x=\left(\frac{1}{e}\right)$,we apply the logarithmic test.

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$$Ln\frac{u_n}{u_{n+1}} = Ln\left[\left(\frac{n+1}{n}\right)^n e\right]$$

$$= 1 + n\,Ln\frac{n}{n+1}$$

$$= 1 + n\,Ln\frac{1}{1+\frac{1}{n}}$$

$$= 1 - n\,Ln\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\right],\ \left(\frac{1}{n} < 1\right)$$

$$= 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \ldots..$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \ldots..$$

$$n\,Ln\frac{n}{n+1} = \frac{1}{2} - \frac{1}{3n} + \ldots.. \to \frac{1}{2} < 1$$

Therefore the series diverges if $x = \left(\frac{1}{e}\right)$.Thus the given series converges if $0 < x < \frac{1}{e}$ and diverges if $x \geq \left(\frac{1}{e}\right)$.

1.14. Leibnitz's test for Alternating series:

An alternating series

$\sum(-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots.$,where all $u_n > 0$, converges

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if

(i) $u_n > u_{n+1}$ for every n (i.e. $\{u_n\}$ is a monotonically decreasing sequence)

(ii) $\lim_{n\to\infty} u_n = 0$

Note:

(i) If a finite number of terms is neither alternating nor monotonic, the series converges if it is an alternating series with the conditions of the test satisfied for n greater than some M.

(ii) If, s_n is the sum of the first n-terms of the alternating series satisfying the conditions of Leibnitz's test, then it can be shown that $|s_n - l| < u_{n+1}$.

Thus the upper bound for the error in approximating the sum by n terms is given by the first unused term.

Example: -17

Test the convergence of the series

$$\sum (-1)^n \sin\frac{1}{n}$$

Solution:

This is an alternating series $\sum (-1)^n u_n$, when

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$$u_n = \sin\frac{1}{n} \text{ and } u_n \to 0 \text{ as } n \to \infty .$$

Further, $\{u_n\}$ is a monotonically decreasing sequence, for

$$\sin\frac{1}{n+1} < \sin\frac{1}{n} , \ (n \geq 1)$$

Therefore, by Leibnitz test, the series is convergent.

Example: -18

Test the convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - ......(0 < x < 1).$$

Solution:

This is an alternating series $\sum(-1)^{n-1} u_n$,

Where

$$u_n = \frac{x^n}{1+x^n} \to 0 \quad (x^n \to 0 \text{ when } 0 < x < 1)$$

Further,

$$\begin{aligned} u_n - u_{n+1} &= \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}} \\ &= \frac{x^n + x^{2n+1} - x^{n+1} - x^{2n+1}}{(1+x^n)(1+x^{n+1})} \\ &= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} > 0, \ (x < 1) \end{aligned}$$

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Therefore, the terms are monotonically decreasing. The given series converges by Leibnitz's test.

1.15. Absolute and Conditional Convergence

An infinite series $\sum u_n$ is said to be *absolutely convergent* if $\sum |u_n|$ is convergent.

If $\sum u_n$ convergent and if $\sum |u_n|$ diverges, then we say that $\sum u_n$ is conditionally convergent.

Theorem: -1.2

Absolute convergence implies convergence. That is if $\sum |u_n|$ converges, then $\sum u_n$ converges.

Proof:

Since for each n,

$$-|u_n| \le u_n \le |u_n|$$

we have ,

$$0 \le u_n + |u_n| \le 2|u_n|$$

If $\sum |u_n|$ converges, then $\sum 2|u_n|$ converges and by the comparison test $\sum (u_n + |u_n|)$ converges.

Therefore,

$$\sum (u_n + |u_n| - |u_n|) = \sum u_n \text{ converges.}$$

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Note: If $\sum u_n$ is divergent, $\sum |u_n|$ is divergent.

Example:-19

Test the convergence of the series

$$\sum (-1)^{n-1} \frac{n\pi^n}{2^{2n}+1} \ .$$

Solution:

This is an alternating series $\sum (-1)^{n-1} u_n$,where

$$u_n = \frac{n\pi^n}{2^{2n}+1}$$

Hence the absolute series is $\sum u_n$.

$$u_n = \frac{n\pi^n}{2^{2n}+1} < \frac{n\pi^n}{2^{2n}} = v_n \quad \text{,say}$$

Now $v_n^{1/n} = n^{1/n} \left(\frac{\pi}{e^2} \right) \to \frac{\pi}{e^2} < 1.$

Therefore by the root test, $\sum v_n$ converges and by the comparison test $\sum u_n$ converges. That is, the given series is absolutely convergent. Since absolute convergence implies convergence, the given series is convergent.

Note:

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Suggestions for choosing an appropriate test:

(i) If the series is an alternating series, use Leibnitz's test. One can consider the absolute series and determine convergence by using the results that absolute convergence implies convergence. The last statement holds if the series contains positive and negative terms in some order.

(ii) If the nth term contains factorials, the ratio test is suggested.

(iii) The difference of finite powers of n in the numerator and denominator is suggestive of a comparision test.

(iv) If the exponent of simple functions of n contain (without the occurrence of factorials) the root test is suggested.

(v) For power series, when the ratio or root test fails, the logarithmic is suggested if $Lim \frac{u_n}{u_{n+1}}$ involves e, otherwise Rabbe's test is suggested.

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Exercises

Test the convergence of the following series

1. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+....$

2. $\sum \frac{2n^3+5}{4n^5+1}$

3. $\sum \frac{n^2}{3^n}$

4. $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+...$

5. $\sum \frac{2^n}{n^n}$

6. $\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n \ (x>0)$

7. $\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+....$

8. $\sum \frac{n^n}{n!}x^n$

9. $\sum \left(\frac{n}{n+1}\right)^{-3}x^{n-1}$

10. $\sum \frac{\cos n\pi}{n+1}$ (Leibnitz test)

11. $\sum \left(\frac{n}{n+1}\right)^{n^2}$

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12. $\sum \sin \frac{1}{n}$

13. $\sum \frac{2^{2n}(n!)^2}{2n!} x^n \ (x > 0)$

14. $\sum_{2}^{\infty} \frac{1}{n(\ln n)^p}$ (Integral test)

15. $\sum \frac{n^2+1}{3^n}$

16. $\sum \frac{(x+1)^n}{3^n n^2} \ (x > -1)$

17. $\sum \frac{1}{\sqrt{n} + \ln n}$

18. $\sum n\left(\frac{1}{2}\right)^n$

19. $\sum_{3}^{\infty} \frac{1}{n \ln n (\ln \ln n)}$ (Integral test)

Answers:

1. Convergent for $p > 2$ and divergent for $p \leq 2$.

2. Convergent.

3. Convergent.

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4. Convergent for $x^2 \leq 1$ and divergent for $x^2 > 1$.

5. Convergent.

6. Convergent for $x < 1$ and divergent for $x \geq 1$.

7. Convergent.

8. Convergent for $x < \frac{1}{\varepsilon}$ and divergent for $x \geq \frac{1}{\varepsilon}$.

9. Convergent for $x \leq 1$ and divergent for $x > 1$.

10. Convergent.

11. Convergent.

12. Divergent.

13. Convergent for $x < 1$ and divergent for $x \geq 1$.

14. Convergent for $p > 1$ and divergent for $p \leq 1$.

15. Divergent.

16. Convergent for $x \leq 2$ and divergent for $x > 2$.

17. Divergent.

18. Convergent.

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19. Convergent.

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Chapter-2

Fourier Series

Fourier series are series of sine and cosine terms. They arise in the representation of periodic functions. They play an important role in solving P.D.E.

2.1. Periodic function:

A function is called *periodic* if it is defined for all real x and if there is a positive number T such that $f(x+T)=f(x)$ for all real x.

It follows that $f(x+nT)=f(x)$ for all x and all integers.

A function $f(x)$ with period 2π can be expanded in the following trigonometric series

$$f(x)=a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+.........$$
$$+b_1\sin x+b_2\sin 2x+b_3\sin 3x+...... \quad (2.1.1)$$
$$=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

This series is called the Fourier series and its coefficients are called Fourier coefficients.

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All terms on the R.H.S of equation (2.1.1) are of period 2π and $f(x)$ is also of period 2π.

We first present how the Fourier coefficients can be evaluated for any given function $f(x)$, before the conditions under which such an expansion is valid are formally stated.

To evaluate $a_0, a_1, a_2, a_3, \ldots$ and $b_1, b_2, \ldots$ the following integrals, involving sine and cosine functions are useful.

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx\, dx = \int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = 0 \;, n \neq 0.$

2. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx$

$$= \int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\, dx$$

$$= 0 \;, m \neq n$$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx$

$$= \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} \left[\cos(m+n)x + \cos(m-n)x \right] dx$$

$$= \frac{1}{2} \left[\frac{1}{m+n} \cos(m+n)x + \frac{1}{m-n} \cos(m-n)x \right]_{\alpha}^{\alpha+2\pi}$$

$$= 0$$

$$\left[\cos(m+n)x(\alpha+2\pi) = \cos(m+n)\alpha \right]$$

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4. $\int_{\alpha}^{\alpha+2\pi} \cos mx \sin nx\, dx = 0$.

5. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\, dx = \int_{\alpha}^{\alpha+2\pi} \sin^2 nx\, dx$

$$= \pi, n \neq 0$$

$$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\, dx = \int_{\alpha}^{\alpha+2\pi} \frac{1+\cos 2nx}{2} dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2n}\sin 2x \right]_{\alpha}^{\alpha+2\pi}$$

$$= \frac{1}{2}[2\pi + 0] = \pi$$

In addition to these properties of integrals involving sine and cosine functions, we often need these trigonometric functions for particular arguments.

(i) $\sin(2n+1)\frac{\pi}{2} = \cos n\pi = (-1)^n$ and

(ii) $\sin n\pi = \cos(2n+1)\frac{\pi}{2} = 0$ for all n.

2.2. Fourier Coefficients:

The Fourier coefficient in (2.1.1) are given by

$$a_0 = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx$$

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and $b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx$

These are known as *Euler's formula*.

To obtain these formulae, we assume that term by term integration is valid.

Integrating on both sides of (2.1.1),

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = a_0 \int_{\alpha}^{\alpha+2\pi} dx + \sum_{n=1}^{\infty} \left[a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx \right]$$

$$= 2\pi a_0$$

(later integrals being zero)

This gives the first of the Euler's formula.

Multiplying both sides of (2.1.1) by $\cos nx$ and integrating

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx$$

$$= a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx + \sum_{m=1}^{\infty} \left[a_m \int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx + b_m \int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx \, dx \right]$$

$$= a_n \int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx$$

$$= \pi a_n$$

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This is the second of the formulae.

Similarly, multiplying by $\sin nx$ and integrating,

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$$

$$= a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx$$

$$+ \sum_{m=1}^{\infty}\left[a_m \int_{\alpha}^{\alpha+2\pi} \cos mx \sin nx\,dx + b_m \int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\,dx \right]$$

$$= b_n \int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx$$

$$= \pi b_n$$

Giving the last coefficient of the Euler's formulae.

Even and Odd functions:

If is an even function in the interval $(-\pi, \pi)$, then

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$,since the integral is an odd function. Thus we get only the cosine series. Similarly, if $f(x)$ is an odd function, we get only the sine series.

2.3. Validity of the Fourier series:

It is a simple matter to calculate the Fourier coefficients for any given function with period 2π if the integrals exist. The question is whether the infinite

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series converges to $f(x)$ for each x of the interval of interest.

We state below the conditions under which such a convergence occurs. Most problems arising in the application of Fourier series satisfy these conditions.

If a function $f(x)$ is continuous in $(a, a+2\pi)$,except possibly at a finite number of points and these discontinuities are finite jumps, and left hand and right hand derivatives exists at each point in $(a, a+2\pi)$,then the Fourier series represented by (1) with the coefficients as in (2) converges to $f(x)$ at each x except at the points of discontinuities at which the sum of the series is equal to the average of the left and right limits.

Note:

Discontinuities with in the interval of definition of $f(x)$ and at the end points can be figured out easily.

The given function defined in the interval $(a, a+2\pi)$ may not be periodic and is generally not of much concern in many applications of Fourier series. What happens outside the interval $(a, a+2\pi)$ is of no

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relevance for applications. The Fourier series expansion assumes the function to be periodic with period 2π. Graph of the function in $(a-2\pi, a), (a+2\pi, a+4\pi)$ etc., repeats itself as in $(a, a+2\pi)$.Thus if $f(a+2\pi) \neq f(a)$ for a function whose Fourier series is developed, the series converges to the average of $f(a)$ and $f(a+2\pi)$, $\frac{[f(a)+f(a+2\pi)]}{2}$ by the above condition of validity of the Fourier series. This is brought out in the following examples.

Example-1

Find the Fourier series representation of

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi & 0 < x < \pi \end{cases} \text{ in } (-\pi, \pi)$$

and hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$.

Solution:

The Fourier representation of $f(x)$ can be represented as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

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$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \pi\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \pi\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\sin nx\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \pi\sin nx\,dx$$

$$= -\frac{1}{n}\left[\cos nx\right]_0^{\pi}$$

$$= \frac{1}{n}\left[1-\cos n\pi\right]$$

$$= \frac{1}{n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, & if \quad n \text{ is even} \\ \frac{2}{n}, & if \quad n \text{ is odd} \end{cases} .$$

Thus

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$$f(x)=\frac{\pi}{2}+2\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+.....\right].$$

Putting $x=0$,the series gives $\frac{\pi}{2}$.

$f(x)$, as given has a finite discontinuity at $x=0$,thus ,

$$\frac{1}{2}\left[f\left(0^-\right)+f\left(0^+\right)\right]=\frac{1}{2}[0+\pi]=\frac{\pi}{2}$$

Putting $x=\frac{\pi}{2}$,

$$\frac{\pi}{2}+2\left[\frac{\sin\frac{\pi}{2}}{1}+\frac{\sin\frac{3\pi}{2}}{3}+\frac{\sin\frac{5\pi}{2}}{5}+.....\right]=\pi$$

That is $2\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.........\right]=\frac{\pi}{2}$.

Example-2

Find a Fourier series to represent $x-x^2$ from $x=-\pi$ to $x=\pi$ and hence show that

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.......=\frac{\pi^2}{12}$$

and

$$1+\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.......=\frac{\pi^2}{6}.$$

Solution:

Here

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$$x - x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)dx = -\frac{\pi^2}{3}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)\cos nx\,dx \\
&= -\frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx \\
&= -\frac{2}{\pi}\left[\frac{1}{n}\left(x^2 \sin nx\right)_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin nx\,dx\right] \\
&= \frac{4}{\pi n}\left[-\frac{1}{n}\left(x\cos nx\right)_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] \\
&= -\frac{4}{\pi n^2}\pi\cos n\pi + \frac{4}{\pi n^2}\left(\sin nx\right)_0^{\pi} \\
&= -\frac{4}{n^2}(-1)^n \\
&= \frac{4}{n^2}(-1)^{n+1}
\end{aligned}$$

($x\cos nx$ is an odd function $x^2\cos nx$ is an even function)

Thus

$$a_1 = \frac{4}{1^2};\ a_2 = -\frac{4}{2^2};\ a_3 = \frac{4}{3^2};\ a_4 = -\frac{4}{4^2}.$$

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$$\pi b_n = \int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \int_{-\pi}^{\pi} \left(x - x^2\right)\sin nx\,dx$$

$$= 2\int_{0}^{\pi} x\sin nx\,dx$$

$$= 2\left[-\frac{1}{n}\left(x\cos nx\right)_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$$

$$= -\frac{2}{n}\pi\cos n\pi = (-1)^{n+1}\frac{2\pi}{n}$$

Thus $b_1 = \frac{2}{1}\,; b_2 = -\frac{2}{2}; b_3 = \frac{2}{3}; b_4 = -\frac{2}{4}$,

Therefore,

$$x - x^2 =$$

$$-\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right]$$

$$+2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right]$$

To establish the second part of the problem, put $x = 0$, to get

$$4\left[\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + ...\right] = \frac{\pi^2}{3}$$

giving the desired result.

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Now, the Fourier series, when $x=\pi$ or $x=-\pi$ is equal to

$$-\frac{\pi^2}{3}+4\left[-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-.......\right]$$
$$=-\frac{\pi^2}{3}-4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.......\right]$$

The graph of the function $f(x)=x(1-x)$ in $-\pi<x<\pi$ and periodic with period 2π. The periodic function is discontinuous at $x=-3\pi,-\pi,\pi,3\pi,......$ Thus the series at $x=\pi$ converges to

$$\frac{1}{2}\left[f(\pi^-)+f(\pi^+)\right]=\frac{1}{2}\left[\pi(1-\pi)-\pi(1+\pi)\right]$$
$$=-\pi^2$$

Therefore,

$$\frac{-\pi^2}{3}-4\left(\sum\frac{1}{n^2}\right)x=-\pi^2$$

That is ,

$$\sum\frac{1}{n^2}=\frac{\pi^2}{6}.$$

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Example-3

Find the Fourier series to represent the function $f(x)$ given by

$f(x)=\begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi < x \le 2\pi \end{cases}$ and hence show that

$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......=\frac{\pi^2}{8}$.

Solution:

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

Where

$$2\pi a_0=\int_0^{2\pi} f(x)\,dx$$

$$=\int_0^{\pi} x\,dx+\int_{\pi}^{2\pi}(2\pi-x)\,dx \text{ giving } a_0=\frac{\pi}{2}.$$

$$=\pi^2$$

$$\pi a_n=\int_0^{2\pi} f(x)\cos nx\,dx$$

$$=\int_0^{\pi} x\cos nx\,dx+\int_{\pi}^{2\pi}(2\pi-x)\cos nx\,dx$$

$$=\frac{1}{n}(x\sin nx)_0^{\pi}-\frac{1}{n}\int_0^{\pi}\sin nx\,dx+\frac{2\pi}{n}(\sin nx)_{\pi}^{2\pi}$$

$$-\frac{1}{n}(x\sin nx)_{\pi}^{2\pi}+\frac{1}{n}\int_{\pi}^{2\pi}\sin nx\,dx$$

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$$= \left[\frac{1}{n^2} \left(\cos nx \right)_0^{\pi} - \frac{1}{n^2} \left(\cos nx \right)_{\pi}^{2\pi} \right]$$
$$= \frac{1}{n^2} \left[\left(-1 \right)^n - 1 \right] - \frac{1}{n^2} \left[1 - \left(-1 \right)^n \right]$$
$$= \frac{2}{n^2} \left[\left(-1 \right)^n - 1 \right]$$

Thus,

$$a_1 = -\frac{4}{\pi 1^2}$$
$$; a_2 = 0;$$
$$a_3 = -\frac{4}{\pi 3^2};$$
$$a_4 = 0; a_5 = -\frac{4}{\pi 5^2}$$

$$\pi b_n = \int_0^{\pi} x \sin nx \, dx + \int_{\pi}^{2\pi} \left(2\pi - x \right) \sin nx \, dx$$

$$= -\frac{1}{n} \left(x \cos nx \right)_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx$$
$$- \frac{1}{n} \left(\left(2\pi - x \right) \cos nx \right)_{\pi}^{2\pi} - \frac{1}{n} \int_0^{2\pi} \cos nx \, dx$$
$$= -\frac{\pi}{n} \left(-1 \right)^n + \frac{1}{n} \pi \left(-1 \right)^n$$
$$= 0$$

Thus,

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$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+.....\right]$$

$$f(\pi)=\pi=\frac{\pi}{2}-\frac{4}{\pi}\left[-1-\frac{1}{3^2}-\frac{1}{5^2}-.....\right],$$

which gives

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......=\frac{\pi^2}{8}.$$

Example-4

Find the Fourier series expansion of the function

$$f(x)=\begin{cases} x & -\frac{\pi}{2}\le x\le\frac{\pi}{2} \\ \pi-x & \frac{\pi}{2}\le x\le\frac{3\pi}{2} \end{cases}$$

and hence show that

$$1+\frac{1}{3^2}+\frac{1}{3^2}+.......=\frac{\pi^2}{8}.$$

Solution:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where

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$$2\pi a_0 = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\,dx \;.$$

$$= 0$$

$$\pi a_n = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nx dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\cos nx dx$$

(The first function integrated is an odd function)

$$= \frac{1}{n}\left[(\pi - x)\sin nx\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \frac{1}{n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin nx\,dx$$

$$= \frac{1}{n}\left[-\frac{\pi}{2}\sin\frac{3n\pi}{2} - \frac{\pi}{2}\sin\frac{n\pi}{2}\right] - \frac{1}{n^2}\left[\cos nx\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= -\frac{\pi}{2n} 2\sin n\pi \cos\frac{n\pi}{2} - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)$$

$$= -\frac{2}{n^2}\sin n\pi \sin\left(-\frac{n\pi}{2}\right) = 0$$

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$$\pi b_n = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin nx dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\sin nx dx$$

$$= 2\int_0^{\frac{\pi}{2}} x\sin nx dx - \frac{1}{n}\Big[(\pi - x)\cos nx\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos nx\, dx$$

$$= -\frac{2}{n}\left[\, x\cos nx\right]_0^{\frac{\pi}{2}} + \frac{2}{n}\int_0^{\frac{\pi}{2}} \cos nx\, dx$$

$$-\frac{1}{n}\left[-\frac{\pi}{2}\cos\frac{3n\pi}{2} - \frac{\pi}{2}\cos\frac{n\pi}{2}\right] - \frac{1}{n^2}\left[\, \sin nx\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= -\frac{2}{n}\frac{\pi}{2}\cos\frac{n\pi}{2} + \frac{2}{n^2}\left[\, \sin nx\right]_0^{\frac{\pi}{2}}$$

$$+\frac{\pi}{2n}2\cos n\pi\cos\frac{n\pi}{2} - \frac{1}{n^2}\left[\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$= -\frac{\pi}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}$$

$$+\frac{\pi}{n}\cos n\pi\cos\frac{n\pi}{2} - \frac{2}{n^2}\cos n\pi\sin\frac{n\pi}{2}$$

$$= \frac{\pi}{n}\cos\frac{n\pi}{2}\left[(-1)^n - 1\right] + \frac{2}{n^2}\sin\frac{n\pi}{2}\left[1 - (-1)^n\right]$$

Now $\pi b_n = 0$ if is n even.

If n is odd and equal to $2m+1.(m = 0,1,.....)$.

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$$\pi b_{2m+1} = \frac{\pi}{2m+1}\cos\frac{(2m+1)\pi}{2}(-2) + \frac{2}{(2m+1)^2}\sin\frac{(2m+1)\pi}{2}2$$

$$= \frac{4}{(2m+1)^2}\sin\frac{(2m+1)\pi}{2}$$

Then ,

$$b_1 = \frac{4}{\pi};\ b_3 = -\frac{4}{\pi 3^2};\ b_5 = \frac{4}{\pi 5^2};\ b_7 = -\frac{4}{\pi 7^2},......$$

Therefore,

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + ...\right]$$

When $x = \frac{\pi}{2};\ f(x) = \frac{\pi}{2}$,

$$\frac{\pi}{2} = \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{3^2} + ...\right)$$

Therefore,

$$1 + \frac{1}{3^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{8}\ .$$

Example-5

Find the Fourier series expansion of the function $f(x) = \cos sx,\ -\pi < x < \pi$,where s is a non-zero function. Hence show that

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$$\cot\theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 2^2\pi^2} + \frac{2\theta}{\theta^2 - 3^2\pi^2} + \dots..$$

Solution:

$$\cos sx = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$2\pi a_0 = \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \int_{-\pi}^{\pi} \cos sx\,dx \quad ;$$

$$= \frac{2}{s}\sin n\pi$$

giving

$$a_0 = \frac{1}{s\pi}\sin s\pi\,.$$

$$\pi a_n = \int_{-\pi}^{\pi} \cos sx \cos nx dx$$

$$= 2\int_0^{\pi} \cos sx \cos nx dx$$

$$= \int_0^{\pi} \left[\cos(s+n)x + \cos(s-n)x\right] dx$$

$$= \frac{\sin(s+n)\pi}{s+n} + \frac{\sin(s-n)\pi}{s-n}$$

and

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$$\pi b_n = \int_{-\pi}^{\pi} \cos sx \sin nx dx = 0 \ \text{,(odd integrand)}$$

Therefore ,

$$\cos sx = \frac{\sin s\pi}{s\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin(s+n)\pi}{s+n} + \frac{\sin(s-n)\pi}{s-n}\right]\cos nx$$

is the required expansion.

The expression in the brackets can be simplified as

$$\frac{1}{s^2-n^2}\left[(s-n)(\sin s\pi\cos n\pi + \cos s\pi \sin n\pi)\right] + (s+n)(\sin s\pi \cos n\pi - \cos s\pi \sin n\pi)$$

$$= \frac{1}{s^2-n^2}\left[2s\sin s\pi\cos n\pi\right]$$

$$= \frac{(-1)^n 2s\sin s\pi}{s^2-n^2}$$

When $x = \pi$,

$$\cos s\pi = \frac{\sin s\pi}{s\pi} + \sum_{n=1}^{\infty}\frac{2s\pi\sin s\pi}{\pi^2 s^2 - \pi^2 n^2}$$

With $s\pi = \theta$,

$$\cot\theta = \frac{1}{\theta} + \sum_{n=1}^{\infty}\frac{2\theta}{\theta^2 - n^2\pi^2}$$

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2.4. Function with any period T :

If $f(x)$ is given in the interval (α,β) with period $T=\beta-\alpha$, a change of variable converts the problem to one with period 2π .

Let

$$t=\frac{2\pi}{T}x, \quad x=\frac{tT}{2\pi}$$

When $x=\alpha$, $t=\frac{2\pi\alpha}{T}=a$, say, and

when

$$x=\beta=\alpha+T, \ t=\frac{2\pi(\alpha+T)}{T}=a+2\pi .$$

Thus we have to represent $f\left(\frac{tT}{2\pi}\right)=g(t)$ as

$$a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$

and obtain the Fourier coefficient as

$$a_0=\frac{1}{2\pi}\int_a^{a+2\pi} f\left(\frac{tT}{2\pi}\right)dt=\frac{1}{T}\int_\alpha^\beta f(x)dx$$

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$$a_n = \frac{1}{\pi}\int_{a}^{a+2\pi} f\left(\frac{tT}{2\pi}\right)\cos nt\,dt$$
$$= \frac{2}{T}\int_{\alpha}^{\beta} f(x)\cos\frac{2\pi nx}{T}\,dx$$

and

$$b_n = \frac{1}{\pi}\int_{a}^{a+2\pi} f\left(\frac{tT}{2\pi}\right)\sin nt\,dt$$
$$= \frac{2}{T}\int_{\alpha}^{\beta} f(x)\sin\frac{2\pi nx}{T}\,dx$$

The Fourier series representation with these coefficients is

$$f(x) = a_0 + \sum_{1}^{\alpha}\left(a_n\cos\frac{2\pi nx}{T} + b_n\sin\frac{2\pi nx}{T}\right)$$

In the practical development of the Fourier series, it is convenient to work in the original variable. Thus ,with $T = \beta - \alpha$.

$$f(x) = a_0 + \sum\left(a_n\cos\frac{2\pi nx}{T} + b_n\sin\frac{2\pi nx}{T}\right)$$

$$a_0 = \frac{1}{T}\int_{\alpha}^{\beta} f(x)\,dx$$

$$a_n = \frac{2}{T}\int_{\alpha}^{\beta} f(x)\cos\frac{2\pi nx}{T}\,dx$$

$$b_n = \frac{2}{T}\int_{\alpha}^{\beta} f(x)\sin\frac{2\pi nx}{T}\,dx$$

$T = 2\pi$ gives the formulae of section 12.2 with $\beta = \alpha + T = \alpha + 2\pi$.

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Example-6

Obtain the Fourier series for $f\left(x\right)=\pi x,\ 0\le x\le 2$

.Hence show that

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......=\frac{\pi}{4}.$

Solution:

$$f\left(x\right)=a_0+\sum\left(a_n\cos n\pi x+b_n\sin n\pi x\right)$$

$$a_0=\frac{1}{T}\int_0^2 f\left(x\right)dx=\frac{\pi}{2}\int_0^2 x\,dx=\pi$$

$$a_n=\frac{2}{T}\int_0^2 f\left(x\right)\cos n\pi x\,dx$$

$$=\pi\int_0^2 x\cos n\pi x\,dx$$

$$=\pi\left[\frac{1}{n\pi}\left(x\sin n\pi x\right)_0^2-\frac{1}{n\pi}\int_0^2 \sin n\pi x\,dx\right]$$

$$=\pi\left[\frac{1}{n^2\pi^2}\left(\cos n\pi x\right)_0^2\right]=0$$

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$$b_n = \frac{2}{T}\int_0^2 f(x)\sin\pi x\,dx$$
$$= \pi\int_0^2 x\sin n\pi x\,dx$$
$$= \pi\left[-\frac{1}{n\pi}\left(x\cos n\pi x\right)_0^2 + \frac{1}{n\pi}\int_0^2 \cos n\pi x\,dx\right]$$
$$= \pi\left[-\frac{2}{n\pi}\left(\cos 2n\pi\right) + \frac{1}{n^2\pi^2}\left(\sin n\pi x\right)_0^2\right] = \frac{2}{n}$$

When $x = \frac{1}{2}, f(x) = \frac{\pi}{2}$ .Thus

$$\frac{\pi}{2} = \pi - 2\left(\sin\frac{\pi}{2} + \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{3} + ...\right)$$
$$= \pi - 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right)$$

Therefore

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

Example-7

Find the Fourier series of the periodic function of period 2, where

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 2x & 0 < x < 1 \end{cases}$$

and show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8} \quad \text{and} \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

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Solution:

Here $T = 2$ and

$$f(x) = a_0 + \sum \left(a_n \cos n\pi x + b_n \sin n\pi x \right).$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left[-\int_{-1}^{0} dx + \int_{0}^{1} 2x\,dx \right]$$

$$= -\frac{1}{2}(0+1) + \frac{1}{2}(1-0) = 0$$

$$a_n = \frac{2}{2}\int_{-1}^{1} f(x)\cos n\pi x\,dx$$

$$= \left[-\int_{-1}^{0} \cos n\pi x\,dx + 2\int_{0}^{1} x\cos n\pi x\,dx \right]$$

$$= -\frac{1}{n\pi}\left(\sin n\pi x\right)_{-1}^{0} + \frac{2}{n\pi}\left(x \sin n\pi x\right)_{0}^{1}$$

$$- \frac{2}{n\pi}\int_{0}^{1} \sin n\pi x\,dx$$

$$= \frac{2}{n^2\pi^2}\left(\cos n\pi x\right)_{0}^{1}$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1 \right]$$

$$a_1 = -\frac{4}{1^2\pi^2}\ ;\ a_2 = 0\ ;\ a_3 = \frac{4}{3^2\pi^2}\ ;$$

$$a_4 = 0\ ;\ a_5 = -\frac{4}{5^2\pi^2}$$

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$$b_n = \frac{2}{2}\int_{-1}^{1} f(x)\sin n\pi x\, dx$$
$$= \left[-\int_{-1}^{0} \sin n\pi x\, dx + 2\int_{0}^{1} x\sin n\pi x\, dx \right]$$
$$= \frac{1}{n\pi}(\cos n\pi x)_{-1}^{0} - \frac{2}{n\pi}\left[(x\cos n\pi x)_0^1 - \int_0^1 \cos n\pi x\, dx \right]$$
$$= \frac{1}{n\pi}\left[1-(-1)^n\right] - \frac{2}{n\pi}\left[(-1)^n\right] - \frac{2}{n^2\pi^2}(\sin n\pi x)_0^1$$
$$= \frac{1}{n\pi}\left[1-3(-1)^n\right]$$

$$b_1 = \frac{4}{\pi}\ ;\ b_2 = -\frac{2}{2\pi}\ ;\ b_3 = \frac{4}{3\pi}\ ;$$
$$b_4 = -\frac{2}{4\pi}\ ;\ b_5 = \frac{4}{5\pi} ; b_6 = -\frac{2}{6\pi}$$

Thus,

$$f(x) = -\frac{4}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + ... \right)$$
$$+ \frac{2}{\pi}\left[2\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{2\sin 3\pi x}{3} \right.$$
$$\left. - \frac{\sin 4\pi x}{4} + \frac{2\sin 5\pi x}{5} ...... \right]$$

When $x = 0$, the series converges to

$$\frac{\left[f(0+1) + f(0-1) \right]}{2} = \frac{-1+0}{2} = \frac{-1}{2}.$$

Therefore,

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$$\frac{-1}{2}=-\frac{4}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+......\right)$$

$$\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+......=\frac{\pi^2}{8}$$

When $x=\frac{1}{2}$, $f=1$,giving

$$1=\frac{2}{\pi}\left(2-\frac{2}{3}+\frac{2}{5}-\frac{2}{7}+......\right)$$

$$\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......=\frac{\pi}{4}.$$

2.5. Half Range Expansions

Often it becomes necessary to obtain a Fourier sine or cosine series expansion of a function in the interval $(0,a)$.This can be achieved by treating $(0,a)$ as a half range of $(-a,a)$ and defining $f(x)$ suitably in the other half range of $(-a,0)$ by making the function even or odd according to the need of a cosine or sine series.

$$f(x)=a_0+\sum\left(a_n\cos\frac{n\pi x}{a}+b_n\sin\frac{n\pi x}{a}\right),\ (-a<x<a).$$

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Suppose we want a cosine series, $f(x)$ being given in $(0,a)$ .Then we extend $f(x)$ to $(-a,0)$ so that $f(x)$ is an even function on $(-a,a)$ .That is $f(-x)=f(x)$.

Then the Fourier coefficients are given by

$$a_0=\frac{1}{T}\int_{-a}^{a} f(x)dx$$
$$=\frac{2}{2a}\int_{0}^{a} f(x)dx$$
$$=\frac{1}{a}\int_{0}^{a} f(x)dx$$

$$a_n=\frac{2}{T}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}dx$$
$$=\frac{2}{a}\int_{0}^{a} f(x)\cos\frac{n\pi x}{a}dx$$

$b_n=0$, since $f(x)\sin\frac{n\pi x}{a}$ is an odd function.

Similarly, if we want a sine series ,we extend $f(x)$ to $(-a,0)$ as an odd function in $(-a,a)$.

Then $a_0=0$; $a_n=0$ and

$$b_n=\frac{2}{T}\int_{-a}^{a} f(x)\sin\frac{n\pi x}{a}dx$$
$$=\frac{2}{a}\int_{0}^{a} f(x)\sin\frac{n\pi x}{a}dx$$

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Note:

1. $f(x)$ has the same value for each $x \in (0, a)$, whether we use even (cosine series) or odd (sine series) expansion. The difference occurs in the other half of $(-a, a)$. for $f(-x) = f(x)$ an even expansion and $f(-x) = -f(x)$ for the odd expansion.

2. Suppose we want to expand $\sin^3 x + \sin 2x \cos x$ as a sine series in the half range $(0, \pi)$.

3. $\sin^3 x + \sin 2x \cos x$

$$= 3\frac{\sin x - \sin^3 x}{4} + \frac{1}{2}(\sin x + \sin 3x)$$

$$= \frac{5}{4}\sin x + \frac{1}{4}\sin 3x$$

is the required expansion.

Example-8

Find the half range cosine series for

$$f(x) = \begin{cases} kx & 0 \le x \le \frac{l}{2} \\ k(l - x) & \frac{l}{2} \le x \le l \end{cases}$$

and hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

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Solution:

Since Period $=2l$ so the half period is l.

$$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\left[k\int_0^{\frac{l}{2}} x\,dx + k\int_{\frac{l}{2}}^{l}(l-x)\,dx\right]$$

$$= \frac{k}{l}\left[\frac{l^2}{8} + l\left(l - \frac{l}{2}\right) - \frac{1}{2}\left(l^2 - \frac{l^2}{4}\right)\right] = \frac{kl}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[k\int_0^{\frac{l}{2}} x\cos\frac{n\pi x}{l}\,dx + k\int_{\frac{l}{2}}^{l}(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

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$$= \frac{2k}{l}\left[\frac{l}{n\pi}\left(x\sin\frac{n\pi x}{l} \right)_0^{\frac{l}{2}} - \frac{l}{n\pi}\int_0^{\frac{l}{2}} \sin\frac{n\pi x}{l}\,dx \right.$$

$$\left. + \frac{l}{n\pi}\left((l-x)\sin\frac{n\pi x}{l} \right)_{\frac{l}{2}}^{l} - \frac{1}{n\pi}\int_{\frac{l}{2}}^{l} -\sin\frac{n\pi x}{l}\,dx] \right]$$

$$= \frac{2k}{l}\left[\frac{l}{n\pi}\frac{l}{2}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi x}{l} \right)_0^{\frac{l}{2}} \right.$$

$$\left. + \frac{l}{n\pi}\left(-\frac{l}{2}\sin\frac{n\pi}{2} \right) - \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi x}{l} \right)_{\frac{l}{2}}^{l} \right]$$

$$= \frac{2k}{l}\left[\frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1 \right) \right.$$

$$\left. - \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2} \right) \right]$$

$$= \frac{2k}{l}\frac{l^2}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2} \right]$$

$$= \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi \right]$$

$a_n = 0$, If is odd. If n is even and $= 2m$

$$a_{2m} = \frac{2kl}{4m^2\pi^2}\left[2\cos m\pi - 1 - \cos 2m\pi \right]$$

$$= \frac{kl}{m^2\pi^2}\left[\cos m\pi - 1 \right]$$

$$= \frac{kl}{m^2\pi^2}\left[(-1)^m - 1 \right]$$

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$$a_2 = -\frac{2kl}{1^2\pi^2}$$
$$a_4 = 0$$
$$a_6 = -\frac{2kl}{3^2\pi^2}$$
$$a_8 = 0$$
$$a_{10} = -\frac{2kl}{5^2\pi^2}, \ldots\ldots$$

$$f(x) = \frac{kl}{4} - \frac{2kl}{\pi^2}\left(\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{6\pi x}{l} + \frac{1}{5^2}\cos\frac{10\pi x}{l} + \ldots\ldots\right)$$

When $x = \frac{l}{2}, f(x) = \frac{kl}{2}$. Therefore

$$\frac{kl}{2} = \frac{kl}{4} - \frac{2kl}{\pi^2}\left(-\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \ldots\ldots\ldots\right)$$

Which gives

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots = \frac{\pi^2}{8}.$$

Example-9

Obtain the Fourier series expansion of $x\sin x$ as a cosine series in $(0, \pi)$ and show that

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$$1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-\frac{2}{7.9}+......=\frac{\pi}{2}.$$

Solution:

$x\sin x = a_0 + \sum a_n \cos nx$,

Where

$$\pi a_0 = \int_0^{\pi} x\sin x\, dx$$
$$= -\left(x\sin x\right)_0^{\pi} + \int_0^{\pi} \cos x\, dx$$
$$= \pi$$

Therefore $a_0 = 1$.

$$\frac{\pi}{2} a_n = \int_0^{\pi} x\sin x \cos nx dx$$
$$= \frac{1}{2}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx\ ,(n \neq 1)$$
$$= \frac{1}{2}\left(-\frac{1}{n+1}\right)\left[x\cos(n+1)x\right]_0^{\pi}$$
$$+\frac{1}{2}\left(\frac{1}{n-1}\right)\left[x\cos(n-1)x\right]_0^{\pi}$$
$$+\frac{1}{2(n+1)}\int_0^{\pi} \cos(n+1)x\, dx$$
$$-\frac{1}{2(n-1)}\int_0^{\pi} \cos(n-1)x\, dx$$

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$$= \frac{-1}{2(n+1)} \pi \cos(n+1)\pi + \frac{1}{2(n-1)} \pi \cos(n-1)\pi$$

$$= \frac{\pi}{2}\left[\frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1}\right]$$

$$= (-1)^{n+1} \frac{2}{n^2-1}\left(\frac{\pi}{2}\right)$$

$$a_2 = -\frac{2}{1.3}\,; a_3 = \frac{2}{2.4}\ ; a_4 = -\frac{2}{3.5}\,; a_5 = \frac{2}{4.6}; \ldots\ldots$$

And if $n = 1$,

$$\frac{\pi}{2} a_1 = \int_0^{\pi} x \sin x \cos x dx$$

$$= \frac{1}{2}\int_0^{\pi} x \sin 2x dx$$

$$= -\frac{1}{2}.\frac{1}{2}(x\cos 2x)_0^{\pi} + \frac{1}{4}\int_0^{\pi} \cos 2x dx$$

$$= -\frac{\pi}{4}$$

$$\Rightarrow a_1 = -\frac{1}{2}$$

$$x \sin x = 1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x$$

$$+ \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \ldots..$$

When $x = \frac{\pi}{2}$,

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$$1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-\frac{2}{7.9}+......=\frac{\pi}{2}.$$

Exercises-1

1. Find the Fourier series to represent the function

$f(x)=|\sin x|$, $-\pi<x<\pi$.

2. Find the Fourier series of the function $f(x)=x^2$ in $(-\pi,\pi)$ and deduce that

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.....=\frac{\pi^2}{12} \text{ and}$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....=\frac{\pi^2}{6}.$$

3. Find the Fourier series for the function

$$f(x)=\begin{cases}1 & \frac{-\pi}{2}<x<\frac{\pi}{2}\\ -1 & \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$$ and hence show that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....=\frac{\pi}{4}.$$

4. Find the Fourier series for $f(x)=|x|$,$-\pi\le x\le\pi$ and hence show that $1+\frac{1}{3^2}+\frac{1}{5^2}+.....=\frac{\pi^2}{8}$.

5. Find a Fourier series representation of the function $f(x)=e^x$, $-\pi<x<\pi$ and hence derive a series for

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$$\frac{\pi}{\sinh \pi}.$$

6. Find the Fourier series expansion of

$$f(x)=\begin{cases}0 & -\pi<x<0\\ \sin x & 0<x<\pi\end{cases}$$ and hence show that

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}.....=\frac{(\pi-2)}{4}.$$

Answers:

1. $\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+......\right).$

2. $\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+......\right).$

3. $\frac{4}{\pi}\left(\cos x-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}-\frac{\cos 7x}{7}+......\right).$

4. $\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+......\right).$

5.

$$\frac{2\sinh \pi}{\pi}\left[\left(\frac{1}{2}-\frac{1}{1^2+1}\cos x+\frac{1}{2^2+1}\cos 2x-\frac{1}{2^3+1}\cos 3x+......\right)\right.$$
$$\left.+\left(\frac{1}{1^2+1}\sin x-\frac{2}{2^2+1}\sin 2x+\frac{3}{3^2+1}\sin 3x+......\right)\right]$$

$$\frac{\pi}{\sinh \pi}=2\left(\frac{1}{5}-\frac{1}{10}+\frac{1}{17}+....\right)$$

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6. $\frac{1}{\pi}+\frac{\sin x}{2}-\frac{2}{\pi}\sum_{1}^{\infty}\frac{\cos 2nx}{4n^2-1}$.

Exercises-2

1.Find the Fourier series expansion of $f(x)$ with period 3, where $f(x)=2x-x^2$ in $(0,3)$.Hence show that $\sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

2. Find the Fourier series corresponding to the function

$$f(x)=\begin{cases}2 & -2\le x<0\\ x & 0<x<2\end{cases}$$ and hence show that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+......=\frac{\pi^2}{8} \text{ and } 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......=\frac{\pi}{4}.$$

3. A periodic square wave has a period 4 whose function is given by $f(x)=\begin{cases}0 & -2<t<1\\ k & -1<t<1\\ 0 & 1<t<2\end{cases}$

Find its Fourier series and hence show that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......=\frac{\pi}{4}.$$

4. If $f(x)=\begin{cases}\pi x & 0\le x<1\\ \pi(2-x) & 1\le x\le 2\end{cases}$ with period 2 ,find the Fourier series of $f(x)$.

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Answers:

1. $$-\frac{9}{\pi^2}\left(\cos\frac{2\pi x}{3}+\frac{1}{2^2}\cos\frac{4\pi x}{3}+\frac{1}{3^2}\cos\frac{6\pi x}{3}+....\right)+\frac{3}{\pi}\left(\sin\frac{2\pi x}{3}+\frac{1}{2}\sin\frac{4\pi x}{3}+\frac{1}{3}\sin\frac{6\pi x}{3}+....\right).$$

2. $$\frac{3}{2}-\frac{4}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+....\right)-\frac{2}{\pi}\left(\sin\frac{\pi x}{2}+\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+....\right).$$

3.

$$\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi t}{2}-\frac{1}{3}\cos\frac{3\pi t}{2}+\frac{1}{5}\cos\frac{5\pi t}{2}-\frac{1}{7}\cos\frac{7\pi t}{2}....\right)$$

4. $$\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos\pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+....\right).$$

Exercises-3

1.Represent $f(x)=\begin{cases}0 & 0<x<\frac{l}{2}\\ 1 & \frac{l}{2}<x<l\end{cases}$ by the cosine series.

2.If $f(x)=\begin{cases}x & 0<x<\frac{\pi}{2}\\ \pi-x & \frac{\pi}{2}<x<\pi\end{cases}$,Show that

$$f(x)=\frac{4}{\pi}\left(\sin x-\frac{1}{3^2}\sin 3x+\frac{1}{5^2}\sin 5x-.....\right)$$
$$=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x+\frac{1}{3^2}\cos 6x+\frac{1}{5^2}\cos 10x+....\right)$$

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3.Expand $f(x)=\begin{cases}\frac{1}{4}-x & 0<x<\frac{1}{2}\\ x-\frac{3}{4} & \frac{1}{2}<x<1\end{cases}$

as the Fourier series of sine terms.

4.Find the half range cosine series for $\sin x$ in $(0,\pi)$.

5.Find the half range sine series for x^2 in $(0,\pi)$.

Answers:

1.

$$\frac{1}{2}-\frac{2}{\pi}\left(\cos\frac{\pi x}{l}-\frac{1}{3}\cos\frac{3\pi x}{l}+\frac{1}{5}\cos\frac{5\pi x}{l}-\frac{1}{7}\cos\frac{7\pi x}{l}+.....\right).$$

3.

$$\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}+\frac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x+......$$

4. $$\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{4.1^2-1}+\frac{\cos 4x}{4.2^2-1}+\frac{\cos 6x}{4.3^2-1}+\frac{\cos 8x}{4.4^2-1}+.....\right)$$

5.

$$\left(\frac{2\pi}{1}-\frac{8}{\pi^3}\right)\sin x+\left(-\frac{2\pi}{3}\right)\sin 2x+\left(\frac{2\pi}{3}-\frac{8}{\pi 3^3}\right)\sin 3x+\left(-\frac{2\pi}{4}\right)\sin 4x+\left(\frac{2\pi}{3}-\frac{8}{\pi 5^3}\right)\sin 5x+....$$

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Chapter-3

Fourier Transform

3.1. Fourier Transform:

Fourier transform of a function $f(x)$ is defined as

$$F(s)=F\left[f(x)\right]=\int_{-\infty}^{\infty} f(x)e^{isx}dx \qquad (3.1.1)$$

The integral exists for all function $f(x)$ which are continuous in R except at most at a finite number of points and discontinuities, if any, are finite jump discontinuities and $f(x)$ is piecewise differentiable.

The inverse Fourier transform of $F(s)$ is defined as

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \qquad (3.1.2)$$

3.1.1 Fourier Sine Transform:

Fourier Sine Transform is defined as

$$F_s(s)=\int_0^{\infty} f(x)\sin sx\,dx$$

and its inverse as

$$f(x)=\frac{2}{\pi}\int_0^{\infty} F_s(s)\sin sx\,ds$$

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3.1.2 Fourier Cosine Transform:

Fourier Cosine Transform is defined as

$$F_c(s) = \int_0^\infty f(x)\cos sx\,dx$$

and its inverse as

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\,ds$$

Finite Fourier sine transform over $0 < x < L$ is defined as

$$F_s(x) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

and its Inverse as

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s(x)\sin\frac{n\pi x}{L}.$$

Finite Fourier cosine transform over $0 < x < L$ is defined as

$$F_c(x) = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

With the corresponding inverse as

$$f(x) = \frac{1}{L}F_c(0) + \frac{2}{L}\sum_{n=1}^{\infty} F_c(x)\cos\frac{n\pi x}{L}$$

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3.2. Properties of Fourier Transform:

1. **Linear**:

$$F\left[af(x)+bg(x)\right]=aF\left[f(x)\right]+bF\left[g(x)\right]$$

2. **Change of scale property:**

If $F(s)$ is any one of the Fourier Transform of $f(x)$

,then

$$F\left[f(ax)\right]=\int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

$$=\int_{-\infty}^{\infty} f(t)\frac{e^{\frac{ist}{a}}}{a}dt$$

$$=\frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{\frac{isx}{a}}dx$$

$$=\frac{1}{a}F\left(\frac{s}{a}\right)$$

3. **Shifting property:**

If $F(s)=F\left[f(x)\right]$, then

$$F\left[f(x-a)\right]=\int_{-\infty}^{\infty} f(t)e^{is(t+a)}dt,\ \left[x-a=t\right]$$

$$=e^{ias}\int_{-\infty}^{\infty} f(x)e^{isx}dx=e^{ias}F(s)$$

4. **Modulation:**

If $F(s)=F\left[f(x)\right]$, Then

$$F\left[f(x)\cos ax\right]=\frac{1}{2}\left[F(s+a)+F(s-a)\right]$$

Proof:

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$$F\left[f(x)\cos ax\right]=\int_{-\infty}^{\infty} f(x)\frac{e^{iax}+e^{-iax}}{2}e^{isx}dx$$

$$=\frac{1}{2}\left[\int_{-\infty}^{\infty}\left(f(x)e^{i(s+a)x}+f(x)e^{i(s-a)x}\right)dx\right]$$

$$=\frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx+\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}dx\right]$$

$$=\frac{1}{2}\left[F(s+a)+F(s-a)\right]$$

Similarly, it can be shown that

$$F_s\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_s(s+a)+F_s(s-a)\right]$$

$$F_c\left[f(x)\sin ax\right]=\frac{1}{2}\left[F_s(s+a)-F_s(s-a)\right]$$

$$F_s\left[f(x)\sin ax\right]=\frac{1}{2}\left[F_c(s-a)-F_c(s+a)\right]$$

$$F_s\left[f(x)\sin ax\right]=\int_{0}^{\infty} f(x)\sin ax\sin sx\,dx$$

$$=\frac{1}{2}\int_{0}^{\infty} f(x)\left[\cos(s-a)x-\cos(s+a)x\right]dx$$

$$=\frac{1}{2}\left[F_c(s-a)-F_c(s+a)\right]$$

Example-1

Find the Fourier transform of

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$$f(x)=\begin{cases}1-x^2 & ,|x|\le 1\\ 0 & ,|x|>1\end{cases}.$$

Solution:

$$F\left[f(x)\right]=\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$=\int_{-1}^{1}\left(1-x^2\right)e^{isx}dx$$

$$=\frac{1}{is}\left[e^{isx}\left(1-x^2\right)\right]_{-1}^{1}+\frac{1}{is}\int_{-1}^{1}e^{isx}2xdx$$

$$=\frac{2}{i^2s^2}\left[e^{isx}x\right]_{-1}^{1}-\frac{2}{i^2s^2}\int_{-1}^{1}e^{isx}dx$$

$$=-\frac{2}{s^2}\left[e^{is}+e^{-is}\right]+\frac{2}{s^2}\frac{1}{is}\left[e^{isx}x\right]_{-1}^{1}$$

$$=-\frac{4}{s^2}\left[\frac{e^{is}+e^{-is}}{2}\right]-\frac{2i}{s^3}\left[e^{is}-e^{-is}\right]$$

$$=-\frac{4}{s^2}\cos s+\frac{4}{s^3}\sin s$$

$$=\frac{4}{s^3}\left(\sin s-s\cos s\right)$$

Example-2

Find the Fourier transform of

$$f(x)=\begin{cases}1 & ,|x|\le a\\ 0 & ,|x|>a\end{cases}$$

and hence evaluate

$$\int_{0}^{\infty}\frac{\sin x}{x}dx.$$

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Solution:

$$F\left[f\left(x\right)\right]=\int_{-\infty}^{\infty} f\left(x\right)e^{isx}dx$$

$$=\int_{-a}^{a} e^{isx}dx=\frac{1}{is}\left[e^{ias}-e^{-ias}\right]$$

$$=\frac{2}{s}\sin as$$

Now

$$f\left(x\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{s}\sin as e^{-isx}ds$$

$$=\begin{cases}1, & -a<x<a\\ 0, & else\,where\end{cases}$$

At $x=0$, we know

$$f\left(x\right)=1.$$

$$f\left(0\right)=1=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}ds$$

Thus

$$\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin ax}{x}dx=1 \text{ with } a=1.$$

We have

$$\int_{0}^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}.$$

Example-3

Find the Fourier sine and cosine transform of

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$$f(x)=\begin{cases}1, & 0\le x\le a\\ 0, & x>a\end{cases}.$$

Solution:

$$F_s(s)=\int_0^\infty f(x)\sin sx\,dx$$
$$=\int_0^a \sin sx\,dx=\frac{1-\cos as}{s}$$

$$F_c(s)=\int_0^\infty \cos sx\,dx=\frac{1}{s}\sin as$$

Example-4

Find the Fourier cosine transform of $e^{-x^2}, 0\le x<\infty$.

Solution:

$$F\left(e^{-x^2}\right)=\int_{-\infty}^{\infty} e^{-x^2}e^{isx}dx$$
$$=\int_{-\infty}^{\infty} e^{-\left(x-\frac{is}{2}\right)^2-\frac{s^2}{4}}dx$$
$$=e^{-\frac{s^2}{4}}\int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}e^{-\frac{s^2}{4}}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\left(\cos sx+i\sin sx\right)dx=\sqrt{\pi}e^{-\frac{s^2}{4}}$$

On equating the real parts; it gives

$$\int_{-\infty}^{\infty} e^{-x^2}\cos sxdx=\sqrt{\pi}e^{-\frac{s^2}{4}}$$

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$$F_c\left[f(x)\right]=\int_0^\infty f(x)\cos sx\,dx$$

Now
$$=\int_0^\infty e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{2}\int_0^\infty e^{-x^2}\cos sx\,dx=\frac{\sqrt{\pi}}{2}e^{-\frac{s^2}{4}}$$

Example-5

Find the Fourier sine transform of

$$f(x)=2x,0<x<4.$$

Solution:

$$F_s(x)=\int_0^4 f(x)\sin\frac{n\pi x}{4}dx$$

$$=2\int_0^4 x\sin\frac{n\pi x}{4}dx$$

$$=-\frac{8}{n\pi}\left[x\cos\frac{n\pi x}{4}\right]_0^4+\frac{8}{n\pi}\int_0^4\cos\frac{n\pi x}{4}dx$$

$$=-\frac{8}{n\pi}\left[4\cos n\pi\right]+\frac{32}{n^2\pi^2}\left[\sin\frac{n\pi x}{4}\right]_0^4$$

$$=-\frac{32}{n\pi}(-1)^n$$

Example-6

Find the cosine transform of

$$f(x)=\begin{cases}\cos x, & 0<x<1\\ 0 & , \ x>1\end{cases}.$$

Solution:

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$$F_c(s)=\int_0^1 \cos x\cos sx\,dx$$
$$=\frac{1}{2}\int_0^1\left[\cos(s+1)x+\cos(s-1)x\right]dx$$
$$=\frac{1}{2}\frac{1}{s+1}\left[\sin(s+1)x\right]_0^1+\frac{1}{2}\frac{1}{s-1}\left[\sin(s-1)x\right]_0^1$$
$$=\frac{1}{2}\left[\frac{\sin(s+1)}{s+1}+\frac{\sin(s-1)}{s-1}\right]$$

Example-7

Find the finite Fourier sine transform of

$$f(x)=\frac{x}{\pi},\ 0<x<\pi.$$

Solution:

$$F_s(n)=\int_0^\pi \frac{x}{\pi}\sin\frac{n\pi x}{\pi}\,dx$$
$$=\frac{1}{\pi}\left[-\frac{1}{n}\left(x\cos nx\right)_0^\pi+\frac{1}{n}\int_0^\pi \cos nx\,dx\right]$$
$$=-\frac{1}{n\pi}\pi\cos n\pi=\frac{(-1)^{n-1}}{n}$$

Example-8

Find the finite cosine transform of

$$f(x)=\frac{x^2}{2\pi}-\frac{\pi}{6},\ 0\le x\le\pi\cdot$$

Solution:

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$$F_c(n)=\int_0^{\pi}\left(\frac{x^2}{2\pi}-\frac{\pi}{6}\right)\cos nx\,dx$$

$$=\frac{1}{n}\left[\left(\left(\frac{x^2}{2\pi}-\frac{\pi}{6}\right)\sin nx\right)_0^{\pi}\right]-\frac{1}{n}\int_0^{\pi}\frac{x}{\pi}\sin nx\,dx$$

$$=\frac{1}{\pi n^2}\left(x\cos nx\right)_0^{\pi}-\frac{1}{\pi n^2}\int_0^{\pi}\cos nx\,dx$$

$$=\frac{(-1)^n}{n^2}$$

$$\Rightarrow F_c(0)=\int_0^{\pi}\left(\frac{x^2}{2\pi}-\frac{\pi}{6}\right)dx=0$$

Example-9

Find the inverse of the finite sine transform

$$F_s(n)=\frac{1+\cos n\pi}{n\pi}\quad ,0<x<\pi$$

Solution:

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}F_s(n)\sin\frac{n\pi x}{\pi}$$

$$=\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1+\cos n\pi}{n}\sin nx$$

Example-10

Solve the integral equation

$$\int_0^{\infty}f(x)\cos sx\,dx=\begin{cases}1-s & ,0<s<1\\ 0 & ,s>1\end{cases}.$$

Solution:

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The problem states that the Fourier cosine transform of $f(x)$ is

$$F_c(s)=\int_0^\infty f(x)\cos sx\,dx$$
$$=\begin{cases}1-s & ,0<s<1\\ 0 & ,s>1\end{cases}$$

Thus the inverse is

$$f(x)=\frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\,ds$$
$$=\frac{2}{\pi}\int_0^1 (1-s)\cos sx\,ds$$
$$=\frac{2}{\pi}\left[\frac{1}{x}(1-s)\sin sx\right]_0^1+\frac{2}{\pi x}\int_0^1 \sin sx\,ds$$
$$=-\frac{2}{\pi x^2}(\cos sx)_0^1=\frac{2}{\pi x^2}(1-\cos sx)$$
$$=\frac{4}{\pi x^2}\sin^2\frac{x}{2}$$

Example-11

Find the finite Fourier cosine transform of

$$f(x)=x\ ,0<x<4.$$

Solution:

$$F_s(n)=\int_0^4 f(x)\cos\frac{n\pi x}{4}dx$$
$$=\int_0^4 x\cos\frac{n\pi x}{4}dx$$

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$$= \frac{4}{n\pi}\left(x\sin\frac{n\pi x}{4} \right)_0^4 - \frac{4}{n\pi}\int_0^4 \sin\frac{n\pi x}{4}dx$$

$$= \frac{16}{n^2\pi^2}\left(\cos\frac{n\pi x}{4} \right)_0^4$$

$$= \frac{16}{n^2\pi^2}\left[(-1)^n - 1 \right]$$

3.3. Solution of Using Fourier transform:

Number of independent variables in a PDE can be reduced by one by an application of an appropriate Fourier transform. The range of the independent variable suggests the use of finite or infinite transform. Further considerations are given below.

Suppose we want to eliminate a second order derivative $\frac{\partial^2 u}{\partial x^2}$ using infinite sine transform.(range of x is 0 to ∞)

$$\int_0^\infty u_{xx}\sin sxdx = \left[u_x \sin sx \right]_0^\infty - s\int_0^\infty u_x \cos sxdx$$

$$= 0 - s\left[u\cos sx \right]_0^\infty - s^2\int_0^\infty u\sin sxdx$$

$$= s\,u|_{x=0} - s^2 U_s(s)$$

Where

$U_s(s) = \int_0^\infty u\sin sxdx$,sine transform of u .

If $u|_{x=0}$ is given ,the above involves only sine transform.

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On other hand

$$\int_0^\infty u_{xx}\cos sx\,dx = [u_x \cos sx]_0^\infty + s\int_0^\infty u_x \sin sx\,dx$$

$$= -u_x\big|_{x=0} + s(u\sin sx)_0^\infty - s^2\int_0^\infty u\cos sx\,dx$$

$$= -u_x\big|_{x=0} - s^2 U_c(s)$$

Where $U_c(s)$ is the cosine transform of u.Thus the following points are to be considered to decide which transform to be used.

(i) Use the complex Fourier transform $\int_{-\infty}^{\infty} ue^{isx}dx$ if the domain of x is infinite.

(ii) If the domain is semi-infinite ,that is $(0,\infty)$.

Use

(a) Sine transform if u at $x=0$ is known.

(b) Cosine transform if u_x at $x=0$ is known.

(iii) If the domain is finite, use **(a)** Sine transform if u is known at both ends. (b) Cosine transform if u_x is known at both ends.

Example-12

Solve the BVP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\ ,-\infty < x < \infty,\ t > 0$$

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and $u(x,0)=f(x)$.

Solution:

Applying the complex form of Fourier transform

$$U(s,t)=\int_{-\infty}^{\infty} u(x,t)e^{isx}dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}e^{isx}dx=\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}e^{isx}dx$$

$$=\left(u_x e^{isx}\right)_{-\infty}^{\infty}-is\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}e^{isx}dx$$

On assuming $u_x \to 0$ as $x \to \infty$ at both ends.

It gives

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{isx}dx=-is\left(ue^{isx}\right)_{-\infty}^{\infty}+i^2s^2\int_{-\infty}^{\infty} ue^{isx}dx$$

$$\Rightarrow \frac{dU}{dt}=-s^2U$$

Assuming $u \to 0$ as $x \to \pm\infty$.

Solving the ordinary differential equation ,It gives

$U=a.e^{-s^2t}$ (Where a is constant)

Then $U(s,0)=\int_{-\infty}^{\infty} f(x)e^{isx}dx=a$

Thus

$$U(s,t)=e^{-s^2t}\int_{-\infty}^{\infty} f(\theta)e^{is\theta}d\theta$$

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The inverse transform of U is given by

$$U(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-s^2t}\left(\int_{-\infty}^{\infty} f(\theta)e^{is\theta}d\theta\right)e^{-isx}dx$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\theta)\int_{-\infty}^{\infty} e^{-s^2t}e^{is(\theta-x)}ds.d\theta$$

Now

$$-s^2t+is(\theta-x)=-t\left[s^2-\frac{is}{t}(\theta-x)\right]$$
$$=-t\left[\left(s-\frac{i}{2t}(\theta-x)\right)^2-\frac{i^2}{4t^2}(\theta-x)^2\right]$$

Therefore

$$U(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\theta)\int_{-\infty}^{\infty} e^{-t\left[\left(s-\frac{i}{2t}(\theta-x)\right)^2+\frac{(\theta-x)^2}{4t^2}\right]}ds.d\theta$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\theta)e^{-\frac{(\theta-x)^2}{4t^2}}\int_{-\infty}^{\infty} e^{-y^2}\frac{dy}{\sqrt{t}}.d\theta$$
$$with\ s-\frac{i}{2t}(\theta-x)=\frac{y}{\sqrt{t}}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\theta)e^{-\frac{(\theta-x)^2}{4t^2}}\frac{\sqrt{\pi}}{\sqrt{t}}d\theta \quad ,$$
$$\left(\int_{-\infty}^{\infty} e^{-y^2}dy=\sqrt{\pi}\right)$$
$$=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\theta)e^{-\frac{(\theta-t)^2}{4t^2}}d\theta$$

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is the desired solution.

Example-13

Solve $\frac{\partial u}{\partial t}=c\frac{\partial u}{\partial x^2}$ $,x>0,t>0$

$u(0,t)=u_0$ $,t>0$ and $u(x,0)=0,x>0$.

Applying Fourier sine transform

$$U_S(t)=\int_0^\infty U\sin sxdx$$

$$\frac{dU_S}{dt}=c\int_0^\infty u_{xx}\sin sxdx$$

$$=c\left(u_x\sin sx\right)_0^\infty-cs\int_0^\infty u_x\cos sx\,dx$$

$$=-cs\left(u_x\cos sx\right)_0^\infty-cs^2\int_0^\infty u\sin sx\,dx \quad as\ u_x\to 0\ as\ x\to\infty$$

$$=csu_0-cs^2U,$$

$$as\ u\big|_{x=0}=u_0\ \ and\ \ u\to 0\ \ as\ x\to\infty$$

Solving this linear first order ordinary differential equation,

$$\frac{dU_s}{dt}+cs^2U_s=csu_0$$

$$I.F=e^{\int cs^2dt}=e^{cs^2t}$$

$$U_se^{cs^2t}=k+\int esu_0e^{cs^2t}dt$$

$$=k+\frac{u_0}{s}e^{cs^2t}$$

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But $U_s = 0$ when $t = 0$ ($u = 0$ for all x at $t = 0$)

Therefore, $0 = k + \frac{u_0}{s}$

$$U_s = -\frac{u_0}{s} e^{-cs^2 t} + \frac{u_0}{s}$$
$$= \frac{u_0}{s}\left(1 - e^{-cs^2 t}\right)$$

Finally, the solution of the given problem is obtained by inversion as

$$u(x,t) = u_0 \int_0^{\infty} \frac{1}{s}\left(1 - e^{-cs^2 t}\right) \sin sxds$$
$$= u_0 \left[\frac{\pi}{2} - \int_0^{\infty} \frac{1}{s} e^{-cs^2 t} \sin sxds\right] . \left[\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}\right]$$

Example-14

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 \le x \le \pi, t > 0$

$u(0,t) = 0, u(\pi,t) = 0$ for $t > 0$ and $u(x,0) = 2x$.

Using finite sine transforms over $[0,\pi]$ with

$$u_n = \int_0^{\pi} u \sin nxdx,$$

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$$\frac{du_n}{dt} = \int_0^{\pi} u_{xx} \sin nx dx$$

$$= \left(u_x \sin nx\right)_0^{\pi} - n\int_0^{\pi} u_x \cos nx dx$$

$$= -n\left(u_x \cos nx\right)_0^{\pi} - n^2 \int_0^{\pi} u \sin nx dx$$

$$= -n^2 U_n$$

Since $u = 0$ at both ends.(given)

Solving this ordinary differential equation,

$$\frac{dU_n}{dt} = -n^2 U_n \;, U_n = ce^{-n^2 t}$$

But $U_n\left(0\right) = \int_0^{\pi} 2x \sin nx dx = -\frac{2\pi}{n}\left(-1\right)^{n-1}$

$$U_n = \frac{2\pi}{n}\left(-1\right)^{n-1} e^{-n^2 t}$$

Finally ,inverting for the desired solution,

$$U\left(x,t\right) = \frac{2}{\pi}\sum_1^{\infty} U_n\left(t\right)\sin nx$$

$$= \frac{2}{\pi} 2\pi \sum_1^{\infty} \frac{\left(-1\right)^{n-1}}{n} e^{-n^2 t} \sin nx$$

$$= 4\sum_1^{\infty} \frac{\left(-1\right)^{n-1}}{n} e^{-n^2 t} \sin nx$$

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Example-15

Solve $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$ $, 0 < x < a, t > 0$

$u(0,t) = u(a,t) = 0$ for all $t > 0$ and

$u(x,0) = \alpha x(a-x)$ and $u_t(x,0) = 0, 0 < x < a$.

Using finite sine transform, (u is known at the ends)

$$U_n = \int_0^a u(x,t)\sin\frac{n\pi x}{a}dx$$

$$\int_0^a u_{tt}\sin\frac{n\pi x}{a}dx = c^2\int_0^a u_{xx}\sin\frac{n\pi x}{a}dx$$

$$\frac{d^2}{dt^2}\int_0^a u\sin\frac{n\pi x}{a}dx = c^2\left(u_x \sin n\pi x\right)_0^a - c^2\frac{n\pi}{a}\int_0^a u_x\cos\frac{n\pi x}{a}dx$$

That is,

$$\frac{d^2U_n}{\partial t^2} = -c^2\frac{n\pi}{a}\left(u\cos\frac{n\pi x}{a}\right)_0^a - c^2\frac{n^2\pi^2}{a^2}\int_0^a \sin\frac{n\pi x}{a}dx$$

$$= -c^2\frac{n^2\pi^2}{a^2}U_n$$

Solving, $U_n = A\cos\frac{cn\pi}{a}t + B\sin\frac{cn\pi}{a}t$

$$\frac{dU_n}{dt} = -A\frac{cn\pi}{a}\sin\frac{cn\pi}{a}t + B\frac{cn\pi}{a}\cos\frac{cn\pi}{a}t$$

Since $u_t(x,0) = 0$ for $0 < x < a, U_n(0) < 0$ .This gives

$B = 0$. Therefore ,

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$$U_n = A\cos\frac{cn\pi}{a}t$$

When $t=0$,

$$U_n(0)=\int_0^a \alpha x(a-x)\sin\frac{n\pi x}{a}dx$$

$$=-\frac{\alpha a}{n\pi}\left[x(a-x)\cos\frac{n\pi x}{a}\right]_0^a+\frac{\alpha a}{n\pi}\int_0^a(a-2x)\cos\frac{n\pi x}{a}dx$$

$$=-\frac{\alpha a^2}{n^2\pi^2}\left[x(a-2x)\sin\frac{n\pi x}{a}\right]_0^a-\frac{\alpha a^2}{n^2\pi^2}\int_0^a(-2)\sin\frac{n\pi x}{a}dx$$

$$=-\frac{2\alpha a^2}{n^2\pi^2}.\frac{a}{n\pi}\left[\cos\frac{n\pi x}{a}\right]_0^a$$

$$=-\frac{2\alpha a^3}{n^3\pi^3}\left[(-1)^n-1\right]$$

When $t=0, U_n(0)=\frac{2\alpha a^3}{n^3\pi^3}\left[1-(-1)^n\right]$.

Therefore $U_n(t)=\frac{2\alpha a^3}{n^3\pi^3}\left[1-(-1)^n\right]\cos\frac{cn\pi}{a}t$.

Inserting this for the desired solution,

$$u(x,t)=\frac{2}{a}\sum_1^\infty\frac{2\alpha a^3}{n^3\pi^3}\left[1-(-1)^n\right]\cos\frac{cn\pi}{a}t\sin\frac{n\pi x}{a}$$

$$=\frac{4\alpha a^2}{\pi^3}\sum_1^\infty\frac{1}{n^3}\left[1-(-1)^n\right]\cos\frac{cn\pi}{a}t\sin\frac{n\pi x}{a}$$

$$=\frac{8\alpha a^2}{\pi^3}\left[\cos\frac{c\pi t}{a}\sin\frac{\pi x}{a}+\frac{1}{3^2}\cos\frac{3c\pi t}{a}\sin\frac{3\pi x}{a}+\frac{1}{5^2}\cos\frac{5c\pi t}{a}\sin\frac{5\pi x}{a}+.....\right]$$

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Example-16

Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, 0<x<\pi, 0<y<\pi$.

$u(0,y)=u(\pi,y)=0$ for $0<y<\pi$ and $u(x,0)=0$ and

$u(x,\pi)=u_0$ for $0<x<\pi$.

Using the finite sine transform

$$U_n=\int_0^\pi u(x,y)\sin nxdx$$

$$\int_0^\pi u_{xx}\sin nxdx+\int_0^\pi u_{yy}\sin nxdx=0$$

That is,

$$(u_x\sin nx)_0^\pi-n\int_0^\pi u_x\cos nxdx+\frac{d^2}{dy^2}\int_0^\pi u\sin nxdx=0.$$

That is, $-n(u\cos nx)_0^\pi-n^2\int_0^\pi u\sin nxdx+\frac{d^2U_n}{dy^2}=0$.

giving $\frac{d^2u_n}{dy^2}=n^2U_n$.

Solving this

$U_n=a\cosh ny+b\sinh ny$ [or $ae^{ny}+be^{-ny}$]

$u(x,0)=0$ for all x .That is $U_n(0)=0$.This condition gives $a=0$ .Further $u(x,\pi)=u_0$.

Therefore ,

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$$U_n(\pi)=\int_0^{\pi} u_0 \sin nx dx = \frac{u_0}{n}\left[1-(-1)^n\right]$$

This condition gives

$b\sinh n\pi = \frac{u_0}{n}\left[1-(-1)^n\right]$, therefore ,

$$U_n(y)=\frac{u_0}{n}\frac{1-(-1)^n}{\sinh n\pi}\sinh ny$$

Inverting for the solution of the given problem,

$$u(x,y)=\frac{2}{\pi}\sum_1^{\infty}\frac{u_0}{n}\frac{1-(-1)^n}{\sinh n\pi}\sinh ny \sin nx$$
$$=\frac{4u_0}{\pi}\sum_1^{\infty}\frac{\sinh(2n-1)y}{n\sinh(2n-1)\pi}\sin(2n-1)x$$

Example-17

Solve, using an appropriate Fourier transform,

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2},\ 0<x<\pi,\ t>0$$

$u_x=0$ when $x=0$ and $x=\pi$ for $t>0$ and

$u(x,0)=f(x), 0<x<\pi$.

As $u_x=0$ at both ends of the finite interval, we use the finite Fourier cosine transform

$$U_n(t)=\int_0^{\pi} u(x,t)\cos nx dx$$

From the given equation

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$$\int_0^\pi \frac{\partial u}{\partial t}\cos nx dx = k\int_0^\pi \frac{\partial^2 u}{\partial x^2}\cos nx dx$$

$$\frac{d}{dt}\int_0^\pi u\cos nx dx = \left(u_x \cos nx\right)_0^\pi + kn\int_0^\pi u_x \sin nx dx$$

$$\frac{dU_n}{dt} = kn\left(u\sin nx\right)_0^\pi - kn^2\int_0^\pi u\cos nx dx$$

$$= -kn^2 U_n$$

$(u_x = 0$ at $x = 0, \pi$ for all $t)$

$U_n = ae^{-kn^2 t}$, a is a constant of integration.

Since $u(x,0) = f(x)$ at $t = 0$,

$$U_n = \int_0^\pi f(x)\cos nx dx = H(n) \text{ say}$$

When $t = 0$, $U_n = H(n)$ giving $a = H(n)$. Thus

$U_n = H(n)e^{-kn^2 t}$.

Inverting this,

$$u(x,t) = \frac{2}{\pi}u_0 + \frac{2}{\pi}\sum_1^\infty U_n H\cos nx$$

$$= \frac{b}{\pi}H(0) + \frac{2}{\pi}\sum_1^\infty H(n)e^{-kn^2 t}\cos nx$$

Where $H(n) = \int_0^\infty f(x)\cos nx dx$.

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Exercises-1

1. Find the Fourier transform of

$$f(x)=\begin{cases} x & for\ |x|<a \\ 0 & elsewhere \end{cases}.$$

2. Find the Fourier transform of

$$f(x)=\begin{cases} x & for\ \alpha<x<\beta \\ 0 & elsewhere \end{cases}.$$

3. Find the Fourier cosine transform of

$$f(x)=\begin{cases} x & 0<x<1 \\ 2-x & 1<x\le 2 \\ 0 & x>2 \end{cases}.$$

4. Find the Inverse Fourier transform of $e^{-|s|a}$.

5. Find the Inverse cosine transform of

$$F(s)=\begin{cases} a-\dfrac{s}{2} & s<2a \\ 0 & s\ge 2a \end{cases}.$$

6. Find the Fourier sine transform of $f(x)=\dfrac{1}{x}$.

Answers:

1. $\dfrac{2i}{s^2}(\sin as - as\cos as)$

2. $\dfrac{i}{s}\left(\alpha e^{is\alpha}-\beta e^{is\beta}\right)+\dfrac{1}{s^2}\left(e^{is\beta}-e^{is\alpha}\right)$

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3. $\frac{2\cos s}{s^2}(1-\cos s)$

4. $\frac{a}{\pi\left(a^2+x^2\right)}$

5. $2\frac{\sin^2 ax}{\pi x^2}$

6. $\frac{\pi}{2}$

Exercises-2

1. Show that the solution of

$u_t = u_{xx}, x>0, t>0,\ u_x(0,t)=0$ for $t>0$ and

$u(x,0)=\begin{cases} x & 0<x\le 1 \\ 0 & x>1 \end{cases}$ is

$\frac{2}{\pi}\int_0^\infty\left[\frac{\sin s}{s}+\frac{\cos s-1}{s^2}\right]e^{-s^2 t}\cos sx dx$.

2. Show that the solution of

$u_t = c^2 u_{xx}, x>0, t>0,\ u(x,0)=0,\ u_x(0,t)=-k$ is

$u(x,t)=k\int_0^\infty \frac{1}{s^2}\cos sx\left(1-e^{-cs^2 t}\right)ds$.

3. Show that the solution of

$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}=0,\ -\infty<x<\infty, y\ge 0$, where the partial

derivatives of u go to 0 as $x\to\pm\infty$ and

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$u(x,0) = f(x)$, $\frac{\partial u}{\partial y}(x,0) = 0$ is

$\frac{1}{\pi}\int_{-\infty}^{\infty} F(s)\cos s^2 y e^{-isx} ds$, where $F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$.

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Chapter-4

Matrices

4.1. Introduction

To find the history of Linear algebra, it is essential that first, we determine what linear algebra is.

Linear algebra is the branch of mathematics generally deals with linear equations such as $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ Linear maps such as

$$\left(x_1, x_2, ..., x_n\right) \mapsto a_1x_1 + a_2x_2 + ... + a_nx_n .$$

And their representation in vector space and through matrix representations.

Linear algebra applies to almost all areas of mathematics. For instance, the Linear fundamental algebra is in defining the necessary objects such as lines, planes and rotations through geometrical representation. It is also used in most sciences and engineering areas because it allows many natural phenomena through modelling and efficiently compute such models. It is also the study of a particular algebraic structure called a vector space. Secondly, it is the study of linear sets of equations and their transformation properties. Finally, it

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is the branch of mathematics linked with investigating the properties of finite-dimensional vector space and linear mapping between such spaces and plays a central role in modern mathematics which is of importance in the field of engineering and physical, social and behavioural science.

In this chapter, we shall introduce one of the vital parts of linear algebra, i.e. a matrix or a rectangular array of numbers together with the standard matrix operations which is generally used in dealing with a linear system of equation. Matrices are come across frequently in many areas of mathematics, engineering and the physical and social sciences typically when data is given in the tabular form.

4.2. Matrices

An $m \times n$ matrix A is a rectangular array of numbers, real or complex with $m-$rows and $n-$columns.

We shall write a_{ij} for the number that appears in ith row and jth column of A.This is called the (i, j) entry of A.

We can either write A in the extended form

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$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ or as } \left[a_{ij}\right]_{m\times n}.$$

Here the subscripts m and n tells us the respective number of rows and columns of A.

Explicit examples of matrices are

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 7 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 3.4 & 6 \\ \sqrt{3} & \frac{4}{5} & -1 \end{bmatrix}.$$

Example-1

Find the extended form of the matrix $\left[(-1)^i j - i\right]_{2\times 3}$.

Solution:

The (i, j) entry of the matrix is $(-1)^i j - i$, where $i = 1, 2$ and $j = 1, 2, 3$.

So the matrix is $\begin{bmatrix} 0 & -3 & -4 \\ -1 & 0 & 1 \end{bmatrix}$.

Two matrices A and B are said to be equal, i.e. $A = B$, if they have the same number of rows and columns and

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the entries of A being similar to the entries of the matrix B.

More briefly, two matrices are equal if they look exactly alike.

4.3. Some special matrices:

We shall discuss here some particular types of matrices that are generally used in the study of matrices,

(i) An $1\times n$ matrix or $n-$row vector of A has a single row $A=\left[a_{11}a_{12}.........a_{1n}\right]$.

(ii) An $m\times 1$ matrix or $m-$column vector of B has just one column

$$B=\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}.$$

(iii) A matrix is said to be a *square* if it has the same number of rows and columns.

(iv) A matrix whose all entries are zero termed as the *zero matrix*. The zero matrix is denoted by 0_{mn} or simply 0.

Similarly, Sometimes 0_{nn} is written as 0_n.

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For example,

0_{23} is the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

0_{33} is the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(v) The identity $n \times n$ matrix has 1's on the principal diagonal, that is from the top left to bottom right and zeros elsewhere. Thus it has the form

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \vdots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The identity matrix is written as I_n or simply I .

(vi) A square matrix is said to be an *upper triangular* matrix, if all the entries below the principal diagonal are all zero. Similarly, a matrix is said to be *lower triangular* if all entries above the principal diagonal are zero.

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For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$ are upper triangular

and lower triangular respectively.

(vii) A square matrix in which all the non-zero elements are on the *principal diagonal* and the remaining components above and below the *principal diagonals* are zeros is called a *diagonal matrix*.

A diagonal matrix is said to be a *scalar matrix* if all the elements on the *principal diagonal* of the *diagonal matrix* are all equal.

Example-2

The matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ respectively

are the diagonal and the scalar matrix.

Note: Diagonal matrices have much simpler algebraic properties than the general square matrices.

4.4. Operation with matrices

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We shall now introduce some addition and multiplication properties that can be performed on matrices.

Our object in doing so is to develop a systematic means of performing calculations with matrices.

(i) Addition and subtraction:

Let A and B two $m \times n$ matrices namely a_{ij} and b_{ij} for their respective (i, j) entries.

Define the sum $A+B$ to be the $m \times n$ matrix whose (i, j) entries is $a_{ij}+b_{ij}$. Thus to form the matrix $A+B$, we simply add the corresponding entries of A and B.

Similarly, the difference $A-B$ is the $m \times n$ matrix whose (i, j) entry is $a_{ij}-b_{ij}$. However $A+B$ and $A-B$ are not defined. If A and B do not have the same number of rows and columns.

(ii) Scalar Multiplication:

By a scalar, we mean a number opposed to a matrix or array of numbers. Let c be a scalar and A be an $m \times n$ matrix. Then the scalar multiple cA is the $m \times n$ matrix whose (i, j) entry is ca_{ij}. Thus to form cA, we have to

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multiply every entry of A by the scalar c. The matrix $(-1)A$ is usually written as $-A$.

It is called the negative of A since it has the property that $A+(-A)=0$.

Example-3

If $A=\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

Then

$$3A+2B=\begin{bmatrix} 5 & 13 & 2 \\ -6 & -4 & 7 \end{bmatrix} \text{and}$$

$$2A-3B=\begin{bmatrix} -1 & 0 & -3 \\ -4 & 6 & -4 \end{bmatrix}.$$

(iii) Matrix multiplication:

Consider a pair of 2×2 matrices

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

The matrix multiplication can be done by using a simple rule, i.e. row times column rule.

For example:

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The $(1,2)$ entry arises from multiplying corresponding entries of row1 of A and column 2 of B and then adding the resulting number. i.e.

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}\begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix} = a_{11}b_{21} + a_{12}b_{22}.$$

Similarly, other multiplication can also be done from a row of A and a column of B.

Next, we define the product AB where A is an $m \times n$ matrix and B is an $n \times p$ matrix.

The rule is that the (i, j) entry of AB is obtained by multiplying the corresponding row entries of A with the column entries of B and then adding up to get the resulting product. It is called the row times-column rule.

For example:

Consider a row i of matrix A as $\begin{bmatrix} a_{i1} & \dots & a_{in} \end{bmatrix}$ and a column j of matrix B as $\begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$.

Hence the (i, j) entry of AB can be expressed as $a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$.

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In more concisely, It can be written by using the summation notation as $\sum_{k=1}^{n} a_{ik} b_{kj}$.

Note:

Matrix multiplications make sense. If the number of columns of matrix A is equals the number of rows of the matrix B . i.e. the product of an $m \times p$ matrix and $p \times n$ matrix is an $m \times n$ matrix.

Example-4

Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 5 & -1 \end{bmatrix}$.

Since, A is an 2×3 matrix and B is an 3×3 matrix , we see that product AB is defined and is a 2×3 matrix.

Using the row times-column rule, we quickly find that

$$AB = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 16 & -2 \end{bmatrix}.$$

However, the product BA is not defined.

Example-5

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Let $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$.

In this case, both AB and BA are defined, but these matrices are different;

$$AB=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0_{22}$$

and $BA=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$.

Note:

The matrix multiplication is not commutative that is AB and BA may be different when both are defined. Similarly, the product of two non-zero matrices can also be zero, which shows that the theory of division in matrices will face some difficulties.

(iv) Power of the matrix:

Let A be an $n\times n$ matrix, then the mth power of A, is defined by the equations

$A^0=I_n$ and $A^{m+1}=A^m A$, Where m is a non-negative integer.

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We do not attempt to define the negative powers at this junction.

Example-6

Let $A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, Then $A^2=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^3=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $A^4=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I_2$.

Note:

The higher power of A do not lead to new matrices.

In this example: A has just four distinct powers, $A^0=I_2, A^1=A, A^2$ and A^3.

(v) The transpose of a matrix:

If A is an $m\times n$ matrix, then the transpose of A, i.e. A^T is the $n\times m$ matrix whose (i,j) entry equals the (j,i) entry of A. Thus the columns of A becoming the rows of A^T.

Example-7

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If $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$,

Then the transpose of A can be defined as $A^T=\begin{bmatrix} a_1 & a_1 & a_1 \\ b_1 & b_2 & b_3 \end{bmatrix}$.

A matrix A is called *symmetric*, when $A=A^T$, i.e. it equals to its transpose.

On the other hand

If A^T equals $-A$ then A is said to be *skew-symmetric*.

For example:

The matrices $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ respectively are the *symmetric* and *skew-symmetric* matrix.

Note:

(i) Symmetric and skew-symmetric matrices are must be square matrices.

(ii) Symmetric matrices can be reduced to a diagonal matrix in a real sense.

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4.5. The law of matrix algebra:

We shall now list several properties which are satisfied by the matrix operations defined above.

These properties will allow us to manipulate the matrix systematically using the properties of matrix addition and multiplication.

Theorem: 4.1

Let A, B and C are matrices and c, d are scalars, by considering matrix addition and scalar multiplication.

The following addition and multiplication matrix operation properties are satisfied.

(i) $A+B=B+A$ (Commutative law)

(ii) $(A+B)+C=A+(B+C)$ (Associative law of addition)

(iii) $A+0=A$

(iv) $(AB)C=A(BC)$ (Associative law of multiplication)

(v) $AI=A=IA$

(vi) $A(B+C)=AB+AC$ (Distributive law)

(vii) $(A+B)C=AC+BC$ (Distributive law)

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(viii) $A-B=A+(-1)B$

(ix) $(cd)A=c(dA)$

(x) $c(AB)=(cA)B=A(cB)$

(xi) $c(A+B)=cA+cB$

(xii) $(c+d)A=cA+dA$

(xiii) $(A+B)^T=A^T+B^T$

(xiv) $(AB)^T=B^TA^T$

4.6. The inverse of a square matrix:

An $n\times n$ square matrix A is said to be *invertible*, if $|A|\neq 0$ and there is an $n\times n$ matrix B such that

$$AB=I_n=BA.$$

If a matrix A is invertible Then B is called an inverse of A.

A matrix which is *not invertible* is called a *singular* matrix, while an invertible matrix is said to be *non-singular*.

Example-8

Show that the matrix $\begin{bmatrix}1 & 3\\ 3 & 9\end{bmatrix}$ is not invertible.

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Solution:

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an inverse of the matrix $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$, then it should satisfy

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Which gives a set of linear equations with no solutions:

$$a + 3c = 1$$
$$b + 3d = 0$$
$$3a + 9c = 0$$
$$3b + 9d = 1$$

Since the first and third equation contradicts each other so the matrix is not invertible.

Example-9

Show that $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is invertible and find an inverse of A if it exists.

Solution:

Suppose that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an inverse of A.

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Write out the product AB and set it equal to I_2 which gives a system of linear equations that has a solution

$$\begin{aligned} a + 2c &= 1 \\ b + 2d &= 0 \\ c &= 0 \\ d &= 1 \end{aligned}$$

Which gives a unique solution

$$a = 1, b = -2, c = 0, d = 1.$$

Thus the matrix $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ can be checked for the inverse of A.

Next we need to verify that BA is equal to I_2.

Now

$$BA = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\Rightarrow B$ is an inverse of A.

$\Rightarrow A$ is invertible.

Now next question is that if a square matrix is *invertible*, then how can we find an inverse.

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We now present some crucial facts about the inverse of matrices.

Theorem-4.2

A square matrix has at most one inverse.

Proof:

Suppose that a square matrix A has two inverses B_1 and B_2 .

Then

$$AB_1 = AB_2 = I = B_1A = B_2A$$

The idea of the proof is to consider the product $(B_1A)B_2$.

Since $B_1A = I$ so it equals $IB_2 = B_2$.

On the other hand, by the associative law, it also equals $B_1(AB_2)$ which equals $B_1I = B_1$. Therefore $B_1 = B_2$.

Note: From now on, we shall write A^{-1} for the unique inverse of an invertible matrix A.

Theorem-4.3

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(a) If A is an invertible matrix then A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.

(b) If A and B are invertible matrices of the same size then AB is invertible and

$$\left(AB\right)^{-1} = B^{-1}A^{-1}.$$

Proof:

(a) Indeed, we have $A.A^{-1} = I = A^{-1}A$ which show that A is an inverse of A^{-1}.

Since A^{-1} cannot have more than one inverse, Therefore its inverse must be A.

(b) To prove the assertion, we need only to check that $B^{-1}A^{-1}$ is an inverse of AB.

Now $\left(AB\right)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1}$,

Now by two applications of the associative law;

the latter equals $AIA^{-1} = AA^{-1} = I$.

Similarly $\left(B^{-1}A^{-1}\right)\left(AB\right) = I$

Since the inverses are unique, so $\left(AB\right)^{-1} = B^{-1}A^{-1}$.

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Lemma-4.1

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix

Then $(AB)^T = B^T A^T$ (1)

and if α and β are scalars,

Then $(\alpha A + \beta B)^T = \alpha A^T + \beta B^T$ (2)

Proof:

From definition:

$$\left((AB)^T\right)_{ij} = (AB)_{ji}$$
$$= \sum_k A_{jk} B_{ki}$$
$$= \sum_k \left(B^T\right)_{ik} \left(A^T\right)_{kj}$$
$$= \left(B^T A^T\right)_{ij}$$

$$\left((\alpha A + \beta B)^T\right)_{ij} = (\alpha A + \beta B)_{ji}$$
$$= (\alpha A)_{ji} + (\beta B)_{ji}$$
$$= \alpha (A)_{ji} + \beta (B)_{ji}$$
$$= \alpha \left(A^T\right)_{ij} + \beta \left(B^T\right)_{ij}$$
$$= \alpha A^T + \beta B^T$$

Example-10

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Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$, Then show that A is symmetric.

Example-11

Let $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$, then show that A is skew-symmetric.

Definition:

Let A be an $m \times n$ matrix, Then A^T denotes the $n \times m$ matrix which is defined as follows $\left(A^T\right)_{ij} = A_{ji}$.

The transpose of a matrix has the following important property.

There is a particular matrix called I and defined by

$I_{ij} = \delta_{ij}$.

Where δ_{ij} is the *Kronecker symbol* defined by

$$\delta_{ij} = \begin{cases} 1, & If \quad i = j \\ 0, & If \quad i \neq j \end{cases}$$

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It is said to be an identity matrix because it is a multiplicative identity in the following sense.

Lemma-4.2

Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix. Then $AI_n = A$. Next if I_m is an $m \times m$ identity matrix. Then It follows that $I_m A = A$.

Proof:

$$\left(AI_n\right)_{ij} = \sum_k A_{ik}\delta_{kj} = A_{ij}$$

and so $AI_n = A$.

The other case is left as an exercise.

4.7. Procedure for finding an inverse of a matrix:

Suppose A is an $n \times n$ matrix. To find A^{-1} if it exists.

From the augmented $n \times 2n$ matrix, $\left[A \vdots I\right]$ and then do row operation until you obtain an $n \times 2n$ matrix of the form $\left[I \vdots B\right]$.

If possible, it finds $B = A^{-1}$.

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Note: The matrix A has no inverse when it is impossible to do row operations and end up with one like $[I \vdots B]$.

Example-12

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$,

Then Find A^{-1}.

Solution: Consider the augmented matrix

$$[A \vdots I] \approx \begin{bmatrix} 1 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 1 & -1 & 1 & \vdots & 0 & 1 & 0 \\ 1 & 1 & -1 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Now upon performing suitable row operations, the 3×3 matrix on the left becomes an identity matrix, and it yields after some computation as

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & 1 & -1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Which gives the inverse of A, as the matrix on the right

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$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

It can be verified by multiplying the inverse with the matrix A that reduces to identity I_n.

$$\text{i.e.} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example-13

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$, Then find A^{-1}.

Solution: First set up an augmented matrix $[A \vdots I]$.

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 1 & 0 & 0 \\ 1 & 0 & 2 & \vdots & 0 & 1 & 0 \\ 3 & 1 & -1 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Step-I $R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$

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$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -2 & 0 & \vdots & -1 & 1 & 0 \\ 0 & -5 & -7 & \vdots & -3 & 0 & 1 \end{bmatrix}$$

Step-II $R_3 \rightarrow -2R_3 - 5R_2$

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -10 & 0 & \vdots & -5 & 5 & 0 \\ 0 & 0 & 14 & \vdots & 1 & 5 & -2 \end{bmatrix}$$

Step-III $R_1 \rightarrow -7R_1 + R_3$

$$\begin{bmatrix} -7 & -14 & 0 & \vdots & -6 & 5 & -2 \\ 0 & -10 & 0 & \vdots & -5 & 5 & 0 \\ 0 & 0 & 14 & \vdots & 1 & 5 & -2 \end{bmatrix}$$

Step-IV $R_1 \rightarrow \left(\frac{-7}{5}\right)R_2 + R_1$

$$\begin{bmatrix} -7 & 0 & 0 & \vdots & 1 & -2 & -2 \\ 0 & -10 & 0 & \vdots & -5 & 5 & 0 \\ 0 & 0 & 14 & \vdots & 1 & 5 & -2 \end{bmatrix}$$

Step-V $R_1 \rightarrow \left(\frac{-1}{7}\right)R_1, R_2 \rightarrow \left(\frac{-1}{10}\right)R_2$ and

$$R_3 \rightarrow \left(\frac{1}{14}\right)R_3$$

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$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \vdots & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{bmatrix}$$

Therefore, the inverse is

$$\begin{bmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{bmatrix}$$

Example-14

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix}$,Then find A^{-1}.

Solution: The augmented matrix $[A \vdots I]$ is of the form

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 1 & 0 & 0 \\ 1 & 0 & 2 & \vdots & 0 & 1 & 0 \\ 2 & 2 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

and proceed to do row operations attempting to obtain $[I \vdots A^{-1}]$.

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Step-I $R_2 \to R_2 + (-1)R_1, R_3 \to R_3 + (-2)R_1$

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -2 & 0 & \vdots & -1 & 1 & 0 \\ 0 & -2 & 0 & \vdots & -2 & 0 & 1 \end{bmatrix}$$

Step-II $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -2 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & -1 & 1 \end{bmatrix}$$

At this point, It can be seen that there will be non-existence of inverse due to the presence of zeros in the last row of the left half of the augmented matrix $[A \vdots I]$.

Thus there will be no way to obtain an identity matrix I on the left side of the augmented matrix implies no inverse of A exist.

Theorem-4.4

If A is an $n \times n$ matrix, r and s are non-negative integers, then

(i) $A^r.A^s = A^{r+s}$

(ii) $(A^r)^s = A^{rs}$

(iii) $A^0 = I_n$

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(iv) $A^r.A^s = \underbrace{A....A}_{r\,times}\underbrace{A....A}_{s\,times} = \underbrace{A....A}_{r+s\,times} = A^{r+s}$

Theorem-4.5

Let A be an $m\times n$ matrix and 0_{mn} be the zero matrix. Let B be an $n\times n$ square matrix, 0_n and I_n be the zero and identity matrices, then

(i) $A+0_{mn} = 0_{mn}+A = A$.

(ii) $B0_n = 0_n B = 0_n$.

(iii) $BI_n = I_n B = B$.

Example-15

Let $A=\begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$

We see that

$$A+0_{23} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix}$$

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$$B0_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$$

$$BI_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} = B$$

Similarly

$$0_{23} + A = A,\ 0_2 B = 0_2,\ I_2 B = B$$

4.8. Matrix multiplication in terms of columns:

(a) Consider the product AB, where A is an $m \times n$ matrix and B is an $n \times r$ matrix.

Let the columns of B be the matrices $B_1, B_2, ...B_r$, write the matrix B as $\left[B_1, B_2, ...B_r\right]$. Thus $AB = A\left[B_1, B_2, ...B_r\right]$.

Matrix multiplication implies that the columns of the products are $AB_1, AB_2, ..., AB_r$.

We can write

$$AB = \left[AB_1, AB_2, ..., AB_r\right].$$

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For example: Suppose $A=\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}$ and

$$B=\begin{bmatrix}4 & 1 & 3\\0 & 2 & -1\end{bmatrix}$$

Then

$$AB=\left[\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix},\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}\right].$$
$$=\begin{bmatrix}8 & 2 & 6\\4 & 11 & -2\end{bmatrix}$$

(b) The matrix product AB, where B is a column matrix.

Consider the general case, where A is an $m\times n$ matrix and B is an $n\times 1$ matrix.

Write A in terms of its columns $[A_1, A_2, ...A_n]$, Then

$$AB=[A_1, A_2, ...A_n]\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}.$$

Matrix multiplication gives $AB=b_1A_1+b_2A_2+....+b_nA_n$

.

As for vectors, the expression $b_1A_1+b_2A_2+....+b_nA_n$ is called a linear combination of $A_1, A_2, ...A_n$.

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It is computed by performing the scalar multiples and then adding the corresponding elements of the resulting matrices.

For example: Suppose

$$A=\begin{bmatrix} 2 & 3 & 1 \\ -4 & 8 & 5 \end{bmatrix} \text{ and } B=\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

Then

$$\begin{aligned} AB &= 3\begin{bmatrix} 2 \\ -4 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -12 \end{bmatrix} - \begin{bmatrix} 6 \\ 16 \end{bmatrix} + \begin{bmatrix} 5 \\ 25 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}. \end{aligned}$$

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Chapter-5

System of Linear equations

In this chapter, we shall discuss an important part of linear algebra i.e. Linear system of equation and determine the solution properties of linear system of equations.

5.1. Introduction

Consider a general system of m-linear equations in n-variables using matrix notation as follows

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{aligned}$$

Represent each side as a column matrix

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If the left hand side of the equation can be written as a product of the matrix coefficients A and a column matrix of variables X i.e. AX and the column matrix

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of constants be B, Then the system of linear equation can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Thus we can write the above system of equations in matrix form as

$$AX = B.$$

Example-1

The system of linear equations

$$\begin{aligned} 3x_1 + 2x_2 - 5x_3 &= 7 \\ x_1 - 8x_2 + 4x_3 &= 9 \\ 2x_1 + 6x_2 - 7x_3 &= -2 \end{aligned}$$

can be written as

$$\begin{bmatrix} 3 & 2 & -5 \\ 1 & -8 & 4 \\ 2 & 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}$$

We now use this notation and the properties of matrices to examine the sums and the scalar multiples of solutions to the systems of linear equations.

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Example-2

The matrix form of the pair of linear equations

$$2x_1 - 3x_2 + 5x_3 = 1$$
$$-x_1 + x_2 - x_3 = 4$$

is $\begin{bmatrix} 2 & -3 & 5 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

5.2. Special Matrices

5.2.1. Idempotent and Nilpotent Matrices:

A square matrix A is said to be *Idempotent* if $A^2 = A$.

A square matrix A is said to be *nilpotent* if there is a positive integer p such that $A^p = 0$.

The least integer p such that $A^p = 0$ is called the degree of *nilpotency* of the matrix.

5.2.2. Elementary matrices

We now introduce a very useful class of matrices called *elementary matrices*.

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An *elementary matrix* is one that can be obtained from the *identity matrix* I_n through a single elementary row operation.

An $n \times n$ matrix is called *elementary* if it is obtained from the identity matrix I_n in one of the three ways.

(a) Interchange rows i and j, where $i \neq j$.

(b) Insert a scalar c as the (i, j) entry, where $i \neq j$.

(c) Put a non-zero scalar c in the (i,i) position.

Illustration:

Consider the following three row operations T_1, T_2 *and* T_3 on I_3. (One representing each kind of row operation) They lead to the three elementary matrices E_1, E_2 and E_3.

Consider $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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Elementary row operation	Corresponding Elementary Matrix
T_1: Interchange rows 2 and 3 of I_3	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
T_2: Multiply row 2 of I_3 by 5	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
T_3: Add 2 times rows of I_3 to row 2	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Remark:

Suppose we want to perform a row operation T on an $m \times n$ matrix A. Let E be the elementary matrix obtained from I_n through the operation T. This row operation can be performed by multiplying A by E.

Note: Every elementary matrix is square and Invertible.

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Example-3

Let $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ be an 3×3 matrices .

Consider the 3 row operation as stated above.

Let us show that the corresponding elementary matrices can indeed be used to perform these operations.

Interchange rows 2 and 3 of I_3 :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}=\begin{bmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Multiply row 2 by 5 of I_3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}=\begin{bmatrix} a_1 & a_2 & a_3 \\ 5b_1 & 5b_2 & 5b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Add twice of row 1 to row 2 of I_3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1+2a_1 & b_2+2a_2 & b_3+2a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

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Remark:

Every elementary matrix is square and invertible.

Theorem-5.1

If A and B are row equivalent matrices and A is invertible then B is invertible.

Theorem-5.2

Let A be any $m \times n$ matrix, then there exist elementary $m \times m$ matrices $E_1, E_2, ...E_k$ such that the matrix $E_k E_{k-1} .....E_1 A$ is in reduced echolen form.

Example-4

Consider the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$.

We easily put this in reduced row echolen form B by applying successively the row operations

$$R_1 \leftrightarrow R_2, \quad \left(\frac{1}{2}\right) R_1, \quad R_1 - \left(\frac{1}{2}\right) R_2 \ .$$

$$A \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = B$$

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Hence

$$E_3E_2E_1A = B,$$

Where $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$.

Theorem-5.3

Let A be an $n \times n$ matrix. Then the following statements about the matrix A are equivalent, that is ,each one implies all the others.

(a) A is invertible.

(b) The system $AX = 0$ has only the trivial solution.

(c) The reduced row echolen form of A is I_n.

(d) A is a product of elementary matrices.

Proof:

We shall establish the logical implication $(a) \to (b)$, $(b) \to (c)$, $(c) \to (d)$ and $(d) \to (a)$.

This will serve to establish the equivalence of the four statements.

If **(a)** holds then A^{-1} exists.

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Thus if we multiply both sides of equation $AX = 0$ on the left by A^{-1}.

We get $A^{-1}AX = A^{-1}.0$, so that $X = A^{-1}.0 = 0$, and the only solution of the linear system is the trivial one.

Thus **(b)** holds.

If **(b)** holds, then we know that the number of pivots of A in reduced row echolen form is n. since A is $n \times n$, this must mean that I_n is the reduced row echolen form of A so that **(c)** holds.

If **(c)** holds, then the theorem (5.2) shows that there are elementary matrices $E_1, E_2 ..... E_k$ such that

$$E_k E_{k-1} ..... E_1 A = I_n .$$

Since elementary matrices are invertible, so is $E_k E_{k-1} ..... E_1$ and thus

$$A = \left(E_k E_{k-1} ..... E_1 \right)^{-1} = E_1^{-1} E_2^{-1} .... E_k^{-1}$$

So that **(d)** is true.

Finally, **(d)** implies **(a)** since a product of elementary matrices is always *invertible*.

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5.3. A procedure for finding the inverse of a matrix

As an application of the ideas in this section, we shall describe an efficient method of computing the inverse of an invertible matrix.

Suppose that A is an invertible $n \times n$ matrix then there exist elementary $n \times n$ matrices $E_1, E_2.....E_k$ such that $E_k E_{k-1}.....E_1 A = I_n$.

Therefore

$$\begin{aligned} A^{-1} &= I_n A^{-1} \\ &= \left(E_k E_{k-1}.....E_1 A\right) A^{-1} . \\ &= \left(E_k E_{k-1}.....E_1\right) I_n \end{aligned}$$

This means that the row operations which reduce A to its reduced row echolen form will automatically transform I_n to A^{-1}.

It is the crucial observation which enables us to compute A^{-1}.

The procedure for computing A^{-1} starts with the partitioned matrix $\left[A \vdots I_n\right]$ and then puts it in reduced row echolen form.

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If A is invertible, then the reduced row echolen form will be

$$\left[I_n \vdots A^{-1} \right].$$

Remark:

If the procedure applied to a matrix that is not invertible, it will be impossible to reach a reduced row echolen form of the above type that is one with on the left. Thus the procedure will also detect non-invertibility of a matrix.

Example-5

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Put the matrix $\left[A \vdots I_3 \right]$ in reduced row echolen form, using elementary row operations as described above:

$$\begin{bmatrix} 2 & -1 & 0 & \vdots & 1 & 0 & 0 \\ -1 & 2 & -1 & \vdots & 0 & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

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$$\Rightarrow\begin{bmatrix} 2 & \frac{-1}{2} & 0 & \vdots & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & \vdots & 0 & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \vdots & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{3} & \vdots & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \vdots & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -1 & \frac{4}{3} & \vdots & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Which is in reduced row echolen form.

Therefore

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

It can be verified by checking that

$$AA^{-1} = I_3 = A^{-1}A.$$

Theorem-5.4

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If A and B are row equivalent matrices and is invertible then B is invertible.

Proof:

Suppose A and B are row equivalent. There exists a sequential row operation $T_1, T_2, ...T_n$ such that $B = T_n.T_{n-1}.......T_1(A)$.

Let the elementary matrices of these operations be $E_1, \mathrm{E}_2, ...\mathrm{E}_n$.

Thus $B = E_n.\mathrm{E}_{n-1}.......\mathrm{E}_1 A$

The matrices $A, E_1, \mathrm{E}_2, ...\mathrm{E}_n$ are all invertible.

Repeatedly applying the property of matrix inverse of a product to the following expression, we get

$$
\begin{aligned}
A^{-1}E_1^{-1}\mathrm{E}_2^{-1}...\mathrm{E}_n^{-1} &= (E_1\mathrm{A})^{-1}\mathrm{E}_2^{-1}...\mathrm{E}_n^{-1} \\
&= (\mathrm{E}_2 E_1 \mathrm{A})^{-1}\mathrm{E}_3^{-1}...\mathrm{E}_n^{-1} \\
&= (\mathrm{E}_n\mathrm{E}_{n-1}....\mathrm{E}_2 E_1 \mathrm{A})^{-1} = B^{-1}
\end{aligned}
$$

Thus B is invertible and the inverse is given by $B^{-1} = A^{-1}E_1^{-1}\mathrm{E}_2^{-1}...\mathrm{E}_n^{-1}$.

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5.4. Application to Linear Systems

5.4.1 Non-Homogenous linear system

Consider a set of m linear equations in n unknowns $x_1, x_2, ..., x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_n \end{aligned}$$

By a solution of the linear system, we shall mean an n-column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

such that the scalars $x_1, x_2, ..., x_n$ satisfies all the equations of system.

The set of all solutions is called the *general solution* of the linear system. This is normally given in the form of a single column vector containing a number of arbitrary quantities.

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A linear system with *no solution* is said to be inconsistent.

5.4.2. Elementary Transformation and Row Operations

The transformation called *elementary transformation*, that can be used to change a system of linear equations into another system of linear equations that has the same solution.These transformations are used to solve the system of linear equations by eliminating variables.

In a matrix, such types of operations are called *elementary row operations*.

It is not necessary to write down the variables $x_1, x_2, x_3, ...$ at each stage.

Elementary transformation	**Row operations**
1.Interchange two equations 2.Multiply both sides of an equation by a nonzero constant.	1.Interchange two rows of a matrix. 2.Multiply the elements of a row by a non-zero constants

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3.Add multiple of one equation to another equation.	3.Add a multiple of the elements of one row to the corresponding elements of another row.

System of equations that are related through *elementary transformations* is called *equivalent systems*.

Matrices that are related through *elementary row operations* are called *row equivalent matrices*.

Remark: Elementary transformations preserve solutions since the order of the equations does not affect the solution.

Example-6

Solve the system

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 + 3x_2 + x_3 = 3$$
$$x_1 - x_2 - 2x_3 = -6$$

Solution:

Elementary Transformation:

Step-I

Eliminate x_1 from 2nd and 3rd equations i.e.

Equation(2) + (-2) x equation (1)

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Equation (3) +(-1)x equation (1)

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ 2x_2 - 3x_3 &= -8 \end{aligned}$$

Step-II

Elliminate x_2 from first and third equations i.e. Equation(1)+(-1)Equation(2)

$$\begin{aligned} x_1 + \quad 2x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -5x_3 &= -10 \end{aligned}$$

Elliminate x_3 from first and second equation

$$\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Matrix method

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

$$\approx \begin{matrix} R_2 + (-2)R_1 \\ R_3 + (-1)R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

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$$\approx \begin{matrix} R_1+(-1)R_2 \\ R_3+(2)R_2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

$$\approx \left(-\frac{1}{5}\right)R_3 \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\approx \begin{matrix} R_1+(-2)R_3 \\ R_2+R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x_1=-1 \\ x_2=1 \\ x_3=2 \end{matrix}$$

Example-7

Solve the system

$$\begin{aligned} x_1-2x_2+4x_3&=12 \\ 2x_1-x_2+5x_3&=18 \\ -x_1+3x_2-3x_3&=-8 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{bmatrix}$$

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$$\approx \begin{matrix} R_2+(-2)R_1 \\ R_3+R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\approx \left(\frac{1}{3}\right)R_2 \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\approx \begin{matrix} R_1+(2)R_2 \\ R_3+(-1)R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$\approx \frac{1}{2}R_3 \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\approx \begin{matrix} R_1+(2)R_3 \\ R_2+R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Hence the general solution is

$$x_1 = 2, x_2 = 1, x_3 = 3.$$

5.4.3. Echelon Form and Reduced Row Echelon Form

The method of Gaussian Ellimination involves an echolen form of the augmented matrix of the system of equations.

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An echolen form satisfies the first three of the conditions of the R.E.F.

Definition:

A Matrix is in ***row echolen form*** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first non zero element of each row is 1.This element is called a leading 1.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.(this implies that all the elements below a leading 1 are zero.)

Example-8

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 6 & 2 & 5 & 2 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

The method of Gauss-Jordan Ellimination involves a Row Reduced Echolen Form of the augmented matrix of the system of equations.

An echolen form satisfies the first four of the conditions of the R.R.E.F.

Now we will discuss the reduced echolen form in mor general form.

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Definition:

A matrix is in ***row reduced echolen form***, if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first non-zero elenments of each row is i.this element is called leading 1.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other element in a column that contains a leading 1 are zero.

Example-9

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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The difference between a R.R.E.F and an R.E.F is that the element above and below a leading 1 are zero in a R.R.E.F while only the element below the leading 1 need be zero in an R.E.F.

Remark:

The Row Reduced Echolen form of a matrix is unique. The method of Gauss Jordan Ellimination is an important systematic way for arriving at the row reduced echolen form.

5.5. Solving Linear Systems via Gaussian Elimination

5.5.1. The Homogeneous Case

Theorem-5.5 A homogenous system of linear equations in n-variables always has the solution $x_1 = 0, x_2 = 0, ...x_n = 0$.This solution is called the trivial solution.

Example-10

$$x_1 + 2x_2 - 5x_3 = 0$$
$$-2x_1 - 3x_2 + 6x_3 = 0$$

Theorem-5.6

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A homogenous system of linear equations that has more variables than equations has many solutions.one of these solutions is the trivial solution.

Example-11

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$x_1 + 3x_3 = 0$$
$$x_2 - 4x_3 = 0$$

$$\Rightarrow \begin{matrix} x_1 = -3x_3 \\ x_2 = 4x_3 \end{matrix}$$

$$\Rightarrow \; x_3 = r, x_1 = -3r, x_2 = 4r$$

$\Rightarrow$ For $r = 0$ it implies $x_1 = x_2 = x_3 = 0$

5.5.2. The Non-Homogeneous Case

If A is the matrix of co-efficient of a system of n-equations in n- variables that has a unique solutions then it is row equivalent to I_n.

$$\Rightarrow [A : B] \approx [I_n : X]$$

$$[A : B_1 B_2 .... B_n] \approx [I_n : X_1 X_2 ..... X_n]$$

Leading to the solutions $X_1, X_2, ... X_n$.

5.6. Criteria for Consistency and Uniqueness

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5.6.1. Gaussian Elimination

In this section, we introduce another elimination method called Gaussian Ellimination. Different methods are suitable for different occasion.it is important to choose the best method for the purpose in mind.

Gaussian Ellimination:

1. Write down the augmented matrix of the system of linear equations.
2. Find an echolen form of the augmented matrix using elementary row operations. This is done by creating leading $1's$ then zeros below each leading 1,column by column, starting with the first column.
3. Write down the system of equations corresponding to the echolen form
4. Use back substitution to arrive at the solution.

Theorem-5.7

(i) A linear system is consistent if and only if all the entries on the right hand sides of those equation in echolen form which contain no unknowns are zero.

(ii) If the system is consistent, the non-pivotal unknowns can be given arbitrary values; the general solution is then obtained by using back substitution to solve for the pivotal unknowns.

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(iii) The system has a unique solution if and only if all unknowns are pivotal.

Remarks:

An important feature of Gauss elimination is that it constitutes a practical algorithm for solving linear systems which can easily be implemented in one of the standard programing language.

Example-11

Solve the system using Gauss Ellimination method

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$
$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$
$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix}$$

$$\approx \begin{matrix} R_2 + R_1 \\ R_3 + (-2)R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{bmatrix}$$

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$$\approx \quad R_3 + (-2)R_2 \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\approx \frac{1}{2}R_3 \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The corresponding system of equation is

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$
$$x_3 + 3x_4 = 1$$
$$x_4 = 2$$

By back substitutions, we get

$$x_3 = 1 - 6 = -5$$

$$x_1 + 2x_2 = 10 \Rightarrow x_1 = -2x_2 + 10$$

Let $x_2 = r$, the systems has many solutions.

The solutions are

$$x_1 = -2r + 10, x_2 = r, x_3 = -5, x_4 = 2 .$$

Example-12

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$
$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$
$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

The back substitutions can also be performed using matrices.

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The final matrix is then the reduced echolen form of the systems.

5.6.3. Gaussian-Jordan Elimination

Gauss Jordan method of solving a system of linear equation using matrices involves creating and in certain location of matrices. These numbers are created in a systematic manner column by column.

Gauss-Jordan elimination is used to solve system of n-equations in n-variables that has a unique solution.

i.e. $\left[A:B\right]\approx\left[I_n:X\right]$ can be used for a system that gives unique solution.

Now we will discuss the method in its more general setting where the number of equations can differ from the number of variables and where there can be a unique solution, many solution or no solution.

Gauss Jordan Elimination

1. Write down the augmented matrix of the system of linear equations.
2. Derive the reduced echolen form of the augmented matrix using elementary row operation. This is done by creating leading $1's$,then zeros above and below

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each leading 1.column by column, starting with the first column.

3. Write down the system of equations corresponding to the reduced echolen form. These system gives the solution.

Example-13

Use Gauss Elimination method to find the Reduced echolen form of the following matrix

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution:

Step-I

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \approx R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} (3) & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

This non zero element is called a pivot.

Step-II

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Create a 1 in the pivot location by multiplying the pivot row by $\frac{1}{Pivot}$

$$\approx \frac{1}{3}R_1 \quad \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Step-III

Create zero elsewhere in the pivot column by adding suitable multiples of the pivot row to all other rows of the matrix

$$\approx R_3 + (-4)R_1 \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

Step-IV

Cover the pivot row and all rows above it.

Repeat Step-I and II for the remaining submatrix.

Repeat step-III for the whole matrix. Continue this until the Reduced Echolen Form is reached.

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & \left(2_{pivot}\right) & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \approx \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

$$\begin{matrix} \approx R_1 + R_2 \\ R_3 + (-2) R_2 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & \left(1_{Pivot}\right) & -6 \end{bmatrix}$$

$$\begin{matrix} \approx R_1 + (-2) R_3 \\ R_2 + R_3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

Which is in Reduced Echolen Form.

Example-14

Solve If Possible

$$3x_1 - 3x_2 + 3x_3 = 9$$
$$2x_1 - x_2 + 4x_3 = 7$$
$$3x_1 - 5x_2 - x_3 = 7$$

Example-15

Solve If Possible

$$x_1 + 2x_2 - x_3 + 3x_4 = 4$$
$$2x_1 + 4x_2 - 2x_3 + 7x_4 = 10$$
$$-x_1 - 2x_2 + x_3 - 4x_4 = -6$$

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Example-16

Solve If Possible

$$x_1 + x_2 + 5x_3 = 3$$
$$x_2 + 3x_3 = -1$$
$$x_1 + 2x_2 + 8x_3 = 3$$

5.7. Homogenous linear system:

A very important type of linear system occurs, when all the scalars on the right hand sides of the equation equals zero.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$$
$$\dots \quad \dots$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0$$

Such a system is called homogenous. It will always have the trivial solution $x_1 = 0, x_2 = 0, ...x_n = 0$.Thus a homogenous linear system is always consistent.

Remark:

The interesting question about a homogenous linear system is whether it has any nontrivial solutions. The answer is easily read off from the echolen form.

Theorem-5.8

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A homogenous linear system has a *non-trivial* solution if and only if the number of pivots in echolen form is less than the number of unknowns.

Note: If the number of pivots is r, then the $n-r$ non-pivotal unknowns can be given arbitrary values, so there will be a *non-trivial solution* whenever $n-r>0$. On the other hand, if $n=r$ then none of the unknowns can be given arbitrary values and there is a *unique solution* namely the trivial one.

Corollary-5.1

A homogenous linear system of m - equations in n unknowns always has a non-trivial solution if $m<n$.

For if r is the number of pivots then $r \le m < n$.

Example-17

For which value of the parameter t does the following homogenous system have non-trivial solution

$$\begin{aligned} 6x_1 - x_2 + x_3 &= 0 \\ tx_1 \qquad + x_3 &= 0 \\ x_2 + tx_3 &= 0 \end{aligned}$$

It satisfies to find the number of pivotal unknowns.

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We proceed to put the linear system in echolen form by applying to it successively the operations $\frac{1}{6}(1)$, $(2)-t(1)$, $(2)\leftrightarrow(3)$ and $(3)-\frac{t}{6}(2)$.

$$x_1-\left(\frac{1}{6}\right)x_2+\left(\frac{1}{6}\right)x_3=0$$
$$x_2+tx_3=0$$
$$\left(1-\frac{t}{6}-\frac{t^2}{6}\right)x_3=0$$

The number of pivots will be less than 3. The number of unknowns, precisely when $1-\frac{t}{6}-\frac{t^2}{6}$ equals zero that is when $t=2$ *or* $t=-3$.

These are the only values of t for which the linear system has non-trivial solution.

5.8. Elementary row operations and row echolen form

Suppose now that we wish to solve the linear system with matrix form $AX=B$ using elementary row operations.

The first step is to identify the augmented matrix $M=[A\vdots B]$.Then we put M in row echolen form, using

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row operations. From this we can determine if the original linear system is consistent. For this to be true in the row echolen form of M, the scalars in the last column which lie below the final pivot must all be zero. To find the general solution of a consistent system, we convert the row echolen matrix back to a linear system and use back substitution to solve it.

Example-18

Consider a linear system

$$\begin{aligned} x_1+3x_2+3x_3+2x_4 &= 1 \\ 2x_1+6x_2+9x_3+5x_4 &= 5 \\ -x_1-3x_2+3x_3 \qquad &= 5 \end{aligned}$$

The augmented matrix here is

$$\begin{bmatrix} 1 & 3 & 3 & 2 & \vdots & 1 \\ 2 & 6 & 9 & 5 & \vdots & 5 \\ -1 & -3 & 3 & 0 & \vdots & 5 \end{bmatrix}$$

Now we can convert it in to row echolen form

$$\begin{bmatrix} 1 & 3 & 3 & 2 & \vdots & 1 \\ 0 & 0 & 1 & \frac{1}{3} & \vdots & 1 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

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Since the bottom right hand entry of the matrix is 0, so the linear system is consistent.

Thus the linear system corresponding to the last matrix is

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 &= 1 \\ x_3 + \frac{1}{3}x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

Hence the general solution given by back substitution is

$$x_1 = -2 - c - 3d, x_2 = d, x_3 = 1 - \frac{c}{3}, x_4 = c.$$

Where c and d are arbitrary scalars.

Theorem-5.9

Let $AX = B$ be a linear system of equation in n-unknowns with augmented matrix

(i) The linear system is *consistent* if and only if the matrices A and M have the same number of pivots in *row echolen form*.

(ii) If the linear system is *consistent* and r denotes the number of pivots of A in row echolen form. Then the $n-r$ unknowns that corresponds to columns of

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A not containing a pivot can be given arbitrary values. Thus the system has a *unique solution*.

Proof:

For the linear system to be consistent, the row echolen form of M must have only zero entries in the last column below the final pivot but this is just the condition for A and M to have the same number of pivots.

Finally, if the linear system is *consistent*, the unknowns corresponding to columns that do not contain pivots may be given arbitrary values and the remaining unknowns found by back substitution.

5.9. Comparison of Gaussian and Gaussian - Jordan Elimination

The method of G.E is in general more efficient than G.J.E is that it involves fewer operations of addition and multiplications.it is during the back substitution that G.E picks up this advantage.

5.9.1.Comparison of Gauss-Jordan Elimination and Gaussian Elimination

Consider an $n\times(n+1)$ matrix and assume that there are no row interchanges. Note that when an element is

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known to become 1 or a zero there are no arithmetic operations involve, substitution is used. Thus for example there are no operations involve in the location where a leading 1 is created. We get starting with the $n\times(n+1)$ augmented matrix of the system.

$$\text{n-rows}\begin{bmatrix} * & * & \ldots\ldots & * \\ * & * & \ldots\ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots\ldots & * \end{bmatrix} \underset{n-multiplications}{\approx} \begin{bmatrix} 1 & * & \ldots\ldots & * \\ * & * & \ldots\ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots\ldots & * \end{bmatrix}$$

$$\underset{\substack{n-multiplications \\ n-1\ adds\ (per\ rows)}}{\approx} \begin{bmatrix} 1 & * & \ldots\ldots & * \\ 0 & * & \ldots\ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \ldots\ldots & * \end{bmatrix} \underset{(n-1)-multiplications}{\approx}$$

$$\begin{bmatrix} 1 & * & \ldots\ldots & * \\ 0 & 1 & \ldots\ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \ldots\ldots & * \end{bmatrix} \underset{\substack{n-1\ multiplications \\ n-1\ adds\ (per\ rows)}}{\approx} \begin{bmatrix} 1 & 0 & \ldots\ldots & ** \\ 0 & 1 & \ldots\ldots & ** \\ \vdots & \vdots & \vdots & \vdots\vdots \\ 0 & 0 & \ldots\ldots & 1* \end{bmatrix} \approx$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \underset{1-multiplications}{\approx} \begin{bmatrix} 1 & 0 & \ldots\ldots & ** \\ 0 & 1 & \ldots\ldots & ** \\ \vdots & \vdots & \vdots & \vdots\vdots \\ 0 & 0 & \ldots\ldots & 1* \end{bmatrix}$$

$$\underset{\substack{1\ multiplications \\ n-1\ adds\ (per\ rows)}}{\approx} \begin{bmatrix} 1 & 0 & \ldots\ldots & 0* \\ 0 & 1 & \ldots\ldots & 0* \\ \vdots & \vdots & \vdots & \vdots\vdots \\ 0 & 0 & \ldots\ldots & 1* \end{bmatrix}$$

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Total number of multiplications

$$=\left[n+(n-1)+....+1\right]+(n-1)\left[n+(n-1)+....+1\right]$$

$$=n\left[n+(n-1)+....+1\right]=n\left[\frac{n(n+1)}{2}\right]=\frac{n^3}{2}+\frac{n^2}{2}$$

Total number of additions

$$=(n-1)\left[n+(n-1)+....+1\right]$$

$$=(n-1)\left[\frac{n(n+1)}{2}\right]=\frac{n^3}{2}-\frac{n}{2}$$

Note: $\left[n+(n-1)+....+1\right]=\frac{n(n+1)}{2}$

5.10. Method of LU decomposition

Definition:

Let A be a square matrix that can be factored in to the form $A=LU$, where U is an upper triangular matrix and L is a lower triangular matrix. This factorisation is called an LU decomposition of A.

Remark:

Not every matrix has an LU decomposition and when it exists it is not unique.

The method that now introduce can be used to solve a system of linear equations if A has an LU decomposition.

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Method of LU decomposition

Let $AX = B$ be a system of n-equation in n-variables where A has LU decomposition $A = LU$. The system thus can be written as

$$LUX = B$$

The method involves writing this system as two subsystems one of which is lower triangular and the other upper triangular

$$UX = Y$$

$$LY = B$$

Observe that substituting for Y from the first equation in to the second gives the original system

$$LUX = B.$$

In practice, we first solve $LY = B$ for Y and then solve $UX = Y$ to get the solution X.

5.11. Construction of a LU decomposition of a Matrix

1. Use row operations to arrive at U.
   (The operations must involve addition multiples of rows to rows.in general, if row interchanges are required to arrive at U an LU form does not exist)
2. The diagonal elements of L are $1's$.The non-zero elements of L correspond to row operations.

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3. The row operation $R_k + cR_j$ implies that $l_{kj} = -c$.

2.4.9. Solution of $AX = B$

1. Find the LU decomposition of A. (If A has no LU decomposition. The method is not applicable.)
2. Solve $LY = B$ by forward substitution.
3. Solve $UX = Y$ by back substitution.

Example-19

Solve the following system of equations using LU decomposition

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 12 \\ -x_1 + 5x_2 - 3x_3 &= -12 \\ 4x_1 - 8x_2 + 23x_3 &= 58 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & -3 & 4 \\ -1 & 5 & -3 \\ 4 & -8 & 23 \end{bmatrix}$$

$$\approx \begin{matrix} R_2 + R_1 \\ R_3 - 4R_1 \end{matrix} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 2 & 1 \\ 0 & 4 & 7 \end{bmatrix}$$

$$\approx R_3 - 2R_2 \begin{bmatrix} 1 & -3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

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These row operations lead to the following LU decomposition of A

$$A=\begin{bmatrix}1&0&0\\-1&1&0\\4&2&1\end{bmatrix}\begin{bmatrix}1&-3&4\\0&2&1\\0&0&5\end{bmatrix}$$

We again solve the given system $LUX=B$. By solving the two sub systems $LY=B$ and $UX=Y$.

$$LY=B\Rightarrow\begin{bmatrix}1&0&0\\-1&1&0\\4&2&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}12\\-12\\58\end{bmatrix}$$

This lower triangular system has solution

$$x_1=1, x_2=-1, x_3=2.$$

The solution to the given system is

$$x_1=1, x_2=-1, x_3=2.$$

Exercise-1

1.Determine the matrix of coefficient and augmented matrix of each of the following systems of equations.

(a) $\begin{aligned}x_1+3x_2&=7\\2x_1-5x_2&=-3\end{aligned}$

(b) $\begin{aligned}-x_1+3x_2-5x_3&=-3\\2x_1-2x_2+4x_3&=8\\x_1+3x_2\qquad\quad&=6\end{aligned}$

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(c)
$$\begin{aligned} 5x_1 + 2x_2 - 4x_3 &= 8 \\ 4x_2 + 3x_3 &= 0 \\ x_1 \qquad - x_3 &= 7 \end{aligned}$$

2.The following systems of equations all have unique solutions. Solve these systems using the method of Gauss Jordan elimination with matrices.

(a)
$$\begin{aligned} x_1 - 2x_2 &= -8 \\ 2x_1 - 3x_2 &= -11 \end{aligned}$$

(b)
$$\begin{aligned} x_1 \qquad + x_3 &= 3 \\ 2x_2 - 2x_3 &= -4 \\ x_2 - 2x_3 &= 5 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - x_2 + 3x_3 &= 3 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 3 \end{aligned}$$

3.Solve (If possible) each of the following systems of three equations in three variables using the method of Gauss Jordan elimination.

(a)
$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 2x_1 - 8x_2 + 11x_3 &= 7 \\ x_1 + 6x_2 + 7x_3 &= 3 \end{aligned}$$

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(b) $$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ 2x_1 + 3x_2 + x_3 &= 18 \\ -x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

(c) $$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

Answers:

1.

(a) $\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 7 \\ 2 & -5 & -3 \end{bmatrix}.$

(b) $\begin{bmatrix} -1 & 3 & -5 \\ 2 & -2 & 4 \\ 1 & 3 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 & -5 & -3 \\ 2 & -2 & 4 & 8 \\ 1 & 3 & 0 & 6 \end{bmatrix}.$

(c) $\begin{bmatrix} 5 & 2 & -4 \\ 0 & 4 & 3 \\ 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 5 & 2 & -4 & 8 \\ 0 & 4 & 3 & 0 \\ 1 & 0 & -1 & 7 \end{bmatrix}$

2. (a) $x_1 = 2, x_2 = 5$

(b) $x_1 = 10, x_2 = -9, x_3 = -7$

(c) $x_1 = 1, x_2 = 4, x_3 = 2$

3.

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(a) $x_1 = 2, x_2 = -1, x_3 = 1$

(b) $x_1 = 3 - 2r, x_2 = 4 + r, x_3 = r$

(c) $x_1 = 4 - 3r, x_2 = 1 - 2r, x_3 = r$.

Exercise-2

4. Solve the systems of equations using Gaussian Ellimination.

5.(a) $$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 - x_2 + x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 14 \end{aligned}$$.

(b) $$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 - 2x_2 + 5x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 + 2x_3 &= 1 \end{aligned}$$.

(c) $$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 - 2x_5 &= 1 \\ 2x_1 - x_2 - x_3 + 3x_4 - x_5 &= 3 \\ -x_1 - x_2 + 5x_3 \qquad - 4x_5 &= -3 \end{aligned}$$.

5. Solve ths systems using the method of LU decomposition.

(a) $$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 - x_2 + 3x_3 &= 3. \\ x_1 - x_2 - 4x_3 &= -7 \end{aligned}$$

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(b) $\begin{array}{l} 3x_1 - x_2 + x_3 = 10 \\ -3x_1 + 2x_2 + x_3 = -8 \\ 9x_1 + 5x_2 - 33x_3 = 24 \end{array}$.

(c) $\begin{array}{l} -2x_1 \qquad + 3x_3 = 3 \\ -14x_1 + 3x_2 + 5x_3 = 11 \\ 8x_1 + 9x_2 - 11x_3 = 7 \end{array}$.

(d) $\begin{array}{l} 2x_1 - 3x_2 + x_3 = -5 \\ 4x_1 - 5x_2 + 6x_3 = 2 \\ -10x_1 + 19x_2 + 9x_3 = 55 \end{array}$.

Answers:

4. (a) $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

(b) No solution.

(c) $x_1 = 2 + 2r - s - t,\ x_2 = 1 + 3r + s - 3t,$ $x_3 = r, x_4 = s, x_5 = t$

5. (a) $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix}, X = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$

(b) $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 8 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -22 \end{bmatrix}, X = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$

(c) $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix}, U = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}, X = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$

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$$\textbf{(d)}\ L=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -5 & 4 & 1\end{bmatrix}, U=\begin{bmatrix}2 & -3 & 1\\ 0 & 1 & 4\\ 0 & 0 & -2\end{bmatrix}, X=\begin{bmatrix}-43\\ -24\\ 9\end{bmatrix}.$$

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Chapter-6

Vector Spaces

The vector space R^n is a set of elements called vectors on which two operations namely addition and scalar multiplication have been defined.

6.1. The Notion of a Vector Space

Definition:

A vector space is a set V of elements called vectors having operation of addition and scalar multiplication defined on it that satisfy the following conditions:

Let $u, v, w \in V$ and $c \,\&\, d$ are scalars

Closure axiom:

1.The sum $u+v$ exists and is an element of V.(V is closed under addition)

2. cu is an element of V.(V is closed under scalar multiplication)

Addition axiom:

3. $u+v=v+u$ (Commutative)

4. $u+(v+w)=(u+v)+w$ (Associative)

5.There exist an element u of V called the zero vector denoted $\overline{0}$ such that $u+\overline{0}=u$.

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6.For every element u of V there exist an element called negative of u denoted by $-u$ such that $u+(-u)=\overline{0}$.

Scalar Multiplication axiom:

7. $c(u+v)=cu+cv$

8. $(c+d)u=cu+du$

9. $c(du)=(cd)u$

10. $1u=u$

The two most common set of scalars used in vector spaces are the set of real numbers and the set of complex numbers.

Examples of vector space

The vector spaces are then called real and complex vector space.

1. **1.** Vector space of matrices M_{mn}
2. **2.** The set of real $m\times n$ matrices M_{mn} , is a vector space over $\mathbb{R}$.
3. **3.** Vector spaces of functions
4. **4.** The set of all functions from a vector space over $\mathbb{R}$.
5. **5.** The set of all functions having the real numbers as their domain with operations of pointwise addition and scalar multiplication is a vector space.

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6. The complex vector space $\mathbb{C}^n$ over $\mathbb{C}$.

Theorem-6.1

Let V be a vector space. v be a vector in V . $\overline{0}$ the zero vector of V , c scalar, and 0 the zero scalar. Then

(a) $0v = \overline{0}$

(b) $c\overline{0} = \overline{0}$

(c) $(-1)v = -v$

(d) If $cv = 0$ then either $c = 0$ or $v = 0$.

Proof:

(a)

$$0v + 0v = (0+0)v \quad (axiom\,8)$$
$$= 0v$$

Add the –ve of $0v$,namely $-0v$ to both sides of this equation

$$(0v+0v)+(-0v) = 0v+(-0v)$$
$$\Rightarrow 0v+(0v+(-0v)) = \overline{0} \qquad (Axiom: 4\,\&\,5)$$
$$\Rightarrow 0v+\overline{0} = \overline{0} \qquad (Axiom: 6)$$
$$\Rightarrow 0v = \overline{0} \qquad (Axiom: 5)$$

(c)

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$$(-1)v + v = (-1)v + 1v \quad (Axiom:10)$$
$$= [(-1)+1]v \quad (Axiom:8)$$
$$= 0v = \overline{0} \quad \text{(Property of } Scalar\ 0)$$

Thus $(-1)v$ is the negative of v. (Axiom:6)

6.2. Subspaces

Definition:

Let V be a vector space and U be a non-empty subset of V .If U is a vector space under the operations of addition and scalar multiplication of V then it is called a subspace of V .

U is a subspace if it is closed under addition and under scalar multiplication. It then inherits the other vector space properties from V .

Example-1

Consider the subset W of R^3 consisting of vectors of the form (a,a,b), where the first two components are the same.

If we add two such vectors (a,a,b) and (c,c,d),

We get $(a+c, \mathrm{a}+c, \mathrm{b}+d)$, a vector with identical first components.

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If we multiply (a,a,b) by a scalar k ,we get (ka,ka,kb) , again a vector with identical first components.

$\Rightarrow$ W is closed under addition and scalar multiplication.

Hence It is a subspace of R^3 .

Example-2

Consider the subset W of R^3 consisting of vectors of the form (a,a^2,b) ,where the second component is the square of the first.

On adding two such vectors (a,a^2,b) and (c,c^2,d),

We get $(a+c,a^2+c^2,b+d)$.

The second component of the vector is not a square of the first.

$\Rightarrow$ The vector is not in W .

$\Rightarrow$ W is not closed under addition.

Hence It is not a subspace of R^3.

Example-3

Prove that the set U of 2×2 diagonal matrices is a subspace of the vector space M_{22} of 2×2 matrices.

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Solution:

Vector addition:

Let $u=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $v=\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$

We get $u+v=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}+\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}=\begin{bmatrix} a+p & 0 \\ 0 & b+q \end{bmatrix}\in U$.

$u+v$ Is a diagonal matrix and is thus an element of U

.

U is closed under addition.

Let c be a scalar.

We get $cu=c\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}=\begin{bmatrix} ca & 0 \\ 0 & cb \end{bmatrix}$

cu is a 2×2 diagonal matrix.

Thus U is closed under multiplication.

U is a subspace of M_{22}.

Example-4

Let P_n denote the set of real polynomial function of $\deg \le n$. Prove that P_n is a vector space, if addition and multiplication are defined on polynomial in a pointwise manner.

Left to the reader.

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Theorem-6.2

Let U be a subspace of a vector space V. U contains the zero vector of V.

Proof:

Let u be an arbitrary vector in U and be the zero vector of V.

Let 0 be the zero scalar.

Since we know $0v = \overline{0}$, since U is closed under scalar multiplication,

this means that is $\overline{0}$ in U.

Remarks:

This theorem tells us for example that all subspaces of R^3 contain $(0,0,0)$. This means that all subspaces of 3-space pass through the origin. This theorem can sometimes be used as a quick check to show that contain subsets cannot be subspaces.

If a given subset does not contain the zero vector it cannot be a subspace.

Example-5

Let W be the set of vectors of the form $(a,a,a+2)$.

Show that $(a,a,a+2)$ is not a subspace of R^3.

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Solution:

We check to see If $(0,0,0)$ is in W .

Is there a value of a for which $(a,a,a+2)$ is $(0,0,0)$.

On equating $(a,a,a+2)=0$ to $(0,0,0)$,we get

$(a,a,a+2)=0$

Equating corresponding components, we get $a=0$ and $a+2=0$.

This system of equation has no solution.

Thus $(0,0,0)$ is not an element of W . W is not a subspace.

6.3. Linear Combinations

Definition:

Let $v_1, v_2, ...v_n$ be vectors in a vector space V .The vector in is a linear combination of $v_1, v_2, ...v_n$, If there exist scalars $c_1, c_2, ...c_n$ such that can be written as $v = c_1v_1 + c_2v_2 + ... + c_n v_n$

Example-6

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The $(7,3,2)$ vector is a linear combination of the vectors $(1,3,0)$ and $(2,-3,1)$ because it can be written as $(7,3,2)=3(1,3,0)+2(2,-3,1)$

Example-7

The vector $(3,4,2)$ is not a linear combination of $(1,1,0)$ and $(2,3,0)$ because ther are no values of c_1 and c_2 for which $(3,4,2)=c_1(1,1,0)+c_2(2,3,0)$ is true.

Example-8

Determine whether the vector $(8,0,5)$ is a linear combination of the vectors $(1,2,3),(0,1,4)$ and $(2,-1,1)$.

Example-9

Determine whether the vector $(4,5,5)$ is a linear combination of the vectors $(1,2,3),(-1,1,4)$ and $(3,3,2)$.

6.4. Spanning a Vector space

Definition:

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Let $v_1, v_2, \ldots v_m$ be m vectors in a vector space V. These vectors span V if every vector in V can be expressed as a linear combination of them.

Example-10

$(1,0,0),(0,1,0)$ and $(0,0,1)$ span R^3 because we can write an arbitrary vector (x,y,z) of R^3 as the linear combination

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.

The vectors

$(1,0,0),(0,1,0)$ and

$(1,1,1)$ also span R^3 because we can write

$(x,y,z) = (x-z)(1,0,0) + (y-z)(0,1,0) + z(0,0,1)$.

The vectors $(1,0,0),(0,2,0)$ and $(3,4,0)$ do not span R^3 because a vector (x,y,z) for which $c \neq 0$ cannot be written as a linear combination of these vectors.

Similarly, the vectors $(1,1,0)$ and $(0,0,1)$ span the subspace of R^3 consisting of vectors of the form (a,a,b) because we can write

$(a,a,b) = a(1,1,0) + b(0,0,1)$.

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Example-11

Show that the vectors $(1,2,0),(0,1,-1)$ and $(1,1,2)$ span R^3 .

Solution:

We now determine whether an arbitrary vector of R^3 is a linear combination of given vectors.

Let (x,y,z) be an arbitrary element of R^3.

We have to determine whether we can write

$$(x,y,z)=c_1(1,2,0)+c_2(0,1,-1)+c_3(1,1,2)$$

$$\Rightarrow (x,y,z)=(c_1+c_3,2c_1+c_2+c_3,-c_2+2c_3)$$

Thus

$$c_1+c_3=x$$

$$2c_1+c_2+c_3=y$$

$$-c_2+2c_3=z$$

Using Gauss Jordan elimination, it is found that

$c_1=3x-y-z$, $c_2=-4x+2y+z$ $c_3=-2x+y+z$

The vectors $(1,2,0),(0,1,-1)$ and $(1,1,2)$ thus span R^3.

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6.5. Generating a vector space

Theorem-6.3

Let $v_1, v_2, ...v_m$ be vectors in a vector space V .Let U be the set consisting of all linear combination of $v_1, v_2, ...v_m$ .Then U is subspace of V spanned by these vectors $v_1, v_2, ...v_m$.

U is said to be the vector space generated by $v_1, v_2, ...v_m$. It is denoted as $Span[v_1, v_2, ...v_m]$.

Proof

Let $u_1 = a_1v_1 + a_2v_2 ... + a_mv_m$ and

$$u_2 = b_1v_1 + b_2v_2 ... + b_mv_m$$

be an arbitrary elements of U ,then

$$\begin{aligned} u_1 + u_2 &= (a_1v_1 + a_2v_2 ... + a_mv_m) + (b_1v_1 + b_2v_2 ... + b_mv_m) \\ &= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 ... + (a_m + b_m)v_m \end{aligned}$$

$u_1 + u_2$ is a linear combination of $v_1, v_2, ...v_m$.

Thus $u_1 + u_2$ is in U.

U is closed under vector addition.

Let c be an arbitrary scalar. Then

$$\begin{aligned} cu_1 &= c(a_1v_1 + a_2v_2 ... + a_mv_m) \\ &= ca_1v_1 + ca_2v_2 ... + ca_mv_m \end{aligned}$$

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is a linear combination of $v_1, v_2, \ldots v_m$.

Therefore cu_1 is in U .

It implies U is closed under scalar multiplication.

Thus U is a subspace of V .

By the definition of U ,every vector in U can be written as a linear combination of $v_1, v_2, \ldots v_m$.

Thus $v_1, v_2, \ldots v_m$ $Span U$.

Example-12

Consider the vectors $(-1,5,3)$ and $(2,-3,4)$ in R^3 .Let $U = Span[(-1,5,3),(2,-3,4)]$. U will be a subspace of R^3 consisting of all vectors of the form $c_1(-1,5,3)+c_2(2,-3,4)$.

Example-13

Let v_1 and v_2 be two vectors in the vector space R^3 . The subspace $Span[v_1, v_2]$ generated by v_1 and v_2 is the set of all vectors of the form

$c_1v_1 + c_2v_2$.

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In general, this space is the plane defined by $v_1 \, and \, v_2$ .If $v_1 \, and \, v_2$ are collinear, Then the space will be the line defined by these vectors.

Example-14

Let $v_1 \, and \, v_2$ span a subspace of a vector space U of a vector space V. Let $k_1 \, and \, k_2$ be non-zero scalars. Show that $k_1 v_1 \, and \, k_2 v_2$ also span U .

Solution:

Let v be a vector in U. Since $v_1 \, and \, v_2$ Span U so there exist scalars a and b

such that $v = a_1 v_1 + a_2 v_2$.

We can write

$$v = \frac{a_1}{k_1}\left(k_1 v_1\right) + \frac{a_2}{k_2}\left(k_2 v_2\right)$$

Thus the vectors $k_1 v_1 \, and \, k_2 v_2 \, Span U$.

Example-15

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Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space M_{22} of 2×2 matrices.

Example-16

Show that the function $h(x)=4x^2+3x-7$ lies in the space $Span\{f,g\}$ generated by $f(x)=2x^2-5$ and $g(x)=x+1$.

6.6. Linear dependence and independence

Definition:

(a) The set of vectors $\{v_1,v_2,...v_m\}$ in a vector space V is said to be linearly dependent if There exist scalars $c_1,c_2,...,c_m$ not all zero such that $c_1v_1+c_2v_2...+c_mv_m=\overline{0}$.

(b) The set of vectors $\{v_1,v_2,...v_m\}$ is linearly independent if $c_1v_1+c_2v_2...+c_mv_m=\overline{0}$ can only be satisfied when $c_1=c_2=...=c_m=0$.

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Example-17

The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is seen to be Linearly independent in R^3 under this definition

$$c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(0,0,0)$$

can only be satisfied if

$$c_1=0, c_2=0 \text{ and } c_3=0.$$

Example-18

Consider the set $\{(1,2,3),(5,1,0),(2,0,0)\}$.

The identity

$$c_1(1,2,3)+c_2(5,1,0)+c_3(2,0,0)=(0,0,0)$$

leads to

$$c_1=0, c_2=0 \text{ and } c_3=0.$$

These vectors are Linearly independent in R^3 .

Example-19

Consider the set $\{(4,1,0),(2,1,3),(0,1,2)\}$.

It can be seen that

$$1.(4,1,0)-2(2,1,3)+3(0,1,2)=(0,0,0).$$

The vectors are linearly dependent in R^3.

Example-20

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Determine whether the set $\{(1,2,0),(0,1,-1),(1,1,2)\}$ is linearly independent in R^3 .

Example: -21

Show that the set $\{x^2+1, 3x-1, -4x+1\}$ linearly independent in P_2.(b) Show that the set $\{x+1, x-1, -x+5\}$ is linearly dependent in P_1 .

Theorem-6.4

A set consisting of two or more vectors in a vector space is linearly dependent Iff it is possible to express one of the vector as a linear combination of other vectors.

Proof:

Let the set $\{v_1, v_2, ...v_m\}$ be linearly dependent. Therefore, there exist scalars $c_1, c_2, ..., c_m$ not all zeros such that

$$c_1v_1 + c_2v_2 ... + c_mv_m = \overline{0}.$$

Assume $c_1 \neq 0$.

The proceeding identity can be rewritten

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$$v_1 = \left(-\frac{c_2}{c_1}\right)v_2 \ldots + \left(-\frac{c_m}{c_1}\right)v_m$$

Thus v_1 is a linear combination of $v_2, \ldots v_m$.

Conversely:

Assume that v_1 is a linear combination of $v_2, \ldots v_m$ .therefore there exists scalars $d_2, d_3, \ldots d_m$ such that

$$v_1 = d_2 v_2 \ldots + d_m v_m$$

Rewrite this equation as

$$1.v_1 + (-d_2)v_2 \ldots + (-d_m)v_m = \overline{0}$$

Thus the set $\{v_1, v_2, \ldots v_m\}$ is linearly dependent.

Linear Dependence of $\{v_1, v_2\}$

The set $\{v_1, v_2\}$ is Linearly dependent iff it is possible to write one vector as a scalar multiple of other vector.

Let $v_2 = cv_1$ it implies v_1, v_2 are collinear.

On the other hand, $\{v_1, v_2\}$ is linearly independent iff it is not possible to write one vector as a multiple of the other.

Theorem-6.5

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Let V be a vector space. Any set of vectors in V that contains the zero vector is linearly dependent.

Proof:

Consider the set $\left\{\overline{0}, v_2 ..., v_m\right\}$,which contains the zero vector.

Let us examine the identity

$$c_1\overline{0} + c_2 v_2 ... + c_m v_m = \overline{0},$$

which shows that the identity is true for

$$c_1 = c_2 = ... = c_m = 0 \text{ . (not all zero)}$$

Thus the set of vectors is linearly dependent.

Theorem-6.6

Let the set $\left\{v_1, v_2, ...v_m\right\}$ be linearly dependent in a vector space V. Any set of vectors in V that contains these vectors will also be linearly dependent.

Proof:

Since the set $\left\{v_1, v_2, ...v_m\right\}$ is linearly dependent there exist scalars $c_1, c_2, ..., c_m$ not all zero such that

$$c_1 v_1 + c_2 v_2 ... + c_m v_m = \overline{0}.$$

Consider the set of vectors $\left\{v_1, v_2, ...v_m, v_{m+1}, ...v_n\right\}$ which contains the given vectors.

There are scalars not all zero namely

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$c_1, c_2, ..., c_m, 0, 0..0$, such that

$$c_1v_1 + c_2v_2 ... + c_mv_m + 0.v_{m+1} + ... + 0.v_n = \overline{0}$$

It implies $\{v_1, v_2, ...v_m, v_{m+1}, ...v_n\}$ are linearly dependent.

Example-22

Let the set $\{v_1, v_2\}$ be linearly independent. Prove that $\{v_1 + v_2, v_1 - v_2\}$ is also linearly independent.

6.7. Properties of Bases

Theorem-6.7

Let the vectors $v_1, v_2, ...v_n$ span a vector space V. Each vector in V can be expressed uniquely as a linear combination of these vectors if and only if the vectors are linearly independent.

Proof:

(a) Assume that $v_1, v_2, ...v_n$ are linear independent.

Let v be a vector in V.

Since $v_1, v_2, ...v_n$ span V.

We can express v as a linear combination of these vectors.

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Suppose that we can write

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n \text{ and}$$

$$v = b_1v_1 + b_2v_2 + ... + b_nv_n$$

Then

$$a_1v_1 + a_2v_2 + ... + a_nv_n = b_1v_1 + b_2v_2 + ... + b_nv_n$$

It implies

$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + ... + (a_n - b_n)v_n = \overline{0}$$

Since $v_1, v_2, ...v_n$ are linearly independent it implies

$$a_1 - b_1 = 0, a_2 - b_2 = 0, ..., a_n - b_n = 0.$$

Implies

$$a_1 = b_1, a_2 = b_2, ..., a_n = b_n.$$

Hence there is only one way of expressing as a linear combination of the vectors $v_1, v_2, ...v_n$.

(b) Let v be a vector in V .

Assume that v can be written in only one way as a linear combination of $v_1, v_2, ...v_n$.

Note that

$$0v_1 + 0v_2 + ... + 0v_n = \overline{0}.$$

This must be the only way $\overline{0}$ can be written as a linear combination of $v_1, v_2, ...v_n$.

Thus

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$$c_1v_1 + c_2v_2 + ... + c_nv_n = \overline{0}$$

can only be satisfied when

$$c_1 = 0, c_2 = 0, ..., c_n = 0.$$

It implies $v_1, v_2, ...v_n$ are linearly independent.

6.8. Basis and Dimensions

Definition:

A finite set of vectors $\{v_1, v_2, ...v_n\}$ is called a basis for vector space V if the set span V and is linearly independent.

Each vector in V can be expressed uniquely as a linear combination of the vectors in a basis.

Example-23

The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ spans R^3 and is linearly independent. It implies it is a standard basis for R^3.

Example-24

The set $\{(1,2,0),(0,1,-1),(1,1,2)\}$ also span R^3 and is linearly independent implies it is a basis for R^3.

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Example-24

The set $\{x^2+1, 3x-1, -4x+1\}$ spans P_2 and is linearly independent. It is a basis for P_2 .

Theorem-6.8

Let $\{v_1, v_2, ...v_n\}$ be a basis for a vector space V . If $\{w_1, w_2, ...w_m\}$ is a set of more than n-vectors in V then this set is linearly dependent.

Proof:

Consider the identity

$$c_1 w_1 + c_2 w_2 + ... + c_n w_m = \overline{0} \qquad (5.8.1)$$

We shall show that values of $c_1, c_2, ..., c_m$ not all zeros exists satisfying the identity. Thus proving that the vectors are linearly dependent.

The set $\{v_1, v_2, ...v_n\}$ is a basis for V . Thus each of the vectors $w_1, w_2, ...w_m$ can be expressed as a linear combination of $v_1, v_2, ...v_n$.

Let

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$$w_1 = a_{11}v_1 + a_{12}v_2 + ... + a_{1n}v_n$$
$$w_2 = a_{21}v_1 + a_{22}v_2 + ... + a_{2n}v_n$$
$$......................................$$
$$w_m = a_{m1}v_1 + a_{m2}v_2 + ... + a_{mn}v_n$$

Substituting for $w_1, \mathrm{w}_2, ...\mathrm{w}_m$ in equation (5.8.1) ,we get

$$c_1\left(a_{11}v_1 + a_{12}v_2 + ... + a_{1n}v_n\right) + c_2\left(a_{21}v_1 + a_{22}v_2 + ... + a_{2n}v_n\right) + ... + c_n\left(a_{m1}v_1 + a_{m2}v_2 + ... + a_{mn}v_n\right) = \overline{0}$$

Rearranging we get

$$v_1\left(c_1a_{11} + c_2a_{21} + ... + c_na_{m1}\right) + v_2\left(c_1a_{12} + c_2a_{22} + ... + c_na_{m2}\right) + ... + v_n\left(c_1a_{1n} + c_2a_{2n} + ... + c_na_{mn}\right) = \overline{0}$$

Since $v_1, v_2, ...v_n$ are linearly independent this identity can be satisfied only if the coefficient are all zero. thus

$$c_1a_{11} + c_2a_{21} + ... + c_na_{m1} = 0$$
$$c_1a_{12} + c_2a_{22} + ... + c_na_{m2} = 0$$
$$....................................$$
$$c_1a_{1n} + c_2a_{2n} + ... + c_na_{mn} = 0$$

Thus finding $c's$ that satisfying (5.8.1) reduces to finding solution to this system of $n-$equations in $m-$variables.

Since $m > n$, the number of variables is greater than the number of equation. We know that such a

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system of homogenous equation has many solutions. there are therefore nonzero values of $c's$ that satisfies (5.8.1).

Thus the set $\{w_1, w_2, ...w_m\}$ are linearly dependent.

Theorem-6.9

All bases for a vector space V have the same number of vectors.

Proof

Let $\{v_1, v_2, ...v_n\}$ and $\{w_1, w_2, ...w_m\}$ be two bases for V.

If we interpret $\{v_1, v_2, ...v_n\}$ as a basis for V and $\{w_1, w_2, ...w_m\}$ as a set of Linearly independent vectors in V then the previous theorem tells us that $m \le n$.

Conversely, If we interpret $\{w_1, w_2, ...w_m\}$ as a basis for V and $\{v_1, v_2, ...v_n\}$ as a set of linearly independent vectors in V then $n \le m$.

Thus $n = m$, proving that both bases consists of the same number of vectors.

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Definition

If a vector space V has a basis consisting of $n-$ vectors then the dimension of V is said to be n.we write $\dim(V)$ for the dimension of V.

Example-25

Consider the set of vectors $\{(1,2,3),(-2,4,1)\}$ in R^3. These vectors generates a subspace V of R^3 consisting of all vectors of the form $v = c_1(1,2,3) + c_2(-2,4,1)$.

The vectors $(1,2,3)$ *and* $(-2,4,1)$ span this subspace.

Furthermore, since the second vector is not a scalar multiple of the first vector, so the vectors are linearly independent.

Therefore, $\{(1,2,3),(-2,4,1)\}$ is a basis for V.

Thus $\dim(V) = 2$.

We know that V is in fact a plane through the origin.

Theorem-6.10

(a) The origin is a subspace of R^3. The dimension of this subspace is define to be zero.

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(b) The one dimensional subspaces of R^3 are lines through the origin.

(c) The two dimensional subspace of R^3 are planes through the origin.

Proof:

(a) Let V be the set $\{(0,0,0)\}$ consisting of a single element, the zero vector of R^3.

Let c be an arbitrary scalar.

Since $(0,0,0)+(0,0,0)=(0,0,0)$

and $c(0,0,0)=(0,0,0)$

V is closed under addition and scalar multiplication.

It is thus a subspace of R^3.

The dimension of this subspace is defined to be zero.

(b) Let V be a basis for a one dimensional subspace V of R^3. Every vector in V is thus of the form cv for some scalar c .We know that these vectors form a line through the origin.

(c) Let $\{v_1, v_2\}$ be a basis for a two dimensional subspace V of R^3.Every vector in V is of the form $c_1v_1 + c_2v_2$. V is thus a plane through the origin.

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Theorem-6.11

Let V be a vector space of dimension n.

(a) If $S=\{v_1,v_2,...v_n\}$ is a set of n-linearly independent vectors in V, Then S is a basis for V.

(b) If $S=\{v_1,v_2,...v_n\}$ is a set of n-vectors that span V. Then S is a basis for V.

Example-26

Prove that the set $\{(1,3,-1),(2,1,0),(4,2,1)\}$ is a basis for R^3.

Theorem-6.12

Let V be a vector space of dimension n. Let $\{v_1,v_2,...v_m\}$ be a set of m linearly independent vectors in V. Where $m<n$. Then there exists vectors $v_{m+1},v_{m+2},...v_n$ such that $\{v_1,v_2,...v_m,v_{m+1},v_{m+2},...v_n\}$ is a basis of V.

Proof:

Since $m<n$, Then $\{v_1,v_2,...v_m\}$ cannot be a basis .

Thus there exist a vector v_{m+1} in V which does not lie in the subspace generated by $v_1,v_2,...v_m$.

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The set $\{v_1, v_2, ...v_m, v_{m+1}\}$ will be linearly independent.

Now if $m+1=n$,

then $\{v_1, v_2, ...v_m, v_{m+1}\}$ is a basis of V.

If $m+1<n$,there will be a vector v_{m+2} that does not lie in the subspace generated by $\{v_1, v_2, ...v_m, v_{m+1}, v_{m+2}\}$.

If $m+2=n$,then $\{v_1, v_2, ...v_m, v_{m+1}, v_{m+2}\}$ is a basis for V.

One continues adding vectors thus until a basis $\{v_1, v_2, ...v_m, v_{m+1}, v_{m+2}, ...v_n\}$ of V is found.

Exercises-1

Vector Space

1.(i) Prove that the set W of all vectors in R^3 of the form $a(1,2,3)$,where a is a real number is a vector space.

(ii) Let U be the set of all vectors in R^3 that are perpendicular to a vector u.Prove that that U is a vector space.

(iii) Let W be the set of all 2×2 matrices having every element a positive number. Prove that W is closed under addition, but is not closed under scalar multiplication. Thus it is not a vector space.

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(iv) Let U be the set of all constant functions with operations of pointwise addition and scalar multiplication, having the real numbers as their domain. Is U a vector space?

(v) Prove that the following sets are not vector spaces. **(a)** The set of all integers. **(b)** The set of all positive numbers.

Subspaces of R^n.

2.(i) Consider the sets of vectors of the following form. Prove that they are subspaces of R^3.

(a) $(a,3a,5a)$. **(b)** $(a,-a,2a)$.

(c) $(a,b,a+2b)$. **(d)** $(a,b,a-b)$.

(ii) Consider the sets of vectors of the following form. Determine whether the sets are subspaces of R^2 or R^3 . Give the geometrical interpretation of each subspace.

(a) $(a,0)$ **(b)** $(a,2a)$

(c) $(a,1)$ **(d)** $(a,a+3)$

(e) $(a,b,0)$ **(f)** $(a,b,2)$

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(g) $(a,b,2a+3b)$

(iii) Are the following sets subspaces of R^3?The set of all vectors of the form (a,b,c),where

(a) $a+b+c=0$ **(b)** $a+b+c=1$

(c) $ab=0$ **(d)** $ab=5$

(e) $ab=ac$ **(f)** $a=b+c$.

(iv) Which of the following subsets of R^3 are subspaces? The set of all vectors of the form (a,b,c), where a, b and c are

(a) Integers **(b)** Nonnegative real numbers

(c) rational numbers

(iv) Are the following sets subspaces of R^3?The set of all vectors of the form

(a) (a,b^2) **(b)** (a,b^3)

(c) $(a,b)$where $a>0$ **(d)** $(a,b)$where $ab<0$

(e) (a,b),where a is non-positive and b is nonnegative.

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(v) Let U be a subset of R^3.let u_1 and u_2 be vectors in U and a and b be scalars. Prove that U is a subspace of R^3 if and only if $au_1 + bu_2$ is a vector in for all values of a and b.

Subspaces of Matrices

3.Determine which of the following subsets of M_{22} form subspaces.

(a) The subset having diagonal elements zero.

(b) The subset consisting of matrices the sum of whose elements is 6.

(c) The subset of matrices of the form $\begin{bmatrix} a & a^2 \\ b & b^2 \end{bmatrix}$.

(d) The subset of matrices of the form $\begin{bmatrix} a & a+2 \\ b & c \end{bmatrix}$.

(ii) Determine which of the following subsets of M_{nn} forms subspaces.

(a) The subset of symmetric matrices.

(b) The subset of matrices that are not symmetric.

(c) The subset of antisymmetric matrices.

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(d) The subset of invertible matrices.

(iii) Which of the following subsets of M_{23} form subspaces?

(a) The subset of matrices of the form $\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$.

(b) The subset of matrices of the form $\begin{bmatrix} a & 2a & 3a \\ b & 2b & 3b \end{bmatrix}$.

(c) The subset of matrices of the form $\begin{bmatrix} a & 1 & b \\ c & d & e \end{bmatrix}$.

Linear Combination in R^n

4.(i) Determine whether the first vector is a linear combination of the other vectors. If it is, give the combination.

(a) $(-1,7);(1,-1),(2,4)$

(b) $(8,13);(1,2),(2,3)$

(c) $(-1,15);(-1,4),(2,-8)$

(d) $(13,6);(1,3),(4,1)$

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(ii) Determine whether the first vector is a linear combination of the other vectors. If it is ,give the combination.

(a) $(-3,3,7);(1,-1,2),(2,1,0),(-1,2,1)$

(b) $(-2,11,7);(1,-1,0),(2,1,4),(-2,4,1)$

(c) $(2,7,13);(1,2,3),(-1,2,4),(1,6,10)$

(d) $(0,10,8);(-1,2,3),(1,3,1),(1,8,5)$

(e) $(1,4,-3);(1,0,1),(1,1,0),(3,1,2)$

(f) $(1,1,2);(0,1,0),(3,5,6),(1,2,1)$

(iii) Give two vectors that are linear combinations of the following vectors.

(a) $(1,2),(3,-5)$ **(b)** $(-1,0),(3,1),(2,4)$

(c) $(1,-3,5),(0,1,2)$

(d) $(1,2,3),(1,1,1),(0,7,2),(4,3,-2)$

Spanning sets

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5. Show that the following sets of vectors span R^2. Express the vector $(3,5)$ in terms of each spanning set.

(a) $(1,1),(1,-1)$ **(b)** $(1,4),(-2,0)$

(c) $(1,3),(3,10)$ **(d)** $(1,0),(0,1)$

(ii) Show that the following sets of vectors span R^3, Express the vector $(1,3,-2)$ in terms of each spanning set.

(a) $(1,2,3),(-1,-1,0),(2,5,4)$

(b) $(1,3,1),(-1,1,0),(4,1,1)$

(c) $(5,1,3),(2,0,1),(-2,-3,-1)$

(d) $(1,0,0),(0,1,0),(0,0,1)$

(iii) Determine whether the following vector span R^2 ?

(a) $(1,-3),(2,-5)$ **(b)** $(1,1),(-2,1)$

(c) $(-3,1),(3,-1)$ **(d)** $(3,2),(1,1)$

(e) $(2,-1),(-4,2)$

(iv) Determine whether the following vectors span R^3.

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(a) $(1,1,0),(-1,1,0),(0,0,3)$

(b) $(4,0,1),(0,1,0),(0,0,1)$

(c) $(1,2,0),(-1,3,0),(2,5,0)$

(d) $(1,1,1),(2,2,2),(1,0,0)$

Subspaces defined by Spanning sets

6. (i) Give three other vectors in the subspace of R^3 generated by the vectors $(1,2,3)$ and $(1,2,0)$.

(ii) Give three other vectors in the subspace of R^3 generated by the vector $(1,2,3)$ .Sketch the subspace.

(iii) Give three other vectors in the subspaces of R^4 generated by the vector $(1,2,-1,3)$.

(iv) Give three other vectors in the subspace of R^4 generated by the vectors

$(2,1,-3,4),(-3,0,1,5),(4,1,2,0)$.

Spaces of Matrices and Functions

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7. (i) In each of the following, determine whether the first matrix is a linear combination of the matrices that follows:

(a) $\begin{bmatrix} 5 & 7 \\ 5 & -10 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$

(b) $\begin{bmatrix} 7 & 6 \\ -5 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 1 \\ 7 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$

(ii) In each of the following, determine whether the first function is a linear combination of the functions that follows:

(a) $f(x) = 3x^2 + 2x + 9;\ g(x) = x^2 + 1,$
$h(x) = x + 3$

(b) $f(x) = 2x^2 + x - 3;\ g(x) = x^2 - x + 1,$
$h(x) = x^2 + 2x - 2$

(c) $f(x) = x^2 + 4x + 5;\ g(x) = x^2 + x - 1,$
$h(x) = x^2 + 2x + 1$

(iii)

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(a) Is the function $f(x) = x + 5$ in the vector space $Span\{g, h\}$ generated by $g(x) = x + 1$ and $h(x) = x + 3$.

(b) Is the function $f(x) = 3x^2 + 5x + 1$ in the vector space $Span\{g, h\}$ generated by $g(x) = 2x^2 + 3$ and $h(x) = x^2 + 3x - 1$.

(c) Give three other functions in the vector space $Span\{g, h\}$ generated by $g(x) = 2x^2 + 3$ and $h(x) = x^2 + 3x - 1$.

Linear Dependence and Independence

8. (i) Determine whether the following sets of vectors are linearly dependent or independent in .

(a) $\{(-1, 2), (2, -4)\}$

(b) $\{(-1, 3), (2, 5)\}$

(c) $\{(1, -2, 3), (-2, 4, 1), (-4, 8, 9)\}$

(d) $\{(1, 0, 2), (2, 6, 4), (1, 12, 2)\}$

(e) $\{(1, 2, 5), (1, -2, 1), (2, 1, 4)\}$

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(f) $\{(1,1,1),(-4,3,2),(4,1,2)\}$

(ii) **(a)** Prove that the set $\{(1,1),(0,2)\}$ is linearly independent in R^2.

(b) Prove that the set $\{(1,1,2),(0,-1,2),(0,0,5)\}$ is linearly independent in R^3.

(c) Prove that the set $\{(3,-2,4,5),(0,2,3,-4),(0,0,2,7),(0,0,0,4)\}$ is linearly independent in R^4.

(iii) Consider the following matrix, which is in reduced echolen form.

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Show that the row vectors form a linearly independent set. Is the set of nonzero row vectors of any matrix in reduced echolen form linearly independent?

Linear Dependence in Matrix and Function Spaces.

9.(i) Determine whether the following sets of matrices are linearly dependent in M_{23}.

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(a) $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 2\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\3 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 4\end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix},\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix},\begin{bmatrix}2 & 1\\4 & 2\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix},\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix},\begin{bmatrix}1 & 2\\5 & 3\end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix}2 & 4\\0 & 1\end{bmatrix},\begin{bmatrix}0 & -2\\8 & 3\end{bmatrix},\begin{bmatrix}-3 & -7\\4 & 0\end{bmatrix}\right\}$

(ii) Determine whether the following sets of functions are linearly dependent in P_2.

(a) $\{f, g, h\}$ Where $f(x) = 2x^2 + 1, g(x) = x^2 + 4x,$
$h(x) = x^2 - 4x + 1$

(b) $\{f, g, h\}$ Where $f(x) = x^2 + 3, g(x) = x + 1,$
$h(x) = 2x^2 - 3x + 3$

(c) $\{f, g, h\}$ Where $f(x) = x^2 + 3x - 1, g(x) = x + 3,$
$h(x) = 2x^2 - x + 1$

(d) $\{f, g, h\}$ Where $f(x) = -x^2 + 2x - 5, g(x) = 5x - 1,$
$h(x) = 7$

10. General Vector Spaces

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(i) Let $\{v_1, v_2\}$ be any vectors in a vector space V .Show that the set $\{v_1, v_2, av_1 + bv_2\}$ is linearly dependent for all values of scalars a and b .

(ii) Let $\{v_1, v_2\}$ be linearly independent in a vector space V . Show that if a vector v_3 is not of the form $av_1 + bv_2$ then the set $\{v_1, v_2, v_3\}$ is linearly independent.

(iii) Let the set $\{v_1, v_2\}$ linearly dependent in a vector space V . Prove that $\{v_1 + v_2, v_1 - v_2\}$ is also linearly dependent.

(iv) Prove that every subset of a linearly independent set is linearly independent. Is every subset of a linearly dependent set linearly dependent?

(v) Let v_1, v_2 and v_3 be vectors in R^3. What can you say about the linear dependence or independence of these vectors in the following cases?

(a) $Span\{v_1, v_2, v_3\}$ is R^3.

(b) $Span\{v_1, v_2, v_3\}$ is the same space as $Span\{v_1, v_2\}$.

(c) $Span\{v_1, v_2, v_3\}$ is the same space as $Span\{v_1\}$.

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Chapter-7

Eigenvalues and Eigenvectors

7.1. Eigen value and Eigen vectors

Eigen values and Eigen vectors are special scalars and vectors associated with matrices. These are used in many branches of the natural and social sciences and engineering.

General application of Eigen vectors are

a. To rank pages in the search engine Google.

b. To use in demography to predicts long terms trend.

c. To use in meteorology with an example of weather prediction for Tel Aviv.

d. To study of oscillating system.

We commence our discussion with the definition of an Eigen value and Eigen vector.

Definition:

Let A be an $n \times n$ matrix. A scalar λ is called an Eigen value of A if there exist a non-zero vector X in R^n such that $AX = \lambda X$.

The vector X is called an Eigen vector corresponding to λ .

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Let us look at the geometrical significance of an Eigen vector that corresponds to a non-zero Eigen value.

Geometrical significance

The vector AX is in the same or opposite direction as X, depending on the sign of λ.

7.2. Computation of Eigen values and Eigen vectors

Let A be an $n \times n$ matrix with Eigen value λ and corresponding Eigen vector X.

Thus

$$AX = \lambda X .$$

This equation can be re written as

$$AX - \lambda X = \overline{0}$$

$$\Rightarrow (A - \lambda I_n) X = \overline{0}$$

This matrix equation represents a system of homogenous linear equations having matrix of coefficient $(A - \lambda I_n)$.

Which implies $X = \overline{0}$ is a solution to this system.

However, Eigen vector have been defined to be nonzero vectors.

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Further, the non-zero solutions to this system of equation can only exist if the matrix of coefficient is singular,

$$|A-\lambda I_n|=0.$$

Hence, Solving the equation $|A-\lambda I_n|=0$ for λ leads to all the Eigen values of A.

On expanding the determinant $|A-\lambda I_n|$,we get a polynomial in λ.

This polynomial is called the characteristic polynomial of A.

The equation $|A-\lambda I_n|=0$ is called the characteristic equation of A.

The Eigen values are then substitute back in to the equation $(A-\lambda I_n)X=\overline{0}$ to find the corresponding Eigen vectors.

Example-1

Find the Eigen values and Eigen vectors of the matrix

$$A=\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}.$$

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Theorem-7.1

Let A be an $m \times n$ matrix and λ an Eigen value of A. The set of all Eigen vectors corresponding to λ, together with the zero vector is a subspace of R^n.

This subspace is called the Eigen space of λ.

Proof:

Let V be the set of all Eigen vectors corresponding to λ ,together with the zero vector.

In order to show that V is a subspace, we have to show that it is closed under vector addition and scalar multiplication.

Let X_1 and X_2 be vectors in V and c be a scalar.

Then $AX_1 = \lambda X_1$ and $AX_2 = \lambda X_2$.

Hence

$$AX_1 + AX_2 = \lambda X_1 + \lambda X_2$$

$$\Rightarrow A(X_1 + X_2) = \lambda(X_1 + X_2)$$

Thus $X_1 + X_2$ is an Eigen vector corresponding to .V is closed under addition.

Further, Since

$$AX_1 = \lambda X_1$$

$$\Rightarrow cAX_1 = c\lambda X_1$$

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$$\Rightarrow A(cX_1) = \lambda(cX_1)$$

Therefore cX_1 is an Eigen vector corresponding to λ.

V is closed under scalar multiplication.

Thus V is a subspace of R^n.

Example-2

Find the Eigen value and corresponding Eigen spaces of the matrix

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Remark:

If an Eigen value occurs as a k times repeated roots of the characteristic equation, we say that it is of multiplicity k.

Example-3

Let A be an $n \times n$ matrix with Eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding Eigen vectors $X_1, X_2, ..., X_n$. Prove that if $c \neq 0$, then the Eigen values of cA are $c\lambda_1, c\lambda_2, ..., c\lambda_n$ with corresponding Eigen vectors $X_1, X_2, ..., X_n$.

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7.3. Characteristic Polynomial

Let $A=\left[a_{ij}\right]$ be an n-square matrix. The matrix $M=A-tI_n$, where I_n is the n-square identity matrix and t is an intermediate, may be obtained by substituting t down the diagonal of A.

The negative of M is the matrix tI_n-A and its determinant

$$\Delta(t)=\det(tI_n-A)=(-1)^n\det(A-tI_n)$$

Which is a polynomial in t of degree n is called the characteristic polynomial of A.

7.3.1.Caylay-Hamilton theorem

Every matrix A is a root of its characteristic polynomial i.e. Every matrix A satisfies its own characteristic equations.

Remark:

Suppose $A=\left[a_{ij}\right]$ is a triangular matrix then $|tI-A|$ is a triangular matrix with diagonal entries $t-a_{ii}$ and hence

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$$\Delta(t) = \det(tI - A)$$
$$= (t - a_{11})(t - a_{22})\,.........(t - a_{nn})$$

It implies the roots of $\Delta(t)$ are the diagonal elements of A.

7.4. Minimal Polynomials

Let A be any square matrix. Let $J(A)$ denote the collection of all polynomials $f(t)$ for which A is a root i.e. for which $f(A) = 0$.

The set $J(A)$ is not empty. Since the Caylay Hamilton theorem tells us that the characteristic polynomial $\Delta_A(t)$ of A belongs to $J(A)$.

Let $m(t)$ denote the monic polynomial of lowest degree in $J(A)$.(Such a polynomial $m(t)$ exists and is unique)

We shall call $m(t)$ the polynomial of the matrix A.

Remark:

A polynomial $f(t) \neq 0$ is monic if its leading coefficient equals one.

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Theorem-7.2

The minimal polynomial $m(t)$ of a matrix (linear operator) A divides every polynomials that has A as a zero.

In particular, $m(t)$ divides the characteristic polynomial $\Delta(t)$ of A.

There is an even stronger relationship between $m(t)$ and $\Delta(t)$.

Theorem-7.3

The characteristic polynomial $\Delta(t)$ and the minimal polynomial $m(t)$ of a matrix A have the same irreducible factors.(It does not say that $m(t)=\Delta(t)$)

Theorem-7.4

A scalar λ is an Eigen value of the matrix A if and only if λ is a root of the polynomial of A.

Example-4

Find the minimal polynomial $m(t)$ of

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$$A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$$

Solution:

Here we have

$$tr(A) = 2 + 7 - 4 = 5$$

$$A_{11} + A_{22} + A_{33} = 2 - 3 + 8 = 7$$

$$|A| = 3$$

Hence

$$\begin{aligned} \Delta(t) &= t^3 - 5t^2 + 7t - 3 \\ &= (t-1)^2 (t-3) \end{aligned}.$$

The minimal polynomial $m(t)$ must divides $\Delta(t)$.

Also each irreducible factor of $\Delta(t)$, that is $(t-3)$ *and* $(t-1)$ must also be a factor of $m(t)$.

Thus $m(t)$ is exactly one of the following:

$$f(t) = (t-3)(t-1) \text{ and } g(t) = (t-3)(t-1)^2.$$

We know by Caylay-Hamilton theorem:

$$g(A) = \Delta(A) = 0$$

Hence we need only test $f(t)$,we have

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$$f(A) = (A - I)(A - 3I)$$
$$= \begin{bmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 4 & -15 \\ 1 & 2 & -7 \end{bmatrix}.$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$f(t) = m(t)$$
$$= (t-3)(t-1)$$
$$= t^2 - 4t + 3$$

is the minimal polynomial of A.

7.5. Properties of Eigen values and Eigen vectors

Theorem: -7.5

Let A be a square matrix over the complex field $\mathbb{C}$.Then A has at least one Eigen value.

Theorem: -7.6

Suppose $v_1, \mathrm{v}_2, ..., \mathrm{v}_n$ are non-zero Eigen vectors of a matrix A belonging to distinct Eigen values

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$\lambda_1, \lambda_2, ..., \lambda_n$. Then $v_1, v_2, ..., v_n$ are linearly independent.

Theorem: -7.7

Suppose the characteristic *polynomial* of $\Delta(t)$ an n-square matrix A is a product of n-distinct factors, say $\Delta(t) = (t - a_1)(t - a_2).........(t - a_n)$. Then A is similar to the diagonal matrix $D = diag(a_1, a_2, ...a_n)$.

Definition:

If λ is an Eigen value of a matrix A then the Algebraic multiplicity of λ is defined to be the multiplicity of λ as a root of the characteristic polynomial of A.
The geometric multiplicity of λ is defined to be the dimension of its Eigen space i.e. $\dim E_\lambda$.

Theorem: -7.8

The geometric multiplicity of an Eigen value λ of a matrix A does not exceed its Algebraic multiplicity i.e. $G.M \le A.M$.

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Theorem: -7.9

Let A be a real symmetric matrix then each root λ of its characteristic polynomial is real.

Theorem: -7.10

Similar matrices have the same Eigen values.

Proof:

Let A and B be similar matrices.

Hence there exists a matrix C such that $B = C^{-1}AC$.

The chacteristic polynomial of B is $|B - \lambda I|$.

Substituting for B and using the multiplicative properties of determinants, we get

$$|B - \lambda I| = |C^{-1}AC - \lambda I| = |C^{-1}(A - \lambda I)C|$$
$$= |C^{-1}||A - \lambda I||C|$$
$$= |A - \lambda I||C^{-1}||C|$$
$$= |A - \lambda I||C|^{-1}|C|$$
$$= |A - \lambda I||C^{-1}C|$$
$$= |A - \lambda I||I|$$
$$= |A - \lambda I|$$

This implies The characteristic polynomial of A and B are identical.

$\Rightarrow$ Their eigen values are the same.

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Definition:

A square matrix A is said to be diagonalizable if there exist a matrix C such that $D = C^{-1}AC$ is a diagonal matrix.

Theorem: -7.11

Let A be an $m \times n$ matrix.

(a) If A has $n-$linearly independent Eigen vectors then it is diagonalizable.

The matrix C whose column consists of $n-$ linearly independent Eigen vectors can be used in a similarity transformation to give a diagonal matrix D. The diagonal elements of D will be the Eigen values.

(b) If A is diagonalizable then it has $n-$linearly independent Eigen vectors.

Proof:

(a)

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Let A have Eigen values $\lambda_1, \lambda_2, \lambda_3......\lambda_n$ (which need not be distinct) with corresponding linearly independent Eigen vectors $v_1, v_2, v_3......v_n$.

Let C be the matrix having $v_1, v_2, v_3......v_n$ as column vectors i.e. $C = [v_1, v_2, v_3......v_n]$.

Since $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$,....., $Av_n = \lambda_n v_n$.

Matrix multiplication in terms of columns gives

$$\begin{aligned} AC &= A[v_1, v_2, v_3......v_n] \\ &= [Av_1, Av_2, Av_3......, Av_n] \\ &= [\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3......, \lambda_n v_n] \\ &= [v_1, v_2, v_3......, v_n]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots \lambda_n \end{bmatrix} = C\begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots \lambda_n \end{bmatrix} \end{aligned}$$

Since the columns of C are linearly independent, C is non-singular.

Thus

$$C^{-1}AC = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots \lambda_n \end{bmatrix}.$$

Therefore, if an matrix A has $n-$linearly independent Eigen vectors, then the Eigen vectors can

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be used as the column of a matrix C that diagonalizes the matrix A.

The diagonal matrix has the Eigen values of A as diagonal elements.

(b) The converse is proved by retracing the above steps. Commence with the assumption that C is a matrix $\left[v_1, v_2, v_3 ...... v_n\right]$ that diagonalizes A.

Thus there exist scalars $\gamma_1, \gamma_2, ..., \gamma_n$ such that

$$C^{-1}AC = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \ddots \, \gamma_n \end{bmatrix}$$

Retracing these steps we arrive at the conclusion that

$$Av_1 = \gamma_1 v_1 \ , Av_2 = \gamma_2 v_2 , ..., Av_n = \gamma_n v_n$$

Thus $\left[v_1, v_2, v_3 ...... v_n\right]$ are the Eigen vectors of A.

Since C is non-singular these vectors (Column vectors of C) are Linearly independent.

Thus if an $n \times n$ matrix A is diagonalizable then it has $n-$linearly independent Eigen vectors.

Example: -5

(a) Show that the following matrix A is diagonalizable.

(b) Find the diagonal matrix D that is similar to A.

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(c) Determine the similarity transformation that diagonalises A

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}.$$

Note:

If A is similar to a diagonal matrix D under the transformation $C^{-1}AC$ then it can be shown that

$$A^k = CD^kC^{-1}.$$

This results can be used to compute A^k.

Let us derive this result and then apply it

$$D^k = \left(C^{-1}AC\right)^k$$
$$= \left(C^{-1}AC\right)\left(C^{-1}AC\right)..........\left(C^{-1}AC\right)$$
$$= C^{-1}A^kC$$

This leads to

$$A^k = CD^kC^{-1}.$$

Example: -6

Compute A^9 for the following matrix A

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}.$$

This technique is used in solving equation called difference equation.

Remark:

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Not every matrix is diagonalizable.

Example: -7

Show that the following matrix $A=\begin{bmatrix}5 & -3\\3 & -1\end{bmatrix}$ is not diagonalizable.

Jordan Block

(a) Consider the following two r-square matrix where $a \neq 0$.

$$J(\lambda:r)=\begin{bmatrix}\lambda & 1 & 0 & \vdots & 0 & 0\\0 & \lambda & 1 & \vdots & 0 & 0\\. & \cdots & . & \cdots & . & .\\0 & 0 & 0 & \vdots & \lambda & 1\\0 & 0 & 0 & \vdots & 0 & \lambda\end{bmatrix}$$ and

$$A=\begin{bmatrix}\lambda & a & 0 & \vdots & 0 & 0\\0 & \lambda & a & \vdots & 0 & 0\\. & \cdots & . & \cdots & . & .\\0 & 0 & 0 & \vdots & \lambda & a\\0 & 0 & 0 & \vdots & 0 & \lambda\end{bmatrix}$$

The matrix called a Jordan block has λ's on the diagonal,1's on the super diagonal and 0's elsewhere.

One can show that $f(t)=(t-\lambda)^r$

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is both the chacteristic and minimal polynomial of both $J(\lambda : r)$ and A.

(b) Consider an arbitrary monic polynomial

$$f(t) = t^n + a_{n-1}t^{n-1} + ... + a_1 t + a_0$$

Let $c(f)$ be the n-square matrix with 1's on the sub-diagonal (Consisting of the entries below the diagonal entries),the negative of the coefficient in the last column and 0's elsewhere as follows:

$$c(f) = \begin{bmatrix} 0 & 0 & \vdots & 0 & -a_0 \\ 1 & 0 & \vdots & 0 & -a_1 \\ 0 & 1 & \vdots & 0 & -a_2 \\ . & \cdots & . & \cdots & . \\ 0 & 0 & . & 1 & -a_{n-1} \end{bmatrix}$$

Then $c(f)$ is called the companion matrix of the polynomial $f(t)$.

Moreover the minimal polynomial $m(t)$ and characteristic polynomial $\Delta(t)$ of the companion

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matrix $c(f)$ are both equal to the original polynomial $f(t)$.

7.6. Characteristic polynomial of Block Matrices

Suppose M is a block triangular matrix, Say $M = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$, Where A_1 *and* A_2 are square matrices.

Then $tI - M$ is also a block triangular matrix., with diagonal blocks $tI - A_1$ and $tI - A_2$.

Thus

$$|tI - M| = \begin{bmatrix} tI - A_1 & -B \\ 0 & tI - A_2 \end{bmatrix} = |tI - A_1|.|tI - A_2|.$$

That is , The characteristic polynomial of M is the product of the characteristic polynomial of the diagonal blocks A_1 *and* A_2.

Theorem: -7.12

Suppose M is Block Triangular matrix with diagonal blocks $A_1, A_2, ..., A_r$ .Then the characteristic polynomial of M is the product of the characteristic polynomial of the diagonal blocks A_i's that is

$$\Delta_m(t) = \Delta_{A_1}(t)\Delta_{A_2}(t)..........\Delta_{A_r}(t).$$

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Example: -8

Consider the matrix

$$M = \begin{bmatrix} 9 & -1 & \vdots & 5 & 7 \\ 8 & 3 & \vdots & 2 & -4 \\ . & . & \cdots & . & . \\ 0 & 0 & \vdots & 3 & 6 \\ 0 & 0 & \vdots & -1 & 8 \end{bmatrix}$$

Then M is a block triangular matrix with diagonal blocks

$$A = \begin{bmatrix} 9 & -1 \\ 8 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 6 \\ -1 & 8 \end{bmatrix}.$$

Here $tr(A) = 9 + 3 = 12, |A| = 27 + 8 = 35$ and so

$$\Delta_A(t) = t^2 - 12t + 35 = (t-5)(t-7)$$

$$tr(B) = 3 + 8 = 11, |B| = 24 + 6 = 30$$

And

$$\Delta_B(t) = t^2 - 11t + 30 = (t-5)(t-6)$$

Accordingly, the characteristic polynomial of M is the product

$$\Delta_m(t) = \Delta_A(t)\Delta_B(t) = (t-5)^2(t-6)(t-7).$$

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7.7. Minimal Polynomial and Block diagonalization

Theorem: -7.13

Suppose M is a Block diagonal matrix with diagonal blocks $A_1, A_2, \ldots\ldots, A_r$ .Then the minimal polynomial of M is equal to the least common multiple (LCM) of the minimal polynomial of the diagonal blocks A_i.

Remark:

We emphasize that this theorem applies to Block Diagonal matrix whereas the analogous previous theorem on characteristic polynomial applies to Block Triangular matrices.

Example: -9

Find the characteristic polynomial $\Delta(t)$ and the minimal polynomial $m(t)$ of the Block Diagonal Matrix

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$$M=\begin{bmatrix} 2 & 5 & \vdots & 0 & 0 & \vdots & 0 \\ 0 & 2 & \vdots & 0 & 0 & \vdots & 0 \\ \ldots & . & \vdots & . & \ldots & \vdots & . \\ 0 & 0 & \vdots & 4 & 2 & \vdots & 0 \\ 0 & 0 & \vdots & 3 & 5 & \vdots & 0 \\ \ldots & . & \vdots & . & \ldots & \vdots & . \\ 0 & 0 & \vdots & 0 & 0 & \vdots & 7 \end{bmatrix}=diag\left(A_1,A_2,A_3\right)$$

Where

$$A_1=\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}, A_2=\begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} \text{ and } A_3=[7].$$

Then $\Delta(t)$ is the product of the characteristic polynomials $\Delta_1(t), \Delta_2(t)$ and $\Delta_3(t)$ of A_1, A_2 *and* A_3 respectively.

One can show that

$$\Delta_1(t)=(t-2)^2$$
$$\Delta_2(t)=(t-2)(t-7)$$
$$\Delta_3(t)=(t-7)$$

Thus

$$\Delta(t)=(t-2)^3(t-7)^2. \text{ (as expected } \deg\Delta(t)=5\text{)}$$

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The minimal polynomial $m_1(t)$, $m_2(t)$ and $m_3(t)$ of the diagonal blocks A_1, A_2 *and* A_3 respectively are equal to the characteristic polynomials that is

$$m_1(t) = (t-2)^2$$
$$m_2(t) = (t-2)(t-7) .$$
$$m_3(t) = (t-7)$$

But $m(t)$ is equal to the Least common multiple of $m_1(t)$, $m_2(t)$ and $m_3(t)$.

Thus

$$m(t) = (t-2)^2 (t-7).$$

7.8. Triangular Form

Let T be a linear operator on an n-dimensional vector space V. Suppose T can be represented by the triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & & \dots \\ & & & a_{nn} \end{bmatrix}$$

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Then the characteristic polynomial $\Delta(t)$ of T is a product of linear factors; that is

$$\begin{aligned}\Delta(t) &= \det(tI - A) \\ &= (t - a_{11})(t - a_{22}).......(t - a_{nn})\end{aligned}.$$

The converse is also true.

Theorem: -7. 14

Let $T : V \to V$ be a linear operator whose characteristic polynomial factors in to linear polynomials then there exist a basis of V in which T is represented by a triangular matrix.

Theorem: -7.15 (Alternate form)

Let A be a square matrix whose characteristic polynomial factors in to linear polynomials. Then A is similar to a triangular matrix, i.e. there exist an invertible matrix P such that $P^{-1}AP$ is triangular.

Remark

An operator T can be brought in to triangular form if it can be represented by a triangular matrix.

In this case, the Eigen values of T are precisely those entries appearing on the main diagonal.

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Example: -10

Let A be a square matrix over the complex field $\mathbb{C}$. Suppose λ is an eigen value of A^2 such that $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an Eigen value of A.

Solution:

By theorem-6.15, A and A^2 are similar respectively to triangular matrices of the form

$$B = \begin{bmatrix} \mu_1 & * & \dots & * \\ & \mu_2 & \dots & * \\ & & & \dots \\ & & & \mu_n \end{bmatrix} \quad \text{and}$$

$$B^2 = \begin{bmatrix} \mu_1^{\,2} & * & \dots & * \\ & \mu_2^{\,2} & \dots & * \\ & & & \dots \\ & & & \mu_n^{\,2} \end{bmatrix}$$

Since similar matrices have the same Eigen values $\lambda = \mu^2$, for some i. Hence $\mu_i = \sqrt{\lambda}$ or $\mu_i = -\sqrt{\lambda}$ is an Eigen value of A.

7.9. Jordan Canonical Form

An operator T can be put in to Jordan canonical form if its characteristic and minimal polynomial factor in to linear polynomials. This is always true if k is the complex field $\mathbb{C}$.In any case ,we can always extend the

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base field k to a field in which the characteristic and minimal polynomial do factor in to linear factors.

Thus in broad sense, every operator has a Jordan canonical form.

Analogously: Every matrix is similar to a matrix in Jordan canonical form.

The following theorem describes the Jordan canonical form J of a linear operator T.

Theorem: -7.16

Let $T:V\rightarrow V$ be a linear operator whose characteristic and minimal polynomials are respectively

$$\Delta(t)=(t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}......(t-\lambda_r)^{n_r} \text{ and}$$

$$m(t)=(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}......(t-\lambda_r)^{m_r},$$

where the λ_i's are distinct scalars. Then T has a block diagonal matrix representation J in which each diagonal entry is a Jordan block $J_{ij}=J(\lambda_i)$.

For each λ_{ij},the corresponding J_{ij} have the following properties,

(a) There is at least one J_{ij} of order m_i,all other J_{ij} are of order $\leq m_i$.

(b) The sum of the orders of the J_{ij} is n_i.

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(c) The number of J_{ij} is equals the geometric multiplicity of λ_i.

(d) The number of J_{ij} of each possible order is uniquely determined by T.

Example: -11

Suppose that the characteristic and minimal polynomials of an operator T are respectively

$$\Delta(t)=(t-2)^4(t-5)^3 \text{ and } m(t)=(t-2)^2(t-5)^3.$$

Then the Jordan canonical form of T is one of the following block diagonal matrices.

$$diag\left(\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix},\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix},\begin{bmatrix}5 & 1 & 0\\0 & 5 & 1\\0 & 0 & 5\end{bmatrix}\right) \text{ or}$$

$$diag\left(\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix},[2],[2],\begin{bmatrix}5 & 1 & 0\\0 & 5 & 1\\0 & 0 & 5\end{bmatrix}\right).$$

The first matrix occurs if T has two independent Eigen vectors belonging to the Eigen value 2,and the second matrix occurs if T has three independent Eigen vectors belonging to 2.

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7.10. Rational Canonical Form

This form exists even when the minimal polynomials cannot be factored in to linear polynomials. (Recall that this is not the case for the Jordan canonical form).

Theorem: -7.17

Let $T: V \to V$ be a linear operator with minimal polynomial

$$m(t) = \left(f_1(t)\right)^{m_1}, \left(f_2(t)\right)^{m_2}, \ldots., \left(f_s(t)\right)^{m_s}$$

Where $f_i(t)$ are distinct monic irreducible polynomials. Then T has a unique block diagonal matrix representation

$$M = diag\left(c_{11}, c_{12}, \ldots.., c_{1r}, \ldots..c_{s1}, c_{s1}, \ldots c_{sr}\right)$$

Where the c_{ij} are the companion matrices of the polynomials $\left(f_i(t)\right)^{n_{ij}}$,where

$$m_1 = n_{11} \geq n_{12} \geq \ldots\ldots. \geq n_{1r}, \ldots\ldots.., m_s = n_{s1} \geq n_{s2} \geq \ldots\ldots. \geq n_{sr}$$

The above matrix representation of T is called its rational canonical form.

The polynomials $\left(f_i(t)\right)^{n_{ij}}$ are called the elementary divisors of T

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Example-12

Let V be a vector space of dimension 8 over the rational field Q and let T be a linear operator on V whose minimal polynomials is

$$m(t) = f_1(t).f_2(t)^2$$
$$= \left(t^4 - 4t^3 + 6t^2 - 4t - 7\right)(t-3)^2$$

and $\dim V = 8$.

The characteristic polynomial

$$\Delta(t) = f_1(t).f_2(t)^4$$
$$= \left(t^4 - 4t^3 + 6t^2 - 4t - 7\right)(t-3)^4.$$

Also the rational canonical form M of T must have one block the companion matrix of $f_1(t)$ and one block companion matrix of $f_2(t)^2$.

There are two possibilities:

(a) $diag\left[c\left(t^4 - 4t^3 + 6t^2 - 4t - 7\right), c(t-3)^2, \mathrm{c}(t-3)^2\right]$

.

(b) $diag\left[c\left(t^4 - 4t^3 + 6t^2 - 4t - 7\right), c(t-3)^2, \mathrm{c}(t-3), \mathrm{c}(t-3)\right]$

.

That is

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$$\textbf{(a)}\ diag\left(\begin{bmatrix}0&0&0&7\\1&0&0&4\\0&1&0&-6\\0&0&1&4\end{bmatrix},\begin{bmatrix}0&-9\\1&6\end{bmatrix},\begin{bmatrix}0&-9\\1&6\end{bmatrix}\right)$$

$$\textbf{(b)}\ diag\left(\begin{bmatrix}0&0&0&7\\1&0&0&4\\0&1&0&-6\\0&0&1&4\end{bmatrix},\begin{bmatrix}0&-9\\1&6\end{bmatrix},[3],[3]\right).$$

Exercises

1.Determine the characteristic polynomials, eigen values and corresponding eigen spaces of the given 2×2 matrices.

(a) $\begin{bmatrix}5&4\\1&2\end{bmatrix}$ **(b)** $\begin{bmatrix}1&-2\\1&4\end{bmatrix}$

(c) $\begin{bmatrix}5&2\\-8&-3\end{bmatrix}$ **(d)** $\begin{bmatrix}2&1\\-1&4\end{bmatrix}$

(e) $\begin{bmatrix}2&-4\\-1&2\end{bmatrix}$

2. Determine the characteristic polynomials, eigen values and corresponding eigen spaces of the given 3×3 and 4×4 matrices.

(a) $\begin{bmatrix}3&2&-2\\-3&-1&3\\1&2&0\end{bmatrix}$ **(b)** $\begin{bmatrix}1&-2&2\\-2&1&2\\-2&0&3\end{bmatrix}$

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(c) $\begin{bmatrix} 15 & 7 & -7 \\ -1 & 1 & 1 \\ 13 & 7 & -5 \end{bmatrix}$ **(d)** $\begin{bmatrix} 4 & 2 & -2 & 2 \\ 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & -3 & 5 \end{bmatrix}$

3.Show that the following matrix has no real eigen values and thus no eigen vectors. Interpret your results geometrically $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

4.Show that the following 2×2 matrices satisfy their characteristic equations.

(a) $\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$ **(d)** $\begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}$

Answers:

1.(a) $\lambda^2 - 7\lambda + 6; 6, 1; r\begin{bmatrix} 4 \\ 1 \end{bmatrix}, s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $\lambda^2 - 5\lambda + 6; 2, 3; r\begin{bmatrix} -2 \\ 1 \end{bmatrix}, s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(c) $\lambda^2 - 2\lambda + 6; 1; r\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(d) $\lambda^2 - 6\lambda + 9;\ \ 3; r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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(e) $\lambda^2 - 4\lambda; 0, 4, 1; r\begin{bmatrix} 2 \\ 1 \end{bmatrix}, s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2.

(a) $-\lambda^3 + 2\lambda^2 + \lambda - 2; 1, -1, 2; r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $(1-\lambda)^2(3-\lambda); 1, 3; r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(c) $(1-\lambda)(2-\lambda)(8-\lambda); 1, 2, 8; r\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(d) $(2-\lambda)(2-\lambda)(4-\lambda)(6-\lambda); 2, 4, 6;$
$r\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, t\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, p\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

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About the Author

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