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Marko Kostić

SELECTED TOPICS IN ALMOST PERIODICITY

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Preface

The theory of almost periodic functions is unavoidable in the world of mathematics. The main purpose of this monograph, entitled "Selected Topics in Almost Periodicity", is to present the recent research results of the author in this field.

In the existing literature, there are numerous research articles dealing with the almost periodic (automorphic) properties and asymptotically almost periodic (automorphic) properties of the abstract Volterra integro-differential equations in Banach spaces, degenerate or non-degenerate in the time variable. Special attention has been paid to fractional integro-differential equations and inclusions, primarily from their invaluable importance in modeling real-world phenomena appearing in physics, chemistry, biology, economy, aerodynamics, etc. This is probably the first research monograph considering uniformly recurrent solutions and c-almost periodic solutions of abstract Volterra integro-differential equations as well as various generalizations of almost periodic functions in Lebesgue spaces with variable coefficients. In our support, it is also worth noting that this is probably the first research monograph considering multi-dimensional almost periodic type functions and their generalizations in adequate detail. Although there might still be a few things to arrange better, we have tried to aggregate many complicated and miscellaneous parts into a stable, compact unity.

This monograph is composed of the introductory chapter and two parts, which are further divided into chapters, sections and subsections. As in my previously published monographs [629–633], the numbering of theorems, propositions, lemmas, corollaries, definitions, etc., is done by chapter and section; we sort the reference list in alphabetical order (the notation with basic function spaces of one real variable is also made). The reader should be familiar with the fundamentals of functional analysis and integration theory, the basic theory of abstract differential equations in Banach spaces, the basic theory of vector-valued almost periodic functions and the vector-valued almost automorphic functions.

Conventional wisdom says you should know your target audience. Concerning the groups of people the book would interest, we wish to mention experts in the fields of almost periodicity and almost automorphy, researchers in abstract partial differential equations, experts from all areas of functional analysis, master students specializing in functional analysis and PhD students in mathematics. We have tried in the reference list to avoid any form of plagiarism. Although it contains more than 680 pages, and around 1100 titles in the reference list, the book is not intended to be a thorough and exhaustive study.

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Loznica/Novi Sad, July 2021 Marko Kostić

Notation

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$: the natural numbers, integers, rationals, reals, complexes.
For any $s \in \mathbb{R}$, we denote $\lfloor s \rfloor = \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil = \inf\{l \in \mathbb{Z} : s \leq l\}$.

$\operatorname{Re} z, \operatorname{Im} z$: the real and imaginary part of a complex number $z \in \mathbb{C}$; $|z|$: the module of z, $\arg(z)$: the argument of $z \in \mathbb{C} \setminus \{0\}$.

$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ $(z_0 \in \mathbb{C}, r > 0)$.

$\Sigma_\alpha = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$, $\alpha \in (0, \pi]$.

$\operatorname{card}(G)$: the cardinality of G.

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

$\mathbb{N}_n = \{1, \ldots, n\}$.

$\mathbb{N}_n^0 = \{0, 1, \ldots, n\}$.

$\mathbb{R}^n$: the real Euclidean space, $n \geq 2$.
If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index, then we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. $f^{(\alpha)} := \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$; $D^\alpha f := (-i)^{|\alpha|} f^{(\alpha)}$.
If (X, τ) is a topological space and $F \subseteq X$, then the interior, the closure, the boundary, and the complement of F with respect to X are denoted by $\operatorname{int}(F)$ (or F°), $\overline{F}$, ∂F and F^c, respectively.
If Z is a vector space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then for each non-empty subset F of Z by $\operatorname{span}(F)$ we denote the smallest linear subspace of Z which contains F.

X: a complex Banach space.

$L(X, Y)$: the space of all continuous linear mappings between complex Banach spaces X and Y, $L(X) = L(X, X)$.

X^*: the dual space of X.

A: a linear operator on X.

$\mathcal{A}$: a multivalued linear operator on X (MLO).
If F is a subspace of X, then we denote by $\mathcal{A}_{|F}$ the part of $\mathcal{A}$ in F.

$\chi_\Omega(\cdot)$: the characteristic function, defined to be identically one on Ω and zero elsewhere.

$\Gamma(\cdot)$: the Gamma function.
If $\alpha > 0$, then $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$; $g_0(t) \equiv$ the Dirac delta distribution.
If $1 \leq p < \infty$, $(X, \|\cdot\|)$ is a complex Banach space, and $(\Omega, \mathcal{R}, \mu)$ is a measure space, then $L^p(\Omega, X, \mu)$ denotes the space which consists of those strongly μ-measurable functions $f : \Omega \to X$ such that $\|f\|_p := (\int_\Omega \|f(\cdot)\|^p \, d\mu)^{1/p}$ is finite; $L^p(\Omega, \mu) \equiv L^p(\Omega, \mathbb{C}, \mu)$.

$L^\infty(\Omega, X, \mu)$: the space which consists of all strongly μ-measurable, essentially bounded functions.

$\|f\|_\infty = \operatorname{ess\,sup}_{t \in \Omega} \|f(t)\|$, the norm of a function $f \in L^\infty(\Omega, X, \mu)$.

$L^p(\Omega : X) \equiv L^p(\Omega, X) \equiv L^p(\Omega, X, \mu)$, if $p \in [1, \infty]$ and $\mu = m$ is the Lebesgue measure; $L^p(\Omega) \equiv L^p(\Omega : \mathbb{C})$.

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$L^p_{loc}(\Omega : X)$: the space consisting of those Lebesgue measurable functions $u(\cdot)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in L^p(\Omega' : X)$; $L^p_{loc}(\Omega) \equiv L^p_{loc}(\Omega : \mathbb{C})$ $(1 \le p \le \infty)$.
Assume that $I = \mathbb{R}$ or $I = [0, \infty)$. By $C_b(I : X)$ we denote the space consisting of bounded continuous functions from I into X; $C_0(I : X)$ denotes the closed subspace of $C_b(I : X)$ consisting of functions vanishing as the absolute value of the argument tends to plus infinity. By $\mathrm{BUC}(I : X)$ we denote the space consisting of all bounded uniformly continuous functions from I to X. Equipped with the sup-norm, $C_b(I : X)$, $C_0(I : X)$ and $\mathrm{BUC}(I : X)$ are Banach spaces.

$C^k(\Omega : X)$: the space of k-times continuously differentiable functions $(k \in \mathbb{N}_0)$ from a non-empty subset $\Omega \subseteq \mathbb{C}$ into X; $C(\Omega : X) \equiv C^0(\Omega : X)$.

$\mathcal{D} = C_0^\infty(\mathbb{R})$, $\mathcal{E} = C^\infty(\mathbb{R})$ and $\mathcal{S} = \mathcal{S}(\mathbb{R})$: the Schwartz spaces of test functions. If $\emptyset \neq \Omega \subseteq \mathbb{R}$, then by $\mathcal{D}_\Omega$ we denote the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\mathrm{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$.

$\mathcal{D}' := L(\mathcal{D}, \mathbb{C})$: the space consisting of all scalar-valued distributions. If $k \in \mathbb{N}$, $p \in [1, \infty]$ and Ω is an open non-empty subset of $\mathbb{R}^n$, then $W^{k,p}(\Omega : X)$ stands for the Sobolev space of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every $i \in \mathbb{N}_k^0$ and for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$, one has $D^\alpha u \in L^p(\Omega : X)$.

$W^{k,p}_{loc}(\Omega : X)$: the space of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in W^{k,p}(\Omega' : X)$.

$\mathcal{F}$, $\mathcal{F}^{-1}$: the Fourier transform and its inverse transform, respectively.

$L^1_{loc}([0, \infty))$, resp. $L^1_{loc}([0, \tau))$: the space of scalar-valued locally integrable functions on $[0, \infty)$, resp. $[0, \tau)$.

J_t^α: the Riemann–Liouville fractional integral of order $\alpha > 0$.

D_t^α: the Riemann–Liouville fractional derivative of order $\alpha > 0$.

$\mathbf{D}_t^\alpha$: the Caputo fractional derivative of order $\alpha > 0$.

$D_{t,+}^\gamma$: the Weyl–Liouville fractional derivative of order $\gamma \in (0, 1]$.

$E_{\alpha,\beta}(z)$: the Mittag-Leffler function $(\alpha > 0, \beta \in \mathbb{R})$; $E_\alpha(z) \equiv E_{\alpha,1}(z)$.

$\Psi_\gamma(t)$: the Wright function $(0 < \gamma < 1)$.

$\mathrm{supp}(f)$: the support of the function $f(t)$.

$L^{p(x)}(\Omega : X)$: the Lebesgue space with variable exponent $p(x)$. Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $1 \le p < \infty$.

$P_c(I : X)$: the space of all continuous c-periodic functions $f : I \to X$ $(c > 0)$.

$\mathrm{AP}(I : X)$: the Banach space consisting of all almost periodic functions from the interval I into X, equipped with the sup-norm.

$\mathrm{UR}(I : X)$: the collection of all uniformly recurrent functions from the interval I into X.

$\mathrm{AAP}(I : X)$: the Banach space consisting of all asymptotically almost periodic functions from the interval I into X, equipped with the sup-norm.

$\mathrm{AUR}(I : X)$: the collection of all asymptotically uniformly recurrent functions from the interval I into X.

$\mathrm{AP}(I \times Y : X)$: the set consisting of all almost periodic functions $f : I \times Y \to X$.
$\mathrm{UR}(I \times Y : X)$: the set consisting of all uniformly recurrent functions $f : I \times Y \to X$.
$\mathrm{AAP}(I \times Y : X)$: the set consisting of all asymptotically almost periodic functions $f : I \times Y \to X$.
$\mathrm{AUR}(I \times Y : X)$: the set consisting of all asymptotically uniformly recurrent functions $f : I \times Y \to X$.
$\mathrm{AP}_{\odot_g}(I \times Y : X)$: the collection of all two-parameter $\odot_g$-almost periodic functions $f : I \times Y \to X$.
$\mathrm{AP}_{\odot_g,b}(I \times Y : X)$: the collection of all two-parameter $\odot_g$-almost periodic functions on bounded sets.
$\mathrm{UR}_b(I \times Y : X)$: the collection of all two-parameter uniformly recurrent functions on bounded sets.
$e - W^p_{\mathrm{ap},\mathbf{K}}(I \times Y : X)$: the collection of all equi-Weyl p-almost periodic functions $f : I \times Y \to X$.
$W^p_{\mathrm{ap},\mathbf{K}}(I \times Y : X)$: the collection of all Weyl p-almost periodic functions $f : I \times Y \to X$.
$W^p_{0,\mathbf{K}}(I \times Y : X)$: the collection of all Weyl p-vanishing functions $f : I \times Y \to X$.
$e - W^p_{0,\mathbf{K}}(I \times Y : X)$: the collection of all equi-Weyl p-vanishing functions $f : I \times Y \to X$.
$L^p_S(I : X)$: the space of all Stepanov p-bounded functions.
$L^{p(x)}_S(I : X)$: the space of all Stepanov $p(x)$-bounded functions.
$\mathrm{APS}^p(I : X)$: the Banach space of all Stepanov p-almost periodic functions from I into X, equipped with the Stepanov norm.
$\mathrm{AAPS}^p(I : X)$: the Banach space of all asymptotically Stepanov p-almost periodic functions $f : I \to X$, equipped with the Stepanov norm.
$\mathrm{AURS}^p(I : X)$: the collection of all asymptotically Stepanov p-uniformly recurrent functions $f : I \to X$.
$\mathrm{AAPS}^p(I \times Y : X)$: the vector space consisting of all Stepanov p-almost periodic functions $f : I \times Y \to X$.
$\mathrm{APS}^{p(x)}(I : X)$: the space of all Stepanov $p(x)$-almost periodic functions $f : I \to X$.
$\mathrm{AAPS}^{p(x)}(I : X)$: the space of all asymptotically Stepanov $p(x)$-almost periodic functions $f : I \to X$.
$\mathrm{AURS}^{p(x)}(I : X)$: the collection of all asymptotically Stepanov $p(x)$-uniformly recurrent functions $f : I \to X$.
$\mathrm{AAPS}^{p(x)}(I \times Y : X)$: the vector space consisting of all asymptotically Stepanov $p(x)$-almost periodic functions $f : I \times Y \to X$.
$e - W^p_{\mathrm{ap}}(I : X)$: the collection of all equi-Weyl p-almost periodic functions $f : I \to X$.
$W^p_{\mathrm{ap}}(I : X)$: the collection of all Weyl p-almost periodic functions $f : I \to X$.
$W^p_0([0,\infty) : X)$ and $e - W^p_0([0,\infty) : X)$: the collections consisting of all Weyl p-vanishing functions and equi-Weyl p-vanishing functions, respectively.

$\mathrm{B}^p(I:X)$ and $B^p(I:X)$: the sets consisting of all Besicovitch–Doss p-almost periodic functions $f: I \to X$ and all Besicovitch-p-almost periodic functions $f: I \to X$, respectively.

$\mathrm{D}^p(I:X)$: the class consisting of all Doss p-almost periodic functions $f: I \to X$.

$\mathrm{ANP}_0(I:X)$: the linear span of almost anti-periodic functions $f: I \to X$.

$\mathrm{ANP}(I:X)$: the linear closure of $\mathrm{ANP}_0(I:X)$ in $\mathrm{AP}(I:X)$.

$\mathrm{AS}(\mathbb{R}:X)$ and $\mathrm{AS}_c(\mathbb{R}:X)$: the Banach spaces consisting of all almost automorphic functions and compactly almost automorphic functions, respectively, equipped with the sup-norm.

$W^p\mathrm{AS}(\mathbb{R}:X)$: the vector space consisting of all Weyl p-almost automorphic functions.

$B^p\mathrm{AS}(\mathbb{R}:X)$: the vector space consisting of all Besicovitch p-almost automorphic functions.

$\mathcal{P}_{p,k}(I:X)$: the vector space consisting of all Bloch (p,k)-periodic functions.

$Q-\mathrm{AAP}(I:X)$: the set consisting of all quasi-asymptotically almost periodic functions from I into X.

$Q-\mathrm{AUR}(I:X)$: the set consisting of all quasi-asymptotically uniformly recurrent functions from I into X.

$S^pQ-\mathrm{AAP}(I:X)$: the set consisting of all Stepanov p-quasi-asymptotically almost periodic functions from I into X.

$S^{p(x)}Q-\mathrm{AAP}(I:X)$, $S^{p(x)}Q-\mathrm{AUR}(I:X)$ and $S^{p(x)}\mathrm{SAP}_\omega(I:X)$: the set consisting of all Stepanov $p(x)$-quasi-asymptotically almost periodic functions from I into X, the set consisting of all Stepanov $p(x)$-quasi-asymptotically uniformly recurrent functions from I into X and the set consisting of all Stepanov $p(x)$-asymptotically ω-periodic functions, respectively.

$\mathcal{SB}_k(I:X)$: the space of all semi-Bloch k-periodic functions from I into X.

$\mathcal{SANP}(I:X)$: the space consisting of all semi-anti-periodic functions from I into X.

$(e-)W_{\mathrm{ap}}^{(p,\phi,F)}(I:X)$: the collection of all (equi)-Weyl (p,ϕ,F)-almost periodic functions $f: I \to X$.

$(e-)W_{\mathrm{ap}}^{(p,\phi,F)_i}(I:X)$: the collection of all (equi)-Weyl $(p,\phi,F)_i$-almost periodic functions $f: I \to X$ ($i=1,2$).

$(e-)W_{\mathrm{ap}}^{[p,\phi,F]}(I:X)$: the collection of all (equi)-Weyl $[p,\phi,F]$-almost periodic functions $f: I \to X$.

$(e-)W_{\mathrm{ap}}^{[p,\phi,F]_i}(I:X)$: the collection of all (equi)-Weyl $[p,\phi,F]_i$-almost periodic functions $f: I \to X$ ($i=1,2$).

$W^{p(x)}_{\phi,F,0}([0,\infty):X)$ and $e-W^{p(x)}_{\phi,F,0}([0,\infty):X)$ [$W^{p(x);1}_{\phi,F,0}([0,\infty):X)$ and $e-W^{p(x);1}_{\phi,F,0}([0,\infty):X)$/$W^{p(x);2}_{\phi,F,0}([0,\infty):X)$ and $e-W^{p(x);2}_{\phi,F,0}([0,\infty):X)$]: the sets consisting of all Weyl (p,ϕ,F)-vanishing functions and equi-Weyl (p,ϕ,F)-vanishing functions [Weyl $(p,\phi,F)_1$-vanishing functions and equi-Weyl $(p,\phi,F)_1$-vanishing functions/Weyl $(p,\phi,F)_2$-vanishing functions and equi-Weyl $(p,\phi,F)_2$-vanishing functions].

$\mathrm{UR}_{\omega,c}(I:X)$, $\mathrm{AP}_{\omega,c}(I:X)$, $\mathrm{AS}_{\omega,c}(I:X)$ and $\mathrm{AS}_{\omega,c;\mathbf{c}}(I:X)$: the space of all (ω,c)-uniformly recurrent functions, the space of all (ω,c)-almost periodic functions, the space of all (ω,c)-almost automorphic functions and the space of all compactly (ω,c)-almost automorphic functions, respectively.

$S^p\mathrm{UR}_{\omega,c}(I:X)$, $S^p\mathrm{AP}_{\omega,c}(I:X)$ and $S^p\mathrm{AS}_{\omega,c}(I:X)$: the space of all Stepanov (p,ω,c)-uniformly recurrent functions, the space of all Stepanov (p,ω,c)-almost periodic functions and the space of all Stepanov (p,ω,c)-almost automorphic functions, respectively.

$\mathrm{AS}^p\mathrm{UR}_{\omega,c}(I:X)$, $\mathrm{AS}^p\mathrm{AP}_{\omega,c}(I:X)$ and $\mathrm{AS}^p\mathrm{AS}_{\omega,c}(I:X)$: the space of all asymptotically Stepanov (p,ω,c)-uniformly recurrent functions, the space of all asymptotically Stepanov (p,ω,c)-almost periodic functions and the space of all asymptotically Stepanov (p,ω,c)-almost automorphic functions, respectively.

$\mathrm{UR}_{\omega,c,i}(I:X)$ and $\mathrm{AP}_{\omega,c,i}(I:X)$: the space of all (ω,c)-uniformly recurrent functions of type i and the space of all (ω,c)-almost periodic functions of type i, respectively ($i=1,2$).

$\mathrm{UR}_c(I:X)$: the set consisting of all c-uniformly recurrent functions from the interval I into X.

$\mathrm{AP}_c(I:X)$: the set consisting of all c-almost periodic functions from the interval I into X.

$\mathcal{SP}_{c,i}(I:X)$: the set of all semi-c-periodic functions of type i, where $i=1,2$.

$\mathcal{SP}_{c,i,+}(I:X)$: the set of all semi-$c$-periodic functions of type i_+, where $i=1,2$.

$e-W_{\mathrm{ur}}^{(p(x),\phi,F)}(I:X)$: the set of all equi-Weyl $(p(x),\phi,F)$-uniformly recurrent functions.

$e-W_{\mathrm{ur}}^{(p(x),\phi,F)_1}(I:X)$: the set of all equi-Weyl $(p(x),\phi,F)_1$-uniformly recurrent functions.

$e-W_{\mathrm{ur}}^{(p(x),\phi,F)_2}(I:X)$: the set of all equi-Weyl $(p(x),\phi,F)_2$-uniformly recurrent functions.

$e-W_{\mathrm{ur}}^{[p(x),\phi,F]}(I:X)$: the set of all equi-Weyl $[p(x),\phi,F]$-uniformly recurrent functions.

$e-W_{\mathrm{ur}}^{[p(x),\phi,F]_1}(I:X)$: the set of all equi-Weyl $[p(x),\phi,F]_1$-uniformly recurrent functions.

$e-W_{\mathrm{ur}}^{[p(x),\phi,F]_2}(I:X)$: the set of all equi-Weyl $[p(x),\phi,F]_2$-uniformly recurrent functions.

$Q-\mathrm{AUR}_{\mathbf{B}}(I\times Y:X)$: the set consisting of all quasi-asymptotically uniformly recurrent, uniformly on $\mathbf{B}$ functions from $I\times Y$ into X.

$S^p\mathrm{UR}_{\omega,c,2}([0,\infty):X)$ and $S^p\mathrm{AP}_{\omega,c,2}([0,\infty):X)$: the collection of all Stepanov (p,ω,c)-uniformly recurrent functions of type 2 and the collection of all Stepanov (p,ω,c)-almost periodic functions of type 2, respectively.

$S^{p(x)}\mathrm{UR}_{\omega,c,2}([0,\infty):X)$ and $S^{p(x)}\mathrm{AP}_{\omega,c,2}([0,\infty):X)$: the collection of all Stepanov $(p(x),\omega,c)$-uniformly recurrent functions of type 2 and the collection of all Stepanov $(p(x),\omega,c)$-almost periodic functions of type 2, respectively.

$\mathrm{PAP}_{0;\omega,c,i}(\mathbb{R} \times Y : X)$: the space of (ω, c, i)-pseudo ergodic vanishing functions ($i = 1, 2$).

$\mathrm{AP}_{\omega,c,i}(\mathbb{R} \times Y : X)$, resp. $\mathrm{AS}_{\omega,c,i}(\mathbb{R} \times Y : X)$: the space of all (ω, c, i)-almost periodic, resp. (ω, c, i)-almost automorphic, functions ($i = 1, 2$).

$\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$, resp. $\mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$: the space of all (ω, c)-pseudo-almost periodic, resp. (ω, c)-pseudo-almost automorphic, functions.

$\mathrm{PAP}_{\omega,c,i}(\mathbb{R} \times Y : X)$, resp. $\mathrm{PAA}_{\omega,c,i}(\mathbb{R} \times Y : X)$: the space of all (ω, c, i)-pseudo-almost periodic, resp. (ω, c, i)-pseudo-almost automorphic, functions.

$\mathcal{B}_{\mathrm{AP}_{\omega,c}}$: the space of smooth (ω, c)-almost periodic functions defined on $\mathbb{R}$.

$\mathcal{B}'_{\mathrm{AP}_{\omega,c}}$: the space of (ω, c)-almost periodic distributions.

$\mathcal{B}'_{0+}$: the space of bounded distributions vanishing at infinity.

$\mathcal{B}'_{\mathrm{aap}}([0, \infty))$: the space of asymptotically almost periodic Schwartz distributions.

$(e-)W^{p}_{\mathrm{ap};c}(I : X)$: the collection of all (equi-)Weyl (p, c)-almost periodic functions.

$\mathrm{SAP}_{\omega;c}(I : X)$ and $\mathrm{SAP}_{c}(I : X)$: the sets of all S-asymptotically (ω, c)-periodic functions and S_c-asymptotically periodic functions ($\omega \in I$, $c \in \mathbb{C} \setminus \{0\}$).

$S^{p(x)}Q - \mathrm{AAP}_{c}(I : X)$: the set consisting of all Stepanov $p(x)$-quasi-asymptotically c-almost periodic functions from I into X.

$S^{p}Q - \mathrm{AAP}_{c}(I : X)$: the set consisting of all Stepanov p-quasi-asymptotically c-almost periodic functions from I into X.

$Q - \mathrm{AAP}_{c}(I : X)$: the collection of all quasi-asymptotically c-almost periodic functions from I into X, respectively.

$Q - \mathrm{AAP}_{c;\mathcal{F}}(I \times Y : X)$: the collection consisting of all quasi-asymptotically c-almost periodic functions $f : I \times Y \to X$ on $\mathcal{F}$.

Introduction

The class of almost periodic functions was introduced by the Danish mathematician H. Bohr [196] (1925), the younger brother of the Nobel Prize-winning physicist N. Bohr, and later generalized by many others. Let $I = \mathbb{R}$ or $I = [0, \infty)$, let $(X, \|\cdot\|)$ be a complex Banach space, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if

$$\|f(t+\tau) - f(t)\| \leq \epsilon, \quad t \in I.$$

By $\vartheta(f, \epsilon)$ we denote the set of all ϵ-periods for $f(\cdot)$. We say that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \epsilon)$.

The class of almost automorphic functions was introduced by the American mathematician S. Bochner [188] (1955). A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if and only if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to X$ such that

$$\lim_{n\to\infty} f(t + a_n) = g(t) \quad \text{and} \quad \lim_{n\to\infty} g(t - a_n) = f(t),$$

pointwise for $t \in \mathbb{R}$. Any almost periodic function is almost automorphic, but the converse statement is not true in general. The theory of almost periodic functions and the theory of almost automorphic functions are still very active fields of investigations of numerous authors, full of open problems, conjectures, hypotheses, and possibilities for further expansions.

There is an enormous literature devoted to the study of almost periodic solutions and almost automorphic solutions of the abstract first-order differential equations. The notion of an almost periodic strongly continuous semigroup was introduced by H. Bart and S. Goldberg in [119] (1978) but some particular results concerning the almost periodicity of individual orbits of strongly continuous semigroups were given already by C. Foias, S. Zaidman [447] (1961), V. Zhikov [1097, 1098] (1966, 1968) and A. I. Perov, T. K. Hai [835] (1972); see also the survey article [840] by V. Q. Phóng as well as the reference list of [631] and the articles [841, 842] obtained in a collaboration of V. Q. Phóng and Yu I. Lyubich.

The notion of an almost periodic cosine operator function was introduced by I. Cioranescu [298] (1986) and after that received considerable attention from many authors. The existence and uniqueness of almost periodic type solutions of the (abstract) second-order differential equations have been investigated in many research articles by now, using the theory of cosine operator functions or other methods (see e. g., [81, 93, 102, 155, 530, 866, 943, 1062, 1068]). We will describe here the main ideas and results of the recent investigation [371] by T. Diagana, J. H. Hassan and S. A. Messaoudi, who analyzed the existence of asymptotically almost periodic mild solutions

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to a class of second-order hyperbolic integro-differential equations of Gurtin–Pipkin type in separable Hilbert spaces. Let H be a separable Hilbert space, and let A be a positive self-adjoint operator in H such that there exists a positive real constant $\omega > 0$ such that $\|Au\| \geq \omega\|u\|$ for all $u \in H$. Assume further that a function $f : [0,\infty) \times H \to H$ is asymptotically almost periodic in the first variable, uniformly on compacts of H in the second variable, and a non-increasing differentiable relaxation function $g : [0,\infty) \to [0,\infty)$ satisfies $g(0) > 0$ and $\int_0^\infty g(s)\,ds < 1$. Under certain extra assumptions, the authors have proved that the abstract Volterra integro-differential equation

$$u''(t) + A^2u(t) - \int_{-\infty}^{t} g(t-s)A^2u(s)\,ds = f(t,u(t)), \quad t \geq 0,$$

accompanied by the initial conditions $u(-t) = u_0(t)$ for $t \geq 0$ and $u'(0) = u_1$, has an asymptotically almost periodic mild solution. The main strategy used is a transformation of such a system into a first-order linear evolution equation whose solutions are governed by exponentially decaying strongly continuous semigroups; an interesting application was made in the study of Kirchhoff plate equation with infinite memory. For almost periodic type solutions of abstract differential equations with integer-order derivatives, we want also to recommend [54, 61, 80, 81, 111, 115, 134, 151, 249–251] and [420, 550, 687, 799, 819, 839, 1001, 1093, 1095, 1096].

The study of almost periodic type solutions of the abstract Volterra integro-differential equations was initiated by J. Prüss in [857, Section 11.4], where the author has analyzed the almost periodic solutions, Stepanov almost periodic solutions and asymptotically almost periodic solutions of the following abstract integro-differential equation

$$u'(t) = \int_0^\infty A_0(s)u'(t-s)\,ds + \int_0^\infty dA_1(s)u(t-s) + f(t), \quad t \in \mathbb{R};$$

here $A_0 \in L^1([0,\infty) : L(Y,X))$, $t \mapsto A_1(t) \in L(Y,X)$, $t \geq 0$ is locally of bounded variation, X and Y are Banach spaces such that Y is densely and continuously embedded into X. Almost immediately after that, Q.-P. Vu [1006] has investigated the almost periodicity of the abstract Cauchy problem

$$u'(t) = Au(t) + \int_0^\infty dBu(\tau)u(t-\tau) + f(t), \quad t \in \mathbb{R},$$

where A is a closed linear operator acting on a Banach space X, $(B(t))_{t\geq 0}$ is a family of closed linear operators on X and $f : \mathbb{R} \to X$ is continuous.

It is very difficult and unpleasant to say precisely who was the first to study the almost periodic solutions of the abstract fractional differential equations. For example,

J. Mu, Y. Zhoa and L. Peng [798] have recently investigated the periodic solutions and S-asymptotically periodic solutions to fractional evolution equation

$$D_{t,+}^{\gamma}u(t) = -Au(t) + g(t), \quad t \in \mathbb{R}$$

and its semilinear analogue

$$D_{t,+}^{\gamma}u(t) = -Au(t) + g\big(t, u(t)\big), \quad t \in \mathbb{R},$$

where $D_{t,+}^{\gamma}$ denotes the Weyl–Liouville fractional derivative of order $\gamma \in (0,1)$, A is the infinitesimal generator of an exponentially decaying strongly continuous semigroup of operators and $g : \mathbb{R} \times X \to X$ satisfies certain assumptions (see also the article [23] by R. Agarwal, B. de Andrade and C. Cuevas as well as the recent articles [138] by P. Bedi, A. Kumar, T. Abdeljawad, A. Khan and [224] by D. Brindle, G. M. N'Guérékata, where the authors have analyzed S-asymptotically ω-periodic mild solutions for fractional differential equations with Hilfer derivatives and Riemann–Liouville derivatives). Later, the author of this monograph extended the results of J. Mu, Y. Zhoa and L. Peng to the abstract fractional differential inclusion

$$D_{t,+}^{\gamma}u(t) \in -\mathcal{A}u(t) + g(t), \quad t \in \mathbb{R},$$

and its semilinear analogue

$$D_{t,+}^{\gamma}u(t) \in -\mathcal{A}u(t) + g\big(t, u(t)\big), \quad t \in \mathbb{R},$$

where $\mathcal{A}$ is a closed multivalued linear operator satisfying condition (P) below. The obtained results enable one to examine the almost periodic type solutions of the following fractional Poisson heat equations:

$$\begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t, m(x)v(t,x)), & t \in \mathbb{R}, x \in \Omega; \\ v(t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

$$\begin{cases} \mathbf{D}_t^{\gamma}[m(x)v(t,x)] = \Delta v(t,x) + bv(t,x), & t \geq 0, x \in \Omega; \\ v(t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega; \\ m(x)v(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

and the following fractional semilinear equation with higher-order differential operators in the Hölder space $X = C^{\alpha}(\overline{\Omega})$:

$$\begin{cases} \mathbf{D}_t^{\gamma}u(t,x) = -\sum_{|\beta|\leq 2m} a_{\beta}(t,x)D^{\beta}u(t,x) - \sigma u(t,x) + f(t, u(t,x)), & t \geq 0, x \in \Omega; \\ u(0,x) = u_0(x), & x \in \Omega; \end{cases}$$

see [631] for more details. Let us also recall that R. Ponce [854] has investigated the bounded mild solutions of the following non-degenerate fractional integro-differential equation

$$D_{t,+}^{\gamma}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t,u(t)), \quad t \in \mathbb{R}, \tag{1}$$

where A is a closed linear operator, $a \in L^1([0,\infty))$ is a scalar-valued kernel and $f(\cdot,\cdot)$ satisfies some Lipschitz type conditions. In particular, almost periodic solutions of (1) have been analyzed. Furthermore, S. Abbas, V. Kavitha and R. Murugesu have recently analyzed Stepanov-like (weighted) pseudo-almost automorphic solutions to the following fractional order abstract integro-differential equation:

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t,u(t),Ku(t)), \quad t \in \mathbb{R},$$

where

$$Ku(t) = \int_{-\infty}^{t} k(t-s)h(s,u(s))\,ds, \quad t \in \mathbb{R},$$

$1 < \alpha < 2$, A is a sectorial operator with domain and range in X, of negative sectorial type $\omega < 0$, the function $k(t)$ is exponentially decaying, the functions $f : \mathbb{R}\times X\times X \to X$ and $h : \mathbb{R}\times X \to X$ are Stepanov-like weighted pseudo-almost automorphic in time for each fixed elements of $X \times X$ and X, respectively, satisfying some extra conditions [9]. For more details about almost periodic type solutions of the abstract fractional differential equations, see the reference list of [631] and the articles [22, 24, 256, 340, 701, 774, 1039].

As we can see from the above, many results concerning the existence and uniqueness of almost periodic type solutions and almost automorphic type solutions to the abstract (semilinear) fractional differential equations have recently been given by numerous authors. In almost all these results (in the linear setting, quite exceptional are some examples and results presented by S. Zaidman [1067, Examples 4, 5, 7, 8; pp. 32–34], which have been employed by many authors so far, for various purposes; we will also use these examples to illustrate our results about the existence and uniqueness of almost periodic type solutions of the abstract integro-differential equations), the basic key is to investigate the invariance of certain kinds of generalized almost periodicity and generalized almost automorphicity under the actions of the infinite convolution product

$$t \mapsto \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

and the finite convolution product

$$t \mapsto \int_0^t R(t-s)f(s)\,ds, \quad t \geq 0.$$

Here, it is commonly assumed that $(R(t))_{t\geq 0} \subseteq L(X,Y)$ is a non-degenerate strongly continuous operator family between the Banach spaces X and Y which exponentially or, at least, polynomially decays as $t \to +\infty$. In [631], we have investigated the case in which $(R(t))_{t>0} \subseteq L(X,Y)$ is a degenerate strongly continuous operator family which decays exponentially or polynomially as $t \to +\infty$, but we have allowed $(R(t))_{t>0}$ to have a removable singularity at zero; by that we basically mean that there exists a number $\zeta \in (0,1)$ such that the operator family $(t^{\zeta}R(t))_{t\geq 0}$ is well defined and strongly continuous at the point $t = 0$. The integral generator of $(R(t))_{t\geq 0}$ is not single-valued in the degenerate case and this is the main reason why we have employed the multivalued linear approach to the abstract degenerate integro-differential equations in [631], which is also followed in this monograph. For the theory of abstract degenerate differential equations of the first order, mention should be made of the research monographs [245] by R. W. Caroll and R. W. Showalter, [431] by A. Favini, A. Yagi, [853] by M. V. Plekhanova, V. E. Fedorov and [965] by G. A. Sviridyuk, V. E. Fedorov. The well-posedness of the abstract degenerate Cauchy problem

$$Bu(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, t \in [0,\tau),$$

where $0 < \tau \leq \infty$, $t \mapsto f(t)$, $t \in [0,\tau)$ is a continuous mapping, $a \in L^1_{\mathrm{loc}}([0,\tau))$ and A, B are closed linear operators, has been thoroughly analyzed in the monograph [633], which provides the reader a valuable information about the abstract degenerate Volterra integro-differential equations (for the scalar-valued Volterra integro-differential equations, we refer the reader to the monograph [488] by G. Gripenberg, S. O. Londen, O. J. Staffans).

We will say just a few words about periodic solutions of the abstract degenerate Volterra integro-differential equations. In [114], V. Barbu and A. Favini have analyzed the 1-periodic solutions of the abstract degenerate differential equation $(d/dt)(Bu(t)) = Au(t)$, $t \geq 0$, accompanied by the initial condition $(Bu)(0) = (Bu)(1)$, by using P. Grisvard's sum of operators method and some results from investigation of J. Prüss [858] in the non-degenerate case. The authors reduced the above problem to $v'(t) \in \mathcal{A}v(t)$, $t \geq 0$, $v(0) = v(1)$, where the multivalued linear operator $\mathcal{A}$ is given by $\mathcal{A} = AB^{-1}$. The main problem is whether the inclusion $1 \in \rho(\mathcal{A})$ holds or not; recall that J. Prüss [858] has proved that $1 \in \rho(A)$ if and only if $2\pi i\mathbb{Z} \subseteq \rho(A)$ and $\sup(\{\|(2\pi in - A)^{-1}\| : n \in \mathbb{Z}\}) < \infty$, provided that A generates a non-degenerate strongly continuous semigroup. Applications are given to the Poisson heat equation in

$H^{-1}(\Omega)$ and $L^2(\Omega)$, as well as to some systems of ordinary differential equations. On the other hand, C. Lizama and R. Ponce [727] have analyzed the existence of 2π-periodic solutions to the following abstract inhomogeneous linear equation:

$$\frac{d}{dt}\big(Bu(t)\big) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad t \geq 0, \tag{2}$$

subjected with the initial condition $(Bu)(0) = (Bu)(2\pi)$. The authors also considered the maximal regularity of (2) in periodic Besov, Triebel–Lizorkin and Lebesgue vector-valued function spaces.

There is an enormous literature concerning periodic solutions for various classes of (abstract) non-degenerate Volterra integro-differential equations. Regarding the classical theory of partial differential equations with integer-order derivatives, we would like to recommend for the reader the references and work quoted in the introductory part of the fourth chapter of the monograph [859] by B. I. Ptashnic, where the following has been emphasized:

1. The ω-periodic solutions in time for the linear wave equation and the following weakly nonlinear wave equation

   $$u_{tt}(t,x) - u_{xx}(t,x) = \epsilon f(t,x,u,u_t,u_x,\epsilon), \quad t \geq 0,\ 0 \leq x \leq \pi,$$

   accompanied by the boundary conditions $u(t,0) = u(t,\pi) = 0$, was analyzed by O. Vejvoda [998] in 1964 ($\epsilon > 0$ is a sufficiently small real parameter). If $\omega \in 2\pi\mathbb{Q}$ and $\omega > 0$, then the existence of ω-periodic solutions for the both classes of wave equations was proved; on the other hand, if $\omega \notin 2\pi\mathbb{Q}$ and $\omega > 0$, then the situation is much more complicated and the author was proved the existence of ω-periodic solutions for a corresponding linear wave equation, only, provided that $\omega = 2\pi\alpha$ and there exist positive real numbers $c > 0$ and $\gamma > 0$ such that

   $$\left|\alpha - \frac{m}{k}\right| > \frac{c}{k^\gamma}.$$

   After that, in 1965, Ya. Gavlova investigated the existence and uniqueness of periodic solutions for the following weakly nonlinear telegraph equation

   $$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t,x) + \epsilon f(t,u,u_t,u_x,\epsilon),$$

   accompanied by the boundary conditions $u(t,0) = u(t,\pi) = 0$, where $a, b, c \in \mathbb{R}$ are certain constants and $\epsilon > 0$ is a sufficiently small real parameter.
2. In 1972, A. Azis and M. Gorak investigated the existence and uniqueness of periodic solutions in the time variable and space variable for the following quasilinear hyperbolic second-order equation:

   $$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y,u,u_x,u_y);$$

in 1971, N. Krylovoi and O. Vejvoda investigated the existence and uniqueness of ω-periodic solutions in the time variable for the following equation:

$$u_{tt} + u_{xxxx} = g(t,x) + \epsilon f(t,x,u,u_x,u_{xx},u_t,\epsilon),$$

accompanied by the boundary conditions

$$u(t,0) = u(t,2\pi) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0.$$

Six years later, in 1977, M. Kopachkovoi and O. Vejvoda analyzed the existence and uniqueness of ω-periodic solutions in the time variable for the following nonlinear equation:

$$u_{tt} + u_{xxxx} - \epsilon u_{xx} \int_0^{\pi} u^2(x,\xi)\,d\xi = g(t,x) + \epsilon^2 F(u)(t,x),$$

which appears in the study of beam vibrations with the effect of elongation. See also the research articles by B. P. Tkach [979], T. I. Kiguradze [603–606], M. F. Kulagina [680, 681] and M. F. Kulagina, E. A. Mikishanina [682]. For nonlinear KdV equations, we refer the reader to the research article [1007] by P. A. Vuillermot.

Furthermore, the Bohr almost periodic solutions to boundary value problems for systems of partial differential equations that arise in solving certain problems for inhomogeneous media have been investigated in the research articles [161] by L. C. Berselli, L. Bisconti, [162] by L. C. Berselli, M. Romito and [1002] by E. V. Vetchanin, E. A. Mikishanina. Regarding the existence and uniqueness of Bohr almost periodic solutions of the Navier–Stokes type equations, the reader may consult [52, 99, 107, 271, 302, 423, 438, 535, 547, 549, 564, 565] and [581, 610, 614, 677, 742, 744, 969, 990, 1046].

For the periodic solutions of abstract first-order differential equations, we refer the reader to the research monographs [234] by T. A. Burton, [721] by J. H. Liu, G. M. N'Guerekata, N. V. Minh and [1061] by T. Yoshizawa and to the research articles [201, 235, 419, 420, 474, 545, 613, 702, 724, 736, 883, 925]; concerning the abstract second-order differential equations in Hilbert spaces, it should be also noted that the existence and uniqueness of periodic solutions for the following equations:

$$\begin{aligned} &u_{tt} + (A+\gamma I)u(t) = F(t,u(t)), \quad t \geq 0 \ (\gamma \in \mathbb{R}); \\ &u_{tt} + A^2 u(t) = F(t,u(t),u'(t)), \quad t \geq 0; \\ &u_{tt}(t) + 2\alpha u_t(t) + Au(t) = g(t) + F(t,u(t)), \quad t \geq 0, \end{aligned}$$

were analyzed by I. Strashkraby, O. Vejvoda (1973), V. Lovicar (1977) and K. Masudy (1966), respectively (A is a positive self-adjoint operator in a Hilbert space H). Concerning the semilinear wave equations, we refer the reader to the research articles [53] by H. Amann, E. Zehnder and [911] by B. Scarpellini, P. A. Vuillermot.

The study of differential equations with discontinuous arguments was initiated by A. D. Myshkis [805] in 1977. The analysis of asymptotically anti-periodic solutions for nonlinear differential first-order equations with piecewise constant argument carried out by W. Dimbour and V. Valmorin [381] has recently been reconsidered and extended of asymptotically Bloch periodic solutions for nonlinear fractional differential inclusions with piecewise constant argument by M. Kostić and D. Velinov in [664]. We have analyzed the following fractional differential Cauchy inclusion with piecewise constant argument:

$$\mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + A_0 u(\lfloor t \rfloor) + g(t, u(\lfloor t \rfloor)), \quad t > 0; \quad u(0) = u_0,$$

where $A_0 \in L(X)$, $g : [0, \infty) \times X \to X$ is a given function, and $\mathbf{D}_t^{\gamma} u(t)$ denotes the Caputo fractional derivative of order γ, taken in a weak sense (cf. the paragraph preceding Definition 3.1.22). It is also worth noting that A. Chávez, S. Castillo and M. Pinto [263] have analyzed the existence of a unique almost automorphic solution for the following differential equation with a piecewise constant argument:

$$y'(t) = A(t)y(t) + B(t)y(\lfloor t \rfloor) + f(t, y(t), y(\lfloor t \rfloor)), \quad t \in \mathbb{R}, \tag{3}$$

where $A(t)$ and $B(t)$ are almost automorphic $p \times p$ complex matrices and $f : \mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p \to \mathbb{C}^p$ is an almost automorphic function satisfying a condition of Lipschitz type. The study carried out in [263] leans heavily on the use of results on discontinuous almost automorphic functions, exponential dichotomies and the Banach fixed point theorem. The almost periodic solutions of (3) were considered for the first time by R. Yuan and J. Hong in [1065] (1997); for more details about differential equations with a piecewise constant argument (DEPCA), the reader may consult the articles [308] by K. L. Cooke and J. Wiener, [920] by S. M. Shah and J. Wiener, [1026] by J. Wiener, as well as the articles [33, 286, 287, 801, 826, 849, 1063] and the list of references cited therein.

There is a vast amount of articles in the existing literature which consider almost automorphic type solutions for various classes of integro-differential equations. Let us only mention our analysis (the joint work with Prof. G. M. N'Guérékata [496]) of the following abstract multi-term fractional differential inclusion:

$$\begin{aligned} &\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) \in \mathcal{A}\mathbf{D}_t^{\alpha} u(t) + f(t), \quad t \geq 0, \\ &u^{(k)}(0) = u_k, \quad k = 0, \ldots, \lceil \alpha_n \rceil - 1, \end{aligned}$$

where $n \in \mathbb{N} \setminus \{1\}$, $A_1, \ldots, A_{n-1}$ are bounded linear operators on a Banach space X, $\mathcal{A}$ is a closed multivalued linear operator on X, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, $f(\cdot)$ is an X-valued function, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α

[133, 630]. Many excellent examples have been presented in the monograph [364] by T. Diagana; see also the following monographs:

1. [56] by M. Amerio and G. Prouse for almost periodic solutions of functional equations,
2. [86] by L. N. Argabright, J. G. de Lamadrid for almost periodic measures,
3. [104, 105] by M. Baake, U. Grimm for applications of almost periodic functions in crystallography,
4. [172] by P. H. Bezandry and T. Diagana for almost periodic solutions of stochastic differential equations,
5. [203] by A. Böttcher, I. Yu. Karlovich and I. M. Spitkovsky for factorization of almost periodic matrix functions (cf. also the article [202] by A. Böttcher for the problematic regarding corona theorem for almost periodic functions of several real variables and the articles [192] by P. Boggiatto, C. Ferández, A. Galbis, [611] by Y. H. Kim for problematic concerning Gabor systems and almost periodic functions),
6. [258] by Y.-K. Chang, G. M. N'Guerekata and R. Ponce for Bloch (p,k)-periodic functions, anti-periodic functions and their applications,
7. [269] by D. N. Cheban for asymptotically almost periodic solutions of linear and nonlinear equations (cf. also the recent article [463] by C. A. Gallegos, H. R. Henríquez and the references cited therein),
8. [409] by E. Yu. Emel'anov for weakly almost periodic C_0-semigroups,
9. [538] by Y. Hino, T. Naito, N. V. Minh and J. S. Shin and [494] by G. M. N'Guérékata for spectral analysis of almost periodic functions and Massera type theorems [752],
10. [541] by R. Hsu for weakly almost periodic functions,
11. [596] by V. Kh. Kharasakhal for almost periodic solutions of ordinary differential equations,
12. [616] by A. Yu. Kolesov, E. F. Mishchenko and N. Kh. Rozov for asymptotic methods of investigation of periodic solutions to nonlinear hyperbolic equations,
13. [895] by A. M. Samoilenko, B. P. Tkach for numerical–analytical methods in the theory of periodic solutions to equations with partial derivatives,
14. [956] by G. Tr. Stamov for almost periodic solutions of impulsive differential equations (see also the research monographs [112] by D. Bainov, P. Simeonov, [832] by N. A. Perestyuk, V. A. Plotnikov, A. M. Somoilenko, N. V. Skripnik, [946] by X. Song, H. Gno, X. Shi, [957] by I. Stamova, G. Stamov, and the research articles [862] by L. Qi, R. Yuan, [894] by A. M. Samoilenko, N. A. Perestyuk, [897] by A. M. Samoilenko, S. I. Trofimchuk, [1036] by Z. Xia, D. Wang, [1057] by P. Yang, Y.-R. Wang and M. Fečkan, [1059] by A. F. Yenicerioglu, V. Yazici, C. Yazici, and the survey article [1018] by J. R. Wang, M. Fečkan and Y. Zhou),
15. [986] by D. U. Umbetzhanov for almost multiperiodic solutions of partial differential equations,
16. [999] by O. Vejvoda (with L. Herrmann, V. Lovicar as contributors) for time-periodic solutions of partial differential equations.

Concerning almost periodic solutions and almost automorphic solutions of the abstract functional integro-differential equations, we also refer the reader to [5, 25, 30, 415–417, 529, 1062]; for almost periodic solutions and almost automorphic solutions of abstract nonlinear integro-differential equations, see [177, 193, 678], the reference lists in the articles [296, 712, 1051, 1052] and the monographs [442, 631]. For semilinear Cauchy inclusions, we can also recommend the monograph [577] by M. Kamenskii, V. Obukhovskii and P. Zecca with another approach obeyed (for semilinear Cauchy equations, see also the monographs [252] by T. Cazenave, A. Haraux and [532] by D. Henry as well as the paper [534] by M. Hieber, N. Kajiwara, K. Kress, P. Tolksdorf).

Concerning the existence and uniqueness of almost periodic type solutions of inhomogeneous evolution equations of first order, the notions of hyperbolic evolution systems and Green's functions are incredible important; for more details on the subject, we refer the reader to P. Acquistapace [14], P. Acquistapace, B. Terreni [15], Y.-H. Chang, J.-S. Chen [257], T. Diagana [364], K. Khalil [584], R. Schnaubelt [912], V. V. Zhikov [1095, 1096] and the list of references in [631]. Let us recall that a family $\{U(t,s) : t \geq s, t,s \in \mathbb{R}\}$ of bounded linear operators on X is said to be an evolution system if and only if the following holds:

(a) $U(s,s) = I$, $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$ and $t,r,s \in \mathbb{R}$,
(b) $\{(\tau,s) \in \mathbb{R}^2 : \tau > s\} \ni (t,s) \mapsto U(t,s)x$ is continuous for any fixed element $x \in X$.

If the family $A(\cdot)$ satisfies the following condition introduced by P. Acquistapace and B. Terreni in [15] (with $\omega = 0$):

(H1) There is a real number $\omega \geq 0$ such that the family of closed linear operators $A(t)$, $t \in \mathbb{R}$ acting on X satisfies $\overline{\Sigma_\phi} \subseteq \rho(A(t) - \omega)$,

$$\|R(\lambda : A(t) - \omega)\| = O((1 + |\lambda|)^{-1}), \quad t \in \mathbb{R}, \lambda \in \overline{\Sigma_\phi}, \quad \text{and}$$
$$\|(A(t) - \omega)R(\lambda : A(t) - \omega)[R(\omega : A(t)) - R(\omega : A(s))]\| = O(|t-s|^\mu |\lambda|^{-\nu}),$$

for any $t,s \in \mathbb{R}, \lambda \in \overline{\Sigma_\phi}$, where $\phi \in (\pi/2, \pi)$, $0 < \mu, \nu \leq 1$ and $\mu + \nu > 1$,

then we have the existence of an evolution system $U(\cdot,\cdot)$ generated by $A(\cdot)$, satisfying the following properties:

1. $U(\cdot, s) \in C^1((s,\infty) : L(X))$ for all $s \in \mathbb{R}$,
2. $\partial_t U(t,s) = A(t)U(t,s)$, $s \in \mathbb{R}$, $t > s$,
3. $\|A(t)^k U(t,s)\| \leq \text{Const.} \cdot (t-s)^{-k}$, $t > s$, $k \in \mathbb{N}_0$,
4. $\|A(t)U(t,s)R(\omega : A(s))\| \leq \text{Const.}$, $t > s$,
5. $\|U(t,s)(\omega - A(s))^\alpha x\| \leq \text{Const.} \cdot (\mu - \alpha)^{-1}(t-s)^{-\alpha}\|x\|$, for $0 < t - s \leq 1$, $k = 0, 1$, $0 \leq \alpha < \nu$, $x \in D((\omega - A(s))^\alpha)$,
6. $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$, for $s \in \mathbb{R}$, $t > s$, $x \in D(A(s))$ and $A(s)x \in \overline{D(A(s))}$.

In many concrete situations, it is very difficult to verify the validity of the following non-trivial condition:

(H2) The evolution system $U(\cdot,\cdot)$ generated by $A(\cdot)$ is hyperbolic (or, equivalently, has exponential dichotomy), i. e., there exist a family of projections $(P(t))_{t\in\mathbb{R}} \subseteq L(X)$, being uniformly bounded and strongly continuous in t, and constants $M', \omega > 0$ such that the following holds, with $Q := I - P$ and $Q(\cdot) := I - P(\cdot)$:
  (a) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$,
  (b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (here we set $U_Q(s,t) = U_Q(t,s)^{-1}$),
  (c) $\|U(t,s)P(s)\| \leq M'e^{-\omega(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq M'e^{-\omega(t-s)}$ for all $t \geq s$.

If the choice $P(t) = I$ for all $t \in \mathbb{R}$ is possible, then $U(\cdot,\cdot)$ is called exponentially stable. Furthermore, we say that $U(\cdot,\cdot)$ is (bounded) exponentially bounded if and only if there exist real constants $M > 0$ and $(\omega = 0)$ $\omega \in \mathbb{R}$ such that $\|U(t,s)P(s)\| \leq Me^{-\omega(t-s)}$ for all $t \geq s$.

The associated Green's function $\Gamma(\cdot,\cdot)$, defined by

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s, t, s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s, t, s \in \mathbb{R}, \end{cases}$$

satisfies

$$\|\Gamma(t,s)\| \leq M'e^{-\omega|t-s|}, \quad t, s \in \mathbb{R},$$

where M' is the constant appearing in the formulation of (H2). If the function $f : \mathbb{R} \to X$ is continuous, then the function

$$u(t) := \int_{-\infty}^{+\infty} \Gamma(t,s)f(s)\,ds, \quad t \in \mathbb{R}$$

is a unique mild solution of the abstract Cauchy problem

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{4}$$

i. e., $u(\cdot)$ is a unique bounded continuous function on $\mathbb{R}$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau, \quad t \geq s;$$

see e. g. [912] and [364, Lemma 9.11, p. 234]. Furthermore, if the function $f : [0,\infty) \to X$ is continuous, then we say that the function

$$u(t) := U(t,0)x + \int_0^t U(t,s)f(s)\,ds, \quad t \geq 0.$$

is a mild solution of the abstract Cauchy problem

$$u'(t) = A(t)u(t) + f(t), \quad t > 0; \quad u(0) = x. \tag{5}$$

We would like to emphasize the following issues with regards to the nonautonomous differential equations:

1. The almost periodic and almost automorphic solutions of the abstract Cauchy problems (4)–(5) and their semilinear analogues have been investigated in a great number of research papers. Without going into full details, we will only refer the reader to the research monographs [364] by T. Diagana, [631] by M. Kostić, the articles [117] by M. Baroun, L. Maniar, R. Schnaubelt, [116] by M. Baroun, K. Ezzinbi, K. Khalil, L. Maniar and the list of references therein.
2. Concerning the applications of evolution systems in the theory of the second-order nonautonomous differential equations, mention should be made of the paper [1070] by D. A. Zakora (almost periodic solutions of such equations have been investigated in [982], as well).

Furthermore, we want to mention the following issues (we continue the numbering):

3. Positive almost periodic solutions for various classes of abstract Volterra integro-differential equations have been extensively studied by now; see, e. g., the articles [31] by E. Ait Dads, K. Ezzinbi, [313] by S.-O. Corduneanu, [418] by K. Ezzinbi, M. A. Hachimi, [708] by Y. Li, T. Zhang, [855] by K. R. Prasad, Md. Khuddush, [1056] by G. Yang, L. Yao, [1086] by T.-W.-T. Zhang, [1101] by Q. Zhou, J. Shao.
4. The inequalities for almost periodic means and almost periodic solutions for certain classes of convolution equations have been considered by S.-O. Corduneanu in her papers [312] and [314], respectively.
5. The almost periodic functions on time scales, the almost automorphic functions on time scales and their applications to the abstract Volterra integro-differential equations have recently been considered by numerous mathematicians (for time scale calculus, we warmly recommend the monograph [186] by M. Bochner and A. Peterson). For more details about this topics, we refer the reader to the research articles [27, 361–363, 498, 499, 561, 704–706, 725, 726, 781, 1009–1014], the recent research monograph [1015] by C. Wang, R. P. Agarwal, D. O'Regan, R. Sakthivel and the references cited therein.

6. It would be really troublesome to quote here all relevant references concerning the almost periodic traveling wave solutions and the almost automorphic traveling wave solutions for various classes of nonlinear partial differential equations. The reader may consult, e. g., [40, 113, 132, 276, 533, 713, 928–930, 1088] and the references cited therein.
7. For recent results about higher-order differential operators on metric graphs, generalized trigonometric polynomials and almost periodic functions, we refer the reader to the papers [685] by P. Kurasov, J. Muller and [686] by P. Kurasov, R. Suhr.
8. The Bass and topological stable ranks for algebras of almost periodic functions on the real line have recently been analyzed by R. Mortini, R. Rupp [794] and R. Mortini, A. Sasane [795].
9. The order of magnitude of Fourier coefficients for almost periodic functions has been investigated by A. Train, R. Jain and W. Carlson [980]; among many other research studies concerning the Fourier analysis of almost periodic functions, we can also recommend the article [502] by A. P. Guinand.
10. The nonlinear equations in some spaces of almost periodic functions have been investigated by D. Bugajewski, X.-X. Ganb and P. Kasprzak in [230].

The notion of almost periodicity can be simply generalized to the case in which $I = \mathbb{R}^n$. Suppose that $F : \mathbb{R}^n \to X$ is a continuous function. Then we say that $F(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l)$ such that

$$\|F(\mathbf{t}+\tau) - F(\mathbf{t})\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n. \tag{6}$$

This is equivalent to saying that for any sequence $(\mathbf{b}_n)$ in $\mathbb{R}^n$ there exists a subsequence $(\mathbf{a}_n)$ of $(\mathbf{b}_n)$ such that $(F(\cdot+\mathbf{a}_n))$ converges in $C_b(\mathbb{R}^n : X)$. Any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic and it is also well known that $F(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$ which converges uniformly to $F(\cdot)$; let us recall that a trigonometric polynomial in $\mathbb{R}^n$ is any linear combination of functions like $\mathbf{t} \mapsto e^{i\langle\lambda,\mathbf{t}\rangle}$, $\mathbf{t} \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. Using the above clarifications, we can simply prove that a continuous function $F : \mathbb{R}^n \to X$ is almost periodic if and only if any of the following equivalent conditions holds:

(i) for every $j \in \mathbb{N}_n$ and $\epsilon > 0$, there exists a finite real number $l > 0$ such that every interval $I \subseteq \mathbb{R}$ of length l contains a point $\tau_j \in I$ such that

$$\|F(t_1, t_2, \ldots, t_j + \tau_j, \ldots, t_n) - F(t_1, t_2, \ldots, t_j, \ldots, t_n)\| \leq \epsilon, \quad \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n; \tag{7}$$

(ii) for every $\epsilon > 0$, there exists a finite real number $l > 0$ such that, for every $j \in \mathbb{N}_n$ and for every interval $I \subseteq \mathbb{R}$ of length l, there exists a point $\tau_j \in I$ such that (7) holds;

(iii) for every $\epsilon > 0$, there exists a finite real number $l > 0$ such that every interval $I \subseteq \mathbb{R}$ of length l contains a point $\tau \in I$ such that, for every $j \in \mathbb{N}_n$, (7) holds with the number τ_j replaced by the number τ therein.

Any almost periodic function $F : \mathbb{R}^n \to X$ is almost periodic with respect to each of the variables but the converse statement is not true since the function $(t_1, t_2) \mapsto \cos(t_1 t_2)$, $t_1, t_2 \in \mathbb{R}$ is almost periodic with respect to the both variables t_1 and t_2 but not almost periodic with respect to (t_1, t_2); observe that, in (7), the inequality must be satisfied for all $\mathbf{t} \in \mathbb{R}^n$. Furthermore, for every almost periodic function $F(\cdot)$ and for every real number $\epsilon > 0$, there exists $l > 0$ such that for each $\mathbf{t}_0 \in \{(t, t, \ldots, t) : t \in \mathbb{R}\}$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \{(t, t, \ldots, t) : t \in \mathbb{R}\}$ such that (6) holds (see Subsection 6.1.2 for more details). Any almost periodic function $F(\cdot)$ is bounded, the mean value

$$M(F) := \lim_{T \to +\infty} \frac{1}{(2T)^n} \int\limits_{s+K_T} F(\mathbf{t})\, d\mathbf{t}$$

exists and it does not depend on $s \in \mathbb{R}^n$; here, $K_T := \{\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : |t_i| \leq T \text{ for } 1 \leq i \leq n\}$. The Bohr–Fourier coefficient $F_\lambda \in X$ is defined by

$$F_\lambda := M(e^{-i\langle \lambda, \cdot \rangle} F(\cdot)), \quad \lambda \in \mathbb{R}^n.$$

The Bohr spectrum of $F(\cdot)$, defined by

$$\sigma(F) := \{\lambda \in \mathbb{R}^n : F_\lambda \neq 0\},$$

is at most a countable set.

The almost periodic functions of two real variables are also investigated by A. S. Besicovitch in the classic monograph [166]. Here we would like to note that the results established in [166] can be straightforwardly generalized to the almost periodic functions of several real variables. For example, if t_i is a fixed variable from the set $\{t_1, \ldots, t_n\}$, then the function $t_i \mapsto F(t_1, \ldots, t_i, \ldots, t_n)$, $t_i \in \mathbb{R}$ is almost periodic for every fixed real numbers $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ so that the mean value

$$M_{t_i}\{F(t_1, \ldots, t_n)\} := \lim_{T_i \to +\infty} \frac{1}{2T_i} \int\limits_{-T_i}^{T_i} F(t_1, \ldots, t_i, \ldots, t_n)\, dt_i$$

exists. Considering $M_{t_i}\{F(t_1, \ldots, t_n)\}$ as a function of the variables $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$, it can be easily shown that it is almost periodic in $\mathbb{R}^{n-1}$. Therefore, we can calculate the repeated mean value

$$(M_{t_j} \circ M_{t_i})\{F(t_1, \ldots, t_n)\} := \lim_{T_j \to +\infty} \frac{1}{2T_j} \int\limits_{-T_j}^{T_j} M_{t_i}\{F(t_1, \ldots, t_n)\}\, dt_j$$

for any fixed real numbers from the set $\{t_1, \ldots, t_n\} \setminus \{t_i, t_j\}$. If we fix these numbers in advance, we can apply [166, Corollary, p. 63] to the almost periodic function

$$F_{ij}(t_i, t_j) := F(t_1, \ldots, t_i, \ldots, t_j, \ldots, t_n), \quad (t_i, t_j) \in \mathbb{R}^2,$$

in order to see that

$$(M_{t_j} \circ M_{t_i})\{F(t_1, \ldots, t_n)\} \equiv (M_{t_i} \circ M_{t_j})\{F(t_1, \ldots, t_n)\}.$$

Inductively, we easily see that, for every finite tuple of different variables $(t_{i_1}, \ldots, t_{i_l})$, where $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, and for every permutation $\sigma : \{i_1, \ldots, i_l\} \to \{i_1, \ldots, i_l\}$, we have

$$(M_{t_{i_1}} \circ \cdots \circ M_{t_{i_l}})\{F(t_1, \ldots, t_n)\} = (M_{t_{\sigma(i_1)}} \circ \cdots \circ M_{t_{\sigma(i_l)}})\{F(t_1, \ldots, t_n)\}.$$

By $AP(\mathbb{R}^n : X)$ and $AP_\Lambda(\mathbb{R}^n : X)$ we denote respectively the Banach space consisting of all almost periodic functions $F : \mathbb{R}^n \to X$, equipped with the sup-norm, and its subspace consisting of all almost periodic functions $F : \mathbb{R}^n \to X$ such that $\sigma(F) \subseteq \Lambda$. As is well known, for every almost periodic function $F \in AP_\Lambda(\mathbb{R}^n : X)$, we can always find a sequence (P_k) of trigonometric polynomials in $\mathbb{R}^n$ which uniformly converges to $F(\cdot)$ on $\mathbb{R}^n$ and satisfies $\sigma(P_k) \subseteq \Lambda$ for all $k \in \mathbb{N}$; see, e. g., [824, Chapter 1, Section 2.3]. The Wiener algebra $APW(\mathbb{R}^n : X)$ is defined as the set of all almost periodic functions $F : \mathbb{R}^n \to X$ whose Fourier series converges absolutely; $APW_\Lambda(\mathbb{R}^n : X) \equiv APW(\mathbb{R}^n : X) \cap AP_\Lambda(\mathbb{R}^n : X)$. It is well known that the Wiener algebra is a Banach algebra with respect to the Wiener norm $\|F\| := \sum_{\lambda \in \mathbb{R}^n} |F_\lambda|$, $F \in APW(\mathbb{R}^n : X)$ as well as that $APW(\mathbb{R}^n : X)$ is dense in $AP(\mathbb{R}^n : X)$.

We are obliged to say that the theory of almost periodic functions of several real variables has not attracted so much attention of the authors compared with the theory of almost periodic functions of one real variable by now. In support of our investigation of the multi-dimensional almost periodicity, we would like to present the following indicative examples.

Example 1. Suppose that a closed linear operator A generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space X whose elements are certain complex-valued functions defined on $\mathbb{R}^n$. Then it is well known that, under certain assumptions, the function

$$u(t, x) = (T(t)u_0)(x) + \int_0^t [T(t-s)f(s)](x)\, ds, \quad t \geq 0,\ x \in \mathbb{R}^n, \tag{8}$$

is a unique classical solution of the abstract Cauchy problem

$$u_t(t, x) = Au(t, x) + F(t, x), \quad t \geq 0,\ x \in \mathbb{R}^n; \quad u(0, x) = u_0(x),$$

where $F(t,x) := [f(t)](x)$, $t \geq 0$, $x \in \mathbb{R}^n$. In many concrete situations (for example, this holds for the Gaussian semigroup on $\mathbb{R}^n$; see [82, Example 3.7.6] and the results obtained by M. U. Mazumder in his doctoral dissertation [758]), there exists a kernel $(t,y) \mapsto E(t,y)$, $t > 0$, $y \in \mathbb{R}^n$ which is integrable on any set $[0,T] \times \mathbb{R}^n$ $(T > 0)$ and satisfies

$$[T(t)f(s)](x) = \int_{\mathbb{R}^n} F(s, x-y)E(t,y)\,dy, \quad t > 0, s \geq 0, x \in \mathbb{R}^n.$$

If this is the case, let us fix a positive real number $t_0 > 0$. Regarding the inhomogeneous part in Eq. (8), we would like to note that the almost periodic behavior of the function

$$x \mapsto u_{t_0}(x) \equiv \int_0^{t_0} [T(t_0 - s)f(s)](x)\,ds, \quad x \in \mathbb{R}^n$$

depends on the almost periodic behavior of the function $F(t,x)$ in the space variable x. The most intriguing case is that in which the function $F(t,x)$ is Bohr almost periodic with respect to the variable $x \in \mathbb{R}^n$, uniformly in the variable t on compact subsets of $[0,\infty)$. Then the function $u_{t_0}(\cdot)$ is likewise Bohr almost periodic, which follows from the estimate

$$\begin{aligned}\left|u_{t_0}(x+\tau) - u_{t_0}(x)\right| &\leq \int_0^{t_0}\int_{\mathbb{R}^n} \left|F(s, x+\tau-y) - F(s,x-y)\right| \cdot \left|E(t_0,y)\right|\,dy\,ds \\ &\leq \epsilon \int_0^{t_0}\int_{\mathbb{R}^n} \left|E(t_0,y)\right|\,dy\,ds\end{aligned}$$

and corresponding definitions.

Example 2. In this example, we perceive an interesting feature of the famous d'Alembert formula, which has been used in [1067, Example 5] in a slightly different context (for almost periodic functions of one real variable). Let $a > 0$; then it is well known that the regular solution of the wave equation $u_{tt} = a^2 u_{xx}$ in domain $\{(x,t) : x \in \mathbb{R}, t > 0\}$, equipped with the initial conditions $u(x,0) = f(x) \in C^2(\mathbb{R})$ and $u_t(x,0) = g(x) \in C^1(\mathbb{R})$, is given by the d'Alembert formula

$$u(x,t) = \frac{1}{2}[f(x-at) + f(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds, \quad x \in \mathbb{R},\ t > 0.$$

Let us suppose that the functions $f(\cdot)$ and $g^{[1]}(\cdot) \equiv \int_0^{\cdot} g(s)\,ds$ are almost periodic. Then the solution $u(x,t)$ can be extended to the whole real line in the time variable and it is almost periodic in $(x,t) \in \mathbb{R}^2$. To verify this, fix a positive real number $\epsilon > 0$. Then there exists a finite real number $l > 0$ such that any subinterval I of $\mathbb{R}$ of length l contains a point $\tau \in I$ such that

$$|f(x+\tau) - f(x)| + |g^{[1]}(x+\tau) - g^{[1]}(x)| < \epsilon, \quad x \in \mathbb{R}. \tag{9}$$

Furthermore, we have $(x, t, \tau_1, \tau_2 \in \mathbb{R})$

$$\begin{aligned}
&\big|u(x+\tau_1, t+\tau_2) - u(x,t)\big| \\
&\quad \le \frac{1}{2}\big|f\big((x-at) + (\tau_1 - a\tau_2)\big) - f(x-at)\big| \\
&\qquad + \frac{1}{2}\big|f\big((x+at) + (\tau_1 + a\tau_2)\big) - f\big([x+at+(\tau_1+a\tau_2)] - (\tau_1 + a\tau_2)\big)\big| \\
&\qquad + \frac{1}{2a}\big|g^{[1]}\big((x-at) + (\tau_1 - a\tau_2)\big) - g^{[1]}(x-at)\big| \\
&\qquad + \frac{1}{2a}\big|g^{[1]}\big((x+at) - (\tau_1 - a\tau_2)\big) - g^{[1]}(x+at)\big|.
\end{aligned} \tag{10}$$

Let $(t_1, t_2) \in \mathbb{R}^2$. Then the interval $[-t_1 - at_2 - (l/2), -t_1 - at_2 + (l/2)]$ contains a point τ' and the interval $[t_1 - at_2 - (l/2), t_1 - at_2 + (l/2)]$ contains a point τ'' such that Eq. (9) holds with the number τ replaced therein with any of the numbers τ', τ''. Setting $\tau_1 := (\tau'' - \tau')/2$ and $\tau_2 := (-\tau_1 - \tau_2)/2a$, it follows that $|\tau_1 - t_1| \le l/2$ and $|\tau_2 - t_2| \le l/2a$, so that the final conclusion simply follows from the corresponding definition and (10).

Now we will remind the reader of several important investigations of multi-dimensional almost periodic functions carried out so far:

1. Problems of Nehari type and contractive extension problems for matrix-valued (Wiener) almost periodic functions of several real variables have been considered by L. Rodman, I. M. Spitkovsky and H. J. Woerdeman in [875], where the authors proved a generalization of the famous Sarason's theorem [900]. In their analysis, the notion of a halfspace in $\mathbb{R}^n$ plays an important role: a non-empty subset $S \subseteq \mathbb{R}^n$ is said to be a halfspace if and only if the following four conditions hold:
   (i) $\mathbb{R}^n = S \cup (-S)$;
   (ii) $\{0\} = S \cap (-S)$;
   (iii) $S + S \subseteq S$;
   (iv) $\alpha \cdot S \subseteq S$ for $\alpha \ge 0$.
   For any halfspace S we can always find a linear bijective mapping $D : \mathbb{R}^n \to \mathbb{R}^n$ such that $S = DE_n$, where E_n is a very special halfspace defined on [873, p. 3190]. In [873, Theorem 1.3], L. Rodman and I. M. Spitkovsky have proved that, if S is a halfspace and $\Lambda \subseteq S$, $0 \in \Lambda$ and $\Lambda + \Lambda \subseteq \Lambda$, then $AP_\Lambda(\mathbb{R}^n : \mathbb{C})$ and $APW_\Lambda(\mathbb{R}^n : \mathbb{C})$ are Hermitian rings.
2. Let us recall that a subset Λ of $\mathbb{R}^n$ is called discrete if and only if any point $\lambda \in \Lambda$ is isolated in Λ. By $\mathcal{V}_\Lambda$ we denote the vector space of all finite complex-valued trigonometric polynomials $\sum_{\lambda \in \Lambda} c(\lambda)e^{-\pi i \lambda \cdot}$ whose frequencies λ belong to Λ. The

space of mean-periodic functions with the spectrum Λ, denoted by $\mathcal{C}_\Lambda$, is defined as the closure of the space $\mathcal{V}_\Lambda$ in the Fréchet space $C(\mathbb{R}^n)$. Clearly, $AP_\Lambda(\mathbb{R}^n : \mathbb{C})$ is contained in $\mathcal{C}_\Lambda$ but the converse statement is not true, in general. The problem of describing structure of closed discrete sets Λ for which the equality $AP_\Lambda(\mathbb{R}^n : \mathbb{C}) = \mathcal{C}_\Lambda$ holds was proposed by J.-P. Kahane in 1957 [574]. For more details about this interesting problem, we refer the reader to the survey article [765] by Y. Meyer; for more details about mean-periodic functions, see also the lectures by J.-P. Kahane [575].

3. In 1971, B. Basit [120] observed that there exists a complex-valued almost periodic function $f : \mathbb{R}^2 \to \mathbb{C}$ such that the function $F : \mathbb{R}^2 \to \mathbb{C}$, defined by $F(x,y) := \int_0^x f(t,y)\,dt$, $(x,y) \in \mathbb{R}^2$, is bounded but not almost periodic. This result was recently reconsidered by S. M. A. Alsulami in [47, Theorem 2.2], who proved that for a complex-valued almost periodic function $f : \mathbb{R}^2 \to \mathbb{C}$, the boundedness of the function $F(\cdot)$ in the whole plane implies its almost periodicity, provided that there exists a complex-valued almost periodic function $g : \mathbb{R}^2 \to \mathbb{C}$ such that $f_x(x,y) = g_y(x,y)$ is a continuous function in the whole plane. This result was proved with the help of an old result of L. H. Loomis [731] which states that, for a bounded complex-valued function $f : \mathbb{R}^n \to \mathbb{C}$, the almost periodicity of all its partial derivatives of the first order implies the almost periodicity of $f(\cdot)$ itself (see also the article [320] by G. Crombez for related results on arbitrary locally compact Hausdorff groups). Let us observe that the above-mentioned result of S. M. A. Alsulami can be straightforwardly extended, with the same proof, to the almost periodic functions $f : \mathbb{R}^n \to \mathbb{C}$; in actual fact, if the function $f : \mathbb{R}^n \to \mathbb{C}$ is almost periodic, the function $F(x_1, x_2, \ldots, x_n) := \int_0^{x_1} f(t, x_2, \ldots, x_n)\,dt$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is bounded and there exist almost periodic functions $G_i : \mathbb{R}^n \to \mathbb{C}$ such that $F_{x_i}(x_1, x_2, \ldots, x_n) = (G_i)_{x_1}(x_1, x_2, \ldots, x_n)$ is a continuous function on $\mathbb{R}^n$, for $2 \leq i \leq n$, then the function $F : \mathbb{R}^n \to \mathbb{C}$ is almost periodic.
4. In [597–601], Yu. Kh. Khasanov has investigated the approximations of uniformly almost periodic functions of two variables by partial sums of Fourier sums and Marcinkiewicz sums in the uniform metric, provided certain conditions (see also the recent paper [609] by J.-G. Kim). For previous work as regards the summability of double Fourier series, we also refer the reader to the papers [749] by I. Marcinkewisz and [1100] by L. V. Zhizhiashvily.
5. In [688, 689], M. A. Latif and M. I. Bhatti have investigated several important questions concerning almost periodic functions defined on $\mathbb{R}^n$ with values in locally convex spaces and fuzzy number type spaces (see also the articles [137] by B. Bede, S. G. Gal, [827] by Y. L. Park, I. H. Jung, M. J. Lee, [926] by S. Shen, Y. Li and the monograph [493] by G. M. N'Guérékata); almost periodic functions defined on $\mathbb{R}^n$ with values in p-Fréchet spaces, where $0 < p < 1$, have been investigated in [497] by G. M. N'Guérékata, M. A. Latif and M. I. Bhatti.

Concerning applications made so far, we recall the following:

1. The problem of the existence of almost periodic solutions for the system of linear partial differential equations

$$\sum_{j=1}^{n} L_{ij}u_j = f_i, \quad 1 \le i \le n,$$

   on $\mathbb{R}^m$, where L_{ij} is an arbitrary linear partial differential operator on $\mathbb{R}^m$, has been analyzed by G. R. Sell [915, 916]. He has extended the results obtained in the article [941] by Y. Sibuya, where the author has analyzed the almost periodic solutions of Poisson's equation. Sibuya's results have been also improved, in another direction, in the recent article [800] by È. Muhamadiev and M. Nazarov, where the authors have relaxed the assumption of the usual boundedness into boundedness in the sense of distributions.
2. The almost periodic solutions of the (semilinear) systems of ordinary differential equations have been analyzed in [442, Chapter 8] with the help of fixed point theorems. Furthermore, Liu Bao-Ping and C. V. Pao have investigated the almost periodic wave plane solutions of certain classes of coupled nonlinear reaction-diffusion equations [718]; in their approach, a solution $u(t,x)$ of such a system, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, is almost periodic in $\mathbb{R}^{n+1}$ and satisfies the requirement that $u(t,x)$ is almost periodic in the time variable $t \in \mathbb{R}$ and periodic in each spatial variable (see [718, Theorem 2]).
3. In his doctoral dissertation [46], S. M. A. Alsulami has considered the question whether the boundedness of solutions of the system of partial first-order differential equations

$$u_s(s,t) = Au(s,t) + f_1(s,t), \quad u_t(s,t) = Bu(s,t) + f_2(s,t); \quad (s,t) \in \mathbb{R}^2, \tag{11}$$

   implies the almost periodicity of solutions to (11). He has analyzed this question in the finite-dimensional setting and the infinite-dimensional setting, by using two different techniques; in both cases, A and B are bounded linear operators acting on the pivot space X.
4. In [952–954], G. Spradlin has provided several interesting results and applications regarding almost periodic functions of several real variables. The existence of positive homoclinic-type solutions of the equation

$$-\Delta u + u = H(\mathbf{t})f(u),$$

   where $H(\cdot)$ is almost periodic and the first integral of $f(\cdot)$ satisfies certain superquadraticity and critical growth conditions, has been analyzed in [954, Theorem 1.2]. The equations of type

$$-\epsilon^2\Delta u + H(\mathbf{t})u = f(u), \tag{12}$$

arise in the study of the nonlinear Schrödinger equations ($\epsilon > 0$). A qualitative analysis of solutions of (12) has been carried out in [953], provided the almost periodicity of the function $H(\cdot)$ and several other non-trivial assumptions.

5. The existence and uniqueness of almost periodic solutions for a class of boundary value problems for hyperbolic equations have been investigated by B. I. Ptashnic and P. I. Shtabalyuk in [860] (cf. also the sixth chapter in the monograph [859] by B. I. Ptashnic). In the region $D_p = (0, T) \times \mathbb{R}^p$ ($T > 0$, $p \in \mathbb{N}$), the authors have analyzed the well-posedness of the following initial value problem:

$$Lu \equiv \sum_{s=0}^{n} \sum_{|\alpha|=2s} a_\alpha \frac{\partial^{2n} u(t,x)}{\partial t^{2n-2s} \partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}} = 0, \tag{13}$$

$$\left.\frac{\partial^{j-1} u}{\partial t^{j-1}}\right|_{t=0} = \varphi_j(x), \quad \left.\frac{\partial^{j-1} u}{\partial t^{j-1}}\right|_{t=T} = \varphi_{j+n}(x) \quad (1 \le j \le n). \tag{14}$$

The basic assumption employed in [860] is that Eq. (13) is Petrovsky hyperbolic, i. e., that for each $\mu = (\mu_1, \mu_2, \ldots, \mu_p) \in \mathbb{R}^p$ all λ-zeros of the equation

$$\sum_{s=0}^{n} \sum_{|\alpha|=2s} a_\alpha \lambda^{2n-2s} \mu_1^{\alpha_1} \mu_2^{\alpha_2} \cdots \mu_p^{\alpha_p} = 0$$

are real. The basic function space used is the Banach space $C_B^q(\overline{D^p})$ consisting of all q-times continuously differentiable functions $u(t,x)$ in $\overline{D^p}$ that are Bohr almost periodic in variables $x_1, x_2, \ldots, x_p$, uniformly in $t \in [0, T]$, equipped with the norm

$$\|u\|_{C_B^q(\overline{D^p})} := \sup_{0 \le |s| \le q} \sup_{(t,x) \in \overline{D^p}} \frac{\partial^{|s|} u(t,x)}{\partial t^{s_0} \partial x_1^{s_1} \cdots \partial x_p^{s_p}};$$

by $C_B^q(\mathbb{R}^p)$ the authors have designated the subspace of $C_B^q(\overline{D^p})$ consisting of those functions which do not depend on the variable t. The existence and uniqueness of solutions of the initial value problem (13)–(14) have been investigated in the space $C_B^{2n}(\overline{D^p})$, under the assumption that $\varphi_j(x) \in C_B^r(\mathbb{R}^p)$ and $r \in \mathbb{N}$ is sufficiently large. If $M_p = \{\mu_k : k \in \mathbb{Z}^p\}$ is the union of spectrum of all functions $\varphi_1(x), \ldots, \varphi_{2n}(x)$, the solutions $u(t,x)$ of problem (13)–(14) have been found in the form

$$u(t,x) = \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x\rangle}, \quad \mu_k \in M_p,$$

where the functions $u_k(t)$ satisfy certain conditions and have the form [860, (8), p. 670]. The uniqueness of solutions of problem (13)–(14) has been considered in [860, Theorem 1], while the existence of solutions of problem (13)–(14) has been considered in [860, Theorem 2] (see also the research articles [932, 933] by P. I. Shtabalyuk).

Suppose now that $\omega \in \mathbb{R}^p \setminus \{0\}$ and $C \in \mathbb{R}$. We want to observe here that the assumption $\varphi_j \in AP_\Lambda(\mathbb{R}^n : \mathbb{C})$ for all $j \in \mathbb{N}_{2n}$, where

$$\Lambda := \{(x_1, \ldots, x_p) \in \mathbb{R}^p; x_1\omega_1 + \cdots + x_p\omega_p = C\},$$

implies that the solution $u(t,x)$ of problem (13)–(14) is (ω, e^{iC})-periodic in the space variable x; see Section 7.2 for the notion. This follows from the computation ($t \in (0,T)$, $x \in \mathbb{R}^p$)

$$\begin{aligned} u(t, x+\omega) = \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x+\omega\rangle} &= \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x\rangle} e^{i\langle \mu_k, \omega\rangle} \\ &= e^{iC} \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x\rangle} = cu(t,x). \end{aligned}$$

6. The class of vector-valued remotely almost periodic functions defined on $\mathbb{R}^n$ was introduced by F. Yang and C. Zhang in [1054] (2011). In the same paper, the authors have provided several applications in the study of the existence and uniqueness of remotely almost periodic solutions for parabolic boundary value problems. A bounded continuous function $F : \mathbb{R}^n \to X$ is said to be remotely almost periodic if and only if for each $\epsilon > 0$ the set of all vectors $\tau \in \mathbb{R}^n$ for which

$$\limsup_{|\mathbf{t}| \to +\infty} |F(\mathbf{t} + \tau) - F(\mathbf{t})| < \epsilon$$

is relatively dense in $\mathbb{R}^n$ (the vector τ is called a remotely ϵ-translation vector of $F(\cdot)$); furthermore, if $\emptyset \neq \Omega \subseteq \mathbb{R}^m$, then a bounded continuous continuous function $F : \mathbb{R}^n \times \Omega \to X$ is said to be remotely almost periodic in $\mathbf{t} \in \mathbb{R}^n$ and uniform on compact subsets of Ω if and only if $F(\cdot, y)$ is remotely almost periodic for each $y \in \Omega$ and is uniformly continuous on $\mathbb{R}^n \times K$ for any compact subset $K \subseteq \Omega$. The following statements hold in the scalar-valued case (see, e. g., [1054, Proposition 2.1–Proposition 2.3]):
(i) If $F(\cdot)$, resp. $F(\cdot;\cdot)$, is remotely almost periodic and the function $\partial F/\partial \mathbf{t}_i(\cdot)$, resp. $\partial F/\partial \mathbf{t}_i(\cdot;\cdot)$, is uniformly continuous on $\mathbb{R}^n$, then the function $\partial F/\partial \mathbf{t}_i$, resp. $\partial F/\partial \mathbf{t}_i(\cdot;\cdot)$, is remotely almost periodic, as well $(1 \leq i \leq n)$;
(ii) if the functions $F_1(\cdot), \ldots, F_k(\cdot)$ are remotely almost periodic ($k \in \mathbb{N}$), then for each $\epsilon > 0$ the set of their common ϵ-translation vectors is relatively dense in $\mathbb{R}^n$;
(iii) if the functions $H_1(\cdot), \ldots, H_k(\cdot)$ are remotely almost periodic ($k \in \mathbb{N}$) and $(H_1(t), \ldots, H_k(t)) \in \Omega$ for all $t \in \mathbb{R}$, then for every remotely almost periodic function $F : \mathbb{R} \times \Omega \to \mathbb{C}$ we see that the function

$$t \mapsto F(H_1(t), \ldots, H_k(t), t), \quad t \in \mathbb{R}$$

is remotely almost periodic.

In [1054, Proposition 2.4–Proposition 2.6], the authors have examined the existence and uniqueness of remotely almost periodic solutions of multi-dimensional heat equations, while the main results of the third section of this paper are concerned with the existence and uniqueness of remotely almost periodic type solutions of the certain types of parabolic boundary value problems (see also [1053] and [1055], where the authors have investigated almost periodic type solutions and slowly oscillating type solutions for various classes of parabolic Cauchy inverse problems). Regarding applications of remotely almost periodic functions, mention should be made of the research articles [1085] by S. Zhang and D. Piao, where the authors have investigated the time remotely almost periodic viscosity solutions of Hamilton–Jacobi equations, and [1081] by C. Zhang and L. Jiang, where the authors have investigated remotely almost periodic solutions for a class of systems of differential equations with piecewise constant argument. Some results about remotely almost periodic solutions of ordinary differential equations will be given in Chapter 5; for more details about slowly oscillating type functions and remotely c-almost periodic type functions in $\mathbb{R}^n$, where $c \in \mathbb{C} \setminus \{0\}$, see Chapter 9.

Furthermore, let us recall that H. Bart and S. Goldberg have proved in [119] that, for every function $f \in AP([0,\infty) : X)$, there exists a unique almost periodic function $\mathbb{E}f : \mathbb{R} \to X$ such that $\mathbb{E}f(t) = f(t)$ for all $t \geq 0$ (see also the paper [427] by S. Favarov and O. Udodova, where the authors have investigated the extensions of almost periodic functions defined on $\mathbb{R}^n$ to the tube domains in $\mathbb{C}^n$, and the paper [157] by J. F. Berglund, where the author has investigated the extensions of almost periodic functions in topological groups and semigroups). We will investigate the extensions of multi-dimensional almost periodic functions in Theorem 6.1.37 and Corollary 6.1.38 (cf. also Remark 4.2.98) following the method proposed by W. M. Ruess and W. H. Summers in [881].

The boundedness and almost periodicity in time for certain classes of evolution variational inequalities, positive boundary value problems for symmetric hyperbolic systems and nonlinear Schrödinger equations have been investigated in the third chapter and the fourth chapter of the important research monograph [824] by A. A. Pankov (for almost periodic properties of Schrödinger equations and Schrödinger type operators, see [95, 96, 98, 325, 326, 346, 347, 383, 828, 886] and the research monograph [580] by Y.Karpeshina). Spatially Besicovitch almost periodic solutions for certain classes of nonlinear second-order elliptic equations, first-order hyperbolic systems, single higher-order hyperbolic equations and nonlinear Schrödinger equations have been investigated in the fifth chapter of this monograph.

It is worth mentioning that G. Spradlin constructed, in [952], an almost periodic infinitely differentiable almost periodic function $G : \mathbb{R}^n \to \mathbb{R}$ with no local minimum (it can be simply shown that this situation cannot occur in the one-dimensional case because any almost periodic function $g : \mathbb{R} \to \mathbb{R}$ must have infinitely many local minima); this important peculiarity of almost periodic functions of several real variables was perceived 25 years ago. The construction of an almost periodic function

$G : \mathbb{R}^n \to \mathbb{R}$ with no local minimum, established in [952], is very complicated and the proof of the main result of this paper, [952, Theorem 1.0], contains almost eight pages including some preliminaries. It can be easily proved, by observing that the function $G(x, y)$ is strictly positive, that the function $(x, y) \mapsto H(x, y) \equiv \int_0^x G(t, y)\,dt$ is bounded and not almost periodic in the plane. As already mentioned, the existence of a complex-valued almost periodic function $H(x, y)$ with these properties was clarified by B. Basit (1971) with a piece of very obscure evidence, not including the smoothness of $G(x, y)$ or its non-negativity.

At the end of manuscript [952], G. Spradlin asked the following questions:

1. The almost periodic function $F : \mathbb{R}^2 \to \mathbb{R}$ constructed in the proof of [952, Theorem 1.0] has an absolute maximum at the point $(0, 0)$. Does there exist an almost periodic function $F : \mathbb{R}^n \to \mathbb{R}$ with no local minimum or maximum?
2. Does there exist a real analytic almost periodic function $F : \mathbb{R}^n \to \mathbb{R}$ with no local minimum or maximum?
3. Is it true that a continuously differentiable almost periodic function $F : \mathbb{R}^n \to \mathbb{R}$ have a critical point?
4. Does there exist a quasi-periodic function $F : \mathbb{R}^n \to \mathbb{R}$ with no local minimum (local minimum or maximum)?

To the best of the author's knowledge, these questions are still open. Concerning almost periodic functions of several real variables, we also refer the reader to the research monographs [309] by C. Corduneanu, [442] by A. M. Fink, [874] by L. Rodman, I. M. Spitkovsky, H. J. Woerdeman, [1067] by S. Zaidman and the research article [936] by M. A. Shubin. The reader also may consult the paper [187] by S. Bochner, which concerns the extensions of the Riesz theorem to the analytic functions of several real variables and the almost periodic functions of several real variables.

In this research monograph, we present several recent results concerning various generalizations of almost periodic functions. The organization and main ideas of the monograph, which consists of the introductory chapter and two parts, can be described as follows. The first chapter is devoted to the recapitulation of basic concepts we will need later on. We reconsider linear and multivalued linear operators in Banach spaces, integration and strongly continuous semigroups in Banach spaces as well as the basic definitions and results from fractional calculus and the theory of abstract degenerate Volterra integro-differential equations. After introducing these basic concepts, in Subsection 1.1.1 we recall the main definitions and results about Lebesgue spaces with variable exponents $L^{p(x)}$.

Part I consists of four chapters (Chapter 2–Chapter 5). Chapter 2 is divided into four sections; Section 2.1, Section 2.2 and Section 2.3 are of introductory character and we recollect there the basic definitions and results about almost periodic functions and almost automorphic functions. Stepanov μ-pseudo-almost periodic functions and related applications are analyzed in Subsection 2.2.1 (this is a major part of our recent joint research study [585] with K. Khalil and M. Pinto), composition principles for Weyl almost periodic functions are analyzed in Subsection 2.2.2, Proposition 2.3.1 and the conclusion clarified in Example 2.3.6 are the only new contributions of ours given in these sections. The main aims of Section 2.4, which considers uniformly recurrent functions and $\odot_g$-almost periodic functions, will be explained within itself.

Various classes of generalized almost periodic functions in the Lebesgue spaces with variable exponents have been analyzed in Chapter 3, which is broken down into Section 3, Section 3.2 and Section 3.3. Section 3 consists of seven subsections. In our joint papers with T. Diagana [372, 373], we have recently introduced and analyzed several important classes of (asymptotically) Stepanov almost periodic functions and (asymptotically) Stepanov almost automorphic functions in the Lebesgue spaces with variable exponents (see also the earlier papers [375, 376] by T. Diagana and M. Zitane). The material of Subsection 3.1.1, Subsection 3.1.2 and Subsection 3.1.3 is taken from [372].

The classes introduced by H. Weyl [1025] and A. S. Kovanko [669] are enormously large compared with the class of Stepanov almost periodic functions; the main purpose of papers [638] and [647] has been to initiate the study of generalized (asymptotical) almost periodicity that intermediates Stepanov and Weyl concept. In these papers, we have introduced the class of Stepanov p-quasi-asymptotically almost periodic functions and proved that this class contains all asymptotically Stepanov p-almost periodic functions and makes a subclass of the class consisting of all Weyl p-almost periodic functions ($p \in [1,\infty)$), taken in the sense of Kovanko's approach [669].

The main aim of Subsection 3.1.4–Subsection 3.1.7 is to continue the research studies raised in [435] and [645–647] by investigating several various classes of asymptotically Weyl almost periodic functions in Lebesgue spaces with variable exponents $L^{p(x)}$. The first important novelty is that we analyze here, for the first time, the class of (equi-)Weyl p-almost periodic functions from the point of view of the theory of Lebesgue spaces with variable exponent $L^{p(x)}$. The second important novelty can be briefly explained as follows: in the case that the functions $\phi(x)$ and $F(l,t)$ satisfy $\phi(x) \neq x$ or $F(l,t) \neq l^{(-1)/p}$, the (p,ϕ,F)-classes of Weyl almost periodic functions and the $[p,\phi,F]$-classes of Weyl almost periodic functions, introduced and systematically analyzed in Section 3.1.4, seem to be not considered elsewhere in the existing literature even in the case that the function $p(x)$ has a constant value $p \geq 1$. The motivation for introduction of these classes of Weyl almost periodic functions comes from the fact that the gap between the Stepanov almost periodicity and the Weyl almost periodicity is enormously large, as mentioned above: we have actually tried to further profile and contour the class of (equi-)Weyl almost periodic functions here. For example, it is well known that the characteristic function of the interval $[0,1/2]$ is equi-Weyl p-almost periodic for any $p \in [1,\infty)$ but not Stepanov almost periodic (see Example 3.1.34 below for more details); we obtain here the better results about the equi-Weyl almost periodicity of this function by showing that this function is equi-Weyl (p,x,ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$ such that $\lim_{l\to+\infty} \psi(l) = +\infty$ as well as that this function is equi-Weyl (p, x, l^{σ})-almost periodic if and only if $\sigma \neq 0$. Besides the above-mentioned novelties, and compared with many other research results established in the previous papers, we would like to mention here only one more major novelty with regards to our new way of approaching the problem of the invariance of the (equi-)Weyl p-almost periodicity under the actions of infinite convolution product (see, e. g., Theorem 3.1.46).

In Definition 3.1.23–Definition 3.1.25, we introduce the classes of (equi-)Weyl (p, ϕ, F)-almost periodic functions and (equi-)Weyl $(p, \phi, F)_i$-almost periodic functions, where $i = 1, 2$. The main aim of Proposition 3.1.27 is to clarify some inclusions between these spaces provided that the function $\phi(\cdot)$ is convex and satisfies certain extra conditions. In order to ensure the translation invariance of generalized Weyl almost periodic functions with variable exponent, in Definition 3.1.29–Definition 3.1.31 we introduce the classes of (equi-)Weyl $[p, \phi, F]$-almost periodic functions and (equi-)Weyl $[p, \phi, F]_i$-almost periodic functions, where $i = 1, 2$. Several useful comments about these spaces have been provided in Remark 3.1.32. In Example 3.1.34–Example 3.1.35, we focus our attention on the following special case: $p(x) \equiv p \in [1, \infty)$, $\phi(x) = x$ and $F(l, t) = l^{(-1)/p\sigma}$, $\sigma \in \mathbb{R}$, which is the most important for the investigations of generalized almost periodicity which stands between the Stepanov and Weyl concepts. In Subsection 3.1.5, we introduce and analyze various types of Weyl ergodic components with variable exponent and asymptotically Weyl almost periodic functions with variable exponent. The introduced classes of generalized (asymptotically) Weyl almost periodic functions are new even in the case that the function $p(x)$ has a constant value $p \geq 1$ and $\phi(x) \neq x$ or $F(l, t) \neq l^{(-1)/p(t)}$. From the application point of view, Subsection 3.1.6 is very important because we examine there the invariance of generalized Weyl almost periodicity with variable exponent under the action of convolution products and the convolution invariance of Weyl almost periodic functions with variable exponent. In order to do that, we basically follow the method proposed in the proof of Theorem 3.1.46. In Subsection 3.1.7, we face the situation in which the exponent $p(x) \equiv p \in [1, \infty)$ is constant and the solution operator family $(R(t))_{t>0} \subseteq L(X, Y)$ has a certain growth order around the points zero and plus infinity, providing also some illustrative applications in the qualitative analysis of solutions to the abstract degenerate fractional differential equations with Weyl–Liouville derivatives or Caputo derivatives. The material of these subsections is taken from [649].

Section 3.2 is broken down into three subsections. In Subsection 3.2.1, we analyze Stepanov uniformly recurrent functions in the Lebesgue spaces with variable exponents. Doss almost periodic functions and Doss uniformly recurrent functions in Lebesgue spaces with variable exponents are investigated in Subsection 3.2.2, while the invariance of generalized Doss almost periodicity with variable exponent under the actions of convolution products is investigated in Subsection 3.2.3.

Section 3.3 is broken down into six subsections. Subsection 3.3.1 introduces the notion of several different types of generalized (equi-)Weyl almost periodicity in

Lebesgue spaces with variable exponents. The spaces introduced in Definition 3.3.1–Definition 3.3.3 may not be translation invariant, in general, which is not the case with the spaces introduced in Definition 3.3.5–Definition 3.3.7. The main aim of Subsection 3.3.1 is to explain without proofs how the structural results and characterizations established for generalized (equi-)Weyl almost periodic functions in [649] can be straightforwardly extended for the corresponding classes of generalized (equi-)Weyl uniformly recurrent functions. In Definition 3.3.8, we introduce the class of quasi-asymptotically uniformly recurrent functions (it is worth noting that some classes of generalized Stepanov and Weyl $p(x)$-almost periodic type functions and $p(x)$-uniformly recurrent type functions have not been considered elsewhere even for the constant coefficients $p(x) \equiv p \in [1, \infty)$). Proposition 3.3.9 shows that any asymptotically uniformly recurrent function is quasi-asymptotically uniformly recurrent; the converse statement is generally false, as a class of very simple counterexamples shows. In Proposition 3.3.10, we prove that the sum of a quasi-asymptotically uniformly recurrent function and a continuous function vanishing at infinity is again quasi-asymptotically uniformly recurrent. In Theorem 3.3.14, we revisit [647, Theorem 2.5] once more and examine some extra conditions under which a quasi-asymptotically uniformly recurrent function is (asymptotically) uniformly recurrent. Subsection 3.3.3 introduces and investigates several different classes of Stepanov quasi-asymptotically uniformly recurrent type functions in the Lebesgue spaces with variable exponents. The notion introduced in this subsection, in which we reconsider and slightly improve several known results from [647] in our new framework, is new even for the constant coefficients $p(x) \equiv p \in [1, \infty)$, and can be used to intermediate the concepts of the quasi-asymptotical almost periodicity (quasi-asymptotical uniform recurrence, S-asymptotical ω-periodicity) and its Stepanov generalizations with constant exponents. In Proposition 3.3.23, we reconsider the assertion of [372, Proposition 4.5] for the Stepanov quasi-asymptotically uniformly recurrent functions (see also Corollary 3.3.24 and Proposition 3.3.25). Any Stepanov p-quasi-asymptotically almost periodic function is Weyl p-almost periodic, and clearly, any (quasi-)asymptotically almost periodic function is Stepanov p-quasi-asymptotically almost periodic for any finite exponent $p \geq 1$ (see [647, Proposition 2.12]); as observed here, the same holds for the related concepts of quasi-asymptotical uniform recurrence. The main objective in Proposition 3.3.26 is to state and prove a general result in this direction. In Subsection 3.3.4, we clarify the main composition principles for the class of quasi-asymptotically uniformly recurrent functions. Our main contributions are given in Subsection 3.3.5, where we examine the invariance of generalized quasi-asymptotical uniform recurrence with variable exponents under the actions of convolution products. Some applications to the abstract Volterra integro-differential equations are presented in Subsection 3.3.6. The material of Section 3.2 and Section 3.3 is taken from our recent papers obtained in a coauthorship with W.-S. Du [657, 658].

The definitions and basic properties of (ω, c)-periodic and (ω, c)-pseudo periodic functions were introduced and analyzed by E. Alvarez, A. Gómez and M. Pinto in [48, 49], motivated by some known results regarding the qualitative properties of solutions to Mathieu's linear differential equation

$$y''(t) + [a - 2q\cos 2t]y(t) = 0,$$

arising in modeling of railroad rails and seasonally forced population dynamics ($\omega > 0$, $c \in \mathbb{C} \setminus \{0\}$); see also [11, 12]. The linear delayed equations can have (ω, c)-periodic solutions, as well (see, e. g., [49, Example 2.5]). The notions of anti-periodicity and Bloch periodicity are special cases of the notion of an (ω, c)-periodicity.

The authors of [49] have analyzed the existence and uniqueness of mild (ω, c)-periodic solutions to the abstract semilinear integro-differential equation (1). Furthermore, E. Alvarez, S. Castillo and M. Pinto have analyzed in [48] the existence and uniqueness of mild (ω, c)-pseudo periodic solutions to the abstract semilinear differential equation of the first order:

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where A generates a strongly continuous semigroup. The authors have proved the existence of positive (ω, c)-pseudo periodic solutions to the Lasota-Wazewska equation with (ω, c)-pseudo periodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geq 0.$$

This equation describes the survival of red blood cells in the blood of an animal (see, e. g., M. Wazewska-Czyzewska and A. Lasota [1024]). Concerning the applications to time-varying impulsive differential equations, mention should be made of the article [1019] by J. R. Wang, L. Ren and Y. Zhou; cf. also the article [19] by M. Agaoglou, M. Fečkan, A. P. Panagiotidou, the article [782] by G. Mophou, G. M. N'Guérékata and the article [700] by M. Li, J. R. Wang and M. Fečkan.

Chapter 4 consists of Section 4.1 and Section 4.2. In Section 4.1, we analyze various types of (ω, c)-almost periodic functions, (ω, c)-uniformly recurrent functions and (compactly) (ω, c)-almost automorphic functions. The classes of (ω, c)-uniformly recurrent functions of type i and (ω, c)-almost periodic functions of type i ($i = 1, 2$) are introduced and analyzed in Subsection 4.1.1. Composition principles for (ω, c)-almost periodic type functions are analyzed in Subsection 4.1.2. The classes of (ω, c)-pseudo-almost periodic functions, (ω, c)-pseudo-almost automorphic functions and related applications are studied in Subsection 4.1.4. Subsection 4.1.5 introduces and investigates (ω, c)-almost periodicity (resp. asymptotic (ω, c)-almost periodicity) in the setting of Schwartz–Sobolev distributions (for simplicity, we will consider only scalar-valued distributions because the extensions to the vector-valued case are straightforward); in the next subsection, we apply our abstract theoretical results in the study of the existence of distributional (ω, c)-almost periodic solutions of linear differential systems. In [857, Chapter II], J. Prüss has analyzed abstract non-scalar Volterra equations. Applications have been given in the analysis of viscoelastic Timoshenko beam model, Midlin–Timoshenko plate model and viscoelastic Kirchhoff plate model, with the corresponding materials being non-synchronous, as well as in the analysis of some problems of linear thermoviscoelasticity and electrodynamics. In Subsection 4.1.7, we initiate the study of asymptotically (ω, c)-almost periodic type solutions of abstract degenerate non-scalar Volterra equations.

The organization and main ideas of Section 4.2, comprising seven subsections, is given as follows. The notion of c-almost periodicity and the notion of c-uniform recurrence, where $c \in \mathbb{C} \setminus \{0\}$, are introduced in Definition 4.2.2 and Definition 4.2.4, respectively (in the case $c = 1$, we recover the usual notions of almost periodicity and uniform recurrence, while in the case $c = -1$, we recover the usual notions of almost anti-periodicity and uniform anti-recurrence); the main idea is the use of difference $f(\cdot+\tau)-cf(\cdot)$ in place of the usually considered difference $f(\cdot+\tau)-f(\cdot)$. After that, in Definition 4.2.5 and Proposition 4.2.6, we introduce the notion of semi-c-periodicity and prove some necessary and sufficient conditions for a continuous function $f : I \to X$ to be semi-c-periodic. Proposition 4.2.11 is crucially important in our analysis because it states that there does not exist a c-uniformly recurrent function $f : I \to X$ if $|c| \neq 1$. The invariance of c-almost type periodicity under the actions of convolution products is also analyzed here. The composition theorems for c-almost periodic type functions are analyzed in Subsection 4.2.1 (the structural results in this subsection are given without proofs, which can be deduced similarly as in our previous research studies; it is also worth noting that we present numerous indicative examples and comments about the problems considered). In Subsection 4.2.2, we present some applications of our abstract results in the analysis of the existence and uniqueness of c-almost periodic type solutions to the abstract (semilinear) Volterra integro-differential inclusions. The class of semi-c-periodic functions with general parameter $c \in \mathbb{C} \setminus \{0\}$ is introduced and analyzed in Subsection 4.2.3; the main result of this subsection is Theorem 4.2.45 which states that the notions of c-periodicity and semi-c-periodicity are equivalent for $|c| \neq 1$. The material of Section 4.1 and Section 4.2 is obtained in a coauthorship with Prof. M. Pinto, M. T. Khalladi, A. Rahmani and D. Velinov [586–590].

Let $p > 0$ and $k \in \mathbb{R}$. Recall that a bounded continuous function $f : I \to X$ is said to be Bloch (p, k)-periodic, or Bloch periodic with period p and Bloch wave vector or Floquet exponent k if and only if $f(x + p) = e^{ikp} f(x)$, $x \in I$, with $p > 0$ and $k \in \mathbb{R}$. The study of Bloch (p, k)-periodic functions is an important subject of applied functional analysis. The Bloch periodic functions are widely used in biology, physics, probability, modeling, solid mechanics [92] and many other areas (see the papers [204] by R. F. Boukadia, C. Droz, M. N. Ichchou, W. Desmet, [246] by G. Carta, M. Brun, [259, 260] by Y. K. Chang, Y. Wei, [307] by M. Collet, M. Ouisse, M. Ruzzene, M. Ichchou, [521] by M. F. Hasler, [522] by M. F. Hasler, G. M. N'Guérékata, [579] by D. Karličić, M. Cajić, T. Chatterjee, S. Adhikari, [664] by M. Kostić, D. Velinov, [679] by I. Krichever, S. P. Novikov, [690] by V. Laude, R. P. Moiseyenko, S. Benchabane, N. F. Declercq, [691] by M. J. Leamy, the forthcoming monograph [258] by Y.-K. Chang, G. M. N'Guerekata, R. Ponce and the references cited therein; for the importance of almost periodicity in mechanics, electron propagation theory and mathematical physics; see also [97, 144] and [950]). As is well known, the notion of an anti-periodic function is a special case of the notion of a Bloch (p, k)-periodic function (a bounded continuous function $f : I \to X$ is said to $f(\cdot)$ is anti-periodic if and only if there exists $p > 0$ such that $f(x + p) = -f(x)$, $x \in I$; any such function needs to be periodic of period $2p$). For more details about anti-periodic type functions and their applications, we refer the reader to [42, 280, 381, 501, 664, 720, 722] and the references cited therein. Semi-Bloch k-periodic functions are investigated in Subsection 4.2.4 (the results are obtained in a coauthorship with B. Chaouchi, S. Pilipović and D. Velinov [262]). The genesis of paper [262] is motivated by reading the research article [69] by J. Andres and D. Pennequin, where the authors have introduced and analyzed the class of semi-periodic functions (sequences) and related applications to differential (difference) equations; see also [70]. Of course, a semi-periodic function is nothing else but a semi-c-periodic function with $c = 1$.

The class of S-asymptotically ω-periodic functions, introduced by H. Henríquez et al. [531] for case $I = \mathbb{R}$ and M. Kostić [647] for the case $I = [0, \infty)$, are reconsidered in Subsection 4.2.6, where we introduce the class of S-asymptotically (ω, c)-periodic functions. Quasi-asymptotically c-almost periodic functions and related composition principles are investigated in Subsection 4.2.7.

Several notes and appendices are provided in the final chapter of Part I, where we particularly analyze recurrent strongly continuous semigroups of operators. The organization of each chapter and section of Part II will be self-explaining.

Concerning drawbacks of the monograph, we wish to emphasize the following:

1. Although this is probably the first research monograph within the field of almost periodicity where the results from the theory of Lebesgue spaces $L^{p(x)}$ have been employed, the use of constant coefficients $p(x) \equiv p \in [1, \infty)$ is unquestionably the best and we need to put maximum effort into getting new applications of almost periodic functions in Lebesgue spaces with variable exponents (from the theoretical point of view, the use of constant coefficients does not give ground to a great extent).
2. We feel it is our duty to say that the approach used for the introduction of the notion in Definition 3.1.23–Definition 3.1.25 and Definition 3.1.29–Definition 3.1.31 is exploited multiple times in the remainder of the book, for various types of generalized almost periodicity; this could be a bit tedious and monotonous for the reader. For the sake of better readability, we have decided to repeat some equations with the Lipschitz type conditions, the convolution products used and the consequences of the Jensen integral inequality several times throughout the book. Also, there are some glaring omissions in time scales direction research since the almost periodic and almost automorphic topic are also important in hybrid domains (the author of monograph is really not an expert in this field; see also the

corresponding part of [631, Section 2.16] for more details about the subject and applications given so far).

3. Part II investigates multi-dimensional almost periodic type functions and their applications (this topic has not been well presented in any published research monograph by now). But many structural results exhibited in Part I are very special consequences of the corresponding results from Part II; we keep all such results from Part I for the sake of better exposition and further popularization of almost periodic functions of one real variable (there is no need to say that the one-dimensional setting is very important from the application point of view).

Finally, we would like to note that a great deal of the introductory part and some notes and appendices to Part I and Part II have recently been published in our joint survey article with W.-S. Du and M. Pinto [400].

1 Preliminaries

1.1 Linear operators and integration in Banach spaces, strongly continuous semigroups and fixed point theorems

In this section, we recollect some essential facts about vector-valued functions, closed operators, integration and strongly continuous semigroups in Banach spaces. We also recall the basic fixed point theorems we will employ later on. In Subsection 1.1.1, we explore the basic definitions and results about the Lebesgue spaces with variable exponents $L^{p(x)}$.

Vector-valued functions, closed operators

Generally, by $(X, \|\cdot\|)$ we denote a Banach space over the field of complex numbers. If $(Y, \|\cdot\|_Y)$ is another Banach space over the field of complex numbers, then by $L(X, Y)$ we denote the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X, X)$. The topologies on $L(X, Y)$ and X^*, the dual space of X, are introduced in the usual way. If not stated otherwise, by I we denote the identity operator on X. If X and Y are two Banach spaces such that Y is continuously embedded in X, then we write $Y \hookrightarrow X$.

We say that a linear operator $A : D(A) \to X$ is closed if and only if the graph of the operator A, defined by $G_A := \{(x, Ax) : x \in D(A)\}$, is a closed subset of $X \times X$. The null space and range of A are denoted by $N(A)$ and $R(A)$, respectively. Let us recall that a linear operator $A : D(A) \to X$ is closed if and only if, for every sequence (x_n) in $D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$, the following hold: $x \in D(A)$ and $Ax = y$; a linear operator A is called closable if and only if there exists a closed linear operator B such that $A \subseteq B$. If F is a linear submanifold of X, then we define the part of A in F by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$.

The power A^n of A is defined inductively ($n \in \mathbb{N}_0$); set $D_\infty(A) := \bigcap_{n\geqslant 1} D(A^n)$. For a closed linear operator A acting on X, we introduce the adjoint A^* of $X^* \times X^*$ by

$$A^* := \{(x^*, y^*) \in X^* \times X^* : x^*(Ax) = y^*(x) \text{ for all } x \in D(A)\}.$$

In the case that A is densely defined, then A^* is single-valued, closed and also known as the adjoint operator of A. Assuming $\alpha \in \mathbb{C} \smallsetminus \{0\}$, A and B are linear operators, we define the operators αA, $A+B$ and AB in the usual way. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set, for every $\alpha > 0$,

$$g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha), \quad t > 0,$$

$g_0(t) \equiv$ the Dirac δ-distribution and $0^\zeta := 0$. Set $\Sigma_\alpha := \{z \in \mathbb{C} \smallsetminus \{0\} : |\arg(z)| < \alpha\}$, $\alpha \in (0, \pi]$.

By $C(\Omega : X)$ we denote the space consisting of all continuous functions $f : \Omega \to X$, where $\emptyset \neq \Omega \subseteq \mathbb{C}^n$ ($n \in \mathbb{N}$); $C(\Omega) \equiv C(\Omega : \mathbb{C})$. Let $0 < \tau \leqslant \infty$ and $a \in L^1_{\mathrm{loc}}([0, \tau))$. Then

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we say that the function $a(t)$ is a kernel on $[0,\tau)$ if and only if for each $f \in C([0,\tau))$ the assumption $\int_0^t a(t-s)f(s)\,ds = 0$, $t \in [0,\tau)$ implies $f(t) = 0$, $t \in [0,\tau)$. If $s \in \mathbb{R}$ and $n \in \mathbb{N}$, we define $\lfloor s\rfloor := \sup\{l \in \mathbb{Z} : s \geqslant l\}$, $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \leqslant l\}$, $\mathbb{N}_n := \{1,\dots,n\}$ and $\mathbb{N}_n^0 := \{0,1,\dots,n\}$. If X, $\mathrm{Y} \neq \emptyset$, put $\mathrm{Y}^{\mathrm{X}} := \{f \mid f : \mathrm{X} \to \mathrm{Y}\}$.

Let $I = \mathbb{R}$ or $I = [0,\infty)$. By $C_b(I : X)$ we denote the space consisting of all bounded continuous functions from I into X; the symbol $C_0(I : X)$ denotes the vector subspace of $C_b(I : X)$ consisting of those functions $f : I \to X$ such that $\lim_{|t|\to\infty} \|f(t)\| = 0$. By $\mathrm{BUC}(I : X)$ we denote the space consisting of all bounded uniformly continuous functions from I to X; $C_b(I) \equiv C_b(I : \mathbb{C})$, $C_0(I) \equiv C_0(I : \mathbb{C})$ and $\mathrm{BUC}(I) \equiv \mathrm{BUC}(I : \mathbb{C})$. Equipped with the sup-norm, $C_b(I : X)$, $C_0(I : X)$ and $\mathrm{BUC}(I : X)$ are Banach spaces.

Regarding analytical functions with values in Banach spaces and locally convex spaces, we refer the reader to [82] and [633] (for almost periodic functions and almost automorphic functions with values in locally convex spaces and general vector topological spaces, we refer the reader to [631, Section 3.11] and the references cited therein).

Integration in Banach spaces

The following definition is elementary.

Definition 1.1.1.

(i) A function $f : I \to X$ is said to be simple if and only if there exist $k \in \mathbb{N}$, elements $z_i \in X$, $1 \leqslant i \leqslant k$ and Lebesgue measurable subsets Ω_k, $1 \leqslant i \leqslant k$ of I, such that $m(\Omega_i) < \infty$, $1 \leqslant i \leqslant k$ and

$$f(t) = \sum_{i=1}^{k} z_i \chi_{\Omega_i}(t), \quad t \in I. \tag{1.1}$$

(ii) A function $f : I \to X$ is said to be measurable if and only if there exists a sequence (f_n) in X^I such that, for every $n \in \mathbb{N}$, $f_n(\cdot)$ is a simple function and $\lim_{n\to\infty} f_n(t) = f(t)$ for a. e. $t \in I$.

(iii) Let $-\infty < a < b < \infty$ and $a < \tau \leqslant \infty$. A function $f : [a,b] \to X$ is said to be absolutely continuous if and only if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that, for any finite collection of open subintervals (a_i,b_i), $1 \leqslant i \leqslant k$ of $[a,b]$ with $\sum_{i=1}^k (b_i - a_i) < \delta$, we have $\sum_{i=1}^k \|f(b_i) - f(a_i)\| < \varepsilon$; a function $f : [a,\tau) \to X$ is said to be absolutely continuous if and only if for every $\tau_0 \in (a,\tau)$, the function $f_{|[a,\tau_0]} : [a,\tau_0] \to X$ is absolutely continuous.

If $f : I \to X$ and (f_n) is a sequence of measurable functions satisfying $\lim_{n\to\infty} f_n(t) = f(t)$ for a. e. $t \in I$, then the function $f(\cdot)$ is measurable too. The Bochner integral of a simple function $f : I \to X$, $f(t) = \sum_{i=1}^k z_i \chi_{\Omega_i}(t)$, $t \in I$ is defined by

$$\int_I f(t)\,dt := \sum_{i=1}^{k} z_i m(\Omega_i).$$

The definition of Bochner integral does not depend on the representation (1.1), as easily approved.

We say that a measurable function $f : I \to X$ is Bochner integrable if and only if there exists a sequence of simple functions (f_n) in X^I such that $\lim_{n\to\infty} f_n(t) = f(t)$ for a. e. $t \in I$ and

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\, dt = 0; \tag{1.2}$$

if this is the case, the Bochner integral of $f(\cdot)$ is defined by

$$\int_I f(t)\, dt := \lim_{n\to\infty} \int_I f_n(t)\, dt.$$

This definition does not depend on the choice of a sequence of simple functions (f_n) in X^I satisfying $\lim_{n\to\infty} f_n(t) = f(t)$ for a. e. $t \in I$ and (1.2). It is well known that $f : I \to X$ is Bochner integrable if and only if $f(\cdot)$ is measurable and the function $t \mapsto \|f(t)\|$, $t \in I$ is integrable. For any Bochner integrable function $f : [0,\infty) \to X$, we have $\int_0^\infty f(t)\, dt = \lim_{\tau\to+\infty} \int_0^\tau f_{|[0,\tau]}(t)\, dt$.

The space of all Bochner integrable functions from I into X is denoted by $L^1(I : X)$; endowed with the norm $\|f\|_1 := \int_I \|f(t)\|\, dt$, $L^1(I : X)$ is a Banach space. It is said that a function $f : [0,\infty) \to X$ is locally (Bochner) integrable if and only if $f(\cdot)_{|[0,\tau]}$ is Bochner integrable for every $\tau > 0$. The space of all locally integrable functions from $[0,\infty)$ into X is denoted by $L^1_{\mathrm{loc}}([0,\infty) : X)$. If $f : [a,b] \to X$ is Bochner integrable, where $-\infty < a < b < +\infty$, then the function $F(t) := \int_a^t f(s)\, ds$, $t \in [a,b]$ is absolutely continuous and $F'(t) = f(t)$ for a. e. $t \in [a,b]$. Basically, we will not distinguish a function and its restriction to any subinterval of its domain.

Theorem 1.1.2.

(i) *(The dominated convergence theorem) Suppose that (f_n) is a sequence of Bochner integrable functions from X^I and that there exists an integrable function $g : I \to \mathbb{R}$ such that $\|f_n(t)\| \leqslant g(t)$ for a. e. $t \in I$ and $n \in \mathbb{N}$. If $f : I \to X$ and $\lim_{n\to\infty} f_n(t) = f(t)$ for a. e. $t \in I$, then $f(\cdot)$ is Bochner integrable, $\int_I f(t)\, dt = \lim_{n\to\infty} \int_I f_n(t)\, dt$ and $\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\, dt = 0$.*

(ii) *(The Fubini theorem) Let I_1 and I_2 be segments in $\mathbb{R}$ and let $I = I_1 \times I_2$. Suppose that $F : I \to X$ is measurable and $\int_{I_1} \int_{I_2} \|f(s,t)\|\, dt\, ds < \infty$. Then $f(\cdot,\cdot)$ is Bochner integrable, the repeated integrals $\int_{I_1} \int_{I_2} f(s,t)\, dt\, ds$ and $\int_{I_2} \int_{I_1} f(s,t)\, ds\, dt$ exist and equal to the integral $\int_I f(s,t)\, ds\, dt$.*

Suppose now that $1 \leqslant p < \infty$ and $(\Omega, \mathcal{R}, \mu)$ is a measure space. By $L^p(\Omega : X)$ we denote the space of all strongly μ-measurable functions $f : \Omega \to X$ such that $\|f\|_p := (\int_\Omega \|f(\cdot)\|^p\, d\mu)^{1/p}$ is finite. The space $L^\infty(\Omega : X)$ consists of all strongly μ-measurable, essentially bounded functions; this space is a Banach space equipped with the norm

$\|f\|_\infty := \operatorname{ess\,sup}_{t\in\Omega} \|f(t)\|$, $f \in L^\infty(\Omega : X)$. Let us recall that we identify functions that are equal μ-almost everywhere on Ω. The famous Riesz–Fischer theorem states that $(L^p(\Omega : X), \|\cdot\|_p)$ is a Banach space for all $p \in [1,\infty]$; furthermore, $(L^2(\Omega : X), \|\cdot\|_2)$ is a Hilbert space. If $\lim_{n\to\infty} f_n = f$ in $L^p(\Omega : X)$, then there exists a subsequence (f_{n_k}) of (f_n) such that $\lim_{k\to\infty} f_{n_k}(t) = f(t)$ μ-almost everywhere. If the Banach space X is reflexive, then $L^p(\Omega : X)$ is reflexive for all $p \in (1,\infty)$ and its dual is isometrically isomorphic to $L^{p/(p-1)}(\Omega : X)$. We refer the reader to [82] and [631] for more details about the absolutely continuous functions. The space consisting of all X-valued functions that are absolutely continuous on any closed subinterval of $[0,\infty)$ will be denoted by $AC_{\mathrm{loc}}([0,\infty) : X)$.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ $(n,\ k \in \mathbb{N})$. By $C^k(\Omega : X)$ we denote the space of k-times continuously differentiable functions $f : \Omega \to X$. The space $L^p_{\mathrm{loc}}(\Omega : X)$ for $1 \leqslant p \leqslant \infty$ is defined in the usual way $(T,\ \tau > 0)$; $L^p_{\mathrm{loc}}(\Omega) \equiv L^p_{\mathrm{loc}}(\Omega : \mathbb{C})$.

Assume now that $k \in \mathbb{N}$ and $p \in [1,\infty]$. Then the Sobolev space $W^{k,p}(\Omega : X)$ consists of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every $i \in \mathbb{N}_k^0$ and for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant k$, we have $D^\alpha u \in L^p(\Omega, X)$. At this place, the derivative D^α is taken in the sense of distributions. By $W^{k,p}_{\mathrm{loc}}(\Omega : X)$ we denote the space of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in W^{k,p}(\Omega' : X)$.

We will use the following simple lemma.

Lemma 1.1.3. *Let $-\infty < a < b < \infty$, let $1 \leqslant p' < p'' < \infty$, and let $f \in L^{p''}([a,b] : X)$. Then $f \in L^{p'}([a,b] : X)$ and*

$$\left[\frac{1}{b-a}\int_a^b \|f(s)\|^{p'}\,ds\right]^{1/p'} \leqslant \left[\frac{1}{b-a}\int_a^b \|f(s)\|^{p''}\,ds\right]^{1/p''}.$$

Strongly continuous semigroups in Banach spaces

An operator family $(T(t))_{t\geqslant 0} \subseteq L(X)$ is said to be a strongly continuous semigroup if and only if the following holds:

(i) $T(0) = I$,
(ii) $T(t+s) = T(t)T(s)$, $t, s \geqslant 0$, and
(iii) the mapping $t \mapsto T(t)x$, $t \geqslant 0$ is continuous for every fixed $x \in X$.

The linear operator

$$A := \left\{(x,y) \in X \times X : \lim_{t\to 0+} \frac{T(t)x - x}{t} = y\right\} \tag{1.3}$$

is said to be the infinitesimal generator of $(T(t))_{t\geqslant 0}$. A strongly continuous semigroup (group) $(T(t))_{t\geqslant 0}$ is also said to be C_0-semigroup; if condition (i) is neglected, then the operator $T(0)$ is a projection and then we say that $(T(t))_{t\geqslant 0}$ is a degenerate C_0-semigroup.

In both cases, degenerate and non-degenerate, we know that there exist finite constants $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|T(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$. The famous Hille–Yosida theorem states that a linear operator A generates a non-degenerate strongly continuous semigroup $(T(t))_{t\geqslant 0}$ satisfying the estimate $\|T(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for some finite constants $M \geqslant 1$ and $\omega \geqslant 0$ if and only if A is closed, densely defined, $(\omega, \infty) \subseteq \rho(A)$ and

$$\|(\lambda - A)^{-n}\| \leqslant \frac{M}{(\lambda - \omega)^n}, \quad \lambda > \omega,\ n \in \mathbb{N}.$$

If not stated otherwise, we will always assume that a C_0-semigroup $(T(t))_{t\geqslant 0}$ is non-degenerate.

If $(T(t))_{t\in\mathbb{R}} \subseteq L(X)$ satisfies (i), (ii) for all $t,\ s \in \mathbb{R}$ and (iii) for $t \in \mathbb{R}$, then we say that $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group, C_0-group for short. Similarly as above, if condition (i) is neglected, then we say that $(T(t))_{t\in\mathbb{R}}$ is a degenerate strongly continuous group, degenerate C_0-group for short. The infinitesimal generator of $(T(t))_{t\in\mathbb{R}}$ is defined through (1.3); in the degenerate case, the infinitesimal generator is a closed multivalued linear operator on X (see Section 1.2 below).

For more details about the theory of strongly continuous semigroups, the reader may consult the monographs [410, 630, 633, 829] and the references quoted therein; for the theory of integrated semigroups and C-regularized semigroups, we refer the reader to [82, 348, 629, 630, 1092] and the references quoted therein.

Fixed point theorems

In this part, we remind the reader of the Banach contraction principle and its well known generalizations, the Bryant fixed point theorem and the Meer–Keeler fixed point theorem; for further information about the fixed point theory, the reader may consult the monographs [26] and [480].

Let (E, d) be a metric space. Then $T : E \to E$ is called a contraction mapping on E if and only if there exists a constant $q \in [0, 1)$ such that $d(T(x), T(y)) \leqslant qd(x, y)$ for all $x,\ y \in E$. We have the following.

Theorem 1.1.4 (The Banach contraction principle, 1922). *Let (E, d) be a complete metric space, and let $T : E \to E$ be a contraction mapping. Then T admits a unique fixed point x in X (i. e. $T(x) = x$).*

Theorem 1.1.5 (The Bryant fixed point theorem, 1968). *Let (E, d) be a complete metric space, and let $T : E \to E$ satisfy the condition that there is an integer $n \in \mathbb{N}$ such that $T^n : E \to E$ is a contraction mapping. Then T has a unique fixed point x in E.*

In their remarkable paper [759], A. Meir and E. Keeler have introduced the notion of a weakly uniformly strict contraction and proved the following fixed point theorem.

Theorem 1.1.6 (A. Meir and E. Keeler, 1969). *Suppose that (E, d) is a complete metric space and $T : E \to E$ satisfies that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in E$ we have*

$$\varepsilon \leqslant d(x, y) \leqslant \varepsilon + \delta \quad \textit{implies } d(Tx, Ty) < \varepsilon.$$

Then the mapping T has a unique fixed point ξ, and moreover, $\lim_{n\to+\infty} T^n x = \xi$ for any $x \in E$.

1.1.1 Lebesgue spaces with variable exponents $L^{p(x)}$

The monograph [377] by L. Diening, P. Harjulehto, P. Hästüso and M. Ruzicka is of invaluable importance in the study of Lebesgue spaces with variable exponents.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}$ be a nonempty subset and let $M(\Omega : X)$ be the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Furthermore, let $\mathcal{P}(\Omega)$ denote the vector space of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, we set

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geqslant 0,\ 1 \leqslant p(x) < \infty, \\ 0, & 0 \leqslant t \leqslant 1,\ p(x) = \infty, \\ \infty, & t > 1,\ p(x) = \infty, \end{cases}$$

and

$$\rho(f) := \int_{\Omega} \varphi_{p(x)}(\|f(x)\|)\, dx. \tag{1.4}$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent by

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \Big\}.$$

Equivalently,

$$L^{p(x)}(\Omega : X) = \{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \};$$

see, e. g., [377, p. 73]. For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ in the following way (see the doctoral dissertation of W. A. J. Luxemburg [739] for further information):

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\{\lambda > 0 : \rho(f/\lambda) \leqslant 1\}.$$

Equipped with the above norm, $L^{p(x)}(\Omega : X)$ becomes a Banach space (see, e. g., [377, Theorem 3.2.7] for the scalar-valued case), coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geqslant 1$ is a constant function. For any $p \in M(\Omega)$, we set

$$p^- := \operatorname{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname{esssup}_{x\in\Omega} p(x).$$

Define

$$C_+(\Omega) := \{p \in M(\Omega) : 1 < p^- \leqslant p(x) \leqslant p^+ < \infty \text{ for a. e. } x \in \Omega\}$$

and

$$D_+(\Omega) := \{p \in M(\Omega) : 1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty \text{ for a. e. } x \in \Omega\}.$$

For $p \in D_+(\Omega)$, the space $L^{p(x)}(\Omega : X)$ behaves nicely, with almost all fundamental properties of the Lebesgue space with constant exponent $L^p(\Omega : X)$ being retained; in this case, we know that the function $\rho(\cdot)$ given by (1.4) is modular in the sense of [377, Definition 2.1.1], and that

$$L^{p(x)}(\Omega : X) = \{f \in M(\Omega : X) : \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty\}.$$

Furthermore, if $p \in C_+(\Omega)$, then $L^{p(x)}(\Omega : X)$ is uniformly convex and thus reflexive ([421]).

We will use the following lemma (see, e. g., [377, Lemma 3.2.20, (3.2.22); Corollary 3.3.4; p. 77] for the scalar-valued case).

Lemma 1.1.7.
(i) *(The Hölder inequality) Let $p, q, r \in \mathcal{P}(\Omega)$ such that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leqslant 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) *Let Ω be of a finite Lebesgue's measure and let $p, q \in \mathcal{P}(\Omega)$ be such that $q \leqslant p$ a. e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$.*
(iii) *Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leqslant \|g\| \leqslant \|f\|$ a. e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leqslant \|f\|_{p(x)}$.*

We will use the following simple lemma, whose proof can be omitted.

Lemma 1.1.8. *Suppose that $f \in L^{p(x)}(\Omega : X)$ and $A \in L(X, Y)$. Then $Af \in L^{p(x)}(\Omega : Y)$ and $\|Af\|_{L^{p(x)}(\Omega:Y)} \leqslant \|A\| \cdot \|f\|_{L^{p(x)}(\Omega:X)}$.*

For additional details upon Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to [375, 376, 421, 797, 815, 891].

1.2 Multivalued linear operators

This section aims to present a brief synopsis of definitions and results from the theory of multivalued linear operators. For more details, we refer to the monograph [321] by R. Cross.

Suppose that X and Y are two Banach spaces. A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) if and only if the following hold:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;
(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then we say that $\mathcal{A}$ is an MLO in X. Let us recall that, for every $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear submanifold of Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$ and $N(\mathcal{A}) := \mathcal{A}^{-1}0 := \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ (we call that the range and kernel space of $\mathcal{A}$, respectively). The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It follows that $\mathcal{A}^{-1}$ is an MLO in X, and that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i. e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective.

Assuming that $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs, we define its sum $\mathcal{A} + \mathcal{B}$ by $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. Clearly, $\mathcal{A} + \mathcal{B}$ is likewise an MLO.

Suppose now that $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ are two MLOs, where Z is a complex Banach space. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. $\mathcal{B}\mathcal{A} : X \to P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$.

The integer powers of an MLO $\mathcal{A} : X \to P(X)$ are defined inductively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set

$$D(\mathcal{A}^n) := \{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\}$$

and

$$\mathcal{A}^n x := (\mathcal{A}\mathcal{A}^{n-1})x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

Assume that $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are two MLOs. Then the inclusion $\mathcal{A} \subseteq \mathcal{B}$ is equivalent to saying that $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

It is said that an MLO operator $\mathcal{A} : X \to P(Y)$ is closed if and only if for any sequences (x_n) in $D(\mathcal{A})$ and (y_n) in Y such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ the suppositions $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Assume that $\mathcal{A} : X \to P(Y)$ is an MLO. Then $\overline{\mathcal{A}} : X \to P(Y)$ is an MLO as well, so that any MLO has a closed linear extension, in contrast to the usually analyzed single-valued linear operators.

Let $\mathcal{A}$ be an MLO in X and $C \in L(X)$. The C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$. If $C = I$, then we say that $\rho(\mathcal{A}) \equiv \rho_C(\mathcal{A})$ is the resolvent set of $\mathcal{A}$ and the mapping $\lambda \mapsto R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ ($\lambda \in \rho(\mathcal{A})$). For the generalized resolvent equations and the analytical properties of C-resolvents of multivalued linear operators, we refer the reader to [633].

Suppose now that $(-\infty, 0] \subseteq \rho(\mathcal{A})$ and that there exist finite numbers $M \geqslant 1$ and $\beta \in (0,1]$ such that

$$\|R(\lambda : \mathcal{A})\| \leqslant M(1 + |\lambda|)^{-\beta}, \quad \lambda \leqslant 0.$$

Then there are two positive numbers $c > 0$ and $M_1 > 0$ such that the resolvent set of $\mathcal{A}$ contains an open region $\Omega = \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant (2M_1)^{-1}(c - \operatorname{Re}\lambda)^{\beta},\ \operatorname{Re}\lambda \leqslant c\}$ of complex plane around the half-line $(-\infty, 0]$, where we have the estimate $\|R(\lambda : \mathcal{A})\| = O((1 + |\lambda|)^{-\beta})$, $\lambda \in \Omega$. Let Γ' be the upwards oriented curve $\{\xi \pm i(2M_1)^{-1}(c - \xi)^{\beta} : -\infty < \xi \leqslant c\}$. Following A. Favini and A. Yagi [431], we define the fractional power

$$\mathcal{A}^{-\theta} := \frac{1}{2\pi i} \int_{\Gamma'} \lambda^{-\theta}(\lambda - \mathcal{A})^{-1}\, d\lambda \in L(X),$$

for $\theta > 1 - \beta$. Set $\mathcal{A}^{\theta} := (\mathcal{A}^{-\theta})^{-1}$ ($\theta > 1 - \beta$). Then the semigroup properties $\mathcal{A}^{-\theta_1}\mathcal{A}^{-\theta_2} = \mathcal{A}^{-(\theta_1+\theta_2)}$ and $\mathcal{A}^{\theta_1}\mathcal{A}^{\theta_2} = \mathcal{A}^{\theta_1+\theta_2}$ hold for $\theta_1,\ \theta_2 > 1 - \beta$ (it is worth recalling that the fractional power $\mathcal{A}^{\theta}$ is not generally injective and the meaning of $\mathcal{A}^{\theta}$ is understood in the MLO sense for $\theta > 1 - \beta$).

For any $\theta \in (0,1)$, the vector space

$$X_{\mathcal{A}}^{\theta} := \Big\{x \in X : \sup_{\xi > 0} \xi^{\theta}\big\|\xi(\xi + \mathcal{A})^{-1}x - x\big\| < \infty\Big\},$$

endowed with the norm

$$\|\cdot\|_{X_{\mathcal{A}}^{\theta}} := \|\cdot\| + \sup_{\xi > 0} \xi^{\theta}\big\|\xi(\xi + \mathcal{A})^{-1} \cdot - \cdot\big\|,$$

is a Banach space.

We will use conditions (P) and (QP) henceforth:

(P) There exist finite constants $c,\ M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Psi_c := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant -c(|\operatorname{Im}\lambda| + 1)\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leqslant M(1 + |\lambda|)^{-\beta}, \quad \lambda \in \Psi.$$

(QP) There exist finite numbers $0 < \beta \leqslant 1$, $0 < d \leqslant 1$, $M > 0$ and $0 < \eta' < \eta'' \leqslant 1$ such that

$$\Psi_{d,\pi\eta''/2} := \{\lambda \in \mathbb{C} : |\lambda| \leqslant d \text{ or } \lambda \in \overline{\Sigma_{\pi\eta''/2}}\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leqslant M(1 + |\lambda|)^{-\beta}, \quad \lambda \in \Psi_{d,\pi\eta''/2}.$$

Hence, the resolvent set of a multivalued linear operator $\mathcal{A}$ satisfying (QP) can be strictly contained in an acute angle. In the single-valued linear case, the class of almost sectorial operators $A = \mathcal{A}$ satisfying condition (P) is crucially important; for more details about almost sectorial operators and their applications, we refer the reader to the papers [833] by F. Periago, [834] by F. Periago and B. Straub, the monographs [630, 631] and the references cited therein.

1.3 Fractional calculus and solution operator families

Fractional calculus and fractional differential equations play an important role in various fields of theoretical and applied science, such as engineering, physics, chemistry, mechanics, electricity, economics, control theory and image processing. For further information about fractional calculus and fractional differential equations, we refer the reader to the monographs S. Abbas, M. Benchohra, G. M. N'Guérékata [6, 7], K. Diethelm [378], C. Goodrich, A. C. Peterson [476], A. A. Kilbas, H. M. Srivastava, J. J. Trujillo [607], V. Kiryakova [612], F. Mainardi [746], S. G. Samko, A. A. Kilbas, O. I. Marichev [892] and M. Kostić [629–633], and to the doctoral dissertation of E. Bazhlekova [133].

Suppose that $\alpha > 0$, $m = \lceil \alpha \rceil$ and $I = (0, T)$ for some $T \in (0, \infty]$. Then the Riemann–Liouville fractional integral J_t^α of order α is defined by

$$J_t^\alpha f(t) := (g_\alpha * f)(t), \quad f \in L^1(I : X),\ t \in I.$$

The Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined for those functions $u \in C^{m-1}([0,\infty) : X)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0,\infty) : X)$, by

$$\mathbf{D}_t^\alpha u(t) := \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right)\right].$$

It is worth noticing that the existence of Caputo fractional derivative $\mathbf{D}_t^{\alpha}u$ for $t \geqslant 0$ implies the existence of Caputo fractional derivative $\mathbf{D}_t^{\zeta}u$ for $t \geqslant 0$ and any $\zeta \in (0,\alpha)$. At some places, we will use a slightly weakened notion of Caputo fractional derivatives, as explicitly emphasized.

The Riemann–Liouville fractional derivative D_t^{α} of order α is defined for those functions $f \in L^1(I : X)$ satisfying $g_{m-\alpha} * f \in W^{m,1}((0,\infty) : X)$, by

$$D_t^{\alpha} f(t) := \frac{d^m}{dt^m} J_t^{m-\alpha} f(t), \quad t \in I.$$

The Riemann–Liouville fractional integrals and derivatives satisfy the following equalities:

$$J_t^{\alpha} J_t^{\beta} f(t) = J_t^{\alpha+\beta} f(t), \quad D_t^{\alpha} J_t^{\alpha} f(t) = f(t),$$

for $f \in L^1(I : X)$ and

$$J_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t)$$

for any $f \in L^1(I : X)$ with $g_{m-\alpha} * f \in W^{m,1}(I : X)$.

The Weyl–Liouville fractional derivative $D_{t,+}^{\gamma} u(t)$ of order $\gamma \in (0,1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that

$$t \mapsto \int_{-\infty}^{t} g_{1-\gamma}(t-s) u(s)\, ds, \quad t \in \mathbb{R},$$

is a well-defined continuously differentiable mapping, by

$$D_{t,+}^{\gamma} u(t) := \frac{d}{dt} \int_{-\infty}^{t} g_{1-\gamma}(t-s) u(s)\, ds, \quad t \in \mathbb{R}.$$

Set $D_{t,+}^{1} u(t) := -(d/dt)u(t)$. For more details about the subject, the reader may consult the article [798].

The Mittag-Leffler functions and the Wright functions play an incredible role in fractional calculus. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C};$$

set, for short, $E_{\alpha}(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$.

The asymptotic behavior of the entire function $E_{\alpha,\beta}(z)$ is given by the following important result (see, e. g., [1028, Theorem 1.1]).

Theorem 1.3.1. *Let* $0 < \sigma < \frac{1}{2}\pi$. *Then, for every* $z \in \mathbb{C} \setminus \{0\}$ *and* $m \in \mathbb{N} \setminus \{1\}$,

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_{s} Z_s^{1-\beta} e^{Z_s} - \sum_{j=1}^{m-1} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O(|z|^{-m}),$$

where Z_s *is defined by* $Z_s := z^{1/\alpha} e^{2\pi i s/\alpha}$ *and the first summation is taken over all those integers s satisfying* $|\arg(z) + 2\pi s| < \alpha(\frac{\pi}{2} + \sigma)$.

Let $\gamma \in (0,1)$. Then the Wright function $\Phi_\gamma(\cdot)$ is defined by

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1 - \gamma - \gamma n)}, \quad z \in \mathbb{C}.$$

Let us recall that $\Phi_\gamma(\cdot)$ is an entire function and that:
(i) $\Phi_\gamma(t) \geqslant 0$, $t \geqslant 0$,
(ii) $\int_0^\infty e^{-\lambda t} \gamma s t^{-1-\gamma} \Phi_\gamma(t^{-\gamma} s)\, dt = e^{-\lambda^\gamma s}$, $\lambda \in \mathbb{C}_+$, $s > 0$, and
(iii) $\int_0^\infty t^r \Phi_\gamma(t)\, dt = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$.

The asymptotic expansion of the Wright function $\Phi_\gamma(\cdot)$, as $|z| \to \infty$ in the sector $|\arg(z)| \leqslant \min((1-\gamma)3\pi/2, \pi) - \varepsilon$, is given by

$$\Phi_\gamma(z) = Y^{\gamma - 1/2} e^{-Y} \left(\sum_{m=0}^{M-1} A_m Y^{-M} + O(|Y|^{-M}) \right),$$

where $Y = (1-\gamma)(\gamma^\gamma z)^{1/(1-\gamma)}$, $M \in \mathbb{N}$ and A_m are certain real numbers (see, e. g., [133]).

Solution operator families

Suppose now that $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$.

We will use the following general definition.

Definition 1.3.2 ([633]). Suppose $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.
(i) Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)} \subseteq L(Y,X) \times L(X)$ if and only if the mappings $t \mapsto R_1(t)y$, $t \geqslant 0$ and $t \mapsto R_2(t)x$, $t \in [0,\tau)$ are continuous for every fixed $x \in X$ and $y \in Y$, and the following conditions hold:

$$\left(\int_0^t a(t-s) R_1(s)y\, ds, R_1(t)y - k(t)C_1 y \right) \in \mathcal{A}, \quad t \in [0,\tau),\ y \in Y \quad \text{and} \tag{1.5}$$

$$\int_0^t a(t-s) R_2(s)y\, ds = R_2(t)x - k(t)C_2 x, \quad \text{whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}. \tag{1.6}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ if and only if (1.5) holds.

(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ if and only if (1.6) holds.

Let us recall that $R(R_1(0) - k(0)C_1) \subseteq \mathcal{A}0$ and, if $a(t)$ is a kernel on $[0,\tau)$, then $R_2(t)\mathcal{A}$ is single-valued for any $t \in [0,\tau)$ and $R_2(t)y = 0$ for any $y \in \mathcal{A}0$ and $t \in [0,\tau)$.

Definition 1.3.3 ([633]). Suppose that $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ if and only if $(R(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized C-uniqueness family having $\mathcal{A}$ as subgenerator, $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in [0,\tau))$.

If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geqslant 0$, then we also say that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family. We will accept a similar terminology for mild (a,k)-regularized C_1-existence families and mild (a,k)-regularized C_2-uniqueness families.

Suppose that $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then we have

$$\begin{aligned}&(a * R_2)(s)R_1(t)y - R_2(s)(a * R_1)(t)y\\&\quad = k(t)(a * R_2)(s)C_1 y - k(s)C_2(a * R_1)(t)y, \quad t \in [0,\tau),\ y \in Y.\end{aligned}$$

The integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$) is defined by

$$\mathcal{A}_{\mathrm{int}} := \left\{(x,y) \in X \times X : R_2(t)x - k(t)C_2 x = \int_0^t a(t-s)R_2(s)y\,ds,\ t \in [0,\tau)\right\};$$

we define the integral generator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ in the same way as above. The integral generator $\mathcal{A}_{\mathrm{int}}$ is an MLO in X which extends any subgenerator of $(R_2(t))_{t\in[0,\tau)}$ $((R(t))_{t\in[0,\tau)})$ in the set theoretical sense; furthermore, the assumption $R_2(t)C_2 = C_2R_2(t)$, $t \in [0,\tau)$ implies that $C_2^{-1}\mathcal{A}_{\mathrm{int}}C_2 = \mathcal{A}_{\mathrm{int}}$ so that $C^{-1}\mathcal{A}_{\mathrm{int}}C = \mathcal{A}_{\mathrm{int}}$ for resolvent families.

Concerning the vector-valued Laplace transform, we can recommend for the reader the monograph [82] by W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander (cf. also [633, 1040]). The following condition on a scalar-valued function $k(t)$ will be used:

(P1) $k(t)$ is Laplace transformable, i. e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$. Put $\operatorname{abs}(k) := \inf\{\operatorname{Re}\lambda : \tilde{k}(\lambda) \text{ exists}\}$, and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

We have the following ([633]).

Theorem 1.3.4. *Suppose $\mathcal{A}$ is a closed* MLO *in X, $C_1 \in L(Y,X)$, $C_2 \in L(X)$, C_2 is injective, $\omega_0 \geqslant 0$ and $\omega \geqslant \max(\omega_0, \operatorname{abs}(|a|), \operatorname{abs}(k))$.*

(i) *Let $(R_1(t), R_2(t))_{t\geqslant 0} \subseteq L(Y,X) \times L(X)$ be strongly continuous, and let the family $\{e^{-\omega t}R_i(t) : t \geqslant 0\}$ be equicontinuous for $i = 1,2$.*

(a) *Suppose $(R_1(t), R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective, $R(C_1) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$,*

$$\tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1}C_1 y = \int_0^\infty e^{-\lambda t}R_1(t)y\,dt, \quad y \in Y, \tag{1.7}$$

$$\left\{\frac{1}{\tilde{a}(z)} : \operatorname{Re} z > \omega,\ \tilde{k}(z)\tilde{a}(z) \neq 0\right\} \subseteq \rho_{C_1}(\mathcal{A}), \tag{1.8}$$

and

$$\tilde{k}(\lambda)C_2 x = \int_0^\infty e^{-\lambda t}\big[R_2(t)x - (a * R_2)(t)y\big]\,dt \quad \text{whenever } (x,y) \in \mathcal{A}. \tag{1.9}$$

(b) *Let* (1.8) *hold, and let* (1.7) *and* (1.9) *hold for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. Then $(R_1(t), R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$.*

(ii) *Let $(R_1(t))_{t\geqslant 0}$ be strongly continuous, and let the family $\{e^{-\omega t}R_1(t) : t \geqslant 0\}$ be equicontinuous. Then $(R_1(t))_{t\geqslant 0}$ is a mild (a,k)-regularized C_1-existence family with a subgenerator $\mathcal{A}$ if and only if for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, one has $R(C_1) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$ and*

$$\tilde{k}(\lambda)C_1 y \in \big(I - \tilde{a}(\lambda)\mathcal{A}\big)\int_0^\infty e^{-\lambda t}R_1(t)y\,dt, \quad y \in Y.$$

(iii) *Let $(R_2(t))_{t\geqslant 0}$ be strongly continuous, and let the family $\{e^{-\omega t}R_2(t) : t \geqslant 0\}$ be equicontinuous. Then $(R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized C_2-uniqueness family with a subgenerator $\mathcal{A}$ if and only if for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective and* (1.9) *holds.*

Theorem 1.3.5. *Let* $(R(t))_{t\geqslant 0} \subseteq L(X)$ *be a strongly continuous operator family such that there exists* $\omega \geqslant 0$ *satisfying the condition that the family* $\{e^{-\omega t}R(t) : t \geqslant 0\}$ *is equicontinuous, and let* $\omega_0 > \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k))$. *Suppose that* $\mathcal{A}$ *is a closed* MLO *in* X *and* $C\mathcal{A} \subseteq \mathcal{A}C$.

(i) *Assume that* $\mathcal{A}$ *is a subgenerator of the global* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (1.5) *for all* $x = y \in X$. *Then, for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, *the operator* $I - \tilde{a}(\lambda)\mathcal{A}$ *is injective,* $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$, *and*

$$\tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in X,\ \operatorname{Re}\lambda > \omega_0,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0, \tag{1.10}$$

$$\left\{\frac{1}{\tilde{a}(\lambda)} : \operatorname{Re}\lambda > \omega_0,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\right\} \subseteq \rho_C(\mathcal{A}), \tag{1.11}$$

and $R(s)R(t) = R(t)R(s),\ t,\ s \geqslant 0$.

(ii) *Assume* (1.10)–(1.11). *Then* $\mathcal{A}$ *is a subgenerator of the global* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (1.5) *for all* $x = y \in X$ *and* $R(s)R(t) = R(t)R(s)$, $t,\ s \geqslant 0$.

Abstract degenerate Volterra integro-differential equations with nonlocal or impulsive conditions have received much attention recently. See, e. g., the research articles [699] by F. Li, J. Liang, H. K. Xu, [709] by J. Liang, Y. Mu, T. J. Xiao, [1091] by Z. H. Zhao, Y. K. Chang and the list of references quoted in the monograph [258]. Mention should be made of the research monograph [1102] by Y. Zhou as well.

Part I: **Almost periodic type functions and solutions to integro-differential equations**

In this part, which consists of Chapters 2–5, we investigate vector-valued almost periodic type functions and almost periodic type solutions of the abstract Volterra integro-differential equations in Banach spaces, which could be degenerate or non-degenerate in the time variable. Special attention is paid to the analysis of various classes of abstract semilinear fractional Cauchy inclusions.

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2 Almost periodic type functions

2.1 Almost periodic functions and asymptotically almost periodic functions

As already mentioned, the notion of almost periodicity was introduced by the famous Danish mathematician H. Bohr around 1925 [196] and later generalized by many others (cf. [34, 309, 311, 364, 442, 492, 493, 538, 697, 934, 1067] for more details on the subject). Here we would like to note that P. Bohl [194, 195] (and E. Esclangon [411–413]) had created the theory of quasi-periodic functions (an important class of almost periodic functions). In his dissertation [194], P. Bohl laid the theoretical foundations for the study of quasi-periodic functions; after that, in the research article [195], he significantly advanced this theory, proving several essential theorems for quasi-periodic functions.

Suppose that $I = \mathbb{R}$ or $I = [0, \infty)$, and $f : I \to X$ is continuous. For any given real number $\varepsilon > 0$, we say that a real number $\tau > 0$ is an ε-period for $f(\cdot)$ if and only if

$$\|f(t+\tau) - f(t)\| \leqslant \varepsilon, \quad t \in I. \tag{2.1}$$

By $\vartheta(f, \varepsilon)$ we denote the set of all ε-periods for $f(\cdot)$. It is said that $f(\cdot)$ is almost periodic if and only if for each $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \varepsilon)$. It is said that $f(\cdot)$ is weakly almost periodic if and only if for each $x^* \in X^*$ the function $x^*(f(\cdot))$ is almost periodic. Any weakly almost periodic function $f \in BUC(I : X)$ with a relatively compact range in X is almost periodic; see, e. g., [82, Proposition 4.5.12].

By $AP(I : X)$ we denote the space consisting of all almost periodic functions from the interval I into X; equipped with the sup-norm, $AP(I : X)$ is a Banach space. This space contains the space $P_c(I : X)$ consisting of all continuous functions $f : I \to X$ of period $c > 0$; by $P(I : X)$ we denote the space consisting of all continuous functions $f : I \to X$ for which there exists $c > 0$ such that $f(\cdot)$ is of period c.

It is well known that the space $AP(I : X)$ is the closure of the set of all trigonometric polynomials in the space $BUC(I : X)$. If α and β are real numbers such that $\beta \neq 0$ and α/β is an irrational real number, then the trigonometric polynomial $t \mapsto f_{\alpha,\beta}(t) \equiv e^{i\alpha t} + e^{i\beta t}$, $t \in \mathbb{R}$ is not periodic but it is almost periodic. The almost periodicity of this mapping has been proved by H. Bohr as follows. Clearly, $\alpha \neq 0$ and the number p_1/p_2 is irrational, where $p_1 = 2\pi/|\alpha|$ and $p_2 = 2\pi/|\beta|$. As a consequence of general theorem in the theory of Diophantine approximations, for any given number $\varepsilon > 0$ in advance, we can find the existence of two sequences (a_n) and (b_n) of arbitrarily large integers such that $|a_n p_1 - b_n p_2| < \varepsilon$ for all $n \in \mathbb{N}$ and that the sequences $(a_{n+1} - a_n)$ and $(b_{n+1} - b_n)$ are bounded. If $n \in \mathbb{N}$ and τ is any real number between $a_n p_1$ and $b_n p_2$, then τ will be $(2(|\alpha| + |\beta|)\varepsilon)$-period of the function $f_{\alpha,\beta}(\cdot)$, as it can be simply approved. Using this fact and the boundedness of sequences $(a_{n+1} - a_n)$ and $(b_{n+1} - b_n)$, we obtain the required.

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Before proceeding, we would like to recommend the articles [83, 84] by V. V. Arestov, [85] by V. V. Arestov, P. Yu. Glazyrina, [391] by M. Donovski and the references quoted therein for the basic results concerning inequalities for trigonometric polynomials and their (fractional) derivatives.

For the sequel, we need some preliminaries from [119]. The translation semigroup $(W(t))_{t\geqslant 0}$ on $AP([0,\infty) : X)$, defined by $[W(t)f](s) := f(t+s)$, $t \geqslant 0$, $s \geqslant 0$, $f \in AP([0,\infty) : X)$ is consisting solely of surjective isometries $W(t)$ $(t \geqslant 0)$ and can be extended to a C_0-group $(W(t))_{t\in\mathbb{R}}$ of isometries on $AP([0,\infty) : X)$, where $W(-t) := W(t)^{-1}$ for $t > 0$. Moreover, the mapping $\mathbb{E} : AP([0,\infty) : X) \to AP(\mathbb{R} : X)$, defined by

$$[\mathbb{E}f](t) := \big[W(t)f\big](0), \quad t \in \mathbb{R},\ f \in AP\big([0,\infty) : X\big),$$

is a linear surjective isometry and $\mathbb{E}f(\cdot)$ is the unique almost periodic extension of a function $f(\cdot)$ from $AP([0,\infty) : X)$ to the whole real line. Let us recall that $[\mathbb{E}(Bf)] = B(\mathbb{E}f)$ for all $B \in L(X)$ and $f \in AP([0,\infty) : X)$.

In the following theorem, we collect the fundamental properties of almost periodic vector-valued functions; by c_0 we denote the Banach space of all numerical sequences tending to zero, equipped with the sup-norm.

Theorem 2.1.1. *Let $f \in AP(I : X)$. Then the following hold:*

(i) $f \in BUC(I : X)$*;*

(ii) *if $g \in AP(I : X)$, $h \in AP(I : \mathbb{C})$, α, $\beta \in \mathbb{C}$, then $\alpha f + \beta g$ and $hf \in AP(I : X)$;*

(iii) *Bohr's transform of $f(\cdot)$,*

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\, ds,$$

exists for all $r \in \mathbb{R}$ and

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_\alpha^{t+\alpha} e^{-irs} f(s)\, ds$$

for all $\alpha \in I$, $r \in \mathbb{R}$. The element $P_r(f)$ is called the Bohr coefficient or the Bohr–Fourier coefficient of $f(\cdot)$;

(iv) *if $P_r(f) = 0$ for all $r \in \mathbb{R}$, then $f(t) = 0$ for all $t \in I$;*

(v) *Bohr's spectrum $\sigma(f) := \{r \in \mathbb{R} : P_r(f) \neq 0\}$ is at most countable;*

(vi) *if X does not contain an isomorphic copy of c_0, $I = \mathbb{R}$ and $g(t) = \int_0^t f(s)\, ds$ $(t \in \mathbb{R})$ is bounded, then $g \in AP(\mathbb{R} : X)$;*

(vii) *if $(g_n)_{n\in\mathbb{N}}$ is a sequence in $AP(I : X)$ and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to g, then $g \in AP(I : X)$;*

(viii) *if $f' \in BUC(I : X)$, then $f' \in AP(I : X)$;*

(ix) *(Spectral synthesis) $f \in \overline{\operatorname{span}\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\}}$;*

(x) *$R(f)$ is relatively compact in X;*

(xi) *(Supremum formula) we have*

$$\|f\|_\infty = \sup_{t\geq t_0}\|f(t)\|, \quad t_0 \in I;$$

(xii) *(Convolution invariance) if* $I = \mathbb{R}$ *and* $g \in L^1(\mathbb{R})$*, then* $g * f \in AP(\mathbb{R} : X)$*, where*

$$(g * f)(t) = \int_{-\infty}^{\infty} g(t-s)f(s)\, ds, \quad t \in \mathbb{R};$$

(xiii) *if* $n \in \mathbb{N}$ *and* $f_1 \in AP(I : X_1), \ldots, f_n \in AP(I : X_n)$*, then* $(f_1, \ldots, f_n) \in AP(I : X_1 \times \cdots \times X_n)$*. Here,* X_i *is a complex Banach space for all* $i = 1, \ldots, n$*;*

(xiv) *if* $f_1 \in AP(I : X_1), \ldots, f_n \in AP(I : X_n)$*, then for each* $\varepsilon > 0$ *there exists a common relatively dense set* $\vartheta(f_1, \ldots, f_n, \varepsilon)$ *of* ε*-periods for any of these functions. Here,* X_i *is a complex Banach space for all* $i = 1, \ldots, n$*;*

(xv) *(Bochner's criterion) Let* $I = \mathbb{R}$*. Then* $f(\cdot)$ *is almost periodic if and only if for any real sequence* (b_n) *there exists a subsequence* (a_n) *of* (b_n) *such that* $(f(\cdot + a_n))$ *converges in* $BUC(\mathbb{R} : X)$.

In connection with the point (vi), it should be noted that the necessary and sufficient condition for X to contain c_0 is given in [82, Theorem 4.6.14]; the importance of such condition has been recognized already by H. Bohr and later employed frequently (see, e. g., the formulation of Kadet's theorem [82, Theorem 4.6.11]). In [572], M. I. Kadets has also shown that a Banach space X contains c_0 (sometimes we also say for such a Banach space X that is perfect) if and only if for any $f \in AP(\mathbb{R} : X)$ we see that the boundedness of the first anti-derivative of $f(\cdot)$ implies its almost periodicity. Let us recall that A. I. Perov and T. K. Hai have proved, in [835, Theorem 1], that any classical solution of the abstract Cauchy problem $u'(t) = iAu(t)$, $t \in \mathbb{R}$ is almost periodic provided that the space X is perfect, the bounded linear operator $A \in L(X)$ is completely continuous (i. e., it maps every relatively weakly compact subset of X into a relatively compact subset of X) and generates a uniformly bounded C_0-group of operators. In the general case, this result does not hold if the underlying Banach space X is not perfect. For more details regarding the anti-derivatives of almost periodic functions, we also refer the reader to the research article [566] by R. A. Johnson.

Example 2.1.2 ([55]). Suppose that $X := l_\infty$ and (λ_n) is a sequence of positive real numbers such that $\lim_{n\to\infty} \lambda_n = 0$. Then the function $f : \mathbb{R} \to X$, given by $f(t) := (\lambda_n \cos(\lambda_n t))$, $t \in \mathbb{R}$, is almost periodic, its first anti-derivative $F(t) = (\sin(\lambda_n t))$, $t \in \mathbb{R}$, is bounded but not almost periodic.

By $AP_\Lambda(I : X)$, where Λ is a non-empty subset of I, we denote the vector subspace of $AP(I : X)$ consisting of all functions $f \in AP(I : X)$ satisfying the requirement that $\sigma(f) \subseteq \Lambda$; $AP_\Lambda(I : X)$ is a closed subspace of $AP(I : X)$ and therefore a Banach space. In [978], M. F. Timan and Yu. Kh. Khasanov have proved an analogue of the Jackson theorem for almost periodic functions with an arbitrary spectrum (see, e. g., [1066] and the references cited therein); for numerous equivalent criteria stating the necessary and sufficient conditions for the almost periodicity of a given function, we refer the reader to [631] and references quoted therein. It is well known that if $f : I \to X$ is a continuous c-periodic function, where $c > 0$, then $\sigma(f) \subseteq 2\pi\mathbb{Z}/c$.

In [446, Corollary 3], A. Fischer has proved that, for every almost periodic nonconstant function $f : \mathbb{R} \to X$, we have $\operatorname{diam}(R(f)) > \|f - P_0(f)\|_\infty$, where $\operatorname{diam}(X')$ denotes the diameter of subset $X' \subseteq X$; in the same paper, he has established several interesting results concerning the class of almost periodic vector-valued functions which can be approximated, with an arbitrarily accuracy given in advance, by continuous periodic functions uniformly on $\mathbb{R}$. We will also recall the following property of almost periodic functions which is important in the analysis of the existence of positive almost periodic solutions for a class of hematopoiesis models in mathematical biology.

Example 2.1.3 (see [388, Lemma 1.3(g)]). Suppose that $f : \mathbb{R} \to X$ and $g : \mathbb{R} \to \mathbb{R}$ are almost periodic functions and $\inf_{t\in\mathbb{R}} g(t) > 0$. Then the function

$$F(t) := \int_{-\infty}^{t} e^{-\int_s^t g(s)\,ds} f(s)\,ds, \quad t \in \mathbb{R}$$

is almost periodic.

In the case that $I = [0,\infty)$, the notion of asymptotical almost periodicity was introduced by A. S. Kovanko [668] in 1929 and later rediscovered, in a slightly different form, by M. Fréchet [453] in 1941 (for comprehensive information about the subject, we refer to [129, 130, 236, 269, 364, 492, 493, 881, 882, 955, 961, 1042, 1078]). Following M. Fréchet, a function $f \in C_b(I : X)$ is said to be asymptotically almost periodic if and only if for every $\varepsilon > 0$ we can find numbers $l > 0$ and $M > 0$ such that every subinterval of I of length l contains, at least, one number τ such that $\|f(t+\tau)-f(t)\| \leqslant \varepsilon$ provided $|t|,\ |t+\tau| \geqslant M$. The space consisting of all asymptotically almost periodic functions from I into X is denoted by $\mathrm{AAP}(I : X)$. It is well known that (see W. M. Ruess, W. H. Summers [880–882] for the case that $I = [0,\infty)$ and C. Zhang [1077, 1078] for the case that $I = \mathbb{R}$) the following statements are equivalent:

(i) $f \in \mathrm{AAP}(I : X)$.
(ii) There exist uniquely determined functions $g \in \mathrm{AP}(\mathbb{R} : X)$ and $\phi \in C_0(I : X)$ such that $f = g + \phi$.

The functions g and ϕ from (ii) are called the principal and corrective terms of the function f, respectively. If there exist functions $g \in P(\mathbb{R} : X)$ (of period $c > 0$) and $\phi \in C_0(I : X)$ such that $f = g + \phi$, then we say that $f(\cdot)$ is asymptotically periodic (asymptotically c-periodic).

By $C_0(I \times Y : X)$ we denote the space of all continuous functions $h : I \times Y \to X$ such that $\lim_{|t|\to\infty} h(t,y) = 0$ uniformly for y in any compact subset of Y. A continuous function $f : I \times Y \to X$ is called uniformly continuous on bounded sets, uniformly for $t \in I$ if and only if for every $\varepsilon > 0$ and every bounded subset K of Y there exists a number $\delta_{\varepsilon,K} > 0$ such that $\|f(t,x)-f(t,y)\| \leqslant \varepsilon$ for all $t \in I$ and all $x, y \in K$ satisfying the requirement that $\|x-y\| \leqslant \delta_{\varepsilon,K}$. If $f : I\times Y \to X$, then we define $\hat{f} : I\times Y \to L^p([0,1] : X)$ by $\hat{f}(t,y) := f(t+\cdot,y)$, $t \geqslant 0$, $y \in Y$.

The following definition and related composition principle can be found, e. g., in [631].

Definition 2.1.4. Let $1 \leqslant p < \infty$.

(i) A function $f : I \times Y \to X$ is called almost periodic if and only if $f(\cdot,\cdot)$ is bounded, continuous and for every $\varepsilon > 0$ and every compact $K \subseteq Y$ there exists $l(\varepsilon,K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\varepsilon,K)$ contains a number τ with the property that $\|f(t+\tau,y) - f(t,y)\| \leqslant \varepsilon$ for all $t \in I$, $y \in K$. The collection of such functions will be denoted by $\mathrm{AP}(I \times Y : X)$.

(ii) A function $f : I \times Y \to X$ is said to be asymptotically almost periodic if and only if it is bounded continuous and admits a decomposition $f(t) = g(t) + q(t)$, $t \in I$, where $g \in \mathrm{AP}(\mathbb{R}\times Y : X)$ and $q \in C_0(I\times Y : X)$. Denote by $\mathrm{AAP}(I\times Y : X)$ the vector space consisting of all such functions.

Theorem 2.1.5.

(i) *Let* $f \in \mathrm{AP}(I \times Y : X)$ *and* $h \in \mathrm{AP}(I : Y)$. *Then the mapping* $t \mapsto f(t,h(t))$, $t \in I$ *belongs to the space* $\mathrm{AP}(I : X)$.

(ii) *Let* $f \in \mathrm{AAP}(I \times Y : X)$ *and* $h \in \mathrm{AAP}(I : Y)$. *Then the mapping* $t \mapsto f(t,h(t))$, $t \geqslant 0$ *belongs to the space* $\mathrm{AAP}(I : X)$.

Let us recall that $f(\cdot)$ is anti-periodic if and only if there exists $p > 0$ such that $f(x+p) = -f(x)$, $x \in I$. Any such function needs to be periodic, as it can be easily proved. Given $\varepsilon > 0$, we call $\tau > 0$ an ε-antiperiod for $f(\cdot)$ if and only if

$$\|f(t+\tau)+f(t)\| \leqslant \varepsilon, \quad t \in I.$$

By $\vartheta_{ap}(f,\varepsilon)$ we denote the set of all ε-antiperiods for $f(\cdot)$. The notion of an almost anti-periodic function was introduced by D. N. Cheban [270] in 1980 (see also D. N. Cheban, I. N. Cheban [273]) and rediscovered 40 years later by M. Kostić and D. Velinov in [666] (see [666, Definition 2.1]):

Definition 2.1.6. It is said that $f(\cdot)$ is almost anti-periodic if and only if for each $\varepsilon > 0$ the set $\vartheta_{ap}(f,\varepsilon)$ is relatively dense in $[0,\infty)$.

We know that any almost anti-periodic function is almost periodic. Denote by $\mathrm{ANP}_0(I : X)$ the linear span of almost anti-periodic functions from I into X. Then [666, Theorem 2.3] implies that $\mathrm{ANP}_0(I : X)$ is a linear subspace of $\mathrm{AP}(I : X)$ and that the linear closure of $\mathrm{ANP}_0(I : X)$ in $\mathrm{AP}(I : X)$, denoted by $\mathrm{ANP}(I : X)$, satisfies

$$\mathrm{ANP}(I : X) = \mathrm{AP}_{\mathbb{R}\setminus\{0\}}(I : X). \tag{2.2}$$

Later, we will generalize the notion of almost anti-periodicity by introducing the notion of c-almost periodicity (see Section 4.2).

Within the theory of topological dynamical systems, the notion of recurrence plays an important role; for more details, the reader may consult the research monographs [358] by J. de Vries and [408] by T. Eisner et al. Following A. Haraux and P. Souplet [511], we introduce the following notion.

Definition 2.1.7. It is said that a continuous function $f : I \to X$ is uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to\infty} \sup_{t\in I} \| f(t+\alpha_n) - f(t) \| = 0. \tag{2.3}$$

It is well known that any almost periodic function is uniformly recurrent, while the converse statement is not true in general. It is worth noting that the convergence of the above limit is uniform in the variable $t \in \mathbb{R}$, so that the notion of a uniformly recurrent function should not be mistakenly identified with the notion of a recurrent function in the continuous Bebutov system [136], where the author has analyzed the usual Fréchet space $C(\mathbb{R})$ and the topology of uniform convergence on compact sets (cf. also Subsection 2.3.9 in the monograph [163] by G. Bertotti and I. D. Mayergoyz, the paper [328] by L. I. Danilov and references cited therein for further information in this direction). A uniformly recurrent function is also called pseudo-periodic by H. Bohr (see [197, Part. II, p. 32]), which has been accepted by many other authors later on; a recurrent function in the continuous Bebutov system is also called uniformly Poisson-stable motion by M. V. Bebutov.

Let us recall that the notion of a pseudo-almost periodic function was introduced by C. Zhang in the doctoral dissertation [1074] (cf. also [1075, 1076]). Henceforth, $\mathrm{PAP}_0(\mathbb{R} : X)$ stands for the space consisting of all pseudo-ergodic components, i. e., the bounded continuous functions $\Phi : \mathbb{R} \to X$ such that

$$\lim_{l\to\infty} \frac{1}{2l} \int_{-l}^{l} \|\Phi(s)\|\, ds = 0.$$

Regarding the space $\mathrm{PAP}_0(\mathbb{R} : \mathbb{C})$, it should be recalled that $f \in \mathrm{PAP}_0(\mathbb{R} : \mathbb{C})$ if and only if $f \cdot f \in \mathrm{PAP}_0(\mathbb{R} : \mathbb{C})$.

We say that a continuous function $f : \mathbb{R} \to X$ is pseudo-almost periodic if and only if it admits a decomposition $f = g + q$, where $g \in \mathrm{AP}(\mathbb{R} : X)$ and $q \in \mathrm{PAP}_0(\mathbb{R} : X)$. It is well known that, if such a decomposition exists, then it must be unique. The space consisting of all pseudo-almost periodic functions will be denoted by $\mathrm{PAP}(\mathbb{R} : X)$.

Example 2.1.8. Define

$$f(t) := \frac{1}{2t} \int_{-t}^{t} s|\sin s|^{s^N}\, ds, \quad t \in \mathbb{R},$$

where $N > 6$. From [32, Example p. 1143] we know that $\lim_{t\to+\infty} f(t) = 0$ and therefore $\cdot|\sin\cdot|^{\cdot^N} \in PAP_0(\mathbb{R} : \mathbb{C})$ for $N > 6$.

For more details about pseudo-almost periodic functions and pseudo-almost periodic solutions of partial differential equations, see the book [365] by T. Diagana, the doctoral dissertation of C. Zhang [1074], the monograph [1079] by the same author and the recent article [1047] by P. T. Xuan, S. L. The and T. T. H. Vu.

The representation of functions by trigonometric series is an old mathematical problem. In 1927, A. S. Besicovitch [167] proved that there exist infinitely many trigonometric series convergent to a bounded, continuous function $f : \mathbb{R} \to \mathbb{C}$, which is of bounded variation in every finite interval; hence, an everywhere convergent trigonometric series need not be almost periodic, in general (see also [171, 714, 989] and references quoted therein). Here we would like to note that D. E. Menshov [761] proved, in 1941, that there exists a sequence Λ of integers such that every measurable function $f : \mathbb{R} \to \mathbb{C}$ admits a representation

$$f(t) = \sum_{k\in\Lambda} c_k e^{ikt},$$

which converges for a. e. $t \in \mathbb{R}$ ($c_k \in \mathbb{Z}$, $k \in \Lambda$). This is the famous Menshov representation theorem and any sequence of integers Λ satisfying the above requirement is called a Menshov spectrum. For further information on the Menshov spectra, we refer the reader to the research articles [626–628] by G. Kozma, A. Olevskii and references cited therein.

Before we switch to the next section, we would like to emphasize the following:

1. The Hartman almost periodic functions on topological groups have been analyzed in the article [306] by G. Cohen, V. Losert and the research article [750] by G. Maresh, R. Winkler.
2. Among many other research papers, almost periodic functions on Banach algebras have been studied by M. Daws [333–335], J. Duncan and A. Ülger [404], M. Filali and M. S. Monfared [441], H. S. Mustafayev [803] and J. C. Quigg [863] (for almost periodic functions on quasigroups, pseudocompact grupoids and universal algebras, see the research studies [288–290] by M. M. Choban and D. I. Pavel).
3. For (sub-)harmonic almost periodic functions and Hardy spaces of almost periodic functions, the reader may consult the research articles [428, 429] by S. Yu. Favorov, A. V. Rakhnin, [599] by Yu. Kh. Khasanov, [767] by J. P. Milaszewicz and [821] by R. Owens, while for (holomorphic) semi-almost periodic functions, the reader may consult the research articles [225–227] by A. Brudnyi, D. Kinzebulatov

and [901] by D. Sarason; we also refer the reader to the interesting paper [723] by K. Liu, Y. Wei and P. Yu which concerns generalized Yang's conjecture for transcendental entire functions.

4. The study of almost periodic transformation groups was initiated by D. Montgomery in [780], with the main results given for the Euclidean space $\mathbb{R}^3$; see also the papers by B. L. Brechner [218], N. E. Foland [448] and masters' thesis by A. P. Wu [1033] for more details about the subject.
5. Certain types of non-classical almost periodic function spaces have been introduced and analyzed by [128] by B. Basit and C. Zhang; concerning the almost periodic solutions of partial differential equations, differential and pseudodifferential operators in the spaces of almost periodic functions, we should also mention the monograph article [938] by M. A. Shubin (cf. also [935]) and the research articles [173, 174] by M. Biroli.

2.2 Stepanov, Weyl and Besicovitch classes

Suppose that $1 \leqslant p < \infty$, $l > 0$ and $f,\ g \in L^p_{\mathrm{loc}}(I : X)$, where $I = \mathbb{R}$ or $I = [0, \infty)$. We define the Stepanov 'metric' by

$$D^p_{S_l}[f(\cdot), g(\cdot)] := \sup_{x \in I}\left[\frac{1}{l}\int_x^{x+l} \|f(t) - g(t)\|^p\, dt\right]^{1/p}.$$

Then, for every two numbers $l_1,\ l_2 > 0$, there exist two positive real constants $k_1,\ k_2 > 0$ independent of $f,\ g$, such that

$$k_1 D^p_{S_{l_1}}[f(\cdot), g(\cdot)] \leqslant D^p_{S_{l_2}}[f(\cdot), g(\cdot)] \leqslant k_2 D^p_{S_{l_1}}[f(\cdot), g(\cdot)].$$

Furthermore, there exists

$$D^p_W[f(\cdot), g(\cdot)] := \lim_{l \to \infty} D^p_{S_l}[f(\cdot), g(\cdot)] \tag{2.4}$$

in $[0, \infty]$. The distance appearing above is called the Weyl distance of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl 'norm' of $f(\cdot)$ are defined by

$$\|f\|_{S^p_l} := D^p_{S_l}[f(\cdot), 0] \quad \text{and} \quad \|f\|_{W^p} := D^p_W[f(\cdot), 0],$$

respectively.

Before proceeding further, we would like to note that it is not clear how we can define the Stepanov distance by considering a general variable exponent $p \in \mathcal{P}(I)$ in place of the constant coefficient $p \geqslant 1$ above; moreover, it is not clear whether the formula (2.4) can be generalized in this context.

In the sequel of this section, we assume that $l_1 = l_2 = 1$. It is said that a function $f \in L^p_{\mathrm{loc}}(I : X)$ is Stepanov p-bounded, S^p-bounded for short, if and only if

$$\|f\|_{S^p} := \sup_{t \in I} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

Equipped with the above norm, the space $L^p_S(I : X)$ consisting of all S^p-bounded functions is a Banach space.

Example 2.2.1. Let $p \geqslant 1$. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(s) := \begin{cases} k, & \text{if } k \leqslant s \leqslant k + k^{-p} \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the function $f(\cdot)$ is neither continuous nor bounded but for each $t \in \mathbb{R}$ we have

$$\begin{aligned} \int_{[t,t+1]} |f(s)|^p \, ds &\leqslant \int_{[\lfloor t \rfloor, \lfloor t \rfloor + 2]} |f(s)|^p \, ds \\ &= \sum_{k=\lfloor t \rfloor}^{\lfloor t \rfloor + 1} \int_{[k,k+k^{-p}] \cap [k,k+1]} |f(s)|^p \, ds \\ &= \sum_{k=\lfloor t \rfloor}^{\lfloor t \rfloor + 1} \int_{[k,k+k^{-p}]} k^p \, ds = 2. \end{aligned}$$

Hence, $f(\cdot)$ is Stepanov p-bounded.

A function $f \in L^p_S(I : X)$ is said to be Stepanov p-almost periodic, S^p-almost periodic shortly, if and only if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I, \ s \in [0,1], \tag{2.5}$$

is almost periodic. We say that the function $f \in L^p_S(I : X)$ is asymptotically Stepanov p-almost periodic if and only if there exist two locally p-integrable functions $g : \mathbb{R} \to X$ and $q : I \to X$ satisfying the following conditions:

(i) g is S^p-almost periodic,
(ii) $\hat{q}$ belongs to the class $C_0(I : L^p([0,1] : X))$,
(iii) $f(t) = g(t) + q(t)$ for all $t \in I$.

Recall, if $f(\cdot)$ is an (asymptotically) almost periodic function, then $f(\cdot)$ is also (asymptotically) Stepanov p-almost periodic for $1 \leqslant p < \infty$. The converse statement is false, however [696].

Example 2.2.2. Assume that $\alpha,\ \beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well-defined irrational number. Then the functions

$$f_{\alpha,\beta}(t) := \sin\left(\frac{1}{2+\cos\alpha t+\cos\beta t}\right), \quad t \in \mathbb{R}, \tag{2.6}$$

and

$$g_{\alpha,\beta}(t) := \cos\left(\frac{1}{2+\cos\alpha t+\cos\beta t}\right), \quad t \in \mathbb{R}, \tag{2.7}$$

are Stepanov p-almost periodic but not almost periodic ($1 \leqslant p < \infty$). The case $\alpha = 1$ and $\beta = \sqrt{2}$ has been further analyzed by A. Nawrocki in [810], who proved with the help of Liouville's theorem and some results from the theory of continuous fractions [810, Theorem 1, Theorem 2] that

$$\lim_{t\to+\infty} \frac{t^{-2-\varepsilon}}{2+\cos t+\cos\sqrt{2}t} = 0, \quad \varepsilon > 0,$$

and

$$\lim_{t\to+\infty} \frac{t^{-2}}{2+\cos t+\cos\sqrt{2}t}$$

does not exist. Recall, the function $t \mapsto 1/(2+\cos t+\cos\sqrt{2}t)$, $t \in \mathbb{R}$ is well defined, continuous and unbounded.

Denote by $\mathrm{APS}^p(I : X)$ and $\mathrm{AAPS}^p(I : X)$ the space consisting of all S^p-almost periodic functions $f : I \to X$ and the space consisting of all asymptotically S^p-almost periodic functions $f : I \to X$, respectively. The Bochner theorem asserts that any uniformly continuous function which is also Stepanov p-almost periodic needs to be almost periodic ($1 \leqslant p < \infty$); the Bochner theorem for Stepanov p-almost periodic functions has been established by Z. Hu and A. B. Mingarelli in [543, Theorem 1] (see also [602], where Yu. Kh. Khasanov and E. Safarzoda have analyzed the approximations of Stepanov almost periodic functions by means of Marcinkiewicz; for the formulation of Bochner criterion in convex bornological spaces, see the research article [991] by V. Valmorin).

Before we introduce the notion of a Stepanov p-almost periodic function $f : I \times Y \to X$, let us note that M. Baake, A. Haynes and D. Len have recently considered the Birkhoff-type averaging almost periodic functions along exponential sequences and proved, in [106, Theorem 5.2], that for any almost periodic function $f : \mathbb{R} \to X$, for any real number $\alpha \in (-\infty,-1) \cup (1,\infty)$ and for a. e. $x \in \mathbb{R}$, one has

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\alpha^n x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,ds.$$

An analogue of this result for scalar-valued Stepanov p-almost periodic functions has been considered in [106, Theorem 4.8] and can be straightforwardly extended to Stepanov p-almost periodic vector-valued functions ($1 \leqslant p < \infty$).

Definition 2.2.3. Let $1 \leqslant p < \infty$. A function $f : I \times Y \to X$ is called Stepanov p-almost periodic if and only if $\hat{f} : I \times Y \to L^p([0,1] : X)$ is almost periodic.

Recall that a bounded continuous function $f : I \times Y \to X$ is asymptotically almost periodic if and only if for every $\varepsilon > 0$ and every compact $K \subseteq Y$ there exist $l(\varepsilon, K) > 0$ and $M(\varepsilon, K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\varepsilon, K)$ contains a number τ with the property that $\|f(t+\tau, y) - f(t, y)\| \leqslant \varepsilon$ provided $|t|$, $|t+\tau| \geqslant M(\varepsilon, K)$, $y \in K$. The notion of an asymptotically Stepanov p-almost periodic function $f(\cdot,\cdot)$ is introduced in [631] for case $I = [0, \infty)$ as follows.

Definition 2.2.4. Let $1 \leqslant p < \infty$. A function $f : I \times Y \to X$ is said to be asymptotically S^p-almost periodic if and only if $\hat{f} : I \times Y \to L^p([0,1] : X)$ is asymptotically almost periodic. The collection of such functions will be denoted by $\mathrm{AAPS}^p(I \times Y : X)$.

Let $\omega \in I$. Then we say that a bounded continuous function $f : I \to X$ is S-asymptotically ω-periodic if and only if $\lim_{|t|\to\infty} \|f(t+\omega) - f(t)\| = 0$. Denote by $\mathrm{SAP}_\omega(I : X)$ the space consisting of all such functions. A Stepanov p-bounded function $f(\cdot)$ is said to be Stepanov p-asymptotically ω-periodic if and only if

$$\lim_{|t|\to\infty} \int_t^{t+1} \|f(s+\omega) - f(s)\|^p \, ds = 0.$$

If we denote by $S^p\mathrm{SAP}_\omega(I : X)$ the space consisting of all such functions, then we see that $\mathrm{SAP}_\omega(I : X) \subseteq S^p\mathrm{SAP}_\omega(I : X)$ and the inclusion is strict (for more details, see H. R. Henríquez [527] and H. R. Henríquez, M. Pierri, P. Táboas [531]).

The (Stepanov) quasi-asymptotically almost periodic functions have been analyzed in [647]. For our further work, it will be necessary to recall the following definition.

Definition 2.2.5. Suppose that $1 \leqslant p < \infty$, $I = [0, \infty)$ or $I = \mathbb{R}$.

(i) A bounded continuous function $f : I \to X$ is said to be quasi-asymptotically almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|f(t+\tau) - f(t)\| \leqslant \varepsilon, \quad \text{provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

Denote by $Q-\mathrm{AAP}(I : X)$ the set consisting of all quasi-asymptotically almost periodic functions from I into X.

(ii) Let us assume that $f \in L_S^p(I : X)$. Then it is said that $f(\cdot)$ is Stepanov p-quasi-asymptotically almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\int_t^{t+1} \|f(s+\tau) - f(s)\|^p \, ds \leqslant \varepsilon^p, \quad \text{provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

Denote by $S^p Q - \mathrm{AAP}(I : X)$ the set consisting of all Stepanov p-quasi-asymptotically almost periodic functions from I into X.

Let us recall that for each number $p \in [1, \infty)$ we see that $Q - \mathrm{AAP}(I : X) \subseteq S^p Q - \mathrm{AAP}(I : X)$ and that any asymptotically Stepanov p-almost periodic function is Stepanov p-quasi-asymptotically almost periodic. Furthermore, if $1 \leqslant p \leqslant q < \infty$, then $S^q Q - \mathrm{AAP}(I : X) \subseteq S^p Q - \mathrm{AAP}(I : X)$ and for any function $f \in L_S^p(I : X)$, we see that $f(\cdot)$ is Stepanov p-quasi-asymptotically almost periodic if and only if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by (2.5), is quasi-asymptotically almost periodic. It is said that $f(\cdot)$ is Stepanov quasi-asymptotically almost periodic if and only if $f(\cdot)$ is Stepanov 1-quasi-asymptotically almost periodic. Any asymptotically almost periodic function $f : I \to X$ is quasi-asymptotically almost periodic. Furthermore, we have $\mathrm{SAP}_\omega(I : X) \subseteq Q - \mathrm{AAP}(I : X)$ and $S^p \mathrm{SAP}_\omega(I : X) \subseteq S^p Q - \mathrm{AAP}(I : X)$.

Let $1 \leqslant p < \infty$. Now we recall the notions of the Besicovitch-p-almost periodic functions and the Besicovitch–Doss-p-almost periodic functions (see also the article [327] by L. I. Danilov for the corresponding notion in complete metric spaces, as well as the landmark paper [748] by J. Marcinkiewicz and M. A. Picardello's article [844]). If $f \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$, then we define

$$\|f\|_{\mathcal{M}^p} := \limsup_{t \to +\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s)\|^p \, ds \right]^{1/p};$$

if $f \in L^p_{\mathrm{loc}}([0, \infty) : X)$, then

$$\|f\|_{\mathcal{M}^p} := \limsup_{t \to +\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s)\|^p \, ds \right]^{1/p}.$$

In any case, $\|\cdot\|_{\mathcal{M}^p}$ is a seminorm on the space $\mathcal{M}^p(I : X)$ consisting of those $L^p_{\mathrm{loc}}(I : X)$-functions $f(\cdot)$ for which $\|f\|_{\mathcal{M}^p} < \infty$. Denote $K_p(I : X) := \{f \in \mathcal{M}^p(I : X) : \|f\|_{\mathcal{M}^p} = 0\}$ and

$$M_p(I : X) := \mathcal{M}^p(I : X)/K_p(I : X).$$

The seminorm $\|\cdot\|_{\mathcal{M}^p}$ on $\mathcal{M}^p(I : X)$ induces the norm $\|\cdot\|_{M^p}$ on $M^p(I : X)$ under which $M^p(I : X)$ is complete so that $(M^p(I : X), \|\cdot\|_{M^p})$ is a Banach space.

Now we are able to introduce the following notion.

Definition 2.2.6. Let $1 \leqslant p < \infty$. We say that a function $f \in L^p_{\mathrm{loc}}(I : X)$ is Besicovitch-p-almost periodic if and only if there exists a sequence of X-valued trigonometric polynomials converging to $f(\cdot)$ in $(M^p(I : X), \| \cdot \|_{M^p})$.

The vector space consisting of all Besicovitch-p-almost periodic functions is denoted by $B^p(I : X)$. It is well known that $B^p(I : X)$ is a closed subspace of $M^p(I : X)$ and therefore a Banach space equipped with the norm $\| \cdot \|_{M^p}$.

The Besicovitch class can be equivalently introduced in a Bohr-like manner, by using the notion of satisfactorily uniform sets (see e. g. [166] and [67, Definition 5.10, Definition 5.11]). We will not use this approach henceforth.

We define the Besicovitch 'distance' of functions $f,\ g \in L^p_{\mathrm{loc}}(I : X)$ by

$$D_{B^p}[f(\cdot), g(\cdot)] := \|f - g\|_{\mathcal{M}^p};$$

the Besicovitch 'norm' of $f \in L^p_{\mathrm{loc}}(I : X)$ is defined by

$$\|f\|_{B^p} := D_{B^p}[f(\cdot), 0] := \|f\|_{\mathcal{M}^p}.$$

We say that $f(\cdot)$ is Besicovitch p-bounded if and only if $\|f\|_{\mathcal{M}^p} < \infty$. Recall that

$$\|f - g\|_{\infty} \geqslant D_{S^p_l}[f(\cdot), g(\cdot)] \geqslant D_{W^p}[f(\cdot), g(\cdot)] \geqslant D_{B^p}[f(\cdot), g(\cdot)],$$

for $1 \leqslant p < \infty$, $l > 0$ and $f,\ g \in L^p_{\mathrm{loc}}(I : X)$, and that the assumption $\|f\|_{\mathcal{M}^p} = 0$ does not imply $f = 0$ a. e. on I. For more details about absolute convergence of Fourier series of Besicovitch almost periodic functions, the reader may consult [600, 601] and references therein.

The notion of a vector-valued Besicovitch–Doss-p-almost periodic function is introduced in [631] following the fundamental analyses of R. Doss [394, 395].

Definition 2.2.7. Let $1 \leqslant p < \infty$. It is said that $f \in L^p_{\mathrm{loc}}(I : X)$ is Besicovitch–Doss-p-almost periodic if and only if the following conditions hold:
(i) (B^p-boundedness) We have $\|f\|_{\mathcal{M}^p} < \infty$.
(ii) (B^p-continuity) We have

$$\lim_{\tau \to 0} \limsup_{t \to +\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s + \tau) - f(s)\|^p \, ds \right]^{1/p} = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{\tau \to 0+} \limsup_{t \to +\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s + \tau) - f(s)\|^p \, ds \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

(iii) (Doss-p-almost periodicity) For every $\varepsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t \to +\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} < \varepsilon,$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t \to +\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} < \varepsilon,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

(iv) For every $\lambda \in \mathbb{R}$, we have

$$\lim_{l \to +\infty} \limsup_{t \to +\infty} \frac{1}{l} \left[\frac{1}{2t} \int_{-t}^{t} \left\| \left(\int_{x}^{x+l} - \int_{0}^{l} \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{l \to +\infty} \limsup_{t \to +\infty} \frac{1}{l} \left[\frac{1}{t} \int_{0}^{t} \left\| \left(\int_{x}^{x+l} - \int_{0}^{l} \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

The vector space consisting of all Besicovitch–Doss-p-almost periodic functions $f : I \to X$ in the sense of Definition 2.2.7 will be denoted by $\mathrm{B}^p(I : X)$. In the case that $X = \mathbb{C}$, an intriguing result of R. Doss says that $\mathrm{B}^p(I : X) = B^p(I : X)$. It is still an unsolved problem whether the equality $\mathrm{B}^p(I : X) = B^p(I : X)$ holds in the vector-valued case.

Before moving to the next subsection, we want also to recommend for the reader the articles [58, 59, 94] and [228, 229], written by a group of Italian mathematicians, for more details about the Besicovitch almost periodic functions.

2.2.1 Stepanov μ-pseudo-almost periodic functions and applications

In this subsection, we provide the main properties of Stepanov μ-ergodic functions and (Stepanov) μ-pseudo-almost periodic functions. We will use the following assumption on the measure μ:

(M) For all $\tau \in \mathbb{R}$, there exist a number $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leqslant \beta \mu(A), \quad \text{provided } A \in \mathcal{B}(\mathbb{R}) \text{ and } A \cap I = \emptyset.$$

By $\mathcal{M}$ we denote a collection consisting of all such measures. In particular, the Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ belongs to $\mathcal{M}$.

Definition 2.2.8 ([181]). Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to X$ is said to be μ-ergodic if and only if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int\limits_{[-r,r]} \|f(t)\|\, d\mu(t) = 0.$$

The space of all such functions is denoted by $\mathcal{E}(\mathbb{R}, X, \mu)$.

Example 2.2.9 ([181]).
(i) Any ergodic function which belongs to the space $\mathrm{PAP}_0(\mathbb{R} : X)$ is nothing else but a μ-ergodic function in the particular case when μ is the Lebesgue measure.
(ii) Let $\rho : \mathbb{R} \to [0, +\infty)$ be a Lebesgue measurable function. We define the positive measure μ on $\mathcal{B}(\mathbb{R})$ by

$$\mu(A) := \int\limits_A \rho(t)\, dt \quad \text{for } A \in \mathcal{B}(\mathbb{R}),$$

where dt denotes the Lebesgue measure. The measure μ is absolutely continuous with respect to dt and the function ρ is called the Radon–Nikodym derivative of μ with respect to dt. In that case $\mu \in \mathcal{M}$ if and only if the function $\rho(\cdot)$ is locally Lebesgue-integrable on $\mathbb{R}$ and satisfies $\int_{\mathbb{R}} \rho(t)\, dt = +\infty$.

The following definition has been introduced in [415].

Definition 2.2.10. Let $\mu \in \mathcal{M}$ and $1 \leqslant p < \infty$. A function $f \in L^p_S(\mathbb{R} : X)$ is said to be Stepanov μ-ergodic (μ-S^p-ergodic, for short) if and only if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int\limits_{[-r,r]} \Bigg(\int\limits_{[t,t+1]} \|f(s)\|^p\, ds \Bigg)^{\frac{1}{p}} d\mu(t) = 0.$$

The space of all such functions is denoted by $\mathcal{E}^p(\mathbb{R}, X, \mu)$.

We need the following lemma from the last-mentioned paper.

Lemma 2.2.11. *Let $1 \leqslant p < \infty$ and $\mu \in \mathcal{M}$ satisfy* (M). *Then the following hold:*
(i) *$\mathcal{E}^p(\mathbb{R}, X, \mu)$ is translation invariant.*
(ii) $\mathcal{E}(\mathbb{R}, X, \mu) \subseteq \mathcal{E}^p(\mathbb{R}, X, \mu)$.

Now we introduce the following definitions.

Definition 2.2.12. Let $\mu \in \mathcal{M}$ and $1 \leqslant p < \infty$. A function $f : \mathbb{R} \times X \to Y$ such that $f(\cdot, x) \in L^p_S(\mathbb{R} : Y)$ for each $x \in X$ is said to be μ-S^p-ergodic in t with respect to x in X if and only if the following hold:

(i) For all $x \in X$, $f(\cdot, x) \in \mathcal{E}^p(\mathbb{R}, Y, \mu)$;
(ii) $f(\cdot,\cdot)$ is S^p-uniformly continuous with respect to the second argument on each compact subset K in X, namely: for every $\varepsilon > 0$, there exists $\delta_{K,\varepsilon} > 0$ such that, for every $x_1, x_2 \in K$, we have

$$\|x_1 - x_2\| \leqslant \delta_{K,\varepsilon} \Rightarrow \left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|^p \, ds \right)^{\frac{1}{p}} \leqslant \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

Denote by $\mathcal{E}^p U(\mathbb{R} \times X, Y, \mu)$ the set of all such functions.

Definition 2.2.13 ([181]). Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is said to be μ-pseudo-almost periodic if and only if $f(\cdot)$ can be decomposed in the form $f = g + \varphi$, where $g \in \mathrm{AP}(\mathbb{R} : X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. The space of all such functions is denoted by $\mathrm{PAP}(\mathbb{R}, X, \mu)$.

If $\mu \in \mathcal{M}$ satisfies (M), then the following hold [181]:
(i) The decomposition of a μ-pseudo-almost periodic in the form $f = g + \varphi$, where $g \in \mathrm{AP}(\mathbb{R} : X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$, is unique.
(ii) $\mathrm{PAP}(\mathbb{R}, X, \mu)$ equipped with the sup-norm is a Banach space.
(iii) $\mathrm{PAP}(\mathbb{R}, X, \mu)$ is translation invariant.

Definition 2.2.14. Let $\mu \in \mathcal{M}$. A function $f : \mathbb{R} \times X \to Y$ such that $f(\cdot, x) \in L^p_S(\mathbb{R} : Y)$ for each $x \in X$ is said to be S^p-μ-pseudo-almost periodic if f can be decomposed in the form $f = g + \varphi$, where $g \in \mathrm{APS}^p(\mathbb{R} \times X : Y)$ and $\varphi \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu)$. The space of all such functions will be denoted by $\mathrm{PAPS}^p U(\mathbb{R}, X, \mu)$.

For the sequel, we need some preliminary results obtained in [584].

Lemma 2.2.15. *Let $1 \leqslant p < +\infty$ and $f : \mathbb{R} \times X \to Y$ be such that $f(\cdot, x) \in L^p_{\mathrm{loc}}(\mathbb{R}, Y)$ for each $x \in X$. Then $f \in \mathrm{APS}^p U(\mathbb{R} \times X, Y)$ if and only if the following hold:*
(i) *For each $x \in X$, $f(\cdot, x) \in \mathrm{APS}^p(\mathbb{R} : Y)$.*
(ii) *f is S^p-uniformly continuous with respect to the second argument on each compact subset K in X in the following sense: for all $\varepsilon > 0$ there exists $\delta_{K,\varepsilon} > 0$ such that for all $x_1, x_2 \in K$ one has*

$$\|x_1 - x_2\| \leqslant \delta_{K,\varepsilon} \Rightarrow \left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|^p \, ds \right)^{\frac{1}{p}} \leqslant \varepsilon \quad \textit{for all } t \in \mathbb{R}.$$

It is clear that Lemma 2.2.15 implies the following.

Proposition 2.2.16. *Let $\mu \in \mathcal{M}$ and $f \in \mathrm{PAPS}^p U(\mathbb{R} \times X, Y, \mu)$, for $1 \leqslant p < +\infty$. Then the following holds:*

(i) *for each* $x \in X$, $f(\cdot, x) \in \mathrm{PAPS}^p(\mathbb{R}, Y, \mu)$*;*
(ii) $f(\cdot,\cdot)$ *is* S^p*-uniformly continuous with respect to the second argument on each compact subset* K *in* X*; namely, for each* $\varepsilon > 0$ *and for each compact set* K *in* X *there exists* $\delta_{K,\varepsilon} > 0$ *such that for all* $x_1, x_2 \in K$*, we have*

$$\|x_1 - x_2\| \leqslant \delta_{K,\varepsilon} \Rightarrow \left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|^p \, ds\right)^{\frac{1}{p}} \leqslant \varepsilon \quad \text{for all } t \in \mathbb{R}. \tag{2.8}$$

Next, we provide some examples of Stepanov μ-pseudo-almost periodic functions of order $1 \leqslant p < \infty$.

Example 2.2.17. Let X be any Banach space and let μ be a measure with the Radon–Nikodym derivative θ defined by $\theta(t) := e^t$ for $t \leqslant 0$ and $\theta(t) := 1$ for $t > 0$. From [181, Example 3.6], we know that the measure μ satisfies the hypothesis (M). Consider the function $\Phi : \mathbb{R} \to \mathbb{R}$ given by $\Phi(t) := \Phi_1(t) + \Phi_2(t)$ with $\Phi_2(t) := \arctan t - (\pi/2)$, $t \in \mathbb{R}$ and

$$\Phi_1(t) := \sum_{n\geqslant 1} \beta_n(t),$$

such that, for every $n \in \mathbb{N}$,

$$\beta_n(t) := \sum_{i \in P_n} H(n^2(t-i)),$$

with $P_n := 3^n(2\mathbb{Z}+1)$ and $H \in C_0^\infty(\mathbb{R} : \mathbb{R})$ with support in $((-1)/2, 1/2)$ such that

$$H \geqslant 0, \quad H(0) = 1 \quad \text{and} \quad \int_{\frac{(-1)}{2}}^{\frac{1}{2}} H(s)\, ds = 1.$$

Then we know that (see [181, Section 5]) the function $\Phi_2(\cdot)$ belongs to the space $\mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$ and $\Phi_1 \in \mathrm{APS}^1(\mathbb{R})$. Furthermore, let $h : X \to X$ be any continuous function. Then the functions $f(t,x) := \Psi(t)h(x)$ and $g(t,x) := \Phi(t)h(x)$ for $t \in \mathbb{R}$ and $x \in X$ define two examples of (purely) Stepanov μ-pseudo-almost periodic X-valued functions. In particular, f is S^1-μ-pseudo-almost periodic and g is S^2-μ-pseudo-almost periodic.

Now we will investigate composition results for μ-pseudo-almost periodic functions in Stepanov sense of order $1 \leqslant p < \infty$. In order to do that, we will use the following lemma, which is a special case of Theorem 6.2.30 below.

Lemma 2.2.18. *Let* $1 \leqslant p < +\infty$ *and* $f \in \mathrm{APS}^p(\mathbb{R} \times X : Y)$. *Assume that* $x \in \mathrm{AP}(\mathbb{R} : X)$. *Then* $f(\cdot, x(\cdot)) \in \mathrm{APS}^p(\mathbb{R} : Y)$.

We have the following result.

Theorem 2.2.19. *Let* $1 \leqslant p < +\infty$ *and* $\mu \in \mathcal{M}$. *If* $f \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu)$ *and* $x \in C_b(\mathbb{R} : X)$ *such that* $K = \overline{\{x(t) : t \in \mathbb{R}\}}$ *is compact in* X*, then* $f(\cdot, x(\cdot)) \in \mathcal{E}^p(\mathbb{R}, Y, \mu)$.

Proof. Let $f \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu)$, and let $K = \overline{\{x(t) : t \in \mathbb{R}\}} \subseteq X$ be a compact subset. Then, for every $\varepsilon > 0$, there exists $\delta_{\varepsilon,K} > 0$ such that (2.8) holds. Since K is compact, there exists a finite subset $\{x_1, \dots, x_n\} \subseteq K$ $(n \in \mathbb{N})$ such that $K \subseteq \bigcup_{i=1}^n B(x_i, \delta_{\varepsilon,K})$. Therefore, for every $t \in \mathbb{R}$, there exists $i(t) \in \{1, \dots, n\}$ such that $\|x(t) - x_{i(t)}\| \leqslant \delta$. Furthermore,

$$\begin{aligned}\left(\int_t^{t+1} \|f(s, x(s))\|_Y^p \, ds\right)^{\frac{1}{p}} &\leqslant \left(\int_t^{t+1} \|f(s, x(s)) - f(s, x_{i(t)})\|_Y^p \, ds\right)^{\frac{1}{p}} \\ &\quad + \left(\int_t^{t+1} \|f(s, x_{i(t)})\|_Y^p \, ds\right)^{\frac{1}{p}} \\ &\leqslant \varepsilon + \sum_{i=1}^n \left(\int_t^{t+1} \|f(s, x_i)\|_Y^p \, ds\right)^{\frac{1}{p}}, \quad t \in \mathbb{R}.\end{aligned}$$

Since $f(\cdot, x_i) \in \mathcal{E}^p(\mathbb{R}, Y, \mu)$ for $i = 1, \dots, n$, we have

$$\begin{aligned}&\frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x(s))\|_Y^p \, ds\right)^{\frac{1}{p}} d\mu(t) \\ &\quad \leqslant \varepsilon + \frac{1}{\mu([-r, r])} \sum_{i=1}^n \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x_i)\|_Y^p \, ds\right)^{\frac{1}{p}} d\mu(t),\end{aligned}$$

for $r > 0$ large enough. Consequently,

$$\limsup_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x(s))\|_Y^p \, ds\right)^{\frac{1}{p}} d\mu(t) \leqslant \varepsilon. \tag{2.9}$$

Since $\varepsilon > 0$ was arbitrary, (2.9) yields

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|f(s, x(s))\|_Y^p \, ds\right)^{\frac{1}{p}} d\mu(t) = 0. \qquad \square$$

Corollary 2.2.20. *Let $\mu \in \mathcal{M}$. Assume that $x \in \mathrm{AP}(\mathbb{R} : X)$ and $f \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu)$. Then $f(\cdot, x(\cdot)) \in \mathcal{E}^p(\mathbb{R}, Y, \mu)$.*

The following lemma is taken from [181].

Lemma 2.2.21. *Let $\mu \in \mathcal{M}$ and $f \in C_b(\mathbb{R} : X)$. Then $f \in \mathcal{E}(\mathbb{R}, X, \mu)$ if and only if for all $\varepsilon > 0$*

$$\lim_{r \to +\infty} \frac{\mu(M_{\varepsilon,r}(f))}{\mu([-r, r])} = 0,$$

where $M_{\varepsilon,r}(f) := \{t \in [-r, r] : \|f(t)\| \geqslant \varepsilon\}$.

The proof of our result related to the composition of S^p-μ-pseudo-almost periodic functions is based on the following lemma due to L. Schwartz [914, p. 109].

Lemma 2.2.22. *Let $\Phi \in C(X : Y)$. Then, for each compact set $K \subseteq X$ and for each $\varepsilon > 0$, there exists $\delta_{K,\varepsilon} > 0$ such that for any $x_1,\ x_2 \in X$, we have*

$$x_1 \in K \quad \text{and} \quad \|x_1 - x_2\| \leqslant \delta_{K,\varepsilon} \Rightarrow \|\Phi(x_1) - \Phi(x_2)\|_Y \leqslant \varepsilon.$$

Theorem 2.2.23. *Let $1 \leqslant p < +\infty$ and $\mu \in \mathcal{M}$. Assume the following:*
(i) *$f : \mathbb{R} \times X \to Y$ is a function such that $f = \tilde{f} + \varphi \in \mathrm{PAPS}^p U(\mathbb{R} \times X, Y, \mu)$ with $\tilde{f} \in \mathrm{APS}^p(\mathbb{R} \times X : Y)$ and $\varphi \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu)$;*
(ii) *$x = x_1 + x_2 \in \mathrm{PAP}(\mathbb{R}, X, \mu)$, where $x_1 \in \mathrm{AP}(\mathbb{R} : X)$ and $x_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$;*
(iii) *for every bounded subset $B \subseteq X$, we have $\sup_{x \in B} \|f(\cdot, x)\|_{S^p} < \infty$.*

Then $f(\cdot, x(\cdot)) \in \mathrm{PAPS}^p(\mathbb{R}, Y, \mu)$.

Proof. We have the following decomposition:

$$\begin{aligned} f(t, u(t)) &= \underbrace{\tilde{f}(t, x_1(t))} + \underbrace{[f(t, x(t)) - f(t, x_1(t))]} + \underbrace{\varphi(t, x_1(t))} \\ &:= \tilde{F}(t) + F(t) + \Psi(t), \quad t \in \mathbb{R}. \end{aligned}$$

Using Lemma 2.2.18, it follows that $\tilde{F} \in \mathrm{APS}^p(\mathbb{R} : Y)$; furthermore, Corollary 2.2.20 shows that $\Psi \in \mathcal{E}^p(\mathbb{R}, Y, \mu)$. Now, it suffices to prove that $F \in \mathcal{E}^p(\mathbb{R}, Y, \mu)$. In view of Lemma 2.2.21, we have

$$\lim_{r \to +\infty} \frac{\mu(M_{\varepsilon,r}(x_2))}{\mu([-r, r])} = 0, \quad \varepsilon > 0.$$

Let $\varepsilon > 0$. Then, for $r > 0$ large enough, we have

$$\begin{aligned} &\frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|F(s)\|_Y^p \, ds \right)^{\frac{1}{p}} d\mu(t) \\ &\leqslant \frac{1}{\mu([-r, r])} \int_{M_{\varepsilon,r}(x_2)} \left(\int_t^{t+1} \|F(s)\|_Y^p \, ds \right)^{\frac{1}{p}} d\mu(t) \\ &\quad + \frac{1}{\mu([-r, r])} \int_{[-r,r] \setminus M_{\varepsilon,r}(x_2)} \left(\int_t^{t+1} \|F(s)\|_Y^p \, ds \right)^{\frac{1}{p}} d\mu(t) \\ &\leqslant \|F\|_{BS^p} \frac{\mu(M_{\varepsilon,r}(x_2))}{\mu([-r, r])} \\ &\quad + \frac{1}{\mu([-r, r])} \int_{[-r,r] \setminus M_{\varepsilon,r}(x_2)} \left(\int_t^{t+1} \|f(s, x(s)) - f(s, x_1(s))\|^p \, ds \right)^{\frac{1}{p}} d\mu(t). \qquad (2.10) \end{aligned}$$

Let $K := \overline{\{x_1(t) : t \in \mathbb{R}\}}$. From $x_1 \in AP(\mathbb{R} : X)$, we assert that K is a compact subset of X. Define

$$\Phi : X \to PAPS^p(\mathbb{R}, Y) \quad \text{through } x \mapsto f(\cdot, x).$$

Since $f \in PAPS^p U(\mathbb{R} \times X, Y, \mu)$, using Proposition 2.2.16 we may deduce that the restriction of Φ on any compact K of X is uniformly continuous, which is equivalent to saying that the function Φ is continuous on X. If we apply Lemma 2.2.22 to Φ, we see that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $t \in \mathbb{R}$ and $\xi_1, \xi_2 \in X$, we have

$$\xi_1 \in K \quad \text{and} \quad \|\xi_1 - \xi_2\| \leqslant \delta \Rightarrow \left(\int_t^{t+1} \|f(s, \xi_1) - f(s, \xi_2)\|_Y^p \, ds \right)^{\frac{1}{p}} \leqslant \varepsilon.$$

Since $x(t) = x_1(t) + x_2(t)$ and $x_1(t) \in K$, we have

$$\begin{aligned} &t \in \mathbb{R} \quad \text{and} \quad \|x_2(s)\| \leqslant \delta \quad \text{for } s \in [t, t+1] \\ &\Rightarrow \left(\int_t^{t+1} \|f(s, x(s)) - f(s, x_1(s))\|_Y^p \, ds \right)^{\frac{1}{p}} \leqslant \varepsilon. \end{aligned}$$

Therefore, by the fact that $x_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$, we have

$$\limsup_{r \to +\infty} \frac{\mu(M_{\delta,r}(x_2))}{\mu([-r, r])} = 0.$$

Using (2.10), we obtain

$$\limsup_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|F(s)\|_Y^p \, ds \right)^{\frac{1}{p}} d\mu(t) \leqslant \varepsilon \quad \text{for all } \varepsilon > 0.$$

Consequently,

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \left(\int_t^{t+1} \|F(s)\|_Y^p \, ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

□

Keeping in mind Theorem 2.2.23, we obtain the following corollary.

Corollary 2.2.24. *Let $1 \leqslant p < +\infty$ and $\mu \in \mathcal{M}$. Assume that $f : \mathbb{R} \times X \to Y$ satisfies the following:*
(i) $f = \tilde{f} + \varphi \in PAPS^p U(\mathbb{R} \times X, Y, \mu)$ *with* $\tilde{f} \in APS^p U(\mathbb{R} \times X, Y)$ *and* $\varphi \in \mathcal{E}^p U(\mathbb{R} \times X, Y, \mu)$*;*
(ii) $x = x_1 + x_2 \in PAP(\mathbb{R}, X, \mu)$*, where* $x_1 \in AP(\mathbb{R} : X)$ *and* $x_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$*;*

(iii) *there exists a non-negative Stepanov p-bounded function* $L(\cdot)$ *such that*

$$\|f(t,x) - f(t,y)\|_Y \leqslant L(t)\|x - y\|, \quad x,\ y \in X, \quad t \in \mathbb{R}.$$

Then $f(\cdot, x(\cdot)) \in \mathrm{PAPS}^p(\mathbb{R}, Y, \mu)$.

Now we will apply our theoretical results in the qualitative analysis of bounded solutions for various kinds of abstract semilinear evolution inclusions in Banach spaces. Consider the following semilinear evolution inclusion:

$$D_{t,+}^{\gamma} u(t) \in \mathcal{A}u(t) + f(t, u(t)),\ t \in \mathbb{R}, \tag{2.11}$$

where $D_{t,+}^{\gamma}$ denotes the Riemann–Liouville fractional derivative of order $\gamma \in (0,1)$, $f : \mathbb{R} \times X \to X$ is Stepanov μ-pseudo-almost periodic in $t \in \mathbb{R}$ and satisfies certain properties with respect to $x \in X$. We assume that a closed multivalued linear operator $\mathcal{A}$ satisfies condition (P). Let $(R_\gamma(t))_{t>0}$ be the operator family considered in [631]. Then we know that

$$\|R_\gamma(t)\| = O(t^{\gamma-1}), \quad t \in (0,1] \quad \text{and} \quad \|R_\gamma(t)\| = O(t^{-\gamma-1}), \quad t \geqslant 1.$$

It is said that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (2.11) if and only if

$$u(t) = \int_{-\infty}^{t} R_\gamma(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}. \tag{2.12}$$

Now we would like to present the following application of Theorem 2.2.23. First of all, we will impose the following hypotheses on $f(\cdot,\cdot)$:

(P2) There exists $L \geqslant 0$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ satisfying the requirement that

$$\varepsilon \leqslant \|x - y\| < \varepsilon + \delta \quad \text{implies} \ \sup_{t\in\mathbb{R}} \left(\int_{t+1}^{t} \|f(s,x) - f(s,y)\|^p\, ds \right)^{\frac{1}{p}} < L\varepsilon,$$

for all $x,\ y \in X$.

(P3) For every bounded subset $B \subseteq X$, we have

$$\sup_{x\in B} \sup_{t\in\mathbb{R}} \left(\int_{t}^{t+1} \|f(s,x)\|^p\, ds \right)^{\frac{1}{p}} < \infty.$$

In the next example, we will show that a function satisfying (P2) is not necessarily a strict contraction. On the other hand, Theorem 1.1.6 implies that this function has a unique fixed point.

Example 2.2.25.

(i) Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \frac{|x|}{1+|x|}, \quad x \in \mathbb{R}.$$

Let $\varepsilon > 0$ and $\delta > 0$ such that $\varepsilon \leqslant |x-y| < \delta + \varepsilon$ for all $x,\ y \in \mathbb{R}$. Then we have

$$\begin{aligned}\big|g(x) - g(y)\big| &= \left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| \\ &\leqslant \frac{||x|-|y||}{1+|x|+|y|+|x|\,|y|} \\ &\leqslant \frac{||x|-|y||}{1+|x|+|y|}. \end{aligned} \tag{2.13}$$

Moreover, by our assumptions and (2.13), we have ($\delta \equiv \varepsilon$)

$$\big|g(x) - g(y)\big| < \frac{(\varepsilon+\delta)}{1+\varepsilon} := \frac{(\varepsilon+\varepsilon^2)}{1+\varepsilon} = \varepsilon.$$

Thus, (P2) holds with $L = 1$ and $f(\cdot,\cdot) \equiv g(\cdot)$. If we assume that $g(\cdot)$ is a strong contraction, then there exists a constant $K \in (0,1)$ such that $|g(x)-g(0)| = g(x) \leqslant K|x|$, which implies that $1/|x| \leqslant K$. Therefore, by letting $x \to 0$, we obtain a contradiction. Hence, $g(\cdot)$ is not a strict contraction.

(ii) Let $X := L^2(\Omega)$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is any bounded open set. Define the function $f : \mathbb{R}\times X \to X$ by

$$f(t,\varphi)(x) := K(t)\frac{\|\varphi\|}{1+\|\varphi\|}Q(x) + H(t,x), \quad t \in \mathbb{R},\ x \in \Omega,$$

where $K : \mathbb{R} \to (0,\infty)$, $Q \in X$, $Q \geqslant 0$ with $\|K\|_{BS^p}\|Q\| \geqslant 1$ and $H : \mathbb{R}\times\Omega \to [0,\infty)$. The function f satisfies (P2) but it is not a strict contraction. In fact, let $\varphi_1,\ \varphi_2 \in X$ and $\varepsilon,\ \delta > 0$ such that $\varepsilon \leqslant \|\varphi_1 - \varphi_2\| < \varepsilon + \delta$. Then a straightforward calculation yields

$$\begin{aligned}\big|f(t,\varphi_1)(x) - f(t,\varphi_2)(x)\big| &= K(t)\left|\frac{\|\varphi_1\|}{1+\|\varphi_1\|} - \frac{\|\varphi_2\|}{1+\|\varphi_2\|}\right|Q(x) \\ &\leqslant K(t)\frac{|\|\varphi_1\| - \|\varphi_2\||}{1+\|\varphi_1\|+\|\varphi_2\|+\|\varphi_1\|\,\|\varphi_2\|}Q(x) \\ &< K(t)\frac{\delta+\varepsilon}{1+\varepsilon}Q(x) \\ &:= K(t)\frac{\varepsilon^2+\varepsilon}{1+\varepsilon}Q(x) \\ &= K(t)Q(x)\varepsilon,\ t \in \mathbb{R},\ x \in \Omega. \end{aligned} \tag{2.14}$$

Moreover, by our assumption and (2.14), we get

$$\begin{aligned}\|f(t,\varphi_1)-f(t,\varphi_2)\|^2 &= \int_\Omega \big|f(t,\varphi_1)(x)-f(t,\varphi_2)(x)\big|^2\,dx\\ &< K(t)^2\int_\Omega Q(x)^2\,dx\,\varepsilon^2\\ &= K(t)^2\|Q\|^2\,\varepsilon^2.\end{aligned}$$

Finally,

$$\left(\int_t^{t+1}\|f(s,\varphi_1)-f(s,\varphi_2)\|^p\,ds\right)^{\frac{1}{p}} < \|K\|_{BS^p}\|Q\|\,\varepsilon \quad \text{for all } t\in\mathbb{R}.$$

This proves the result with $L := \|K\|_{BS^p}\|Q\|$. To show that $f(\cdot,\cdot)$ is not necessarily a strict contraction, assume the converse, i. e., there exists $0<\tilde{K}<1$ such that for all $\varphi_1,\ \varphi_2\in X$ we have

$$\big\|f(t,\varphi_1)-f(t,\varphi_2)\big\| \leqslant \tilde{K}\|\varphi_1-\varphi_2\|, \quad t\in\mathbb{R}.$$

Set, for every $n\in\mathbb{N}$, $\varphi_n(x) := \exp(-n\sqrt{|x|})$ for all $x\in\Omega$. It is clear that $(\varphi_n)_n\subseteq X$ and $\lim_{n\to\infty}\|\varphi_n\| = 0$. Therefore,

$$K(t)\frac{\|\varphi_n\|}{1+\|\varphi_n\|}\|Q\| \leqslant \tilde{K}\|\varphi_n\| \quad \text{for all } n\geqslant 1,\ t\in\mathbb{R},$$

which implies that

$$K(t)\frac{1}{1+\|\varphi_n\|}\|Q\| \leqslant \tilde{K} \quad \text{for all } n\geqslant 1,\ t\in\mathbb{R}.$$

Now, by letting $n\to+\infty$, we obtain

$$K(t)\|Q\| \leqslant \tilde{K} \quad \text{for all } t\in\mathbb{R}.$$

Thus, if we pass to the S^p-norm, we obtain

$$1 \leqslant \|K\|_{BS^p}\|Q\| \leqslant \tilde{K},$$

which contradicts the assumption on $\tilde{K}$.

Now we will state the following result.

Proposition 2.2.26. *Let* $\mu\in\mathcal{M}$ *satisfy* (M) *and* $1<p<\infty$. *Suppose that* $f(\cdot,u(\cdot))\in \mathrm{PAPS}^p(\mathbb{R},X,\mu)$. *Then the mapping* $F_0(\cdot)$ *given by*

$$(F_0u)(t) := \int_{-\infty}^{t} R_\gamma(t-s)f\big(s,u(s)\big)\,ds, \quad t\in\mathbb{R}$$

maps $\mathrm{PAP}(\mathbb{R},X,\mu)$ *into* $\mathrm{PAP}(\mathbb{R},X,\mu)$.

Proof. For any $u \in \mathrm{PAP}(\mathbb{R}, X, \mu)$, we define

$$(F_0 u)(t) := \int_{-\infty}^{t} R_\gamma(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Since $s \mapsto f(s, u(s)) \in L_S^p(\mathbb{R} : X)$, we obtain

$$\begin{aligned}
\|F_0 u(t)\| &\leqslant \int_{-\infty}^{t} \|R_\gamma(t-s)\| \cdot \|f(s, u(s))\|\, ds \\
&= \int_0^{\infty} \|R_\gamma(s)\| \cdot \|f(t-s, u(t-s))\|\, ds \\
&= \int_0^{1} \|R_\gamma(s)\| \cdot \|f(t-s, u(t-s))\|\, ds + \int_1^{\infty} \|R_\gamma(s)\| \cdot \|f(t-s, u(t-s))\|\, ds \\
&\leqslant \left(\int_0^1 s^{q(\gamma-1)}\, ds \right)^{\frac{1}{q}} \left(\int_0^1 \|f(t-s, u(t-s))\|^p\, ds \right)^{\frac{1}{p}} \\
&\quad + \sum_{k \geqslant 1} k^{-\gamma-1} \left(\int_k^{k+1} \|f(t-s, u(t-s))\|^p\, ds \right)^{\frac{1}{p}} \\
&= \left(1 - q(\gamma-1)\right)^{\frac{1}{q}} \left(\int_0^1 \|f(t-s, u(t-s))\|^p\, ds \right)^{\frac{1}{p}} \\
&\quad + S_\gamma \left(\int_k^{k+1} \|f(t-s, u(t-s))\|^p\, ds \right)^{\frac{1}{p}} \\
&\leqslant \left(\left(1 - q(\gamma-1)\right)^{\frac{1}{q}} + S_\gamma \right) \|f(\cdot, u(\cdot))\|_{S^p}, \quad t \in \mathbb{R}.
\end{aligned}$$

Thus, $F_0(\cdot)$ is well defined. Furthermore, from the fact that $s \mapsto f(s, u(s)) \in \mathrm{PAPS}^p(\mathbb{R}, X, \mu)$, we obtain by definition that $f(s, u(s)) = \tilde{f}(s, u(s)) + \varphi(s, u(s))$ with some $s \mapsto \tilde{f}(s, u(s)) \in \mathrm{APS}^p(\mathbb{R} : X)$ and $s \mapsto \varphi(s, u(s)) \in \mathcal{E}^p(\mathbb{R}, X, \mu)$. Clearly,

$$(F_0 u)(t) = \int_{-\infty}^{t} R_\gamma(t-s) \tilde{f}(s, u(s))\, ds + \int_{-\infty}^{t} R_\gamma(t-s) \varphi(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Let $\varepsilon > 0$, since $s \mapsto \tilde{f}(s, u(s)) \in \mathrm{APS}^p(\mathbb{R} : X)$, there exists $l_\varepsilon > 0$ such that each interval of length l_ε contains an element τ such that

$$\left(\int_t^{t+1} \|\tilde{f}(s+\tau, u(s+\tau)) - \tilde{f}(s, u(s))\|^p\, ds \right)^{\frac{1}{p}} < \varepsilon / \left(\left(1 - q(\gamma-1)\right)^{\frac{1}{q}} + S_\gamma \right)$$

uniformly in $t \in \mathbb{R}$. Hence, by the Hölder inequality, we have

$$\begin{aligned}
&\|F_0u(t+\tau) - F_0u(t)\| \\
&\quad \leqslant \int_0^\infty \|R_\gamma(s)\| \cdot \|\tilde{f}(t+\tau-s, u(t+\tau-s)) - \tilde{f}(t-s, u(t-s))\|\, ds \\
&\quad = \int_0^1 \|R_\gamma(s)\| \cdot \|\tilde{f}(t+\tau-s, u(t+\tau-s)) - \tilde{f}(t-s, u(t-s))\|\, ds \\
&\qquad + \int_1^\infty \|R_\gamma(s)\| \cdot \|\tilde{f}(t+\tau-s, u(t+\tau-s)) - \tilde{f}(t-s, u(t-s))\|\, ds \\
&\quad \leqslant \Bigl(\int_0^1 s^{q(\gamma-1)}\, ds\Bigr)^{\frac{1}{q}} \Bigl(\int_0^1 \|\tilde{f}(t+\tau-s, u(t+\tau-s)) - \tilde{f}(t-s, u(t-s))\|^p\, ds\Bigr)^{\frac{1}{p}} \\
&\qquad + \sum_{k\geqslant 1} k^{-\gamma-1} \Bigl(\int_k^{k+1} \|\tilde{f}(t+\tau-s, u(t+\tau-s)) - \tilde{f}(t-s, u(t-s))\|^p\, ds\Bigr)^{\frac{1}{p}} \\
&\quad = (1-q(\gamma-1))^{\frac{1}{q}} \Bigl(\int_0^1 \|\tilde{f}(t+\tau-s, u(t+\tau-s)) - \tilde{f}(t-s, u(t-s))\|^p\, ds\Bigr)^{\frac{1}{p}} \\
&\qquad + S_\gamma \Bigl(\int_k^{k+1} \|\tilde{f}(t+\tau-s, u(t+\tau-s)) - \tilde{f}(t-s, u(t-s))\|^p\, ds\Bigr)^{\frac{1}{p}} \\
&\quad \leqslant \varepsilon, \quad t \in \mathbb{R}.
\end{aligned}$$

Therefore, it suffices to prove that $t \mapsto \int_{-\infty}^t R_\gamma(t-s)\varphi(s, u(s))\, ds \in \mathcal{E}(\mathbb{R}, X, \mu)$. Indeed, let $r > 0$. Then, by the Hölder inequality, we obtain

$$\begin{aligned}
\frac{1}{\mu([-r,r])} \int_{-r}^r \|F_0u(t)\|\, d\mu(t) &\leqslant \frac{1}{\mu([-r,r])} \int_{-r}^r \int_{-\infty}^t \|R_\gamma(t-s)\| \cdot \|f(s, u(s))\|\, ds\, d\mu(t) \\
&= \frac{1}{\mu([-r,r])} \int_{-r}^r \int_0^\infty \|R_\gamma(s)\| \cdot \|f(t-s, u(t-s))\|\, ds\, d\mu(t) \\
&= \frac{1}{\mu([-r,r])} \int_{-r}^r \int_0^1 \|R_\gamma(s)\| \cdot \|f(t-s, u(t-s))\|\, ds\, d\mu(t) \\
&\quad + \frac{1}{\mu([-r,r])} \int_{-r}^r \int_1^\infty \|R_\gamma(s)\| \cdot \|f(t-s, u(t-s))\|\, ds\, d\mu(t) \\
&\leqslant \frac{(1-q(\gamma-1))^{\frac{1}{q}}}{\mu([-r,r])} \int_{-r}^r \Bigl(\int_0^1 \|f(t-s, u(t-s))\|^p\, ds\Bigr)^{\frac{1}{p}} d\mu(t)
\end{aligned}$$

$$+\frac{S_\gamma}{\mu([-r,r])}\int_{-r}^{r}\left(\int_{k}^{k+1}\|f(t-s,u(t-s))\|^p\,ds\right)^{\frac{1}{p}}d\mu(t)$$
$$\to 0 \quad \text{as } r\to+\infty.$$

By Lemma 2.2.11(i), the set $\mathcal{E}^p(\mathbb{R},X,\mu)$ is translation invariant. Hence, the result follows immediately. □

Theorem 2.2.27. *Let* $1<p<+\infty$ *and* $\mu\in\mathcal{M}$ *satisfy* (M). *Assume that* $f\in \mathrm{PAPS}^pU(\mathbb{R}\times X,X,\mu)$ *such that* (P2) *and* (P3) *hold with* $LS_\gamma^q\leqslant 1$, *where*

$$S_\gamma^q:=\sum_{k\geqslant 1}k^{-\gamma-1}+\left(1-q(\gamma-1)\right)^{\frac{1}{q}}.$$

Then the inclusion (2.11) *has a unique* μ*-pseudo-almost periodic mild solution given by the integral representation* (2.12).

Proof. Let $u\in \mathrm{PAP}(\mathbb{R},X,\mu)$. By (P3) and Theorem 6.2.30, we see that the function $s\mapsto f(s,u(s))$ belongs to the space $\mathrm{PAPS}^p(\mathbb{R},X,\mu)$. Then, by Proposition 2.2.26, $F_0(\cdot)$ maps $\mathrm{PAP}(\mathbb{R},X,\mu)$ into itself. It suffices to prove that $F_0(\cdot)$ has a unique fixed point in $\mathrm{PAP}(\mathbb{R},X,\mu)$ using Theorem 1.1.6. Let $\varepsilon>0$, and let $L>0$ and $\delta>0$ be determined from (P2). Let $u,\ v\in \mathrm{PAP}(\mathbb{R},X,\mu)$ satisfy $\varepsilon\leqslant\|u(t)-v(t)\|<\varepsilon+\delta$ for all $t\in\mathbb{R}$. Then the hypothesis (P2) yields

$$\begin{aligned}
\|F_0u(t)-F_0v(t)\| &\leqslant \int_{-\infty}^{t}\|R_\gamma(t-s)\|\cdot\|f(s,u(s))-f(s,v(s))\|\,ds\\
&=\int_0^\infty\|R_\gamma(s)\|\cdot\|f(t-s,u(t-s))-f(t-s,u(t-s))\|\,ds\\
&=\int_0^1\|R_\gamma(s)\|\cdot\|f(t-s,u(t-s))-f(t-s,v(t-s))\|\,ds\\
&\quad+\int_1^\infty\|R_\gamma(s)\|\cdot\|f(t-s,u(t-s))-f(t-s,v(t-s))\|\,ds\\
&\leqslant\left(\int_0^1 s^{q(\gamma-1)}\,ds\right)^{\frac{1}{q}}\left(\int_0^1\|f(t-s,u(t-s))-f(t-s,v(t-s))\|^p\,ds\right)^{\frac{1}{p}}\\
&\quad+\sum_{k\geqslant 1}k^{-\gamma-1}\left(\int_k^{k+1}\|f(t-s,u(t-s))-f(t-s,v(t-s))\|^p\,ds\right)^{\frac{1}{p}}\\
&=\left(1-q(\gamma-1)\right)^{\frac{1}{q}}\left(\int_0^1\|f(t-s,u(t-s))-f(t-s,v(t-s))\|^p\,ds\right)^{\frac{1}{p}}
\end{aligned}$$

$$+\sum_{k\geqslant 1} k^{-\gamma-1}\left(\int_k^{k+1} \|f(t-s,u(t-s))-f(t-s,v(t-s))\|^p\,ds\right)^{\frac{1}{p}}$$
$$< S_\gamma^q L\varepsilon \quad \text{for all } t\in\mathbb{R}.$$

Hence, by the assumption $S_\gamma^q L \leqslant 1$ we obtain

$$\|F_0u-F_0v\|_\infty < \varepsilon.$$

The result follows immediately from Theorem 1.1.6. □

In order to visualize the advantage of Theorem 2.2.27, in the next result we will use the following Lipschitz type assumption:

(Q) There exists a non-negative Stepanov p-bounded function $L(\cdot)$, where $1\leqslant p<\infty$, such that

$$\|f(t,x)-f(t,y)\| \leqslant L(t)\|x-y\|, \quad x,\ y\in X,\ t\in\mathbb{R}.$$

Theorem 2.2.28. *Let* $1<p<+\infty$ *and* $\mu\in\mathcal{M}$ *satisfy* (M). *Assume that* $f\in \mathrm{PAPS}^pU(\mathbb{R}\times X,X,\mu)$ *such that* (Q) *holds with* $\|L\|_{S^p}S_\gamma^q < 1$. *Then the inclusion* (2.11) *has a unique* μ*-pseudo-almost periodic mild solution given by the integral representation* (2.12).

Proof. From Theorem 2.2.27 and Corollary 2.2.24, it suffices to prove that the mapping $F_0(\cdot)$ has a unique fixed point. Indeed, let $u,\ v\in \mathrm{PAP}(\mathbb{R},X,\mu)$. Then, by (Q), we get

$$\begin{aligned}
\|F_0u(t)-F_0v(t)\| &\leqslant \int_{-\infty}^{t} \|R_\gamma(t-s)\|\cdot\|f(s,u(s))-f(s,v(s))\|\,ds\\
&= \int_0^\infty \|R_\gamma(s)\|\cdot\|f(t-s,u(t-s))-f(t-s,v(t-s))\|\,ds\\
&= \int_0^1 \|R_\gamma(s)\|\cdot\|f(t-s,u(t-s))-f(t-s,v(t-s))\|\,ds\\
&\quad+\int_1^\infty \|R_\gamma(s)\|\cdot\|f(t-s,u(t-s))-f(t-s,v(t-s))\|\,ds\\
&\leqslant \left(\int_0^1 s^{q(\gamma-1)}\,ds\right)^{\frac{1}{q}}\left(\int_0^1 \|f(t-s,u(t-s))-f(t-s,v(t-s))\|^p\,ds\right)^{\frac{1}{p}}\\
&\quad+\sum_{k\geqslant 1} k^{-\gamma-1}\left(\int_k^{k+1} \|f(t-s,u(t-s))-f(t-s,v(t-s))\|^p\,ds\right)^{\frac{1}{p}}\\
&\leqslant (1-q(\gamma-1))^{\frac{1}{q}}\left(\int_0^1 L(t-s)^p\,ds\right)^{\frac{1}{p}}\|u-v\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\geqslant 1} k^{-\gamma-1}\Bigg(\int\limits_{k}^{k+1} L(t-s)^{p}\,ds\Bigg)^{\frac{1}{p}}\|u-v\|_{\infty}\\
&\leqslant S_{\gamma}^{q}\|L\|_{S^{p}}\|u-v\|_{\infty},\quad t\in\mathbb{R}.
\end{aligned}
$$

Hence, we obtain

$$\|F_0u - F_0v\|_{\infty} \leqslant S_{\gamma}^{q}\|L\|_{S^{p}}\|u-v\|_{\infty}.$$

Then the result follows from the Banach contraction principle since $S_{\gamma}^{q}\|L\|_{S^{p}} < 1$. □

Remark 2.2.29. It is very important to state that, in Theorem 2.2.27, under the assumptions (P2) and (P3), condition $\|L\|_{S^{p}}S_{\gamma}^{q} = 1$ (i. e. $\|L\|_{S^{p}} = (S_{\gamma}^{q})^{-1}$) yields the existence and uniqueness of μ-pseudo-almost periodic mild solutions to the inclusion (2.11) in view of the Meir–Keeler fixed point theorem. However, the existence and uniqueness result does not hold in the case of consideration of Theorem 2.2.28.

Now we will revisit the fractional Poisson heat equation once more [631].

Example 2.2.30. Of concern is the following semilinear fractional Poisson heat equation in the L^2-setting:

$$
\begin{cases}
D_t^{\gamma}(m(x)v(t,x)) = \Delta v(t,x) + g(t,v(t,x)) + H(t,x), & t\in\mathbb{R},\ x\in\Omega,\\
v(t,x)|_{\partial\Omega} = 0; & t\in\mathbb{R},\ x\in\partial\Omega,
\end{cases}
\tag{2.15}
$$

where $\gamma \in (0,1)$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ an open bounded subset with a sufficiently smooth boundary $\partial\Omega$ and $m \in L^{\infty}(\Omega)$, $m \geqslant 0$. Here, $H : \mathbb{R}\times\Omega \to \mathbb{R}$ is S^p-μ-pseudo-almost periodic function. Let $X = L^2(\Omega)$ be the Lebesgue space of square-integrable functions on Ω. Define the multivalued linear operator $\mathcal{A}$ on X by

$$\mathcal{A}\varphi := \Delta\cdot m(x)^{-1}\varphi,$$

with maximal domain and Δ being the Dirichlet Laplacian. In addition, we set $f : \mathbb{R}\times X \to X$ by

$$f(t,\varphi)(x) := g\big(t,\varphi(x)\big) + H(t,x),\quad t\in\mathbb{R},\ x\in\Omega.$$

Then the problem (2.15) can be modeled through (2.11). It is well known that $\mathcal{A}$ satisfies (P); see [631] and the references therein. Moreover, for $\varphi \in X$, we define

$$g\big(t,\varphi(x)\big) := K(t)\frac{R(x)}{1+\|\varphi\|},\quad t\in\mathbb{R},\ x\in\Omega,$$

where $K : \mathbb{R} \to (0,\infty)$ is S^p-μ-pseudo-almost periodic and $R \in X$, $R \geqslant 0$. Then we have the following.

Lemma 2.2.31. *The function $f(\cdot,\cdot)$ satisfies* (P2) *with* $L = \|K\|_{BS^{p}}\|R\|$.

Proof. Let $\varphi_1, \varphi_2 \in X$, and let $\varepsilon > 0$ and $\delta > 0$ be such that $\varepsilon \leqslant \|\varphi_1 - \varphi_2\| < \varepsilon + \delta$. So,

$$\begin{aligned} \left|f(t,\varphi_1)(x) - f(t,\varphi_2)(x)\right| &= K(t)\left|\frac{R(x)}{1+\|\varphi_1\|} - \frac{R(x)}{1+\|\varphi_2\|}\right| \\ &\leqslant K(t)\frac{|\|\varphi_1\| - \|\varphi_2\||}{1+\|\varphi_1\|+\|\varphi_2\|}R(x), \quad t \in \mathbb{R},\ x \in \Omega. \end{aligned} \tag{2.16}$$

Then, using (2.16), we get

$$\begin{aligned} \int_\Omega \left|f(t,\varphi_1)(x) - f(t,\varphi_2)(x)\right|^2 dx &\leqslant K(t)^2 \frac{\|\varphi_1 - \varphi_2\|^2}{(1+\varepsilon)^2} \int_\Omega R(x)^2\,dx \\ &\leqslant K(t)^2 \frac{(\delta+\varepsilon)^2}{(1+\varepsilon)^2}\|R\|^2 \\ &\leqslant K(t)^2\|R\|^2\varepsilon^2, \quad t \in \mathbb{R} \text{ (by taking } \delta := \varepsilon^2\text{)}. \end{aligned}$$

Hence, for all $\varphi_1, \varphi_2 \in X$, we have

$$\left(\int_t^{t+1} \|f(s,\varphi_1) - f(s,\varphi_2)\|^p\,ds\right)^{\frac{1}{p}} < \|K\|_{S^p}\varepsilon \quad \text{for all } t \in \mathbb{R}.$$

This proves the result with $L := \|K\|_{S^p}\|R\|$. □

Lemma 2.2.32. *The function $f(\cdot,\cdot)$ is Lipschitzian with respect to the second argument with Lipschitz constant $L(\cdot) := K(\cdot)\|R\|$. Moreover, $f(\cdot,\cdot)$ satisfies* (P3).

Proof. Let $\varphi_1, \varphi_2 \in X$. By the proof of Lemma 2.2.31, we have

$$\|f(t,\varphi_1) - f(t,\varphi_2)\| \leqslant K(t)\|R\|\,\|\varphi_1 - \varphi_2\|, \quad t \in \mathbb{R}.$$

This simply completes the proof. □

At this stage we set $p := 2$; then $q = 2$. Furthermore, we take

$$\|K\|_{BS^2}\|R\| = (S_\gamma^2)^{-1}.$$

Then we have the following result.

Theorem 2.2.33. *The inclusion* (2.15) *has a unique μ-pseudo-almost periodic solution.*

Proof. The result follows from Theorem 2.2.27. □

Finally, let us note that, in our joint paper [585] with K. Khalil and M. Pinto, we have also analyzed the existence and uniqueness of μ-pseudo-almost periodic solutions of the following semilinear nonautonomous evolution equation:

$$x'(t) = A(t)x(t) + f(t,x(t)) \quad \text{for } t \in \mathbb{R}.$$

Let $(A(t), D(A(t))), t \in \mathbb{R}$ be a family of linear closed operators on X. Of concern is the following linear Cauchy problem:

$$\begin{cases} u'(t) = A(t)u(t), & t \geqslant s, \\ u(s) = x \in X. \end{cases}$$

Here, we assume that $(A(t), D(A(t))), t \in \mathbb{R}$ generates an evolution family, which solves the problem (6.41), i. e., a two-parameter family $(U(t,s))_{t\geqslant s}$ of linear bounded operators in X such that the map $(t,s) \mapsto U(t,s) \in L(X)$ is strongly continuous, $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = I$ for $t \geqslant s \geqslant r$. A (mild) solution to problem (6.41) is $u(t) = U(t,s)x$ for $t \geqslant s$. In particular, if $A(t)$ is time-independent, i. e., $A(t) = A$ for all $t \in \mathbb{R}$, then $U(t,s) = T(t-s)$, where $(T(t))_{t\geqslant 0}$ is a semigroup of bounded linear operators on X. Notice that, in general, the domains $D(A(t))$ of the operators $A(t)$ are not necessarily dense in X and may change with respect to t. Unlike semigroups, there is no necessary and sufficient spectral criteria for $(A(t), D(A(t))), t \in \mathbb{R}$ to generate an evolution family. If $A(t)$ has a constant domain $D(A(t)) = D, t \in \mathbb{R}$, then we have the following generation result.

(C1) Let $(A(t), D), t \in \mathbb{R}$ be the generators of analytic semigroups $(T^t(\tau))_{\tau\geqslant 0}$ on X of the same type (N, ω); that is, $\|T_t(s)\| \leqslant Ne^{\omega s}$, $s \geqslant 0$ (uniformly in t). Assume that $A(t)$ is invertible for all $t \in \mathbb{R}$, $\sup_{t,s\in\mathbb{R}} \|A(t)A(s)^{-1}\| < \infty$ and there exist constants $\omega \in \mathbb{R}$, $L \geqslant 0$ and $0 < \mu \leqslant 1$ such that

$$\big\|(A(t) - A(s))R(\omega : A(r))\big\| \leqslant L|t-s|^{\mu} \quad \text{for } t,\ s,\ r \in \mathbb{R}.$$

In this case, the map $(t,s) \mapsto U(t,s) \in L(X)$ is continuously differentiable for $t > s$ with respect to the variable t, $U(t,s)$ maps X into $D(A(t))$ and we have $\partial U(t,s)/\partial t = A(t)U(t,s)$. Moreover, $U(t,s)$ and $(t-s)A(t)U(t,s)$ are exponentially bounded.

Given a hyperbolic evolution family $(U(t,s))_{t\geqslant s}$, then its associated Green function is defined by

$$G(t,s) := \begin{cases} U(t,s)P(s), & t,\ s \in \mathbb{R},\ s \leqslant t, \\ -\tilde{U}(t,s)Q(s), & t,\ s \in \mathbb{R},\ s > t. \end{cases}$$

The exponential dichotomy of $(U(t,s))_{t\geqslant s}$ holds in the following case:

(C2) Assume that (C1) holds and the semigroups $(T^t(\tau))_{\tau\geqslant 0}$ are hyperbolic with projections P_t and constants $N,\ \delta > 0$ such that $\|A(t)T^t(\tau)P_t\| \leqslant \psi(\tau)$ and $\|A(t)T^t_Q(\tau)Q_t\| \leqslant \psi(-\tau)$ for $\tau > 0$ and a function ψ such that the mapping $\mathbb{R} \ni s \mapsto \varphi(s) := |s|^{\mu}\psi(s)$ is integrable with $L\|\varphi\|_{L^1(\mathbb{R})} < 1$.

In [585], we have used the following hypotheses:

(H1) The operators $A(t)$, $t \in \mathbb{R}$ generate a strongly continuous evolution family $(U(t,s))_{t\geqslant s}$ on X.

(H2) The evolution family $(U(t,s))_{t\geqslant s}$ has an exponential dichotomy on $\mathbb{R}$ with constants $N,\ \delta > 0$, projections $P(t),\ t \in \mathbb{R}$ and Green's function $G(\cdot,\cdot)$.
(H3) $R(\omega : A(\cdot))$ is almost periodic for some $\omega \in \mathbb{R}$.
(H4) $f(\cdot,\cdot)$ is Lipschitzian in bounded sets with respect to the second argument, i. e., for each $\rho > 0$ there exists a non-negative scalar function $L_\rho(\cdot) \in L_S^p(\mathbb{R})$ (for $1 \leqslant p < \infty$) such that

$$\|f(t,x) - f(t,y)\| \leqslant L_\rho(t)\|x - y\|, \quad x,\ y \in B(0,\rho),\ t \in \mathbb{R}.$$

We have also analyzed the semilinear Cauchy problem

$$x'(t) = A(t)x(t) + f\big(t, x(t)\big) \quad \text{for } t \in \mathbb{R}. \tag{2.17}$$

By a mild solution of (2.17) we mean any continuous function $u : \mathbb{R} \to X$ which satisfies the following variation of constants formula:

$$u(t) = U(t,\sigma)u(\sigma) + \int_\sigma^t U(t,s)f\big(s,u(s)\big)\,ds \quad \text{for all } t \geqslant \sigma.$$

In particular, we have analyzed the existence and uniqueness of μ-pseudo-almost periodic solutions of the following linear inhomogeneous equation:

$$u'(t) = A(t)u(t) + h(t) \quad \text{for all } t \in \mathbb{R}.$$

We have applied our abstract theoretical results in the study of following time-dependent parameters reaction-diffusion equation describing the behavior of a one-species intraspecific competition:

$$\begin{cases} v_t(t,x) = \Delta v(t,x) - a(t)v(t,x) + b(t)v(t,x)^2 + C(t,x), & t \in \mathbb{R},\ x \in \Omega, \\ v(t,x)|_{\partial\Omega} = 0; & t \in \mathbb{R},\ x \in \partial\Omega, \end{cases} \tag{2.18}$$

where
- $\Omega \subseteq \mathbb{R}^n$ ($n \geqslant 1$) is an open bounded subset with a sufficiently smooth boundary.
- Δ is the Dirichlet Laplacian on Ω; here the diffusion parameter equals 1.
- $a \in AP(\mathbb{R} : [0,\infty))$ with $0 < a_0 := \inf_{s\in\mathbb{R}} a(s) \leqslant a(t) \leqslant \sup_{s\in\mathbb{R}} a(s) = a_1 < \infty$. It is assumed that $a(\cdot)$ is Hölder continuous with constant $L = 1$ and exponent $\mu = 1$.
- The nonlinear term $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

  $$g\big(t, v(t,x)\big) = b(t)v^2(t,x) + C(t,x), \quad x \in \overline{\Omega},$$

  where $b \in APS^1(\mathbb{R} : [0,\infty))$.
- $C : \mathbb{R} \times \overline{\Omega} \to (0,\infty)$ is locally integrable with respect to t and continuous with respect to x.

2.2.2 Composition principles for Weyl almost periodic functions

The notion of an (equi-)Weyl p-almost periodic function plays an important role in our investigations (cf. [631, Section 2.3] for more details):

Definition 2.2.34. Let $1 \leqslant p < \infty$ and $f \in L^p_{\mathrm{loc}}(I : X)$.

(i) We say that the function $f(\cdot)$ is equi-Weyl p-almost periodic, $f \in e - W^p_{ap}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x\in I}\left[\frac{1}{l}\int\limits_x^{x+l}\|f(t+\tau)-f(t)\|^p\,dt\right]^{1/p} \leqslant \varepsilon. \tag{2.19}$$

(ii) We say that the function $f(\cdot)$ is Weyl p-almost periodic, $f \in W^p_{ap}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int\limits_x^{x+l}\|f(t+\tau)-f(t)\|^p\,dt\right]^{1/p} \leqslant \varepsilon.$$

It is well known that $\mathrm{APS}^p(I : X) \subseteq e - W^p_{ap}(I : X) \subseteq W^p_{ap}(I : X)$ and $e - W^p_{ap}(I : X) \subseteq B^p(I : X)$. Some results about the integration of equi-Weyl p-almost periodic functions have been established by L. Radová in [865].

In the remainder of this subsection, we will present a few research results obtained recently in [639], which have not been presented in any other research monograph by now.

The following definition is slightly different from the corresponding definitions introduced in [141] and [642] for the class of equi-Weyl p-almost periodic functions, with only one pivot space $X = Y$.

Definition 2.2.35.

(i) A function $F : I\times Y \to X$ is said to be equi-Weyl p-almost periodic in $t \in I$ uniformly with respect to compact subsets of Y if and only if $f(\cdot, u) \in L^p_{\mathrm{loc}}(I : X)$ for each fixed element $u \in Y$ and if for each $\varepsilon > 0$ and each compact K of Y there exist two numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{u\in K}\sup_{x\in I}\left[\frac{1}{l}\int\limits_x^{x+l}\|F(t+\tau,u)-F(t,u)\|^p\,dt\right]^{1/p} < \varepsilon.$$

We denote by $e - W^p_{ap,\mathbf{K}}(I\times Y : X)$ the vector space consisting of all such functions.

(ii) A function $F : I \times Y \to X$ is said to be Weyl p-almost periodic in $t \in I$ uniformly with respect to compact subsets of Y if $f(\cdot, u) \in L^p_{\mathrm{loc}}(I : X)$ for each fixed element $u \in Y$ and if for each $\varepsilon > 0$ and each compact K of Y we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ satisfying the requirement that there exists a finite number $l(\varepsilon, \tau) > 0$ such that

$$\sup_{u \in K} \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \|F(t+\tau, u) - F(t, u)\|^p \, dt \right]^{1/p} < \varepsilon, \quad l \geqslant l(\varepsilon, \tau).$$

We denote by $W^p_{ap,\mathbf{K}}(I \times Y : X)$ the collection of all such functions.

The following definition is well known in the case that $X = Y$ (cf. [642]).

Definition 2.2.36. Let $q : [0, \infty) \times Y \to X$ be such that $q(\cdot, u) \in L^p_{\mathrm{loc}}([0, \infty) : X)$ for each fixed element $u \in Y$.

(i) It is said that $q(\cdot, \cdot)$ is Weyl p-vanishing uniformly with respect to compact subsets of Y if and only if for each compact set K of Y we have

$$\lim_{t \to \infty} \lim_{l \to \infty} \sup_{\xi \geqslant 0, u \in K} \left[\frac{1}{l} \int_\xi^{\xi+l} \|q(t+s, u)\|^p \, ds \right]^{1/p} = 0.$$

(ii) It is said that $q(\cdot, \cdot)$ is equi-Weyl p-vanishing uniformly with respect to compact subsets of Y if and only if for each compact set K of Y we have

$$\lim_{l \to \infty} \lim_{t \to \infty} \sup_{\xi \geqslant 0, u \in K} \left[\frac{1}{l} \int_\xi^{\xi+l} \|q(t+s, u)\|^p \, ds \right]^{1/p} = 0.$$

We denote by $W^p_{0,\mathbf{K}}(I \times Y : X)$ and $e - W^p_{0,\mathbf{K}}(I \times Y : X)$ the classes consisting of all Weyl p-vanishing functions, uniformly with respect to compact subsets of Y and all equi-Weyl p-vanishing functions, uniformly with respect to compact subsets of Y, respectively.

Similarly, for the class of (equi-)Weyl p-almost periodic functions, we have the following result which is not comparable with [141, Theorem 3] in the case of consideration of equi-Weyl p-almost periodic functions, with $I = \mathbb{R}$ and $X = Y$.

Theorem 2.2.37. *Suppose that the following conditions hold:*

(i) $F \in (e-)W^p_{ap,\mathbf{K}}(I \times Y : X)$ *with* $p > 1$, *and there exist a number* $r \geqslant \max(p, p/(p-1))$ *and a function* $L_F \in L^r_S(I)$ *such that*

$$\|F(t, x) - F(t, y)\| \leqslant L_F(t) \|x - y\|_Y, \quad t \in I,\ x,\ y \in Y. \tag{2.20}$$

(ii) $x \in (e-)W^p_{ap}(I : Y)$, *and there exists a set* $E \subseteq I$ *with* $m(E) = 0$ *such that* $K := \{x(t) : t \in I \setminus E\}$ *is relatively compact in* Y.

(iii) *For every $\varepsilon > 0$, there exist two numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a number $\tau \in I'$ such that*

$$\sup_{t\in I, u\in K}\left[\frac{1}{l}\int_t^{t+l}\|F(s+\tau,u)-F(s,u)\|^p\,ds\right]^{1/p} \leqslant \varepsilon \tag{2.21}$$

and

$$\sup_{t\in I}\left[\frac{1}{l}\int_t^{t+l}\|x(s+\tau)-x(s)\|_Y^p\,ds\right]^{1/p} \leqslant \varepsilon \tag{2.22}$$

in the case of consideration of equi-Weyl p-almost periodic functions, resp., there exists a finite number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a number $\tau \in I'$ satisfying the requirement that there exists a number $l(\varepsilon,\tau) > 0$ so that (2.21)–(2.22) *hold for all numbers $l \geqslant l(\varepsilon,\tau)$, in the case of consideration of Weyl p-almost periodic functions.*

Then $q := pr/(p+r) \in [1,p)$ and $F(\cdot,x(\cdot)) \in (e-)W_{ap}^q(I : X)$.

Proof. Without loss of generality, we may assume that $X = Y$. Since the function $L_F(\cdot)$ is Stepanov r-bounded, equivalently, Weyl r-bounded, the measurability and S^p-boundedness of the function $F(\cdot,x(\cdot))$ follow similarly to the proof of [729, Theorem 2.2]. Applying the Hölder inequality and an elementary calculation involving the estimate (2.20) and condition (ii), we see that, for every $t,\ \tau \in I$ and $l > 0$,

$$\begin{aligned}
&\frac{1}{l}\int_t^{t+l}\|F(s+\tau,x(s+\tau))-F(s,x(s))\|^q\,ds\\
&\quad\leqslant \frac{1}{l}\Bigg[\left(\int_t^{t+l}L_F^r(s+\tau)\,ds\right)^{1/r}\left(\int_t^{t+l}\|x(s+\tau)-x(s)\|^p\,dt\right)^{1/p}\\
&\qquad+\left(\int_t^{t+l}\|F(s+\tau,x(s))-F(s,x(s))\|^q\,ds\right)^{1/q}\Bigg]\\
&\quad\leqslant \frac{1}{l}\Bigg[\left(\int_t^{t+l}L_F^r(s+\tau)\,ds\right)^{1/r}\left(\int_t^{t+l}\|x(s+\tau)-x(s)\|^p\,dt\right)^{1/p}\\
&\qquad+\left(\int_t^{t+l}\left(\sup_{u\in K}\|F(s+\tau,u)-F(s,u)\|\right)^q\,ds\right)^{1/q}\Bigg].
\end{aligned}$$

The remaining part of the proof is almost the same for both classes of functions, equi-Weyl p-almost periodic functions and Weyl p-almost periodic functions; because of that, we will consider only the first class up to the end of proof. Let $\varepsilon > 0$ be given. By (iii), there exist two numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a number $\tau \in I'$ such that (2.21)–(2.22) hold. Since the validity of (2.21)–(2.22) with given numbers $l > 0$ and $\tau \in I$ implies the validity of (2.21)–(2.22) with numbers nl and $\tau \in I$ ($n \in \mathbb{N}$), we may assume that the number $l > 0$ is as large as we want to be. Then, due to Lemma 1.1.3, we obtain the existence of a finite number $M > 0$ such that

$$\frac{1}{l}\left(\int_t^{t+l} L_F^r(s+\tau)\,ds\right)^{1/r} \leqslant Ml^{(1/r)-1}\|L_F\|_{W^r}, \quad t \in I,$$

and

$$\frac{1}{l}\left(\int_t^{t+l} L_F^r(s+\tau)\,ds\right)^{1/r}\left(\int_t^{t+l}\|x(s+\tau)-x(s)\|^p\,dt\right)^{1/p}$$
$$\leqslant Ml^{(1/p)+(1/r)-1}\|L_F\|_{W^r} = l^{(1/q)-1}\|L_F\|_{W^r} \leqslant \|L_F\|_{W^r}, \quad t \in I.$$

For the estimation of the term

$$\frac{1}{l}\left(\int_t^{t+l}\left(\sup_{u\in K}\|F(s+\tau,u)-F(s,u)\|\right)^q ds\right)^{1/q}, \quad t \in I,$$

we can use the trick employed for proving [729, Lemma 2.1]. Since K is totally bounded, there exist an integer $k \in \mathbb{N}$ and a finite subset $\{x_1,\ldots,x_k\}$ of K such that $K \subseteq \bigcup_{i=1}^k B(x_i,\varepsilon)$, where $B(x,\varepsilon) := \{y \in X : \|x-y\| \leqslant \varepsilon\}$. Applying Minkowski's inequality and a simple argumentation similar to that used in the proof of the above-mentioned lemma, we get the existence of a finite positive real number $c_q > 0$ such that

$$\frac{1}{l}\left(\int_t^{t+l}\left(\sup_{u\in K}\|F(s+\tau,u)-F(s,u)\|\right)^q ds\right)^{1/q}$$
$$\leqslant \frac{c_q}{l}\left[\varepsilon\left(\int_t^{t+l}[L_F^q(s+\tau)+L_F^q(s)]\,ds\right)^{1/q}\right.$$
$$\left.+\sum_{i=1}^k\left(\int_t^{t+l}\|F(s+\tau,x_i)-F(s,x_i)\|^q\,ds\right)^{1/q}\right].$$

The term $\frac{1}{l}(\int_t^{t+l}[L_F^q(s+\tau)+L_F^q(s)]\,ds)^{1/q}$ can be estimated by using Lemma 1.1.3 in the following way:

$$\leqslant \frac{1}{l}\left(\int_{t+\tau}^{t+l+\tau} L_F^q(s)\,ds\right)^{1/q} + \frac{1}{l}\left(\int_t^{t+l} L_F^q(s)\,ds\right)^{1/q}$$
$$\leqslant Ml^{(-1/r)+(1/q)-1}\|L_F\|_{W^r}l^{1/r} \leqslant M\|L_F\|_{W^r}, \quad t \in I.$$

Similarly, using Lemma 1.1.3 and (iii), we get

$$\frac{1}{l}\sum_{i=1}^{k}\left(\int_{t}^{t+l}\|F(s+\tau,x_i)-F(s,x_i)\|^q\,ds\right)^{1/q}$$
$$\leqslant \frac{1}{l}l^{(1/q)-(1/p)}\sum_{i=1}^{k}\left(\int_{t}^{t+l}\|F(s+\tau,x_i)-F(s,x_i)\|^p\,ds\right)^{1/p} \leqslant \varepsilon l^{(1/q)-1}, \quad t\in I.$$

This completes the proof of theorem. □

Remark 2.2.38. To the best knowledge of the author, it is not known whether the assumptions $F \in W^p_{ap}(I\times Y : X)$ and $x \in W^p_{ap}(I : Y)$ imply the validity of condition (iii), as for the class of Stepanov p-almost periodic functions.

The following result for the class of Weyl p-almost periodic functions can be also deduced with the help of argumentation contained in [729] (compare with Theorem 2.4.49, where we will analyze the Stepanov class).

Theorem 2.2.39. *Suppose that p, $q \in [1,\infty)$, $r \in [1,\infty]$, $1/p = 1/q+1/r$ and the following conditions hold:*
(i) $F \in W^p_{ap,\mathbf{K}}\mathrm{AP}(I\times Y : X)$ *and there exists a function* $L_F \in L^r_S(I)$ *such that* (2.20) *holds.*
(ii) $x \in W^q_{ap}\mathrm{AP}(I : Y)$*, and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{x(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* Y.
(iii) *For every* $\varepsilon > 0$*, there exists a finite number* $L > 0$ *such that any interval* $I' \subseteq I$ *of length* L *contains a number* $\tau \in I'$ *satisfying the requirement that there exists a number* $l(\varepsilon,\tau) > 0$ *so that* (2.21) *holds for all numbers* $l \geqslant l(\varepsilon,\tau)$ *and* (2.22) *holds for all numbers* $l \geqslant l(\varepsilon,\tau)$*, with the number* p *replaced by* q *therein.*

Then $F(\cdot,x(\cdot)) \in W^p_{ap}\mathrm{AP}(I : X)$.

After proving Theorem 2.2.37, the subsequent composition principle for asymptotically (equi-)Weyl p-almost periodic functions follows almost immediately; cf. also [642, Theorem 3.4] for a similar result in this direction.

Theorem 2.2.40. *Suppose that* $p > 1$, $r \geqslant \max(p, p/(p-1))$, $q = pr/(p+r)$*, and the conditions* (i)–(iii) *of* Theorem 2.2.37 *hold with the interval* $I = [0,\infty)$ *and the functions* $F(\cdot,\cdot)$, $x(\cdot)$ *replaced therein with the functions* $G(\cdot,\cdot)$, $y(\cdot)$*. Suppose, further, that the following hold:*
(i) *The function* $Q := F-G : [0,\infty)\times Y \to X$ *belongs to the class* $(e-)W^{q'}_{0,\mathbf{K}}([0,\infty)\times Y : X)$ *for some number* $q' \in [1,\infty)$.
(ii) *The function* $z : [0,\infty) \to Y$ *belongs to the class* $(e-)W^{q''}_0([0,\infty) : Y)$ *for some number* $q'' \in [1,\infty)$.
(iii) $x(t) = y(t) + z(t)$ *for a. e.* $t \geqslant 0$*, and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{x(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* Y.

Then the mapping $t \mapsto F(t,x(t))$, $t \geqslant 0$ *belongs to the class* $(e-)W^{q}_{ap}([0,\infty) : X) + (e-)W^{q'}_{0}([0,\infty) : X) + (e-)W^{q'''}_{0}([0,\infty) : X)$, *provided* $q''' \in [1,\infty)$ *and* $1/r + 1/q'' = 1/q'''$.

Proof. It is clear that $F(t,x(t)) = [G(t,x(t)) - G(t,y(t))] + G(t,y(t)) + Q(t,x(t))$, $t \geqslant 0$. By Theorem 2.2.37, we know that $G(\cdot,y(\cdot)) \in (e-)W^{q}_{ap}([0,\infty) : X)$. Keeping in mind (i) and (iii), the function $t \mapsto Q(t,x(t))$, $t \geqslant 0$ belongs to the class $(e-)W^{q'}_{0}([0,\infty) : X)$ by definition (see the notions of classes $W^{p}_{0,\mathbf{K}}(I \times Y : X)$ and $e - W^{p}_{0,\mathbf{K}}(I \times Y : X)$ introduced in Definition 2.2.36). Therefore, it suffices to show that the mapping $t \mapsto G(t,x(t)) - G(t,y(t))$, $t \geqslant 0$ belongs to the class $(e-)W^{q'''}_{0}([0,\infty) : X)$. But this follows similarly to the proof of [642, Theorem 3.4], with the exponents p, q, r replaced therein with the exponents q''', q'', r, respectively. □

An analogue of [642, Theorem 3.4] for the class of asymptotically Weyl p-almost periodic functions can be also deduced by means of Theorem 2.2.39.

2.3 Almost automorphic type functions

Suppose that $f : \mathbb{R} \to X$ is continuous. As already mentioned in the introductory part, we say that $f(\cdot)$ is almost automorphic if and only if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a mapping $g : \mathbb{R} \to X$ such that

$$\lim_{n\to\infty} f(t + a_n) = g(t) \quad \text{and} \quad \lim_{n\to\infty} g(t - a_n) = f(t), \tag{2.23}$$

pointwise for $t \in \mathbb{R}$. If this is the case, we have $f \in C_b(\mathbb{R} : X)$ and the limit function $g(\cdot)$ is bounded on $\mathbb{R}$ but not necessarily continuous on $\mathbb{R}$. If the convergence of limits appearing in (2.23) is uniform on compact subsets of $\mathbb{R}$, then we say that $f(\cdot)$ is compactly almost automorphic. The vector space consisting of all almost automorphic, resp., compactly almost automorphic functions, is denoted by $AA(\mathbb{R} : X)$, resp., $AA_{\mathbf{c}}(\mathbb{R} : X)$. By Bochner's criterion [364], any almost periodic function is compactly almost automorphic. The converse statement is not true, however [443]. Recall that P. R. Bender proved in his doctoral dissertation [149] that an almost automorphic function $f(\cdot)$ is compactly almost automorphic if and only if it is uniformly continuous (1966, Iowa State University).

The almost automorphy of a function $f : \mathbb{R} \to X$ can be also introduced in the following equivalent way: A function $f : \mathbb{R} \to X$ is said to be almost automorphic if and only if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) such that

$$\lim_{m\to\infty} \lim_{n\to\infty} f(t + a_n - a_m) = f(t), \quad t \in \mathbb{R}.$$

An interesting example of an almost automorphic function that is not compactly almost automorphic is given by W. A. Veech [993, 994]

$$f(t) := \frac{2 + e^{it} + e^{it\sqrt{2}}}{|2 + e^{it} + e^{it\sqrt{2}}|}, \quad t \in \mathbb{R}. \tag{2.24}$$

Let $I = \mathbb{R}$ or $I = [0, \infty)$. A continuous function $f : I \to X$ is said to be asymptotically (compactly) almost automorphic if and only if there exist a function $q \in C_0(I : X)$ and a (compactly) almost automorphic function $h : \mathbb{R} \to X$ such that $f(t) = h(t) + q(t)$, $t \in I$. Any asymptotically almost periodic function $f : I \to X$ is asymptotically (compactly) almost automorphic. Asymptotically almost periodic functions and asymptotically (compactly) almost automorphic functions form closed subspaces of $C_b(\mathbb{R} : X)$ equipped with the sup-norm.

For the sake of completeness, we will include the proof of following simple proposition.

Proposition 2.3.1.
(i) *Suppose that* $f \in AA(\mathbb{R} : \mathbb{C})$ *and* $g \in AA(\mathbb{R} : X)$. *Then* $fg \in AA(\mathbb{R} : X)$.
(ii) *Suppose that* $f \in AA_{\mathbf{c}}(\mathbb{R} : \mathbb{C})$ *and* $g \in AA_{\mathbf{c}}(\mathbb{R} : X)$. *Then* $fg \in AA_{\mathbf{c}}(\mathbb{R} : X)$.

Proof. Suppose that (b_n) is a given real sequence. Then there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to X$ such that (2.23) holds pointwise for $t \in \mathbb{R}$, with the function $g(\cdot)$ replaced therein with the function $h_1(\cdot)$. Furthermore, there exist a subsequence (a_{n_k}) of (a_n) and a map $h_2 : \mathbb{R} \to \mathbb{C}$ such that $\lim_{k\to\infty} f(t + a_{n_k}) = h_2(t)$ and $\lim_{k\to\infty} h_2(t - a_{n_k}) = f(t)$, pointwise for $t \in \mathbb{R}$. This simply implies that $\lim_{k\to\infty} f(t + a_{n_k})g(t + a_{n_k}) = h_1(t)h_2(t)$ and $\lim_{k\to\infty} h_1(t - a_{n_k})h_2(t - a_{n_k}) = f(t)g(t)$, pointwise for $t \in \mathbb{R}$, finishing the proof of (i). The proof of (ii) follows from (i) and the fact that the pointwise product of two bounded uniformly continuous functions is a uniformly continuous function. □

Example 2.3.2 ([46]). Suppose that the function $f : \mathbb{R} \to X$ is almost periodic (almost automorphic), $A \in L(X)$ and there exists a unique bounded classical solution $u(\cdot)$ of the abstract Cauchy problem $u'(t) = Au(t) + f(t)$, $t \in \mathbb{R}$. Then $u(\cdot)$ is almost periodic (almost automorphic).

Let $p \in [1, \infty)$. Then a function $f \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ is said to be Stepanov p-almost automorphic (see, e. g., G. M. N'Guérékata and A. Pankov [500], and V. Casarino [249–251] for a slightly different approach) if and only if for every real sequence (a_n), there exist a subsequence (a_{n_k}) and a function $g \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \int_t^{t+1} \left\| f(a_{n_k} + s) - g(s) \right\|^p ds = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_t^{t+1} \left\| g(s - a_{n_k}) - f(s) \right\|^p ds = 0$$

for each $t \in \mathbb{R}$; a function $f \in L^p_{\mathrm{loc}}(I : X)$ is called asymptotically Stepanov p-almost automorphic if and only if there exist an S^p-almost automorphic function $g : \mathbb{R} \to X$ and a function $q \in L^p_S(I : X)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : X))$. Any Stepanov p-almost automorphic function $f(\cdot)$ has to be Stepanov p-bounded. Furthermore, if $1 \leqslant p \leqslant q < \infty$ and a function $f(\cdot)$ is (asymptotically)

Stepanov q-almost automorphic, then $f(\cdot)$ is (asymptotically) Stepanov p-almost automorphic. We say that a function $f(\cdot)$ is (asymptotically) Stepanov almost automorphic if and only if $f(\cdot)$ is (asymptotically) Stepanov 1-almost automorphic. Let us recall that any uniformly continuous Stepanov almost periodic (automorphic) function $f(\cdot)$ is almost periodic (automorphic). The vector space consisting of all S^p-almost automorphic functions, resp., asymptotically S^p-almost automorphic functions, will be denoted by $\mathrm{AAS}^p(\mathbb{R}:X)$, resp., $\mathrm{AAAS}^p([0,\infty):X)$. By the (asymptotical) Stepanov almost automorphy we mean (asymptotical) Stepanov 1-almost automorphy. Recall that the (asymptotical) Stepanov p-almost periodicity of $f(\cdot)$ for some $p \in [1,\infty)$ implies the (asymptotical) Stepanov p-almost automorphy of $f(\cdot)$.

Example 2.3.3 ([387]). Let $\varepsilon \in (0,1/2)$, and let $f(t) := \sin(1/(2+\cos n+\cos\sqrt{2}n))$, provided that $n \in \mathbb{Z}$ and $t \in (n-\varepsilon, n+\varepsilon)$. Otherwise, we define $f(t) := 0$. Then for each $p \in [1,\infty)$ we see that $f(\cdot)$ is S^p-almost automorphic.

Let us recall that any uniformly continuous Stepanov almost periodic (automorphic) function $f(\cdot)$ is almost periodic (automorphic); see [386, Theorem 3.3]. The following lemma can be deduced by using an elementary argumentation involving [560, Proposition 3.1], the above-mentioned theorem and a simple observation that any uniformly continuous function $q \in C_0(I : L^p([0,1]:X))$ belongs to the space $C_0(I:X)$.

Lemma 2.3.4. *Let $f : I \to X$ be uniformly continuous and $p \in [1,\infty)$.*
(i) *If $f(\cdot)$ is asymptotically Stepanov p-almost periodic, then $f(\cdot)$ is asymptotically almost periodic.*
(ii) *If $f(\cdot)$ is asymptotically Stepanov p-almost automorphic, then $f(\cdot)$ is asymptotically almost automorphic.*

The concepts of Weyl almost automorphy and Weyl pseudo-almost automorphy were introduced by S. Abbas [4] in 2012.

Definition 2.3.5. Let $p \geqslant 1$. Then we say that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}:X)$ is Weyl p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{\mathrm{loc}}(\mathbb{R}:X)$ such that

$$\lim_{k\to+\infty}\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f(t+s_{n_k}+x)-f^*(t+x)\right\|^p\,dx=0 \tag{2.25}$$

and

$$\lim_{k\to+\infty}\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f^*(t-s_{n_k}+x)-f(t+x)\right\|^p\,dx=0 \tag{2.26}$$

for each $t \in \mathbb{R}$. The set of all such functions is denoted by $W^p\mathrm{AA}(\mathbb{R}:X)$.

We continue by providing the following example.

Example 2.3.6. Assume that $1 \leqslant p < \infty$. Let $(a_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence in $\mathbb{R}$ such that $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to-\infty} a_n = -\infty$. Let $(b_n)_{n\in\mathbb{Z}}$ be a real sequence. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) := b_n$ if $t \in [a_n, a_{n+1})$ for some $n \in \mathbb{Z}$, which can be written as a countable sums of step functions. Let us recall that the Stepanov p-almost automorphy of some special kinds of function $f(\cdot)$ has been considered in [387] and [500].

Define the functions $P_1 : \mathbb{R} \to \mathbb{Z}$ and $P_2 : \mathbb{R} \to \mathbb{Z}$ by $P_1(t) := n$ if and only if $t \in [a_n, a_{n+1})$ and $P_2(t) := m$ if and only if $t+1 \in [a_m, a_{m+1})$ $(t \in \mathbb{R})$. These functions are well defined, single valued, monotonically increasing and we have $P_1(t) \leqslant P_2(t)$ for all $t \in \mathbb{R}$. From the definition, it immediately follows that the function $f(\cdot)$ is S^p-bounded if and only if

$$\sup_{t\in\mathbb{R}}\left[b_{P_1(t)}\cdot(a_{P_1(t)+1}-t)+\sum_{j=P_1(t)+1}^{P_2(t)-1} b_j\cdot(a_{j+1}-a_j)+(t+1-a_{P_2(t)})\cdot b_{P_2(t)}\right]<\infty.$$

Note, if the function $f(\cdot)$ is S^p-bounded, then the continuity of $f(\cdot)$ is equivalent to saying that the function $f(\cdot)$ is constant, which is also equivalent with the almost automorphy of $f(\cdot)$.

Let us examine now the special case $f(x) := \chi_{(0,1/2)}(x)$, $x \in \mathbb{R}$, where $\chi_{(0,1/2)}(\cdot)$ denotes the characteristic function of $(0, 1/2)$. In [631, Example 3.1.3], we have proved that this function is equi-Weyl 1-almost periodic, Weyl 1-almost automorphic and not Stepanov p-almost automorphic $(1 \leqslant p < \infty)$. The analysis carried out in this example also shows that a general function $f(\cdot)$ under the consideration from the first part of this example cannot be Stepanov p-almost automorphic $(1 \leqslant p < \infty)$ if $f(\cdot)$ is constant on some right half ray (ω, ∞) or some left half ray $(-\infty, \omega)$, where $\omega \in \mathbb{R}$.

The set $W^p\mathrm{AA}(\mathbb{R} : X)$, equipped with the usual operations of pointwise addition of functions and multiplication of functions with scalars, has a linear vector structure. As we will see later, the Weyl p-almost automorphicity does not imply the Besicovitch p-boundedness.

The class of Besicovitch p-almost automorphic functions has been analyzed by F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte and M. Smaali in [139]. This class extends the class of Weyl p-almost automorphic functions and now we allow the possible non-existence of limit

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f(t+s_{n_k}+x)-f^*(t+x)\right\|^p\,dx,$$

resp.,

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f^*(t-s_{n_k}+x)-f(t+x)\right\|^p\,dx$$

in (2.25), resp., (2.26).

Definition 2.3.7. Let $p \geqslant 1$. Then we say that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ is Besicovitch p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \|f(t+s_{n_k}+x) - f^*(t+x)\|^p \, dx = 0$$

and

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \|f^*(t-s_{n_k}+x) - f(t+x)\|^p \, dx = 0$$

for each $t \in \mathbb{R}$. The set of all such functions is denoted by $B^p AA(\mathbb{R} : X)$.

We can prove that the set $B^p AA(\mathbb{R} : X)$, equipped with the usual operations, has a linear vector structure. Let us stress once more that it is not clear how we can prove that a Besicovitch p-almost periodic function is Besicovitch p-almost automorphic [631]. For more details about the class of Besicovitch p-pseudo-almost automorphic functions, we refer the reader to [631].

For more details about almost periodic functions (sequences), almost automorphic functions (sequences) and their applications, we refer the reader to [9, 29, 191, 206, 209, 283–285, 369, 509, 620, 927] and [299–301, 366, 367, 461, 466, 637, 725, 726, 766, 850, 1069].

2.4 Almost periodic type functions and densities

We will first describe the main ideas and aims of this section, which consists of three subsections. Albeit the definitions of an almost periodic function and a uniformly recurrent function are quite easy and understandable, the class consisting of all almost periodic functions and the class consisting of all uniformly recurrent functions are sometimes very unpleasant and difficult to deal with. For example, already H. Bohr has marked in his pioneering papers that it is not so satisfactory to introduce the concept of almost periodicity by requiring that for each number $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is unbounded (see, e. g., [198]). A bounded uniformly continuous function $f : I \to \mathbb{R}$ satisfying this property need not be almost periodic, its Bohr–Fourier coefficients cannot be defined in general, and moreover, if two bounded uniformly continuous functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ satisfy this property, then its sum $f + g : I \to \mathbb{R}$ generally does not satisfy this property (see [196, part I, pp. 117–118]). Furthermore, saying that for each number $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is unbounded is equivalent to saying that $f(\cdot)$ is uniformly recurrent; hence, the sum of two bounded, uniformly continuous, uniformly recurrent functions is not uniformly recurrent, in general. Taking into account

Proposition 2.4.31 below, we see that the sum of two bounded, uniformly continuous $\odot_g$-almost periodic functions is not $\odot_g$-almost periodic, in general. This example can be also used for proving the fact that the pointwise product of two bounded uniformly continuous, uniformly recurrent ($\odot_g$-almost periodic) functions is not uniformly recurrent ($\odot_g$-periodic), in general.

The above-mentioned observation of H. Bohr has motivated us to further analyze some very specific examples of generalized almost periodic functions in more detail (see [122] and [824, Appendix 3] for a non-updated list of unsolved problems in the theory). First of all, we recall that B. Basit and H. Güenzler have constructed, in [125, Example 3.2], a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ such that its first anti-derivative $t \mapsto \int_0^t f(s)\,ds$, $t \in \mathbb{R}$ is almost periodic, while the function $f(\cdot)$ itself is not uniformly continuous, not Stepanov almost periodic, not almost automorphic and

$$\sup_{t\in[-2,0]} \left|f(t+\tau) - f(t)\right| \geqslant 1 \quad \text{for all } \tau \geqslant 2. \tag{2.27}$$

The construction concretely goes as follows. Define a continuous 2^{n+1}-periodic function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) := \sin(2^n\pi t)$ for $t \in [2^n - 1, 2^n]$, $f_n(t) := 0$ for $t \in [-2^n, 2^n - 1)$, and extend it 2^{n+1}-periodically to the whole real axis. Then $\operatorname{supp}(f_n) = [2^n - 1, 2^n] + 2^{n+1}\mathbb{Z}$, which simply implies that $\operatorname{supp}(f_n)$ and $\operatorname{supp}(f_m)$ are disjunct sets for each integers n, $m \in \mathbb{N}$ with $n \neq m$. Therefore, the function $f(x) := \sum_{n=1}^{\infty} f_n(x)$, $x \in \mathbb{R}$ is well defined. This function satisfies all above properties, and we will provide a small contribution here by proving that the set $\vartheta(f, \varepsilon)$ is empty for each number $\varepsilon \in (0, 1)$:

△ Suppose that $\tau \in \vartheta(f, \varepsilon)$. Due to (2.27), we have $\tau \in (0, 2)$ so that there exist two possibilities: $\tau \in (0, 1)$ or $\tau \in [1, 2)$. In the first case, there exists a sufficiently large number $n \in \mathbb{N}$ such that $(2^n + 1) - (2^n - 1 + 2^{-n-1}) > \tau$. Let $t = 2^n - 1 + 2^{-n-1}$; then $t + \tau \in (2^n, 2^n + 1)$ and therefore $f(t) = 1$ while $f(t + \tau) = 0$ so that $|f(t + \tau) - f(t)| = 1 > \varepsilon$. In the second case, there exists a sufficiently large number $n \in \mathbb{N}$ such that $\tau > 2^{-n-1}$. In this case, take $t = 2^n - 2^{-n-1}$; then $t + \tau \in (2^n, 2^n + 1)$ and therefore $f(t) = -1$ while $f(t + \tau) = 0$ so that $|f(t + \tau) - f(t)| = 1 > \varepsilon$.

Essentially, the functions $f(\cdot)$ satisfying the requirement that there exists a number $\varepsilon \in (0, 1)$ such that the set $\vartheta(f, \varepsilon)$ is bounded will not occupy our attention henceforth. In connection with the above example, we would like to propose the following question.

Question 2.4.1. Suppose that $f : I \to X$ is a bounded, continuous and Stepanov almost periodic. Is it true that $\vartheta(f, \varepsilon) \neq \emptyset$ ($\vartheta(f, \varepsilon)$ is unbounded) for all $\varepsilon > 0$?

More concretely, assume that α, $\beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well-defined irrational number. Let the function $f_{\alpha,\beta}(\cdot)$ and $g_{\alpha,\beta}(\cdot)$ be given through (2.6) and (2.7), respectively. Is it true that $\vartheta(f, \varepsilon) \neq \emptyset$ ($\vartheta(f, \varepsilon)$ is unbounded) [$\vartheta(g, \varepsilon) \neq \emptyset$ ($\vartheta(g, \varepsilon)$ is unbounded)] for all $\varepsilon > 0$?

We continue by observing that A. Haraux and P. Souplet have proved, in [511, Theorem 1.1], that the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\left(\frac{t}{2^n}\right) dt, \quad t \in \mathbb{R}, \tag{2.28}$$

is uniformly continuous, uniformly recurrent and unbounded. From the argumentation given in the proof of the above-mentioned theorem, it immediately follows that the function $f(\cdot)$ given by (2.28) is neither Besicovitch almost periodic [631] nor asymptotically Stepanov almost automorphic. The reason for that is quite simple: this function is even and enjoys the property that

$$\limsup_{t\to+\infty} \frac{1}{2t} \int_{-t}^{t} f(s)\, ds = +\infty.$$

Since the concepts of H. Weyl and A. S. Besicovitch suggest very general ways of approaching almost automorphicity [631], it is logical to ask whether the function $f(\cdot)$ is Weyl almost automorphic. We will prove the following result.

Theorem 2.4.2. *The function $f(\cdot)$, given by* (2.28), *is Weyl p-almost automorphic for any finite exponent $p \geqslant 1$ and satisfies the requirement that for each number $\tau \in \mathbb{R}$ the function $f(\cdot + \tau) - f(\cdot)$ belongs to the space* $ANP(\mathbb{R} : \mathbb{C})$.

Concerning this contribution, it is worth noting that the unbounded functions $f : \mathbb{R} \to \mathbb{R}$ such that for each number $\tau \in \mathbb{R}$ the function $f(\cdot+\tau)-f(\cdot)$ belongs to the space $AP(\mathbb{R} : \mathbb{C})$ have been analyzed by A. M. Samoilenko and S. I. Trofimchuk in [896] (let us recall that the bounded functions satisfying this condition are always almost periodic due to the famous Loomis theorem [731]; see also the results obtained in the articles [127] by B. Basit and A. J. Pryde, [156] by I. Berg, [689] by M. A. Latif, M. I. Bhatti, [977] by R. Terras and [996, 997] by W. P. Veith). Let us also note that the function $f(\cdot)$, given by (2.28), has been employed by H. Y. Zhao and M. Fečkan in [1089], for proving the fact that for each finite real numbers $M, L > 0$ the set consisting of all almost periodic functions $h : \mathbb{R} \to \mathbb{R}$ such that $|h(t)| \leqslant M$, $t \in \mathbb{R}$ and $|h(t_1) - h(t_2)| \leqslant L|t_1 - t_2|$, $t_1, t_2 \in \mathbb{R}$ is not precompact in $C(\mathbb{R})$.

Furthermore, in [511, Theorem 1.2], A. Haraux and P. Souplet have proved that for each real number $c > 0$ the function $h(\cdot) = \min(c, f(\cdot))$, where $f(\cdot)$ is given by (2.28), is bounded uniformly continuous, uniformly recurrent and not asymptotically almost periodic. Since the function $h(\cdot)$ is uniformly continuous, Lemma 2.3.4(ii) below implies that $h(\cdot)$ is asymptotically Stepanov p-almost automorphic ($p \geqslant 1$) if and only if $h(\cdot)$ is asymptotically almost automorphic. But this is actually not the case because [511, Lemma 2.1] can be improved in the following manner.

Lemma 2.4.3. *Let $\omega : \mathbb{R} \to [0,\infty)$ be Lipschitz continuous and such that the set $\omega([0,+\infty))$ is unbounded. Define, for each finite number $c > \liminf_{t\to+\infty} \omega(t)$, the function $\omega_1 : \mathbb{R} \to [0,\infty)$ by $\omega_1(t) := \min(c, \omega(t))$, $t \in \mathbb{R}$. Then the restriction of the function $\omega_1(\cdot)$ to the non-negative real axis is not asymptotically almost automorphic.*

The proof of Lemma 2.4.3 is almost the same as that of [511, Lemma 2.1]. The only thing worth noticing is that the existence of an almost automorphic function $\omega_1^*(\cdot)$ such that $\lim_{t\to+\infty} |\omega_1(t) - \omega_1^*(t)| = 0$ implies, as in the proof of the above-mentioned lemma, that $\omega_1^* \equiv c$; this follows by using the same arguments, almost directly from the definition of almost automorphicity (we do not use here the fact that the limits in the second part of proof are uniform on $\mathbb{R}$).

We will extend [511, Theorem 1.2] in the following way.

Theorem 2.4.4. *Let the function $f(\cdot)$ be given by (2.28), and let $c > 0$. Then the function $h(t) := \min(c, f(t))$, $t \in \mathbb{R}$ is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

Concerning this contribution, we have made a decision to further analyze the function constructed by H. Bohr on pp. 113–115 of the first part of his landmark trilogy [196]. In actual fact, the results obtained by A. M. Fink in his doctoral dissertation [444] tell us that this function is uniformly continuous (nonexpansive, in fact), uniformly recurrent and not almost periodic. The construction of this function goes as follows. Let $\tau_1 := 1$, $\tau_2 > 2$ and let the sequence $(\tau_n)_{n\in\mathbb{N}}$ of positive real numbers satisfy $\tau_n > 2\sum_{i=1}^{n-1} i\tau_i$ for all $n \in \mathbb{N}$. Let the sequence $(f_n : \mathbb{R} \to \mathbb{R})_{n\in\mathbb{N}}$ be defined as follows. Set $f_1(x) := 1 - |x|$ for $|x| \leqslant 1$ and $f_1(x) := 0$, otherwise. If the functions $f_1(\cdot), \ldots, f_{n-1}(\cdot)$ are already defined, set

$$f_n(x) := f_{n-1}(x) + \sum_{m=1}^{n-1} \frac{n-m}{n}\big[f_{n-1}(x - m\tau_n) + f_{n-1}(x + m\tau_n)\big], \quad x \in \mathbb{R}.$$

Then

$$\big|f_n(x + \tau_n) - f_n(x)\big| \leqslant \frac{1}{n}, \quad n \in \mathbb{N},\ x \in \mathbb{R},$$

and the function

$$f(x) := \lim_{n\to+\infty} f_n(x), \quad x \in \mathbb{R}, \tag{2.29}$$

is well defined, even and satisfies $0 \leqslant f(x) \leqslant 1$ for all $x \in \mathbb{R}$. It is worth observing that this function also satisfies all clarified properties of the function $h(\cdot)$ from Theorem 2.4.4.

Theorem 2.4.5. *The function $f : \mathbb{R} \to \mathbb{R}$, given by (2.29), is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

In Example 2.4.37, we will show that, for some concrete choices of sequences $(\tau_n)_{n\in\mathbb{N}}$, the function $f : \mathbb{R} \to \mathbb{R}$, given by (2.29), is Weyl p-almost automorphic for each finite exponent $p \geqslant 1$. Since any Stepanov p-quasi-asymptotically almost periodic function is Weyl p-almost periodic ($p \geqslant 1$) in the sense of A. S. Kovanko's approach (see [647, Proposition 2.11]), it is quite reasonable to ask the following.

Question 2.4.6. Is it true that the function $f(\cdot)$, given by (2.29), is (equi-)Weyl p-almost periodic for some (each) finite exponent $p \geqslant 1$?

We would like to note that the function used by J. de Vries in [358, point 6., p. 208] can serve as a much simpler example of a bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying all clarified properties of functions examined in Theorem 2.4.4 and Theorem 2.4.5: Let $(p_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that $p_i | p_{i+1}$, $i \in \mathbb{N}$ and $\lim_{i\to\infty} p_i/p_{i+1} = 0$. Define the function $f_i : [-p_i, p_i] \to [0, 1]$ by $f_i(t) := |t|/p_i$, $t \in [-p_i, p_i]$ and extend the function $f_i(\cdot)$ periodically to the whole real axis; the obtained function, denoted by the same symbol $f_i(\cdot)$, is of period $2p_i$ ($i \in \mathbb{N}$). Set

$$f(t) := \sup\{f_i(t) : i \in \mathbb{N}\}, \quad t \in \mathbb{R}. \tag{2.30}$$

We will prove the following.

Theorem 2.4.7. *The function $f : \mathbb{R} \to \mathbb{R}$, given by (2.30), is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

We proceed with much elementary things, by considering a general continuous function $f : I \to X$. Suppose first that there exists a number $\varepsilon > 0$ such that $\vartheta(f, \varepsilon) \neq \emptyset$, say $\tau \in \vartheta(f, \varepsilon)$. Setting $M := \sup_{t\in I, |t|\leqslant\tau} \|f(t)\|$, it can be simply proved by induction that we have $\|f(t)\| \leqslant M + n\varepsilon$ for all $t \in I$ with $|t| \in [n\tau, (n+1)\tau]$ ($n \in \mathbb{N}$). Hence, $\|f(t)\| \leqslant M + |t|\varepsilon/\tau$ for all $t \in I$ with $|t| \in [n\tau, (n+1)\tau]$ ($n \in \mathbb{N}$), so that

$$\|f(t)\| \leqslant M + |t|\varepsilon/\tau, \quad t \in \mathbb{R} \tag{2.31}$$

and the function $f(\cdot)$ is linearly bounded as $|t| \to +\infty$. Furthermore, it is clear that the assumption $\vartheta(f, \varepsilon) \neq \emptyset$ for each $\varepsilon > 0$ implies that $\vartheta(f, \varepsilon)$ is infinite for each $\varepsilon > 0$ and that there does not exist a finite constant M such that the interval $[0, M]$ contains the union of sets $\vartheta(f, \varepsilon)$ when $\varepsilon > 0$; this is a simple consequence of the fact that for each $\varepsilon > 0$ we have $j\vartheta(f, \varepsilon/n) \subseteq \vartheta(f, \varepsilon)$ for all $j = 1, \ldots, n$. Let us observe that a linear function $f : I \to \mathbb{C}$ can serve as an example of a function for which the growth order in (2.31) cannot be improved and for which the assumption $\vartheta(f, \varepsilon) \neq \emptyset$ for each $\varepsilon > 0$ does not imply the existence of a number $\varepsilon_0 > 0$ such that the set $\vartheta(f, \varepsilon_0)$ is unbounded.

To the best of our knowledge, this is the first systematic study of vector-valued uniformly recurrent functions. In this section, we attempt to further profile the sets of

ε-periods of uniformly recurrent functions by introducing the class of $\odot_g$-almost periodic functions, which is simply defined by using the notions of lower and upper (Banach) densities for the subsets of the non-negative real axis (we feel it is our duty to say that we have only partially succeeded in our mission because it is very difficult to practically control and give intrinsic characterizations of ε-periods). The lower and upper (Banach) m_n-densities for the subsets of $\mathbb{N}$, considered recently in [643], are discrete analogues of the lower and upper (Banach) g-densities considered in this section. In the discrete setting, these densities play an important role in the field of linear chaos, for example, in definitions of frequent hypercyclicity and reiterative m_n-distributional chaos of linear continuous operators on Fréchet spaces. In the continuous setting, these densities play an important role in the qualitative analysis of solutions to the abstract (fractional) integro-differential equations in Fréchet spaces; see, e. g., the recent research monograph [632] by the author and references cited therein for a brief introduction to the theory of linear chaos. We generalize the notion of almost periodicity by analyzing several different types of (Stepanov) $\odot_g$-almost periodicity for functions with values in complex Banach spaces. In actual fact, we analyze the lower and upper (Banach) g-densities of sets $\vartheta(f,\varepsilon)$, where $\varepsilon > 0$ and $g : [0,\infty) \to [1,\infty)$ is an increasing mapping satisfying condition (2.33) below.

The organization of section can be briefly described as follows. Subsection 2.4.1 investigates the lower and upper (Banach) g-densities for the subsets of the non-negative real line; in this subsection, we present our first significant contributions, Theorem 2.4.10 and Theorem 2.4.11, in which we transfer the main result of paper [482] by G. Grekos, V. Toma and J. Tomanová to the continuous setting and reconsider the notion and several recent results from [643].

In Subsection 2.4.2, we analyze $\odot_g$-almost periodic functions, uniformly recurrent functions and their Stepanov generalizations. With the notation explained below, we say that a continuous function $f : I \to X$ is $\odot_g$-almost periodic if and only if for each $\varepsilon > 0$ we have $\odot_g(\vartheta(f,\varepsilon)) > 0$; see Definition 2.4.12, in which the symbol $\odot_g$ denotes exactly one of the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\underline{Bd}_{l;gc}$, $\underline{Bd}_{u;gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$. In the paragraph following Definition 2.4.12, we collect the basic properties of $\odot_g$-almost periodic functions and uniformly recurrent functions. The main purpose of Proposition 2.4.13 is to clarify the supremum formula for uniformly recurrent functions; in Proposition 2.4.14, we prove that any almost periodic function $f : I \to X$ is $\odot_g$-almost periodic. All introduced concepts are equivalent in case $g(x) \equiv x$, and reduced then to the concept of almost periodicity (Proposition 2.4.15). After that, in Proposition 2.4.16, we prove that the almost periodicity is equivalent with the $\underline{Bd}_{l;gc}$-almost periodicity and $\underline{Bd}_{u;gc}$-almost periodicity for every increasing mapping $g(\cdot)$ satisfying condition (2.33).

Definition 2.4.20 introduces the notions of asymptotical uniform recurrence and asymptotical $\odot_g$-almost periodicity, while Proposition 2.4.21 restates all results from Subsection 2.4.2 proved by then in this context. We introduce the notion of (asymptotical) Stepanov p-uniform recurrence and (asymptotical) Stepanov $(p,\odot_g)$-almost periodicity in Definition 2.4.22. The main purpose of Theorem 2.4.24 is to show that any asymptotically uniformly recurrent, quasi-asymptotically almost periodic function is asymptotically almost periodic; the Stepanov analogue of this statement is also considered here. Proposition 2.4.26 shows that the uniform recurrence and asymptotical almost automorphicity (asymptotical almost periodicity) implies almost automorphicity (almost periodicity), for the usually considered classes and Stepanov classes. Furthermore, in Theorem 2.4.28 and Proposition 2.4.29, we prove that any uniformly continuous (asymptotically) Stepanov p-uniformly recurrent [(asymptotically) Stepanov $(p, \odot_g)$-almost periodic/Stepanov p-quasi-asymptotically almost periodic] function $f : I \to X$ is asymptotically uniformly recurrent [asymptotically $\odot_g$-almost periodic, quasi-asymptotically almost periodic].

Proposition 2.4.31 indicates that for any (asymptotically) uniformly continuous, uniformly recurrent function we can find an increasing mapping $g : [0, \infty) \to [1, \infty)$ such that (2.33) holds and $f(\cdot)$ is (asymptotically) $\cdot_g$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$ (see also Remark 2.4.32, where we use the densities $\overline{Bd}_{l:gc}(\cdot)$ and $\overline{Bd}_{u:gc}(\cdot)$). In Example 2.4.35, we prove that the compactly almost automorphic function constructed by A. M. Fink in [443] is not asymptotically uniformly recurrent; the proofs of Theorem 2.4.2, Theorem 2.4.4, Theorem 2.4.5 and Theorem 2.4.7 are provided after that.

We investigate the existence and uniqueness of uniformly recurrent and $\odot_g$-almost periodic type solutions of abstract integro-differential equations in Banach spaces in a concise, semi-heuristical manner, paying special attention to the invariance of (asymptotical) uniform recurrence and (asymptotical) $\odot_g$-almost periodicity under the actions of convolution products.

The function sign : $\mathbb{R} \to \{-1, 0, 1\}$ is defined by $\operatorname{sign}(t) := -1$ (0, 1) if and only if $t < 0$ ($t = 0$, $t > 0$); if $c \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then we define $cA := \{ca : a \in A\}$. Let us recall that a function $f : (0, \infty) \to \mathbb{R}$ is called subadditive if and only if $f(x + y) \leqslant f(x) + f(y)$, $x, y > 0$. A continuous version of Fekete's lemma states that, for every measurable subadditive function $f : (0, \infty) \to \mathbb{R}$, the limit $\lim_{t\to+\infty}(f(t)/t)$ exists in $[-\infty, \infty)$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \inf_{t>0} \frac{f(t)}{t};$$

see, e. g., [536, Theorem 6.6.1]. We will use the following simple lemma.

Lemma 2.4.8. *There do not exist $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that*

$$\operatorname{sign}(\cos((n + k)\pi\sqrt{2})) = \operatorname{sign}(\cos(n\pi\sqrt{2})), \quad n \in \mathbb{Z}, |n| \geqslant n_0. \tag{2.32}$$

Proof. Since $\cos(n\pi\sqrt{2}) \neq 0$ for all $n \in \mathbb{Z}$, it is clear that (2.32) is equivalent to saying that $\cos((n + k)\pi\sqrt{2}) \cdot \cos(n\pi\sqrt{2}) > 0$, $n \in \mathbb{Z}$, $|n| \geqslant n_0$. If $k \in \mathbb{N}$ satisfies the above condition and $k\pi\sqrt{2} = 2k_0\pi + a$ for some numbers $k_0 \in \mathbb{Z}$ and $a \in (0, 2\pi)$, then we get from the above: $\cos(n\pi\sqrt{2}+a)\cdot\cos(n\pi\sqrt{2}) > 0$, $n \in \mathbb{Z}$, $|n| \geqslant n_0$. This cannot be satisfied because the set $\{e^{in\pi\sqrt{2}} : n \in \mathbb{Z}, |n| \geqslant n_0\}$ is dense in the unit circle and $\cos x = \operatorname{Re}(e^{ix})$, $x \in \mathbb{R}$. □

2.4.1 Lower and upper (Banach) g-densities

Unless stated otherwise, in this subsection we will always assume that $g : [0,\infty) \to [1,\infty)$ is an increasing mapping satisfying the requirement that there exists a finite number $L \geqslant 1$ such that

$$x \leqslant Lg(x), \quad x \geqslant 0, \tag{2.33}$$

which clearly implies $\liminf_{x\to+\infty} g(x)/x > 0$. If $A \subseteq [0,\infty)$ and $a,\ b \geqslant 0$, then we define $A(a,b) := \{x \in A\ ;\ x \in [a,b]\}$.

For simplicity and better exposition, in this subsection we will use the Lebesgue measure $m(\cdot)$ on the non-negative real line, only, which will be sufficiently enough for our analyses of uniformly continuous $\odot_g$-almost periodic functions; we are obliged to say that the general case is much more complicated and is almost not considered below.

Let us define (cf. [632] and [643] for more details):

(i) The lower g-density of A, denoted for short by $\underline{d}_{gc}(A)$,

$$\underline{d}_{gc}(A) := \liminf_{x\to+\infty} \frac{m(A(0,g(x)))}{x};$$

(ii) the upper g-density of A, denoted for short by $\overline{d}_{gc}(A)$,

$$\overline{d}_{gc}(A) := \limsup_{x\to+\infty} \frac{m(A(0,g(x)))}{x},$$

as well as:

(i) the lower $l;gc$-Banach density of A, denoted for short by $\underline{Bd}_{l;gc}(A)$,

$$\underline{Bd}_{l;gc}(A) := \liminf_{x\to+\infty} \liminf_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x};$$

(ii) the lower $u;gc$-Banach density of A, denoted for short by $\underline{Bd}_{u;gc}(A)$,

$$\underline{Bd}_{u;gc}(A) := \limsup_{x\to+\infty} \liminf_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x};$$

(iii) the (upper) $l;gc$-Banach density of A, denoted for short by $\overline{Bd}_{l;gc}(A)$,

$$\overline{Bd}_{l;gc}(A) := \liminf_{x\to+\infty} \limsup_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x};$$

(iv) the (upper) $u;fc$-Banach density of A, denoted for short by $\overline{Bd}_{u;gc}(A)$,

$$\overline{Bd}_{u;gc}(A) := \limsup_{x\to+\infty} \limsup_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x}.$$

Remark 2.4.9. It is worth noting that, for every set $A \subseteq [0, \infty)$, we have

$$\begin{aligned}
&\liminf_{x\to+\infty} \limsup_{y\to+\infty} \frac{m([I \setminus A](y, y+g(x)))}{x} \\
&\quad = \liminf_{x\to+\infty} \limsup_{y\to+\infty} \left[\frac{g(x) - m(A(y, y+g(x)))}{x} \right] \\
&\quad = \liminf_{x\to+\infty} \left[\frac{g(x)}{x} - \liminf_{y\to+\infty} \frac{m(A(y, y+g(x)))}{x} \right]. \qquad (2.34)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\limsup_{x\to+\infty} \limsup_{y\to+\infty} \frac{m([I \setminus A](y, y+g(x)))}{x} \\
&\quad = \limsup_{x\to+\infty} \left[\frac{g(x)}{x} - \liminf_{y\to+\infty} \frac{m(A(y, y+g(x)))}{x} \right], \qquad (2.35) \\
&\liminf_{x\to+\infty} \frac{m([I \setminus A](0, g(x)))}{x} = \liminf_{x\to+\infty} \left[\frac{g(x)}{x} - \limsup_{x\to+\infty} \frac{m(A(0, g(x)))}{x} \right],
\end{aligned}$$

and

$$\limsup_{x\to+\infty} \frac{m([I \setminus A](0, g(x)))}{x} = \limsup_{x\to+\infty} \left[\frac{g(x)}{x} - \liminf_{x\to+\infty} \frac{m(A(0, g(x)))}{x} \right].$$

Case $g(x) := (1 + |x|)^q$, $x \geqslant 0$ is the most important ($q \geqslant 1$), when we denote the corresponding densities by $\underline{d}_{qc}(A)$, $\overline{d}_{qc}(A)$, $\underline{Bd}_{l;qc}(A)$, $\underline{Bd}_{u;qc}(A)$, $\underline{Bd}_{l;qc}(A)$ and $\underline{Bd}_{l;qc}(A)$. Arguing similarly as in [643, Example 2.1(i)], for each number $q > 1$ we can simply construct a set $A \subseteq [0, \infty)$ such that $\overline{Bd}_{l;qc}(A) = 0$ and $\overline{Bd}_{u;qc}(A) = +\infty$; using the construction given in [643, Example 2.1(ii)], for each number $q > 1$ we can simply construct a set $A \subseteq [0, \infty)$ such that $\overline{d}_{qc}(A) = +\infty$ and $\overline{Bd}_{u;qc}(A) = 0$ so that the case $q > 1$ is not standard. Furthermore, if $q = 1$, then we get the usual concepts of lower and upper Banach densities: in this case, we have the following.

Theorem 2.4.10. *Let $A \subseteq [0, \infty)$. Then we have*

$$\begin{aligned}
&\underline{Bd}_{l;1c}(A) = \underline{Bd}_{u;1c}(A) \\
&\quad = \sup_{x>0} \liminf_{y\to+\infty} \frac{m(A(y, y+x))}{x} = \sup_{x>0} \inf_{y\geqslant 0} \frac{m(A(y, y+x))}{x} := \underline{Bd}_c(A)
\end{aligned}$$

and

$$\begin{aligned}
&\overline{Bd}_{l;1c}(A) = \overline{Bd}_{u;1c}(A) \\
&\quad = \inf_{x>0} \limsup_{y\to+\infty} \frac{m(A(y, y+x))}{x} = \inf_{x>0} \sup_{y\geqslant 0} \frac{m(A(y, y+x))}{x} := \overline{Bd}_c(A). \qquad (2.36)
\end{aligned}$$

Proof. Using the continuous version of Fekete's lemma, for the proof of the first equality in (2.36) it suffices to show that the function

$$F(x) := \limsup_{y\to+\infty} m\big(A(y, y+x)\big), \quad x > 0,$$

is subadditive, i. e., that for each fixed real numbers $x_1,\ x_2 > 0$ we have

$$\lim_{t\to+\infty}\sup_{t\geqslant y} m\big(A(t,t+x_1+x_2)\big) \leqslant \lim_{t\to+\infty}\sup_{t\geqslant y} m\big(A(t,t+x_1)\big) + \lim_{t\to+\infty}\sup_{t\geqslant y} m\big(A(t,t+x_2)\big).$$

This follows immediately if we prove that for each real number $y \geqslant 0$ we have

$$m\big(A(t,t+x_1+x_2)\big) \leqslant \sup_{t\geqslant y} m\big(A(t,t+x_1)\big) + \sup_{t\geqslant y} m\big(A(t,t+x_2)\big).$$

But this is a simple consequence of the fact that for each real number $y \geqslant 0$ we have $t + x_1 \geqslant y$ and

$$m\big(A(t,t+x_1+x_2)\big) \leqslant m\big(A(t,t+x_1)\big) + m\big(A(t+x_1,t+x_1+x_2)\big);$$

see also P. Ribenboim's paper [871]. Since

$$\limsup_{y\to+\infty} \frac{m(A(y,y+x))}{x} \leqslant \sup_{y\geqslant 0} \frac{m(A(y,y+x))}{x} \leqslant \liminf_{x\to+\infty}\sup_{y\geqslant 0} \frac{m(A(y,y+x))}{x},$$

for the proof of (2.36) it remains to be shown that

$$\liminf_{x\to+\infty}\sup_{y\geqslant 0} \frac{m(A(y,y+x))}{x} \leqslant \overline{Bd}_{u;1c}(A). \tag{2.37}$$

For this, we will slightly adapt the arguments proposed in the proof of discrete version of this statement, given in [482]. Define

$$D = \{x \in [0,1] : \forall L > 0\ \exists \text{ interval } I' \subseteq [0,\infty) \text{ s. t. } m(I') \geqslant L \text{ and } m(A\cap I')/m(I') \geqslant x\}.$$

Repeating verbatim the arguments given in [482, Subsection 2.1], we obtain

$$\liminf_{x\to+\infty}\sup_{y\geqslant 0} \frac{m(A(y,y+x))}{x} \leqslant b := \sup D.$$

The proof of (2.36) will be completed if one shows that

$$b \leqslant \inf_{x>0}\bigg(\limsup_{y\to+\infty} \frac{m(A(y,y+x))}{x}\bigg).$$

Suppose the contrary. Then there are a positive real number $x_0 > 0$ and two real numbers $x_1,\ x_2 \in [0,1]$ such that $x_1 < x_2 < b$ and

$$\limsup_{y\to+\infty} m\big(A(y,y+x_0)\big) < x_0x_1.$$

By definition of $\limsup_{y\to+\infty}\cdot$, this implies that there exists a positive real number $y_0 > 0$ such that $m(A(y,y+x_0)) < x_0x_1$ for all $y \geqslant y_0$. We will prove that there exists a sufficiently large number $L > 0$ such that every subinterval $I' \subseteq I$ with $m(I') \geqslant L$

satisfies $m(A \cap I') < x_2 m(I')$, showing that $x_2 \notin D$ and implying the contradiction. To see this, suppose that $I' = [y, y+h]$ for some $h > 0$. Then there exists $q \in \mathbb{N}_0$ such that $qx_0 \leqslant h < (q+1)x_0$ and therefore

$$\begin{aligned} m\big(A(y,y+h)\big) &\leqslant y_0 + m\big(A(y_0,y+h)\big) \leqslant y_0 + \sum_{j=0}^{q} m\big(A\big(y_0 + jx_0, y_0 + (j+1)x_0\big)\big) \\ &\leqslant y_0 + (q+1)x_0x_1 \leqslant y_0 + x_0x_1 + qx_0x_1 < y_0 + x_0x_1 + hx_1 < hx_2, \end{aligned}$$

for any $h > 0$ sufficiently large. The proof of (2.37) follows from (2.34)–(2.35) and (2.36). □

By the proof of Theorem 2.4.10, it follows that for each subset $A \subseteq [0,\infty)$ we have

$$\overline{Bd}_c(I \setminus A) + \underline{Bd}_c(A) = 1. \tag{2.38}$$

Since the case $g(x) \equiv x$ is very special in our analysis, we will also prove the following result which is well known in the discrete case (we then write $\underline{d}_c(A) \equiv \underline{d}_{gc}(A)$ and $\overline{d}_c(A) \equiv \overline{d}_{gc}(A)$).

Theorem 2.4.11. *Let $A \subseteq [0,\infty)$. Then we have*

$$0 \leqslant \underline{Bd}_c(A) \leqslant \underline{d}_c(A) \leqslant \overline{d}_c(A) \leqslant \overline{Bd}_c(A) \leqslant 1.$$

Proof. The only non-trivial parts are $\underline{Bd}_c(A) \leqslant \underline{d}_c(A)$ and $\overline{d}_c(A) \leqslant \overline{Bd}_c(A)$; due to (2.38), it suffices to show that $\overline{d}_c(A) \leqslant \overline{Bd}_c(A)$. Suppose the contrary. Due to (2.36) and definition of $\limsup_{x\to+\infty} \cdot$, it follows that

$$\lim_{t\to+\infty} \sup_{t\geqslant x} \frac{m(A(0,t))}{t} > \inf_{x>0} \sup_{y\geqslant 0} \frac{m(A(y,y+x))}{x}.$$

Since the mapping in the above limit is monotonically decreasing in variable t, we get the existence of positive real numbers $\delta > 0$, $x_0 > 0$ and $y_0 > 0$ such that

$$\frac{m(A(0,y))}{y} \geqslant \frac{m(A(z,z+x_0))}{x_0} + \delta, \quad y \geqslant y_0,\ z \geqslant 0. \tag{2.39}$$

Due to (2.39), we have

$$m\big(A(0,y)\big) \leqslant \sum_{j=0}^{\lfloor y/x_0 \rfloor} m\big(A\big(jx_0,(j+1)x_0\big)\big) \leqslant \big(\lfloor y/x_0 \rfloor + 1\big)\bigg(\frac{m(A(0,y))}{y} - \delta\bigg)x_0,$$

i. e.,

$$\bigg(1 - \frac{x_0}{y}\big(\lfloor y/x_0 \rfloor + 1\big)\bigg)\frac{m(A(0,y))}{y} \leqslant -\delta x_0\big(\lfloor y/x_0 \rfloor + 1\big)/y, \quad y \geqslant y_0.$$

After taking the limits as $y \to +\infty$, we obtain $0 \leqslant -\delta$, which is a contradiction. □

For more details about densities, see also Chapter 5. Let us finally note that, in the combinatorial and additive number theory, the sets with positive upper Banach density play a major role; see, e. g., [468, Section 5.7, Section 5.8]. A great number of results about the lower and upper (Banach) densities, known for the subsets of integers, cannot be so easily reformulated and reconsidered for the subsets of the non-negative real axis. This is not the case with the statements of [643, Proposition 2.5–Proposition 2.7, Corollary 2.2], which can be simply reformulated for (Banach) g-densities; details can be left to the interested reader.

2.4.2 $\odot_g$-Almost periodic functions, uniformly recurrent functions and their Stepanov generalizations

We will always assume henceforth that $g : [0,\infty) \to [1,\infty)$ is an increasing mapping satisfying the requirement that there exists a finite number $L \geqslant 1$ such that (2.33) holds. Let $\odot_g$ denote exactly one of the symbols $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\underline{Bd}_{l;gc}$, $\underline{Bd}_{u;gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$.

We start by introducing the following notion.

Definition 2.4.12. Let $f : I \to X$ be continuous. Then it is said that $f(\cdot)$ is $\odot_g$-almost periodic if and only if for each $\varepsilon > 0$ we have $\odot_g(\vartheta(f,\varepsilon)) > 0$.

We will use hereafter the following fundamental properties of $\odot_g$-almost periodic functions and uniformly recurrent functions, collected as follows (for parts (iv)–(vi), see [166, pp. 3–4]; for parts (vii)–(viii), see [697, p. 3]):

(i) Any constant function is $\odot_g$-almost periodic, and for any $\odot_g$-almost periodic (uniformly recurrent) function $f(\cdot)$ we see that the function $\|f(\cdot)\|$ is $\odot_g$-almost periodic (uniformly recurrent). Any $\odot_g$-almost periodic function is uniformly recurrent.

(ii) Since for each $\varepsilon > 0$ and $c \in \mathbb{C} \setminus \{0\}$ we have $\vartheta(cf,\varepsilon) = \vartheta(f,\varepsilon/|c|)$, the $\odot_g$-almost periodicity of the function $f(\cdot)$ implies the $\odot_g$-almost periodicity of the function $cf(\cdot)$. Similarly, the uniform recurrence of the function $f(\cdot)$ implies the uniform recurrence of the function $cf(\cdot)$.

(iii) The set consisting of all $\odot_g$-almost periodic (uniformly recurrent) functions is translation invariant in the sense that for each $\tau \in I$ and any $\odot_g$-almost periodic (uniformly recurrent) function $f(\cdot)$, the function $f(\cdot+\tau)$ is also $\odot_g$-almost periodic (uniformly recurrent).

(iv) If $(f_n(\cdot))$ is a sequence of $\odot_g$-almost periodic (uniformly recurrent) functions and $(f_n(\cdot))$ converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ is $\odot_g$-almost periodic (uniformly recurrent).

(v) If $X = \mathbb{C}$, $\inf_{x\in I} |f(x)| > m > 0$ and $f(\cdot)$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function, then the function $1/f(\cdot)$ is likewise a bounded $\odot_g$-almost periodic (uniformly recurrent).

(vi) If $f(\cdot)$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function and $g : [0,\infty) \to X$ is continuous, then the mapping $g(\|f(\cdot)\|)$ is bounded and $\odot_g$-almost periodic (uniformly recurrent).

(vii) If $f(\cdot)$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function and $r > 0$, then the function $\|f(\cdot)\|^r$ is bounded and $\odot_g$-almost periodic (uniformly recurrent).

Furthermore, it can be simply shown that:

(viii) If $f : \mathbb{R} \to X$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function and $\psi \in L^1(\mathbb{R})$, then the function $(\psi * f)(\cdot)$ is bounded, uniformly continuous and $\odot_g$-almost periodic (uniformly recurrent).

(ix) If $f : [0,\infty) \to X$ is uniformly recurrent and belongs to the space $C_0([0,\infty) : X)$, then $f \equiv 0$.

(x) If $f : \mathbb{R} \to X$ is $\odot_g$-almost periodic (uniformly recurrent), then the function $\check{f} : \mathbb{R} \to X$, defined by $\check{f}(\cdot) := f(-\cdot)$, is $\odot_g$-almost periodic (uniformly recurrent). If, additionally, $f_{|[0,\infty)}(\cdot) \in C_0([0,\infty) : X)$ or $\check{f}_{|[0,\infty)}(\cdot) \in C_0([0,\infty) : X)$, then $f \equiv 0$.

(xi) If $a \in I$ and the function $f(\cdot)$ is $\odot_g$-almost periodic (uniformly recurrent), then the function $f(\cdot + a) - f(\cdot)$ is $\odot_g$-almost periodic (uniformly recurrent).

For the sake of completeness, we will include short proofs of the following two propositions (the first proposition improves the corresponding result for almost periodic functions; for almost automorphic functions, see [631, Lemma 3.9.9]).

Proposition 2.4.13. *(Supremum formula) Suppose that $f : I \to X$ is uniformly recurrent. Then we have*

$$\sup_{t\in I}\|f(t)\| = \sup_{t\geqslant a}\|f(t)\| \in [0,\infty], \quad a \in I.$$

Proof. Let $a \in I$, $t \in I$ and $\varepsilon > 0$ be fixed. It suffices to show that

$$\|f(t)\| \leqslant \varepsilon + \sup_{s\geqslant a}\|f(s)\|.$$

In order to do that, take any strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and (2.3) holds. Let $n \in \mathbb{N}$ be such that $t + \alpha_n \geqslant a$. Then $\|f(t + \alpha_n) - f(t)\| \leqslant \varepsilon$ and therefore

$$\|f(t)\| \leqslant \varepsilon + \|f(t + \alpha_n)\| \leqslant \varepsilon + \sup_{s\geqslant a}\|f(s)\|,$$

as claimed. □

Proposition 2.4.14. *Any almost periodic function $f : I \to X$ is $\odot_g$-almost periodic.*

Proof. Let us recall that any almost periodic function is uniformly continuous. Using this fact, it can be easily shown that for each $\varepsilon > 0$ there exist two finite constants

$\delta > 0$ and $l > 0$ such that any segment $[y, y + g(x)]$ for $x \geqslant L(1 + l)$ and $y \geqslant 0$ contains the segment $[y, y + x/L]$ (cf. (2.33)) and therefore at least $\lfloor x/Ll \rfloor \geqslant 1$ disjunct intervals of length δ whose elements are ε-periods for $f(\cdot)$; see also [166, Corollary, p. 2]. This clearly implies $\odot_g(\vartheta(f, \varepsilon)) > \delta/Ll > 0$. □

Now we will prove the following.

Proposition 2.4.15. *Let $f : I \to X$ be continuous and $g(x) \equiv x$. Then $f(\cdot)$ is almost periodic if and only if $f(\cdot)$ is $\odot_g$-almost periodic.*

Proof. Having in mind Proposition 2.4.14 and Theorem 2.4.11, it suffices to show that any $\underline{Bd}_c$-almost periodic function $f : I \to X$ is almost periodic. Towards this end, it suffices to show that any set $A \subseteq [0, \infty)$ satisfying $\underline{Bd}_c(A) > 0$ is relatively dense. Otherwise, for every real number $L > 0$, we see that there exists an interval I_L of length L which does not contain any ε-period of $f(\cdot)$. Thus, an unbounded set $\bigcup_{n\in\mathbb{N}} I_{2^n}$ does not contain any ε-period of $f(\cdot)$, which immediately implies that $\underline{Bd}_c(A) = 0$ by definition. □

Concerning the notions of $\underline{Bd}_{l;gc}$-almost periodicity and $\underline{Bd}_{u;gc}$-almost periodicity, the things are pretty clear. In the following proposition, whose discrete analogue has been considered in [643, Proposition 2.4], we will prove that these notions are equivalent with the one of almost periodicity.

Proposition 2.4.16. *Let $f : I \to X$ be continuous and let $g : [0, \infty) \to [1, \infty)$ be an increasing mapping satisfying the requirement that there exists a finite number $L \geqslant 1$ such that* (2.33) *holds. Then $f(\cdot)$ is almost periodic if and only if $f(\cdot)$ is $\underline{Bd}_{l;gc}$-almost periodic if and only if $f(\cdot)$ is $\underline{Bd}_{u;gc}$-almost periodic.*

Proof. Due to Proposition 2.4.14 and the fact that any $\underline{Bd}_{l;gc}$-almost periodic function is $\underline{Bd}_{u;gc}$-almost periodic, it suffices to show that any $\underline{Bd}_{u;gc}$-almost periodic function is almost periodic. Suppose the contrary and fix a number $x > 0$. Then there exists a number $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$, there exists an interval $I_n = [y_n, y_n + 2n + 2g(x)] \subseteq [0, \infty)$ of length $2n + 2g(x)$ such that the set $\vartheta(f, \varepsilon)$ does not meet I_n. Then, for every $n \in \mathbb{N}$, the interval $I_n' = [y_n + n + g(x), y_n + 2n + 2g(x)]$ does not meet $\vartheta(f, \varepsilon)$ and has the length $n + g(x) \geqslant g(x)$. This implies $m(([\vartheta(f, \varepsilon)](y_n + n + g(x), y_n + 2n + 2g(x))) = 0$. Hence, $\liminf_{y\to+\infty} m([\vartheta(f, \varepsilon)](y, y + x)) = 0$, which contradicts the condition $\underline{Bd}_{u;gc}(\vartheta(f, \varepsilon)) > 0$. □

Remark 2.4.17. Let $f : I \to X$ be continuous and let $c \in I \setminus \{0\}$. Define the function $f_c : I \to X$ by $f_c(t) := f(ct)$, $t \in I$. Then we have $|c|\vartheta(f, \varepsilon) \subseteq \vartheta(f_c, \varepsilon)$ for all $\varepsilon > 0$, which simply implies that for any uniformly recurrent function $f(\cdot)$ we see that the function $f_c(\cdot)$ is uniformly recurrent. Due to Proposition 2.4.16 and the corresponding statement for almost periodic functions, the same holds for $\odot_g$-almost periodicity with $\odot_g \in \{\underline{Bd}_{l;gc}, \underline{Bd}_{u;gc}\}$. If $\odot_g$ is one of the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$, then directly from their definitions and the definition of $\odot_g$-almost periodicity we may conclude, keeping in mind the fact that for any Lebesgue measurable subset $A \subseteq [0,\infty)$ the set cA is also Lebesgue measurable with $m(cA) = cm(A)$, that the $\odot_g$-almost periodicity of the function $f(\cdot)$ implies the $\odot_g$-almost periodicity of the function $f_c(\cdot)$ for any $c \in I \smallsetminus \{0\}$ with $|c| \leqslant 1$. Assume now that $\odot_g$ is one of the above four densities and $|c| > 1$. In this case, it is almost inevitable to impose some additional conditions on the function $g(\cdot)$ under which the $\odot_g$-almost periodicity of the function $f(\cdot)$ implies the $\odot_g$-almost periodicity of the function $f_c(\cdot)$. For example, it is very natural to assume additionally that $g(\cdot)$ is continuous, strictly increasing and that there exist two numbers $t_0 > 0$ and $\delta > 0$ such that $|c|g(t) \leqslant g(t/\delta)$ for all $t \geqslant t_0$. For the Banach density $\overline{Bd}_{u;gc}$, the claimed statement then follows from the computation ($x > 0$ satisfies $t = g^{-1}(g(x)/c) \geqslant t_0$)

$$\begin{aligned}\limsup_{y\to+\infty} \frac{m(cA(y,y+g(x)))}{x} &= \limsup_{y\to+\infty} \frac{cm(A(y/c,y/c+(g(x)/c)))}{x}\\ &= \limsup_{y\to+\infty} \frac{m(A(y,y+(g(x)/c)))}{x} = \limsup_{y\to+\infty} \frac{m(A(y,y+g(t)))}{g^{-1}(cg(t))}\\ &= \limsup_{y\to+\infty} \frac{m(A(y,y+g(t)))}{t} \frac{t}{g^{-1}(cg(t))}\\ &\geqslant \delta \limsup_{y\to+\infty} \frac{m(A(y,y+g(t)))}{t}.\end{aligned}$$

For the Banach density $\overline{Bd}_{l;gc}$ and for the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, the claimed statement follows similarly.

Remark 2.4.18 (see also [511, Lemma 2.1]). If $f : \mathbb{R} \to \mathbb{R}$ is a (uniformly) continuous, $\odot_g$-almost periodic (uniformly recurrent) function, $\varepsilon > 0$, $c \in \mathbb{R}$ and $\tau \in \vartheta(f,\varepsilon)$, then $\tau \in \vartheta(\min(c,f),\varepsilon)$ and the function $\min(c,f(\cdot))$ is (uniformly) continuous and $\odot_g$-almost periodic (uniformly recurrent).

Remark 2.4.19. Let $f : \mathbb{R} \to \mathbb{R}$ be an almost periodic function such that there exist two real numbers a and b such that $a < 0 < b$ and an analytic function $F : \{z \in \mathbb{C} : a < \operatorname{Re} z < b\} \to \mathbb{C}$ such that $F(ix) = f(x)$ for all $x \in \mathbb{R}$. Then the function $h : \mathbb{R} \to \mathbb{R}$, defined by $h(x) := \operatorname{sign}(f(x))$, $x \in \mathbb{R}$ is Stepanov p-almost periodic for any finite exponent $p \geqslant 1$. For $p = 1$, this has been proved in [696, Theorem 5.3.1, p. 210], while the general case follows from the consideration given in [631, Example 2.2.3(i)] (we feel it to be our duty to say that we have made small mistakes in the formulations of conditions in [631, Example 2.2.2, Example 2.2.3(ii)] by neglecting the necessary condition on the analytical extensibility of the function $f((-i)\cdot)$ to the strip $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$). The Bochner criterion is essentially employed in the proof of the above-mentioned theorem and we would like to observe here that the above condition on the analytical extensibility of the function $f((-i)\cdot)$ can be neglected in some situations, even for the uniform recurrence and $\odot_g$-almost periodicity. More precisely, let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly recurrent function (an $\odot_g$-almost periodic function) satisfying

$$(\exists L \geqslant 1)\ (\forall \varepsilon > 0)\ (\forall y \in \mathbb{R})\ m\big(\{x \in [y,y+1] : |f(x)| \leqslant \varepsilon\}\big) \leqslant L\varepsilon.$$

Then the function $h(\cdot)$, defined above, is uniformly recurrent ($\odot_g$-almost periodic), which follows from the foregoing arguments.

Now we will introduce the following definition.

Definition 2.4.20.
(i) Suppose that $f \in C(I : X)$. Then we say that the function $f(\cdot)$ is asymptotically uniformly recurrent if and only if there exist a uniformly recurrent function $h : \mathbb{R} \to X$ and a function $\phi \in C_0(I : X)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$.
(ii) Suppose that $f \in C(I : X)$. Then we say that the function $f(\cdot)$ is asymptotically $\odot_g$-almost periodic if and only if there exist an $\odot_g$-almost periodic function $h : \mathbb{R} \to X$ and a function $\phi \in C_0(I : X)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$.

Assume that the function $f : [0,\infty) \to X$ is continuous and the function $h : [0,\infty) \to X$ is continuous. For each $\varepsilon > 0$ and $M > 0$, we define

$$\vartheta_M(f,\varepsilon) := \{\tau > 0 : \|f(t+\tau) - f(t)\| \leqslant \varepsilon,\ t \geqslant M\}.$$

Then it is clear that the assumption $M_1 \leqslant M_2$ implies $\vartheta_{M_1}(f,\varepsilon) \subseteq \vartheta_{M_2}(f,\varepsilon)$. Furthermore, if $\phi \in C_0([0,\infty) : X)$ and $\varepsilon > 0$, then we have the existence of a number $M > 0$ such that

$$\begin{aligned}\|[h+\phi](t+\tau) - [h+\phi](t)\| &\leqslant \|h(t+\tau) - h(t)\| + \|\phi(t+\tau) - \phi(t)\| \\ &\leqslant \|h(t+\tau) - h(t)\| + \frac{\varepsilon}{2}, \quad t \geqslant M,\end{aligned}$$

so that $\vartheta(h,\varepsilon/2) \subseteq \vartheta_M(h+\phi,\varepsilon)$. Therefore, for any asymptotically $\odot_g$-almost periodic function $f : [0,\infty) \to X$ for each $\varepsilon > 0$ there exists $M > 0$ such that $\odot_g(\vartheta_M(f,\varepsilon)) > 0$ (a similar statement holds for the Stepanov classes). In the case that $g(x) \equiv x$, then we also have the converse: if for each $\varepsilon > 0$ there exists $M > 0$ such that $\odot_g(\vartheta_M(f,\varepsilon)) > 0$, then the function $f(\cdot)$ is asymptotically almost periodic; if $\odot_g$ is $\underline{Bd}_{l;gc}$ or $\underline{Bd}_{u;gc}$, then the converse also holds in general case. For the remaining four densities, it seems very conceivable that the converse does not hold in general case.

From this definition and previously proved results in this section, it is clear that we have the following.

Proposition 2.4.21.
(i) *Any asymptotically almost periodic function is asymptotically $\odot_g$-almost periodic, and any asymptotically $\odot_g$-almost periodic function is asymptotically uniformly recurrent.*
(ii) *Let $f : I \to X$ be continuous and $g(x) \equiv x$. Then $f(\cdot)$ is asymptotically almost periodic if and only if $f(\cdot)$ is asymptotically $\odot_g$-almost periodic.*
(iii) *Let $f : I \to X$ be continuous and let $g : [0,\infty) \to [1,\infty)$ be an increasing mapping satisfying the requirement that there exists a finite number $L \geqslant 1$ such that* (2.33) *holds. Then $f(\cdot)$ is asymptotically almost periodic if and only if $f(\cdot)$ is asymptotically $\underline{Bd}_{l;gc}$-almost periodic if and only if $f(\cdot)$ is asymptotically $\underline{Bd}_{u;gc}$-almost periodic.*

Now we have an open door to the introduction of the concepts of (asymptotical) Stepanov $p(x)$-uniform recurrence and (asymptotical) Stepanov $(p(x), \odot_g)$-almost periodicity.

Definition 2.4.22.
(i) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{\mathrm{loc}}(I : X)$ is said to be Stepanov $p(x)$-uniformly recurrent if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (2.5), is uniformly recurrent.
(ii) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{\mathrm{loc}}(I : X)$ is said to be Stepanov $(p(x), \odot_g)$-almost periodic if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (2.5), is $\odot_g$-almost periodic.

If $p(x) \equiv p \in [1, \infty)$, then we also say that the function $f(\cdot)$ is Stepanov p-uniformly recurrent (Stepanov $(p, \odot_g)$-almost periodic).

Definition 2.4.23.
(i) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{\mathrm{loc}}(I : X)$ is said to be asymptotically Stepanov $p(x)$-uniformly recurrent if and only if there exist a Stepanov $p(x)$-uniformly recurrent function $h : \mathbb{R} \to X$ and a function $q \in L^{p(x)}_S(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^{p(x)}([0,1] : X))$.
(ii) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{\mathrm{loc}}(I : X)$ is said to be asymptotically Stepanov $(p(x), \odot_g)$-almost periodic if and only if there exist a Stepanov $(p(x), \odot_g)$-almost periodic function $h : \mathbb{R} \to X$ and a function $q \in L^{p(x)}_S(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^{p(x)}([0,1] : X))$.

If $p(x) \equiv p \in [1, \infty)$, then we also say that the function $f(\cdot)$ is asymptotically Stepanov p-uniformly recurrent (asymptotically Stepanov $(p, \odot_g)$-almost periodic).

We can simply state the analogues of Proposition 2.4.14-Proposition 2.4.16 and Proposition 2.4.21 for the Stepanov classes. Taking into account Proposition 2.4.16 and Proposition 2.4.21(iii), in the remainder of section we will always assume, if not explicitly stated otherwise, that $\odot_g$ denotes exactly one of the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$. Before proceeding any further, we would like to note that we can similarly introduce and analyze the concepts of $\odot_g$-almost anti-periodicity and Stepanov $(p, \odot_g)$-almost anti-periodicity [631].

The following result, which is closely related with [647, Theorem 2.5, Theorem 2.10], plays a significant role in the proof of Theorem 2.4.4.

Theorem 2.4.24.
(i) *Suppose that the function $f : I \to X$ is asymptotically uniformly recurrent and quasi-asymptotically almost periodic. Then the function $f(\cdot)$ is asymptotically almost periodic.*

(ii) *Suppose that* $p \in \mathcal{P}([0,1])$, *the function* $f \in L_S^{p(x)}(I : X)$ *is asymptotically Stepanov* $p(x)$*-uniform recurrent and Stepanov* $p(x)$*-quasi-asymptotically almost periodic. Then the function* $f(\cdot)$ *is asymptotically Stepanov* $p(x)$*-almost periodic.*

Proof. The proof of theorem essentially follows from the argumentation contained in the proof of [631, Theorem 2.5]; for the sake of completeness, we will include all details of proof. Suppose that the function $f : I \to X$ satisfies the assumptions in (i). Then there exist a uniformly recurrent function $h(\cdot)$ and a function $q \in C_0(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\big\|[h(t+\tau) - h(t)] + [q(t+\tau) - q(t)]\big\| \leqslant \varepsilon, \quad \text{provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

Since $f(\cdot)$ is bounded and $q \in C_0(I : X)$, we see that $h(\cdot)$ is bounded. The above implies the existence of a finite number $M_1(\varepsilon, \tau) \geqslant M(\varepsilon, \tau)$ such that

$$\big\|h(t+\tau) - h(t)\big\| \leqslant 2\varepsilon, \quad \text{provided } t \in I \text{ and } |t| \geqslant M_1(\varepsilon, \tau). \tag{2.40}$$

Define the function $H : I \to X$ by $H(t) := h(t+\tau) - h(t)$, $t \in I$. Then the function $H(\cdot)$ is bounded and, due to the property (xi), we see that the function $H(\cdot)$ is uniformly recurrent. Applying supremum formula clarified in Proposition 2.4.13 and (2.40), we get

$$\sup_{t \in I} \big\|H(t)\big\| = \sup_{t \geqslant M_1(\varepsilon,\tau)} \big\|H(t)\big\| = \sup_{t \geqslant M_1(\varepsilon,\tau)} \big\|h(t+\tau) - h(t)\big\| \leqslant 2\varepsilon.$$

Hence, $\|h(t+\tau) - h(t)\| \leqslant 2\varepsilon$ for all $t \in I$ and $h(\cdot)$ is almost periodic by definition, which completes the proof of part (i). For part (ii), observe first that there exist a Stepanov p-uniformly recurrent function $h(\cdot)$ and a function $q \in L_S^p(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : X))$. Repeating verbatim the arguments given in the proof of part (i), with the function $f(\cdot)$ replaced therein with the function $\hat{f}(\cdot)$, we see that the function $\hat{h} : I \to L^p([0,1] : X)$ is asymptotically almost periodic. This simply completes the proof of (ii). □

Example 2.4.25. Define

$$f(t) := \left(\frac{4n^2t^2}{(t^2+n^2)^2}\right)_{n \in \mathbb{N}}, \quad t \geqslant 0.$$

Then $f \in Q-\mathrm{AAA}([0,\infty) : c_0) \cap \mathrm{BUC}([0,\infty) : c_0)$ and $f(\cdot)$ is not asymptotically almost automorphic (see [647, Example 2.6, Theorem 2.5]). Due to Theorem 2.4.24(ii) and Lemma 2.3.4(i), the function $f(\cdot)$ is not asymptotically Stepanov (1-)uniformly recurrent.

The results presented in the subsequent proposition are expected to a certain extent:

Proposition 2.4.26. *Let $p \in \mathcal{P}([0,1])$.*
(i) *If $f : \mathbb{R} \to X$ is uniformly recurrent and asymptotically almost automorphic, then $f(\cdot)$ is almost automorphic.*
(ii) *If $f : I \to X$ is uniformly recurrent and asymptotically almost periodic, then $f(\cdot)$ is almost periodic.*
(iii) *If $f : \mathbb{R} \to X$ is Stepanov $p(x)$-uniformly recurrent and asymptotically Stepanov $p(x)$-almost automorphic, then $f(\cdot)$ is Stepanov $p(x)$-almost automorphic.*
(iv) *If $f : I \to X$ is Stepanov $p(x)$-uniformly recurrent and asymptotically Stepanov $p(x)$-almost periodic, then $f(\cdot)$ is Stepanov $p(x)$-almost periodic.*

Proof. We will prove only (i). Suppose that $f : \mathbb{R} \to X$ is uniformly recurrent and asymptotically almost automorphic. Then there exist a function $h \in AA(\mathbb{R} : X)$, a function $q \in C_0(\mathbb{R} : X)$ and a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (2.3) holds and $f(t) = h(t) + q(t)$ for all $t \in \mathbb{R}$. Fix a number $t \in \mathbb{R}$. Then $\lim_{n\to+\infty} q(t + \alpha_n) = 0$ and, in combination with (2.3), we get

$$\lim_{n\to+\infty} h(t+\alpha_n) = f(t) \quad \text{and} \quad \lim_{n\to+\infty} f(t-\alpha_n) = f(t). \tag{2.41}$$

Since $h(\cdot)$ is almost automorphic, we can extract a subsequence (β_n) of (α_n) such that there exists a mapping $f_1 : \mathbb{R} \to X$ satisfying

$$\lim_{n\to+\infty} h(t+\beta_n) = f_1(t) \quad \text{and} \quad \lim_{n\to+\infty} f_1(t-\beta_n) = h(t) \quad \text{for all } t \in \mathbb{R}. \tag{2.42}$$

The uniqueness of the first limits in (2.41) and (2.42) yields $f_1(t) = f(t)$. Using the uniqueness of the second limits in (2.41) and (2.42), we get $f(t) = h(t)$, which completes the proof of (i). □

Combining Theorem 2.4.24 and Proposition 2.4.26, we may deduce the following.

Corollary 2.4.27. *Let $p \in \mathcal{P}([0,1])$.*
(i) *If $f : I \to X$ is uniformly recurrent and asymptotically almost periodic, then $f(\cdot)$ is almost periodic.*
(ii) *If $f \in L_S^{p(x)}(I : X)$ is Stepanov $p(x)$-uniform recurrent and Stepanov $p(x)$-quasi-asymptotically almost periodic, then $f(\cdot)$ is Stepanov $p(x)$-almost periodic.*

In the following theorem, we reconsider the statements given in Lemma 2.3.4 for the (asymptotical) Stepanov $p(x)$-uniform recurrence and (asymptotical) Stepanov $(p(x), \odot_g)$-almost periodicity.

Theorem 2.4.28. *Let $p \in \mathcal{P}([0,1])$.*
(i) *If the function $h : I \to X$ is uniformly recurrent, $\phi \in C_0(I : X)$ and $f(t) = h(t) + \phi(t)$ for all $t \in I$, then*

$$\{h(t) : t \in I\} \subseteq \overline{\{f(t) : t \in I\}}. \tag{2.43}$$

(ii) *If $h : I \to X$ is uniformly continuous and Stepanov $p(x)$-uniformly recurrent (Stepanov $(p(x), \odot_g)$-almost periodic), then the function $h(\cdot)$ is uniformly recurrent ($\odot_g$-almost periodic).*
(iii) *If $f : I \to X$ is uniformly continuous and asymptotically Stepanov $p(x)$-uniformly recurrent (asymptotically Stepanov $(p(x), \odot_g)$-almost periodic), then the function $f(\cdot)$ is asymptotically uniformly recurrent (asymptotically $\odot_g$-almost periodic).*

Proof. Part (i) can be simply deduced as follows. Let the numbers $t \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. It is clear that there exists a strictly increasing sequence (α_n) of positive real numbers such that $\|h(t) - h(t + \alpha_n)\| < \varepsilon/2$, $n \in \mathbb{N}$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\|h(t) - f(t + \alpha_n)\| \leqslant \|h(t) - h(t + \alpha_n)\| + \|q(t + \alpha_n)\| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This, in turn, implies (2.43). For the proofs of (ii) and (iii), it suffices to consider case $p(x) \equiv 1$. If the function $h : I \to X$ satisfies the requirements of (ii), then for each $\sigma \in (0, 1)$ the function $h_\sigma : I \to X$, given by

$$h_\sigma(t) := \frac{1}{\sigma} \int_t^{t+\sigma} h(s)\, ds, \quad t \in I, \tag{2.44}$$

is continuous and, due to the uniform continuity of $h(\cdot)$, we have the existence of a number $\delta \in (0, 1)$ such that $\|h(t') - h(t'')\| < \varepsilon$, provided t', $t'' \in I$ and $|t' - t''| < \delta$. Therefore, if $\sigma \in (0, \delta)$, then we have

$$\|h_\sigma(t) - h(t)\| \leqslant \frac{1}{\sigma} \int_t^{t+\sigma} \|h(s) - h(t)\|\, ds < \varepsilon, \quad t \in \mathbb{R}, \tag{2.45}$$

and $\lim_{\sigma \to 0+} h_\sigma(t) = h(t)$ uniformly in $t \in I$. By property (iv) from the beginning of section, it suffices to show that for each fixed number $\sigma \in (0, 1)$ the function $h_\delta(\cdot)$ is uniformly recurrent ($\odot_g$-almost periodic). But this follows from the argumentation given on [166, p. 80], where it has been proved that for each number $\varepsilon > 0$ we have $\vartheta(\hat{h}, \sigma\varepsilon) \subseteq \vartheta(h_\sigma, \varepsilon)$. This completes the proof of (ii). To deduce (iii), observe that there exist a Stepanov 1-uniformly recurrent (Stepanov $(1, \odot_g)$-almost periodic) function $h(\cdot)$ and a function $q \in L_S^1(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^1([0, 1] : X))$. Using (i) and the arguments contained in the proof of [560, Proposition 3.1], we see that the both functions $h(\cdot)$ and $q(\cdot)$ are uniformly continuous. This shows that $q \in C_0(I : X)$ and, due to part (ii), $h(\cdot)$ is uniformly recurrent ($\odot_g$-almost periodic). The proof of the theorem is thereby completed. □

In [1042, Proposition 12], R. Xie and C. Zhang have proved that any uniformly continuous function $f \in S^p\mathrm{SAP}_\omega(I : X)$ belongs to the space $\mathrm{AP}_\omega(I : X)$; see [1042] for the notion. As already mentioned, we have $S^p\mathrm{SAP}_\omega(I : X) \subseteq S^p Q - \mathrm{AAP}(I : X)$ and it is reasonable to ask whether we can extend the above result by showing that any uniformly continuous function $f \in S^pQ - \mathrm{AAP}(I:X)$ belongs to the space $Q - \mathrm{AAP}(I:X)$. This is actually the case, as the next proposition shows (an extension to the variable exponent $p \in \mathcal{P}([0,1])$ can be made).

Proposition 2.4.29. *Let $p \in [1,\infty)$, and let $f \in S^pQ - \mathrm{AAP}(I:X)$ be uniformly continuous. Then $f \in Q - \mathrm{AAP}(I:X)$.*

Proof. The proof of proposition is very similar to the proof of Theorem 2.4.28(ii). Clearly, it suffices to consider the case $p = 1$. Define, for every number $\sigma \in (0,1)$, the function $f_\sigma(\cdot)$ by replacing the function $h(\cdot)$ in (2.44) with the function $f(\cdot)$. Then the function $f_\sigma(\cdot)$ is bounded and continuous ($\sigma \in (0,1)$). Furthermore, (2.45) holds with the functions $h_\sigma(\cdot)$ and $h(\cdot)$ replaced therein with the functions $f_\sigma(\cdot)$ and $f(\cdot)$. Due to [647, Theorem 2.13(ii)], it suffices to show that the function $f_\sigma(\cdot)$ is quasi-asymptotically almost periodic for each number $\sigma \in (0,1)$. But this simply follows from the estimate

$$\|f_\sigma(t+\tau) - f_\sigma(t)\| \leqslant \frac{1}{\sigma}\int\limits_t^{t+1}\|f(s+\tau) - f(s)\|\,ds, \quad t \in I,\ \tau \in I,\ \sigma \in (0,1),$$

which can be proved as in [166, p. 80]. □

Remark 2.4.30. The proof of Proposition 2.4.29 considerably shortens the proof of [1042, Proposition 12]. Therefore, the word “Stepanov” in the formulations of Theorem 2.4.4 and Theorem 2.4.5 can be encompassed with the round brackets.

The following proposition will be important in the sequel.

Proposition 2.4.31. *Suppose that the function $f : I \to X$ is uniformly continuous and (asymptotically) uniformly recurrent. Then there exist a finite number $L \geqslant 1$ and an increasing mapping $g : [0,\infty) \to [1,\infty)$ such that (2.33) holds and $f(\cdot)$ is (asymptotically) $\cdot_g$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$.*

Proof. Without loss of generality, we may assume that the equation (2.3) holds with the sequence (α_n) satisfying $\alpha_{n+1} - \alpha_n \geqslant 1$. It suffices to prove the proposition for uniformly recurrent functions. Let $\varepsilon > 0$ be fixed. Due to the uniform continuity of $f(\cdot)$, we see that there exist an integer $n_0 \in \mathbb{N}$ and a finite real number $\delta > 0$ such that the set $\vartheta(f,\varepsilon)$ contains the union of disjunct intervals $[\alpha_n - \delta, \alpha_n + \delta]$ for $n \geqslant n_0$. Let $g : [0,\infty) \to [1,\infty)$ be any increasing mapping such that $g(n) > \alpha_{n+1}$ for all $n \in \mathbb{N}$. Hence, (2.33) holds with some finite number $L \geqslant 1$. Furthermore, if $x \in [n, n+1]$, then the interval $[0, g(x)]$ contains at least $(n-n_0)$ disjunct intervals of length δ whose union belongs to $\vartheta(f,\varepsilon)$. This simply implies that $m([\vartheta(f,\varepsilon)](0,g(x))) \geqslant \delta(n-n_0)$ and therefore $m([\vartheta(f,\varepsilon)](0,g(x)))/x \geqslant \delta(n-n_0)/(n+1)$. This simply implies $\underline{d}_c(\vartheta(f,\varepsilon)) > 0$, so that $f(\cdot)$ is $\underline{d}_{gc}$-almost periodic and therefore $\overline{d}_{gc}$-almost periodic. □

Remark 2.4.32. The proof of Proposition 2.4.31 does not work for the upper $l;gc$-Banach density $\overline{Bd}_{l;gc}(\cdot)$ and the upper $u;gc$-Banach density $\overline{Bd}_{u;gc}(\cdot)$. In general, these densities differ from the densities

$$\overline{Bd}_{l:gc}(A) := \liminf_{x\to+\infty} \sup_{y\geqslant 0} \frac{m(A(y,y+g(x)))}{x}$$

and

$$\overline{Bd}_{u:gc}(A) := \limsup_{x\to+\infty} \sup_{y\geqslant 0} \frac{m(A(y,y+g(x)))}{x},$$

respectively. Repeating verbatim the above arguments, it can be simply proved that for any uniformly continuous, uniformly recurrent function $f : I \to X$ there exist a finite number $L \geqslant 1$ and an increasing mapping $g : [0,\infty) \to [1,\infty)$ such that (2.33) holds and $f(\cdot)$ is $\cdot_g$-almost periodic for $\cdot_g \in \{\overline{Bd}_{l:gc}, \overline{Bd}_{u:gc}\}$.

Remark 2.4.33. By the proof of Proposition 2.4.31, it follows that, for every uniformly continuous, uniformly recurrent function $f_i : I \to X$ $(1 \leqslant i \leqslant n)$, we can find a finite number $L \geqslant 1$ and an increasing mapping $g : [0,\infty) \to [1,\infty)$ such that (2.33) holds and $f_i(\cdot)$ is $\cdot_g$-almost periodic for all $1 \leqslant i \leqslant n$ and $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$.

Keeping in mind the corresponding definitions and Proposition 2.4.31, the next result follows immediately (the previous two remarks can be reformulated in this context as well).

Proposition 2.4.34. *Suppose that $p \in \mathcal{P}([0,1])$, $f : I \to X$ is (asymptotically) Stepanov $p(x)$-uniformly recurrent and $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is uniformly continuous. Then there exist a finite number $L \geqslant 1$ and an increasing mapping $g : [0,\infty) \to [1,\infty)$ such that (2.33) holds and $f(\cdot)$ is (asymptotically) Stepanov $(p(x), \odot_g)$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$.*

It is worth noticing that Proposition 2.4.31 cannot be applied to the compactly almost automorphic functions which are not asymptotically uniformly recurrent, in general. Concerning this problematic, we would like to present the following illustrative example.

Example 2.4.35. Any almost periodic function has to be compactly almost automorphic, while the converse statement is not true, however. The first example of a scalar-valued compactly almost automorphic function which is not almost periodic has been constructed by A. M. Fink (see [443, p. 521]). Set $a_n := \operatorname{sign}(\cos(n\pi\sqrt{2}))$, $n \in \mathbb{Z}$ and define after that the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := \alpha a_n + (1-\alpha)a_{n+1}$ if $t \in [n, n+1)$ for some integer $n \in \mathbb{Z}$ and $t = \alpha n + (1-\alpha)(n+1)$ for some number $\alpha \in (0,1]$. As verified in [443], this function is compactly almost automorphic (therefore, uniformly continuous) but not almost periodic. We will extend this result by showing that the function $f(\cdot)$ is not asymptotically uniformly recurrent. If we suppose the contraposition, then there exists a strictly increasing sequence (τ_n) of positive real numbers tending to plus infinity such that, for every $\varepsilon > 0$, we have the existence of two finite numbers $M > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|f(x+\tau_n) - f(x)\| \leqslant 2\varepsilon, \quad |x| \geqslant M,\ n \geqslant n_0.$$

Let $\varepsilon \in (0,1/2)$ and $n \geqslant n_0$. Then it is clear that there exists $l \in \mathbb{N}$, as large as we want, such that $a_l > 0$ and $a_{l+1} < 0$. Then $f(l+(1/2)) = 0$ and therefore $|f(l+(1/2)+\tau_n)| \leqslant 2\varepsilon$. This clearly implies the existence of an integer $k \in \mathbb{Z}$ such that the number $l+(1/2)+\tau_n$ lies in a certain small neighborhood of number $k+(1/2)$; more precisely, since the linear function connecting the points $(k,-1)$ and $(k+1,1)$ is given by $y = 2x-2k-1$, we get from the above that $|2(l+(1/2)+\tau_n)-2k-1| \leqslant 2\varepsilon$, which simply implies $|\tau_n-(k-l)| \leqslant \varepsilon$ and therefore $\tau_n \in (0,\varepsilon] \cup \bigcup_{k\in\mathbb{N}}[k-\varepsilon,k+\varepsilon]$. Fix now an integer $k \in \mathbb{N}$. We will show that the inclusion $\tau_n \in [k-\varepsilon,k+\varepsilon]$ cannot be true. Otherwise, for each real number $t \in \mathbb{R}$ we have $|f(t+\tau_n)-f(t+k)| \leqslant 2\varepsilon$, which can be easily approved, so that

$$\begin{aligned}\big|f(t+k)-f(t)\big| &\leqslant \big|f(t+k)-f(t+\tau_n)\big| + \big|f(t+\tau_n)-f(t)\big|\\ &\leqslant 2\varepsilon+\varepsilon = 3\varepsilon, \quad |t| \geqslant M.\end{aligned}$$

This contradicts Lemma 2.4.8. Notice also that the argumentation given above shows that, for every $\varepsilon \in (0,1)$, we have $\vartheta(f,\varepsilon)\cap(\varepsilon/2,+\infty) = \emptyset$. Furthermore, for every $\varepsilon \in (0,1)$ and $\tau \in (0,\varepsilon/2]$, we have $|f(t+\tau)-f(t)| \leqslant 2\tau \leqslant \varepsilon$ so that, actually,

$$\forall \varepsilon \in (0,1): \quad \vartheta(f,\varepsilon) = (0,\varepsilon/2].$$

Let us recall that A. M. Fink has constructed in [442, Example 6.1] an odd almost periodic function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_0^t f(s)\,ds \leqslant 0, \quad t \in \mathbb{R}, \quad \int_0^{2^{n-1}} f(s)\,ds \leqslant -n, \quad n \in \mathbb{N}$$

and the function

$$F(t) := e^{\int_0^t f(s)\,ds}, \quad t \in \mathbb{R}$$

is bounded but not almost periodic. The construction goes as follows. For any number $n \in \mathbb{N}\setminus\{1\}$, we define the function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) := (-n)/(2^{n-1}-1)$, $t \in [1,2^{n-1}-1]$, $f_n(0) := 0$, $f_n(2^{n-1}-1) := 0$, $f_n(\cdot)$ is linear on segments $[0,1]$ and $[2^{n-1}-1,2^{n-1}]$; after that, we extend $f_n(\cdot)$ to be odd and periodic of period 2^n. The function $f(t) := \sum_{n=2}^{\infty} f_n(t)$, $t \in \mathbb{R}$ is well defined, odd and satisfies the above-mentioned properties. Furthermore, we have $F(t) \leqslant 1$ for all $t \in \mathbb{R}$ so that the Lagrange mean value theorem directly shows that the function $F(\cdot)$ is Lipschitzian with the Lipschitz constant $\|f\|_\infty$; in particular,

$F(\cdot)$ is uniformly continuous. It could be of some interest to know whether the function $F(\cdot)$ is not uniformly recurrent.

Finally, it should be note that several intriguing examples of functions with almost periodic behavior have been constructed by D. Bugajewski, A. Nawrocki in [232] and M. Vesely in [1000].

Before providing the proofs of Theorem 2.4.2, Theorem 2.4.4, Theorem 2.4.5 and Theorem 2.4.7, we would like to mention one more problem.

Question 2.4.36. Let us recall that the function $f(\cdot)$, given by (2.24), is almost automorphic function and not compactly almost automorphic. We would like to ask whether for each number $\varepsilon \in (0,1)$ we have $\vartheta(f,\varepsilon) \neq \emptyset$ ($\vartheta(f,\varepsilon)$ is unbounded).

Proof of Theorem 2.4.2. We will first prove that for each fixed number $\tau \in \mathbb{R}$ we see that the function $f(\cdot+\tau)-f(\cdot)$ belongs to the space $\mathrm{ANP}(\mathbb{R}:\mathbb{C})$. Towards this end, note that

$$\begin{aligned} f(t+\tau)-f(t) &= \sum_{n=1}^{\infty} \frac{1}{n}\left[\sin^2\frac{t+\tau}{2^n} - \sin^2\frac{t}{2^n}\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{2n}\left[\cos\frac{t}{2^{n-1}} - \cos\frac{t+\tau}{2^{n-1}}\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{2t+\tau}{2^n}\sin\frac{\tau}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\left[\sin\frac{t}{2^{n-1}}\cos\frac{\tau}{2^n} + \cos\frac{t}{2^{n-1}}\sin\frac{\tau}{2^n}\right]\sin\frac{\tau}{2^n}, \quad t \in \mathbb{R}. \end{aligned}$$

Since the functions $t \mapsto \sin\frac{t}{2^{n-1}}$, $t \in \mathbb{R}$ and $t \mapsto \cos\frac{t}{2^{n-1}}$, $t \in \mathbb{R}$ are anti-periodic of anti-period $T = 2^{n-1}\pi$, it follows that the function

$$f_k(t) := \sum_{n=1}^{k} \frac{1}{n}\left[\sin\frac{t}{2^{n-1}}\cos\frac{\tau}{2^n} + \cos\frac{t}{2^{n-1}}\sin\frac{\tau}{2^n}\right]\sin\frac{\tau}{2^n}, \quad t \in \mathbb{R}$$

belongs to the space $\mathrm{ANP}_0(\mathbb{R}:\mathbb{C})$. Moreover, $\lim_{k\to+\infty} f_k(t) = f(t+\tau)-f(t)$ uniformly on $\mathbb{R}$ since

$$\left|\sum_{n=k+1}^{\infty} \frac{1}{n}\left[\sin\frac{t}{2^{n-1}}\cos\frac{\tau}{2^n} + \cos\frac{t}{2^{n-1}}\sin\frac{\tau}{2^n}\right]\sin\frac{\tau}{2^n}\right| \leqslant |\tau| \sum_{n=k+1}^{\infty} \frac{1}{n2^{n-1}}, \quad t \in \mathbb{R}.$$

Especially, due to the fact that $\mathrm{ANP}(\mathbb{R}:\mathbb{C}) = \mathrm{AP}_{\mathbb{R}\setminus\{0\}}(\mathbb{R}:\mathbb{C})$, we have $0 \notin \sigma(f(\cdot+\tau) - f(\cdot))$, i. e.,

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \left|f(s+\tau)-f(s)\right| ds = 0.$$

This readily implies

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \big|f(s+\tau)-f(s)\big|^p\,ds=0,\quad p\geqslant 1,$$

because

$$\big|f(s+\tau)-f(s)\big|^p\leqslant \big|f(s+\tau)-f(s)\big|\cdot\Big(\sup_{x\geqslant 0}\big|f(x+\tau)-f(x)\big|\Big)^{p-1},\quad s\geqslant 0.$$

Taking into account [631, Proposition 2.13.4], we easily see that for each numbers $t,\tau\in\mathbb{R}$ we have

$$\begin{aligned}&\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\big|f(t+\tau+x)-f(t+x)\big|^p\,dx\\&=\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\big|f(t+\tau+x)-f(t+x)\big|^p\,dx=0,\end{aligned}$$

so that the function $f(\cdot)$ is Weyl p-almost automorphic with the limit function $f^*\equiv f$. This completes the proof of Theorem 2.4.2. □

Proof of Theorem 2.4.4. Suppose that the function $h(\cdot)$ is Stepanov quasi-asymptotically almost periodic. It is clear that the function $h(\cdot)$ is asymptotically Stepanov uniform recurrent, so that Theorem 2.4.24(ii) implies that the function $h(\cdot)$ is asymptotically Stepanov almost periodic. Since $h(\cdot)$ is uniformly continuous, Lemma 2.3.4(i) implies that the function $h(\cdot)$ is asymptotically almost periodic. This cannot be true because the restriction of the function $h(\cdot)$ to the non-negative real axis is not asymptotically (Stepanov) almost automorphic by Lemma 2.4.3. □

Proof of Theorem 2.4.5. The function $f(\cdot)$, given by (2.29), satisfies the requirement that for each $\varepsilon>0$ there exists a positive real number $\delta>0$ such that the set $\vartheta(f,\varepsilon)$ contains the set $\bigcup_{n\geqslant\lceil 1/\varepsilon\rceil}[\tau_n-\delta,\tau_n+\delta]$ and $f(x)=f_n(x)$ for all $x\in[-\tau_{n-1},\tau_{n-1}]$ ($n\in\mathbb{N}$). Furthermore, the function $f(\cdot)$ equals zero on arbitrarily long intervals and for each number $\varepsilon\in(0,1)$ we see that the sets $\{x\in\mathbb{R}:f(x)\notin[1-\varepsilon,1+\varepsilon]\}$ and $\vartheta(f,\varepsilon)$ are disjunct (see [444, Example 8, pp. 31–33] for more details). This essentially implies that the function $f(\cdot)$ cannot be asymptotically Stepanov almost automorphic (we will present a direct proof, without appealing to Lemma 2.3.4(ii) and Proposition 2.4.26(iii)). If we suppose the contraposition, then there exist a Stepanov almost automorphic function $h(\cdot)$ and a function $q\in C_0(\mathbb{R}:L^1([0,1]:\mathbb{C}))$ such that $f(t)=h(t)+q(t)$ for a. e. $t\in\mathbb{R}$. Moreover, we have the existence of disjunct intervals $I_n=[b_n,b_n'']\subseteq[0,\infty)$ whose length is strictly greater than n^2 and which satisfy $f(x)=0$ for all $x\in I_n$ ($n\in\mathbb{N}$). Define

$b_n := (b'_n + b''_n)/2$ ($n \in \mathbb{N}$). Then there exist a subsequence (a_n) of (b_n) and a function $g^* \in L^1_{\mathrm{loc}}(\mathbb{R} : \mathbb{C})$ such that

$$\lim_{n\to+\infty} \int_t^{t+1} |f(x+a_n) - q(x+a_n) - g^*(x)|\, dx = 0$$

for all $t \in \mathbb{R}$, and

$$\lim_{n\to+\infty} \int_t^{t+1} |g^*(x-a_n) - [f(x) - q(x)]|\, dx = 0$$

for all $t \geqslant 0$. Let $\varepsilon \in (0, 1/2)$ be given. Then there exists $n_0 \in \mathbb{N}$ such that $n_0/(n_0-1) > 3\varepsilon/2$ and $\int_{\tau_{n_0}}^{1+\tau_{n_0}} |q(x)|\, dx < \varepsilon/8$. Since $1 \geqslant f(x) \geqslant f_n(x) \geqslant n_0/(n_0-1)$ for $x = \tau_{n_0}$, $f_n(x) = 0$ for $x = \tau_{n_0}+1$ and the function $f_n(\cdot)$ is linear on the interval $[\tau_{n_0}, \tau_{n_0}+1]$ (see also [196, part I, p. 115]), the second limit equality with $t = \tau_{n_0}$ easily implies the existence of an integer $n_1 \geqslant n_0$ such that

$$\int_{\tau_{n_0}-a_n}^{1+\tau_{n_0}-a_n} |g^*(x)|\, dx \geqslant \frac{n_0}{2(n_0-1)} - \frac{\varepsilon}{2} > \frac{\varepsilon}{4}, \quad n \geqslant n_1.$$

Returning to the first limit equation, with $t = \tau_{n_0} - a_{n_1}$, and taking into account that $\lim_{m\to\infty} \int_t^{t+1} |q(x+a_m)|\, dx < \varepsilon/8$ for all $m \in \mathbb{N}$ sufficiently large, we obtain the existence of an integer $m_1 \geqslant n_1$ such that

$$\int_{\tau_{n_0}-a_{n_1}+a_m}^{1+\tau_{n_0}-a_{n_1}+a_m} |f(x)|\, dx = \int_{\tau_{n_0}-a_{n_1}}^{1+\tau_{n_0}-a_{n_1}} |f(x+a_m)|\, dx > \frac{\varepsilon}{4} - \frac{\varepsilon}{8} > 0$$

for all $m \geqslant m_1$. But this is simply impossible because for large values of m we see that $[\tau_{n_0} - a_{n_1} + a_m, 1 + \tau_{n_0} - a_{n_1} + a_m]$ is contained in a larger interval where the function $f(\cdot)$ equals zero. If we assume that the function $f(\cdot)$ is Stepanov quasi-asymptotically almost periodic, then the first part of proof of Theorem 2.4.4 shows that the function $f(\cdot)$ is asymptotically Stepanov almost periodic, which cannot be true according to the first part of proof of this theorem. □

Example 2.4.37. Without going into full details, let us only note that the function $f(\cdot)$ considered above can be Weyl p-almost automorphic ($p \geqslant 1$) if the sequence (τ_n) marches rapidly to plus infinity. This follows from the fact that the function $f(\cdot)$ is bounded and belongs to the space $PAP_0(\mathbb{R} : \mathbb{C})$. To explain this in more detail, let a_n denote the number of triangles appearing on the graph of the function $f_n(\cdot)$. Then $a_1 = 1$ and $a_n = (2n-1)a_{n-1}$, $n \in \mathbb{N} \setminus \{1\}$ so that $a_n = (2n-1)!!$, $n \in \mathbb{N}$. The Lebesgue measure of each such triangle cannot exceed 1 so that $\int_{-\infty}^{+\infty} f_n(x)\,dx \leqslant (2n-1)!!$, $n \in \mathbb{N}$. Suppose, for simplicity, that $\lim_{n\to+\infty}(2n-1)!!/\tau_{n-2} = 0$. If $\tau_{n-1} \geqslant l \geqslant \tau_{n-2}$ for some sufficiently large integer $n \in \mathbb{N}$, then

$$\frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} f_n(x)\,dx \leqslant \frac{1}{\tau_{n-2}}\int_{-\infty}^{\infty} f_n(x)\,dx \leqslant \frac{(2n-1)!!}{\tau_{n-2}},$$

so that $\lim_{l\to+\infty}(1/2l)\int_{-l}^{l} f(x)\,dx = 0$, as claimed. Needless to say that, due to Proposition 2.4.31, there exists a suitable function $g(\cdot)$ such that the function $f(\cdot)$ is $\cdot_g$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$ (see also [508, pp. 477–478]).

Proof of Theorem 2.4.7. It is already known that the function $f(\cdot)$ satisfies $\lim_{i\to+\infty} \|f(\cdot+2p_i)-f(\cdot)\|_\infty = 0$, so that $f(\cdot)$ is uniformly recurrent. Keeping in mind Proposition 2.4.29 and arguing as in the proof of Theorem 2.4.4, we see that $f(\cdot)$ is (Stepanov) quasi-asymptotically almost periodic if and only if $f(\cdot)$ is asymptotically almost periodic. By Proposition 2.4.26(ii), this would imply that the function $f(\cdot)$ is almost periodic; this is not the case because the function $f(\cdot)$ is not almost automorphic (asymptotically almost automorphic, equivalently, due to Proposition 2.4.26(i)). If we suppose the contrary, then there exist a subsequence (p_{i_k}) of (p_i) and a function $\omega : \mathbb{R} \to \mathbb{R}$ such that $\lim_{k\to+\infty} f(t+p_{i_k}) = \omega(t)$ and $\lim_{k\to+\infty} \omega(t-p_{i_k}) = f(t)$ for all $t \in \mathbb{R}$. Observe that the function $f_i(\cdot)$ satisfies $f_i(t+p_i) \geqslant 1-\varepsilon$, provided $|t| \leqslant \varepsilon p_i$ and $i \in \mathbb{N}$. Let $t \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then there exists $i_0 \in \mathbb{N}$ such that $|t| \leqslant \varepsilon p_i$ for all integers $i \geqslant i_0$. Therefore, for any integer $i \geqslant i_0$, we have

$$1 \geqslant f(t+p_i) \geqslant f_i(t+p_i) \geqslant 1-\varepsilon,$$

so that $1 = \lim_{i\to+\infty} f(t+p_i) = \lim_{k\to+\infty} f(t+p_{i_k}) = \omega(t)$. Therefore, $\omega(t) \equiv 1$ and returning to the second limit equality we get $f(t) \equiv 1$, which is a contradiction (see also [358, Figure 3.7.3, p. 208]). □

We continue by proposing an interesting result closely connected with our previous analysis of uniformly recurrent functions and the recent researches of I. Area, J. Losada and J. J. Nieto [77, 78] concerning the quasi-periodic properties of fractional integrals and fractional derivatives of scalar-valued periodic functions (see also I. Area, J. Losada, J. J. Nieto [79] and J. M. Jonnalagadda [568] for the discrete analogues). In [631], we have emphasized that the almost periodic properties and the almost automorphic properties of the Riemann–Liouville integrals are very unexplored in the vector-valued case.

Suppose that $\alpha \in (0,1)$ and $T > 0$. In [77, Theorem 1], the authors have proved that the Riemann–Liouville integral $J_t^\alpha f(t) = \int_0^t g_\alpha(t-s)f(s)\,ds$, $t \in \mathbb{R}$ of a non-zero essentially bounded T-periodic function $f : \mathbb{R} \to \mathbb{R}$ cannot be T-periodic. Suppose now that $f : \mathbb{R} \to X$ is a non-zero essentially bounded T-periodic function. Then [78, Lemma 3] continues to holds for $f(\cdot)$, as it can be simply verified, so that the function $J_t^\alpha f(\cdot)$ is S-asymptotically T-periodic. If we suppose that the function $J_t^\alpha f(\cdot)$ is uniformly recurrent (compactly almost automorphic), this would imply by [531, Lemma 3.1] and the arguments used in the proof of [531, Proposition 3.4] that the function $J_t^\alpha f(\cdot)$ is T-periodic. This will be used in the proof of the following proper extension of [78, Theorem 9].

Theorem 2.4.38. *Suppose that $\alpha \in (0,1)$, $T > 0$ and $f : \mathbb{R} \to X$ is a non-zero essentially bounded T-periodic function. Then $J_t^\alpha f(\cdot)$ cannot be uniformly recurrent (almost automorphic).*

Proof. Suppose that $J_t^\alpha f(\cdot)$ is uniformly recurrent (almost automorphic) and $x^* \in X^*$ is an arbitrary functional. Let $\langle x^*, f(\cdot)\rangle = a(\cdot) + ib(\cdot)$, where $a(\cdot)$ and $b(\cdot)$ are real-valued functions. Then it is clear that the function $J_t^\alpha \langle x^*, f(\cdot)\rangle = J_t^\alpha a(\cdot) + iJ_t^\alpha b(\cdot)$ is uniformly recurrent (almost automorphic) because $J_t^\alpha \langle x^*, f(\cdot)\rangle = \langle x^*, J_t^\alpha f(\cdot)\rangle$, which further implies that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are uniformly recurrent (almost automorphic). Let us assume first that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are uniformly recurrent. Since $a(\cdot)$ and $b(\cdot)$ are essentially bounded functions of period T, the above discussion implies that $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are periodic functions of period T. Then we can apply [77, Theorem 1] in order to see that $a(\cdot) \equiv b(\cdot) \equiv 0$. This implies $\langle x^*, f(\cdot)\rangle \equiv 0$ and therefore $f(\cdot) \equiv 0$. The proof is quite similar if we assume that the function $J_t^\alpha f(\cdot)$ is almost automorphic, when the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are also almost automorphic. Since the function $J_t^\alpha f(\cdot)$ is bounded, repeating verbatim the above arguments we may deduce from [78, Theorem 5] that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are asymptotically T-periodic and, in particular, bounded and uniformly continuous. Therefore, the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are compactly almost automorphic. But then we can argue in the same way as for the uniform recurrence to see that $a(\cdot) \equiv b(\cdot) \equiv 0$. □

Applying the trick used in the first part of the proof and the well known fact that a weakly bounded set in a locally convex space is bounded, we may conclude that the statements of [77, Theorem 1, Corollary 2] and [78, Lemma 2, Lemma 3; Proposition 1, Proposition 2; Theorem 2, Theorem 3, Theorem 4, Theorem 8] hold in the vector-valued case (concerning the above-mentioned statements from [78], it seems very plausible that the continuity of the function $f(\cdot)$ in their formulations can be replaced with the essential boundedness). It is clear that it [78, Corollary 1] cannot be reformulated even for the complex-valued functions and, regarding the main structural results established in [77, 78], it remains to be considered whether the statements of [78, Theorem 5, Theorem 6, Theorem 7] hold in the vector-valued case. We will analyze this question elsewhere.

We proceed with some applications of (asymptotically) uniformly recurrent functions and (asymptotically) $\odot_g$-almost periodic functions. We shall mostly be concerned with the invariance of (asymptotical) uniform recurrence and (asymptotical) $\odot_g$-almost periodicity under the actions of convolution products.

Let $f : \mathbb{R} \to X$. We will first investigate the uniformly recurrent and $\odot_g$-almost periodic properties of the function

$$F(t) := \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{2.46}$$

where a strongly continuous operator family $(R(t))_{t>0} \subseteq L(X,Y)$ satisfies certain assumptions. In our recent research studies regarding this question, it is commonly assumed that the function $f(\cdot)$ is Stepanov $p(x)$-bounded for some function $p \in \mathcal{P}([0,1])$. If this is the case, we can simply reformulate the statement of Proposition 3.1.18 below as follows (cf. also [1067, Examples 4, 5, 7, 8; pp. 32–34], which can be simply reformulated for the uniform recurrence and $\odot_g$-almost periodicity).

Proposition 2.4.39. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is Stepanov $p(x)$-bounded and Stepanov $p(x)$-uniformly recurrent (Stepanov $(p(x),\odot_g)$-almost periodic), and the mapping $t \mapsto \check{f}(\cdot - t) \in L^{p(x)}([0,1] : X)$ is continuous, then the function $F : \mathbb{R} \to Y$, given by* (2.46)*, is well defined and uniformly recurrent ($\odot_g$-almost periodic).*

Proof. The function $F(\cdot)$ is well defined due to the computation carried out in the proof of Proposition 3.1.18. The proof of the above-mentioned proposition also shows that, if $\tau \in \mathbb{R}$ is an ε-period of the function $\hat{\check{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$, then the resulting function $F(\cdot)$ satisfies, under given conditions on $(R(t))_{t>0}$, an estimate of the type $\|F(t+\tau) - F(t)\|_Y \leqslant L\varepsilon$, $t \in \mathbb{R}$, where $L \geqslant 1$ is a finite constant independent of t, ε and τ. Hence, the assumption $\odot_g(\vartheta(\hat{\check{f}},\varepsilon)) > 0$ for all $\varepsilon > 0$ implies that $\odot_g(\vartheta(F,\varepsilon)) > 0$ for all $\varepsilon > 0$. Therefore, it remains to be proved that the function $F(\cdot)$ is continuous. But this follows similarly to the proof of [631, Proposition 3.5.3] and our assumption that the mapping $t \mapsto \check{f}(\cdot - t) \in L^{p(x)}([0,1] : X)$ is continuous (see also [373, Proposition 5.1]). □

Remark 2.4.40. In general case $p \in \mathcal{P}([0,1])$, the mapping $t \mapsto \check{f}(\cdot - t) \in L^{p(x)}([0,1] : X)$ is not necessarily continuous (see, e. g., [667, p. 602]). This is always true provided that $p \in D_+([0,1])$.

Basically, case in which the function $f : \mathbb{R} \to X$ is not Stepanov $p(x)$-bounded has not attracted the attention of the authors so far. Keeping in mind our previous results, we would like to state the following proposition with regards to this question (the uniform continuity of the function $\hat{\check{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ has not been assumed above).

Proposition 2.4.41. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$, $\check{f} : \mathbb{R} \to X$ is Stepanov $p(x)$-uniformly recurrent (Stepanov $(p(x),\odot_g)$-almost periodic), there exists a continuous function $P : \mathbb{R} \to [1,\infty)$ such that*

$$\|f(t-\cdot)\|_{L^{p(\cdot)}[0,1]} \leqslant P(t), \quad t \in \mathbb{R} \tag{2.47}$$

and $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying the requirement that for each* $t \in \mathbb{R}$ *we have*

$$\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} P(t-k) < \infty. \tag{2.48}$$

If the function $\hat{\hat{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ *is uniformly continuous, then the function* $F : \mathbb{R} \to Y$*, given by* (2.46)*, is well defined and uniformly recurrent* ($\odot_g$*-almost periodic).*

Proof. We will only outline the most important details for Stepanov $(p, \odot_g)$-almost periodic functions. The function $F(\cdot)$ is well defined since, due to Lemma 1.1.7(i) and the estimates (2.47)–(2.48), we have

$$\begin{aligned}
\int_0^{\infty} \|R(s)\| \|f(t-s)\| \, ds &= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(s)\| \|f(t-s)\| \, ds \\
&= \sum_{k=0}^{\infty} \int_0^1 \|R(s+k)\| \|f(t-s-k)\| \, ds \\
&\leqslant 2 \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}([0,1]:X)} \|f(t-k-\cdot)\|_{L^{p(x)}([0,1]:X)} \\
&\leqslant 2 \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}([0,1]:X)} P(t-k) < \infty,
\end{aligned}$$

for any $t \in \mathbb{R}$. It is clear that our assumptions imply

$$M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} < \infty,$$

so that $\vartheta(f, \varepsilon) \subseteq \vartheta(F, M\varepsilon)$. Since we have assumed that the function $\hat{\hat{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ is uniformly continuous, the arguments contained in the proof of [631, Proposition 2.6.11] can be repeated verbatim in order to see that the function $F(\cdot)$ is continuous. This completes the proof of proposition. □

Proposition 2.4.39 and Proposition 2.4.41 can be simply incorporated in the study of the existence and uniqueness of uniformly recurrent solutions and $\odot_g$-almost periodic solutions of the fractional Cauchy inclusion

$$D_{t,+}^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), \ t \in \mathbb{R}, \tag{2.49}$$

where $D_{t,+}^{\gamma}$ denotes the Riemann–Liouville fractional derivative of order $\gamma \in (0,1]$, $f : \mathbb{R} \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator satisfying condition (P) (see Subsection 3.1.3 and [631] for more details).

Taking into account Proposition 2.4.39 and Proposition 2.4.41, we can simply provide extensions of [631, Proposition 2.6.13, Theorem 2.9.5, Theorem 2.9.7, Theorem

2.9.15], concerning the asymptotical Stepanov p-uniform recurrence/asymptotical Stepanov $(p, \odot_g)$-almost periodicity of the finite convolution product

$$\mathbf{F}(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0.$$

These results can be applied in the qualitative analysis of asymptotically uniformly recurrent/asymptotically $\odot_g$-almost periodic solutions (asymptotically Stepanov p-uniformly recurrent/asymptotically Stepanov $(p, \odot_g)$-almost periodic solutions) of the following abstract Cauchy inclusion:

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + f(t), & t \geqslant 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^\gamma$ denotes the Caputo fractional derivative of order $\gamma \in (0,1]$, $x_0 \in X$, $f : [0,\infty) \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator satisfying condition (P) (see Subsection 3.1.3 and [631] for more details).

The sum of two uniformly recurrent ($\odot_g$-almost periodic) functions need not be uniformly recurrent ($\odot_g$-almost periodic), unfortunately. But it is worth noticing that there exist many concrete situations where this difficulty can be overcome. For example, it is very simple to extend the assertions of [631, Theorem 2.14.7] and [389, Theorem 2.3] for the asymptotical Stepanov $(p, \odot_g)$-almost periodicity. To explain this in more detail, let us observe that the equation appearing on [389, p. 240, l. 5] can be rewritten as

$$\int_{-\infty}^t \Gamma(t,s)f(s)\,ds = \lim_{k\to+\infty} \int_0^k \Gamma(t,t-s)f(t-s)\,ds, \quad t \in \mathbb{R};$$

arguing as in the proof of above-mentioned theorem from [389] we may conclude that for each integer $k \in \mathbb{N}$ the function

$$t \mapsto \int_0^k \Gamma(t,t-s)f(t-s)\,ds, \quad t \in \mathbb{R}$$

is $\odot_g$-almost periodic, provided that the function $f(\cdot)$ is Stepanov $(p, \odot_g)$-almost periodic and Stepanov p-bounded ($p > 1$), while case $p = 1$ follows from the same arguments and the proof of [631, Theorem 2.14.6], when it is necessary to assume that $f(\cdot)$ is Stepanov $(1, \odot_g)$-almost periodic and Stepanov 1-bounded. In both cases, $p > 1$ and $p = 1$, we need to employ the property (iv) to achieve the final results.

We close the subsection with the observation that the results whose proofs lean heavily on the use of Bochner criterion cannot be really reconsidered for uniformly recurrent functions and $\odot_g$-almost periodic functions.

2.4.3 Composition principles for almost periodic type functions and applications

In this subsection, we introduce and analyze the classes of two-parameter (asymptotically) uniformly recurrent functions, two-parameter (asymptotically) $\odot_g$-almost periodic functions and their Stepanov generalizations. Several composition principles are established in this context, which enables one to provide certain applications to the abstract semilinear integro-differential Cauchy problems and inclusions. Since the structural results presented in this subsection can be deduced by uncomplicated modifications of results known in the existing literature, we have decided to provide the main details of the proofs for only two statements, Theorem 2.4.44 and Theorem 2.4.46.

For every $\varepsilon > 0$ and for every bounded set $B \subseteq Y$, we define $\vartheta(F;\varepsilon,B)$ as the set constituted of all numbers $\tau > 0$ such that

$$\|F(t+\tau,y) - F(t,y)\| \leqslant \varepsilon, \quad t \in I,\ y \in B.$$

The following definition is crucial in our analysis.

Definition 2.4.42.

(i) A continuous function $F : I \times Y \to X$ is called uniformly recurrent, resp. $\odot_g$-almost periodic, if and only if for every $\varepsilon > 0$ and every compact $K \subseteq Y$ there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\in I} \|F(t+\alpha_n,y) - F(t,y)\| = 0, \quad y \in K, \tag{2.50}$$

resp. if and only if for every $\varepsilon > 0$ and every compact $K \subseteq Y$ we have $\odot_g(\vartheta(F;\varepsilon,K)) > 0$.

The collection of all two-parameter uniformly recurrent functions, resp. $\odot_g$-almost periodic functions, will be denoted by $\mathrm{UR}(I \times Y : X)$, resp. $\mathrm{AP}_{\odot_g}(I \times Y : X)$.

(ii) A continuous function $F : I \times Y \to X$ is called uniformly recurrent on bounded sets, resp. $\odot_g$-almost periodic on bounded sets, if and only if for every $\varepsilon > 0$ and every bounded set $B \subseteq Y$ there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that (2.50) holds with $K = B$, resp. if and only if for every $\varepsilon > 0$ and every bounded set $B \subseteq Y$ we have $\odot_g(\vartheta(F;\varepsilon,B)) > 0$.

The collection of all two-parameter uniformly recurrent functions on bounded sets, resp. $\odot_g$-almost periodic functions on bounded sets, will be denoted by $\mathrm{UR}_b(I \times Y : X)$, resp. $\mathrm{AP}_{\odot_g,b}(I \times Y : X)$.

(iii) A continuous function $F : I \times Y \to X$ is said to be asymptotically uniformly recurrent, resp. asymptotically $\odot_g$-almost periodic, if and only if $F(\cdot)$ admits a decomposition $F = G + Q$, where $G \in \mathrm{UR}(\mathbb{R} \times Y : X)$, resp. $G \in \mathrm{AP}_{\odot_g}(\mathbb{R} \times Y : X)$, and $Q \in C_0(I \times Y : X)$.

Denote by $\mathrm{AUR}(I \times Y : X)$, resp. $\mathrm{AAP}_{\odot_g}(I \times Y : X)$, the collection consisting of all asymptotically uniformly recurrent functions, resp. asymptotically $\odot_g$-almost periodic functions.

(iv) A continuous function $F : I \times Y \to X$ is said to be asymptotically uniformly recurrent on bounded sets, resp. asymptotically $\odot_g$-almost periodic on bounded sets, if and only if $F(\cdot)$ admits a decomposition $F = G + Q$, where $G \in \mathrm{UR}_b(\mathbb{R} \times Y : X)$, resp. $G \in \mathrm{AP}_{\odot_g,b}(\mathbb{R} \times Y : X)$, and $Q \in C_0(I \times Y : X)$.
Denote by $\mathrm{AUR}_b(I \times Y : X)$, resp. $\mathrm{AAP}_{\odot_g,b}(I \times Y : X)$, the collection consisting of all asymptotically uniformly recurrent functions, resp. asymptotically $\odot_g$-almost periodic functions.

In the contrast to the approach of C. Zhang for almost periodic functions depending on the parameter [1078] (see also [631, Definition 2.1.4]), we do not assume a priori the boundedness of the function $F(\cdot, \cdot)$ in our approach. This is quite reasonable because uniformly recurrent functions and $\odot_g$-almost periodic functions of one real variable need not be bounded, in general. It is worth noticing that introducing parts (ii) and (iv) is motivated by definition of almost periodicity used by T. Diagana in [631, Definition 3.29].

For the Stepanov classes, we will use the following notion (see also [631, Definition 2.2.4, Definition 2.2.5; Lemma 2.2.7]).

Definition 2.4.43. Let $p \in \mathcal{P}([0,1])$.

(i) A function $F : I \times Y \to X$ is called Stepanov $p(x)$-uniformly recurrent/Stepanov $p(x)$-uniformly recurrent on bounded sets (Stepanov $(p(x), \odot_g)$-almost periodic/Stepanov $(p(x), \odot_g)$-almost periodic on bounded sets) if and only if the function $\hat{F} : I \times Y \to L^{p(x)}([0,1] : X)$ is uniformly recurrent/uniformly recurrent on bounded sets ($\odot_g$-almost periodic/$\odot_g$-almost periodic on bounded sets).

(ii) We say that $F : I \times Y \to X$ is asymptotically Stepanov $p(x)$-uniformly recurrent/asymptotically Stepanov $p(x)$-uniformly recurrent on bounded sets (asymptotically Stepanov $(p(x), \odot_g)$-almost periodic/asymptotically Stepanov $(p(x), \odot_g)$-almost periodic on bounded sets) if and only if there exist two functions $G : \mathbb{R} \times Y \to X$ and $Q : I \times Y \to X$ satisfying the requirement that for each $y \in Y$ the functions $G(\cdot, y)$ and $Q(\cdot, y)$ are locally $p(x)$-integrable the following hold:
   (a) $\hat{G} : \mathbb{R} \times Y \to L^{p(x)}([0,1] : X)$ is uniformly recurrent/uniformly recurrent on bounded sets ($\odot_g$-almost periodic/$\odot_g$-almost periodic on bounded sets),
   (b) $\hat{Q} \in C_0(I \times Y : L^{p(x)}([0,1] : X))$,
   (c) $F(t, y) = G(t, y) + Q(t, y)$ for all $t \in I$ and $y \in Y$.

If $p(x) \equiv p \in [1, \infty)$, then we also say that a function $F : I \times Y \to X$ is Stepanov p-uniformly recurrent/Stepanov p-uniformly recurrent on bounded sets etc.

A serious difficulty in our investigations is that, for two given uniformly recurrent functions $f : I \to X$ and $g : I \to X$, the sequence (α_n) for which (2.3) holds need not have a subsequence (α_{n_k}) for which

$$\lim_{k\to\infty} \sup_{t\in\mathbb{R}} \| g(t + \alpha_{n_k}) - g(t) \| = 0;$$

moreover, for given two $\odot_g$-almost periodic functions $f : I \to X$ and $g : I \to X$, the set consisting of their joint ε-periods can be bounded (this cannot occur for almost periodic functions). Now we will slightly improve [631, Theorem 3.30] for uniformly recurrent functions and $\odot_g$-almost periodic functions.

Theorem 2.4.44. *Suppose that $f : I \to Y$ is uniformly recurrent ($\odot_g$-almost periodic) and the range of $f(\cdot)$ is relatively compact, resp. bounded. If $F : I \times Y \to X$ is uniformly recurrent ($\odot_g$-almost periodic), resp. uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets), and there exists a finite constant $L > 0$ such that*

$$\|F(t,x) - F(t,y)\| \leqslant L\|x-y\|_Y, \quad t \in I,\ x,\ y \in Y, \tag{2.51}$$

then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic), providing additionally the following condition: there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity for which (2.3) *holds and* (2.50) *holds with $K = \overline{\{f(t) : t \in I\}}$, resp. for each $\varepsilon > 0$ we have $\odot_g(\vartheta(F; \varepsilon, \overline{\{f(t) : t \in I\}}) \cap \vartheta(f, \varepsilon)) > 0$.*

Proof. The proof of theorem is very similar to the proof of [631, Theorem 3.30] and we will only outline the main details for $\odot_g$-almost periodic functions. Let $\varepsilon > 0$ be given, and let $\tau \in \vartheta(F; \varepsilon/2(1+L), \overline{\{f(t) : t \in I\}}) \cap \vartheta(f, \varepsilon/2(1+L))$. Then $\|f(t+\tau) - f(t)\| \leqslant \varepsilon/2(1+L)$, $t \in I$ and we have

$$\|\mathcal{F}(t+\tau) - \mathcal{F}(t)\| \leqslant L\|f(t+\tau) - f(t)\|_Y + \|F(t+\tau, f(t)) - F(t+\tau, f(t))\|, \quad t \in I.$$

Hence,

$$\|\mathcal{F}(t+\tau) - \mathcal{F}(t)\| \leqslant [L\varepsilon/2(1+L)] + \varepsilon/2(1+L) < \varepsilon, \quad t \in I,$$

which completes the proof. □

Similarly we can prove the following slight extension of [631, Theorem 3.31].

Theorem 2.4.45. *Suppose that $f : I \to Y$ is a bounded uniformly recurrent function (bounded $\odot_g$-almost periodic function). If $F : I \times Y \to X$ is uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets) and uniformly continuous on bounded sets, uniformly for $t \in I$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic), providing additionally the following condition: there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity for which* (2.3) *holds and* (2.50) *holds with $K = \overline{\{f(t) : t \in I\}}$, resp. for each $\varepsilon > 0$ we have $\odot_g(\vartheta(F; \varepsilon, \overline{\{f(t) : t \in I\}}) \cap \vartheta(f, \varepsilon)) > 0$.*

Before proceeding further, it should be observed that the statement of [631, Theorem 3.32] (see also the proof of [442, Theorem 2.11]) can be formulated and slightly extended for uniformly recurrent ($\odot_g$-almost periodic) functions with relatively compact range.

Composition principles for asymptotically almost periodic functions have been analyzed in a great number of research papers. With regards to this question, we will state and give the main details of proof for the following slight extension of [364, Theorem 3.49], only (observe, however, that we can similarly reconsider and slightly extend the statements of [364, Theorems 3.50–3.52]).

Theorem 2.4.46. *Suppose that $h : I \to Y$ is uniformly recurrent ($\odot_g$-almost periodic), the range of $h(\cdot)$ is relatively compact, resp. bounded, $q \in C_0(I : X)$ and $f(t) = h(t) + q(t)$ for all $t \in I$. Suppose, further, $H : I \times Y \to X$ is uniformly recurrent ($\odot_g$-almost periodic), resp. uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets), there exists a finite constant $L > 0$ such that* (2.51) *holds with the function $F(\cdot,\cdot)$ replaced therein with the function $H(\cdot,\cdot)$, and there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity for which* (2.3) *holds with the function $f(\cdot)$ replaced therein with the function $h(\cdot)$ and* (2.50) *holds with the function $f(\cdot)$ replaced therein with the function $h(\cdot)$ and set $K = \overline{\{h(t) : t \in I\}}$, resp. for each $\varepsilon > 0$ we have $\odot_g(\vartheta(H;\varepsilon,\overline{\{h(t) : t \in I\}}) \cap \vartheta(h,\varepsilon)) > 0$. If $f(\cdot)$ has a relatively compact range, $Q \in C_0(I \times Y : X)$ and $F(t,y) = H(t,y) + Q(t,y)$ for all $t \in I$ and $y \in Y$, then the mapping $\mathcal{F}(t) := F(t,f(t))$, $t \in I$ is asymptotically uniformly recurrent (asymptotically $\odot_g$-almost periodic).*

Proof. Due to Theorem 2.4.44, we see that the mapping $t \mapsto H(t,h(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic). Furthermore, we have the decomposition

$$F(t,f(t)) = H(t,h(t)) + [H(t,f(t)) - H(t,h(t))] + Q(t,f(t)), \quad t \in I.$$

Since the function $H(\cdot,\cdot)$ satisfies (2.51), we have

$$\|H(t,f(t)) - H(t,h(t))\| \leqslant L\|f(t) - h(t)\|_Y \leqslant L\|q(t)\|_Y \to 0 \quad \text{as } |t| \to +\infty.$$

The proof of theorem completes the observation that $\lim_{|t|\to+\infty} \|Q(t,f(t))\| = 0$, which follows from definition of space $C_0(I \times Y : X)$ and our assumption that $f(\cdot)$ has a relatively compact range. ☐

Remark 2.4.47. The assumption [364, (3.13)] is superfluous. Furthermore, we note that the assumption that the range of $h(\cdot)$ is relatively compact, resp. bounded, implies that $f(\cdot)$ is bounded; therefore, if we use the space $C_{0,b}(I \times Y : X)$ in place of $C_0(I \times Y : X)$ here, the assumption that $f(\cdot)$ has a relatively compact range is superfluous as well.

Remark 2.4.48. Consider, for simplicity, asymptotically uniformly recurrent functions. The principal part $\mathbf{f}(\cdot)$ of the function $\mathcal{F}(t) = F(t,f(t))$, $t \in I$ satisfies (2.3) with the same sequence (α_n) and the function $\mathbf{f}(\cdot)$ in place of $f(\cdot)$. This holds for all remaining results established in this subsection, and this fact will be of some importance for applications made later on.

Concerning the composition principles for Stepanov almost periodic functions, the most influential paper written by now is the paper [729] by W. Long and H.-S. Ding.

Repeating almost verbatim the arguments given in the proof of [729, Lemma 2.1, Theorem 2.2] (see also [372, Theorem 2.4]), we can deduce the following result (we feel it is our duty to say that the previously proved results are more appropriate for applications in finite-dimensional spaces because condition on relative compactness of range of the function $f(\cdot)$ is almost inevitable to be used; see condition (ii) below).

Theorem 2.4.49. *Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $p \in \mathcal{P}([0,1])$. Suppose that the following conditions hold:*
(i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p(x), \odot_g)$-almost periodic and there exist a function $r(\cdot) \geqslant \max(p(\cdot), (p(\cdot)/(p(\cdot)-1)))$ and a function $L_F \in L_S^{r(x)}(I)$ such that (2.20) holds true.*
(ii) *The function $f : I \to Y$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p(x), \odot_g)$-almost periodic, and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{f(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact in Y.*
(iii) *For every compact set $K \subseteq Y$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \sup_{u\in K} \| F(t+s+\alpha_n, u) - F(t+s,u) \|_{L^{p(s)}[0,1]} = 0 \tag{2.52}$$

and (2.3) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced, respectively, by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^{p(x)}([0,1]:X)}$ therein, resp. for every number $\varepsilon > 0$ and for every compact set $K \subseteq Y$, the set consisting of all positive real numbers $\tau > 0$ such that

$$\sup_{t\in I} \sup_{u\in K} \| F(t+s+\tau, u) - F(t+s,u) \|_{L^{p(s)}[0,1]} < \varepsilon$$

and (2.1) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced, respectively, by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^{p(x)}([0,1]:X)}$ therein.

Set $q(x) := p(x)r(x)/(p(x)+r(x)) \in [1, p(x))$ provided $x \in [0,1]$ and $r(x) < \infty$ and $q(x) := p(x)$ provided $r(x) = +\infty$. Then $q(x) := p(x)r(x)/(p(x)+r(x)) \in [1,p(x))$ provided $x \in [0,1]$, $r(x) < \infty$ and $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-uniformly recurrent, resp. Stepanov $(q(x), \odot_g)$-almost periodic. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $q(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-bounded as well.

In [631, Theorem 2.7.2], we have also considered the value $p = 1$ in Theorem 2.4.49 and the usual condition regarding the existence of a Lipschitz constant $L > 0$ such that (2.51) holds.

Using the foregoing arguments, we can simply deduce the following extension of the above-mentioned theorem.

Theorem 2.4.50. *Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $p \in \mathcal{P}([0,1])$. Suppose that the following conditions hold:*

(i) *The function* $F : I \times Y \to X$ *is Stepanov* $p(x)$*-uniformly recurrent, resp. Stepanov* $(p(x), \odot_g)$*-almost periodic,* $L > 0$ *and* (2.51) *holds.*
(ii) *The same as condition* (ii) *of* Theorem 2.4.49.
(iii) *The same as condition* (iii) *of* Theorem 2.4.49.

Then the function $F(\cdot, f(\cdot))$ *is Stepanov* $p(x)$*-uniformly recurrent, resp. Stepanov* $(p(x), \odot_g)$*-almost periodic. Furthermore, the assumption that* $F(\cdot, 0)$ *is Stepanov* $p(x)$*-bounded implies that the function* $F(\cdot, f(\cdot))$ *is Stepanov* $p(x)$*-bounded, as well.*

Following the analysis of F. Bedouhene, Y. Ibaouene, O. Mellah and P. Raynaud de Fitte [141, Theorem 3] for the class of equi-Weyl p-almost periodic functions and the analysis of W. Long and H.-S. Ding [729], in [639, Theorem 2.1] we have established a new composition principle for the class of Stepanov p-almost periodic functions that is not comparable with [729, Theorem 2.2]. Using the proof of the last-mentioned theorem and the proof of [639, Theorem 2.1], we can slightly generalize Theorem 2.4.50. It is also straightforward to reformulate the statements of [631, Proposition 2.7.3-Proposition 2.7.4], resp. [639, Proposition 2.1], for the asymptotical Stepanov $p(x)$-uniform recurrence and the asymptotical Stepanov $(p(x), \odot_g)$-almost periodicity. Details can be left to the interested reader.

Now we will present two interesting applications of established theoretical results in the analysis of the existence and uniqueness of uniformly recurrent type solutions of the abstract semilinear fractional integro-differential inclusions.

1. In the first application, we will consider the finite-dimensional space $X := \mathbb{C}^n$, where $n \geqslant 2$. Suppose that $c > 0$, $A, B \in \mathbb{C}^{n,n}$ (the space of all complex matrices of format $n \times n$), the matrix B is not invertible, and that the degree of complex polynomial $P(\lambda) := \det(\lambda B - A)$, $\lambda \in \mathbb{C}$ is equal to n and its roots lie in the region $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -c(|\operatorname{Im}\lambda| + 1)\}$. Due to [633, Proposition 2.1.2], we see that the region Ψ from the formulation of condition (P) belongs to the resolvent set of multivalued linear operator $\mathcal{A} = AB^{-1}$ and that

$$(\lambda - AB^{-1})^{-1} = B(\lambda B - A)^{-1}, \quad \lambda \in \Psi.$$

Since the degree of complex polynomial $P(\cdot)$ is equal to n, the above formula simply implies that there exists a positive real constant $M > 0$ such that condition (P) holds with $\beta = 1$, so that the operator $\mathcal{A}$ generates an exponentially decaying strongly continuous degenerate semigroup $(T(t))_{t\geqslant 0}$ which can be analytically extended to a sector around positive real axis (cf. [633] for more details).

Suppose now that $0 < \gamma < 1$ and $\nu > -1$. Define

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu} \int_0^\infty s^\nu \Phi_\gamma(s) T(st^\gamma)x\, ds, \quad t > 0,\ x \in X, \tag{2.53}$$

$$S_\gamma(t) := T_{\gamma,0}(t) \quad \text{and} \quad P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma, \quad t > 0;$$

see also E. Bazhlekova [133] and R.-N. Wang, D.-H. Chen, T.-J. Xiao [1021]. Recall [633] that, in the general case $\beta \in (0,1]$, there exists a finite constant $M_1 > 0$ such that

$$\|S_\gamma(t)\| + \|P_\gamma(t)\| \leqslant M_1 t^{\gamma(\beta-1)}, \quad t > 0, \tag{2.54}$$

and

$$\|S_\gamma(t)\| \leqslant M_1 t^{-\gamma},\ t \geqslant 1 \quad \text{and} \quad \|P_\gamma(t)\| \leqslant M_2 t^{-2\gamma}, \quad t \geqslant 1. \tag{2.55}$$

Set $R_\gamma(t) := t^{\gamma-1}P_\gamma(t)$, $t > 0$. Then (2.54)–(2.55) yield

$$\|R_\gamma(t)\| = O(t^{\gamma-1} + t^{-\gamma-1}), \quad t > 0. \tag{2.56}$$

Consider now the following abstract fractional inclusion:

$$D_+^\gamma \vec{u}(t) \in -\mathcal{A}\vec{u}(t) + F(t, \vec{u}(t)), \quad t \in \mathbb{R}, \tag{2.57}$$

where $D_+^\gamma u(t)$ denotes the Weyl–Liouville fractional derivative of order γ and $F : \mathbb{R} \times X \to X$; after the usual substitution $\vec{v}(t) \in B^{-1}\vec{u}(t)$, $t \in \mathbb{R}$, this inclusion becomes

$$D_+^\gamma [B\vec{v}(t)] = -A\vec{v}(t) + F(t, B\vec{v}(t)), \quad t \in \mathbb{R}.$$

Following J. Mu, Y. Zhoa and L. Peng [798], it will be said that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (2.57) if and only if

$$\vec{u}(t) = \int_{-\infty}^{t} R_\gamma(t-s)F(s, \vec{u}(s))\, ds, \quad t \in \mathbb{R}.$$

For the sequel, fix a strictly increasing sequence (α_n) of positive reals tending to plus infinity. Denote

$$\mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X) := \{\vec{u} \in \mathrm{UR}(\mathbb{R} : X); \vec{u}(\cdot) \text{ is bounded and (2.3) holds with } f = \vec{u}\}.$$

Equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_\infty$, $\mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X)$ becomes a complete metric space.

Now we are able to state the following result.

Theorem 2.4.51. *Suppose that the function $F : \mathbb{R} \times X \to X$ satisfies the requirement that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\in\mathbb{R}} \sup_{y\in B} \|F(t,y)\| \leqslant M_B$. Suppose, further, that the function $F : \mathbb{R} \times X \to X$ is Stepanov p-uniformly recurrent with $p > 1$, and there exist a number $r \geqslant \max(p, p/(p-1))$ and a function $L_F \in L_S^r(I)$ such that $q := pr/(p+r) > 1$ and (2.20) holds with $I = \mathbb{R}$. If*

$$\frac{(\gamma-1)q}{q-1} > -1, \tag{2.58}$$

there exists an integer $n \in \mathbb{N}$ *such that* $M_n < 1$, *where*

$$M_n := \sup_{t\geqslant 0} \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \|R_\gamma(t-x_n)\| \times \prod_{i=2}^{n} \|R_\gamma(x_i - x_{i-1})\| \prod_{i=1}^{n} L_F(x_i)\, dx_1\, dx_2 \cdots\, dx_n,$$

and for every compact set $K \subseteq Y$, (2.52) *holds, then the abstract fractional Cauchy inclusion* (2.57) *has a unique bounded uniformly recurrent solution.*

Proof. Define $\Upsilon : BUR_{(\alpha_n)}(\mathbb{R} : X) \to BUR_{(\alpha_n)}(\mathbb{R} : X)$ by

$$(\Upsilon\vec{u})(t) := \int_{-\infty}^{t} R_\gamma(t-s)F(s,\vec{u}(s))\, ds, \quad t \in \mathbb{R}.$$

Let us firstly show that the mapping $\Upsilon(\cdot)$ is well defined. Suppose that $\vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$. Then $R(\vec{u}) = B$ is a bounded set so that the mapping $t \mapsto F(t,\vec{u}(t))$, $t \in \mathbb{R}$ is bounded due to the prescribed assumption. Applying Theorem 2.4.49, we see that the function $F(\cdot,\vec{u}(\cdot))$ is Stepanov q-uniformly recurrent. Define $q' := q/(q-1)$. Then (2.56) and (2.58) together imply that $\|R_\gamma(\cdot)\| \in L^{q'}[0,1]$ and $\sum_{k=0}^{\infty} \|R_\gamma(\cdot)\|_{L^{q'}[k,k+1]} < \infty$ due to our analysis from [631, Remark 2.6.12]. Applying Proposition 2.4.39, we see that the function

$$t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)F(s,\vec{u}(s))\, ds, \quad t \in \mathbb{R}$$

is bounded, continuous and uniformly recurrent, which shows that $\Upsilon\vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$, as claimed. Furthermore, a simple calculation shows that

$$\|(\Upsilon^n\vec{u}_1) - (\Upsilon^n\vec{u}_2)\|_\infty \leqslant M_n\|\vec{u}_1 - \vec{u}_2\|_\infty, \quad \vec{u}_1,\ \vec{u}_2 \in BUR_{(\alpha_n)}(\mathbb{R} : X),\ n \in \mathbb{N}.$$

Since we have assumed that there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, the well known extension of the Banach contraction principle shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point, finishing the proof of the theorem. □

2. Suppose that a closed multivalued linear operator $\mathcal{A}$ satisfies condition (P) in X, which can be finite-dimensional or infinite-dimensional, with general exponent $\beta \in (0,1]$. Consider the abstract semilinear fractional differential inclusion

$$(DFP)_{f,\gamma,s} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + F(t,u(t)), & t > 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^\gamma$ denotes the Caputo fractional derivative of order γ, $x_0 \in X$ and $F : [0,\infty) \times X \to X$. By a mild solution of $(\mathrm{DFP})_{f,\gamma,s}$, we mean any function $u \in C([0,\infty) : X)$ satisfying

$$u(t) = S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0.$$

In what follows, we will assume that $\lim_{t\to 0+} S_\gamma(t)x_0 = x_0$ so that the mapping $t \mapsto S_\gamma(t)x_0$, $t \geqslant 0$ belongs to the space $C_0([0,\infty) : X)$; see the estimate (2.54). Arguing as in the proof of [364, Theorem 3.46], we may conclude that $\mathcal{X} := \mathrm{BUR}_{(\alpha_n)}([0,\infty) : X) \oplus C_0([0,\infty) : X)$ is a complete metric space equipped with the distance $d(\cdot,\cdot)$ used above. Set, for every $u \in \mathcal{X}$ and $n \in \mathbb{N}$,

$$(\Upsilon_A u)(t) := S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0;$$

$$A_n := \sup_{t\geqslant 0} \int_0^t \int_0^{x_n} \cdots \int_0^{x_2} \|R_\gamma(t-x_n)\| \prod_{i=2}^n \|R_\gamma(x_i - x_{i-1})\| \prod_{i=1}^n L_F(x_i)\,dx_1\,dx_2 \cdots dx_n.$$

Then a simple calculation shows that

$$\|(\Upsilon_A^n u) - (\Upsilon_A^n v)\|_\infty \leqslant A_n \|u-v\|_\infty, \quad u,\ v \in \mathcal{X},\ n \in \mathbb{N}.$$

Keeping in mind [648, Proposition 3.1], Theorem 2.4.46, Remarks 2.4.47–2.4.48 and the proof of [631, Lemma 2.6.3], we can similarly clarify the following result.

Theorem 2.4.52. *Suppose that the function $F : [0,\infty) \times X \to X$ is continuous and satisfies the requirement that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\geqslant 0} \sup_{y\in B} \|F(t,y)\| \leqslant M_B$. Suppose, further, that $H : [0,\infty) \times X \to X$ is uniformly recurrent on bounded sets, there exists a finite constant $L > 0$ such that (2.51) holds with the function $F(\cdot,\cdot)$ replaced therein with the function $H(\cdot,\cdot)$ and $I = [0,\infty)$. Let (2.50) hold with any bounded set $B = K$, and let there exist an integer $n \in \mathbb{N}$ such that $A_n < 1$. If $Q \in C_{0,b}(I \times Y : X)$ and $F(t,y) = H(t,y) + Q(t,y)$ for all $t \geqslant 0$ and $y \in Y$, then the abstract fractional Cauchy inclusion $(\mathrm{DFP})_{f,\gamma,s}$ has a unique mild solution.*

Let Ω be a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a. e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $X := L^p(\Omega)$. Suppose that the operator $A := \Delta - b$ acts on X with the Dirichlet boundary conditions, and that B is the multiplication operator by the function $m(x)$. As explained in [631], we can apply Theorem 2.4.52 with $\mathcal{A} = AB^{-1}$ in the study of existence and uniqueness of asymptotically uniformly recurrent solutions of the semilinear fractional Poisson heat equation

$$\begin{cases} \mathbf{D}_t^\gamma[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,m(x)v(t,x)), & t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

3 Generalized almost periodicity in Lebesgue spaces with variable exponents

3.1 Generalized almost periodicity in Lebesgue spaces with variable exponents. Part I

The main purpose of this section is to investigate generalized asymptotically almost periodic functions in Lebesgue spaces with variable exponents. Suppose that $f : [a,b] \to \mathbb{R}$ is a non-negative Lebesgue-integrable function, where $a, b \in \mathbb{R}$, $a < b$, and $\phi : [0,\infty) \to \mathbb{R}$ is a convex function. Let us recall that the Jensen integral inequality states that

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leqslant \frac{1}{b-a}\int_a^b \phi(f(x))\,dx.$$

Using this integral inequality, we can simply prove that, for every two sequences (a_k) and (x_k) of non-negative real numbers such that $\sum_{k=0}^{\infty} a_k = 1$, we have

$$\phi\left(\sum_{k=0}^{\infty} a_k x_k\right) \leqslant \sum_{k=0}^{\infty} a_k \phi(x_k). \tag{3.1}$$

More generally, we have the following (see, e. g., [399, Theorem 1.1]).

Lemma 3.1.1. *Let (Ω, Λ, μ) be a measure space with $0 < \mu(\Omega) < +\infty$ and let $\phi : I \to \mathbb{R}$ be a convex function defined on an open interval I in $\mathbb{R}$. If $f : \Omega \to I$ satisfies $f,\ \phi \circ f \in L(\Omega, \Lambda, \mu)$, then we have*

$$\phi\left(\frac{1}{\mu(\Omega)}\int_\Omega f d\mu\right) \leqslant \frac{1}{\mu(\Omega)}\int_\Omega \phi(f)\,d\mu. \tag{3.2}$$

If $\phi(\cdot)$ is strictly convex on I, then the equality in (3.2) holds if and only if $f(\cdot)$ is constant almost everywhere on Ω; furthermore, if $\phi : [0,\infty) \to \mathbb{R}$ is a concave function, then the above inequalities reverse.

3.1.1 Almost periodic and asymptotically almost periodic type solutions with variable exponents $L^{p(x)}$

Before proceeding further, we need to recall the recently introduced notions of $S^{p(x)}$-boundedness and (asymptotical) Stepanov $p(x)$-almost periodicity:

Definition 3.1.2 ([372]). Let $p \in \mathcal{P}([0,1])$ and let $I = \mathbb{R}$ or $I = [0,\infty)$. A function $f \in M(I : X)$ is said to be Stepanov $p(x)$-bounded (or $S^{p(x)}$-bounded) if and only if $f(\cdot + t) \in$

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$L^{p(x)}([0,1] : X)$ for all $t \in I$, and the sup-norm of Bochner transform satisfies $\sup_{t\in I} \|f(\cdot + t)\|_{p(x)} < \infty$; more precisely,

$$\|f\|_{S^{p(x)}} := \sup_{t\in I} \inf\left\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(x+t)\|}{\lambda}\right) dx \leqslant 1\right\} < \infty.$$

The collection of such functions will be denoted by $L_S^{p(x)}(I : X)$.

From Definition 3.1.2 it follows that the space $L_S^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in L_S^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in L_S^{p(x)}(I : X)$. This is not the case with the notion introduced by T. Diagana and M. Zitane in [375, 376]. In the second part of the following definition, we extend the notion of asymptotical Stepanov $p(x)$-almost periodicity introduced in the case $I = [0, \infty)$ to the general case of interval I (see also [372, Proposition 4.12]).

Definition 3.1.3 ([372]).
(i) Let $p \in \mathcal{P}([0,1])$ and let $I = \mathbb{R}$ or $I = [0, \infty)$. A function $f \in L_S^{p(x)}(I : X)$ is said to be Stepanov $p(x)$-almost periodic (Stepanov $p(x)$-a. p., for short) if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is almost periodic. The collection of such functions will be denoted by $\mathrm{APS}^{p(x)}(I : X)$.
(ii) Let $p \in \mathcal{P}([0,1])$. Then a function $f \in L_S^{p(x)}(I : X)$ is said to be asymptotically Stepanov $p(x)$-almost periodic (Stepanov $p(x)$-a. p., for short) if and only if there exist two locally p-integrable functions $g : \mathbb{R} \to X$ and $q : I \to X$ satisfying the following conditions:
   (i) g is $S^{p(x)}$-almost periodic,
   (ii) $\hat{q}$ belongs to the class $C_0(I : L^{p(x)}([0,1] : X))$,
   (iii) $f(t) = g(t) + q(t)$ for all $t \in I$.
   The collection of such functions will be denoted by $\mathrm{AAPS}^{p(x)}(I : X)$.

As in the case of Stepanov $p(x)$-boundedness, the space $\mathrm{APS}^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in \mathrm{APS}^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in \mathrm{APS}^{p(x)}(I : X)$. A similar statement holds for the space $\mathrm{AAPS}^{p(x)}([0, \infty) : X)$.

We will extend [375, Definition 3.10] in the following way (in this paper, the authors have considered the case $I = \mathbb{R}$ and $p \in C_+(\mathbb{R})$; we can extend the notion introduced in [375, Definition 3.11] in the same way):

Definition 3.1.4. Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $p \in \mathcal{P}(I)$. Then it is said that a measurable function $f : I \to X$ belongs to the space $BS^{p(x)}(I : X)$ if and only if

$$\|f\|_{\mathbf{S}^{p(x)}} := \sup_{t\in I} \inf\left\{\lambda > 0 : \int_t^{t+1} \varphi_{p(x)}(\|f(x)\|/\lambda)\, dx \leqslant 1\right\} < \infty.$$

Equipped with the norm $\|\cdot\|_{S^{p(x)}}$, the space $L_S^{p(x)}(I : X)$ consisting of all S^p-bounded functions is a Banach space, which is continuously embedded in $L_S^1(I : X)$, for any $p \in \mathcal{P}([0,1])$. Furthermore, it can be easily shown that $\mathrm{APS}^{p(x)}(I : X)$ ($\mathrm{AAPS}^{p(x)}(I : X)$ with $I = [0,\infty)$) is a closed subspace of $L_S^{p(x)}(I : X)$ and therefore is a Banach space itself, for any $p \in \mathcal{P}([0,1])$.

If $p \in \mathcal{P}([0,1])$, then Lemma 1.1.7(ii) yields $L^{p(x)}([0,1] : X) \hookrightarrow L^1([0,1] : X)$, where the symbol $\hookrightarrow$ stands for a "continuous embedding", so that $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$.

We have the following.

Proposition 3.1.5. *Suppose $p \in \mathcal{P}([0,1])$. Then the following continuous embeddings hold:*
(i) $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$, *and*
(ii) $\mathrm{APS}^{p(x)}(I : X) \hookrightarrow \mathrm{APS}^1(I : X)$ *and* $\mathrm{AAPS}^{p(x)}([0,\infty) : X) \hookrightarrow \mathrm{AAPS}^1([0,\infty) : X)$.

Similarly, the following holds.

Proposition 3.1.6. *Suppose $p \in D_+([0,1])$ and $1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ for a.e. $x \in [0,1]$. Then the following continuous embeddings hold:*
(i) $L_S^{p^+}(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^{p^-}(I : X)$, *and*
(ii) $\mathrm{APS}^{p^+}(I : X) \hookrightarrow \mathrm{APS}^{p(x)}(I : X) \hookrightarrow \mathrm{APS}^{p^-}(I : X)$ *and* $\mathrm{AAPS}^{p^+}([0,\infty) : X) \hookrightarrow \mathrm{AAPS}^{p(x)}([0,\infty) : X) \hookrightarrow \mathrm{AAPS}^{p^-}([0,\infty) : X)$.

Now we will prove that any almost periodic function is $S^{p(x)}$-almost periodic, for any $p \in \mathcal{P}([0,1])$.

Proposition 3.1.7. *Let $p \in \mathcal{P}([0,1])$, and let $f : I \to X$ be almost periodic. Then $f(\cdot)$ is $S^{p(x)}$-almost periodic.*

Proof. To prove that $f(\cdot)$ is $S^{p(x)}$-bounded and $\|f\|_{L_S^{p(x)}} \leqslant \|f\|_\infty$, it suffices to show that, for every $t \in \mathbb{R}$, we have

$$[\|f\|_\infty, \infty) \subseteq \left\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(x+t)\|}{\lambda}\right) dx \leqslant 1\right\}. \tag{3.3}$$

For $\lambda \geqslant \|f\|_\infty$, we have $\|f(x+t)\|/\lambda \leqslant 1$, $t \in I$. It is obvious that, in this case,

$$\varphi_{p(x)}\left(\frac{\|f(x+t)\|}{\lambda}\right) \leqslant 1, \quad t \in I,$$

so that the integrand does not exceed 1; as a matter of fact, by definition of $\varphi_{p(x)}(\cdot)$, we only need to observe that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+x)\|/\lambda)^{p(x)} \leqslant 1^{p(x)} = 1$, $t \in I$. Hence, (3.3) holds. Using the uniform continuity of $f(\cdot)$ and a similar argumentation, we can show that the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is uniformly continuous. The almost periodicity of the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ can be proved in a direct way: for $\varepsilon > 0$ given as above, there is a finite number $l > 0$ such that any subinterval I' of I of length l contains a number $\tau \in I'$ such that $\|f(t+\tau) - f(t)\| \leqslant \varepsilon$, $t \in I$. After that, it suffices to observe that, for this $\varepsilon > 0$, we can choose the same length $l > 0$ and the same ε-almost period τ from I' ensuring the validity of the inequality

$$\|\hat{f}(t+\tau+\cdot) - \hat{f}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leqslant \varepsilon, \quad t \in I:$$

in order to see that the last inequality holds true, we only need to prove that, for every $t \in I$, we have

$$[\varepsilon, \infty) \subseteq \left\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(t+\tau+x) - f(t+x)\|}{\lambda}\right) dx \leqslant 1\right\}.$$

Indeed, if $\lambda \geqslant \varepsilon$, then $\|f(t+\tau+x) - f(t+x)\|/\lambda \leqslant 1$, $t \in I$ and the integrand cannot exceed 1: this simply follows from definition of $\varphi_{p(x)}(\cdot)$ and observation that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+\tau+x) - f(t+x)\|/\lambda)^{p(x)} \leqslant 1^{p(x)} = 1$, $t \in I$. The proof of the proposition is thereby complete. □

We can similarly prove the following proposition.

Proposition 3.1.8. *Let $p \in \mathcal{P}([0,1])$, and let $f : I \to X$ be asymptotically almost periodic. Then $f(\cdot)$ is asymptotically $S^{p(x)}$-almost periodic.*

Taking into account Proposition 3.1.5(ii) and the method employed in the proof of Proposition 3.1.7, we can state the following.

Proposition 3.1.9. *Assume that $p \in \mathcal{P}([0,1])$ and $f \in L_S^{p(x)}(I : X)$. Then the following holds:*
(i) $L^\infty(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$.
(ii) $\mathrm{AP}(I : X) \hookrightarrow \mathrm{APS}^{p(x)}(I : X) \hookrightarrow \mathrm{APS}^1(I : X)$ *and* $\mathrm{AAP}(I : X) \hookrightarrow \mathrm{AAPS}^{p(x)}(I : X) \hookrightarrow \mathrm{AAPS}^1(I : X)$.
(iii) *The continuity (uniform continuity) of $f(\cdot)$ implies continuity (uniform continuity) of $\hat{f}(\cdot)$.*

In the general case, we have the following.

Proposition 3.1.10. *Assume that $p,\ q \in \mathcal{P}([0,1])$ and $p \leqslant q$ a. e. on $[0,1]$. Then we have:*
(i) $L_S^{q(x)}(I : X) \hookrightarrow L_S^{p(x)}(I : X)$.
(ii) $\mathrm{APS}^{q(x)}(I : X) \hookrightarrow \mathrm{APS}^{p(x)}(I : X)$ *and* $\mathrm{AAPS}^{q(x)}(I : X) \hookrightarrow \mathrm{AAPS}^{p(x)}(I : X)$.
(iii) *If $p \in D_+([0,1])$, then*

$$L^\infty(I : X) \cap \mathrm{APS}^{p(x)}(I : X) = L^\infty(I : X) \cap \mathrm{APS}^1(I : X)$$

and

$$L^\infty(I : X) \cap \mathrm{AAPS}^{p(x)}(I : X) = L^\infty(I : X) \cap \mathrm{AAPS}^1(I : X).$$

Proof. We will prove only (iii) for almost periodicity. Keeping in mind Proposition 3.1.6(ii), it suffices to assume that $p(x) \equiv p > 1$. Then, clearly, $L^{\infty}(I : X) \cap APS^{p}(I : X) \subseteq L^{\infty}(I : X) \cap APS^{1}(I : X)$ and it remains to be proved the opposite inclusion. So, let $f \in L^{\infty}(I : X) \cap APS^{1}(I : X)$. The required conclusion follows from the elementary definitions and the next simple calculation, which is valid for any $t, \tau \in \mathbb{R}$:

$$\begin{aligned}\left[\int_{t}^{t+1} \|f(\tau+s)-f(s)\|^{p}\, ds\right]^{1/p} &\leqslant \left[\int_{t}^{t+1} (2\|f\|_{\infty})^{p-1}\|f(\tau+s)-f(s)\|\, ds\right]^{1/p} \\ &= (2\|f\|_{\infty})^{(p-1)/p}\left[\int_{t}^{t+1} \|f(\tau+s)-f(s)\|\, ds\right]^{1/p}. \qquad \square\end{aligned}$$

Remark 3.1.11. Recall that $APS^{p(x)}(I : X)$ can be strictly contained in $APS^{1}(I : X)$, even in the case that $p(x) \equiv p > 1$ is a constant function. The already employed example of H. Bohr and E. Følner shows that $AAPS^{p}(I : X)$ can be strictly contained in $AAPS^{1}(I : X)$ for $p > 1$ (see e. g. [526, Lemma 1]).

Remark 3.1.12. Proposition 3.1.7 and Proposition 3.1.8 can be simply deduced by using Proposition 3.1.10(ii) and the equalities $AP(I : X) = APS^{\infty}(I : X) \cap C(I : X)$, $AAP(I : X) = AAPS^{\infty}(I : X) \cap C([0,\infty) : X)$, which can be proved almost trivially.

Now we would like to present the following example.

Example 3.1.13. Let us note that, for every trigonometric polynomial $f : \mathbb{R} \to \mathbb{R}$, the function $F(\cdot) := \operatorname{sign}(f(\cdot))$ is Stepanov 1-almost periodic. Since $F \in L^{\infty}(\mathbb{R})$, Proposition 3.1.10(iii) shows that the function $F(\cdot)$ is Stepanov p-almost periodic for any $p \geqslant 1$, while Proposition 3.1.9(i) shows that the function $F(\cdot)$ is Stepanov $p(x)$-bounded for any $p \in \mathcal{P}([0,1])$. Due to Proposition 3.1.6(ii), we have $F \in APS^{p(x)}(\mathbb{R} : \mathbb{C})$ for any $p \in D_{+}([0,1])$.

Suppose now that $f(x) := \sin x + \sin \sqrt{2}x$, $x \in \mathbb{R}$ and $p(x) := 1 - \ln x$, $x \in [0,1]$. We will prove that $F \notin APS^{p(x)}(\mathbb{R} : \mathbb{C})$. In actual fact, it is sufficient to show that, for every $\lambda \in (0, 2/e)$ and for every $l > 0$, we can find an interval $I \subseteq \mathbb{R}$ of length $l > 0$ such that, for every $\tau \in I$, there exists $t \in \mathbb{R}$ such that

$$\begin{aligned}\int_{0}^{1} \left(\frac{1}{\lambda}\right)^{1-\ln x} \big|&\operatorname{sign}[\sin(x+t+\tau) + \sin \sqrt{2}(x+t+\tau)] \\ &- \operatorname{sign}[\sin(x+t) + \sin \sqrt{2}(x+t)]\big|^{1-\ln x}\, dx = \infty.\end{aligned}$$

Let $\lambda \in (0, 2/e)$ and $l > 0$ be given. Take arbitrarily any interval $I \subseteq \mathbb{R} \setminus \{0\}$ of length l and after that take arbitrarily any number $\tau \in I$. Since $(1/\lambda)^{1-\ln x} \geqslant 1/x$, $x \in [0,1]$ and $1 - \ln x \geqslant 1$, $x \in [0,1]$, a continuity argument shows that it is enough to prove the existence of a number $t \in \mathbb{R}$ such that

$$[\sin(t+\tau) + \sin \sqrt{2}(t+\tau)] \cdot [\sin t + \sin \sqrt{2}t] < 0. \tag{3.4}$$

If $\sin\tau + \sin\sqrt{2}\tau > 0$ ($\sin\tau + \sin\sqrt{2}\tau < 0$), then we can take $t \sim 0-$ ($t \sim 0+$). Hence, we assume henceforward $\sin\tau + \sin\sqrt{2}\tau = 0$ and $\tau \neq 0$. There exist two possibilities:

$$\tau \in \frac{2\mathbb{Z}\pi}{1+\sqrt{2}} \setminus \{0\} \quad \text{or} \quad \tau \in \frac{(2\mathbb{Z}+1)\pi}{\sqrt{2}-1}.$$

In the first case, take $t_0 = \pi/(\sqrt{2}-1)$. Then an elementary argumentation shows that $\tau + t_0 \notin (2\mathbb{Z}\pi)/(1+\sqrt{2}) \cup ((2\mathbb{Z}+1)\pi)/(\sqrt{2}-1)$ so that $\sin(t_0+\tau) + \sin\sqrt{2}(t_0+\tau) \neq 0$. If $\sin(t_0+\tau)+\sin\sqrt{2}(t_0+\tau) > 0$ ($\sin(t_0+\tau)+\sin\sqrt{2}(t_0+\tau) < 0$), then for t satisfying (3.4) we can take any number belonging to a small left/right interval around t_0 for which $\sin t + \sin\sqrt{2}t < 0$ ($\sin t + \sin\sqrt{2}t > 0$). In the second case, there exists an integer $m \in \mathbb{Z}$ such that $\tau = (2m+1)\pi/(\sqrt{2}-1)$ and we can take $t_0 = (-2m+1)\pi/(\sqrt{2}-1)$. Then $\tau + t_0 = (2\pi)/(\sqrt{2}-1)$ and $\sin(t_0+\tau) + \sin\sqrt{2}(t_0+\tau) \neq 0$, so that we can use a trick similar to that used in the first case. Let us only mention in passing that, with the notion introduced in [373], also the function $F(\cdot)$ cannot be $S^{p(x)}$-almost automorphic.

The situation is quite different if we consider the situation in which $f(x) := \sin x$, $x \in \mathbb{R}$. Then $F(\cdot)$ is Stepanov $p(x)$-almost periodic for any $p \in \mathcal{P}([0,1])$. Strictly speaking, it can be easily shown that the mapping $\hat{F} : \mathbb{R} \to L^{p(x)}[0,1]$ is continuous and $\|F(t+\tau+\cdot) - F(t+\cdot)\|_{L^{p(x)}[0,1]} = 0$ for all $t \in \mathbb{R}$ and $\tau \in 2\pi\mathbb{Z}$. This, in turn, implies the claimed statement.

3.1.2 Generalized two-parameter almost periodic type functions and composition principles

Assume that $I = \mathbb{R}$ or $I = [0,\infty)$. The notion of (asymptotical) Stepanov $p(x)$-almost periodicity for the functions depending on two parameters is introduced as follows:

Definition 3.1.14. Let $p \in \mathcal{P}([0,1])$.

(i) A function $f : I \times Y \to X$ is called Stepanov $p(x)$-almost periodic, $S^{p(x)}$-almost periodic for short, if and only if $\hat{f} : I \times Y \to L^{p(x)}([0,1] : X)$ is almost periodic. The vector space consisting of all such functions will be denoted by $\mathrm{APS}^{p(x)}(I \times Y : X)$.

(ii) A function $f : I \times Y \to X$ is said to be asymptotically $S^{p(x)}$-almost periodic if and only if it admits a decomposition $f(t,y) = g(t,y) + q(t,y)$, $t \in I$, where $g \in \mathrm{APS}^{p(x)}(\mathbb{R} \times Y : X)$ and $q \in C_0(I \times Y : X)$. The vector space consisting of all such functions will be denoted by $\mathrm{AAPS}^{p(x)}(I \times Y : X)$.

The proof of following proposition is very similar to the proof of [631, Lemma 2.2.7] and therefore omitted (for simplicity, we will not consider case $I = \mathbb{R}$ here).

Proposition 3.1.15. *Let $p \in \mathcal{P}([0,1])$. Suppose that $\hat{f} : [0,\infty) \times Y \to L^{p(x)}([0,1] : X)$ is an asymptotically almost periodic function. Then there are two functions $g : \mathbb{R} \times Y \to X$*

and $q : [0,\infty) \times Y \to X$ *satisfying the requirement that for each* $y \in Y$ *the functions* $g(\cdot,y)$ *and* $q(\cdot,y)$ *are Stepanov* $p(x)$*-bounded, and that the following holds:*
(i) $\hat{g} : \mathbb{R} \times Y \to L^{p(x)}([0,1] : X)$ *is almost periodic,*
(ii) $\hat{q} \in C_0([0,\infty) \times Y : L^{p(x)}([0,1] : X))$,
(iii) $f(t,y) = g(t,y) + q(t,y)$ *for all* $t \geqslant 0$ *and* $y \in Y$.

Moreover, for every compact set $K \subseteq Y$, *there exists an increasing sequence* $(t_n)_{n\in\mathbb{N}}$ *of positive reals such that* $\lim_{n\to\infty} t_n = \infty$ *and* $g(t,y) = \lim_{n\to\infty} f(t+t_n,y)$ *for all* $y \in Y$ *and a. e.* $t \geqslant 0$.

In [631, Theorem 2.7.1, Theorem 2.7.2], we have slightly improved the important composition principle attributed to W. Long, S.-H. Ding [729, Theorem 2.2]. Further refinements for $S^{p(x)}$-almost periodicity can be deduced similarly, appealing to Lemma 1.1.7(i)–(iii) and the arguments employed in the proof of [729, Theorem 2.2].

Theorem 3.1.16. *Let* $I = \mathbb{R}$ *or* $I = [0,\infty)$, *and let* $p \in \mathcal{P}([0,1])$. *Suppose that the following conditions hold:*
(i) $F \in \mathrm{APS}^{p(x)}(I \times Y : X)$ *and there exist a function* $r \in \mathcal{P}([0,1])$ *such that* $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/(p(\cdot)-1))$ *and a function* $L_F \in L_S^{r(x)}(I)$ *such that* (2.20) *holds;*
(ii) $u \in \mathrm{APS}^{p(x)}(I : Y)$, *and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{u(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* Y; *here,* $m(\cdot)$ *denotes the Lebesgue measure.*

Define $q \in \mathcal{P}([0,1])$ *by* $q(x) := p(x)r(x)/(p(x)+r(x))$, *if* $x \in [0,1]$ *and* $r(x) < \infty$, $q(x) := p(x)$, *if* $x \in [0,1]$ *and* $r(x) = +\infty$. *Then* $q(x) \in [1, p(x))$ *for* $x \in [0,1]$, $r(x) < \infty$ *and* $F(\cdot, u(\cdot)) \in \mathrm{APS}^{q(x)}(I : X)$.

For the asymptotical two-parameter Stepanov $p(x)$-almost periodicity, we can deduce the following composition principles with $X = Y$; the proof is very similar to those of [631, Proposition 2.7.3, Proposition 2.7.4] established in the case of constant functions p, q, r and the interval $I = [0,\infty)$.

Proposition 3.1.17. *Let* $p \in \mathcal{P}([0,1])$. *Suppose that the following conditions hold:*
(i) $g \in \mathrm{APS}^{p(x)}(\mathbb{R} \times X : X)$, *there exist a function* $r \in \mathcal{P}([0,1])$ *such that* $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/(p(\cdot)-1))$ *and a function* $L_g \in L_S^{r(x)}(\mathbb{R})$ *such that* (2.20) *holds with the function* $f(\cdot,\cdot)$ *replaced by the function* $g(\cdot,\cdot)$ *therein.*
(ii) $v \in \mathrm{APS}^{p(x)}(\mathbb{R} : X)$, *and there exists a set* $\mathrm{E} \subseteq \mathbb{R}$ *with* $m(\mathrm{E}) = 0$ *such that* $K = \{v(t) : t \in \mathbb{R} \setminus \mathrm{E}\}$ *is relatively compact in* X.
(iii) $f(t,x) = g(t,x) + q(t,x)$ *for all* $t \in I$ *and* $x \in X$, *where* $\hat{q} \in C_0(I \times X : L^{q(x)}([0,1] : X))$ *with* $q(\cdot)$ *defined as above;*
(iv) $u(t) = v(t) + \omega(t)$ *for all* $t \geqslant 0$, *where* $\hat{\omega} \in C_0(I : L^{p(x)}([0,1] : X))$.
(v) *There exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{u(t) : t \in I \setminus E'\}$ *is relatively compact in* X.

Then $f(\cdot, u(\cdot)) \in \mathrm{AAPS}^{q(x)}(I : X)$.

3.1.3 Generalized (asymptotical) almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$: action of convolution products

Throughout this subsection, which has also appeared as a part of [633], we assume that $p \in \mathcal{P}([0,1])$ and a multivalued linear operator $\mathcal{A}$ fulfills condition (P). Then we know that the degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ satisfies the estimate $\|T(t)\| \leqslant M_0 e^{-ct} t^{\beta-1}$, $t > 0$ for some finite constant $M_0 > 0$. Furthermore, $(T(t))_{t>0}$ is given by

$$T(t)x = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda - \mathcal{A})^{-1} x \, d\lambda, \quad t > 0,\ x \in X,$$

where Γ is the upwards oriented curve $\lambda = -c(|\eta| + 1) + i\eta$ $(\eta \in \mathbb{R})$. For any $0 < \gamma < 1$ and $\nu > -\beta$, we define the operator family $(T_{\gamma,\nu}(t))_{t>0}$ through (2.53). Set, as before, $S_\gamma(t) := T_{\gamma,0}(t)$ and $P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma$, $t > 0$. Then $(S_\gamma(t))_{t>0}$ is a subordinated (g_γ, I)-regularized resolvent family generated by $\mathcal{A}$, which is generally not strongly continuous at zero. By our analysis from [633], we know that there exists a finite constant $M_1 > 0$ such that

$$\|S_\gamma(t)\| + \|P_\gamma(t)\| \leqslant M_1 t^{\gamma(\beta-1)}, \quad t > 0,$$

and that there exists a finite constant $M_2 > 0$ such that

$$\|S_\gamma(t)\| \leqslant M_2 t^{-\gamma}, \quad t \geqslant 1 \quad \text{and} \quad \|P_\gamma(t)\| \leqslant M_2 t^{-2\gamma}, \quad t \geqslant 1.$$

Set $R_\gamma(t) := t^{\gamma-1} P_\gamma(t)$, $t > 0$.

We will first investigate infinite convolution products. Keeping in mind composition principles clarified in the previous subsection, it is almost straightforward to reformulate some known results concerning semilinear analogues of the above inclusions (see e. g. [631, Theorem 2.7.6–Theorem 2.7.9; Theorem 2.9.10–Theorem 2.9.11; Theorem 2.9.17–Theorem 2.9.18]); because of that, this question will not be examined here for the sake of brevity.

We start by stating the following generalization of [636, Proposition 2.11] (the reflexion at zero keeps the spaces of Stepanov p-almost periodic functions unchanged, which may or may not be the case with the spaces of Stepanov $p(x)$-almost periodic functions).

Proposition 3.1.18. *Suppose that p, $q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic, then the function $G : \mathbb{R} \to Y$, given by* (2.46), *is well defined and almost periodic.*

Proof. Without loss of generality, we may assume that $X = Y$. It is clear that, for every $t \in \mathbb{R}$, we see that $G(t) = \int_0^\infty R(s)g(t-s)\,ds$ and the last integral is absolutely convergent due to Lemma 1.1.7(i) and $S^{p(x)}$-boundedness of the function $\check{g}(\cdot)$:

$$\begin{aligned}\int_0^\infty \|R(s)\|\|g(t-s)\|\,ds &= \sum_{k=0}^\infty \int_k^{k+1} \|R(s)\|\|g(t-s)\|\,ds \\ &= \sum_{k=0}^\infty \int_0^1 \|R(s+k)\|\|g(t-s-k)\|\,ds \\ &\leqslant 2\sum_{k=0}^\infty \|R(\cdot+k)\|_{L^{q(\cdot)}([0,1]:X)}\|g(t-k-\cdot)\|_{L^{p(\cdot)}([0,1]:X)} \\ &\leqslant 2M\sup_{t\in\mathbb{R}}\|\check{g}(\cdot-t)\|_{L^{p(\cdot)}([0,1]:X)},\end{aligned}$$

for any $t \in \mathbb{R}$. Let a number $\varepsilon > 0$ be fixed. Then there is a finite number $l > 0$ such that any subinterval I of $\mathbb{R}$ of length l contains a number $\tau \in I$ such that $\|\check{g}(t-\tau+\cdot) - \check{g}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leqslant \varepsilon$, $t \in \mathbb{R}$. Invoking Lemma 1.1.7(i) and this fact, we get

$$\begin{aligned}\|G(t+\tau)-G(t)\| &\leqslant \int_0^\infty \|R(r)\|\cdot\|g(t+\tau-r)-g(t-r)\|\,dr \\ &= \sum_{k=0}^\infty \int_k^{k+1} \|R(r)\|\cdot\|g(t+\tau-r)-g(t-r)\|\,dr \\ &= \sum_{k=0}^\infty \int_0^1 \|R(r+k)\|\cdot\|g(t+\tau-r-k)-g(t-r-k)\|\,dr \\ &\leqslant 2\sum_{k=0}^\infty \|R(\cdot+k)\|_{L^{q(x)}[0,1]}\|g(t+\tau-\cdot-k)-g(t-\cdot-k)\|_{L^{p(x)}[0,1]} \\ &= 2\sum_{k=0}^\infty \|R(\cdot+k)\|_{L^{q(\cdot)}[0,1]}\|\check{g}(\cdot-t-\tau+k)-\check{g}(\cdot-t+k)\|_{L^{p(\cdot)}[0,1]} \\ &\leqslant 2\varepsilon\sum_{k=0}^\infty \|R(\cdot+k)\|_{L^{q(\cdot)}[0,1]} = 2M\varepsilon, \quad t \in \mathbb{R},\end{aligned}$$

which clearly implies that the set of all ε-periods of $G(\cdot)$ is relatively dense in $\mathbb{R}$. It remains to be proved the uniform continuity of $G(\cdot)$. Since $\hat{\check{g}}(\cdot)$ is uniformly continuous, we have the existence of a number $\delta \in (0,1)$ such that

$$\|\check{g}(\cdot-t') - \check{g}(\cdot-t)\|_{L^{p(x)}[0,1]} < \varepsilon, \quad \text{provided } t,\ t' \in \mathbb{R} \text{ and } |t-t'| < \delta. \tag{3.5}$$

For any $\delta' \in (0,\delta)$, the above computation with $\tau = \delta' = t' - t$ and (3.5) together imply that, for every $t \in \mathbb{R}$,

$$\begin{aligned}\|G(t+\delta') - G(t)\| &\leqslant 2\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(\cdot)}[0,1]}\|\check{g}(\cdot - t' + k) - \check{g}(\cdot - t + k)\|_{L^{p(\cdot)}[0,1]}\\ &\leqslant 2\varepsilon\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(\cdot)}[0,1]} = 2M\varepsilon.\end{aligned}$$

This completes the proof of the proposition. □

Example 3.1.19.

(i) Suppose that $\beta \in (0,1)$ and $(R(t))_{t>0} = (T(t))_{t>0}$ is a degenerate semigroup generated by $\mathcal{A}$. Let us recall that there exists a finite constant $M > 0$ such that $\|T(t)\| \leqslant Mt^{\beta-1}$, $t \in (0,1]$ and $\|T(t)\| \leqslant Me^{-ct}$, $t \geqslant 1$. Let $p_0 > 1$ be such that

$$\frac{p_0}{p_0-1}(\beta-1) \leqslant -1,$$

let $p \in \mathcal{P}([0,1])$, and let $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$. Assume that we have constructed a function $\check{g} \in \mathrm{APS}^{p(x)}(\mathbb{R} : X)$ such that $\check{g} \notin \mathrm{APS}^{p}(\mathbb{R} : X)$ for all $p \geqslant p_0$ (Question: Could we manipulate here somehow the construction established in [199, Example, p. 70]?). Then, in our concrete situation, [636, Proposition 2.11] cannot be applied since

$$\frac{p}{p-1}(\beta-1) \leqslant -1, \quad p \in [1,p_0).$$

Now we will briefly explain that $\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$, showing that Proposition 3.1.18 is applicable. Strictly speaking, for $k = 0$, $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$ by our assumption, while, for $k \geqslant 1$, it can be simply shown that $\|R(\cdot+k)\|_{L^{q(x)}[0,1]} \leqslant Me^{-ck}$ so that $\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$, as claimed.

(ii) By a mild solution of problem obtained by replacing the MLO $\mathcal{A}$ with the MLO $-\mathcal{A}$ in (2.49), we mean the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)g(s)\,ds$, $t \in \mathbb{R}$ (cf. also [798, Lemma 6]). Let $p \in \mathcal{P}([0,1])$, and let $\|R_\gamma(\cdot)\|_{L^{q(x)}[0,1]} < \infty$. Then, for $k \in \mathbb{N}$, we have

$$\|R_\gamma(\cdot+k)\|_{L^{q(x)}[0,1]} \leqslant M_2k^{-1-\gamma}.$$

Hence, $\sum_{k=0}^{\infty}\|R_\gamma(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$ and we can apply Proposition 3.1.18.

The results obtained for the infinite convolution product can be simply incorporated in the study of existence and uniqueness of almost periodic solutions of the following abstract Cauchy differential inclusion of first order

$$u'(t) \in \mathcal{A}u(t) + g(t), \quad t \in \mathbb{R},$$

and the abstract Cauchy relaxation differential inclusion (2.49) with the MLO $\mathcal{A}$ replaced therein with $-\mathcal{A}$. It is also clear that Proposition 3.1.18 can be used to study the existence and uniqueness of almost periodic solutions of the following abstract integral inclusion

$$u(t) \in \mathcal{A} \int_{-\infty}^{t} a(t-s)u(s)\,ds + g(t), \quad t \in \mathbb{R},$$

where $a \in L^1_{\mathrm{loc}}([0,\infty))$, $a \neq 0$, $\check{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic and $\mathcal{A}$ is a closed multivalued linear operator on X; see, e. g., [631].

In the following proposition, whose proof is very similar to that of [373, Proposition 3.12], we state some invariance properties of generalized asymptotical almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$ under the action of finite convolution products (see also [631, Proposition 2.7.5, Lemma 2.9.3] for similar results). This proposition generalizes [636, Proposition 2.13] provided that $p > 1$ in its formulation.

Proposition 3.1.20. *Suppose that $p \in \mathcal{P}([0,1])$, $q \in D_+([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying the requirement that, for every $t \geqslant 0$, we have*

$$m_t := \sum_{k=0}^{\infty} \|R(\cdot + t + k)\|_{L^{q(x)}[0,1]} < \infty.$$

Suppose, further, that $\check{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic, $q \in L_S^{p(x)}([0,\infty) : X)$ and $f(t) = g(t) + q(t)$, $t \geqslant 0$. Let r_1, $r_2 \in \mathcal{P}([0,1])$ and the following hold:

(i) *For every $t \geqslant 0$, the mapping $x \mapsto \int_0^{t+x} R(t+x-s)q(s)\,ds$, $x \in [0,1]$ belongs to the space $L^{r_1(x)}([0,1] : X)$ and we have*

$$\lim_{t\to+\infty} \left\| \int_0^{t+x} R(t+x-s)q(s)\,ds \right\|_{L^{r_1(x)}[0,1]} = 0.$$

(ii) *For every $t \geqslant 0$, the mapping $x \mapsto m_{t+x}$, $x \in [0,1]$ belongs to the space $L^{r_2(x)}[0,1]$ and we have*

$$\lim_{t\to+\infty} \|m_{t+x}\|_{L^{r_2(x)}[0,1]} = 0.$$

Then the function $H(\cdot)$, given by

$$H(t) := \int_0^{t} R(t-s)f(s)\,ds, \quad t \geqslant 0,$$

is well defined, bounded and belongs to the class $\mathrm{APS}^{p(x)}(\mathbb{R} : X) + S_0^{r_1(x)}([0,\infty) : X) + S_0^{r_2(x)}([0,\infty) : X)$, with the meaning clear.

Remark 3.1.21. In [636, Remark 2.14], we have examined the conditions under which the function $H(\cdot)$ defined above is asymptotically almost periodic, provided that the function $g(\cdot)$ is S^p-almost periodic for some $p \in [1,\infty)$. The interested reader may try to analyze similar problems with the function $\check{g}(\cdot)$ being $S^{p(x)}$-almost periodic for some $p \in \mathcal{P}([0,1])$.

In order to describe how Proposition 3.1.20 can be applied in concrete situations, we need the following weakened definition of Caputo fractional derivatives of order $\gamma \in (0,1)$. The Caputo fractional derivative $\mathbf{D}_t^\gamma u(t)$ is defined for those functions $u : [0,T] \to X$ for which $u_{|(0,T]}(\cdot) \in C((0,T] : X)$, $u(\cdot)-u(0) \in L^1((0,T) : X)$ and $g_{1-\gamma} * (u(\cdot)-u(0)) \in W^{1,1}((0,T) : X)$, by

$$\mathbf{D}_t^\gamma u(t) = \frac{d}{dt}\big[g_{1-\gamma} * \big(u(\cdot) - u(0)\big)\big](t), \quad t \in (0,T].$$

We will use the following definition.

Definition 3.1.22 (cf. [633, Section 3.5] for more details). By a classical solution of the abstract fractional Cauchy problem

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + f(t), & t > 0, \\ u(0) = x_0, \end{cases}$$

we mean any function $u \in C([0,\infty) : X)$ satisfying the requirement that the function $\mathbf{D}_t^\gamma u(t)$ is well defined on any finite interval $(0,T]$ and belongs to the space $C((0,T] : X)$, and that $u(0) = u_0$ and $\mathbf{D}_t^\gamma u(t) - f(t) \in \mathcal{A}u(t)$ for $t > 0$.

Applying [633, Theorem 3.5.3], we see that the unique classical solution of $(\mathrm{DFP})_{f,\gamma}$ is given by the formula

$$u(t) = S_\gamma(t)x_0 + \int_0^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s)\, ds, \quad t \geqslant 0.$$

Suppose that $x_0 \in X$ belongs to the domain of continuity of $(S_\gamma(t))_{t>0}$ (by that, we mean that $\lim_{t\to 0+} S_\gamma(t)x_0 = x_0$; this holds in the case that $x \in D((-\mathcal{A})^\theta)$ with $1 \geqslant \theta > 1-\beta$ or that $x \in X_{\mathcal{A}}^\theta$ with $1 > \theta > 1-\beta$). Then the function $t \mapsto S_\gamma(t)x_0$, $t \geqslant 0$ is continuous and tends to zero as $t \to +\infty$. Keeping this in mind and imposing some additional conditions of the function $f(\cdot)$, we can straightforwardly apply Proposition 3.1.20. This proposition can be also applied in the qualitative properties of solutions to the following inhomogeneous abstract Cauchy problem of third order:

$$\alpha u'''(t) + u''(t) - \beta A u(t) - \gamma A u'(t) = f(t), \quad \alpha,\ \beta,\ \gamma > 0,\ t \geqslant 0,$$

appearing in the theory of dynamics of elastic vibrations of flexible structures [337] (see also [336]).

Finally, we will present some noteworthy applications. Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded subset with smooth boundary $\partial\Omega$ and let $1 < p < \infty$. Among many other statements, we can make use of Proposition 3.1.20 to establish the existence and uniqueness of asymptotically $S^{p(x)}$-almost automorphic solutions to the damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \frac{\partial}{\partial t}(m(x)\frac{\partial u}{\partial t}) + (2\omega m(x) - \Delta)\frac{\partial u}{\partial t} + (A(x;D) - \omega\Delta + \omega^2 m(x))u(x,t) = f(x,t), \\ \quad t \geqslant 0,\ x \in \Omega; \\ u = \frac{\partial u}{\partial t} = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(0,x) = u_0(x), \quad m(x)[(\frac{\partial u}{\partial t})(x,0) + \omega u_0] = m(x)u_1(x),\ x \in \Omega, \end{cases}$$

where $m(x) \in L^\infty(\Omega)$, $m(x) \geqslant 0$ a. e. $x \in \Omega$, Δ is the Dirichlet Laplacian in $L^2(\Omega)$, acting on its maximal domain, $H_0^1(\Omega) \cap H^2(\Omega)$, and $A(x;D)$ is a second-order linear differential operator on Ω with continuous coefficients on $\overline{\Omega}$; see, e. g., [431, Example 6.1] and [631] for further details.

Notice that we can also consider the existence and uniqueness of asymptotically $S^{p(x)}$-almost periodic solutions to the following fractional damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \mathbf{D}_t^\gamma(m(x)\mathbf{D}_t^\gamma u) + (2\omega m(x) - \Delta)\mathbf{D}_t^\gamma u + (A(x;D) - \omega\Delta + \omega^2 m(x))u(x,t) = f(x,t), \\ \quad t \geqslant 0,\ x \in \Omega; \\ u = \mathbf{D}_t^\gamma u = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(0,x) = u_0(x), \quad m(x)[\mathbf{D}_t^\gamma u(x,0) + \omega u_0] = m(x)u_1(x),\ x \in \Omega. \end{cases}$$

3.1.4 (p, ϕ, F)-Classes and $[p, \phi, F]$-classes of Weyl almost periodic functions

Throughout this subsection, we assume the following general conditions:

(A) $I = \mathbb{R}$ or $I = [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$, $p \in \mathcal{P}(I)$ and $F : (0,\infty) \times I \to (0,\infty)$.

(B) The same as (A) with the assumption $p \in \mathcal{P}(I)$ replaced by $p \in \mathcal{P}([0,1])$ therein.

We introduce the notions of an (equi-)Weyl (p,ϕ,F)-almost periodic function and an (equi-)Weyl $(p,\phi,F)_i$-almost periodic function, where $i = 1,2$, as follows (see [631] for case that $p(x) \equiv p \in [1,\infty)$, $\phi(x) = x$ and $F(l,t) = l^{(-1)/p}$, when we deal with the usually considered (equi-)Weyl p-almost periodic functions, as well as to [372, Remark 4.13] for case that $\phi(x) = x$ and $F(l,t) = l^{(-1)/p(t)}$).

Definition 3.1.23. Suppose that condition (A) holds, $f : I \to X$ and $\phi(\|f(\cdot+\tau) - f(\cdot)\|) \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl (p,ϕ,F)-almost periodic, $f \in e-W_{ap}^{(p,\phi,F)}(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers

$l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{(p,\phi,F,\tau)} := \sup_{t\in I}\big[F(l,t)\big[\phi\big(\big\|f(\cdot+\tau)-f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big] \leqslant \varepsilon. \tag{3.6}$$

(ii) It is said that the function $f(\cdot)$ is Weyl (p,ϕ,F)-almost periodic, $f \in W_{ap}^{(p,\phi,F)}(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{(p,\phi,F,\tau)} := \lim_{l\to\infty}\sup\sup_{t\in I}\big[F(l,t)\big[\phi\big(\big\|f(\cdot+\tau)-f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big] \leqslant \varepsilon. \tag{3.7}$$

Definition 3.1.24. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot+\tau)-f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $(p,\phi,F)_1$-almost periodic, $f \in e-W_{ap}^{(p,\phi,F)_1}(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{(p,\phi,F,\tau)_1} := \sup_{t\in I}\big[F(l,t)\phi\big[\big(\big\|f(\cdot+\tau)-f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $(p,\phi,F)_1$-almost periodic, $f \in W_{ap}^{(p,\phi,F)_1}(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{(p,\phi,F,\tau)_1} := \lim_{l\to\infty}\sup\sup_{t\in I}\big[F(l,t)\phi\big[\big(\big\|f(\cdot+\tau)-f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big] \leqslant \varepsilon.$$

Definition 3.1.25. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot+\tau)-f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $(p,\phi,F)_2$-almost periodic, $f \in e-W_{ap}^{(p,\phi,F)_2}(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{(p,\phi,F,\tau)_2} := \sup_{t\in I}\phi\big[F(l,t)\big[\big(\big\|f(\cdot+\tau)-f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $(p,\phi,F)_2$-almost periodic, $f \in W_{ap}^{(p,\phi,F)_2}(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{(p,\phi,F,\tau)_2} := \lim_{l\to\infty}\sup\sup_{t\in I}\phi\big[F(l,t)\big[\big(\big\|f(\cdot+\tau)-f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big] \leqslant \varepsilon.$$

Before we go any further, we would like to that the above definitions are incapable of being compared to each other: for example, in Definition 3.1.23, we calculate the value of $\phi(\|f(\cdot+\tau)-f(\cdot)\|)_{L^{p(\cdot)}[t,t+l]}$, while in Definition 3.1.24 we first calculate the value of $\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(\cdot)}[t,t+l]}$ and after that we apply the function $\phi(\cdot)$.

If $i = 1, 2$ and $F(l,t) = \psi(l)^{(-1)/p(t)}$ for some function $\psi : (0,\infty) \to (0,\infty)$ and all $t \in I$, then we also say that the function $f(\cdot)$ is (equi-)Weyl (p,ϕ,ψ)-almost periodic, resp. (equi-)Weyl $(p,\phi,\psi)_i$-almost periodic, when the corresponding class of functions is also denoted by $(e-)W_{ap}^{(p,\phi,\psi)}(I : X)$, resp. $(e-)W_{ap}^{(p,\phi,\psi)_i}(I : X)$. There is no need to say that the above classes coincide in the case that $\phi(x) \equiv x$.

Example 3.1.26.

(i) If $\phi(0) = 0$, then any continuous periodic function $f : I \to X$ belongs to any of the above introduced function spaces. If $\phi(0) > 0$, then a constant function cannot belong to any of the function spaces introduced in Definition 3.1.25, while the function spaces introduced in Definition 3.1.23–Definition 3.1.24 can contain constant functions (see also Remark 3.1.28(iii)).

(ii) If $\phi(x) = x$ and $p(x) \equiv p \in [1,\infty)$, then any Stepanov p-bounded function $f : I \to X$ belongs to any of the above introduced function spaces with $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1/p$; in particular, if $f(\cdot)$ is Stepanov $p(x)$-bounded and $p \in D_+(I)$, then $f(\cdot)$ belongs to any of the above introduced function spaces with $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1/p^+$. This simply follows from the inequality

$$\left(\int_t^{t+l} \|f(s+\tau)-f(s)\|^p\, ds\right)^{1/p} \leqslant \sum_{k=0}^{\lfloor l\rfloor}\left(\int_{t+k}^{t+k+1} \|f(s+\tau)-f(s)\|^p\, ds\right)^{1/p},$$

which is valid for any $t,\ \tau \in I$, $l > 0$, and a simple argumentation. Suppose now that $I = \mathbb{R}$ or $I = [0,\infty)$, $p \in \mathcal{P}(I)$ and $f \in BS^{p(x)}(I : X)$. A similar line of reasoning shows that $f(\cdot)$ belongs to any of the above introduced function spaces provided that

(a) $p \in D_+(I)$ and $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1/p^+$, or

(b) $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1$, in the general case. For this, it is only worth noting that we have $\varphi_{p(x)}(t/l^\sigma) \leqslant (1/l^\sigma)\varphi_{p(x)}(t)$ for any $t \geqslant 0$ and $l \geqslant 1$.

(iii) If X does not contain an isomorphic copy of the sequence space c_0, $\phi(x) = x$ and $F(l,t) \equiv F(t)$, where $\lim_{t\to+\infty} F(t) = +\infty$, then there is no trigonometric polynomial $f(\cdot)$ and function $p \in \mathcal{P}(\mathbb{R})$ such that $f \in e - W_{ap}^{(p,x,F)}(\mathbb{R} : X)$. If we suppose the contrary, then using the fact that the space $L^{p(x)}[t,t+l]$ is continuously embedded into the space $L^1[t,t+l]$ with the constant of embeddings less than or equal to $2(1+l)$ (see, e. g., [377, Corollary 3.3.4]), where $t \in \mathbb{R}$ and $l > 0$, we see that for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq \mathbb{R}$ of length L contains a point $\tau \in I'$ such that

$$\sup_{t\in\mathbb{R}}\left[F(t)\left\|f(\cdot+\tau)-f(\cdot)\right\|_{L^1[t,t+l]}\right] \leqslant 2\varepsilon(1+l). \tag{3.8}$$

Let such numbers $l > 0$ and $\tau \in \mathbb{R}$ be fixed. By (3.8), we see that the mapping

$$t \mapsto f_1(t) \equiv \int_t^{t+l} \|f(s+\tau) - f(s)\| \, ds, \quad t \geqslant 0$$

belongs to the space $C_0([0,\infty) : \mathbb{C})$. On the other hand, the mapping $s \mapsto \|f(s+\tau) - f(s)\|$, $s \in \mathbb{R}$ is almost periodic and satisfies $\int_0^t \|f(s+\tau) - f(s)\| \, ds < \infty$, so that the mapping

$$t \mapsto f_2(t) \equiv \int_0^{t} \|f(s+\tau) - f(s)\| \, ds, \quad t \in \mathbb{R}$$

is almost periodic by Theorem 2.1.1(vi). By the translation invariance, the same holds for the mapping $f_1(\cdot) = f_2(\cdot + \tau) - f_2(\cdot)$. Since $f_1 \in C_0([0,\infty) : \mathbb{C})$, we get $f_1 \equiv 0$, so that $\|f(s+\tau) - f(s)\| = 0$ for all $s \geqslant 0$ and $f(\cdot)$ is periodic, which is a contradiction. Based on the conclusion obtained in this part, we will not examine the question whether, for a given number $\varepsilon > 0$ and an equi-Weyl (p,ϕ,F)-almost periodic function or an equi-Weyl $(p,\phi,F)_i$-almost periodic function $(i = 1,2)$, we can find a trigonometric polynomial $P(\cdot)$ such that $\|P-f\|_{(p,\phi,F)} < \varepsilon$ or $\|P-f\|_{(p,\phi,F)_i} < \varepsilon$ $(i = 1,2)$, where

$$e - \|f\|_{(p,\phi,F)} := \sup_{t\in I}\big[F(l,t)\big[\phi\big(\|f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big],$$
$$e - \|f\|_{(p,\phi,F)_1} := \sup_{t\in I}\big[F(l,t)\phi\big[\big(\|f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big],$$

and

$$e - \|f\|_{(p,\phi,F)_2} := \sup_{t\in I}\phi\big[F(l,t)\big[\big(\|f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big].$$

For the usually considered class of equi-Weyl p-almost periodic functions, where $1 \leqslant p < \infty$, the answer to the above question is affirmative (see, e. g., [631, Theorem 2.3.2]). Observe also that the sub-additivity of the function $\phi(\cdot)$ implies the sub-additivity of functions $e - \|\cdot\|_{(p,\phi,F)}$ and $e - \|\cdot\|_{(p,\phi,F)_i}$, where $i = 1,2$; since the limit superior is also a sub-additive operation, the same holds for the functions $\|\cdot\|_{(p,\phi,F)}$ and $\|\cdot\|_{(p,\phi,F)_i}$, where $i = 1,2$, defined as above (cf. the second parts of Definition 3.1.23–Definition 3.1.25, as well as Definition 3.1.29–Definition 3.1.31 below).

In the case that the function $\phi(\cdot)$ is convex and $p(x) \equiv 1$, we have the following result.

Proposition 3.1.27. *Suppose that $p(x) \equiv 1$, $f : I \to X$, $\|f(\cdot+\tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I, as well as condition*

(C) *$\phi(\cdot)$ is convex and there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(lx) \leqslant \varphi(l)\phi(x)$ for all $l > 0$ and $x \geqslant 0$*

holds. Set $F_1(l,t) := F(l,t)l[\varphi(l)]^{-1}$, $l > 0$, $t \in I$ *and* $F_2(l,t) := l^{-1}\varphi(F(l,t)l)$, $l > 0$, $t \in I$. *Then we have:*
(i) $f \in (e-)W_{ap}^{(1,\phi,F)} \Rightarrow f \in (e-)W_{ap}^{(1,\phi,F_1)_1}$.
(ii) $f \in (e-)W_{ap}^{(1,\phi,F_2)} \Rightarrow f \in (e-)W_{ap}^{(1,\phi,F)_2}$.

Proof. To prove (i), suppose that $f \in (e-)W_{ap}^{(1,\phi,F)}$. Then the assumption (C) and the Jensen integral inequality together imply

$$\begin{aligned}\phi(\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]}) &= \phi(l\cdot l^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]})\\ &\leqslant \varphi(l)\phi(l^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]})\\ &\leqslant \varphi(l)l^{-1}[\phi(\|f(\cdot+\tau)-f(\cdot)\|)]_{L^1[t,t+l]}.\end{aligned}$$

This simply yields $f \in (e-)W_{ap}^{(1,\phi,F_1)_1}$. To prove (ii), suppose that $f \in (e-)W_{ap}^{(1,\phi,F_2)}$. Then the assumption (C) and the Jensen integral inequality together imply

$$\begin{aligned}\phi(F(l,t)\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]}) &= \phi(F(l,t)l\cdot l^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]})\\ &\leqslant \varphi(F(t,l)l)l^{-1}[\phi(\|f(\cdot+\tau)-f(\cdot)\|)]_{L^1[t,t+l]}.\end{aligned}$$

This simply yields $f \in (e-)W_{ap}^{(1,\phi,F)_2}$. □

Before we go any further, let us recall that any equi-Weyl p-almost periodic function needs to be Weyl p-almost periodic, while the converse statement does not hold in general. On the other hand, it is not true that an equi-Weyl (p,ϕ,ψ)-almost periodic function, resp. equi-Weyl $(p,\phi,\psi)_i$-almost periodic function, is Weyl (p,ϕ,ψ)-almost periodic, resp. Weyl $(p,\phi,\psi)_i$-almost periodic; moreover, an unrestrictive choice of the function $\psi(\cdot)$ allows us to work with a substantially large class of quasi-almost periodic functions: As it can be simply approved, any Stepanov p-almost periodic function $f(\cdot)$ is equi-Weyl (p,ϕ,ψ)-almost periodic with $p(x) \equiv p \in [1,\infty)$, $\psi(l) \equiv 1$, $\phi(x) = x$; on the other hand, any continuous Stepanov p-almost periodic function $f(\cdot)$ which is not periodic cannot be Weyl $(p,x,1)$-almost periodic, for example. Let us explain the last fact in more detail. If we suppose the contrary, then for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (3.7) holds with $p(x) \equiv p \in [1,\infty)$, $\psi(l) \equiv 1$ and $\phi(x) = \varphi(x) = x$. This simply implies that for each $\varepsilon > 0$ we can find a strictly increasing sequence (l_n) of positive real numbers tending to infinity such that for each $t \in I$ and $n \in \mathbb{N}$ we have $\int_{t+l_n}^{t}\|f(x+\tau)-f(x)\|^p\,dx \leqslant \varepsilon$ for each $\varepsilon > 0$; hence, $\int_I \|f(x+\tau)-f(x)\|^p\,dx \leqslant \varepsilon$ and therefore $\int_I \|f(x+\tau)-f(x)\|^p\,dx = 0$. This yields $f(x+\tau) = f(x)$, $x \in I$, which is a contradiction to our preassumption.

Remark 3.1.28.
(i) It is clear that, if $f(\cdot)$ is an (equi-)Weyl (p,ϕ,F)-almost periodic function, resp. (equi-)Weyl $(p,\phi,F)_1$-almost periodic function, and $F(l,t) \geqslant F_1(l,t)$ for every $l > 0$

and $t \in I$, then $f(\cdot)$ is (equi-)Weyl (p, ϕ, F_1)-almost periodic, resp. (equi-)Weyl $(p, \phi, F_1)_1$-almost periodic. Furthermore, if $f(\cdot)$ is an (equi-)Weyl $(p, \phi, F)_2$-almost periodic function, then $f(\cdot)$ is an (equi-)Weyl $(p, \phi, F_1)_2$-almost periodic function provided that $F(l, t) \geqslant F_1(l, t)$ for every $l > 0$, $t \in I$ and $\phi(\cdot)$ is monotonically increasing, or $F(l, t) \leqslant F_1(l, t)$ for every $l > 0$, $t \in I$ and $\phi(\cdot)$ is monotonically decreasing.

(ii) If $f(\cdot)$ is an (equi-)Weyl (p, ϕ, F)-almost periodic function, resp. (equi-)Weyl $(p, \phi, F)_i$-almost periodic function, $\phi_1(\cdot)$ is measurable and $0 \leqslant \phi_1 \leqslant \phi$, then Lemma 1.1.7(iii) shows that $f(\cdot)$ is (equi-)Weyl (p, ϕ_1, F)-almost periodic, resp. (equi-)-Weyl $(p, \phi_1, F)_i$-almost periodic, where $i = 1, 2$.

(iii) Regarding the first parts in the above definitions, it is worth noticing that we do not allow the number $l > 0$ to be sufficiently large: in some concrete situations, it is crucial to allow the number $l > 0$ to be sufficiently small; we will explain this fact by two illustrative examples. First, let us consider Definition 3.1.23(i). Suppose that $p(x) \equiv p \in [1, \infty)$ and there exists an absolute constant $c > 0$ such that for each $l > 0$ and $\tau \in I$ we have

$$\sup_{t \in I} \phi(\|f(\cdot + \tau) - f(\cdot)\|)_{L^{p(x)}[t,t+l]} \leqslant c.$$

Then it simply follows that the function $f(\cdot)$ is equi-Weyl (p, ϕ, ψ)-almost periodic provided that $\lim_{l \to 0+} \psi(l) = +\infty$. Second, suppose that $f \in L^{\infty}(I : X)$. Then $f(\cdot)$ is equi-Weyl $(p, x, 1)$-almost periodic for any $p \in \mathcal{D}(I)$, which can be simply approved by considering the case of constant coefficient $p(x) \equiv p^{+}$ and the choice $l = l(\varepsilon) = \varepsilon$.

In order to ensure the translation invariance of Weyl spaces with variable exponent, we need to follow a slightly different approach [372, 373].

Definition 3.1.29. Suppose that condition (B) holds, $f : I \to X$ and $\phi(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|) \in L^{p(x)}([0, 1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $[p, \phi, F]$-almost periodic, $f \in e - W_{ap}^{[p,\phi,F]}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{[p,\phi,F,\tau]} := \sup_{t \in I} \big[F(l, t)\big[\phi\big(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $[p, \phi, F]$-almost periodic, $f \in W_{ap}^{[p,\phi,F]}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{[p,\phi,F,\tau]} := \limsup_{l \to \infty} \sup_{t \in I} \big[F(l, t)\big[\phi\big(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] \leqslant \varepsilon.$$

Definition 3.1.30. Suppose that condition (B) holds, $f : I \to X$ and $\|f(\cdot l + t + \tau) - f(t + \cdot l)\| \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $[p,\phi,F]_1$-almost periodic, $f \in e-W_{ap}^{[p,\phi,F]_1}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{[p,\phi,F,\tau]_1} := \sup_{t\in I}\big[F(l,t)\phi\big[\big(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $[p,\phi,F]_2$-almost periodic, $f \in W_{ap}^{[p,\phi,F]_2}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{[p,\phi,F,\tau]_1} := \limsup_{l\to\infty}\sup_{t\in I}\big[F(l,t)\phi\big[\big(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] \leqslant \varepsilon.$$

Definition 3.1.31. Suppose that condition (B) holds, $f : I \to X$ and $\|f(\cdot l+t+\tau)-f(t+\cdot l)\| \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $[p,\phi,F]_2$-almost periodic, $f \in e-W_{ap}^{[p,\phi,F]_2}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{[p,\phi,F,\tau]_2} := \sup_{t\in I}\phi\big[F(l,t)\big[\big(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $[p,\phi,F]_2$-almost periodic, $f \in W_{ap}^{[p,\phi,F]_2}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{[p,\phi,F,\tau]_2} := \limsup_{l\to\infty}\sup_{t\in I}\phi\big[F(l,t)\big[\big(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] \leqslant \varepsilon.$$

Remark 3.1.32.

(i) Let $p \in \mathcal{P}([0,1])$, let $I = \mathbb{R}$ or $I = [0,\infty)$, and let a function $f \in L_S^{p(x)}(I : X)$ be Stepanov $p(x)$-almost periodic. Then it readily follows that $f(\cdot)$ is equi-Weyl $[p,\phi,F]$-almost periodic with $\phi(x) \equiv x$ and $F(l,t) \equiv 1$.

(ii) In the case that $p(x) \equiv p \in [1,\infty)$, it can be simply shown that the class of (equi)-Weyl $[p,\phi,[l/\psi(l)]^{1/p}]$-almost periodic functions, resp. (equi)-Weyl $[p,\phi,[l/\psi(l)]^{1/p}]_2$-almost periodic functions, coincides with the class of (equi)-Weyl (p,ϕ,ψ)-almost periodic functions, resp. (equi)-Weyl $(p,\phi,\psi)_2$-almost periodic functions. It is clear that the class of (equi)-Weyl $[p,\phi,[l/\psi(l)]^{1/p}]_1$-almost periodic functions and the class of (equi)-Weyl $(p,\phi,\psi)_1$-almost periodic functions coincide provided that $\phi(cx) = c\phi(x)$ for all $c,\ x \geqslant 0$.

(iii) It can be simply verified that the validity of condition
(D) For any $\tau_0 \in I$ there exists $c > 0$ such that

$$\frac{F(l,t)}{F(l,t+\tau_0)} \leqslant c, \quad t \in I,\ l > 0,$$

implies that the spaces $(e-)W_{ap}^{[p,\phi,F]}(I:X)$ and $(e-)W_{ap}^{[p,\phi,F]_1}(I:X)$ are translation invariant; this particularly holds provided the function $F(l,t)$ does not depend on the variable t. Furthermore, the space $(e-)W_{ap}^{[p,\phi,F]_2}(I:X)$ is translation invariant provided condition
(D)$'$ For any $\tau_0 \in I$ there exists $c > 0$ such that

$$\phi(F(l,t)x) \leqslant c\phi(F(l,t+\tau_0)x), \quad x \geqslant 0,\ t \in I,\ l > 0.$$

(iv) If $p,\ q \in \mathcal{P}([0,1])$ and $q(x) \leqslant p(x)$ for a. e. $x \in [0,1]$, then Lemma 1.1.7(ii) shows that any (equi)-Weyl $[p,\phi,F]$-almost periodic function is (equi)-Weyl $[q,\phi,F]$-almost periodic. Furthermore, condition $x,\ y \geqslant 0$ and $x \leqslant cy$ implies $\phi(x) \leqslant c\phi(y)$, resp. $x,\ y \geqslant 0$ and $x \leqslant cy$ implies $\phi(F(l,t)x) \leqslant c\phi(F(l,t)y)$ for all $l > 0$ and $t \in I$, ensures that any (equi)-Weyl $[p,\phi,F]_1$-almost periodic function is (equi)-Weyl $[q,\phi,F]_1$-almost periodic, resp. any (equi)-Weyl $[p,\phi,F]_2$-almost periodic function is (equi)-Weyl $[q,\phi,F]_2$-almost periodic.
(v) It is clear that, if $f(\cdot)$ is an (equi)-Weyl $[p,\phi,F]$-almost periodic function, resp. (equi)-Weyl $[p,\phi,F]_1$-almost periodic function, and $F(l,t) \geqslant F_1(l,t)$ for every $l > 0$ and $t \in I$, then $f(\cdot)$ is (equi)-Weyl $[p,\phi,F_1]$-almost periodic, resp. (equi)-Weyl $[p,\phi,F_1]_1$-almost periodic. Furthermore, any (equi)-Weyl $[p,\phi,F]_2$-almost periodic function is (equi)-Weyl $[p,\phi,F]_2$-almost periodic provided that $F(l,t) \geqslant F_1(l,t)$ for every $l > 0$, $t \in I$ and $\phi(\cdot)$ is monotonically increasing, or $F(l,t) \leqslant F_1(l,t)$ for every $l > 0$, $t \in I$ and $\phi(\cdot)$ is monotonically decreasing.
(vi) If $f(\cdot)$ is an (equi-)Weyl $[p,\phi,F]$-almost periodic function, $\phi_1(\|f(\cdot l+t+\tau)-f(t+\cdot l)\|)$ is measurable for any $\tau \in I$, $t \in I$, $l > 0$, and $0 \leqslant \phi_1 \leqslant \phi$, then Lemma 1.1.7(iii) shows that $f(\cdot)$ is an (equi)-Weyl $[p,\phi_1,F]$-almost periodic. Furthermore, if $0 \leqslant \phi_1 \leqslant \phi$, only, and $f(\cdot)$ is an (equi-)Weyl $[p,\phi,F]_i$-almost periodic function, then $f(\cdot)$ is an (equi-)Weyl $[p,\phi_1,F]_i$-almost periodic function, where $i = 1,2$.

In the case that the function $\phi(\cdot)$ is convex and $p(x) \equiv 1$, we have the following proposition which can be shown following the lines of the proof of Proposition 3.1.27.

Proposition 3.1.33. *Suppose that $\phi(\cdot)$ is convex, $p(x) \equiv 1$, $f : I \to X$ and $\|f(\cdot l + t + \tau) - f(t + \cdot l)\| \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t \in I$ and $l > 0$. Then the following holds:*
(i) $f \in (e-)W_{ap}^{[1,\phi,F]} \Rightarrow f \in (e-)W_{ap}^{[1,\phi,F]_1}$.
(ii) *If condition* (C) *holds, then* $f \in (e-)W_{ap}^{[1,\phi,\varphi\circ F]} \Rightarrow f \in (e-)W_{ap}^{[1,\phi,F]_2}$.

Regarding Proposition 3.1.27 and Proposition 3.1.33, it should be observed that the reverse inclusions and inequalities can be obtained assuming the condition

(C)′ $\phi(\cdot)$ is concave and there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(lx) \geqslant \varphi(l)\phi(x)$ for all $l > 0$ and $x \geqslant 0$.

It is clear that any (equi-)Weyl p-almost periodic function $f(\cdot)$ is (equi-)Weyl (p,ϕ,ψ)-almost periodic with $p(x) \equiv p \in [1,\infty)$, $\phi(x) = x$, $\psi(l) = l$. Concerning this observation, we wish to present two illustrative examples.

Example 3.1.34. Let us recall (J. Stryja [962]; see, e. g., Example 4.27 in the survey article [67] by J. Andres, A. M. Bersani, R. F. Grande) that the function $g(\cdot) := \chi_{[0,1/2]}(\cdot)$ is equi-Weyl p-almost periodic for any $p \in [1,\infty)$ but not Stepanov almost periodic. Since for each l, $\tau \in \mathbb{R}$ we have

$$\left(\sup_{t\in\mathbb{R}} \int_t^{t+l} |f(x+\tau) - f(x)|^p \, dx \right)^{1/p} \leqslant 1,$$

it can be easily shown that the function $g(\cdot)$ is equi-Weyl (p,x,ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$ such that $\lim_{l\to+\infty} \psi(l) = +\infty$; moreover, for each $\varepsilon \in (0,1/2)$ we can always find $t \in \mathbb{R}$ such that

$$\int_t^{t+1} |f(x+\tau) - f(x)|^p \, dx > \varepsilon, \quad \tau > \varepsilon.$$

Hence, the function $g(\cdot)$ cannot be equi-Weyl (p,x,l^0)-almost periodic. Taking into account Remark 3.1.28(iii) and the above conclusions, we see that $g(\cdot)$ is equi-Weyl (p,x,l^σ)-almost periodic if and only if $\sigma \neq 0$.

Example 3.1.35. Let us recall (J. Stryja [962]; see also [67, Example 4.29] and [631]) that the Heaviside function $g(\cdot) := \chi_{[0,\infty)}(\cdot)$ is not equi-Weyl 1-almost periodic but it is Weyl p-almost periodic for any number $p \in [1,\infty)$. Furthermore, it is not difficult to see that for each real number $\tau \in \mathbb{R}$ we have

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+l} |f(x+\tau) - f(x)|^p \, dx \right)^{1/p} = |\tau|^{1/p}$$

for any real number $l > |\tau|$. This implies that the function $g(\cdot)$ is Weyl (p,x,ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$ such that $\lim_{l\to+\infty} \psi(l) = +\infty$ and that $g(\cdot)$ cannot be Weyl (p,x,ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$ such that $\limsup_{l\to+\infty} [\psi(l)]^{-1} > 0$; in particular, $g(\cdot)$ is Weyl (p,x,l^σ)-almost periodic if and only if $\sigma > 0$. On the other hand, the function $g(\cdot)$ cannot be equi-Weyl (p,x,ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$; in actual fact, if we suppose contrary, then Eq. (3.6) is violated with $|\tau|^{1/p} > \varepsilon\psi(l)^{1/p}$. See also [631, Example 2.11.15–Example 2.11.17].

3.1.5 Weyl ergodic components with variable exponents

Unless stated otherwise, in this subsection we assume that $p \in \mathcal{P}([0,\infty))$, $\phi : [0,\infty) \to [0,\infty)$ and $F : (0,\infty) \times [0,\infty) \to (0,\infty)$. In the following three definitions, we extend the notion of an (equi-)Weyl p-vanishing function introduced in [645], where the case $p(x) \equiv p \in [1,\infty)$, $F(l,t) \equiv l^{(-1)/p}$ and $\phi(x) \equiv x$ has been considered.

Definition 3.1.36.
(i) It is said that a function $q : [0,\infty) \to X$ is equi-Weyl (p,ϕ,F)-vanishing if and only if $\phi(\|q(t+\cdot)\|) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l>0$ and

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \sup_{x\geqslant 0}\big[F(l,t)\big\|\phi\big(\|q(t+v)\|\big)\big\|_{L^{p(v)}[x,x+l]}\big] = 0. \tag{3.9}$$

(ii) It is said that a function $q : [0,\infty) \to X$ is Weyl (p,ϕ,F)-vanishing if and only if $\phi(q(t+\cdot)) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l>0$ and

$$\lim_{t\to+\infty} \limsup_{l\to+\infty} \sup_{x\geqslant 0}\big[F(l,t)\big\|\phi\big(\|q(t+v)\|\big)\big\|_{L^{p(v)}[x,x+l]}\big] = 0. \tag{3.10}$$

Definition 3.1.37.
(i) It is said that a function $q : [0,\infty) \to X$ is equi-Weyl $(p,\phi,F)_1$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l>0$ and

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \sup_{x\geqslant 0}\big[F(l,t)\phi\big(\|q(t+v)\|_{L^{p(v)}[x,x+l]}\big)\big] = 0. \tag{3.11}$$

(ii) It is said that a function $q : [0,\infty) \to X$ is Weyl $(p,\phi,F)_1$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l>0$ and

$$\lim_{t\to+\infty} \limsup_{l\to+\infty} \sup_{x\geqslant 0}\big[F(l,t)\phi\big(\|q(t+v)\|_{L^{p(v)}[x,x+l]}\big)\big] = 0. \tag{3.12}$$

Definition 3.1.38.
(i) It is said that a function $q : [0,\infty) \to X$ is equi-Weyl $(p,\phi,F)_2$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l>0$ and

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \sup_{x\geqslant 0} \phi\big[F(l,t)\|q(t+v)\|_{L^{p(v)}[x,x+l]}\big] = 0. \tag{3.13}$$

(ii) It is said that a function $q : [0,\infty) \to X$ is Weyl $(p,\phi,F)_2$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l>0$ and

$$\lim_{t\to+\infty} \limsup_{l\to+\infty} \sup_{x\geqslant 0} \phi\big[F(l,t)\|q(t+v)\|_{L^{p(v)}[x,x+l]}\big] = 0. \tag{3.14}$$

Denote by $W^{p(x)}_{\phi,F,0}([0,\infty) : X)$ and $e-W^{p(x)}_{\phi,F,0}([0,\infty) : X)$ $[W^{p(x);1}_{\phi,F,0}([0,\infty) : X)$ and $e-W^{p(x);1}_{\phi,F,0}([0,\infty) : X)/W^{p(x);2}_{\phi,F,0}([0,\infty) : X)$ and $e-W^{p(x);2}_{\phi,F,0}([0,\infty) : X)]$ the sets consisting of all Weyl (p,ϕ,F)-vanishing functions and equi-Weyl (p,ϕ,F)-vanishing functions [Weyl $(p,\phi,F)_1$-vanishing functions and equi-Weyl $(p,\phi,F)_1$-vanishing functions/Weyl $(p,\phi,F)_2$-vanishing functions and equi-Weyl $(p,\phi,F)_2$-vanishing functions], respectively. In the case that $p(x) \equiv p \in [1,\infty)$, $F(l,t) \equiv l^{(-1)/p}$ and $\phi(x) \equiv x$, the above classes coincide and we denote them by $W_0^p([0,\infty) : X)$ and $e - W_0^p([0,\infty) : X)$. These classes are very general and we want only to recall that, for instance, an equi-Weyl p-vanishing function $q(\cdot)$ need not be bounded as $t \to +\infty$ [645].

A great number of very simple examples can be constructed in order to show that, in the general case, the limit

$$\lim_{t\to+\infty} \sup_{x\geqslant 0}\big[F(l,t)\|\phi(\|q(t+v)\|)\|_{L^{p(v)}[x,x+l]}\big]$$

in Eq. (3.9) does not exist for any fixed number $l > 0$; the same holds for Eqs. (3.10)–(3.14). The question when these limits exist is meaningful but it will not be analyzed here.

Furthermore, we have the following observation.

Remark 3.1.39.

(i) Suppose that the function $\phi(\cdot)$ is monotonically increasing and satisfies, for each scalar α, $\beta \geqslant 0$ there exists a finite real number $\pi(\alpha,\beta) > 0$ such that, for every non-negative real numbers x, $y \geqslant 0$, we have

$$\phi(\alpha x + \beta y) \leqslant \pi(\alpha,\beta)\big[\phi(x) + \phi(y)\big].$$

Then (equi-)Weyl (p,ϕ,F)-vanishing functions and (equi-)Weyl $(p,\phi,F)_i$-vanishing functions, where $i = 1,2$, form vector spaces.

(ii) If the function $F(l,t)$ satisfies condition (D), resp. (D)$'$, then the space of (equi-)-Weyl (p,ϕ,F)-vanishing functions and the space of (equi-)Weyl $(p,\phi,F)_1$-vanishing functions, resp. the space of (equi-)Weyl $(p,\phi,F)_2$-vanishing functions, are translation invariant.

In this section, we will not follow the approach obeyed in [372] and previous section, with the principal assumption $p \in \mathcal{P}([0,1])$. With regard to this question, we will present only one illustrative example.

Example 3.1.40. Suppose that $p \in \mathcal{P}([0,1])$. Let us recall that the space of Stepanov $p(\cdot)$-vanishing functions (see [372]), denoted by $S_0^{p(x)}([0,\infty) : X)$, is consisting of those functions $q \in L_S^{p(x)}([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^{p(x)}([0,1] : X))$. The notion of space $S_0^{p(x)}([0,\infty) : X)$ can be extended in many other ways; for example:

(i) Let $\phi : (0,\infty) \to (0,\infty)$ and $G : (0,\infty) \to (0,\infty)$. Then we say that a function $q(\cdot)$ belongs to the space $S_{\phi,G,0}^{p(\cdot)}([0,\infty) : X)$ if and only if $\phi(\|q(t+\cdot)\|) \in L^{p(\cdot)}[0,1]$ for all $t \geqslant 0$ and

$$\lim_{t\to+\infty} G(t)\|\phi(\|q(t+v)\|)\|_{L^{p(v)}[0,1]} = 0.$$

In this part, as well as in parts (ii) and (iii), we will use the 1-periodic extension of the function $p(\cdot)$ to the non-negative real axis, denoted henceforth by $p_1(\cdot)$. Then the class $S^{p(\cdot)}_{\phi,G,0}([0,\infty) : X)$ is contained in the class of equi-Weyl (p_1, ϕ, F)-vanishing functions with a suitable chosen function $F(l,t)$, provided that the function $G(\cdot)$ is monotonically increasing. More precisely, let a number $\varepsilon > 0$ be fixed. Then there exists a sufficiently large real number $t_0 > 0$ such that $\|\phi(q(t+v))\|_{L^{p(v)}[0,1]} < \varepsilon G(t)^{-1}$ for all numbers $t \geqslant t_0$. Since we have assumed that $G(\cdot)$ is monotonically increasing, this implies that, for every $t \geqslant t_0$, $x \geqslant 0$ and $m \in \mathbb{N}_0$, we have

$$\int_0^1 \varphi_{p(v)}\big(\phi(\|q(t+v+\lfloor x\rfloor+m)\|)/[\varepsilon G(t)^{-1}]\big)\,dv \leqslant 1.$$

Using the inequality ($x \geqslant 0$, $l > 0$)

$$\begin{aligned}\int_x^{x+l} &\varphi_{p_1(v)}\big(\phi(\|q(t+v)\|)/[\varepsilon G(t)^{-1}]\big)\,dv\\ &\leqslant \sum_{k=0}^{l}\int_{\lfloor x\rfloor+k}^{\lfloor x\rfloor+k+1}\varphi_{p_1(v)}\big(\phi(\|q(t+v)\|)/[\varepsilon G(t)^{-1}]\big)\,dv,\end{aligned}$$

the above yields

$$\int_x^{x+l} \varphi_{p_1(v)}\big(\phi(\|q(t+v)\|)/[\varepsilon G(t)^{-1}]\big)\,dv \leqslant l+1, \quad \text{i. e.,}$$

$$\int_x^{x+l} \frac{1}{l+1}\varphi_{p_1(v)}\big(\phi(\|q(t+v)\|)/[\varepsilon G(t)^{-1}]\big)\,dv \leqslant 1.$$

Since

$$\varphi_{p_1(v)}\big(\phi(\|q(t+v)\|)/[\varepsilon(l+1)G(t)^{-1}]\big) \leqslant \frac{1}{l+1}\varphi_{p_1(v)}\big(\phi(\|q(t+v)\|)/[\varepsilon G(t)^{-1}]\big),$$

the above implies $\|\phi(\|q(t+v)\|)\|_{L^{p(v)}[x,x+l]} < \varepsilon G(t)^{-1}(1+l)$ for all $t \geqslant t_0$, $x \geqslant 0$ and $l > 0$. Hence, the required conclusion holds provided that there exists a finite real constant $C > 0$ such that

$$\big|F(l,t)G(t)^{-1}(1+l)\big| \leqslant C, \quad l > 0,\ t > 0.$$

(ii) Let $\phi : (0,\infty) \to (0,\infty)$ and $G : (0,\infty) \to (0,\infty)$. Then we say that a function $q(\cdot)$ belongs to the space $S^{p(\cdot)}_{\phi,G,0;1}([0,\infty) : X)$ if and only if $q(t+\cdot) \in L^{p(\cdot)}[0,1]$ for all $t \geqslant 0$ and

$$\lim_{t\to+\infty} G(t)\phi\big(\|q(t+v)\|_{L^{p(v)}[0,1]}\big) = 0.$$

Then the class $S^{p(\cdot)}_{\phi,G,0;1}([0,\infty):X)$ is contained in the class of equi-Weyl $(p_1,\phi,F)_1$-vanishing functions with a suitable chosen function $F(l,t)$, provided that the function $G(\cdot)$ is monotonically increasing. Arguing as in (i), this holds provided that, for example, $\sup\phi^{-1}([0,G(t)^{-1}])<\infty$ and

$$\lim_{l\to+\infty}\limsup_{t\to+\infty}F(l,t)(l+1)\sup\phi^{-1}([0,G(t)^{-1}])=0.$$

(iii) Let $\phi:(0,\infty)\to(0,\infty)$ and $G:(0,\infty)\to(0,\infty)$. Then we say that a function $q(\cdot)$ belongs to the space $S^{p(\cdot)}_{\phi,G,0;2}([0,\infty):X)$ if and only if $q(t+\cdot)\in L^{p(\cdot)}[0,1]$ for all $t\geqslant 0$ and

$$\lim_{t\to+\infty}\phi(G(t)\|\phi(q(t+v))\|_{L^{p(v)}[0,1]})=0.$$

Then the class $S^{p(\cdot)}_{\phi,G,0;2}([0,\infty):X)$ is contained in the class of equi-Weyl $(p_1,\phi,F)_2$-vanishing functions with a suitable chosen function $F(l,t)$, provided that the function $G(\cdot)$ is monotonically increasing. Arguing as in (i), this holds provided that, for example, the function $\phi(\cdot)$ is monotonically increasing, $\sup\phi^{-1}([0,1])<+\infty$ and

$$\lim_{l\to+\infty}\limsup_{t\to+\infty}\phi(F(l,t)G(t)^{-1}(1+l)\sup\phi^{-1}([0,1]))=0.$$

An analogue of Proposition 3.1.27 can be proved for (equi-)Weyl (p,ϕ,F)-vanishing functions and (equi-)Weyl $(p,\phi,F)_i$-vanishing functions, provided that the function $\phi(\cdot)$ is convex and $q(v)\equiv 1$. Furthermore, an analogue of Remark 3.1.28(i)–(ii) can be formulated for (equi-)Weyl (p,ϕ,F)-vanishing functions and (equi-)Weyl $(p,\phi,F)_i$-vanishing functions. Concerning Lemma 1.1.7(ii) and Remark 3.1.32(v), it should be noted that the embedding type result established already in the mentioned [377, Corollary 3.3.4] for scalar-valued functions (see also Lemma 1.1.7(ii)) enables one to see that the following expected result holds true.

Proposition 3.1.41. *Suppose $r,\ p\in\mathcal{P}([0,\infty))$ and $1\leqslant r(x)\leqslant p(x)$ for a. e. $x\geqslant 0$. Let $F_1(l,t)=2\max(l^{\operatorname{essinf}(1/r(x)-1/p(x))},l^{\operatorname{esssup}(1/r(x)-1/p(x))})F(l,t)$ or $F_1(l,t)=2(1+l)F(l,t)$ for all $l>0$ and $t\geqslant 0$. Then we have:*

(i) *If the function $q(\cdot)$ is (equi-)Weyl (r,ϕ,F)-vanishing provided that $q(\cdot)$ is (equi-)Weyl (p,ϕ,F_1)-vanishing.*
(ii) *Suppose that there exists a function $\varphi:[0,\infty)\to[0,\infty)$ such that $\phi(cx)\leqslant\varphi(c)\phi(x)$ for all $c\geqslant 0$ and $x\geqslant 0$. Let $F_2(l,t)=\varphi(2(1+l))F(l,t)$ or $F_1(l,t)=\varphi(2\max(l^{\operatorname{essinf}(1/r(x)-1/p(x))},l^{\operatorname{esssup}(1/r(x)-1/p(x))}))F(l,t)$ for $l>0$ and $t\geqslant 0$. Then the function $q(\cdot)$ is (equi-)Weyl $(r,\phi,F)_1$-vanishing provided that $q(\cdot)$ is (equi-)Weyl $(p,\phi,F_2)_1$-vanishing.*
(iii) *If $\phi(\cdot)$ is monotonically increasing, then the function $q(\cdot)$ is (equi-)Weyl $(r,\phi,F)_2$-vanishing provided that $q(\cdot)$ is (equi-)Weyl $(p,\phi,F_1)_2$-vanishing.*

If $1 \leqslant r \leqslant p$ are constant coefficients, then the choices $F_1(l,t) = l^{1/r-1/p}F(l,t)$ in (i), (iii) and $F_1(l,t) = \varphi(l^{1/r-1/p})F(l,t)$ in (ii) can be made.

We continue by reexamining the conclusions established in [645, Example 4.5, Example 4.6].

Example 3.1.42. Define

$$q(t) := \sum_{n=0}^{\infty} \chi_{[n^2,n^2+1]}(t), \quad t \geqslant 0.$$

Then we know that $\hat{q} \notin C_0([0,\infty) : L^p([0,1] : \mathbb{C}))$ and the function $q(\cdot)$ is equi-Weyl p-almost periodic for any exponent $p \geqslant 1$; see [645, Example 4.5]. In this example, we have proved the estimate

$$\left(\int_x^{x+l} \|q(t+v)\|^p \, dv \right)^{1/p} \leqslant \left(2 + \frac{l}{\sqrt{t}+\sqrt{l}} \right)^{1/p} \leqslant 2 + \left(\frac{l}{\sqrt{t}+\sqrt{l}} \right)^{1/p},$$

for any $x \geqslant 0$, $t \geqslant 0$, $l > 0$, so that the function $q(\cdot)$ is equi-Weyl (p,x,F)-vanishing provided that

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \quad F(l,t)\left[2 + \left(\frac{l}{\sqrt{t}+\sqrt{l}} \right)^{1/p} \right] = 0.$$

In particular, this holds for the function $F(l,t) = l^{\sigma}$, where $\sigma < 0$.

Example 3.1.43. Define

$$q(t) := \sum_{n=0}^{\infty} \sqrt{n}\chi_{[n^2,n^2+1]}(t), \quad t \geqslant 0.$$

Then we know that the function $q(\cdot)$ is not equi-Weyl p-vanishing for any exponent $p \geqslant 1$ and that the function $q(\cdot)$ is Weyl p-vanishing for any exponent $p \geqslant 1$; see [645, Example 4.6]. In this example, we have proved the estimate

$$\left(\int_x^{x+l} \|q(t+v)\|^p \, dv \right)^{1/p} \leqslant (l+t)^{1/2p}, \quad x \geqslant 0,\ t \geqslant 0,\ l > 0,$$

so that the function $q(\cdot)$ is Weyl (p,x,F)-vanishing provided that

$$\lim_{t\to+\infty} \limsup_{l\to+\infty} F(l,t)(l+t)^{1/2p} = 0.$$

In particular, this holds for the function $F(l,t) = l^{\sigma}$, where $\sigma < (-1)/2p$.

We will present one more illustrative example.

Example 3.1.44. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences of positive real numbers such that $(a_n)_{n\in\mathbb{N}}$ is strictly monotonically increasing, $\lim_{n\to+\infty}(a_{n+1}-a_n) = +\infty$ and $\lim_{n\to+\infty}\phi(b_n) = 0$. Let $q : [0,\infty) \to (0,\infty)$ be defined by $q(t) := b_n$ if and only if $t \in [a_{n-1}, a_n)$ for some $n \in \mathbb{N}$, where $a_0 := 0$. If $p \in D_+([0,\infty))$, $l > 0$ and $t > 0$, then we have

$$\begin{aligned}
&\sup_{x\geqslant 0}\bigl[F(l,t)\bigl\|\phi\bigl(q(t+v)\bigr)\bigr\|_{L^{p(v)}[x,x+l]}\bigr] \\
&\quad\leqslant \sup_{x\geqslant 0}\bigl[2(1+l)F(l,t)\bigl\|\phi\bigl(q(t+\cdot)\bigr)\bigr\|_{L^{p^+}[x,x+l]}\bigr] \\
&\quad= \sup_{x\geqslant 0}\bigl[2(1+l)F(l,t)\bigl\|\phi\bigl(q(\cdot)\bigr)\bigr\|_{L^{p^+}[t+x,t+x+l]}\bigr].
\end{aligned}$$

Assume, additionally, that there exists a function $G : (0,\infty) \to (0,\infty)$ such that $F(l,t) \leqslant G(l)$ for all $l > 0$ and $t > 0$. Since we have assumed that $\lim_{n\to+\infty}(a_{n+1}-a_n) = +\infty$, for each number $l > 0$ we have

$$\limsup_{t\to+\infty}\sup_{x\geqslant 0}\bigl[2(1+l)F(l,t)\bigl\|\phi\bigl(q(\cdot)\bigr)\bigr\|_{L^{p^+}[t+x,t+x+l]}\bigr] = 0,$$

because $\lim_{n\to+\infty}\phi(b_n) = 0$ and

$$\bigl\|\phi\bigl(q(\cdot)\bigr)\bigr\|_{L^{p^+}[t+x,t+x+l]} \leqslant l\max\bigl(\phi(b_n),\ \phi(b_{n+1})\bigr),$$

where $n \in \mathbb{N}$ is such that $x+t \leqslant a_n$ and $x+t+l \leqslant a_{n+1}$. Therefore, the function $q(\cdot)$ is equi-Weyl (p,ϕ,F)-vanishing.

In [645], we have introduced a great number of various types of asymptotically Weyl almost periodic function spaces with constant exponent $p \geqslant 1$. In order to relax our exposition, we will introduce here only one general definition of an asymptotically Weyl almost periodic function with variable exponent, which extends the notion introduced in Definition 3.1.3(ii).

Definition 3.1.45. Let $h : I \to X$. Then we say that $h(\cdot)$ is asymptotically Weyl almost periodic with variable exponent if and only if there exist two functions $g : \mathbb{R} \to X$ and $q : I \to X$ such that $h(t) = g(t) + q(t)$ for a. e. $t \in I$, $g(\cdot)$ belongs to some of function spaces introduced in Definition 3.1.23–Definition 3.1.25 or Definition 3.1.29–Definition 3.1.31 and $q(\cdot)$ belongs to some of function spaces introduced in Definition 3.1.36–Definition 3.1.38 (with possibly different functions p, p_1; ϕ, ϕ_1; F, F_1 and the meaning being clear).

Observe that we can also extend the notion of Weyl p-pseudo-ergodic component ($p \geqslant 1$) following the approach obeyed in the previous part of the section and provide certain extensions of [645, Proposition 4.11] in this context.

3.1.6 Weyl almost periodicity with variable exponent and convolution products

In the analyses of (equi-)Weyl (p,ϕ,F)-almost periodic functions and (equi-)Weyl $[p,\phi,F]$-almost periodic functions, we will use the following conditions:

(A1) $I=\mathbb{R}$ or $I=[0,\infty)$, $\psi:(0,\infty)\to(0,\infty)$, $\varphi:[0,\infty)\to[0,\infty)$, $\phi:[0,\infty)\to[0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy)\leqslant\varphi(x)\phi(y)$ for all $x,\ y\geqslant 0$, $p\in\mathcal{P}(I)$.

(B1) The same as (A) with the assumption $p\in\mathcal{P}(I)$ replaced by $p\in\mathcal{P}([0,1])$ therein.

Theorem 3.1.46. *Suppose that condition* (A1) *holds with* $I=\mathbb{R}$, $\breve{g}:\mathbb{R}\to X$ *is (equi-)-Weyl* (p,ϕ,F)*-almost periodic and measurable,* $F_1:(0,\infty)\times I\to(0,\infty)$, $p,\ q\in\mathcal{P}(\mathbb{R})$, $1/p(x)+1/q(x)=1$, $(R(t))_{t>0}\subseteq L(X,Y)$ *is a strongly continuous operator family and* (a_k) *is a sequence of positive real numbers such that* $\sum_{k=0}^{\infty}a_k=1$. *If for every real number* $x,\ \tau\in\mathbb{R}$ *we have*

$$\int_{-x}^{\infty}\|R(v+x)\|\|\breve{g}(v)\|\,dv<\infty, \tag{3.15}$$

and if, for every $t\in\mathbb{R}$ *and* $l>0$, *we have*

$$H(l,x):=\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\|\varphi(\|R(v+x)\|)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]}F(l,-x+lk)^{-1}<\infty, \tag{3.16}$$

$$\int_{t}^{t+l}\varphi_{p(x)}(2l^{-1}H(l,x)F_1(l,t)^{-1})\,dx\leqslant 1, \tag{3.17}$$

resp. if (3.16) *holds and there exists* $l_0>0$ *such that for all* $l\geqslant l_0$ *and* $t\in\mathbb{R}$ *we have* (3.17)*, then the function* $G:\mathbb{R}\to Y$*, given by* (2.46)*, is well defined and (equi-)-Weyl* (p,ϕ,F_1)*-almost periodic.*

Proof. We will prove the theorem only for the class of equi-Weyl (p,ϕ,F)-almost periodic functions. Since $G(x)=\int_{-x}^{\infty}R(v+x)\breve{g}(v)\,dv$, $x\in\mathbb{R}$, the estimate in (3.15) shows that the function $G(\cdot)$ is well defined and that the integral in definition of $G(x)$ converges absolutely ($x\in\mathbb{R}$). Furthermore, the same estimate shows that for each real number τ we have

$$\int_{-x}^{\infty}\|R(v+x)\|\|\breve{g}(v+\tau)\|\,dv=\int_{-(x-\tau)}^{\infty}\|R(v+(x-\tau))\|\|\breve{g}(v)\|\,dv<\infty,$$

so that the integral in definition of $G(x+\tau)-G(x)$ converges absolutely ($x\in\mathbb{R}$). Let $\varepsilon>0$ be a fixed real number. Then we can find two real numbers $l>0$ and $L>0$ such that any interval $I'\subseteq I$ of length L contains a point $\tau\in I'$ such that (3.6) holds for the function $\breve{g}(\cdot)$, with the number τ replaced by the number $-\tau$ therein. Using our assumptions from condition (A1), the Jensen integral inequality applied to the function $\phi(\cdot)$ (see also condition (3.15)), the fact that the functions $\phi(\cdot)$ and $\varphi_{p(x)}(\cdot)$ are monotonically increasing, (3.1) and Lemma 1.1.7(i), we see that for each real number $x \in \mathbb{R}$ the following holds:

$$
\begin{aligned}
&\varphi_{p(x)}(\phi(\|G(x+\tau)-G(x)\|)/\lambda)\\
&\quad\leqslant \varphi_{p(x)}\left(\phi\left(\int_{-x}^{\infty}\|R(v+x)\|\|\check{g}(v+\tau)-\check{g}(v)\|\,dv\right)/\lambda\right)\\
&\quad= \varphi_{p(x)}\left(\phi\left(\sum_{k=0}^{\infty}a_k\int_{-x+kl}^{-x+(k+1)l}a_k^{-1}\|R(v+x)\|\|\check{g}(v+\tau)-\check{g}(v)\|\,dv\right)/\lambda\right)\\
&\quad\leqslant \varphi_{p(x)}\left(\sum_{k=0}^{\infty}a_k\phi\left(\int_{-x+kl}^{-x+(k+1)l}a_k^{-1}\|R(v+x)\|\|\check{g}(v+\tau)-\check{g}(v)\|\,dv\right)/\lambda\right)\\
&\quad\leqslant \varphi_{p(x)}\left(\sum_{k=0}^{\infty}a_k\phi\left(la_k^{-1}\cdot l^{-1}\int_{-x+kl}^{-x+(k+1)l}\|R(v+x)\|\|\check{g}(v+\tau)-\check{g}(v)\|\,dv\right)/\lambda\right)\\
&\quad\leqslant \varphi_{p(x)}\left(l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\int_{-x+kl}^{-x+(k+1)l}\phi(\|R(v+x)\|\|\check{g}(v+\tau)-\check{g}(v)\|)\,dv/\lambda\right)\\
&\quad\leqslant \varphi_{p(x)}\left(l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\int_{-x+kl}^{-x+(k+1)l}\varphi(\|R(v+x)\|)\phi(\|\check{g}(v+\tau)-\check{g}(v)\|)\,dv/\lambda\right)\\
&\quad\leqslant \varphi_{p(x)}\Bigg(2l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\|\varphi(\|R(v+x)\|)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]}\\
&\qquad\times\phi(\|\check{g}(v+\tau)-\check{g}(v)\|)_{L^{p(v)}[-x+kl,-x+(k+1)l]})/\lambda\Bigg)\\
&\quad\leqslant \varphi_{p(x)}\left(2l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\|\varphi(\|R(v+x)\|)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]}\varepsilon F(l,-x+kl)^{-1}/\lambda\right).
\end{aligned}
$$

Let $K \subseteq \mathbb{R}$ be an arbitrary compact set. Since the above computation holds for every real number $\tau \in \mathbb{R}$ and for every arbitrarily large real number $l > 0$, we can find $t \in \mathbb{R}$ such that $K \subseteq [t, t+l]$. Now we get from (3.17) that the function $\phi(\|G(\cdot+\tau)-G(\cdot)\|)$ belongs to the space $L^{p(x)}(K)$ by definition. Condition (3.17) and the above computation also imply that for each real number $t \in \mathbb{R}$ we have

$$\int_{t}^{t+l}\varphi_{p(x)}(\phi(\|G(x+\tau)-G(x)\|)/\lambda)\,dx \leqslant 1,$$

with $\lambda = \varepsilon F_1(l,t)$, which simply implies the final conclusion. □

Remark 3.1.47.
(i) Suppose that $p(x) \equiv p \in [1,\infty)$. Then condition (3.17) can be weakened to

$$\int_t^{t+l} \varphi_{p(x)}\big(l^{-1}H(l,x)F_1(l,t)^{-1}\big)\,dx \leqslant 1, \tag{3.18}$$

resp. there exists $l_0 > 0$ such that for all $l \geqslant l_0$ and $t \in \mathbb{R}$ we have (3.18).
(ii) Suppose that $\phi(x) = \varphi(x) = \psi(x) = x$. Then condition (3.17), resp. (3.18), holds provided that $l \geqslant 1$ and the term in the large brackets in this equation does not exceed $1/l$ or that $0 < l < 1$ and the term in the large brackets in this equation does not exceed 1. Similar comments can be made in the case of the consideration of Theorem 3.1.49 below (see also Corollary 2.3.4).

Corollary 3.1.48. *Suppose that condition* (A1) *holds with $I = \mathbb{R}$, $p(x) \equiv p \geqslant 1$, $1/p + 1/q = 1$, $\check{g} : \mathbb{R} \to X$ is (equi-)Weyl (p,ϕ,F)-almost periodic and measurable, $F_1 : (0,\infty)\times I \to (0,\infty)$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and (a_k) is a sequence of positive real numbers such that $\sum_{k=0}^{\infty} a_k = 1$. If for every real numbers $x,\ \tau \in \mathbb{R}$ we have* (3.15) *and if, for every $t \in \mathbb{R}$ and $l > 0$, we have*

$$H_p(l,x) := \sum_{k=0}^{\infty} a_k\varphi(la_k^{-1})\big\|\varphi(\|R(\cdot)\|)\big\|_{L^q[kl,(k+1)l]}F(l,-x+lk)^{-1} < \infty \tag{3.19}$$

and

$$\int_t^{t+l} \big(l^{-1}H_p(l,x)F_1(l,t)^{-1}\big)^p\,dx \leqslant 1, \tag{3.20}$$

resp. if (3.19) *holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$ and $t \in \mathbb{R}$ we have* (3.20)*, then the function $G : \mathbb{R} \to Y$, given by* (2.46)*, is well defined and (equi-)Weyl (p,ϕ,F_1)-almost periodic.*

Now we will state and prove the following result with regards to the class of (equi-)Weyl $[p,\phi,F]$-almost periodic functions.

Theorem 3.1.49. *Suppose that condition* (B1) *holds with $I = \mathbb{R}$, $g : \mathbb{R} \to X$ is measurable, $\omega : (0,\infty) \to (0,\infty)$, $F : (0,\infty)\times I \to (0,\infty)$, (a_k) is a sequence of positive real numbers such that $\sum_{k=0}^{\infty} a_k = 1$, $(b_k)_{k\geqslant 0}$ is a sequence of positive real numbers, $S : (0,\infty)\times\mathbb{R} \to (0,\infty)$ is a given function, as well as for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that*

$$\sup_{x\in[0,1]} \big[\phi\big(\|g(xl+t-r-k+\tau)-g(xl+t-r-k)\|\big)_{L^{p(r)}[0,1]}\big] \leqslant \omega(\varepsilon)b_kS(l,t) \tag{3.21}$$

for any integer $k \geqslant 0$ and real number $t \in \mathbb{R}$. Suppose, further, that the second inequality in (3.15) *holds, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family. If for every real numbers $t,\ \tau \in \mathbb{R}$, every positive real number $l > 0$ and every real number $x \in [0,1]$ we have*

$$\int_0^\infty \|R(v)\| \|g(xl + t + \tau - v) - g(xl + t - v)\|\, dv < \infty, \tag{3.22}$$

and if, for every $t \in \mathbb{R}$, $x \in [0,1]$ and $l,\ \varepsilon > 0$, we have

$$W(x) := \sum_{k=0}^{\infty} a_k \varphi(a_k^{-1}) \|\varphi(\|R(v + x)\|)\|_{L^{q(v)}[0,1]} b_k < \infty, \tag{3.23}$$

$$\int_0^1 \varphi_{p(x)}(2\varepsilon^{-1} F_1(l,t)^{-1} \omega(\varepsilon) S(l,t) W(x))\, dx \leqslant 1, \tag{3.24}$$

resp. if (3.23) *holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we have* (3.24)*, then the function $G : \mathbb{R} \to Y$, given by* (2.46)*, is well defined and (equi-)Weyl $[p, \phi, F_1]$-almost periodic.*

Proof. We will prove the theorem only for the class of equi-Weyl $[p, \phi, F]$-almost periodic functions. As above, the function $G(\cdot)$ is well defined. Let $\varepsilon > 0$ be a fixed real number. Then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (3.21) holds for any integer $k \geqslant 0$ and any real number $t \in \mathbb{R}$. Using our assumptions from condition (B1), the Jensen integral inequality applied to the function $\phi(\cdot)$ (see also condition (3.22)), the fact that the functions $\phi(\cdot)$ and $\varphi_{p(x)}(\cdot)$ are monotonically increasing, (3.1) and Lemma 1.1.7(i), we see that, for every real numbers $x \in [0,1]$ and $t \in \mathbb{R}$, the following holds:

$$\begin{aligned}
&\varphi_{p(x)}(\phi(\|G(xl + t + \tau) - G(xl + t)\|)/\lambda) \\
&\quad \leqslant \varphi_{p(x)}\left(\phi\left(\int_0^\infty \|R(v)\| \|g(xl + t + \tau - v) - g(xl + t - v)\|\, dv\right)/\lambda\right) \\
&\quad = \varphi_{p(x)}\left(\phi\left(\sum_{k=0}^{\infty} a_k \int_0^1 a_k^{-1} \|R(v + k)\| \|g(xl + t + \tau - v - k) - g(xl + t - v - k)\|\, dv\right)/\lambda\right) \\
&\quad \leqslant \varphi_{p(x)}\left(\sum_{k=0}^{\infty} a_k \int_0^1 \phi(a_k^{-1} \|R(v + k)\| \|g(xl + t + \tau - v - k) - g(xl + t - v - k)\|\, dv)/\lambda\right) \\
&\quad \leqslant \varphi_{p(x)}\left(\sum_{k=0}^{\infty} a_k \varphi(a_k^{-1}) \int_0^1 \varphi(\|R(v + k)\|)\right. \\
&\qquad \left. \times \phi(\|g(xl + t + \tau - v - k) - g(xl + t - v - k)\|)\, dv/\lambda\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \varphi_{p(x)}\Biggl(\frac{2}{\lambda}\sum_{k=0}^{\infty} a_k\varphi(a_k^{-1})\varphi(\|R(v+k)\|)_{L^{q(v)}[0,1]} \\
&\quad \times \phi(\|g(xl+t+\tau-v-k)-g(xl+t-v-k)\|)_{L^{p(v)}[0,1]}\Biggr) \\
&\leqslant \varphi_{p(x)}\Biggl(\frac{2}{\lambda}\sum_{k=0}^{\infty} a_k\varphi(a_k^{-1})\varphi(\|R(v+k)\|)_{L^{q(v)}[0,1]}\omega(\varepsilon)b_kS(l,t)\Biggr).
\end{aligned}
$$

Arguing as in the proof of Theorem 3.1.46, we see from condition (3.24) that the function $\phi(\|G(\cdot l+t+\tau)-G(t+\cdot l)\|)$ belongs to the space $L^{p(\cdot)}([0,1])$ for arbitrary real numbers $\tau,\ t\in\mathbb{R}$ and $l>0$. Condition (3.24) implies that for each real numbers $t\in\mathbb{R}$ and $x\in[0,1]$ we have

$$
\int_0^1 \varphi_{p(x)}(\phi(\|G(xl+t+\tau)-G(xl+t)\|)/\lambda)\,dx\leqslant 1,
$$

with $\lambda=\varepsilon F_1(l,t)^{-1}$, which simply implies the final conclusion. □

Corollary 3.1.50. *Suppose that condition* (B1) *holds with* $I=\mathbb{R}$ *and* $p(x)\equiv p\in[1,\infty)$, $1/p+1/q=1$, $g:\mathbb{R}\to X$ *is measurable,* $\omega:(0,\infty)\to(0,\infty)$, $F:(0,\infty)\times I\to(0,\infty)$, (a_k) *is a sequence of positive real numbers such that* $\sum_{k=0}^{\infty}a_k=1$, $(b_k)_{k\geqslant 0}$ *is a sequence of positive real numbers,* $S:(0,\infty)\times\mathbb{R}\to(0,\infty)$ *is a given function, as well as for each* $\varepsilon>0$ *we can find real numbers* $l>0$ *and* $L>0$ *such that any interval* $I'\subseteq I$ *of length* L *contains a point* $\tau\in I'$ *such that* (3.21) *holds with* $p(r)\equiv p$, *for any integer* $k\geqslant 0$ *and any real number* $t\in\mathbb{R}$. *Suppose, further, that the second inequality in* (3.15) *holds, and* $(R(t))_{t>0}\subseteq L(X,Y)$ *is a strongly continuous operator family. If for every real number* $t,\ \tau\in\mathbb{R}$, *every positive real number* $l>0$ *and every real number* $x\in[0,1]$ *we have* (3.22), *and if, for every* $t\in\mathbb{R}$, $x\in[0,1]$ *and* $l>0$, *we have*

$$
W_p(x):=\sum_{k=0}^{\infty}a_k\varphi(a_k^{-1})\|\varphi(\|R(\cdot)\|)\|_{L^q[x,x+1]}b_k<\infty \tag{3.25}
$$

and

$$
\int_0^1 \varphi_{p(x)}(2F_1(l,t)^{-1}S(l,t)W_p(x))\,dx\leqslant 1, \tag{3.26}
$$

resp. if (3.25) *holds and there exists* $l_0>0$ *such that for all* $l\geqslant l_0$ *and* $t\in\mathbb{R}$ *we have* (3.26), *then the function* $G:\mathbb{R}\to Y$, *given by* (2.46), *is well defined and (equi-)Weyl* $[p,\phi,F_1]$*-almost periodic.*

Concerning Theorem 3.1.49, it should be noted that, in [372, Proposition 6.1], we have analyzed the situation in which the function $\check{g}:\mathbb{R}\to X$ is $S^{p(x)}$-almost periodic and $\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} < \infty$. Then the resulting function $G(\cdot)$ is almost periodic, which cannot be derived from the above-mentioned theorem.

For the class of (equi-)Weyl $(p, \phi, F)_1$-almost periodic functions, we will state the following result.

Theorem 3.1.51. *Suppose that $\check{g} : \mathbb{R} \to X$ is (equi-)Weyl $(p, \phi, F)_1$-almost periodic and measurable, $F_1 : (0, \infty) \times I \to (0, \infty)$, $p,\ q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family and for every real numbers $x,\ \tau \in \mathbb{R}$ we have* (3.15)*. Suppose that, for every real number $t \in \mathbb{R}$ and positive real numbers $l,\ \varepsilon > 0$, there exist two positive real numbers $a > 0$ and $\lambda > 0$ such that $\lambda \leqslant a$, $[0, a] \subseteq \phi^{-1}([0, \varepsilon F(l, t)^{-1}])$,*

$$\sum_{k=0}^{\infty} \|R(v + x)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} \sup \phi^{-1}\big([0, \varepsilon F(l, -x + kl)^{-1}]\big) < \infty \tag{3.27}$$

and the term

$$\int_{t}^{t+l} \varphi_{p(x)}\left(2 \frac{\sum_{k=0}^{\infty} \|R(v + x)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} \sup \phi^{-1}([0, \varepsilon F(l, -x + kl)^{-1}])}{\lambda}\right) dx \tag{3.28}$$

does not exceed 1*, resp.* (3.27) *holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we see that the term in* (3.28) *does not exceed* 1*, then the function $G : \mathbb{R} \to Y$, given by* (2.46)*, is well defined and (equi-)Weyl $(p, \phi, F_1)_1$-almost periodic.*

Proof. As in the proof of Theorem 3.1.46, we see that the function $G(\cdot)$ is well defined and the integrals in definitions of $G(x)$ and $G(x+\tau)-G(x)$ converge absolutely $(x, \tau \in \mathbb{R})$. By Lemma 1.1.7(ii), we see that the function $G(\cdot + \tau) - G(\cdot)$ belongs to the space $L^{p(x)}(K)$ for each compact set $K \subseteq \mathbb{R}$. The remaining part follows similarly to the proof of Theorem 3.1.46, by using condition (3.27), and the estimates

$$\begin{aligned} \|G(x + \tau) - G(x)\| \leqslant 2 \sum_{k=0}^{\infty} &\|R(v + x)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} \\ &\times \|\check{g}(v + \tau) - \check{g}(v)\|_{L^{p(v)}[-x+kl,-x+(k+1)l]} \end{aligned}$$

and

$$\|\check{g}(v + \tau) - \check{g}(v)\|_{L^{p(v)}[-x+kl,-x+(k+1)l]} \leqslant \sup \phi^{-1}\big([0, \varepsilon F(l, -x + kl)^{-1}]\big),$$

and the equivalence relation

$$\begin{aligned} &\phi\big(\|G(\cdot + \tau) - G(\cdot)\|_{L^{p(x)}[t,t+l]}\big) \leqslant \varepsilon F_1(l, t)^{-1} \\ &\quad \Leftrightarrow \|G(\cdot + \tau) - G(\cdot)\|_{L^{p(x)}[t,t+l]} \leqslant \phi^{-1}\big([0, \varepsilon F_1(l, t)^{-1}]\big), \end{aligned}$$

for any $x,\ t,\ \tau \in \mathbb{R}$ and $l > 0$. □

Regarding the class of (equi-)Weyl $[p,\phi,F]_1$-almost periodic functions, we will only state the following result; the proof can be deduced as above and therefore omitted (we can similarly formulate analogues of Corollary 3.1.48 and Corollary 3.1.50, as well as the conclusions from Remark 3.1.47).

Theorem 3.1.52. *Suppose that $g : \mathbb{R} \to X$ is measurable, $\omega : (0,\infty) \to (0,\infty)$, $F : (0,\infty) \times I \to (0,\infty)$, $(b_k)_{k\geqslant 0}$ is a sequence of positive real numbers, $S : (0,\infty) \times \mathbb{R} \to (0,\infty)$ is a given function, as well as for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that*

$$\sup_{x\in[0,1]} \big[\|g(xl+t-r-k+\tau) - g(xl+t-r-k)\|_{L^{p(r)}[0,1]}\big] \leqslant \omega(\varepsilon) b_k S(l,t)$$

for any integer $k \geqslant 0$ and real number $t \in \mathbb{R}$. Suppose, further, that the second inequality in (3.15) holds, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family. If for every real numbers $t,\ \tau \in \mathbb{R}$, every positive real number $l > 0$ and every real number $x \in [0,1]$ we have (3.22), if

$$W_2(x) := \sum_{k=0}^{\infty} \|R(v+x)\|_{L^{q(v)}[0,1]} b_k < \infty, \quad x \in [0,1], \tag{3.29}$$

and if, for every $t \in \mathbb{R}$ and $l,\ \varepsilon > 0$, we have the existence of two positive real numbers $a > 0$ and $\lambda > 0$ such that $\lambda \leqslant a$, $[0,a] \subseteq \phi^{-1}([0,\varepsilon F_1(l,t)^{-1}])$ and

$$\int_0^1 \varphi_{p(x)}\Big(2\frac{\omega(\varepsilon)S(l,t)W_2(x)}{\lambda}\Big)\,dx \leqslant 1, \tag{3.30}$$

resp. if (3.29) holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we have (3.30), then the function $G : \mathbb{R} \to Y$, given by (2.46), is well defined and (equi-)Weyl $[p,\phi,F_1]$-almost periodic.

Remark 3.1.53. The assertions of Theorem 3.1.51, resp. Theorem 3.1.52, can be much simpler formulated provided that:

(A2) The function $\phi : [0,\infty) \to [0,\infty)$ is a monotonically increasing bijection and $p \in \mathcal{P}(\mathbb{R})$, resp.

(B2) The function $\phi : [0,\infty) \to [0,\infty)$ is a monotonically increasing bijection and $p \in \mathcal{P}([0,1])$.

Any of these conditions implies that the function $\phi^{-1} : [0,\infty) \to [0,\infty)$ is a monotonically increasing bijection, as well. If condition (A2), resp. (B2), holds, then the class of (equi-)Weyl $(p,\phi,F)_2$-almost periodic functions, resp. (equi-)Weyl $[p,\phi,F]_2$-almost periodic functions, coincides with the class of (equi-)Weyl $(p,x,F)_2$-almost periodic functions, resp. (equi-)Weyl $[p,x,F]_2$-almost periodic functions.

Regarding the invariance of (equi-)Weyl $(p, \phi, F)_2$-almost periodicity and (equi-)-Weyl $[p, \phi, F]_2$-almost periodicity under the actions of infinite convolution products, we will only state the following analogues of Theorem 3.1.51 and Theorem 3.1.52.

Theorem 3.1.54. *Suppose that $\check{g} : \mathbb{R} \to X$ is (equi-)Weyl $(p, \phi, F)_2$-almost periodic and measurable, $F_1 : (0, \infty) \times I \to (0, \infty)$, p, $q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family and for every real numbers x, $\tau \in \mathbb{R}$ we have* (3.15)*. Suppose that, for every real number $t \in \mathbb{R}$ and positive real numbers l, $\varepsilon > 0$, there exist two positive real numbers $a > 0$ and $\lambda > 0$ such that $\lambda \leqslant a$, $[0, a] \subseteq F(l, t)^{-1}\phi^{-1}([0, \varepsilon])$,*

$$\sum_{k=0}^{\infty} \|R(v + x)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} F(l, -x + kl)^{-1} < \infty \tag{3.31}$$

and the term

$$\int_t^{t+l} \varphi_{p(x)}\left(2\frac{\sum_{k=0}^{\infty} \|R(v + x)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} F(l, -x + kl)^{-1} \sup \phi^{-1}([0, \varepsilon])}{\lambda}\right) dx \tag{3.32}$$

does not exceed 1*, resp.* (3.31) *holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we see that the term in* (3.32) *does not exceed* 1*, then the function $G : \mathbb{R} \to Y$, given by* (2.46)*, is well defined and (equi-)Weyl $(p, \phi, F_1)_2$-almost periodic.*

Theorem 3.1.55. *Suppose that, with the exception of Eq.* (3.30)*, all remaining assumptions from the formulation of* Theorem 3.1.52 *hold. If for every $t \in \mathbb{R}$ and l, $\varepsilon > 0$ we have the existence of two positive real numbers $a > 0$ and $\lambda > 0$ such that $\lambda \leqslant a$, $[0, a] \subseteq F_1(l, t)^{-1}\phi^{-1}([0, \varepsilon])$ and*

$$\int_0^1 \varphi_{p(x)}\left(2\frac{\omega(\varepsilon)S(l, t)W_2(x)}{\lambda}\right) dx \leqslant 1, \tag{3.33}$$

resp. if (3.29) *holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we have* (3.33)*, then the function $G : \mathbb{R} \to Y$, given by* (2.46)*, is well defined and (equi-)Weyl $[p, \phi, F_2]$-almost periodic.*

The invariance of asymptotical Weyl p-almost periodicity under the action of finite convolution product, where the exponent $p \in [1, \infty)$ has a constant value, has been examined in [141], [645, Proposition 5.3, Examples 5.4–5.6] and [435, Proposition 1, Remark 2–Remark 5]. Concerning the invariance of asymptotical Weyl $p(x)$-almost periodicity under the action of finite convolution product, we will state and prove only one proposition. In order to do so, suppose that (see also Definition 3.1.45, where the domain of the function $g(\cdot)$ is the non-negative real axis) there exist two functions $g : \mathbb{R} \to X$ and $q : [0, \infty) \to X$ such that $h(t) = g(t) + q(t)$ for a. e. $t \geqslant 0$, $g(\cdot)$ belongs to some of function spaces introduced in Definition 3.1.23–Definition 3.1.25 or Definition 3.1.29–Definition 3.1.31, with $I = \mathbb{R}$, and $q(\cdot)$ belongs to some of function spaces introduced in Definition 3.1.36–Definition 3.1.38, with $I = [0, \infty)$. The study of qualitative properties of the function

$$t \mapsto H(t) \equiv \int_0^t R(t-s)\big[g(s) + q(s)\big]\,ds, \quad t \geqslant 0,$$

is based on the decomposition

$$H(t) = \int_0^t R(t-s)q(s)\,ds + \left[\int_{-\infty}^t R(t-s)g(s)\,ds - \int_t^\infty R(s)g(t-s)\,ds\right], \quad t \geqslant 0$$

and the use of corresponding results for infinite convolution product. In the following proposition, we will analyze the qualitative properties of functions

$$t \mapsto H_1(t) \equiv \int_t^\infty R(s)g(t-s)\,ds, \quad t \geqslant 0 \tag{3.34}$$

and

$$t \mapsto H_2(t) \equiv \int_0^t R(t-s)q(s)\,ds, \quad t \geqslant 0$$

separately. In the first part of the proposition, we continue our analysis from [373, Proposition 5.2]; our previous results show that case $p(x) \equiv p > 1$ is not simple in the analysis of asymptotical Weyl p-almost periodicity so that we will consider case $p(x) \equiv 1$ in the second part, with the notion introduced in Definition 3.1.36(i) only (cf. also [645, Proposition 5.3(i)]).

Proposition 3.1.56.

(i) *Suppose that* $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ *and* $(R(t))_{t>0} \subseteq L(X, Y)$ *is a strongly continuous operator family. Let the function* $\check{g} : \mathbb{R} \to X$ *be Stepanov* $p(x)$*-bounded and let for each* $t \geqslant 0$ *the series* $\sum_{k=0}^{\infty} \|R(\cdot + t + k)\|_{L^{q(\cdot)}[0,1]} \equiv S(t)$ *be convergent. Then the function* $H_1(\cdot)$*, given by* (3.34)*, is well defined. Furthermore, this function is continuous provided that the Bochner transform* $\hat{\check{g}} : \mathbb{R} \to L^{p(x)}([0,1])$ *is uniformly continuous, while the function* $H_1(\cdot)$ *satisfies* $\lim_{t\to+\infty} H_1(t) = 0$ *provided that* $\lim_{t\to+\infty} S(t) = 0$.

(ii) *Suppose that* $(R(t))_{t>0} \subseteq L(X, Y)$ *is a strongly continuous operator family such that* $\int_0^\infty \|R(s)\|_{L(X,Y)}\,ds < \infty$*. Let the function* $q : [0, \infty) \to Y$ *be equi-Weyl* $(1, x, F)$*-vanishing and let* $F_1 : (0, \infty) \times [0, \infty) \to (0, \infty)$*. If for each* $\varepsilon > 0$ *there* *exists* $l_0 > 0$ *such that for each* $l > l_0$ *there exists* $t_{0,l} > 0$ *such that for each* $t \geqslant t_{0,l}$ *we have*

$$\sup_{x\geqslant 0}\left[F_1(l,t)\int_0^{x+t}\left[\int_{x+t}^{x+t+l}\|R(s-r)\|_{L(X,Y)}\,ds\right]\|q(r)\|_Y\,dr\right] < \varepsilon,$$

and if, additionally, there exists a finite constant $M > 0$ *such that*

$$\frac{F_1(l,t)}{F(l,t)} \leqslant M, \quad l > 0,\ t \geqslant 0, \tag{3.35}$$

then the function $H_2(\cdot)$ *is equi-Weyl* $(1,x,F_1)$*-vanishing.*

Proof. (i): The first part follows from the Stepanov $p(x)$-boundedness of the function $\check{g}(\cdot)$ and the next simple computation

$$\begin{aligned}\left\|\int_t^{\infty}R(s)\check{g}(s-t)\,ds\right\| &= \left\|\sum_{k=0}^{\infty}\int_0^1 R(s+t+k)\check{g}(s+k)\,ds\right\| \\ &\leqslant 2\sum_{k=0}^{\infty}\|R(\cdot+t+k)\|_{L^{q(\cdot)}[0,1]}\sup_{k\in\mathbb{N}_0}\|\check{g}(\cdot+k)\|_{L^{p(\cdot)}[0,1]}.\end{aligned}$$

This computation also shows that $\lim_{t\to+\infty}H_1(t) = 0$ provided that $\lim_{t\to+\infty}S(t) = 0$. For remainder, let us suppose that the function $\hat{\check{g}} : \mathbb{R} \to L^{p(x)}([0,1])$ is uniformly continuous. Let (t_n) be a sequence of non-negative reals converging to a number $t \geqslant 0$. Then we can use the Hölder inequality and the decomposition

$$\begin{aligned}&\int_t^{\infty}R(s)g(t-s)\,ds - \int_{t_n}^{\infty}R(s)g(t_n-s)\,ds \\ &\quad = \int_t^{\infty}R(s)[\check{g}(s-t)-\check{g}(s-t_n)]\,ds + \int_t^{t_n}R(s)\check{g}(s-t)\,ds, \quad n\in\mathbb{N},\end{aligned}$$

in order to see that

$$\begin{aligned}&\left\|\int_t^{\infty}R(s)g(t-s)\,ds - \int_{t_n}^{\infty}R(s)g(t_n-s)\,ds\right\| \\ &\quad\leqslant 2\sum_{k=0}^{\infty}\|R(\cdot+t+k)\|_{L^{q(\cdot)}[0,1]}\sup_{k\in\mathbb{N}_0}\|\check{g}(\cdot+k)-\check{g}(\cdot+k+(t-t_n))\|_{L^{p(\cdot)}[0,1]} \\ &\qquad + 2\|R(\cdot)\|_{L^{q(\cdot)}[0,|t_n-t|]}\|\check{g}(\cdot)\|_{L^{p(\cdot)}[0,1]}, \quad n\in\mathbb{N}.\end{aligned}$$

Since $\|R(\cdot)\|_{L^{q(\cdot)}[0,|t_n-t|]} \to 0$ as $n \to +\infty$ (see, e. g., [377, Lemma 3.2.8(c)]) and the function $\hat{\check{g}} : \mathbb{R} \to L^{p(x)}([0,1])$ is uniformly continuous, the proof of the first part is completed.

(ii): By the proof of [645, Proposition 5.3(i)], we have

$$F_1(l,t)\int_{x+t}^{x+t+l}\|H_2(s)\|_Y\,ds \leqslant F_1(l,t)\int_0^{x+t}\left[\int_{x+t}^{x+t+l}\|R(s-r)\|_{L(X,Y)}\,ds\right]\|q(r)\|_Y\,dr$$
$$+F_1(l,t)\left[\int_0^{\infty}\|R(v)\|_{L(X,Y)}\,dv\right]\cdot\int_{x+t}^{x+t+l}\|q(r)\|_Y\,dr,$$

for any $x \geqslant 0$ and $l > 0$. Our preassumption shows that the first addend is equi-Weyl $(1,x,F_1)$-vanishing. The second addend is likewise equi-Weyl $(1,x,F_1)$-vanishing because we have assumed that the function $q(\cdot)$ is equi-Weyl $(1,x,F)$-vanishing and condition (3.35). □

We round off this subsection by examining the convolution invariance of Weyl almost periodic functions with variable exponent. In order to do that, we shall basically follow the method proposed in the proof of Theorem 3.1.46.

Proposition 3.1.57. *Suppose that $I=\mathbb{R}$, $\psi \in L^1(\mathbb{R})$, $(a_k)_{k\in\mathbb{Z}}$ is a sequence of positive real numbers satisfying $\sum_{k\in\mathbb{Z}} a_k = 1$ and condition* (A1) *holds true. Let $f \in (e-)W_{ap}^{(p,\phi,F)}(\mathbb{R}:X)\cap L^{\infty}(\mathbb{R}:X)$. Then the function*

$$x \mapsto (\psi * f)(x) := \int_{-\infty}^{+\infty}\psi(x-y)f(y)\,dy, \quad x\in\mathbb{R}, \tag{3.36}$$

is well defined and belongs to the space $L^{\infty}(\mathbb{R}:X)$. Furthermore, if $p_1 \in \mathcal{P}(\mathbb{R})$, $F_1 : (0,\infty)\times\mathbb{R}\to(0,\infty)$ and if, for every $t\in\mathbb{R}$ and $l>0$, we have

$$\int_t^{t+l}\varphi_{p_1(x)}\left(2l^{-1}F_1(l,t)\varphi(l)\sum_{k\in\mathbb{Z}}\frac{a_k\|\varphi(a_k^{-1}\psi(x-z))\|_{L^{q(z)}[x-(k+1)l,x-kl]}}{F(l,x-(k+1)l)}\right)dx \leqslant 1, \tag{3.37}$$

then we have $\psi * f \in (e-)W_{ap}^{(p_1,\phi,F_1)}(\mathbb{R}:X)$.

Proof. The proof can be deduced by using the arguments contained in the proof of Theorem 3.1.46, the equalities

$$\|\phi(\|(\psi*f)(\cdot+\tau)-(\psi*f)(\cdot)\|)\|_{L^{p_1(\cdot)}[t,t+l]}$$
$$=\inf\left\{\lambda>0:\int_t^{t+l}\varphi_{p_1(x)}\left(\frac{\phi(\|(\psi*f)(x+\tau)-(\psi*f)(x)\|)}{\lambda}\right)dx\leqslant 1\right\}$$
$$=\inf\left\{\lambda>0:\int_t^{t+l}\varphi_{p_1(x)}\left(\frac{\phi(\|\int_{-\infty}^{+\infty}\psi(y)[f(x+\tau-y)-f(x-y)]\,dy\|)}{\lambda}\right)dx\leqslant 1\right\}$$

and the following computation:

$$
\begin{aligned}
&\int_t^{t+l} \varphi_{p_1(x)}\left(\frac{\phi(\| \int_{-\infty}^{+\infty} \psi(y)[f(x+\tau-y)-f(x-y)]\,dy\|)}{\lambda}\right) dx \\
&\quad \leqslant \int_t^{t+l} \varphi_{p_1(x)}\left(\frac{\phi(\sum_{k\in\mathbb{Z}} a_k \| \int_{kl}^{(k+1)l} a_k^{-1}\psi(y)[f(x+\tau-y)-f(x-y)]\,dy\|)}{\lambda}\right) dx \\
&\quad \leqslant \int_t^{t+l} \varphi_{p_1(x)}\left(\frac{\sum_{k\in\mathbb{Z}} a_k \phi(l^{-1}l\| \int_{kl}^{(k+1)l} a_k^{-1}\psi(y)[f(x+\tau-y)-f(x-y)]\,dy\|)}{\lambda}\right) dx \\
&\quad \leqslant \int_t^{t+l} \varphi_{p_1(x)}\left(\frac{\sum_{k\in\mathbb{Z}} a_k \varphi(l)l^{-1} \int_{kl}^{(k+1)l} \phi(a_k^{-1}\psi(y)\|f(x+\tau-y)-f(x-y)\|)\,dy}{\lambda}\right) dx \\
&\quad \leqslant \int_t^{t+l} \varphi_{p_1(x)}\left(\frac{\sum_{k\in\mathbb{Z}} a_k \varphi(l)l^{-1} \int_{kl}^{(k+1)l} \varphi(a_k^{-1}\psi(y))\phi(\|f(x+\tau-y)-f(x-y)\|)\,dy}{\lambda}\right) dx \\
&\quad = \int_t^{t+l} \varphi_{p_1(x)}\left(\frac{\sum_{k\in\mathbb{Z}} a_k \varphi(l)l^{-1} \int_{x-(k+1)l}^{x-kl} \varphi(a_k^{-1}\psi(x-z))\phi(\|f(z+\tau)-f(z)\|)\,dz}{\lambda}\right) dx \\
&\quad \leqslant \int_t^{t+l} \varphi_{p_1(x)}\Bigg(2\sum_{k\in\mathbb{Z}} a_k \varphi(l)l^{-1} \big\|\varphi\big(a_k^{-1}\psi(x-z)\big)\big\|_{L^{q(z)}[x-(k+1)l,x-kl]} \\
&\qquad \times \frac{\|\phi(\|f(z+\tau)-f(z)\|)\|_{L^{p(z)}[x-(k+1)l,x-kl]}}{\lambda}\Bigg) dx,
\end{aligned}
$$

which is valid for every $t,\ \tau \in \mathbb{R}$ and $l > 0$. □

We can similarly prove the following result for the class of (equi-)Weyl $[p, \phi, F]$-almost periodic functions.

Proposition 3.1.58. *Suppose that $I = \mathbb{R}$, $\psi \in L^1(\mathbb{R})$, $(a_k)_{k\in\mathbb{Z}}$ is a sequence of positive real numbers satisfying $\sum_{k\in\mathbb{Z}} a_k = 1$ and condition (B1) holds true. Let $f \in (e-)W_{ap}^{[p,\phi,F]}(\mathbb{R} : X) \cap L^\infty(\mathbb{R} : X)$. Then the function $(\psi * f)(\cdot)$ defined by (3.36) belongs to the space $L^\infty(\mathbb{R} : X)$. Furthermore, if $p_1 \in \mathcal{P}([0,1])$, $F_1 : (0,\infty) \times \mathbb{R} \to (0,\infty)$ and if, for every $t \in \mathbb{R}$ and $l > 0$, we have*

$$
\int_0^1 \varphi_{p_1(x)}\left(2F_1(l,t)\sum_{k\in\mathbb{Z}} \frac{\|\varphi(la_k^{-1}\psi(xl-(z+k)l))\|_{L^{q(z)}[0,1]}}{F(l,t+kl)}\right) dx \leqslant 1, \tag{3.38}
$$

then we have $\psi * f \in (e-)W_{ap}^{[p_1,\phi,F_1]}(\mathbb{R} : X)$.

In the case of consideration of constant coefficients, the coefficient 2 in Eqs. (3.37) and (3.38) can be neglected. It might be interesting to formulate the corresponding results for the classes of (equi-)Weyl $(p, \phi, F)_i$-almost periodic functions and (equi-)Weyl $[p, \phi, F]_i$-almost periodic functions, where $i = 1, 2$, and to formulate an extension of [645, Proposition 4.3] for Weyl almost periodic functions with variable exponent.

3.1.7 Growth order of solution operator families

In this subsection, we will analyze solution operator families $(R(t))_{t>0} \subseteq L(X, Y)$ which satisfies the condition

$$\|R(t)\|_{L(X,Y)} \leqslant \frac{Mt^{\beta-1}}{1+t^{\gamma}}, \quad t > 0 \text{ for some finite constants } \gamma > 1,\ \beta \in (0,1],\ M > 0, \tag{3.39}$$

or the condition

$$\|R(t)\|_{L(X,Y)} \leqslant Mt^{\beta-1}e^{-ct}, \quad t > 0 \text{ for some finite constants } \beta \in (0,1] \text{ and } c > 0. \tag{3.40}$$

For simplicity, we will analyze only the constant exponents $p(x) \equiv p \in [1, \infty)$ as well the class of (equi-)Weyl (p, ϕ, F)-almost periodic functions and the class of (equi-)Weyl $(p, \phi, F)_i$-almost periodic functions, where $i = 1, 2$. So, let $1/p + 1/q = 1$ and let $(R(t))_{t>0} \subseteq L(X, Y)$ satisfy (3.39) or (3.40). We will additionally assume that $q(\beta - 1) > -1$ provided that $p > 1$, resp. $\beta = 1$, provided that $p = 1$.

In [631, Proposition 2.11.1, Theorem 2.11.4], the author has investigated the estimate (3.39) and case $p(x) \equiv p \in [1, \infty)$, where the resulting function $G(\cdot)$ is also bounded and continuous (see also [435] and [644]). We would like to note that Theorem 3.1.46 provides a new way of looking at the invariance of the (equi-)Weyl p-almost periodicity under the action of infinite convolution product and that the (equi-)Weyl p-almost periodicity in [631, Theorem 2.11.4] can be proved directly from Corollary 3.1.48. Let us explain this in more detail. Let a function $g : \mathbb{R} \to X$ be (equi-)Weyl p-almost periodic. Then the function $G : \mathbb{R} \to Y$, defined through (2.46), is (equi-)Weyl p-almost periodic and we can show this in the following way. It is clear that the function $\check{g}(\cdot)$ is also (equi-)Weyl p-almost periodic. By Corollary 3.1.48, with an arbitrary sequence of positive real numbers such that $\sum_{k=0}^{\infty} a_k = 1$ and the function $\varphi(x) \equiv x$, observing also that the class of (equi-)Weyl p-almost periodic functions is closed under pointwise multiplications with scalars, it suffices to show, by considering the function $(M^{-1}R(t))_{t>0}$ for a sufficiently large real number $M > 0$, that for every real numbers $t \in \mathbb{R}$ and $l > 0$ we have

$$\int_t^{t+l} \Biggl(\sum_{k=0}^{\infty} \Biggl(\int_{kl}^{(k+1)l} \frac{t^{(\beta-1)q}\,dt}{(1+t^{\gamma})^q} \Biggr)^{1/q} \Biggr)^p dx \leqslant \text{Const.}, \tag{3.41}$$

provided that $p > 1$, resp.

$$\int_t^{t+l} \sum_{k=0}^{\infty} \Bigl\| \frac{\cdot^{\beta-1}}{1+\cdot^{\gamma}} \Bigr\|_{L^{\infty}[kl,(k+1)l]} dx \leqslant \text{Const.}, \tag{3.42}$$

provided that $p = 1$. As

$$\int_{kl}^{(k+1)l} \frac{t^{(\beta-1)q}\,dt}{(1+t^{\gamma})^{q}} \leqslant \frac{1}{1+k^{q\gamma}l^{q\gamma}}(k+1)^{(\beta-1)q}l^{(\beta-1)q+1}, \quad k \in \mathbb{N}_0,$$

the estimate (3.41) follows from the inequality $(\beta - 1 + (1/q) - \gamma)p + 1 \leqslant 0$, which is true. The estimate (3.42) is much simpler and follows from the inequality $\gamma > 1$.

With regards to Theorem 3.1.51 and Theorem 3.1.54, we will provide two examples.

Example 3.1.59. Suppose that $\phi(x) = x^{\alpha}$, $x \geqslant 0$, where $\alpha > 0$. If the estimate (3.39) holds, then condition (3.27) holds provided that, for every $x \in \mathbb{R}$ and $l > 0$, we have

$$\sum_{k=0}^{\infty} k^{\beta-1-\gamma}\big[F(l,-x+kl)\big]^{(-1)/\alpha} < \infty,$$

while condition (3.28) holds provided that, for every $t \in \mathbb{R}$ and $l > 0$, we have

$$\int_{t}^{t+l} \left(\left(\frac{1}{1+k^{q\gamma}l^{q\gamma}}(k+1)^{(\beta-1)q}l^{(\beta-1)q+1} \right)^{1/q} \left(\frac{F(l,t)}{F(l,-x+kl)} \right)^{1/\alpha} \right)^{p} dx \leqslant 1,$$

if $p > 1$, resp.

$$\int_{t}^{t+l} \frac{(kl)^{\beta-1}}{1+k^{\gamma}l^{\gamma}} \left(\frac{F(l,t)}{F(l,-x+kl)} \right)^{1/\alpha} dx \leqslant 1,$$

if $p = 1$. If the estimate (3.40) holds, then condition (3.27) holds provided that, for every $x \in \mathbb{R}$ and $l > 0$, we have

$$\sum_{k=0}^{\infty} e^{-ck}k^{\beta-1}\big[F(l,-x+kl)\big]^{(-1)/\alpha} < \infty,$$

while condition (3.28) holds provided that, for every $t \in \mathbb{R}$ and $l > 0$, we have

$$\int_{t}^{t+l} \left(e^{-ck}(kl)^{\beta-1} \left(\frac{F(l,t)}{F(l,-x+kl)} \right)^{1/\alpha} \right)^{p} dx \leqslant 1.$$

Example 3.1.60. Suppose that condition (A2) holds. If the estimate (3.39) holds, then condition (3.31) holds provided that

$$\sum_{k=0}^{\infty} k^{\beta-1-\gamma}F(l,-x+kl)^{-1} < \infty,$$

while condition (3.32) holds provided that, for every $t \in \mathbb{R}$ and $l > 0$, we have

$$\int_{t}^{t+l} \left(\left(\frac{1}{1+k^{q\gamma}l^{q\gamma}}(k+1)^{(\beta-1)q}l^{(\beta-1)q+1} \right)^{1/q} \frac{F(l,t)}{F(l,-x+kl)} \right)^{p} dx \leqslant 1,$$

if $p > 1$, resp.

$$\int_t^{t+l} \frac{(kl)^{\beta-1}}{1+k^\gamma l^\gamma} \frac{F(l,t)}{F(l,-x+kl)}\,dx \leqslant 1,$$

if $p = 1$. If the estimate (3.40) holds, then (3.31) holds provided that, for every $x \in \mathbb{R}$ and $l > 0$, we have

$$\sum_{k=0}^{\infty} e^{-ck} k^{\beta-1} [F(l,-x+kl)]^{-1} < \infty,$$

while condition (3.32) holds provided that, for every $t \in \mathbb{R}$ and $l > 0$, we have

$$\int_t^{t+l} \left(e^{-ck} (kl)^{\beta-1} \frac{F(l,t)}{F(l,-x+kl)} \right)^p dx \leqslant 1.$$

At the end of this section, let us only note that we can incorporate our results in the study of the abstract fractional Cauchy inclusions (2.49) and $(\mathrm{DFP})_{f,\zeta}$, provided that the multivalued linear operator $\mathcal{A}$ satisfies condition (P). Then there exists a strongly continuous operator family $(S_\zeta(t))_{t>0}$ satisfying the estimate of type (3.39), in the case $\zeta \in (0,1)$, or estimate of type (3.40), in the case $\zeta = 1$, such that the unique mild solution of problem $(\mathrm{DFP})_{f,\zeta}$ is given by

$$t \mapsto u(t) \equiv S_\zeta(t)u_0 + \int_0^t S_\zeta(t-s)f(s)\,ds, \quad t \geqslant 0,$$

where u_0 belongs to the continuity set of $(S_\zeta(t))_{t>0}$, i. e., $\lim_{t\to 0+} S_\zeta(t)u_0 = u_0$. Moreover, $\lim_{t\to+\infty} S_\zeta(t)u_0 = 0$ and Proposition 3.1.56 can be straightforwardly applied.

3.2 Generalized almost periodicity in Lebesgue spaces with variable exponents. Part II

In this section, we introduce and analyze Stepanov uniformly recurrent functions, Doss uniformly recurrent functions and Doss almost periodic functions in Lebesgue spaces with variable exponents. We investigate the invariance of these types of generalized almost periodicity in Lebesgue spaces with variable exponents under the actions of convolution products, providing also some illustrative applications to the abstract semilinear integro-differential inclusions in Banach spaces.

The organization of the section can be briefly described as follows. Subsection 3.2.1 investigates the Stepanov uniformly recurrent functions in Lebesgue spaces with variable exponents. The proofs of structural results in this section can be given by employing the slight modifications of the corresponding results from [372] (see also [648]) and therefore omitted. Our main contributions are given in Subsection 3.2.2 and Subsection 3.2.3, where we introduce and analyze several various classes of Doss almost periodic (uniformly recurrent) functions in Lebesgue spaces with variable exponents and the invariance of generalized Doss almost periodicity under the actions of convolution products. The final subsection is reserved for applications of our abstract theoretical results to the abstract semilinear integro-differential inclusions in Banach spaces. In addition to the above, we provide several illustrative examples, remarks and comments about the material presented.

3.2.1 Stepanov uniform recurrence in Lebesgue spaces with variable exponents

First of all, we will introduce the concept of (asymptotical) $S^{p(x)}$-uniform recurrence.

Definition 3.2.1. Let $p \in \mathcal{P}([0,1])$, and let $f : I \to X$ be such that $f \in L^{p(x)}(K : X)$ for any compact set $K \subseteq I$.

(i) We say that $f(\cdot)$ is Stepanov $p(x)$-uniformly recurrent if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is uniformly recurrent. The collection of such functions will be denoted by $\mathrm{URS}^{p(x)}(I : X)$ ($\mathrm{URS}^{p}(I : X)$, if $p(x) \equiv p \in [1,\infty)$).

(ii) We say that $f(\cdot)$ is asymptotically Stepanov $p(x)$-uniformly recurrent if and only if there exist a Stepanov $p(x)$-uniformly recurrent function $h : \mathbb{R} \to X$ and a function $q \in L_S^{p(x)}(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^{p(x)}([0,1] : X))$. The collection of such functions will be denoted by $\mathrm{AURS}^{p(x)}(I : X)$ ($\mathrm{AURS}^{p}(I : X)$, if $p(x) \equiv p \in [1,\infty)$).

The spaces $\mathrm{URS}^{p(x)}(I : X)$ and $\mathrm{AURS}^{p(x)}(I : X)$ are translation invariant, as it can be easily approved. Furthermore, we have the following proposition which can be deduced by using the same argumentation as in the proofs of corresponding structural results concerning Stepanov almost periodicity with variable exponent.

Proposition 3.2.2.

(i) *Suppose* $p \in \mathcal{P}([0,1])$. *Then* $\mathrm{URS}^{p(x)}(I : X) \subseteq \mathrm{URS}^{1}(I : X)$, $\mathrm{AURS}^{p(x)}(I : X) \subseteq \mathrm{AURS}^{1}(I : X)$, $\mathrm{UR}(I : X) \subseteq \mathrm{URS}^{p(x)}(I : X) \subseteq \mathrm{URS}^{1}(I : X)$ *and* $\mathrm{AUR}(I : X) \subseteq \mathrm{AURS}^{p(x)}(I : X) \subseteq \mathrm{AURS}^{1}(I : X)$.

(ii) *Suppose* $p \in D_+([0,1])$ *and* $1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ *for a. e.* $x \in [0,1]$. *Then we have* $\mathrm{URS}^{p^+}(I : X) \subseteq \mathrm{URS}^{p(x)}(I : X) \subseteq \mathrm{URS}^{p^-}(I : X)$ *and* $\mathrm{AURS}^{p^+}(I : X) \subseteq \mathrm{AURS}^{p(x)}(I : X) \subseteq \mathrm{AURS}^{p^-}(I : X)$.

(iii) *Assume that* p, $q \in \mathcal{P}([0,1])$ *and* $p \leqslant q$ *a. e. on* $[0,1]$. *Then we have* $\mathrm{URS}^{q(x)}(I : X) \subseteq \mathrm{URS}^{p(x)}(I : X)$ *and* $\mathrm{AURS}^{q(x)}(I : X) \subseteq \mathrm{AURS}^{p(x)}(I : X)$.

(iv) *If* $p \in D_+([0,1])$, *then*

$$L^{\infty}(I : X) \cap \mathrm{URS}^{p(x)}(I : X) = L^{\infty}(I : X) \cap \mathrm{URS}^{1}(I : X)$$

and

$$L^{\infty}(I:X)\cap \mathrm{AURS}^{p(x)}(I:X)=L^{\infty}(I:X)\cap \mathrm{AURS}^{1}(I:X).$$

We continue by providing two illustrative examples.

Example 3.2.3. Let us recall that H. Bohr and E. Følner have constructed, for any given number $p > 1$, a Stepanov almost periodic function defined on the whole real axis that is Stepanov p-bounded and not Stepanov p-almost automorphic (see [199, Example, pp. 70–73]). We want to observe here that the function $f(\cdot)$ cannot be Stepanov p-uniformly recurrent. Strictly speaking, let us consider case $h_1 = 2$ in the afore-mentioned example. If we suppose the contrary, then the mapping $\hat{f}:\mathbb{R}\to L^p([0,1]:X)$ is uniformly recurrent, which in particular implies that for each number $\varepsilon>0$ there exists an arbitrarily large positive real number $\tau>0$ such that

$$\int_{-3/2}^{3/2}\left|f(s+\tau)-f(s)\right|^{p}\,ds<2\varepsilon^{p},$$

which is in contradiction to the estimate $\int_{-3/2}^{3/2}|f(s+\tau)-f(s)|^p\,ds\geqslant 2^{-p}$ (see [199, p. 73, l.-9–l.-4]).

Example 3.2.4. Define $f(x):=\sin x+\sin\sqrt{2}x$, $x\in\mathbb{R}$ and $p(x):=1-\ln x$, $x\in[0,1]$. We know that the function $\operatorname{sign}(f(\cdot))$ is neither Stepanov $p(x)$-almost periodic nor Stepanov $p(x)$-almost automorphic [372, 373]. Moreover, we have already proved that for every real numbers $\lambda\in(0,2/e)$, $l>0$, every interval $I\subseteq\mathbb{R}\smallsetminus\{0\}$ of length l and every number $\tau\in I$, there exists a number $t\in\mathbb{R}$ such that

$$\begin{aligned}\int_0^1\left(\frac{1}{\lambda}\right)^{1-\ln x}&\big|\operatorname{sign}\big[\sin(x+t+\tau)+\sin\sqrt{2}(x+t+\tau)\big]\\&-\operatorname{sign}\big[\sin(x+t)+\sin\sqrt{2}(x+t)\big]\big|^{1-\ln x}\,dx=\infty.\end{aligned}$$

This implies that the function $\operatorname{sign} f(\cdot)$ cannot be Stepanov $p(x)$-uniformly recurrent, as well.

Now we will state two results about the invariance of uniform recurrence under the actions of infinite convolution products. The first result slightly extends [631, Proposition 2.6.11]; the proof can be given by using the same arguments as in the proof of the above-mentioned proposition, appealing to the Hölder inequality from Lemma 1.1.7(i).

Proposition 3.2.5. *Suppose that $p,\ q\in\mathcal{P}([0,1])$, $1/p(x)+1/q(x)=1$ and $(R(t))_{t>0}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M:=\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]}<\infty$. If $\check{f}:\mathbb{R}\to X$ is $S^{p(x)}$-bounded, $S^{p(x)}$-uniformly recurrent and the Bochner transform of function $\check{f}:\mathbb{R}\to X$ is uniformly continuous, then the function $F:\mathbb{R}\to Y$, given by* (2.46)*, is well defined and uniformly recurrent.*

Using a similar argumentation, we can clarify the following result in which we do not require that the function $\check{f} : \mathbb{R} \to X$ is $S^{p(x)}$-bounded.

Proposition 3.2.6. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is $S^{p(x)}$-uniformly recurrent, the Bochner transform of function $\check{f} : \mathbb{R} \to X$ is uniformly continuous,*

$$\|\check{f}(\cdot - t)\|_{L^{p(x)}[0,1]} \leqslant P(t), \quad t \in \mathbb{R}$$

and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying the requirement that for each $t \in \mathbb{R}$ we have

$$\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} P(t-k) < \infty,$$

then the function $F : \mathbb{R} \to Y$, given by (2.46)*, is well defined and uniformly recurrent.*

Now we will introduce the notion of (asymptotical) Stepanov $p(x)$-uniform recurrence for the functions depending on two parameters; this notion extends the notion introduced in Definition 2.4.42 and Definition 2.4.43, where we have considered the constant coefficient $p(x) \equiv p \in [1,\infty)$.

Definition 3.2.7. Let $p \in \mathcal{P}([0,1])$.

(i) A function $f : I \times Y \to X$ is called Stepanov $p(x)$-uniformly recurrent if and only if $\hat{f} : I \times Y \to L^{p(x)}([0,1] : X)$ is uniformly recurrent.

(ii) A function $f : I \times Y \to X$ is said to be asymptotically $S^{p(x)}$-uniformly recurrent if and only if there exist a Stepanov $p(x)$-uniformly recurrent function $g : [0,\infty) \times Y \to X$ and a function $q \in C_0(I \times Y : X)$ such that $f(t,y) = g(t,y) + q(t,y)$ for all $t \in I$ and $y \in Y$.

A great number of composition principles established for Stepanov $p(x)$-almost periodic functions can be straightforwardly extended for Stepanov $p(x)$-uniformly recurrent functions. For example, we have the following.

Theorem 3.2.8. *Let $p \in \mathcal{P}([0,1])$. Suppose that the following conditions hold:*

(i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent, and there exist a function $r \in \mathcal{P}([0,1])$ and a function $L_f \in L_S^{r(x)}(I)$ such that $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/(p(\cdot)-1))$ and* (2.20) *holds.*

(ii) *The function $f : I \to Y$ is Stepanov $p(x)$-uniformly recurrent and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{f(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact in Y.*

(iii) *For every compact set $K \subseteq Y$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \sup_{u\in K} \|F(t+s+\alpha_n, u) - F(t+s, u)\|_{L^{p(s)}([0,1]:X)} = 0 \tag{3.43}$$

and (2.3) *holds with the function* $f(\cdot)$ *and the norm* $\|\cdot\|$ *replaced, respectively, by the function* $\hat{f}(\cdot)$ *and the norm* $\|\cdot\|_{L^{p(x)}([0,1]:X)}$ *therein.*

Then $q(x) := p(x)r(x)/(p(x)+r(x)) \in [1, p(x))$ *and* $F(\cdot, f(\cdot))$ *is Stepanov* $q(x)$*-uniformly recurrent. Furthermore, the assumption that* $F(\cdot, 0)$ *is Stepanov* $q(x)$*-bounded also implies that the function* $F(\cdot, f(\cdot))$ *is Stepanov* $q(x)$*-bounded.*

We close the subsection with the observation that it is not so difficult to reformulate the statements of [631, Proposition 2.7.3–Proposition 2.7.4] for the asymptotical Stepanov $p(x)$-uniform recurrence.

3.2.2 Doss almost periodicity and Doss uniform recurrence in Lebesgue spaces with variable exponents

Throughout this subsection, we assume that condition (A) holds true. The notion of Doss $p(x)$-almost periodicity has not been introduced so far. Following the approach obeyed for the classes of (equi-)Weyl (p, ϕ, F)-almost periodic functions and (equi-)-Weyl $(p, \phi, F)_i$-almost periodic functions ($i = 1, 2$), we introduce the following notion for Doss classes.

Definition 3.2.9. Suppose that condition (A) holds, $f : I \to X$ and $\phi(\|f(\cdot+\tau) - f(\cdot)\|) \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) A function $f(\cdot)$ is said to be Doss (p, ϕ, F)-almost periodic if and only if for every $\varepsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t\to+\infty}\big[F(t)\big[\phi\big(\|f(\cdot+\tau) - f(\cdot)\|\big)_{L^{p(x)}[-t,t]}\big]\big] < \varepsilon, \tag{3.44}$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t\to+\infty}\big[F(t)\big[\phi\big(\|f(\cdot+\tau) - f(\cdot)\|\big)_{L^{p(x)}[0,t]}\big]\big] < \varepsilon,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

(ii) A function $f(\cdot)$ is said to be Doss (p, ϕ, F)-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\big[F(t)\big[\phi\big(\|f(\cdot+\alpha_n) - f(\cdot)\|\big)_{L^{p(x)}[-t,t]}\big]\big] = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\big[F(t)\big[\phi\big(\|f(\cdot+\alpha_n) - f(\cdot)\|\big)_{L^{p(x)}[0,t]}\big]\big] = 0,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

Definition 3.2.10. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot + \tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) A function $f(\cdot)$ is said to be Doss $(p, \phi, F)_1$-almost periodic if and only if for every $\varepsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t\to+\infty}\big[F(t)\phi\big[\big(\|f(\cdot + \tau) - f(\cdot)\|\big)_{L^{p(x)}[-t,t]}\big]\big] < \varepsilon,$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t\to+\infty}\big[F(t)\phi\big[\big(\|f(\cdot + \tau) - f(\cdot)\|\big)_{L^{p(x)}[0,t]}\big]\big] < \varepsilon,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

(ii) A function $f(\cdot)$ is said to be Doss $(p, \phi, F)_1$-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\big[F(t)\phi\big[\big(\|f(\cdot + \alpha_n) - f(\cdot)\|\big)_{L^{p(x)}[-t,t]}\big]\big] = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\big[F(t)\big[\phi\big(\|f(\cdot + \alpha_n) - f(\cdot)\|\big)_{L^{p(x)}[0,t]}\big]\big] = 0,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

Definition 3.2.11. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot + \tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) A function $f(\cdot)$ is said to be Doss $(p, \phi, F)_2$-almost periodic if and only if for every $\varepsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t\to+\infty}\big[\phi\big[F(t)\big(\|f(\cdot + \tau) - f(\cdot)\|\big)_{L^{p(x)}[-t,t]}\big]\big] < \varepsilon,$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t\to+\infty}\big[\phi\big[F(t)\big(\|f(\cdot + \tau) - f(\cdot)\|\big)_{L^{p(x)}[0,t]}\big]\big] < \varepsilon,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

(ii) A function $f(\cdot)$ is said to be Doss $(p, \phi, F)_2$-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\big[\phi\big[F(t)\big(\|f(\cdot + \alpha_n) - f(\cdot)\|\big)_{L^{p(x)}[-t,t]}\big]\big] = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\big[\phi\big[F(t)\big(\|f(\cdot + \alpha_n) - f(\cdot)\|\big)_{L^{p(x)}[0,t]}\big]\big] = 0,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

The case in which $\phi(x) \equiv x$ and $\psi(t) \equiv (2t)^{(-1)/p}$, $t > 0$ if $I = \mathbb{R}$, resp. $\psi(t) \equiv t^{(-1)/p}$, $t > 0$ if $I = [0, \infty)$, leads to the usual class of Doss p-almost periodic functions [631, 644]. The notion introduced in the above three definitions is rather general; for example, in the case that $p(x) \equiv p \in [1, \infty)$ and $\sigma > 0$, then any essentially bounded function $f(\cdot)$ is Doss $(p, x, t^{-(1+\sigma)/p})$-almost periodic.

Example 3.2.12.

(i) Suppose that $\phi(0) = 0$. Then any continuous periodic function $f : I \to X$ is Doss $(p, \phi, F)_i$-almost periodic for $i = 1, 2$; furthermore, if $\phi(\cdot)$ is locally bounded, then the function $f(\cdot)$ is Doss (p, ϕ, F)-almost periodic.

(ii) Suppose that $f : I \to X$ is almost periodic. Then $f(\cdot)$ is Doss (p, ϕ, F)-almost periodic [Doss $(p, \phi, F)_1$-almost periodic/Doss $(p, \phi, F)_2$-almost periodic] if $\phi(\cdot)$ is continuous, monotonically increasing and $F(\cdot)\|1\|_{L^{p(x)}[-\cdot,\cdot]} \in L^\infty((0, \infty))$ [$\phi(\cdot)$ is monotonically increasing, there exists a continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$, $x,\ y \geqslant 0$ and $F(\cdot)\|1\|_{L^{p(x)}[-\cdot,\cdot]} \in L^\infty((0, \infty))$/$\phi(\cdot)$ is monotonically increasing, there exists a continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$, $x,\ y \geqslant 0$ and $\phi(F(\cdot)\|1\|_{L^{p(x)}[-\cdot,\cdot]}) \in L^\infty((0, \infty))$].

Example 3.2.13. We have already clarified that the function $f : \mathbb{R} \to \mathbb{R}$, given by (2.28), is uniformly continuous, uniformly recurrent and Besicovitch unbounded. Furthermore, we have proved that for each number $\tau \in \mathbb{R}$ we have

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \left|f(s + \tau) - f(s)\right|^p ds = 0, \quad p \geqslant 1,$$

so that the function $f(\cdot)$ is Doss $p(x)$-almost periodic for any function $p \in D_+(\mathbb{R})$.

Example 3.2.14. Let $\zeta \geqslant 1$ and $0^\zeta := 0$. Define the complex-valued function

$$f_\zeta(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^\zeta\left(\frac{t}{2^n}\right), \quad t \in \mathbb{R}.$$

Then the function $f_\zeta(\cdot)$ is Lipschitz continuous and uniformly recurrent. To prove the Lipschitz continuity of the function $f_\zeta(\cdot)$, it suffices to observe that the function $t \mapsto \sin^\zeta(t)$, $t \in \mathbb{R}$ is continuous and

$$\left|\sin^\zeta x - \sin^\zeta y\right| \leqslant \zeta|x - y|, \quad x,\ y \in \mathbb{R}. \tag{3.45}$$

To see that the function $f_\zeta(\cdot)$ is uniformly recurrent, it suffices to see that for each integer $k \in \mathbb{N} \smallsetminus \{1\}$ we have

$$\begin{aligned}&\left|f_\zeta(t + 2^k\pi) - f_\zeta(t)\right| \\ &\quad = \left|\sum_{n=1}^{\infty} \frac{1}{n}\left[\sin^\zeta\left(\frac{t + 2^k\pi}{2^n}\right) - \sin^\zeta\left(\frac{t}{2^n}\right)\right]\right|\end{aligned}$$

$$
\begin{aligned}
&= \left|\sum_{n=1}^{k-1}\frac{1}{n}\left[\sin^{\zeta}\left(\frac{t+2^k\pi}{2^n}\right)-\sin^{\zeta}\left(\frac{t}{2^n}\right)\right]\right| + \left|\sum_{n=k}^{\infty}\frac{1}{n}\left[\sin^{\zeta}\left(\frac{t+2^k\pi}{2^n}\right)-\sin^{\zeta}\left(\frac{t}{2^n}\right)\right]\right| \\
&= \left|\sum_{n=k}^{\infty}\frac{1}{n}\left[\sin^{\zeta}\left(\frac{t+2^k\pi}{2^n}\right)-\sin^{\zeta}\left(\frac{t}{2^n}\right)\right]\right| \leqslant \sum_{n=k}^{\infty}\frac{1}{n}\left|\sin^{\zeta}\left(\frac{t+2^k\pi}{2^n}\right)-\sin^{\zeta}\left(\frac{t}{2^n}\right)\right| \\
&\leqslant \sum_{n=k}^{\infty}\frac{\zeta}{n}2^{k-n}\pi = \frac{2\pi\zeta}{k}, \quad t \in \mathbb{R},
\end{aligned}
$$

where we have applied (3.45) in the last line of computation. In the case that $\zeta = 2l$ for some integer $l \in \mathbb{N}$, we see that the function $f_{\zeta}(\cdot)$ is Besicovitch unbounded. This can be inspected as in the proof of [511, Theorem 1.1], with the additional observation that

$$
\int_0^{2^{k-n}\pi} \sin^{2l} t\,dt = \frac{2}{3}\frac{(2l-1)!!}{(2l)!!}\int_0^{2^{k-n}\pi} \sin^2 t\,dt \quad (k \in \mathbb{N} \setminus \{1\},\ 1 \leqslant n \leqslant k);
$$

here, we have used the well-known recurrent formula

$$
\int_0^{2^{k-n}\pi} \sin^{2l} t\,dt = \frac{2l-1}{2l}\int_0^{2^{k-n}\pi} \sin^{2l-2} t\,dt,
$$

which can be deduced with the help of the partial integration (take $u = \sin^{2l-1} t$ and $dv = \sin t \cdot dt$). We would like to ask whether the function $f_{\zeta}(\cdot)$ is Besicovitch unbounded in the general case and for which functions $p \in D_+(\mathbb{R})$ we see that $f(\cdot)$ is Doss $p(x)$-almost periodic (see also Example 4.2.24).

In order to ensure the translation invariance of generalized Weyl spaces of almost periodic functions, we have analyzed the classes of (equi-)Weyl $[p,\phi,F]$-almost periodic functions and (equi-)Weyl $[p,\phi,F]_i$-almost periodic functions ($i = 1,2$). In this subsection, we will follow a slightly different approach. First of all, for any $\tau_0 \in I$ we set $p_{\tau_0}(\cdot) := p(\cdot+\tau_0)$. Then we have the following.

Theorem 3.2.15. *Suppose that $F_1(\cdot)$ is monotonically decreasing, there exists a function $F_0 : (0,\infty) \to (0,\infty)$ such that $F(xy) \leqslant F_0(x)\cdot F(y)$, $x,\ y > 0$, $\tau_0 \in I$ and*

$$
\limsup_{t\to+\infty} F_0\left(\frac{t}{t+\tau_0}\right) < \infty. \tag{3.46}
$$

Define $f_{\tau_0}(\cdot) := f(\cdot+\tau_0)$. Then the following holds:

(i) *Suppose that $f(\cdot)$ is Doss (p,ϕ,F)-almost periodic, resp. Doss (p,ϕ,F)-uniformly recurrent. Then $f_{\tau_0}(\cdot)$ is Doss (p_{τ_0},ϕ,F_1)-almost periodic, resp. Doss (p_{τ_0},ϕ,F_1)-uniformly recurrent.*
(ii) *Suppose that $f(\cdot)$ is Doss $(p,\phi,F)_1$-almost periodic, resp. Doss $(p,\phi,F)_1$-uniformly recurrent, and $\phi(\cdot)$ is monotonically increasing. Then $f_{\tau_0}(\cdot)$ is Doss $(p_{\tau_0},\phi,F_1)_1$-almost periodic, resp. Doss $(p_{\tau_0},\phi,F_1)_1$-uniformly recurrent.*

(iii) *Suppose that $f(\cdot)$ is Doss $(p,\phi,F)_2$-almost periodic, resp. Doss $(p,\phi,F)_2$-uniformly recurrent, $\phi(\cdot)$ is monotonically increasing, there exists a function $\phi_0 : [0,\infty) \to [0,\infty)$ such that $\phi(xy) \leqslant \phi_0(x)\cdot\phi(y)$, $x,\ y \geqslant 0$ and, in place of condition* (3.46),

$$\limsup_{t\to+\infty}\,(\phi_0\circ F_0)\Big(\frac{t}{t+\tau_0}\Big) < \infty. \tag{3.47}$$

Then $f_{\tau_0}(\cdot)$ is Doss $(p_{\tau_0},\phi_1,F_1)_2$-almost periodic, resp. Doss $(p_{\tau_0},\phi_1,F_1)_2$-uniformly recurrent.

Proof. We will consider only Doss almost periodic functions with variable exponent. Suppose that $\tau \in I$ and (3.44) holds. We need to prove first that $\phi(\|f(\cdot+\tau+\tau_0) - f(\cdot+\tau_0)\|) \in L^{p_{\tau_0}(x)}(K)$ for any $\tau \in I$ and any compact subset K of I. But, this directly follows from the corresponding definitions of the space $L^{p_{\tau_0}(x)}(K)$, the function $p_{\tau_0}(\cdot)$ and an elementary substitution $\cdot \mapsto \cdot+\tau_0$. The statement (i) then follows from the next computation:

$$\begin{aligned}
&\limsup_{t\to+\infty}\Big[F_1(t)\inf\Big\{\lambda>0:\int_0^t \varphi_{p_{\tau_0}(x)}\Big(\frac{\phi(\|f(x+\tau+\tau_0)-f(x+\tau_0)\|)}{\lambda}\Big)\,dx\leqslant 1\Big\}\Big]\\
&=\limsup_{t\to+\infty}\Big[F_1(t)\inf\Big\{\lambda>0:\int_{\tau_0}^{t+\tau_0} \varphi_{p_{\tau_0}(x-\tau_0)}\Big(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\Big)\,dx\leqslant 1\Big\}\Big]\\
&=\lim_{t\to+\infty}\sup_{y\geqslant t}\Big[F_1(y)\inf\Big\{\lambda>0:\int_{\tau_0}^{y+\tau_0} \varphi_{p(x)}\Big(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\Big)\,dx\leqslant 1\Big\}\Big]\\
&\leqslant\limsup_{t\to+\infty}\sup_{y\geqslant t}\Big[F_1\Big(\frac{t}{t+\tau_0}(y+\tau_0)\Big)\\
&\quad\times\inf\Big\{\lambda>0:\int_{\tau_0}^{y+\tau_0} \varphi_{p(x)}\Big(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\Big)\,dx\leqslant 1\Big\}\Big]\\
&\leqslant\limsup_{t\to+\infty}\sup_{y\geqslant t}\Big[F_0\Big(\frac{t}{t+\tau_0}\Big)\\
&\quad\times F(y+\tau_0)\inf\Big\{\lambda>0:\int_{\tau_0}^{y+\tau_0} \varphi_{p(x)}\Big(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\Big)\,dx\leqslant 1\Big\}\Big]\\
&\leqslant\limsup_{t\to+\infty}F_0\Big(\frac{t}{t+\tau_0}\Big)\\
&\quad\times\limsup_{t\to+\infty}\sup_{y\geqslant t}\Big[F(y+\tau_0)\inf\Big\{\lambda>0:\int_{\tau_0}^{y+\tau_0} \varphi_{p(x)}\Big(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\Big)\,dx\leqslant 1\Big\}\Big]\\
&\leqslant\limsup_{t\to+\infty}F_0\Big(\frac{t}{t+\tau_0}\Big)
\end{aligned}$$

$$\times \limsup_{t\to+\infty} \sup_{y\geqslant t}\left[F(y+\tau_0)\inf\left\{\lambda>0: \int_0^{y+\tau_0} \varphi_{p(x)}\left(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\right)dx \leqslant 1\right\}\right]$$

$$\leqslant \limsup_{t\to+\infty} F_0\left(\frac{t}{t+\tau_0}\right)$$

$$\times \limsup_{t\to+\infty} \sup_{y\geqslant t+\tau_0}\left[F(y)\inf\left\{\lambda>0: \int_0^{y} \varphi_{p(x)}\left(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\right)dx \leqslant 1\right\}\right]$$

$$= \limsup_{t\to+\infty} F_0\left(\frac{t}{t+\tau_0}\right)$$

$$\times \limsup_{t\to+\infty} \sup_{y\geqslant t}\left[F(y)\inf\left\{\lambda>0: \int_0^{y} \varphi_{p(x)}\left(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\right)dx \leqslant 1\right\}\right]$$

$$= \limsup_{t\to+\infty} F_0\left(\frac{t}{t+\tau_0}\right)$$

$$\times \limsup_{t\to+\infty} \left[F(t)\inf\left\{\lambda>0: \int_0^{t} \varphi_{p(x)}\left(\frac{\phi(\|f(x+\tau)-f(x)\|)}{\lambda}\right)dx \leqslant 1\right\}\right]$$

$$\leqslant \limsup_{t\to+\infty} F_0\left(\frac{t}{t+\tau_0}\right)\cdot\varepsilon.$$

The proof of (ii) is similar because then we can start from the term

$$\limsup_{t\to+\infty}\left[F_1(t)\phi\left(\inf\left\{\lambda>0: \int_0^{t} \varphi_{p_{\tau_0}(x)}\left(\frac{\|f(x+\tau+\tau_0)-f(x+\tau_0)\|}{\lambda}\right)dx \leqslant 1\right\}\right)\right],$$

use the same computation and the assumption that $\phi(\cdot)$ is monotonically increasing. The proof of (iii) is also similar because, with the obvious change of computation caused by the use of different notion, we can use the same computation and the inequality (see also (3.47))

$$\phi\left(F_0\left(\frac{t}{t+\tau_0}\right)\cdot F(y+\tau_0)\right) \leqslant \phi_0\left(F_0\left(\frac{t}{t+\tau_0}\right)\right)\cdot\phi_1(F(y+\tau_0)). \qquad \square$$

We will include the proof of the next proposition for the sake of completeness.

Proposition 3.2.16. *Suppose that $p(x)\equiv 1$, $f: I\to X$, $\|f(\cdot+\tau)-f(\cdot)\|\in L^1(K)$ for any $\tau\in I$ and any compact subset K of I, as well as condition*

(B)′ *$\phi(\cdot)$ is convex and there exists a function $\varphi:[0,\infty)\to(0,\infty)$ such that $\phi(tx)\leqslant \varphi(t)\phi(x)$ for all $t\geqslant 0$ and $x\geqslant 0$.*

Set $F_1(t):=F(t)t[\varphi(t)]^{-1}$, $t>0$, $F_2(t):=(2t)^{-1}\varphi(2F(t)t)$, $t>0$ provided that $I=\mathbb{R}$, and $F_2(t):=t^{-1}\varphi(F(t)t)$, $t>0$ provided that $I=[0,\infty)$. Then we have:

(i) *If $f(\cdot)$ is Doss $(1,\phi,F)$-almost periodic, resp. Doss $(1,\phi,F)$-uniformly recurrent, then $f(\cdot)$ is Doss $(1,\phi,F_1)_1$-almost periodic, resp. Doss $(1,\phi,F_1)_1$-uniformly recurrent.*
(ii) *If $f(\cdot)$ is Doss $(1,\phi,F_2)$-almost periodic, resp. Doss $(1,\phi,F_2)$-uniformly recurrent, then $f(\cdot)$ is Doss $(1,\phi,F)_2$-almost periodic, resp. Doss $(1,\phi,F)_2$-uniformly recurrent.*

Proof. We will consider only Doss almost periodic functions with variable exponent and case $I=[0,\infty)$. To prove (i), we can use the assumption (B)$'$ and the Jensen integral inequality ($\tau>0$):

$$\begin{aligned}\phi(\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]}) &= \phi(t\cdot t^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]})\\ &\leqslant \varphi(t)\phi(t^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]})\\ &\leqslant \varphi(t)t^{-1}[\phi(\|f(\cdot+\tau)-f(\cdot)\|)]_{L^1[0,t]}.\end{aligned}$$

This simply shows that $f(\cdot)$ is Doss $(1,\phi,F_1)_1$-almost periodic. To prove (ii), suppose that $f(\cdot)$ is Doss $(1,\phi,F_2)$-almost periodic. Then the assumption (B)$'$ and the Jensen integral inequality together imply ($\tau>0$):

$$\begin{aligned}\phi(F(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]}) &= \phi(F(t)t\cdot t^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]})\\ &\leqslant \varphi(F(t))t^{-1}[\phi(\|f(\cdot+\tau)-f(\cdot)\|)]_{L^1[0,t]}.\end{aligned}$$

This simply shows that $f(\cdot)$ is Doss $(1,\phi,F)_2$-almost periodic. □

Remark 3.2.17.
(i) It is clear that, if $f(\cdot)$ is Doss (p,ϕ,F)-almost periodic [Doss (p,ϕ,F)-uniformly recurrent], resp. Doss $(p,\phi,F)_1$-almost periodic [Doss $(p,\phi,F)_1$-uniformly recurrent], and $F(t)\geqslant F_1(t)$ for every $t\in I$, then $f(\cdot)$ is Doss (p,ϕ,F_1)-almost periodic [Doss (p,ϕ,F_1)–uniformly recurrent], resp. Doss $(p,\phi,F_1)_1$-almost periodic [Doss $(p,\phi,F_1)_1$-uniformly recurrent]. Furthermore, if $f(\cdot)$ is Doss $(p,\phi,F)_2$-almost periodic [Doss $(p,\phi,F)_2$-uniformly recurrent], then $f(\cdot)$ is Doss $(p,\phi,F_1)_2$-almost periodic [Doss $(p,\phi,F_1)_2$-uniformly recurrent] provided that $F(t)\geqslant F_1(t)$ for every $t\in I$ and $\phi(\cdot)$ is monotonically increasing, or $F(t)\leqslant F_1(t)$ for every $t\in I$ and $\phi(\cdot)$ is monotonically decreasing.
(ii) If $f(\cdot)$ is Doss (p,ϕ,F)-almost periodic [Doss (p,ϕ,F)-uniformly recurrent], resp. Doss $(p,\phi,F)_i$-almost periodic [Doss $(p,\phi,F)_i$-uniformly recurrent], $\phi_1(\cdot)$ is measurable and $0\leqslant\phi_1\leqslant\phi$, then Lemma 1.1.7(iii) shows that $f(\cdot)$ is Doss (p,ϕ_1,F)-almost periodic [Doss (p,ϕ_1,F)-uniformly recurrent], resp. Doss $(p,\phi_1,F)_i$-almost periodic [Doss $(p,\phi_1,F)_i$-uniformly recurrent], where $i=1,2$.

Example 3.2.18.
(i) Let $p(x)\equiv p\in[1,\infty)$ and $f(x):=\chi_{[0,1/2]}(x)$, $x\in\mathbb{R}$. Then it can be simply shown that for each real number τ such that $|\tau|>1$ we have

$$\int_{-t}^{t}|f(x+\tau)-f(x)|^p\,dx\leqslant\frac{1}{2}+2\int_{0}^{1/2}|f(x)|^p\,dx,\quad t\in\mathbb{R}.$$

This implies that $f(\cdot)$ is Doss $(p, x, t^{-\sigma})$-almost periodic for each real number $\sigma > 0$.

(ii) Let $p(x) \equiv p \in [1, \infty)$ and $f(x) := \chi_{[0,\infty)}(x)$, $x \in \mathbb{R}$. Then it can be simply shown that for each real number τ we have

$$\int_{-t}^{t} |f(x+\tau) - f(x)|^p \, dx = \int_{-t+\tau}^{\tau} |f(x)|^p \, dx + \int_{\tau}^{t+\tau} |f(x)|^p \, dx, \quad t \in \mathbb{R}.$$

Hence,

$$\int_{-t}^{t} |f(x+\tau) - f(x)|^p \, dx \leqslant 2|\tau|, \quad \text{provided } \tau \in \mathbb{R},\ t \geqslant |\tau|,$$

and $f(\cdot)$ is Doss $(p, x, t^{-\sigma})$-almost periodic for each real number $\sigma > 0$.

Concerning embeddings between different Doss almost periodic type spaces with variable exponent, we would like to state the following result.

Proposition 3.2.19. *Let $p,\ q \in \mathcal{P}(I)$ and let $1 \leqslant q(x) \leqslant p(x)$ for a. e. $x \in I$.*

(i) *Suppose that a function $f(\cdot)$ is Doss (p, ϕ, F)-almost periodic, resp. Doss (p, ϕ, F)-uniformly recurrent, and $F_1(t) := F(t)/t$, $t > 0$. Then $f(\cdot)$ is Doss (q, ϕ, F_1)-almost periodic, resp. Doss (q, ϕ, F_1)-uniformly recurrent.*

(ii) *Suppose that a function $f(\cdot)$ is Doss $(p, \phi, F)_1$-almost periodic, resp. Doss $(p, \phi, F)_1$-uniformly recurrent, $\phi(\cdot)$ is monotonically increasing, there exists a function $\varphi : [0, \infty) \to (0, \infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$, $x,\ y \geqslant 0$ and $F_1(t) := F(t)/\varphi(2(1+2t))$, $t > 0$ provided $I = \mathbb{R}$, resp. $F_1(t) := F(t)/\varphi(2(1+t))$, $t > 0$ provided $I = [0, \infty)$. Then $f(\cdot)$ is Doss $(q, \phi, F_1)_1$-almost periodic, resp. Doss $(q, \phi, F_1)_1$-uniformly recurrent.*

(iii) *Suppose that a function $f(\cdot)$ is Doss $(p, \phi, F)_2$-almost periodic, resp. Doss $(p, \phi, F)_2$-uniformly recurrent, there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$, $x,\ y \geqslant 0$ and*

$$\varphi\left(\frac{2F_1(\cdot)(1+2\cdot)}{F(\cdot)}\right) \in L^{\infty}((0,\infty)), \quad \textit{if } I = \mathbb{R},$$
$$\textit{resp. } \varphi\left(\frac{2F_1(\cdot)(1+\cdot)}{F(\cdot)}\right) \in L^{\infty}((0,\infty)), \quad \textit{if } I = [0,\infty).$$

Then $f(\cdot)$ is Doss $(q, \phi, F_1)_2$-almost periodic, resp. Doss $(q, \phi, F_1)_2$-uniformly recurrent.

Proof. We will prove only (iii), for the class of Doss $(p, \phi, F)_2$-almost periodic functions defined on the interval $I = [0, \infty)$. Let the numbers $t,\ \tau > 0$ be given. Then the conclusion simply follows from the calculation

$$\begin{aligned}\phi\big(F_1(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{q(x)}[0,t]}\big) &\leqslant \phi\big(2F_1(t)(1+t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(x)}[0,t]}\big)\\ &= \phi\bigg(\frac{2F_1(t)(1+t)}{F(t)}F(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(x)}[0,t]}\bigg)\\ &\leqslant \varphi\bigg(\frac{2F_1(t)(1+t)}{F(t)}\bigg)\phi\big(F(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(x)}[0,t]}\big),\end{aligned}$$

where we have used Lemma 1.1.7(ii), and the corresponding definition of Doss $(q,\phi,F_1)_2$-almost periodicity. □

3.2.3 Invariance of generalized Doss almost periodicity with variable exponent under the actions of convolution products

In this subsection, we will investigate the invariance of three types of generalized Doss almost periodicity introduced above under the actions of infinite convolution products (for the sake of simplicity, we will not consider here the finite convolution products).

In [644, Theorem 2.1], we have analyzed the invariance of Doss p-almost periodicity under the actions of infinite convolution products, provided that the function $f(\cdot)$ in (2.46) is Stepanov p-bounded ($1 \leqslant p < \infty$). In the formulation of the subsequent result, which is not satisfactory to a certain extent (let us only note that the above-mentioned theorem from [644], which is a unique result in the existing literature concerning this problematic, cannot be deduced from Theorem 3.2.20), we will not use this condition.

Theorem 3.2.20. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all x, $y \geqslant 0$ and $p \in \mathcal{P}(\mathbb{R})$. Suppose, further, $\check{f} : \mathbb{R} \to X$ is Doss (p,ϕ,F)-almost periodic, resp. Doss (p,ϕ,F)-uniformly recurrent, and measurable, $F_1 : (0,\infty) \to (0,\infty)$, $q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and for every real number $x \in \mathbb{R}$ we have (3.15) with the function $\check{g}(\cdot)$ replaced therein with the function $\check{g}(\cdot)$. Suppose that for each $\varepsilon > 0$ there exist an increasing sequence (a_m) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ satisfying the requirement that, for every $t \geqslant t_0(\varepsilon)$, we have*

$$\int_{-t}^{t} \varphi_{p(x)}\Big(2\varphi(a_m)a_m^{-1}F_1(t)\limsup_{m\to+\infty}\big[\big[\varphi(\|R(\cdot+x)\|)\big]_{L^{q(\cdot)}[-x,-x+a_m]}F(t+a_m)^{-1}\big]\Big)\,dx \leqslant 1. \quad (3.48)$$

Then the function $F : \mathbb{R} \to Y$, given by (2.46), is well defined and Doss (p,ϕ,F_1)-almost periodic, resp. Doss (p,ϕ,F_1)-uniformly recurrent.

Proof. We will consider only the class of Doss (p,ϕ,F)-almost periodic functions because the proof for the class of Doss (p,ϕ,F)-uniformly recurrent functions can be deduced quite analogously. Since $F(x) = \int_{-x}^{\infty} R(v+x)\check{f}(v)\,dv$, $x \in \mathbb{R}$, the validity of condition (3.15) shows that the function $F(\cdot)$ is well defined and that the integrals in definitions of $F(x)$ and $F(x+\tau)-F(x)$ converge absolutely ($x \in \mathbb{R}$). Let $\varepsilon > 0$ be fixed, and let the sequences (t_n), (t_n') and (a_m) satisfy the prescribed requirements. Using the facts that the function $\phi(\cdot)$ is continuous and the function $\varphi_{p(x)}(\cdot)$ is monotonically increasing, we have ($x \in \mathbb{R}$, λ, $\tau > 0$):

$$\begin{aligned}
&\varphi_{p(x)}\left(\frac{\phi(\|F(x+\tau)-F(x)\|)}{\lambda}\right)\\
&\leqslant \varphi_{p(x)}\left(\frac{\phi(\int_{-x}^{\infty}\|R(v+x)\|\|\check{f}(v+\tau)-\check{f}(v)\|\,dv)}{\lambda}\right)\\
&\leqslant \varphi_{p(x)}\left(\lim_{m\to+\infty}\frac{\phi(\int_{-x}^{-x+a_m}\|R(v+x)\|\|\check{f}(v+\tau)-\check{f}(v)\|\,dv)}{\lambda}\right)\\
&= \varphi_{p(x)}\left(\lim_{m\to+\infty}\frac{\phi(\int_{-x}^{-x+a_m}a_m a_m^{-1}\|R(v+x)\|\|\check{f}(v+\tau)-\check{f}(v)\|\,dv)}{\lambda}\right)\\
&\leqslant \varphi_{p(x)}\left(\limsup_{m\to+\infty}\frac{\varphi(a_m)a_m^{-1}\int_{-x}^{-x+a_m}\phi(\|R(v+x)\|\|\check{f}(v+\tau)-\check{f}(v)\|)\,dv}{\lambda}\right)\\
&\leqslant \varphi_{p(x)}\left(\limsup_{m\to+\infty}\frac{2\varphi(a_m)a_m^{-1}[\varphi(\|R(v+x)\|)]_{L^{q(v)}[-x,-x+a_m]}}{\lambda}\right.\\
&\qquad\left.\times\frac{[\phi(\|\check{f}(v+\tau)-\check{f}(v)\|)]_{L^{p(v)}[-x,-x+a_m]}}{\lambda}\right),
\end{aligned}$$

where we have also used the Jensen integral inequality and the Hölder inequality. Let $\varepsilon > 0$ be fixed and let $\tau > 0$ be such that (3.44) holds, i. e., there exists $t_1(\varepsilon,\tau) \geqslant 0$ such that

$$[F(t)[\phi(\|\check{f}(\cdot+\tau)-\check{f}(\cdot)\|)_{L^{p(x)}[-t,t]}]] < \varepsilon, \quad t \geqslant t_1(\varepsilon,\tau). \tag{3.49}$$

Suppose that $t \geqslant \max(t_0(\varepsilon), t_1(\varepsilon,\tau))$. Then for each $x \in [-t,t]$ and $m \in \mathbb{N}$ we have $[-x,-x+a_m] \subseteq [-(t+a_m), t+a_m]$ so that the above calculation and (3.49) give

$$\begin{aligned}
&\varphi_{p(x)}\left(\frac{\phi(\|F(x+\tau)-F(x)\|)}{\lambda}\right)\\
&\leqslant \varphi_{p(x)}\left(\limsup_{m\to+\infty}\frac{2\varphi(a_m)a_m^{-1}[\varphi(\|R(v+x)\|)]_{L^{q(v)}[-x,-x+a_m]}\varepsilon/F(t+a_m)}{\lambda}\right).
\end{aligned}$$

Integrating this estimate over the interval $[-t,t]$ and using (3.48) we see that the inequality

$$\int_{-t}^{t}\varphi_{p(x)}\left(\frac{\phi(\|F(x+\tau)-F(x)\|)}{\lambda}\right)dx \leqslant 1$$

holds with $\lambda = \varepsilon/F_1(t)$, which completes the proof in a routine manner. □

We can similarly prove the following results for Doss $(p,\phi,F)_1$-almost periodic functions, resp. Doss $(p,\phi,F)_1$-uniformly recurrent functions, and Doss $(p,\phi,F)_2$-almost periodic functions, resp. Doss $(p,\phi,F)_2$-uniformly recurrent functions; for the sake of brevity, we will only provide descriptions of the proofs since they are very similar to the proof of Theorem 3.2.20.

Theorem 3.2.21. *Suppose that $\phi : [0,\infty) \to [0,\infty)$ is a continuous monotonically increasing bijection and $p \in \mathcal{P}(\mathbb{R})$. Suppose, further, $\check{f} : \mathbb{R} \to X$ is Doss $(p,\phi,F)_1$-almost periodic, resp. Doss $(p,\phi,F)_1$-uniformly recurrent, and measurable, $F_1 : (0,\infty) \to (0,\infty)$, $q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and, for every real number $x \in \mathbb{R}$, we have (3.15). Suppose that for each $\varepsilon > 0$ there exist an increasing sequence (a_m) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ satisfying the requirement that, for every $t \geqslant t_0(\varepsilon)$, we have*

$$\int_{-t}^{t} \varphi_{p(x)}\left(\frac{\limsup_{m\to+\infty}[2[\varphi(\|R(\cdot+x)\|)]_{L^{q(\cdot)}[-x,-x+a_m]}\phi^{-1}(\varepsilon/F(t+a_m))]}{\phi^{-1}(\varepsilon/F_1(t))}\right)dx \leqslant 1. \quad (3.50)$$

Then the function $F : \mathbb{R} \to Y$, given by (2.46), is well defined and Doss $(p,\phi,F_1)_1$-almost periodic, resp. Doss $(p,\phi,F_1)_1$-uniformly recurrent.

Proof. As in the proof of Theorem 3.2.20 above, the function $F(\cdot)$ is well defined and the integrals in definitions of $F(x)$ and $F(x+\tau) - F(x)$ converge absolutely ($x \in \mathbb{R}$). Let $\varepsilon > 0$ be fixed. Then it suffices to show that, for every $t \geqslant t_0(\varepsilon)$, we have ($x \in \mathbb{R}$, λ, $\tau > 0$)

$$\|R(s)[F(x+t+\tau-s) - F(x+t-s)]\|_{L^{p(x)}[-t,t]} \leqslant \phi^{-1}(\varepsilon/F_1(t)).$$

But we can repeat the arguments used in the proof of the above-mentioned theorem, with $\phi(x) \equiv x$, in order to see that

$$\begin{aligned}
&\varphi_{p(x)}\left(\frac{\|F(x+\tau)-F(x)\|}{\lambda}\right)\\
&\quad\leqslant \varphi_{p(x)}\left(\frac{2[\varphi(\|R(v+x)\|)]_{L^{q(v)}[-x,-x+a_m]}[\|\check{f}(v+\tau)-\check{f}(v)\|]_{L^{p(v)}[-x,-x+a_m]}}{\lambda}\right)\\
&\quad\leqslant \varphi_{p(x)}\left(\frac{2[\varphi(\|R(v+x)\|)]_{L^{q(v)}[-x,-x+a_m]}\phi^{-1}(\varepsilon/F(t+a_m))}{\lambda}\right).
\end{aligned}$$

The rest of the proof is clear because we can take $\lambda = \phi^{-1}(\varepsilon/F_1(t))$ and use condition (3.50). □

Theorem 3.2.22. *Suppose that $\phi : [0,\infty) \to [0,\infty)$ is a continuous monotonically increasing bijection and $p \in \mathcal{P}(\mathbb{R})$. Suppose, further, $\check{f} : \mathbb{R} \to X$ is Doss $(p,\phi,F)_2$-almost periodic, resp. Doss $(p,\phi,F)_2$-uniformly recurrent, and measurable, $F_1 : (0,\infty) \to$*

$(0,\infty)$, *$q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and, for every real number $x \in \mathbb{R}$, we have* (3.15). *Suppose that for each $\varepsilon > 0$ there exist an increasing sequence (a_m) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ satisfying the requirement that, for every $t \geqslant t_0(\varepsilon)$, we have*

$$\int_{-t}^{t} \varphi_{p(x)}\left(2F_1(t) \limsup_{m\to+\infty} \frac{[\varphi(\|R(\cdot + x)\|)]_{L^{q(\cdot)}[-x,-x+a_m]}}{F(t+a_m)}\right) dx \leqslant 1. \tag{3.51}$$

Then the function $F : \mathbb{R} \to Y$, given by (2.46)*, is well defined and Doss $(p,\phi,F_1)_2$-almost periodic, resp. Doss $(p,\phi,F_1)_2$-uniformly recurrent.*

Proof. We can use the same trick as above, with $\lambda = \phi^{-1}(\varepsilon)/F_1(t)$. □

Remark 3.2.23.
(i) Suppose that $p(x) \equiv p \in [1,\infty)$. Then we can use the usual Hölder inequality in order to see that the estimates (3.48)–(3.51) can be modified by removing the multiplication with the number 2 therein.
(ii) Although we will not define the notion of Besicovitch–Doss almost periodicity with variable exponent here, we would like to note that the statement of [631, Theorem 2.13.7] and the corresponding part of this result which considers the Doss almost periodicity cannot be so easily reexamined in our framework.

Concerning the convolution invariance of generalized almost periodicity introduced in this subsection, we will clarify the following result (see also [631, Theorem 3.11.26]).

Proposition 3.2.24. *Suppose that $\psi \in L^1(\mathbb{R})$, $-\infty < a < b < +\infty$, $\operatorname{supp}(\psi) \subseteq [a,b]$, $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$, $p,\ q \in \mathcal{P}(\mathbb{R})$ and $1/p(x) + 1/q(x) = 1$. Suppose, further, that the function $f : \mathbb{R} \to X$ is Doss (p,ϕ,F)-almost periodic, resp. Doss (p,ϕ,F)-uniformly recurrent, and essentially bounded. Then the function*

$$x \mapsto (\psi * f)(x) := \int_{-\infty}^{+\infty} \psi(x-y) f(y)\, dy, \quad x \in \mathbb{R}$$

is well defined and essentially bounded. Furthermore, if $p_1 \in \mathcal{P}(\mathbb{R})$, $F_1 : (0,\infty) \to (0,\infty)$ and if, for every $\varepsilon > 0$ there exists a positive real number $t_1(\varepsilon) > 0$ such that

$$\int_{-t}^{t} \varphi_{p_1(x)}\left(2F_1(t)\varphi(b-a) \frac{\|\varphi(|\psi(x-z)|)\|_{L^{q(z)}[x-b,x-a]}}{(b-a)F(t+c)}\right) dx \leqslant 1,$$

where $c = \max(|a|,|b|)$, then the function $\psi * f(\cdot)$ is Doss (p_1,ϕ,F_1)-almost periodic, resp. Doss (p_1,ϕ,F_1)-uniformly recurrent.

Proof. We will give the main details of the proof for the class of Doss (p,ϕ,F)-almost periodic functions, only. For every $x \in \mathbb{R}$ and $\tau \in \mathbb{R}$, we have

$$\begin{aligned}
&\phi(\|(\psi * f)(x+\tau) - (\psi * f)(x)\|) \\
&\leqslant \phi\left((b-a)(b-a)^{-1} \int_a^b |\psi(y)| \cdot \|f(x+\tau-y) - f(x-y)\| \, dy \right) \\
&\leqslant \frac{\varphi(b-a)}{b-a} \int_a^b \phi(|\psi(y)| \cdot \|f(x+\tau-y) - f(x-y)\|) \, dy \\
&= \frac{\varphi(b-a)}{b-a} \int_{x-b}^{x-a} \phi(|\psi(x-z)| \cdot \|f(z+\tau) - f(z)\|) \, dz \\
&\leqslant 2\frac{\varphi(b-a)}{b-a} \|\varphi(|\psi(x-z)|)\|_{L^{q(z)}[x-b,x-a]} \|f(z+\tau) - f(z)\|_{L^{p(z)}[x-b,x-a]},
\end{aligned}$$

where we have used the Jensen integral inequality and the Hölder inequality. The proof can be completed as it has been done in the final part of the proof of Theorem 3.2.20. □

Composition principles for Besicovitch almost periodic functions have been investigated by M. Ayachi and J. Blot in [101]. An issue that will not be addressed in this study is composition principles for Doss almost periodic functions with variable exponents.

Fix now a strictly increasing sequence (α_n) of positive reals tending to plus infinity, and set

$$\mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X) := \{\vec{u} \in \mathrm{UR}(\mathbb{R} : X); \vec{u}(\cdot) \text{ is bounded and } (2.3) \text{ holds with } f = \vec{u}\}.$$

Equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_\infty$, $\mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X)$ is a complete metric space. We have the following.

Theorem 3.2.25. *Suppose that the function $F : \mathbb{R} \times X \to X$ satisfies the requirement that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\in\mathbb{R}} \sup_{y\in B} \|F(t,y)\| \leqslant M_B$. Suppose, further, that $p,\ r \in \mathcal{P}([0,1])$, the function $F : \mathbb{R}\times X \to X$ is Stepanov $p(x)$-uniformly recurrent, $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/(p(\cdot)-1))$ and there exists a function $L_F \in L_S^{r(x)}(I)$ is such that $q(x) := p(x)r(x)/(p(x)+r(x)) > 1$ for a. e. $x \in \mathbb{R}$ and (2.20) holds with $I = \mathbb{R}$. If there exist a positive real number $q' > 0$ and an integer $n \in \mathbb{N}$ such that $(\gamma-1)q' > -1$ and $q(x)/(q(x)-1) \leqslant q'$ for a. e. $x \in \mathbb{R}$, and $M_n < 1$, where*

$$\begin{aligned}
M_n := \sup_{t\geqslant 0} \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \|R_\gamma(t-x_n)\| \\
\times \prod_{i=2}^{n} \|R_\gamma(x_i - x_{i-1})\| \prod_{i=1}^{n} L_F(x_i) \, dx_1 \, dx_2 \cdots \, dx_n,
\end{aligned}$$

and for every compact set $K \subseteq Y$, (3.43) *holds, then the abstract fractional Cauchy inclusion* (2.57) *has a unique bounded uniformly recurrent solution.*

Proof. We will only content ourselves with sketching it. Define $\Upsilon : \mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X) \to \mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X)$ by

$$(\Upsilon\vec{u})(t) := \int_{-\infty}^{t} R_\gamma(t-s)F(s,\vec{u}(s))\,ds, \quad t \in \mathbb{R}.$$

Suppose that $\vec{u} \in \mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X)$. Then $R(\vec{u}) = B$ is a bounded set, so that the mapping $t \mapsto F(t,\vec{u}(t))$, $t \in \mathbb{R}$ is bounded. Applying Theorem 3.2.8, we see that the function $F(\cdot,\vec{u}(\cdot))$ is Stepanov $q(x)$-uniformly recurrent. Define $q'(x) := q(x)/(q(x)-1)$ for a. e. $x \in \mathbb{R}$. Then (2.56) and the prescribed assumptions imply that $\|R_\gamma(\cdot)\| \in L^{q'(x)}[0,1]$ and $\sum_{k=0}^{\infty}\|R_\gamma(\cdot)\|_{L^{q'(x)}[k,k+1]} < \infty$. Applying Proposition 3.2.5, we see that the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)F(s,\vec{u}(s))\,ds$, $t \in \mathbb{R}$ is uniformly recurrent. It can be simply proved that this function is also bounded continuous so that $\Upsilon\vec{u} \in \mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X)$ and the mapping $\Upsilon(\cdot)$ is well defined. A simple calculation shows that

$$\|(\Upsilon^n\vec{u}_1) - (\Upsilon^n\vec{u}_2)\|_\infty \leqslant M_n\|\vec{u}_1 - \vec{u}_2\|_\infty, \quad \vec{u}_1,\ \vec{u}_2 \in \mathrm{BUR}_{(\alpha_n)}(\mathbb{R} : X),\ n \in \mathbb{N}.$$

Since we have assumed that $M_n < 1$, the Bryant fixed point theorem shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point. This completes the proof of theorem. □

3.3 Generalized almost periodicity in Lebesgue spaces with variable exponents. Part III

In this section, we consider the Stepanov and Weyl classes of generalized almost periodic type functions and generalized uniformly recurrent type functions. We investigate the invariance of generalized almost periodicity and generalized uniform recurrence with variable exponents under the actions of convolution products, providing also certain applications.

3.3.1 Generalized Weyl uniform recurrence in Lebesgue spaces with variable exponents $L^{p(x)}$

Throughout this subsection, we will occasionally use conditions (A) and (B). We will first extend the notion introduced in Definition 3.1.23–Definition 3.1.25.

Definition 3.3.1. Suppose that condition (A) holds, $f : I \to X$, and $\phi(\|f(\cdot+\tau) - f(\cdot)\|) \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $(p(x),\phi,F)$-uniformly recurrent, $f \in e-W_{ur}^{(p(x),\phi,F)}(I:X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\sup_{t\in I}\big[F(l_n,t)\big[\phi\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l_n]}\big]\big]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $(p(x),\phi,F)$-uniformly recurrent, $f \in W_{ur}^{(p(x),\phi,F)}(I:X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\limsup_{l\to\infty}\sup_{t\in I}\big[F(l,t)\big[\phi\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big]=0. \tag{3.52}$$

Definition 3.3.2. Suppose that condition (A) holds, $f:I\to X$ and $\|f(\cdot+\tau)-f(\cdot)\|\in L^{p(x)}(K)$ for any $\tau\in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $(p(x),\phi,F)_1$-uniformly recurrent, $f \in e-W_{ur}^{(p(x),\phi,F)_1}(I:X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\sup_{t\in I}\big[F(l_n,t)\phi\big[\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l_n]}\big]\big]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $(p(x),\phi,F)_1$-uniformly recurrent, $f \in W_{ur}^{(p(x),\phi,F)_1}(I:X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\limsup_{l\to\infty}\sup_{t\in I}\big[F(l,t)\phi\big[\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big]=0.$$

Definition 3.3.3. Suppose that condition (A) holds, $f:I\to X$ and $\|f(\cdot+\tau)-f(\cdot)\|\in L^{p(x)}(K)$ for any $\tau\in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $(p(x),\phi,F)_2$-uniformly recurrent, $f \in e-W_{ur}^{(p(x),\phi,F)_2}(I:X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\sup_{t\in I}\phi\big[F(l_n,t)\big[\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l_n]}\big]\big]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $(p(x),\phi,F)_2$-uniformly recurrent, $f \in W_{ur}^{(p(x),\phi,F)_2}(I:X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\limsup_{l\to\infty}\sup_{t\in I}\phi\big[F(l,t)\big[\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+l]}\big]\big]=0.$$

It is clear that the class of (equi-)Weyl $(p(x),\phi,F)$-uniformly recurrent functions, resp. (equi-)Weyl $(p(x),\phi,F)_i$-uniformly recurrent functions, extends the class of (equi-)Weyl $(p(x), \phi, F)$-almost periodic functions, resp. (equi-)Weyl $(p(x), \phi, F)_i$-almost periodic functions ($i = 1, 2$). The case that $p(x) \equiv p$, $\phi(x) \equiv x$ and $F(l, t) = l^{(-1)/p}$ is the most indicative, when we say that the function $f(\cdot)$ is (equi-)Weyl p-uniformly recurrent. The class of (equi-)Weyl p-uniformly recurrent functions has not been considered elsewhere by now.

We have already shown that an equi-Weyl (p, ϕ, ψ)-almost periodic function, resp. equi-Weyl $(p, \phi, \psi)_i$-almost periodic function, does not need to be Weyl (p, ϕ, ψ)-almost periodic, resp. Weyl $(p, \phi, \psi)_i$-almost periodic ($i = 1, 2$). This statement continues to hold for the generalized uniformly recurrent functions introduced above. For example, any continuous Stepanov p-almost periodic function $f(\cdot)$ which is not periodic cannot be Weyl $(p, x, 1)$-uniformly recurrent, while it is always equi-Weyl $(p, x, 1)$-almost periodic.

Example 3.3.4. If X does not contain an isomorphic copy of the sequence space c_0, $\phi(x) = x$ and $F(l, t) \equiv F(t)$, where $\lim_{t\to+\infty} F(t) = +\infty$, then there do not exist a non-periodic trigonometric polynomial $f(\cdot)$ and a function $p \in \mathcal{P}(\mathbb{R})$ such that $f \in e - W_{ur}^{(p,x,F)}(\mathbb{R} : X)$. This can be verified based on the argumentation contained in Example 3.1.26(iii).

Furthermore, the statement of Proposition 3.1.27 and the conclusions established in Remark 3.1.28 can be reformulated for the introduced classes of generalized Weyl uniformly recurrent functions. In order to ensure the translation invariance of generalized Weyl spaces of uniformly recurrent functions with variable exponents, we will follow a slightly different approach based on the already exhibited idea from [372].

Definition 3.3.5. Suppose that condition (B) holds, $f : I \to X$ and $\phi(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|) \in L^{p(x)}([0, 1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $[p(x), \phi, F]$-uniformly recurrent, $f \in e - W_{ur}^{[p(x),\phi,F]}(I : X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \sup_{t\in I} \big[F(l_n, t)\big[\phi\big(\big\|f(\cdot l_n + t + \alpha_n) - f(t + \cdot l_n)\big\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] = 0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $[p(x), \phi, F]$-uniformly recurrent, $f \in W_{ur}^{[p(x),\phi,F]}(I : X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \limsup_{l\to\infty} \sup_{t\in I} \big[F(l, t)\big[\phi\big(\big\|f(\cdot l + t + \alpha_n) - f(t + \cdot l)\big\|\big)_{L^{p(\cdot)}[0,1]}\big]\big] = 0.$$

Definition 3.3.6. Suppose that condition (B) holds, $f : I \to X$ and $\|f(\cdot l + t + \tau) - f(t + \cdot l)\| \in L^{p(x)}([0, 1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $[p(x), \phi, F]_1$-uniformly recurrent, $f \in e - W_{ur}^{[p(x),\phi,F]_1}(I : X)$ for short, if and only if we can find two sequences (l_n) and

(α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\sup_{t\in I}\big[F(l_n,t)\phi\big[\big(\|f(\cdot l_n+t+\alpha_n)-f(t+\cdot l_n)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $[p(x),\phi,F]_2$-uniformly recurrent, $f \in W_{ur}^{[p(x),\phi,F]_2}(I : X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\limsup_{l\to\infty}\sup_{t\in I}\big[F(l,t)\phi\big[\big(\|f(\cdot l+t+\alpha_n)-f(t+\cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big]=0.$$

Definition 3.3.7. Suppose that condition (B) holds, $f : I \to X$ and $\|f(\cdot l+t+\tau)-f(t+\cdot l)\| \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl $[p(x),\phi,F]_2$-uniformly recurrent, $f \in e-W_{ur}^{[p(x),\phi,F]_2}(I : X)$ for short, if and only we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\sup_{t\in I}\phi\big[F(l_n,t)\big[\big(\|f(\cdot l_n+t+\alpha_n)-f(t+\cdot l_n)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl $[p(x),\phi,F]_2$-uniformly recurrent, $f \in W_{ur}^{[p(x),\phi,F]_2}(I : X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n = +\infty$ and

$$\lim_{n\to+\infty}\limsup_{l\to\infty}\sup_{t\in I}\phi\big[F(l,t)\big[\big(\|f(\cdot l+t+\alpha_n)-f(t+\cdot l)\|\big)_{L^{p(\cdot)}[0,1]}\big]\big]=0.$$

The statement of Proposition 3.1.33 and the conclusions established in Remark 3.1.32 can be reformulated for the generalized Weyl uniformly recurrent functions introduced in the above three definitions. All statements regarding the convolution invariance of the generalized Weyl almost periodicity with variable exponents can be straightforwardly reformulated for the generalized Weyl uniformly recurrent functions introduced in this section; we leave it to the reader to make this precise.

3.3.2 Quasi-asymptotically uniformly recurrent type functions with variable exponents

In the following definition, we will extend the notion of quasi-asymptotical almost periodicity.

Definition 3.3.8. We say that a continuous function $f : I \to X$ is quasi-asymptotically uniformly recurrent if and only if there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty}\sup_{|t|\geqslant M_n}\|f(t+\alpha_n)-f(t)\|=0. \tag{3.53}$$

Denote by $Q - \mathrm{AUR}(I : X)$ the set consisting of all quasi-asymptotically uniformly recurrent functions from I into X.

It is expected that the class of quasi-asymptotically uniformly recurrent functions extends the class of asymptotically uniformly recurrent functions. For completeness, we will include all details of the proof of the following proposition.

Proposition 3.3.9. *Suppose that $f : I \to X$ is asymptotically uniformly recurrent. Then $f(\cdot)$ is quasi-asymptotically uniformly recurrent.*

Proof. Let $h \in \mathrm{UR}(\mathbb{R} : X)$, $q \in C_0(I : X)$ and $f = h + q$. By our assumption, we have the existence of a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (2.3) holds with the function $f(\cdot)$ replaced therein with the function $h(\cdot)$. Let $n \in \mathbb{N}$ be fixed. Then we can find a sufficiently large real number $M_n' > 0$ and a sufficiently large integer $n_0 \in \mathbb{N}$ such that $\|q(t)\| \leqslant 1/n$ for $|t| \geqslant M_n'$ and $\|h(t + \alpha_n) - h(t)\| \leqslant 1/n$, $n \geqslant n_0$. Then, for every $t \in \mathbb{R}$ such that $|t| \geqslant M_n := M_n' + \alpha_n$, we have $|t|$, $|t + \alpha_n| \geqslant M_n'$ and

$$\big\|\big[h(t + \alpha_n) - h(t)\big] + \big[q(t + \alpha_n) - q(t)\big]\big\| \leqslant \frac{1}{n} + \big\|q(t + \alpha_n) - q(t)\big\| \leqslant \frac{2}{n}, \ n \geqslant n_0.$$

This simply implies the required assertion. □

Applying the same arguments, we can deduce the following.

Proposition 3.3.10. *Suppose that $f : I \to X$ is quasi-asymptotically uniformly recurrent and $q \in C_0(I : X)$. Then $(f + q)(\cdot)$ is likewise quasi-asymptotically uniformly recurrent.*

The proof of following proposition is simple and also can be omitted.

Proposition 3.3.11. *Suppose that $I = \mathbb{R}$ and $f : I \to X$. Then $f(\cdot)$ is quasi-asymptotically uniformly recurrent (quasi-asymptotically almost periodic, S-asymptotically ω-periodic) if and only if $\check{f}(\cdot)$ is quasi-asymptotically uniformly recurrent (quasi-asymptotically almost periodic, S-asymptotically ω-periodic).*

If $f \in Q - \mathrm{AUR}(\mathbb{R} : X)$ and $\varphi \in L^1(\mathbb{R})$ has a compact support, then it can be easily shown that the convolution $\varphi * f(\cdot) := \int_{\mathbb{R}} \varphi(\cdot - y) f(y)\, dy$ belongs to the class $Q - \mathrm{AUR}(\mathbb{R} : X)$. Furthermore, any quasi-asymptotically almost periodic function is bounded by definition, and this is no longer true for quasi-asymptotically uniformly recurrent functions. In connection with this, we would like to present the following illustrative example.

Example 3.3.12. Let the function $f(\cdot)$ be defined by (2.28). We know that for each real number $c > 0$ the function $h(t) := \min(c, f(t))$, $t \in \mathbb{R}$ is bounded uniformly continuous, uniformly recurrent, and not (Stepanov) p-quasi-asymptotically almost periodic ($p \geqslant 1$). On the other hand, Proposition 3.3.9 shows that the function $h(\cdot)$ is quasi-asymptotically uniformly recurrent.

Furthermore, if $f \in C^1(I : X)$ and $f' \in C_0(I : X)$, then the Lagrange mean value theorem implies that the function $f(\cdot)$ is quasi-asymptotically uniformly recurrent. In particular, the function $f(t) := \ln(1+t)$, $t \geqslant 0$, is quasi-asymptotically uniformly recurrent; on the other hand, it can be simply verified that the function $f(\cdot)$ is not asymptotically uniformly recurrent. The notion of quasi-asymptotical uniform anti-recurrence can be also introduced and analyzed (see also [647, Example 2.3, Remark 2.4]).

Example 3.3.13. The function $f : [0, \infty) \to \mathbb{R}$ given by $f(t) := \sin(\ln(1 + t))$, $t \geqslant 0$, is quasi-asymptotically almost periodic but not asymptotically almost periodic (see [647] and [882, Example 4.1, Theorem 4.2]). Now we will prove that this function cannot be asymptotically uniformly recurrent. Suppose the contrary, and fix a sufficiently small number $\varepsilon > 0$. Then an elementary argumentation shows that there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ such that $|\sin(\ln(t + \alpha_n)) - \sin(\ln t)| \leqslant 2\varepsilon$ for all $t \geqslant t_0(\varepsilon)$ and $n \in \mathbb{N}$. Hence,

$$\left|\sin \frac{\ln(1 + (\alpha_n/t))}{2} \cos \frac{\ln(t(t + \alpha_n))}{2}\right| \leqslant \varepsilon, \quad t \geqslant t_0(\varepsilon),\ n \in \mathbb{N}.$$

Let $n_0(\varepsilon) \in \mathbb{N}$ be such that $\alpha_n \geqslant t_0(\varepsilon)$ for $n \geqslant n_0(\varepsilon)$. Plugging $t = k\alpha_n$, where $1 \leqslant k \leqslant 5$, the above estimate simply implies that there exists a finite constant $c > 0$ such that

$$\left|\cos \frac{\ln(a\alpha_n^2)}{2}\right| \leqslant c\varepsilon, \quad 2 \leqslant a \leqslant 30,\ n \geqslant n_0(\varepsilon).$$

Then we get the existence of a real number $c_\varepsilon > 0$ such that $\lim_{\varepsilon \to 0+} c_\varepsilon = 0$ and

$$\operatorname{dist}\big(a\alpha_n^2, \{\exp((2k + 1)\pi) : k \in \mathbb{N}_0\}\big) \leqslant e^{2c_\varepsilon}, \quad 2 \leqslant a \leqslant 30,\ n \geqslant n_0(\varepsilon).$$

It can be simply proved that this estimate cannot be satisfied simultaneously for $a = 2$ and $a = e^\pi$, which yields a contradiction.

In [647, Theorem 2.5], we have proved that any asymptotically almost automorphic function which is also quasi-asymptotically almost periodic is always asymptotically almost periodic. The arguments contained in the proof of the above-mentioned theorem also show that any asymptotically uniformly recurrent function which is quasi-asymptotically almost periodic is always asymptotically almost periodic and that the following result holds true.

Theorem 3.3.14. *Let* $\mathrm{F}(I : X)$ *be any space of functions* $h : I \to X$ *satisfying the requirement that for each* $\tau \in I$ *the supremum formula holds for the function* $h(\cdot + \tau) - h(\cdot)$, *that is,*

$$\sup_{t \in I} \|h(\cdot + \tau) - h(\cdot)\| = \sup_{t \in I,\ t \geqslant a} \|h(\cdot + \tau) - h(\cdot)\|, \quad a \in I.$$

Then we have:

(i) $[\mathrm{F}(I:X) + C_0(I:X)] \cap Q - \mathrm{AUR}(I:X) \subseteq \mathrm{AUR}(I:X)$.
(ii) $\mathrm{F}(I:X) \cap Q - \mathrm{AUR}(I:X) \subseteq \mathrm{UR}(I:X)$.

Proof. We will include the main details of the proof for the sake of completeness. Let $h \in \mathrm{F}(I:X)$, $q \in C_0(I:X)$ and $f = h+q \in Q-\mathrm{AUR}(I:X)$. By our assumptions, there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that, for every integer $n \in \mathbb{N}$, there exists an integer $n_0 \in \mathbb{N}$ with

$$\big\|[h(t+\alpha_n) - h(t)] + [q(t+\alpha_n) - q(t)]\big\| \leqslant 1/n, \quad \text{for } t \in I,\ |t| \geqslant M_n,\ n \geqslant n_0.$$

Let $n \in \mathbb{N}$ be fixed. Since $q \in C_0(I:X)$, there exists a finite number $M_n' \geqslant M_n$ such that

$$\|h(t+\alpha_n) - h(t)\| \leqslant 2/n, \quad \text{provided } t \in I \text{ and } |t| \geqslant M_n',\ n \geqslant n_0.$$

Define the function $H_n : I \to X$ by $H_n(t) := h(t+\alpha_n) - h(t)$, $t \in I$. Since the supremum formula holds for the function $H_n(\cdot)$, we get

$$\sup_{t\in I}\|H_n(t)\| = \sup_{t\geqslant M_n'}\|H_n(t)\| \leqslant 2/n.$$

Hence, $\lim_{n\to+\infty}\sup_{t\in I}\|h(t+\alpha_n)-h(t)\| = 0$ and $h(\cdot)$ is thus uniformly recurrent, which immediately implies part (i). Part (ii) can be deduced similarly. □

In the following illustrative application of Theorem 3.3.14, we will consider case in which $I = \mathbb{R}$ and $\mathrm{F}(I:X) = AA(I:X)$, the space of all almost automorphic functions from I into X (see [631] for more details).

Example 3.3.15. Set $a_n := \operatorname{sign}(\cos(n\pi\sqrt{2}))$, $n \in \mathbb{Z}$ and define after that the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := \alpha a_n + (1-\alpha)a_{n+1}$ if $t \in [n, n+1)$ for some integer $n \in \mathbb{Z}$ and $t = \alpha n + (1-\alpha)(n+1)$ for some number $\alpha \in (0,1]$. This function is compactly almost automorphic but not almost periodic; furthermore, we have proved that the function $f(\cdot)$ is not asymptotically uniformly recurrent. Using this fact and Theorem 3.3.14, it readily follows that the function $f(\cdot)$ is not quasi-asymptotically uniformly recurrent, as well.

3.3.3 Stepanov classes of quasi-asymptotically uniformly recurrent type functions

Throughout this subsection, we use the following conditions:

$(A)_S$ $I = \mathbb{R}$ or $I = [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$, $p \in \mathcal{P}(I)$, $\mathsf{F} : I \times (0,\infty) \times I \to (0,\infty)$, $F : I \times \mathbb{N} \to (0,\infty)$, $\mathsf{F} : I \to (0,\infty)$ and $\omega \in I$.

$(B)_S$ The same as $(A)_S$ with the assumption $p \in \mathcal{P}(I)$ replaced by $p \in \mathcal{P}([0,1])$ therein.

We first introduce the Stepanov (p,ϕ,F)-classes of quasi-asymptotically uniformly recurrent functions and the Stepanov $(p,\phi,F)_i$-classes of quasi-asymptotically uniformly recurrent functions, where $i=1,2$ and $p\in\mathcal{P}(I)$. In this approach, we may lose the information about the translation invariance of introduced spaces.

Definition 3.3.16. Let $(A)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov (p,ϕ,F)-quasi-asymptotically almost periodic, resp. Stepanov (p,ϕ,F)-quasi-asymptotically uniformly recurrent, if and only if $\phi(\|f(\cdot+\tau)-f(\cdot)\|)\in L^{p(\cdot)}(K)$ for any $\tau\in I$ and any compact set $K\subseteq I$ as well as for each $\varepsilon>0$ there exists a finite number $L(\varepsilon)>0$ such that any interval $I'\subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau\in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon,\tau)>0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \mathsf{F}(t,\varepsilon,\tau)\phi\big(\|f(\cdot+\tau)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+1]}\leqslant\varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty}\sup_{|t|\geqslant M_n} F(t,n)\phi\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+1]}=0. \tag{3.54}$$

(ii) We say that a function $f : I\to X$ is Stepanov (p,ϕ,F)-asymptotically ω-periodic if and only if $\phi(\|f(\cdot+\omega)-f(\cdot)\|)\in L^{p(\cdot)}(K)$ for any compact set $K\subseteq I$ and

$$\lim_{|t|\to\infty}\mathsf{F}(t)\phi\big(\|f(\cdot+\omega)-f(\cdot)\|\big)_{L^{p(\cdot)}[t,t+1]}=0.$$

Definition 3.3.17. Let $(A)_S$ hold.

(i) A function $f : I\to X$ is called Stepanov $(p,\phi,\mathsf{F})_1$-quasi-asymptotically almost periodic, resp. Stepanov $(p,\phi,F)_1$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot+\tau)-f(\cdot)\|\in L^{p(\cdot)}(K)$ for any $\tau\in I$ and any compact set $K\subseteq I$ as well as for each $\varepsilon>0$ there exists a finite number $L(\varepsilon)>0$ such that any interval $I'\subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau\in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon,\tau)>0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)}\mathsf{F}(t,\varepsilon,\tau)\phi\big(\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(\cdot)}[t,t+1]}\big)\leqslant\varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty}\sup_{|t|\geqslant M_n}F(t,n)\phi\big(\|f(\cdot+\alpha_n)-f(\cdot)\|_{L^{p(\cdot)}[t,t+1]}\big)=0.$$

(ii) We say that a function $f : I\to X$ is Stepanov $(p,\phi,\mathsf{F})_1$-asymptotically ω-periodic if and only if $\|f(\cdot+\omega)-f(\cdot)\|\in L^{p(\cdot)}(K)$ for any compact set $K\subseteq I$ and

$$\lim_{|t|\to\infty}\mathsf{F}(t)\phi\big(\|f(\cdot+\omega)-f(\cdot)\|_{L^{p(\cdot)}[t,t+1]}\big)=0.$$

Definition 3.3.18. Let $(A)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov $(p, \phi, \mathsf{F})_2$-quasi-asymptotically almost periodic, resp. Stepanov $(p, \phi, F)_2$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot + \tau) - f(\cdot)\| \in L^{p(\cdot)}(K)$ for any $\tau \in I$ and any compact set $K \subseteq I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t| \geqslant M(\varepsilon,\tau)} \phi\big(\mathsf{F}(t, \varepsilon, \tau)\|f(\cdot + \tau) - f(\cdot)\|_{L^{p(\cdot)}[t,t+1]}\big) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n \to +\infty} \sup_{|t| \geqslant M_n} \phi\big(F(t, n)\|f(\cdot + \alpha_n) - f(\cdot)\|_{L^{p(\cdot)}[t,t+1]}\big) = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov $(p, \phi, \mathsf{F})_2$-asymptotically ω-periodic if and only if $\|f(\cdot + \omega) - f(\cdot)\| \in L^{p(\cdot)}(K)$ for any compact set $K \subseteq I$ and

$$\lim_{|t| \to \infty} \phi\big(\mathsf{F}(t)\|f(\cdot + \omega) - f(\cdot)\|_{L^{p(\cdot)}[t,t+1]}\big) = 0.$$

In the second approach, we will employ condition $(B)_S$ and assume therefore that $p \in \mathcal{P}([0, 1])$. Using the substitution $\cdot \to \cdot + t$, the translation invariance of considered function spaces can be obtained (see, e. g., Remark 3.1.32(iii)).

Definition 3.3.19. Let $(B)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov $[p, \phi, \mathsf{F}]$-quasi-asymptotically almost periodic, resp. Stepanov $[p, \phi, F]$-quasi-asymptotically uniformly recurrent, if and only if $\phi(\|f(\cdot + t + \tau) - f(\cdot + t)\|) \in L^{p(\cdot)}[0, 1]$ for any $\tau, t \in I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t| \geqslant M(\varepsilon,\tau)} \mathsf{F}(t, \varepsilon, \tau)\phi\big(\|f(\cdot + t + \tau) - f(\cdot + t)\|\big)_{L^{p(\cdot)}[0,1]} \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n \to +\infty} \sup_{|t| \geqslant M_n} F(t, n)\phi\big(\|f(\cdot + t + \alpha_n) - f(\cdot + t)\|\big)_{L^{p(\cdot)}[0,1]} = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov $[p, \phi, \mathsf{F}]$-asymptotically ω-periodic if and only if $\phi(\|f(\cdot + t + \omega) - f(\cdot + t)\|) \in L^{p(\cdot)}[0,1]$ for any $t \in I$ and

$$\lim_{|t|\to\infty} \mathsf{F}(t)\phi(\|f(\cdot + t + \omega) - f(\cdot + t)\|)_{L^{p(\cdot)}[0,1]} = 0.$$

Definition 3.3.20. Let $(B)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov $[p, \phi, \mathsf{F}]_1$-quasi-asymptotically almost periodic, resp. Stepanov $[p, \phi, F]_1$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot + t + \tau) - f(\cdot + t)\| \in L^{p(\cdot)}[0,1]$ for any $\tau,\ t \in I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \mathsf{F}(t, \varepsilon, \tau)\phi(\|f(\cdot + t + \tau) - f(\cdot + t)\|_{L^{p(\cdot)}[0,1]}) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} F(t, n)\phi(\|f(\cdot + t + \alpha_n) - f(\cdot + t)\|_{L^{p(\cdot)}[0,1]}) = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov $[p, \phi, \mathsf{F}]_1$-asymptotically ω-periodic if and only if $\|f(\cdot + t + \omega) - f(\cdot + t)\| \in L^{p(\cdot)}[0,1]$ for any $t \in I$ and

$$\lim_{|t|\to\infty} \mathsf{F}(t)\phi(\|f(\cdot + t + \omega) - f(\cdot + t)\|_{L^{p(\cdot)}[0,1]}) = 0.$$

Definition 3.3.21. Let $(B)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov $[p, \phi, \mathsf{F}]_2$-quasi-asymptotically almost periodic, resp. Stepanov $[p, \phi, F]_2$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot + t + \tau) - f(\cdot + t)\| \in L^{p(\cdot)}[0,1]$ for any $\tau,\ t \in I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \phi(\mathsf{F}(t, \varepsilon, \tau)\|f(\cdot + t + \tau) - f(\cdot + t)\|_{L^{p(\cdot)}[0,1]}) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} \phi(F(t, n)\|f(\cdot + t + \alpha_n) - f(\cdot + t)\|_{L^{p(\cdot)}[0,1]}) = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov $[p, \phi, \mathsf{F}]_2$-asymptotically ω-periodic if and only if $\|f(\cdot + t + \omega) - f(\cdot + t)\| \in L^{p(\cdot)}[0,1]$ for any $t \in I$ and

$$\lim_{|t|\to\infty} \phi\big(\mathsf{F}(t)\big\|f(\cdot + t + \omega) - f(\cdot + t)\big\|_{L^{p(\cdot)}[0,1]}\big) = 0.$$

Remark 3.3.22. The notion introduced in the above definitions is rather general. Let us only say the following: suppose that $I = \mathbb{R}$, the function $\phi(\cdot)$ is locally bounded, $\omega \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}}\big[\big\|f(\cdot)\big\|_{L^{p(\cdot-\omega)}[t,t+1]} + \big\|f(\cdot)\big\|_{L^{p(\cdot)}[t,t+1]}\big] < \infty.$$

Then it readily follows that $f(\cdot)$ is Stepanov (p, ϕ, F)-asymptotically ω-periodic for any function $\mathsf{F} \in C_0(\mathbb{R} : X)$.

The notion introduced in the above definitions extends the notion of Stepanov p-quasi-asymptotical almost periodicity and the notion of Stepanov p-asymptotical ω-periodicity ($1 \leqslant p < \infty$). In case that $p(x) \equiv p \in [1, \infty)$, the Stepanov (p, ϕ, F)-classes coincide with the corresponding Stepanov $[p, \phi, F]$-classes of functions. The most intriguing case, without any doubt, is that in which the functions F, F, F are identically equal to one and $\phi(x) \equiv x$; if this is the case and $p \in \mathcal{P}([0,1])$ (see Definition 3.3.19–Definition 3.3.21), then we also say that the function $f : I \to X$ is Stepanov $p(x)$-quasi-asymptotically almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic). In what follows, by $S^{p(x)}Q-\mathrm{AAP}(I : X)$ ($S^{p(x)}Q-\mathrm{AUR}(I : X)$, $S^{p(x)}\mathrm{SAP}_\omega(I : X)$) we denote the collection of all such functions. It can be easily verified that the function $f : I \to X$ is Stepanov $p(x)$-quasi-asymptotically almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic) if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is quasi-asymptotically almost periodic (quasi-asymptotically uniformly recurrent, S-asymptotically ω-periodic). This enables one to transfer the statements of Proposition 3.3.11 and Theorem 3.3.14 to the Stepanov classes (see also [647, Theorem 2.10, Proposition 2.11]) as well as to conclude that $S^{p(x)}\mathrm{SAP}_\omega(I : X) \subseteq S^{p(x)}Q - \mathrm{AAP}(I : X) \subseteq S^{p(x)}Q - \mathrm{AUR}(I : X)$ for any $p \in \mathcal{P}([0,1])$; see also [647, Proposition 2.7].

Unfortunately, the spaces of (Stepanov $p(x)$-) quasi-asymptotically uniformly recurrent type functions are not closed under the operations of pointwise addition and multiplication. For instance, the consideration from [647, Example 2.16–Example 2.18] enables one to see that the following hold:

(i) There exist a continuous periodic function $f : \mathbb{R} \to \mathbb{R}$ and a function $g \in \mathrm{SAP}_2(\mathbb{R} : \mathbb{R})$ such that the function $(f \cdot g)(\cdot)$ is not quasi-asymptotically uniformly recurrent.
(ii) There exist a Stepanov p-almost periodic function $f : \mathbb{R} \to \mathbb{R}$, where the exponent $p \geqslant 1$ can be chosen arbitrarily, and a function $g \in \mathrm{SAP}_4(\mathbb{R} : \mathbb{R})$ such that $(f \cdot g)(\cdot)$ does not belong to the class $S^1Q - \mathrm{AUR}(\mathbb{R} : \mathbb{R})$.

(iii) There exist a continuous periodic function $f : [0, \infty) \to \mathbb{R}$ and a function $g \in SAP_4([0, \infty) : \mathbb{R})$ such that the function $(f + g)(\cdot)$ does not belong to the class $S^1Q - \mathrm{AUR}([0, \infty) : \mathbb{R})$.

We continue by stating the following.

Proposition 3.3.23. *Suppose that $\phi(\cdot)$ is continuous for $t = 0$, $\phi(0) = 0$ and any of the functions $\mathbf{F}$, F, $\mathcal{F}$ is bounded. Then any quasi-asymptotically uniformly recurrent function $f : I \to X$ is Stepanov (p, ϕ, F)-quasi-asymptotically uniformly recurrent, Stepanov $[p, \phi, F]$-quasi-asymptotically uniformly recurrent as well as Stepanov $(p, \phi, F)_i$-quasi-asymptotically uniformly recurrent and Stepanov $[p, \phi, F]_i$-quasi-asymptotically uniformly recurrent ($i = 1, 2$). The same statement holds for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions.*

Proof. We will provide the main details of the proof for the class of Stepanov $[p, \phi, F]$-quasi-asymptotically uniformly recurrent functions. Let (α_n) and (M_n) be the sequences from Definition 3.3.8, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|\phi(t)| = |\phi(t) - \phi(0)| < \varepsilon$, $|t| \leqslant \delta$. Hence, $\sup \phi([0, \delta]) \leqslant \varepsilon$. By our assumption, we have the existence of an integer $n_0 \in \mathbb{N}$ such that

$$\sup_{|t| \geqslant M_n} \|f(t + \alpha_n) - f(t)\| \leqslant \delta, \quad n \geqslant n_0.$$

Let $n \in \mathbb{N}$ with $n \geqslant n_0$ be fixed. Then, for every $t \geqslant M_n' := M_n + 1$, we have $|t + x| \geqslant |t| - 1 \geqslant M_n$, $x \in [0, 1]$. This implies that, for every $t \geqslant M_n'$, $x \in [0, 1]$ and $\lambda \geqslant \varepsilon$, we have $\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda \leqslant 1$, $\varphi_{p(x)}(\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda) \leqslant 1$ and therefore

$$\int_0^1 \varphi_{p(x)}(\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda)\, dx \leqslant 1.$$

Thus,

$$[\varepsilon, \infty) \subseteq \left\{\lambda > 0 : \int_0^1 \varphi_{p(x)}(\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda)\, dx \leqslant 1\right\},$$

which yields

$$\phi(\|f(\cdot + t + \alpha_n) - f(\cdot + t)\|)_{L^{p(\cdot)}[0,1]} \leqslant \varepsilon, \quad n \geqslant n_0.$$

This completes the proof by the boundedness of the function $F(\cdot, \cdot)$. □

As an immediate consequence, we have the following statement.

Corollary 3.3.24. *Let $\omega \in I$ and $p \in \mathcal{P}([0,1])$. Then any quasi-asymptotically almost periodic (quasi-asymptotically uniformly recurrent, S-asymptotically ω-periodic) function $f : I \to X$ is Stepanov $p(x)$-quasi-asymptotically almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic).*

Using the trivial inequalities and Lemma 1.1.7, we can clarify numerous inclusions for the introduced classes of functions. For instance, we can simply deduce the following:

(i) $S^{p(x)}\mathrm{SAP}_\omega(I : X) \subseteq S^1\mathrm{SAP}_\omega(I : X)$, $S^{p(x)}Q - \mathrm{AAP}(I : X) \subseteq S^1Q - \mathrm{AAP}(I : X)$ and $S^{p(x)}Q - \mathrm{AUR}(I : X) \subseteq S^1Q - \mathrm{AUR}(I : X)$.

(ii) Suppose $p \in D_+([0,1])$ and $1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ for a. e. $x \in [0,1]$. Then we have $S^{p^+}\mathrm{SAP}_\omega(I : X) \subseteq S^{p(x)}\mathrm{SAP}_\omega(I : X) \subseteq S^{p^-}\mathrm{SAP}_\omega(I : X)$, $S^{p^+}Q - \mathrm{AAP}(I : X) \subseteq S^{p(x)}Q - \mathrm{AAP}(I : X) \subseteq S^{p^-}Q - \mathrm{AAP}(I : X)$, and $S^{p^+}Q - \mathrm{AUR}(I : X) \subseteq S^{p(x)}Q - \mathrm{AUR}(I : X) \subseteq S^{p^-}Q - \mathrm{AUR}(I : X)$.

(iii) Suppose $p,\ q \in \mathcal{P}([0,1])$ and $p \leqslant q$ a. e. on $[0,1]$. Then we have $S^{q(x)}\mathrm{SAP}_\omega(I : X) \subseteq S^{p(x)}\mathrm{SAP}_\omega(I : X)$, $S^{q(x)}Q - \mathrm{AAP}(I : X) \subseteq S^{p(x)}Q - \mathrm{AAP}(I : X)$ and $S^{q(x)}Q - \mathrm{AUR}(I : X) \subseteq S^{p(x)}Q - \mathrm{AUR}(I : X)$.

These inclusions can be simply transferred and reformulated for the general classes of functions introduced in Definition 3.3.16–Definition 3.3.18 and Definition 3.3.19–Definition 3.3.21; details can be left to the interested reader.

The first part of subsequent result is very similar to Proposition 3.1.27; the proof is based on the use of Jensen integral inequality and therefore is omitted.

Proposition 3.3.25.

(i) *Suppose that $\phi(\cdot)$ is convex, $p(x) \equiv 1$ and $f \in L^1_{\mathrm{loc}}(I : X)$. If $f(\cdot)$ is Stepanov $(p,1,F)$-quasi-asymptotically uniformly recurrent, then $f(\cdot)$ is Stepanov $(p,1,F)_1$-quasi-asymptotically uniformly recurrent. If the function $\phi(\cdot)$ is concave, then the above inclusion reverses.*

(ii) *Suppose that there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$. If $f(\cdot)$ is Stepanov $(p,\phi,F)_1$-quasi-asymptotically uniformly recurrent, resp. Stepanov $[p,\phi,F]_1$-quasi-asymptotically uniformly recurrent, then $f(\cdot)$ is Stepanov $(p,\phi,F_1)_2$-quasi-asymptotically uniformly recurrent, resp. Stepanov $[p,\phi,F_1]_2$-quasi-asymptotically uniformly recurrent, provided that $F = \varphi \circ F_1$.*

Furthermore, the same statements hold for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions.

The basic structural properties of quasi-asymptotically almost periodic functions clarified in [647, Theorem 2.13] can also be formulated in our framework, for the general classes of functions introduced in this subsection. We leave it to the reader to make this explicit.

If $p \in [1,\infty)$, then any Stepanov p-quasi-asymptotically almost periodic function is Weyl p-almost periodic (see [647, Proposition 2.12]). The argumentation used in the proof of this result also shows that any Stepanov p-quasi-asymptotically uniformly recurrent function is Weyl p-uniformly recurrent. In the general case, we will clarify only one result of this type regarding the notion introduced in Definition 3.3.1 and Definition 3.3.16. Before doing so, observe that, if $p \in \mathcal{P}(I)$, $a,\ b,\ c \in I$, $a < b < c$ and $f \in L^{p(x)}[a,c]$, then $f \in L^{p(x)}[a,b]$, $f \in L^{p(x)}[b,c]$ and

$$\|f\|_{L^{p(x)}[a,c]} \leqslant \|f\|_{L^{p(x)}[a,b]} + \|f\|_{L^{p(x)}[b,c]}. \tag{3.55}$$

Proposition 3.3.26. *Suppose that the function $f : I \to X$ is Stepanov (p,ϕ,F)-quasi-asymptotically uniformly recurrent. If $F_1 : (0,\infty)\times I \to (0,\infty)$ satisfies*

$$\lim_{n\to+\infty}\limsup_{l\to+\infty}\sup_{t\in I} F_1(l,t)\left[\frac{1}{F(t,n)} + \cdots + \frac{1}{F(\lfloor t+l\rfloor,n)}\right] < \infty \tag{3.56}$$

and

$$\lim_{l\to+\infty}\sup_{t\in I} F_1(l,t) = 0, \tag{3.57}$$

then the function $f(\cdot)$ is Weyl $(p(x),\phi,F_1)$-uniformly recurrent.

Proof. By our assumption, we have $\phi(\|f(\cdot+\tau) - f(\cdot)\|) \in L^{p(\cdot)}(K)$ for any $\tau \in I$ and any compact set $K \subseteq I$; furthermore, we know that there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that (3.54) holds. We will prove that (3.52) holds with the function $F(\cdot,\cdot)$ replaced therein with the function $F_1(\cdot,\cdot)$. Let $n \in \mathbb{N}$ and $l > 0$ be fixed. If $t \in I$, then there exist four possibilities:

1. $|t| \geqslant M_n$ and $|t+l| \geqslant M_n$;
2. $|t| \geqslant M_n$ and $|t+l| \leqslant M_n$;
3. $|t| \leqslant M_n$ and $|t+l| \geqslant M_n$;
4. $|t| \leqslant M_n$ and $|t+l| \leqslant M_n$.

The consideration is similar for all these cases and we will give the proof for case [1.], only. If $t \geqslant 0$, then we have $t \geqslant M_n$, $t+l \geqslant M_n$ and therefore

$$[F_1(l,t)[\phi(\|f(\cdot+\alpha_n) - f(\cdot)\|)_{L^{p(\cdot)}[t,t+l]}]] \leqslant F_1(l,t)\left[\frac{\varepsilon}{F(t,n)} + \cdots + \frac{\varepsilon}{F(\lfloor t+l\rfloor,n)}\right];$$

see also (3.55). Employing condition (3.56), we immediately get (3.52). If $t \leqslant 0$, then we have $t \leqslant -M_n$ and $t+l \geqslant M_n$ for a sufficiently large $l > 0$ (it suffices to consider only this case because, in (3.52), we operate with $\limsup_{l\to+\infty}\cdot$). We have

$$\begin{aligned}&[F_1(l,t)[\phi(\|f(\cdot+\alpha_n) - f(\cdot)\|)_{L^{p(\cdot)}[t,t+l]}]]\\ &\quad\leqslant [F_1(l,t)[\phi(\|f(\cdot+\alpha_n) - f(\cdot)\|)_{L^{p(\cdot)}[t,-M_n]} + \phi(\|f(\cdot+\alpha_n) - f(\cdot)\|)_{L^{p(\cdot)}[-M_n,M_n]}\end{aligned}$$

$$
\begin{aligned}
&\quad + \phi(\|f(\cdot+\alpha_n)-f(\cdot)\|)_{L^{p(\cdot)}[M_n,t+l]}]] \\
&\leqslant F_1(l,t)\biggl[\biggl(\frac{\varepsilon}{F(t,n)}+\cdots+\frac{\varepsilon}{F(t+\lfloor -t-M_n\rfloor,n)}\biggr) \\
&\quad + \biggl(\frac{\varepsilon}{F(M_n,n)}+\cdots+\frac{\varepsilon}{F(M_n+\lfloor t+l-M_n\rfloor,n)}\biggr)\biggr] \\
&\quad + F_1(l,t)\phi(\|f(\cdot+\alpha_n)-f(\cdot)\|)_{L^{p(\cdot)}[-M_n,M_n]} \\
&\leqslant 2F_1(l,t)\biggl[\frac{\varepsilon}{F(t,n)}+\cdots+\frac{\varepsilon}{F(\lfloor t+l\rfloor,n)}\biggr] \\
&\quad + F_1(l,t)\phi(\|f(\cdot+\alpha_n)-f(\cdot)\|)_{L^{p(\cdot)}[-M_n,M_n]}.
\end{aligned}
$$

Using (3.56)–(3.57), we get (3.52). □

3.3.4 Composition principles for the class of quasi-asymptotically uniformly recurrent functions

In this subsection, we will briefly consider quasi-asymptotically uniformly recurrent functions depending on two parameters and related composition theorems (for the sake of brevity, we will say only a few words about the Stepanov classes). In order to unify several different approaches used in the existing literature (see also Definition 2.4.42–Definition 2.4.43 and Theorem 2.4.44), in this subsection we will assume that $\mathbf{B} \subseteq P(Y)$, where $P(Y)$ denotes the power set of Y; usually, $\mathbf{B}$ denotes the collection of all bounded subsets of Y or all compact subsets of Y.

Definition 3.3.27.
(i) By $C_{0,\mathbf{B}}(I \times Y : X)$ we denote the space of all continuous functions $H : I \times Y \to X$ such that $\lim_{|t|\to+\infty} H(t,y) = 0$ uniformly for y in any subset $B \in \mathbf{B}$.
(ii) A continuous function $F : I \times Y \to X$ is said to be uniformly continuous on $\mathbf{B}$, uniformly for $t \in I$ if and only if for every $\varepsilon > 0$ and for every $B \in \mathbf{B}$ there exists a number $\delta_{\varepsilon,B} > 0$ such that $\|F(t,x) - F(t,y)\| \leqslant \varepsilon$ for all $t \in I$ and all $x, y \in B$ satisfying $\|x-y\| \leqslant \delta_{\varepsilon,B}$.

We continue by introducing the following definition.

Definition 3.3.28. Suppose that $F : I \times Y \to X$ is a continuous function and $\mathbf{B} \subseteq P(Y)$. Then we say that $F(\cdot,\cdot)$ is quasi-asymptotically uniformly recurrent, uniformly on $\mathbf{B}$ if and only if for every $B \in \mathbf{B}$ there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that:

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} \|F(t+\alpha_n,x) - F(t,x)\| = 0, \quad x \in B. \tag{3.58}$$

Denote by $Q-\mathrm{AUR}_{\mathbf{B}}(I \times Y : X)$ the set consisting of all quasi-asymptotically uniformly recurrent, uniformly on $\mathbf{B}$ functions from $I \times Y$ into X.

Using the argumentation employed in the proofs of [364, Theorem 3.30, Theorem 3.31], we may deduce the following results.

Theorem 3.3.29. *Suppose that $\mathbf{B} \subseteq P(Y)$, $R(f) \in \mathbf{B}$, $F \in Q-\mathrm{AUR}_{\mathbf{B}}(I \times Y : X)$ and $f \in Q-\mathrm{AUR}(I : Y)$. If there exist a finite number $L > 0$ such that* (2.51) *holds with a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that* (3.58) *holds with $B = R(f)$ and* (2.3) *holds, then the function $t \mapsto F(t, f(t))$, $t \in I$, belongs to the class $Q-\mathrm{AUR}(I : X)$.*

Theorem 3.3.30. *Suppose that $\mathbf{B} \subseteq P(Y)$, $R(f) \in \mathbf{B}$, $F \in Q-\mathrm{AUR}_{\mathbf{B}}(I \times Y : X)$ and $f \in Q-\mathrm{AUR}(I : Y)$. If $F : I \times Y \to X$ is uniformly continuous on $\mathbf{B}$, uniformly for $t \in I$ and there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that* (3.58) *holds with $B = R(f)$ and* (2.3) *holds, then the function $t \mapsto F(t, f(t))$, $t \in I$ belongs to the class $Q-\mathrm{AUR}(I : X)$.*

Similarly as in Definition 3.3.28, we can introduce the notion of a quasi-asymptotically almost periodic, uniformly on $\mathbf{B}$ function and the notion of a S-asymptotically ω-periodic, uniformly on $\mathbf{B}$ function. It is worth noticing that Theorem 3.3.29 and Theorem 3.3.30 continue to hold in this framework.

In [647, Definition 2.22], we have introduced the notion of a Stepanov p-quasi-asymptotically almost periodic function depending on two parameters ($1 \leqslant p < \infty$); the notion of a Stepanov $p(x)$-quasi-asymptotically almost periodic function (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent function, Stepanov $p(x)$-asymptotically ω-periodic function) can be introduced in a similar fashion. The interested reader should try to extend [647, Theorem 2.23, Theorem 2.24] in this context.

3.3.5 Invariance of generalized quasi-asymptotical uniform recurrence under the actions of convolution products

This subsection investigates the invariance of generalized quasi-asymptotical uniform recurrence under the actions of finite and infinite convolution products. Using the same arguments as in the proofs of [647, Proposition 3.1, Proposition 3.2], we can deduce the validity of the following statement.

Proposition 3.3.31.
(i) *Suppose that $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family and $\int_0^\infty \|R(s)\|\, ds < \infty$. If $f \in Q-\mathrm{AUR}([0,\infty) : X) \cap L^\infty([0,\infty) : X)$, then the function*

$F(\cdot)$, defined by

$$\mathrm{F}(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0, \tag{3.59}$$

belongs to the class $Q - \mathrm{AUR}([0,\infty) : Y) \cap L^{\infty}([0,\infty) : Y)$.

(ii) *Suppose that $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and $\int_0^{\infty} \|R(s)\|\,ds < \infty$. If $f \in Q - \mathrm{AUR}(\mathbb{R} : X) \cap L^{\infty}(\mathbb{R} : X)$, then the function $\mathbf{F}(t)$, defined by* (2.46), *with the function $F(\cdot)$ replaced therein with the function $\mathbf{F}(\cdot)$, belongs to the class $Q - \mathrm{AUR}(\mathbb{R} : Y) \cap L^{\infty}(\mathbb{R} : Y)$.*

We would like to illustrate Proposition 3.3.31 by the following example.

Example 3.3.32. Suppose that $X = H$ is an infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$. In [770], R. K. Miller and R. L. Wheeler have investigated the well-posedness of the following abstract Cauchy problem of non-scalar type

$$x'(t) = Au(t) + \int_0^t b(t-s)(A + aI)u(s)\,ds + f(t), \quad x(0) = x_0; \tag{3.60}$$

here, $b(t)$ is a scalar-valued kernel, $b \in C^1([0,\infty))$, $a \in \mathbb{C}$, $f : [0,\infty) \to H$ is continuous and A is a densely defined, self-adjoint closed linear operator in H. If the assumptions [770, (A1)–(A5)] hold with the coefficients $\alpha = \beta_0 = \beta_1 = 0$, then [770, Theorem 7] implies that there exists a unique residual resolvent $(R(t))_{t\geqslant 0}$ for (3.60) such that $\|R(\cdot)\| \in L^p([0,\infty))$ for $2 \leqslant p < \infty$. Furthermore, if the assumptions [770, (A1)–(A5)] hold with the coefficients $\alpha = \beta_0 = \beta_1 = 0$ and the assumption [770, (A6)] holds provided that $B\sigma(L) \neq \emptyset$ (see [770, p. 273] for the notion), then [770, Theorem 8] implies that there exists a unique residual resolvent $(R(t))_{t\geqslant 0}$ for (3.60) such that $\|R(\cdot)\| \in L^p([0,\infty))$ for $1 \leqslant p < \infty$; if this is the case, then Proposition 3.3.31 is applicable since, due to [770, Theorem 2], the unique solution of (3.60) for all $x_0 \in D(A)$ and $f \in C^1([0,\infty) : X)$ is given by

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0.$$

For some other foundational papers concerning integrability of solution operator families appearing in the theory of abstract Volterra integro-differential equations, we can recommend for the reader [456, 487, 695, 771]. A comprehensive survey of non-updated results can be found in [857, Section 10].

Concerning the invariance of Stepanov quasi-asymptotically almost periodic properties analyzed in the previous subsection, it would be really difficult and rather long to examine all introduced classes. Primarily for this reason, we will focus our attention on the notion introduced in Definition 3.3.19, only.

The following result admits a simple reformulation for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions.

Proposition 3.3.33. *Suppose that* (a_k) *is a sequence of positive real numbers such that* $\sum_{k=0}^{\infty} a_k = 1$, $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ *is a convex monotonically increasing function satisfying* $\phi(xy) \leqslant \varphi(x)\phi(y)$ *for all* $x, y \geqslant 0$, $p, q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ *and* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying*

$$M := \sum_{k=0}^{\infty} a_k \varphi(a_k^{-1})[\varphi(\|R(\cdot + k)\|)]_{L^{q(\cdot)}[0,1]} < \infty. \tag{3.61}$$

Suppose, further, that for every $x \in \mathbb{R}$ *we have* $\int_{-x}^{\infty} \|R(v+x)\|\|\check{f}(v)\|\,dv < \infty$, *and that* $\check{f}(\cdot)$ *is Stepanov* $[p,\phi,F]$*-quasi-asymptotically uniformly recurrent,* $M_1 := \sup_{t\in\mathbb{R}}[\phi(\|f(t-s)\|)]_{L^{p(s)}[0,1]} < \infty$, $F_1 : (0,\infty)\times\mathbb{N} \to (0,\infty)$ *is bounded and satisfies the requirement that there exists a finite real constant* $c > 0$ *such that* $F_1(t,n) \leqslant cF(t,n)$ *for all* $t > 0$ *and* $n \in \mathbb{N}$. *Then the function* $\mathbf{F} : \mathbb{R} \to Y$, *given by* (2.46), *with the function* $F(\cdot)$ *replaced therein with the function* $\mathbf{F}(\cdot)$, *is well defined and Stepanov* $[\infty,\phi,F_1]$*-quasi-asymptotically uniformly recurrent.*

Proof. Since for every $x \in \mathbb{R}$ we have $\int_{-x}^{\infty} \|R(v+x)\|\|\check{f}(v)\|\,dv < \infty$, it can be easily verified that the function $\mathbf{F}(\cdot)$ is well defined and that the integral which defines $\mathbf{F}(x+\tau) - \mathbf{F}(x)$ is absolutely convergent for every $x \in \mathbb{R}$ and $\tau \in \mathbb{R}$. For the rest, let (α_n) and (M_n) be the sequences from Definition 3.3.19, for the function $f(\cdot)$ replaced therein with the function $\check{f}(\cdot)$. Let $\varepsilon > 0$ be given, and let $n_0 \in \mathbb{N}$ be such that $\phi(\|f(t+\alpha_n+\cdot) - f(t+\cdot)\|)_{L^{p(\cdot)}[0,1]} < \varepsilon/F(t,n)$, $n \geqslant n_0$, $|t| \geqslant M_n$. Clearly, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=k_0(\varepsilon)}^{\infty} a_k \varphi(a_k^{-1})[\varphi(\|R(\cdot + k)\|)]_{L^{q(\cdot)}[0,1]} < \varepsilon. \tag{3.62}$$

Let $\alpha_n \geqslant k_0(\varepsilon)$ for $n \geqslant n_1$. Let $n \in \mathbb{N}$ with $n \geqslant \max(n_0, n_1)$ be fixed, and let $|t| \geqslant M_n' := M_n + \alpha_n + 2$. Then for each $x \in [0,1]$ we have (apply the Jensen inequality, (3.1) and the Hölder inequality)

$$\begin{aligned}
&\|\mathbf{F}(t+x+\alpha_n) - \mathbf{F}(t+x)\| \\
&\quad\leqslant \phi\Bigg(\int_0^{\infty} \|R(s)\|\|f(x+t+\alpha_n-s) - f(x+t-s)\|\,ds\Bigg) \\
&\quad= \phi\Bigg(\sum_{k=0}^{\infty} a_k \int_0^1 a_k^{-1}\|R(s+k)\|\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|\,ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{k=0}^{\infty} a_k \phi\left(\int_0^1 a_k^{-1} \|R(s+k)\| \|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\| \, ds \right) \\
&\leqslant \sum_{k=0}^{\infty} a_k \varphi(a_k^{-1}) \int_0^1 \phi(\|R(s+k)\| \|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|) ds \\
&\leqslant 2 \sum_{k=0}^{\infty} a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} \\
&\quad \times [\phi(\|f(x+t+\alpha_n-k-\cdot) - f(x+t-k-\cdot)\|)]_{L^{p(\cdot)}[0,1]},
\end{aligned}
$$

which implies that for $t \leqslant -M_n'$ we have

$$
\begin{aligned}
\|\mathbf{F}(t+x+\alpha_n) - \mathbf{F}(t+x)\| &\leqslant 2 \sum_{k=0}^{\infty} a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} \frac{\varepsilon}{F(t,n)} \\
&\leqslant 2c \sum_{k=0}^{\infty} a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} \frac{\varepsilon}{F_1(t,n)}.
\end{aligned}
$$

If $t \geqslant M_n'$, then we have $\lfloor t - M_n \rfloor \geqslant k_0(\varepsilon)$ and (3.62) implies

$$
\begin{aligned}
&\|\mathbf{F}(t+x+\alpha_n) - \mathbf{F}(t+x)\| \\
&\leqslant 2 \sum_{k=0}^{\lfloor t-M_n \rfloor} a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} \\
&\quad \times [\phi(\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|)]_{L^{p(s)}[0,1]} \\
&\quad + 2 \sum_{k=\lfloor t-M_n \rfloor}^{\lceil t+M_n \rceil} a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} \\
&\quad \times [\phi(\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|)]_{L^{p(s)}[0,1]} \\
&\quad + 2 \sum_{k=\lceil t+M_n \rceil}^{\infty} a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} \\
&\quad \times [\phi(\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|)]_{L^{p(s)}[0,1]} \\
&\leqslant 2 \frac{\varepsilon}{F(t,n)} \left(\sum_{k=0}^{\lfloor t-M_n \rfloor} + \sum_{k=\lceil t+M_n \rceil}^{\infty} \right) a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} + \varepsilon \cdot \varphi(2) \cdot M_1 \\
&\leqslant 2c \frac{\varepsilon}{F_1(t,n)} \left(\sum_{k=0}^{\lfloor t-M_n \rfloor} + \sum_{k=\lceil t+M_n \rceil}^{\infty} \right) a_k \varphi(a_k^{-1}) [\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} + \varepsilon \cdot \varphi(2) \cdot M_1,
\end{aligned}
$$

since

$$
\begin{aligned}
&[\phi(\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|)]_{L^{p(s)}[0,1]} \\
&\quad \leqslant [\phi(\|f(x+t+\alpha_n-k-s)\| + \|f(x+t-k-s)\|)]_{L^{p(s)}[0,1]}
\end{aligned}
$$

$$\leqslant \varphi(2)\left[\frac{1}{2}\phi(\|f(x+t+\alpha_n-k-s)\|)+\frac{1}{2}\phi(\|f(x+t-k-s)\|)\right]_{L^{p(s)}[0,1]}$$
$$\leqslant \varphi(2)\cdot M_1.$$

This simply completes the proof. □

We will also state the following special corollary, which generalizes [647, Proposition 3.4].

Proposition 3.3.34. *Suppose that $q \in \mathcal{P}([0,1])$, $1/p(x)+1/q(x)=1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M := \sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is Stepanov $p(x)$-quasi-asymptotically uniformly recurrent (Stepanov $p(x)$-quasi-asymptotically almost periodic, Stepanov $p(x)$-asymptotically ω-periodic) and $S^{p(x)}$-bounded, then the function $\mathbf{F} : \mathbb{R} \to Y$, given by (2.46), with the function $F(\cdot)$ replaced therein with the function $\mathbf{F}(\cdot)$, is well defined, bounded and quasi-asymptotically uniformly recurrent (quasi-asymptotically almost periodic, S-asymptotically ω-periodic).*

Proof. We will consider the Stepanov $p(x)$-quasi-asymptotically uniformly recurrent functions, only. Since $\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$ and $\check{f}(\cdot)$ is $S^{p(x)}$-bounded, we can apply the same arguments as in the proofs of [372, Proposition 6.1] and [373, Proposition 5.1] in order to see that the function $\mathbf{F}(\cdot)$ is bounded and continuous. The remainder of the proof follows from the computations carried out in the proof of Proposition 3.3.33, with $p(x)=\varphi(x)=\phi(x)=x$ and $F(t,n)=F_1(t,n)=1$. □

The following result regarding the finite convolution product also admits a reformulation for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions.

Proposition 3.3.35. *Suppose that (a_k) is a sequence of positive real numbers such that $\sum_{k=0}^{\infty}a_k = 1$, $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x)+1/q(x)=1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying the requirement that (3.61) holds. Suppose, further, that the mapping $\mathrm{F} : [0,\infty) \to Y$, given by (3.59), is well defined and that $f(\cdot)$ is Stepanov $[p,\phi,F]$-quasi-asymptotically uniformly recurrent,*

$$M_1 := \sup_{t\geqslant 0}\sup_{t\in[0,s]}[\phi(\|f(t-s)\|)]_{L^{p(s)}[0,1]} < \infty,$$

$F_1 : (0,\infty)\times\mathbb{N} \to (0,\infty)$ is bounded and satisfies the requirement that there exists a finite real constant $c>0$ such that $F_1(t,n) \leqslant cF(t,n)$ for all $t>0$ and $n \in \mathbb{N}$. Then the function $\mathrm{F}(\cdot)$ is Stepanov $[\infty,\phi,F_1]$-quasi-asymptotically uniformly recurrent.

Proof. The proof is very similar to the proof of Proposition 3.3.35 and we will only outline two details. Let $\varepsilon>0$ be fixed, and let the numbers $M_n>0$ and $k_0(\varepsilon)$, n_0, $n_1 \in \mathbb{N}$

be as above. Then for each $x \in [0,1]$, $|t| \geqslant M_n' + \alpha_n + 2$ and $n \in \mathbb{N}$ with $n \geqslant \max(n_0, n_1)$ we have

$$\begin{aligned}
&\phi(\|F(x+t+k+\alpha_n) - F(x+t+k)\|) \\
&\quad \leqslant \sum_{k=0}^{\lceil t\rceil} a_k \varphi(a_k^{-1})[\varphi(\|R(\cdot+k)\|)]_{L^{q(\cdot)}[0,1]} \\
&\qquad \times [\phi(\|f(x+t+k+\alpha_n-s) - f(x+t+k-s)\|)]_{L^{p(s)}[0,1]}.
\end{aligned}$$

After that, we can decompose the sum $\sum_{k=0}^{\lceil t\rceil} \cdot$ into two parts:

$$\sum_{k=0}^{\lceil t\rceil} \cdot = \sum_{k=0}^{k_0(\varepsilon)} \cdot + \sum_{k=k_0(\varepsilon)}^{\lceil t\rceil} \cdot,$$

and apply similar arguments. This completes the proof in a routine manner. □

Similarly we can deduce the following extension of [647, Proposition 3.3] (see also the proof of [373, Proposition 5.1]).

Proposition 3.3.36. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M := \sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$. If $f : [0,\infty) \to X$ is Stepanov $p(x)$-quasi-asymptotically almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic), $f(t-\cdot) \in L^{p(x)}[0,t]$ for $0 < t \leqslant 1$ and*

$$\sup_{k\in\mathbb{N}_0} \sup_{t\geqslant 0} \|f(t+k-\cdot)\|_{L^{p(\cdot)}[0,1]} < \infty,$$

then the function $\mathrm{F} : [0,\infty) \to Y$, given by (3.59), is well defined, bounded and quasi-asymptotically almost periodic (quasi-asymptotically uniformly recurrent, S-asymptotically ω-periodic).

Remark 3.3.37. We would like to note that it is very difficult to remove the assumption on the boundedness of the function $f(\cdot)$ in Proposition 3.3.31, resp. the Stepanov $p(x)$-boundedness of functions in Proposition 3.3.34–Proposition 3.3.36, in contrast to our recent research study [648].

3.3.6 Applications to the abstract Volterra integro-differential equations

Concerning possible applications of our theoretical results to the abstract Volterra integro-differential equations in Banach spaces, we would like to say first a few words about the abstract nonautonomous differential equations of first order. In the first part of [647, Section 4], we have investigated the generalized almost periodic properties of the mild solutions to the abstract Cauchy problems

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{3.63}$$

$$u'(t) = A(t)u(t) + f(t), \quad t > 0;\ u(0) = x, \tag{3.64}$$

where the operator family $A(\cdot)$ satisfies certain conditions. In [647, Subsection 4.1], we have investigated the generalized almost periodic properties of the semilinear analogues to the abstract Cauchy problems (3.63)–(3.64).

The statement of [647, Theorem 4.1] can be straightforwardly extended for the inhomogeneities $f \in S^{p(x)}Q - \mathrm{AAP}([0,\infty) : X)$ by replacing the number q in the equation [647, (4.1)] with the function $q(x)$ and using the translation $\cdot \mapsto \cdot + k$ $(1/p(x)+1/q(x) = 1)$ therein; we can also consider the inhomogeneities $f \in S^{p(x)}Q - \mathrm{AUR}([0,\infty) : X)$ which are Stepanov $p(x)$-bounded, by slightly modifying the equation [647, (4.1)] in the formulation of this result. Similar comments can be made for [647, Theorem 4.3]. Concerning semilinear problems, the statements of [647, Theorem 4.6, Theorem 4.7] can be reformulated by replacing the space $Q - \mathrm{AAP}(I : X)$ with the space $BQ - \mathrm{AUR}_{(\alpha_n)}(I : X)$ consisting of all bounded functions $f : I \to X$ which are quasi-asymptotically uniformly recurrent and for which there exists a fixed sequence (α_n) of positive real numbers such that (3.53) holds; equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_\infty$, this space becomes a complete metric space. The conclusions established in [647, Example 2.8] also can be reexamined in this context.

By a mild solution of the abstract semilinear Cauchy inclusion

$$(\mathrm{DFP})_{F,\gamma,s} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + F(t,u(t)), & t > 0, \\ u(0) = x_0, \end{cases}$$

we mean any function $u \in C([0,\infty) : X)$ satisfying

$$u(t) = S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0.$$

Now we are in a position to state the following result.

Theorem 3.3.38. *Suppose that the function $F : \mathbb{R} \times X \to X$ is continuous and satisfies the requirement that for each bounded subset B of X there exist a finite real constant $M_B > 0$ and a sequence (M_n) of positive real numbers such that* (3.58) *holds and* $\sup_{t\in\mathbb{R}} \sup_{x\in B} \|F(t,x)\| \leqslant M_B$*. Let there exist a finite number $L > 0$ such that* (2.51) *holds, and let there exist an integer $n \in \mathbb{N}$ such that $A_n < 1$, where*

$$A_n := \sup_{t\geqslant 0} \int_0^t \int_0^{x_n} \cdots \int_0^{x_2} L^n \|R_\gamma(t-x_n)\| \prod_{i=2}^n \|R_\gamma(x_i - x_{i-1})\|\,dx_1\,dx_2 \cdots dx_n.$$

Then the abstract fractional Cauchy inclusion $(DFP)_{F,\gamma,s}$ has a unique solution which belongs to the space $BQ - \mathrm{AUR}_{(\alpha_n)}([0,\infty) : X)$.

Proof. Set, for every $u \in C_b([0,\infty) : X)$,

$$(\Upsilon u)(t) := S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0.$$

Suppose that $u \in BQ - \mathrm{AUR}_{(\alpha_n)}([0,\infty) : X)$. Then $R(u) = B$ is a bounded set and our assumption implies that the mapping $t \mapsto F(t,u(t))$, $t \in \mathbb{R}$ is bounded. Applying Theorem 3.3.29, we see that the function $F(\cdot,u(\cdot))$ is quasi-asymptotically uniformly recurrent. After that, we can employ Proposition 3.3.31(i) and Proposition 3.3.10 (there is no need to say that we can retain the same sequence (α_n) after applying the above-mentioned statements, with the meaning clear) in order to see that $\Upsilon u \in BQ - \mathrm{AUR}_{(\alpha_n)}([0,\infty) : X)$. Hence, the mapping $\Upsilon(\cdot)$ is well defined. Since

$$\big\|(\Upsilon^n u) - (\Upsilon^n v)\big\|_\infty \leqslant A_n\|u-v\|_\infty, \quad u,\ v \in C_b\big([0,\infty) : X\big),\ n \in \mathbb{N},$$

the well-known extension of the Banach contraction principle shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point. This completes the proof. □

Let Ω be a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a. e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $X := L^p(\Omega)$. Suppose that the operator $A := \Delta - b$ acts on X with the Dirichlet boundary conditions, and that B is the multiplication operator by the function $m(x)$. Then we can apply Theorem 3.3.38 with $\mathcal{A} = AB^{-1}$ in the study of existence and uniqueness of bounded quasi-asymptotically uniformly recurrent solutions of the semilinear fractional Poisson heat equation

$$\begin{aligned}
\mathbf{D}_t^\gamma\big[m(x)v(t,x)\big] &= (\Delta-b)v(t,x) + f\big(t,m(x)v(t,x)\big), \quad t \geqslant 0,\ x \in \Omega;\\
v(t,x) &= 0, \quad (t,x) \in [0,\infty)\times\partial\Omega,\\
m(x)v(0,x) &= u_0(x), \quad x \in \Omega.
\end{aligned}$$

It is also worth noting that we can apply Theorem 3.3.38 in the analysis of existence and uniqueness of bounded quasi-asymptotically uniformly recurrent solutions of the following fractional semilinear equation with higher-order differential operators in the Hölder space $X = C^\alpha(\overline{\Omega})$:

$$\begin{cases}
\mathbf{D}_t^\gamma u(t,x) = -\sum_{|\beta|\leqslant 2m} a_\beta(t,x)D^\beta u(t,x) - \sigma u(t,x) + f(t,u(t,x)), & t \geqslant 0,\ x \in \Omega;\\
u(0,x) = u_0(x), & x \in \Omega,
\end{cases}$$

where $\alpha \in (0,1)$, $m \in \mathbb{N}$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^{4m}, $D^\beta = \prod_{i=1}^n(\frac{1}{i}\frac{\partial}{\partial x_i})^{\beta_i}$, the functions $a_\beta : \overline{\Omega} \to \mathbb{C}$ satisfy certain conditions and $\sigma > 0$ is sufficiently large. For more details, see [631].

Basically, our results on the invariance of generalized quasi-asymptotical almost periodicity and uniform recurrence, established in Subsection 3.3.5, can be applied at any place where the variation of parameters formula takes effect. For our purposes, it will be very important to reexamine [1067, Example 5]. It is well known that the unique regular solution of the wave equation $u_{xx}(x,t) = u_{tt}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial conditions $u(x,0) = f(x)$, $x \in \mathbb{R}$, $u_t(x,0) = g(x)$, $x \in \mathbb{R}$, is given by the famous d'Alembert formula

$$u(x,t) := \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds, \quad x \in \mathbb{R},\ t \geqslant 0. \tag{3.65}$$

Let $t_0 > 0$ be a fixed real number. If the function $f(\cdot)$ is quasi-asymptotically uniformly recurrent, resp. $g(\cdot)$ is quasi-asymptotically uniformly recurrent, then the function $x \mapsto 1/2[f(x+t_0) + f(x-t_0)]$, $x \in \mathbb{R}$, resp.

$$H_{t_0}(x) := \frac{1}{2}\int_{x-t_0}^{x+t_0} g(s)\,ds, \quad x \in \mathbb{R},$$

is likewise quasi-asymptotically uniformly recurrent; this can be shown as in [1067]. Their sum will be quasi-asymptotically uniformly recurrent provided that these functions share the same sequence (α_n) in Definition 3.3.8.

4 (ω, c)-Almost periodic type functions, c-almost periodic type functions and applications

4.1 (ω, c)-Almost periodic type functions and applications

The following notion has recently been introduced and analyzed in the case that $I = \mathbb{R}$; see [48, 49]. Let $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. A continuous function $f : I \to X$ is said to be (ω, c)-periodic if and only if $f(t+\omega) = cf(t)$ for all $t \in I$. The number ω is called c-period of f. The space of all (ω, c)-periodic functions $f : I \to X$ will be denoted with $P_{\omega,c}(I : X)$. Let we note that, by putting $c = 1$, we obtain the space of all ω-periodic functions $f : I \to X$; by putting $c = -1$, we obtain the space of all ω-anti-periodic functions $f : I \to X$; by putting $c = e^{ik\omega}$ we obtain the space of all Bloch (ω, k)-periodic functions.

The following facts about the (ω, c)-periodic functions should be stated at the very beginning (see also [48, 49]):

(i) If $f \in P_{\omega,c}([0,\infty) : X)$, $f(\cdot)$ is not identically equal to zero and $|c| > 1$, then $\limsup_{t\to+\infty} \|f(t)\| = +\infty$; if $f \in P_{\omega,c}(\mathbb{R} : X)$ and $|c| > 1$, then $\lim_{t\to-\infty} f(t) = 0$ and, if $f(\cdot)$ is not identically equal to zero, then $\limsup_{t\to+\infty} \|f(t)\| = +\infty$.

(ii) If $f \in P_{\omega,c}(I : X)$ and $f(x) \neq 0$ for all $x \in I$, then the function $(1/f)(\cdot)$ belongs to the class $P_{\omega,1/c}(I : X)$.

(iii) If $f \in P_{\omega,c}(I : X)$ and $|c| = 1$, then the function is almost periodic. To see this, observe that there exists a real number $k \in \mathbb{R}$ such that $f(x+\omega) = e^{ik\omega} f(x)$, $x \in I$, so that the function $f(\cdot)$ is Bloch (ω, k)-periodic. After that, the conclusion simply follows because the function $e^{-ik\cdot} f(\cdot)$ is periodic. In this case, we can simply compute the Bohr spectrum by using the computation

$$\begin{aligned} P_r(f) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\, ds = \lim_{n\to+\infty} \frac{1}{n\omega} \int_0^{n\omega} e^{-irs} f(s)\, ds \\ &= \lim_{n\to+\infty} \frac{1}{n\omega} \sum_{j=0}^{n-1} \int_{j\omega}^{(j+1)\omega} e^{-irs} f(s)\, ds \\ &= \lim_{n\to+\infty} \frac{1}{n\omega} \sum_{j=0}^{n-1} \int_0^{\omega} e^{-ir(s+j\omega)} c^j f(s)\, ds \\ &= \frac{1}{\omega} \int_0^{\omega} e^{-irs} f(s)\, ds \times \lim_{n\to+\infty} \frac{\sum_{j=0}^{n-1} (ce^{-ir\omega})^j}{n}. \end{aligned}$$

Therefore, if $c = e^{ir\omega}$, then $P_r = 1$; otherwise, we have $P_r = 0$ because:

$$\left| \sum_{j=0}^{n-1} (ce^{-ir\omega})^j \right| = \left| \frac{c^n e^{-irn\omega} - 1}{ce^{-ir\omega} - 1} \right| \leqslant \frac{2}{ce^{-ir\omega} - 1}, \quad n \in \mathbb{N}.$$

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Furthermore, arguing as in the above-mentioned remark, we may deduce that for each $k \in \mathbb{R}$ the existence of a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \|f(\cdot + \alpha_n) - e^{ik\alpha_n} f(\cdot)\|_\infty = 0$$

is equivalent to saying that the function $F(\cdot) := e^{-ik\cdot} f(\cdot)$ is uniformly recurrent.

Due to the argumentation given in the proof of [49, Proposition 2.2], with $I = \mathbb{R}$, we see that the function $f(\cdot)$ is (ω, c)-periodic if and only if the function $c^{-\frac{\cdot}{\omega}} f(\cdot)$ belongs to the space $P_\omega(I : X)$. This statement will play an important role in our further work.

In this section, we will consider three different approaches for introducing the spaces of (ω, c)-almost periodic type functions and their Stepanov generalizations. The first approach is the simplest one and (in the case of consideration of (ω, c)-almost automorphic functions and their Stepanov generalizations, we will always tacitly assume that $I = \mathbb{R}$).

Definition 4.1.1. Let $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. Then it is said that a continuous function $f : I \to X$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(ω, c)-almost automorphic/compactly (ω, c)-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by

$$f_{\omega,c}(t) := c^{-(t/\omega)} f(t), \quad t \in I, \tag{4.1}$$

is uniformly recurrent (almost periodic/almost automorphic/compactly almost automorphic). By $\mathrm{UR}_{\omega,c}(I : X)$, $\mathrm{AP}_{\omega,c}(I : X)$, $\mathrm{AA}_{\omega,c}(I : X)$ and $\mathrm{AA}_{\omega,c;\mathbf{c}}(I : X)$ we denote the space of all (ω, c)-uniformly recurrent functions, the space of all (ω, c)-almost periodic functions, the space of all (ω, c)-almost automorphic functions and the space of all compactly (ω, c)-almost automorphic, respectively.

It is clear that the space $P_{\omega,c}(I : X)$ is contained in any of the above introduced spaces. Since the sum of two uniformly recurrent functions need not be uniformly recurrent, $\mathrm{UR}_{\omega,c}(I : E)$ is not a vector space together with the usual operations of addition of functions and pointwise multiplication of functions with scalars [648]. But $\mathrm{AP}_{\omega,c}(I : E)$, $\mathrm{AA}_{\omega,c}(I : E)$ and $\mathrm{AA}_{\omega,c;\mathbf{c}}(I : E)$ are vector spaces together with the above operations. The class of $(\omega, c, \odot_g)$-almost periodic functions can be also introduced and analyzed but we will skip all related details concerning this class of functions for simplicity.

For positive real numbers c_1, $\omega_1 > 0$ and c_2, $\omega_2 > 0$, we have the identity

$$c_1^{-\frac{t}{\omega_1}} c_2^{-\frac{t}{\omega_2}} = \left(c_1^{\frac{\omega}{\omega_1}} c_2^{\frac{\omega}{\omega_2}}\right)^{\frac{-t}{\omega}}, \quad t \in I.$$

With the help of [631, Theorem 2.1.1(ii)], Proposition 2.3.1 and this equality, we can simply deduce the following.

Proposition 4.1.2. *Suppose that $\omega > 0$, c_1, $\omega_1 > 0$, c_2, $\omega_2 > 0$, $f(\cdot)$ is (ω_1, c_1)-almost periodic ((ω_1, c_1)-almost automorphic/compactly (ω_1, c_1)-almost automorphic), $g(\cdot)$ is (ω_2, c_2)-almost periodic ((ω_2, c_2)-almost automorphic/compactly (ω_2, c_2)-almost automorphic) and the function $f(\cdot)$ or the function $g(\cdot)$ is scalar-valued. Set $c := c_1^{\frac{\omega}{\omega_1}} c_2^{\frac{\omega}{\omega_2}}$. Then the function $fg(\cdot)$ is (ω, c)-almost periodic ((ω, c)-almost automorphic/compactly (ω, c)-almost automorphic).*

We continue by giving some elementary observations.

Remark 4.1.3. If the function $f_{\omega,c}(\cdot)$ is bounded and $|c| < 1$, then we have $\lim_{t\to+\infty} f(t) = 0$; moreover, if $I = \mathbb{R}$, the function $f_{\omega,c}(\cdot)$ is bounded and $|c| > 1$, then we have $\lim_{t\to-\infty} f(t) = 0$.

Remark 4.1.4. In (4.1), one can consider an arbitrary function $c(\cdot)$ in place of the function $c^{-(\cdot/\omega)}$ but then the things become much more complicated. For example, following the examination from the previous remark, it seems reasonable to use the function $c^{-(|\cdot|/\omega)}$ in place of the function $c^{-(\cdot/\omega)}$. We will not follow this approach for simplicity and we will consider here only the asymptotically (ω, c)-almost periodic type functions defined on the non-negative real axis.

It is clear that any (ω, c)-almost periodic function is (ω, c)-uniformly recurrent and compactly (ω, c)-almost automorphic, as well as that any compactly (ω, c)-almost automorphic function is (ω, c)-almost automorphic. Even in the case that $c = 1$ and $\omega > 0$ is arbitrary, there exists a compactly almost automorphic function which is not uniformly recurrent and therefore not almost periodic.

Definition 4.1.5. Let $c \in \mathbb{C}$, $|c| \geqslant 1$ and $\omega > 0$. Then it is said that a continuous function $f : [0, \infty) \to X$ is asymptotically (ω, c)-uniformly recurrent (asymptotically (ω, c)-almost periodic, asymptotically (compactly) (ω, c)-almost automorphic) if and only if there exist an (ω, c)-uniformly recurrent ((ω, c)-almost periodic, (compactly) (ω, c)-almost automorphic) function $h : \mathbb{R} \to X$ and a function $q \in C_0([0, \infty) : X)$ such that $f(t) = h(t) + q(t)$ for all $t \geqslant 0$.

The following facts concerning the introduced classes of functions should be stated:

1. Suppose that $|c| = 1$ and $\omega > 0$. Then we can use Theorem 2.1.1(ii) and Proposition 2.3.1 in order to see that the function $f : I \to X$ is (ω, c)-almost periodic ((compactly) (ω, c)-almost automorphic) if and only if $f(\cdot)$ is almost periodic ((compactly) almost automorphic). In the case that $I = [0, \infty)$, the same assertion holds for the asymptotically (ω, c)-almost periodic functions and the asymptotically (compactly) (ω, c)-almost automorphic functions.
2. Suppose that $|c| > 1$, $\omega > 0$ and $f : I \to X$ is (ω, c)-uniformly recurrent or (ω, c)-almost automorphic. If $f(\cdot)$ is not identically equal to zero, then the supremum formula shows that $f(\cdot)$ is unbounded; moreover, in the case of considera-

tion of (ω, c)-almost automorphicity, the function $f(\cdot)$ is unbounded as $t \to +\infty$ due to Remark 4.1.3. In the case that $I = [0, \infty)$, the same assertion holds for the asymptotically (ω, c)-uniformly recurrent functions and the asymptotically (compactly) (ω, c)-almost automorphic functions. In particular, a constant non-zero function cannot be asymptotically (ω, c)-uniformly recurrent or asymptotically (ω, c)-almost automorphic.
3. Suppose $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and $f : [0, \infty) \to X$ is (ω, c)-almost periodic. Then it is well known that there exists a unique almost periodic function $F_{\omega,c} : \mathbb{R} \to X$ such that $F_{\omega,c}(t) = f_{\omega,c}(t)$, $t \geqslant 0$. Define $F(t) := c^{t/\omega} F_{\omega,c}(t)$, $t \in \mathbb{R}$. Then it simply follows that the function $F(\cdot)$ is a unique (ω, c)-almost periodic function which extends the function $f(\cdot)$ to the whole real line.
4. Let $c \in \mathbb{R}$ and $\omega > 0$. Then, for every (ω, c)-uniformly recurrent ((compactly) (ω, c)-almost automorphic) function $f(\cdot)$, we see that the function $\|f(\cdot)\|$ is (ω, c)-uniformly recurrent ((compactly) (ω, c)-almost automorphic). In the case that $I = [0, \infty)$, then the same assertion holds for the asymptotically (ω, c)-uniformly recurrent functions and the asymptotically (compactly) (ω, c)-almost automorphic functions.
5. The spaces $UR_{\omega,c}(I : X)$, $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$ and $AA_{\omega,c;\mathbf{c}}(I : X)$ are invariant under pointwise multiplications with scalars. In the case that $I = [0, \infty)$, the same holds for the corresponding spaces of asymptotically (ω, c)-almost periodic type functions.
6. The spaces $UR_{\omega,c}(I : X)$, $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$ and $AA_{\omega,c;\mathbf{c}}(I : X)$ are translation invariant. In the case that $I = [0, \infty)$, the same holds for the corresponding spaces of asymptotically (ω, c)-almost periodic type functions.
7. If $I = [0, \infty)$, $|c| \geqslant 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $UR_{\omega,c}(I : X)$ ($AP_{\omega,c}(I : X)/AA_{\omega,c}(I : X)/AA_{\omega,c;\mathbf{c}}(I : X)$) converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ belongs to the space $UR_{\omega,c}(I : X)$ ($AP_{\omega,c}(I : X)/AA_{\omega,c}(I : X)/AA_{\omega,c;\mathbf{c}}(I : X)$). In the case that $I = [0, \infty)$, then the same assertion holds for the asymptotically (ω, c)-almost periodic type function spaces.

For completeness, we will include the most relevant details of the proofs of the following two propositions.

Proposition 4.1.6. *Suppose* $X = \mathbb{C}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, $f : I \to \mathbb{C}$ *and* $\inf_{x \in I} |f(x)| > m > 0$. *Then the following hold:*
(i) *If* $|c| = 1$ *and the function* $f(\cdot)$ *is* (ω, c)*-uniformly recurrent ((*ω, c*)-almost periodic/*(ω, c)*-almost automorphic/compactly* (ω, c)*-almost automorphic), then the function* $(1/f)(\cdot)$ *is* $(\omega, 1/c)$*-uniformly recurrent ((*$\omega, 1/c$*)-almost periodic/*$(\omega, 1/c)$*-almost automorphic/compactly* $(\omega, 1/c)$*-almost automorphic).*
(ii) *If* $|c| \leqslant 1$, $I = [0, \infty)$ *and* $f(\cdot)$ *is* (ω, c)*-uniformly recurrent ((*ω, c*)-almost periodic), then the function* $(1/f)(\cdot)$ *is* $(\omega, 1/c)$*-uniformly recurrent ((*$\omega, 1/c$*)-almost periodic).*

Proof. The proof of (i) essentially follows from the simple argumentation and the conclusions obtained in the point [1.], while the proof of (ii) can be deduced as follows. Suppose that the function $f(\cdot)$ is (ω, c)-almost periodic, i. e., the function $f_{\omega,c}(\cdot)$ is almost periodic. This implies that for each number $\varepsilon > 0$ there exists a finite number $l > 0$ such that any subinterval I' of I contains at least one point τ such that

$$\left|c^{-\frac{t+\tau}{\omega}} f(t+\tau) - c^{-\frac{t}{\omega}} f(t)\right| \leqslant \varepsilon, \quad t \geqslant 0.$$

This implies

$$\left|f(t+\tau) - c^{-\frac{\tau}{\omega}} f(t)\right| \leqslant \varepsilon \left|c^{\frac{t+\tau}{\omega}}\right|, \quad t \geqslant 0.$$

Then the final conclusion is a consequence of the following simple calculation:

$$\begin{aligned}\left|\frac{c^{\frac{t+\tau}{\omega}}}{f(t+\tau)} - \frac{c^{\frac{t}{\omega}}}{f(t)}\right| &= \left|c^{\frac{t}{\omega}}\right| \cdot \left|\frac{f(t+\tau) - c^{-\frac{\tau}{\omega}} f(t)}{f(t+\tau) \cdot f(t)}\right| \\ &\leqslant \frac{\varepsilon}{m^2}\left|c^{\frac{2t+\tau}{\omega}}\right| \leqslant \frac{\varepsilon}{m^2}, \quad t \geqslant 0.\end{aligned}$$

The proof for (ω, c)-uniform recurrence is similar and therefore omitted. □

Proposition 4.1.7. *Suppose that $I = \mathbb{R}$, $f : \mathbb{R} \to X$ satisfies the requirement that the function $f_{\omega,c}(\cdot)$ is a bounded uniformly recurrent (almost periodic, (compactly) almost automorphic) and $c^{-\frac{\cdot}{\omega}}\psi(\cdot) \in L^1(\mathbb{R})$. Then the function $c^{-\frac{\cdot}{\omega}}(\psi * f)(\cdot)$ is bounded uniformly continuous and the function $(\psi * f)(\cdot)$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(compactly) (ω, c)-almost automorphic).*

Proof. For every $x \in \mathbb{R}$, the convolution $(\psi * f)(x)$ is well defined and we have

$$c^{-\frac{x}{\omega}}(\psi * f)(x) = \int_{-\infty}^{\infty} \left[c^{-\frac{x-y}{\omega}} \psi(x-y)\right] \cdot \left[c^{-\frac{y}{\omega}} f(y)\right] dy, \quad x \in \mathbb{R}.$$

Then the corresponding statement follows from the fact that the space of all almost periodic ((compactly) almost automorphic) functions and the space of all bounded uniformly recurrent functions are convolution invariant. □

The following definitions are logical analogues of Definition 4.1.1–Definition 4.1.5 for Stepanov classes.

Definition 4.1.8. Let $p \in \mathcal{P}([0,1])$, $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$. Then it is said that a function $f \in L^{p(x)}_{\mathrm{loc}}(I : X)$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent (Stepanov $(p(x), \omega, c)$-almost periodic/Stepanov $(p(x), \omega, c)$-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by (4.1), is Stepanov $p(x)$-uniformly recurrent (Stepanov $p(x)$-almost periodic/Stepanov $p(x)$-almost automorphic).

By $S^{p(x)}\mathrm{UR}_{\omega,c}(I : X)$, $S^{p(x)}\mathrm{AP}_{\omega,c}(I : X)$ and $S^{p(x)}\mathrm{AA}_{\omega,c}(I : X)$ we denote the space of all Stepanov $(p(x), \omega, c)$-uniformly recurrent functions, the space of all Stepanov

$(p(x), \omega, c)$-almost periodic functions and the space of all Stepanov $(p(x), \omega, c)$-almost automorphic functions, respectively. If $p(x) \equiv p \in [1, \infty)$, then by $S^p\mathrm{UR}_{\omega,c}(I : X)$, $S^p\mathrm{AP}_{\omega,c}(I : X)$ and $S^p\mathrm{AA}_{\omega,c}(I : X)$ we denote the space of all Stepanov (p, ω, c)-uniformly recurrent functions, the space of all Stepanov (p, ω, c)-almost periodic functions and the space of all Stepanov (p, ω, c)-almost automorphic functions, respectively.

Definition 4.1.9. Let $p \in \mathcal{P}([0, 1])$, $c \in \mathbb{C}$, $|c| \geqslant 1$ and $\omega > 0$. Then it is said that a function $f \in L^{p(x)}_{\mathrm{loc}}([0, \infty) : X)$ is asymptotically Stepanov $(p(x), \omega, c)$-uniformly recurrent (asymptotically Stepanov $(p(x), \omega, c)$-almost periodic, asymptotically Stepanov $(p(x), \omega, c)$-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by (4.1), is asymptotically Stepanov $p(x)$-uniformly recurrent (asymptotically Stepanov $p(x)$-almost periodic, asymptotically Stepanov $p(x)$-almost automorphic).

By $\mathrm{AS}^{p(x)}\mathrm{UR}_{\omega,c}(I : X)$, $\mathrm{AS}^{p(x)}\mathrm{AP}_{\omega,c}(I : X)$ and $\mathrm{AS}^{p(x)}\mathrm{AA}_{\omega,c}(I : X)$ we denote the space of all asymptotically Stepanov $(p(x), \omega, c)$-uniformly recurrent functions, the space of all asymptotically Stepanov $(p(x), \omega, c)$-almost periodic functions and the space of all asymptotically Stepanov $(p(x), \omega, c)$-almost automorphic functions, respectively. If $p(x) \equiv p \in [1, \infty)$, then by $\mathrm{AS}^p\mathrm{UR}_{\omega,c}(I : X)$, $\mathrm{AS}^p\mathrm{AP}_{\omega,c}(I : X)$ and $\mathrm{AS}^p\mathrm{AA}_{\omega,c}(I : X)$ we denote the space of all asymptotically Stepanov (p, ω, c)-uniformly recurrent functions, the space of all asymptotically Stepanov (p, ω, c)-almost periodic functions and the space of all asymptotically Stepanov (p, ω, c)-almost automorphic functions, respectively.

The conclusions established in the points [1.–2., 4.–7.] can be simply reformulated for the Stepanov classes. For example, if we consider the point [2.], then we may conclude the following: Suppose that $|c| > 1$, $\omega > 0$ and $f : I \to X$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent or Stepanov $(p(x), \omega, c)$-almost automorphic. If $f(\cdot)$ is not almost everywhere equal to zero, then the function $f(\cdot)$ is not Stepanov $p(x)$-bounded; moreover, in the case of consideration of Stepanov $(p(x), \omega, c)$-almost automorphicity, the function $\hat{f}(\cdot)$ is unbounded as $t \to +\infty$ so that a constant non-zero function cannot be Stepanov $(p(x), \omega, c)$-uniformly recurrent or Stepanov $(p(x), \omega, c)$-almost automorphic.

Essentially, any established result for almost periodic type functions and their Stepanov generalizations can be straightforwardly reformulated for (ω, c)-almost periodic type functions and their Stepanov generalizations (in the sequel, we will try not to consider such statements). For example, using the corresponding statement for the uniformly recurrent functions we can immediately deduce the following.

Proposition 4.1.10. *Let $p \in \mathcal{P}([0, 1])$. If $f : [0, \infty) \to X$ satisfies the requirement that the function $f_{\omega,c}(\cdot)$ is uniformly continuous and asymptotically Stepanov $p(x)$-uniformly recurrent, then the function $f(\cdot)$ is asymptotically (ω, c)-uniformly recurrent.*

Let us only note that the uniform continuity of the function $f_{\omega,c}(\cdot)$ is ensured provided that $|c| \geqslant 1$ and $f(\cdot)$ is a bounded uniformly continuous function. This follows from the fact that, for every two non-negative real numbers $t_1,\ t_2 \geqslant 0$ such that $t_1 < t_2$, the Darboux inequality yields

$$\begin{aligned}\left\|c^{-\frac{t_1}{\omega}} f(t_1) - c^{-\frac{t_2}{\omega}} f(t_2)\right\| &\leqslant \left\|c^{-\frac{t_1}{\omega}}\left[f(t_1) - f(t_2)\right]\right\| + \left\|\left[c^{-\frac{t_1}{\omega}} - c^{-\frac{t_2}{\omega}}\right] f(t_2)\right\| \\ &\leqslant \left\|f(t_1) - f(t_2)\right\| + \frac{1}{\omega}\left(\ln|c| + \pi\right) \cdot |t_1 - t_2| \cdot \|f\|_\infty.\end{aligned}$$

Now we would like to endow the introduced spaces of (asymptotically) (ω, c)-almost periodic type functions with certain norms. We start with the notion introduced in Definition 4.1.1 and Definition 4.1.5. Define

$$\|f\|_{\omega,c} := \sup_{t \in I} \left\|c^{-\frac{t}{\omega}} f(t)\right\|.$$

Proposition 4.1.11. *The spaces* $\mathrm{AP}_{\omega,c}(I : X)$, $\mathrm{AA}_{\omega,c}(I : X)$, $\mathrm{AA}_{\omega,c;\mathbf{c}}(I : X)$, $\mathrm{AAP}_{\omega,c}([0,\infty) : X)$, $\mathrm{AAA}_{\omega,c}([0,\infty) : X)$ *and* $\mathrm{AAA}_{\omega,c;\mathbf{c}}([0,\infty) : X)$, *equipped with the norm* $\|\cdot\|_{\omega,c}$, *are Banach spaces.*

Proof. Denote by $\mathcal{X}$ any of the above spaces. Suppose that $(f_n)_n$ is a Cauchy sequence in $\mathcal{X}$. Hence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m,\ n \geqslant N$, we have $\|f_n - f_m\|_{\omega,c} < \varepsilon$. So, there exist $u_m,\ u_n \in c^{-\frac{\cdot}{\omega}}\mathcal{X}$ (with the meaning clear) such that $f_m(t) = c^{\frac{t}{\omega}} u_m(t)$ and $f_n(t) = c^{\frac{t}{\omega}} u_n(t)$ for all $t \in I$. For $m,\ n \geqslant N$, we have

$$\begin{aligned}\|u_m - u_n\|_\infty &= \sup_{t\in I}\left\|u_m(t) - u_n(t)\right\| \\ &= \sup_{t\in I}\left\|c^{-\frac{t}{\omega}} f_m(t) - c^{-\frac{t}{\omega}} f_n(t)\right\| \\ &= \sup_{t\in I}\left\||c|^{-\frac{t}{\omega}}\left[f_m(t) - f_n(t)\right]\right\| \\ &= \|f_n - f_m\|_{\omega,c} < \varepsilon.\end{aligned}$$

Hence, $(u_n)_n$ is a Cauchy sequence in $c^{-\frac{\cdot}{\omega}}\mathcal{X}$, which is a complete space. Then there exists $u \in c^{-\frac{\cdot}{\omega}}\mathcal{X}$ such that $\lim_{n\to+\infty} u_n = u$. Define $f(t) := c^{\frac{t}{\omega}} u(t)$, $t \in I$. Thus,

$$\begin{aligned}\|f_n - f\|_{\omega,c} &= \sup_{t\in I}\left\||c|^{-\frac{t}{\omega}}\left[f_n(t) - f(t)\right]\right\| \\ &= \sup_{t\in I}\left\||c|^{-\frac{t}{\omega}} c^{\frac{t}{\omega}} u_n(t) - |c|^{-\frac{t}{\omega}} c^{\frac{t}{\omega}} u(t)\right\| \\ &= \sup_{t\in I}\left\|u_n(t) - u(t)\right\| \to 0,\end{aligned}$$

when $n \to \infty$. Hence, $\mathcal{X}$ is a Banach space. □

For any $c \in \mathbb{C} \setminus \{0\}$ and $p \in [1,\infty)$, we denote by $L^p_{S,c}(I : X)$ the space of all functions $f \in L^p_{\mathrm{loc}}(I : X)$ such that

$$\|f\|_{p,\omega,c} := \sup_{t\in I}\left(\int_t^{t+1} |c|^{-\frac{s}{\omega}} f(s)\,ds\right)^{1/p}.$$

Then $(L^p_{S,c}(I : X), \|\cdot\|_{p,\omega,c})$ is a Banach space. Arguing as above, we may conclude that $S^p\mathrm{AP}_{\omega,c}(I : X)$ $(S^p\mathrm{AA}_{\omega,c}(I : X)/\mathrm{AS}^p\mathrm{AP}_{\omega,c}(I : X)$, $\mathrm{AS}^p\mathrm{AA}_{\omega,c}(I : X))$ is a closed subspace of $L^p_{S,c}(I : X)$ and therefore a Banach space itself.

Before proceeding further, we want to recommend for the reader the recent research [1] by L. Abadias, E. Alvarez and R. Grau concerning (ω, c)-periodic mild solutions to non-autonomous abstract differential equations.

4.1.1 (ω, c)-Uniform recurrence of type i and (ω, c)-almost periodicity of type i $(i = 1, 2)$

Suppose temporarily that $f \in P_{\omega,c}(I : X)$ and $n \in \mathbb{N}$. Then we have $f(t + n\omega) = c^n f(t)$, $t \in I$. Setting $\alpha_n = n\omega$, we get for each $\varepsilon > 0$ and $t \in I$

$$\left\|f(t+\alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t)\right\| \leqslant \varepsilon \quad \text{and} \quad \left\|c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t)\right\| \leqslant \varepsilon. \tag{4.2}$$

Equation (4.2) motivates us to introduce the following concepts of (ω, c)-uniform recurrence and (ω, c)-almost periodicity [it is not clear how we can do that for (compact) (ω, c)-almost automorphicity in a satisfactory way].

Definition 4.1.12. Suppose that $f : I \to X$ is continuous, $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.

(i) We say that $f(\cdot)$ is (ω, c)-uniformly recurrent of type 1 (type 2) if and only if there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\in I} \left\|f(t+\alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t)\right\| = 0 \quad \Big(\lim_{n\to+\infty} \sup_{t\in I} \left\|c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t)\right\| = 0\Big).$$

(ii) We say that $f(\cdot)$ is (ω, c)-almost periodic of type 1 (type 2) if and only if for each $\varepsilon > 0$ the set

$$\Big\{\tau > 0 : \sup_{t\in I} \left\|f(t+\tau) - c^{\frac{\tau}{\omega}} f(t)\right\| < \varepsilon\Big\} \quad \Big(\Big\{\tau > 0 : \sup_{t\in I} \left\|c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t)\right\| < \varepsilon\Big\}\Big)$$

is relatively dense in $[0, \infty)$.

By $\mathrm{UR}_{\omega,c,i}(I : X)$ and $\mathrm{AP}_{\omega,c,i}(I : X)$, we denote the space of all (ω, c)-uniformly recurrent functions of type i and the space of all (ω, c)-almost periodic functions of type i, respectively $(i = 1, 2)$.

It is clear that the set $\{n\omega : n \in \mathbb{N}\}$ is relatively dense in $[0, \infty)$. Taking into account this observation, it follows that the space $P_{\omega,c}(I : X)$ is contained in the spaces $\mathrm{UR}_{\omega,c,i}(I : X)$ and $\mathrm{AP}_{\omega,c,i}(I : X)$, for $i = 1, 2$; moreover, $\mathrm{UR}_{\omega,c,i}(I : X) \supseteq \mathrm{AP}_{\omega,c,i}(I : X)$ for

$i = 1, 2$ and it is clear that for any $t \in I$ and $\tau \geqslant 0$ we have

$$\begin{aligned}\left\|c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t)\right\| &= \left\|c^{\frac{-\tau}{\omega}}\left[f(t+\tau) - c^{\frac{\tau}{\omega}} f(t)\right]\right\| \\ &= |c|^{\frac{-\tau}{\omega}}\left\|f(t+\tau) - c^{\frac{\tau}{\omega}} f(t)\right\|.\end{aligned}$$

Therefore, in the case that $|c| = 1$, it simply follows that the (ω, c)-almost periodicity of type 1 (type 2) is equivalent with the usual almost periodicity as well as that the notion of (ω, c)-uniform recurrence of type 1 is equivalent with the notion of (ω, c)-uniform recurrence of type 2.

But, in the case that $|c| \neq 1$, the concepts introduced in Definition 4.1.12 are not satisfactory to a great extent. Before stating the corresponding result which justifies this fact, let us denote by $M_{\omega,c}(I : X)$ the space consisting of all functions $f : I \to X$ such that $c^{-\cdot/\omega} f(\cdot) \in P(I : X)$. It is clear that $M_{\omega,c}(I : X)$ is not a vector space together with the usual operations.

Theorem 4.1.13. *Let $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.*
(i) *Suppose that $|c| > 1$. Then $\mathrm{UR}_{\omega,c,i}(I : X) = \mathrm{AP}_{\omega,c,i}(I : X) = M_{\omega,c}(I : X)$ for $i = 1, 2$.*
(ii) *Suppose that $|c| < 1$ and $I = \mathbb{R}$. Then $\mathrm{UR}_{\omega,c,i}(I : X) = \mathrm{AP}_{\omega,c,i}(I : X) = M_{\omega,c}(I : X)$ for $i = 1, 2$.*

Before giving the proof of Theorem 4.1.13, we will state two lemmas. The first one is simple and follows almost immediately from Definition 4.1.12.

Lemma 4.1.14. *Suppose that $f : I \to X$ is continuous, $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.*
(i) *If $|c| \geqslant 1$, then $\mathrm{UR}_{\omega,c,1}(I : X) \subseteq \mathrm{UR}_{\omega,c,2}(I : X)$ and $\mathrm{AP}_{\omega,c,1}(I : X) \subseteq \mathrm{AP}_{\omega,c,2}(I : X)$.*
(ii) *If $|c| \leqslant 1$, then $\mathrm{UR}_{\omega,c,1}(I : X) \supseteq \mathrm{UR}_{\omega,c,2}(I : X)$ and $\mathrm{AP}_{\omega,c,1}(I : X) \supseteq \mathrm{AP}_{\omega,c,2}(I : X)$.*
(iii) *In the case that $I = [0, \infty)$ and $|c| \geqslant 1$, then $\mathrm{UR}_{\omega,c,2}(I : X) \subseteq \mathrm{UR}_{\omega,c}(I : X)$ and $\mathrm{AP}_{\omega,c,2}(I : X) \subseteq \mathrm{AP}_{\omega,c}(I : X)$.*

Lemma 4.1.15. *Suppose that $I = \mathbb{R}$ and $f : \mathbb{R} \to X$. Then $f(\cdot)$ is (ω, c)-uniformly recurrent of type 1 (type 2) [(ω, c)-almost periodic of type 1 (type 2)] if and only if the function $\check{f}(\cdot)$ is $(\omega, 1/c)$-uniformly recurrent of type 2 (type 1) [(ω, c)-almost periodic of type 2 (type 1)].*

Proof. The proof simply follows by observing that, for every $\tau > 0$ and $\varepsilon > 0$, we have

$$\begin{gathered}\sup_{t\in I}\left\|f(t+\tau) - c^{\frac{\tau}{\omega}} f(t)\right\| < \varepsilon \Leftrightarrow \sup_{t\in I}\left\|f(-t+\tau) - c^{\frac{\tau}{\omega}} f(-t)\right\| < \varepsilon \\ \Updownarrow \\ \sup_{t\in I}\left\|\check{f}(t-\tau) - c^{\frac{\tau}{\omega}} \check{f}(t)\right\| < \varepsilon \Leftrightarrow \sup_{t\in I}\left\|\check{f}(t) - c^{\frac{\tau}{\omega}} \check{f}(t+\tau)\right\| < \varepsilon \\ \Updownarrow \\ \sup_{t\in I}\left\|(1/c)^{-\frac{\tau}{\omega}} \check{f}(t+\tau) - \check{f}(t)\right\| < \varepsilon.\end{gathered}$$

□

Proof of Theorem 4.1.13. Keeping in mind Lemma 4.1.15, it suffices to prove (i). Towards this end, we recognize two cases: $I = [0,\infty)$ and $I = \mathbb{R}$. In the first case, it suffices to show that $UR_{\omega,c,2}([0,\infty) : X) \subseteq M_{\omega,c}([0,\infty) : X)$ and $M_{\omega,c}([0,\infty) : X) \subseteq AP_{\omega,c,1}([0,\infty) : X)$. So, let $f \in UR_{\omega,c,2}([0,\infty) : X)$. This implies that there exist a finite constant $M > 0$ and a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\sup_{t\in I, n\in\mathbb{N}} \left\| c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t) \right\| \leqslant M.$$

Since $f(t) = c^{t/\omega} f_{\omega,c}(t)$, $t \geqslant 0$, the above implies

$$\left\| f_{\omega,c}(t+\alpha_n) - f_{\omega,c}(t) \right\| \leqslant |c|^{-(t/\omega)} M, \quad t \geqslant 0,\ n \in \mathbb{N}.$$

Hence, for every $n \in \mathbb{N}$, we have $\lim_{t\to+\infty}[f_{\omega,c}(t+\alpha_n) - f_{\omega,c}(t)] = 0$. On the other hand, Lemma 4.1.14(iii) shows that, for every $n \in \mathbb{N}$, we see that the function $f_{\omega,c}(\cdot+\alpha_n) - f_{\omega,c}(\cdot)$ is uniformly recurrent; hence, for every $n \in \mathbb{N}$, we have $f_{\omega,c}(\cdot+\alpha_n) \equiv f_{\omega,c}(\cdot)$ and therefore $f_{\omega,c}(\cdot)$ belongs to the space $P([0,\infty) : X)$, as claimed. To see that $M_{\omega,c}([0,\infty) : X) \subseteq AP_{\omega,c,1}([0,\infty) : X)$, suppose that $f_{\omega,c}(t+T) = f_{\omega,c}(t)$ for all $t \geqslant 0$ and some $T > 0$. This simply implies that $f(t+nT) = c^{nT/\omega} f(t)$ for all $n \in \mathbb{N}$ so that $f \in AP_{\omega,c,1}([0,\infty) : X)$ because the set $\{nT : n \in \mathbb{N}\}$ is relatively dense in $[0,\infty)$. Suppose now that $I = \mathbb{R}$. Similarly as above, it follows that $UR_{\omega,c,i}(\mathbb{R} : X) \supseteq AP_{\omega,c,i}(\mathbb{R} : X) \supseteq M_{\omega,c}(\mathbb{R} : X)$ for $i = 1, 2$. Therefore, it suffices to show that $UR_{\omega,c,2}(\mathbb{R} : X) \subseteq M_{\omega,c}(\mathbb{R} : X)$. Let $f \in UR_{\omega,c,2}(\mathbb{R} : X)$. Since the restriction of $f(\cdot)$ on $[0,\infty)$ belongs to the space $UR_{\omega,c,2}([0,\infty) : X)$, it readily follows that there exists a number $T > 0$ such that $f_{\omega,c}(t+T) = f_{\omega,c}(t)$ for all $t \geqslant 0$. To complete the proof, it suffices to prove that this equality holds for all real numbers $t < 0$. Let $\varepsilon > 0$ be fixed. Due to our assumption, we have the existence of an integer $n_0 \in \mathbb{N}$ such that $t + \alpha_n > 0$ and that

$$\begin{gathered} \left\| c^{t/\omega} f_{\omega,c}(t+\alpha_n) - c^{t/\omega} f_{\omega,c}(t) \right\| \leqslant \varepsilon \quad \text{and} \\ \left\| c^{(t+T)/\omega} f_{\omega,c}(t+T+\alpha_n) - c^{(t+T)/\omega} f_{\omega,c}(t+T) \right\| \leqslant \varepsilon, \end{gathered}$$

i. e.,

$$\begin{gathered} \left\| c^{t/\omega} f_{\omega,c}(t+\alpha_n) - c^{t/\omega} f_{\omega,c}(t) \right\| \leqslant \varepsilon \quad \text{and} \\ \left\| c^{t/\omega} f_{\omega,c}(t+\alpha_n) - c^{t/\omega} f_{\omega,c}(t+T) \right\| \leqslant \varepsilon |c|^{-T/\omega}. \end{gathered}$$

This implies

$$\left\| c^{t/\omega} [f_{\omega,c}(t+T) - f_{\omega,c}(t)] \right\| \leqslant \varepsilon \big(1 + |c|^{-T/\omega}\big).$$

Letting $\varepsilon \to 0+$, we get $f_{\omega,c}(t+T) = f_{\omega,c}(t)$, as claimed. □

Corollary 4.1.16. *Suppose that $i = 1, 2$, $|c| < 1$, $\omega > 0$ and $f \in \mathrm{AP}_{\omega,c,i}([0,\infty) : X)$. Then there exists a function $F \in \mathrm{AP}_{\omega,c,i}(\mathbb{R} : X)$ such that $F(t) = f(t)$ for all $t \geqslant 0$ if and only if $f \in M_{\omega,c}([0,\infty) : X)$.*

Furthermore, the points [4., 5., 6., 7.] from the beginning of this section can be restated as follows:

4′. Let $c \in \mathbb{R}$ and $\omega > 0$. Then, for every (ω, c)-uniformly recurrent function $f(\cdot)$ of type 1 (type 2), we see that the function $\|f(\cdot)\|$ is $(\omega, |c|)$-uniformly recurrent of type 1 (type 2).

5′. The spaces $\mathrm{UR}_{\omega,c,i}(I : X)$ and $\mathrm{AP}_{\omega,c,i}(I : X)$ are invariant under pointwise multiplications with scalars ($i = 1, 2$).

6′. The spaces $\mathrm{UR}_{\omega,c,i}(I : X)$ and $\mathrm{AP}_{\omega,c,i}(I : X)$ are translation invariant ($i = 1, 2$).

7′. If $I = [0,\infty)$, $|c| \geqslant 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $\mathrm{UR}_{\omega,c,2}(I : X)$ converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ belongs to the space $\mathrm{UR}_{\omega,c,2}(I : X)$. Furthermore, if $I = [0,\infty)$, $|c| \leqslant 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $\mathrm{UR}_{\omega,c,1}(I : X)$ ($\mathrm{AP}_{\omega,c,1}(I : X)$) converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ belongs to the space $\mathrm{UR}_{\omega,c,1}(I : X)$ ($\mathrm{AP}_{\omega,c,1}(I : X)$).

Now we will prove the following.

Proposition 4.1.17. *Suppose that $i = 1, 2$, $|c| < 1$, $\omega > 0$ and $f \in \mathrm{AP}_{\omega,c,i}(I : X)$. Then the function $f_{\omega,c}(\cdot)$ is bounded and $\lim_{t\to+\infty} f(t) = 0$.*

Proof. By Theorem 4.1.13(ii) and Lemma 4.1.14(iv) it suffices to consider the case $I = [0,\infty)$ and the class $\mathrm{AP}_{\omega,c,1}([0,\infty) : X)$. Let $\varepsilon = 1$. Then there exists a finite number $l > 0$ such that any subinterval I' of $[0,\infty)$ contains a point τ such that $\|c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t)\| < 1$ for all $t \geqslant 0$. Suppose that $t \in [nl, (n+1)l]$ for some $n \in \mathbb{N}$. Then there exists $\tau_n \in [(n-1)l, nl]$ such that $\|c^{\frac{-\tau_n}{\omega}} f(t'+\tau_n) - f(t')\| < 1$ for all $t' \geqslant 0$. In particular, $t - \tau_n = t' \in [0, 2l]$ and the above implies $\|f(t)\| \leqslant (1+M)|c|^{\tau_n/\omega} \leqslant (1+M)[\max_{t''\in[0,2l]} |c|^{-t''/\omega}]|c|^{t/\omega}$, where $M := \sup_{x\in[0,2l]} \|f(x)\|$. This yields the required limit equality. □

Example 4.1.18. Denote the restriction of the function $f(\cdot)$ given by (2.28) to the non-negative real axis by the same symbol. Then Proposition 4.1.17 implies that the function $c^{-\cdot/\omega} f(\cdot)$ cannot belong to the space $\mathrm{AP}_{\omega,c,i}([0,\infty) : \mathbb{C})$ for $i = 1, 2$. On the other hand, it is clear that $c^{-\cdot/\omega} f(\cdot) \in \mathrm{UR}_{\omega,c}([0,\infty) : \mathbb{C}) \subseteq \mathrm{UR}_{\omega,c,i}([0,\infty) : \mathbb{C})$ for $i = 1, 2$.

Corollary 4.1.19. *Suppose that $|c| < 1$ and $\omega > 0$. Then $f \in \mathrm{AP}_{\omega,c,1}([0,\infty) : X)$ if and only if the function $f_{\omega,c}(\cdot)$ is bounded and continuous.*

Proof. Due to Proposition 4.1.17, it suffices to show that the boundedness of function $f_{\omega,c}(\cdot)$ implies $f \in \mathrm{AP}_{\omega,c,1}([0,\infty) : X)$. If so, then we need to prove that for each $\varepsilon > 0$ the set consisting of all positive reals $t > 0$ such that

$$\left\|c^{(t+\tau)/\omega} f_{\omega,c}(t+\tau) - c^{(t+\tau)/\omega} f_{\omega,c}(t)\right\| \leqslant \varepsilon, \quad t \geqslant 0$$

is relatively dense in $[0, \infty)$. But this simply follows from the fact that this set contains a ray $[a(\varepsilon), \infty)$ for a sufficiently large real number $a(\varepsilon) > 0$, which can be proved by using the boundedness of $f_{\omega,c}(\cdot)$ and the simple inequality $|c|^{t/\omega} \leqslant 1$, $t \geqslant 0$. □

Remark 4.1.20. Suppose that $|c| < 1$ and $\omega > 0$. Using Corollary 4.1.19, we can simply prove that $f \in AP_{\omega,c,1}([0, \infty) : X)$ if and only if for every (there exists) strictly increasing sequence (α_n) of positive reals tending to plus infinity such that $\lim_{n\to+\infty} \sup_{t\geqslant 0} \|f(t + \alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t)\| = 0$.

Example 4.1.21. Suppose that $f(t) := 2^{-t}[1+(1/\ln(2+t))]$, $t \geqslant 0$. Due to Corollary 4.1.19, this function belongs to the space $AP_{1,1/2,1}([0, \infty) : \mathbb{C}) \subseteq UR_{1,1/2,1}([0, \infty) : \mathbb{C})$. On the other hand, $f(\cdot)$ does not belong to the space $UR_{1,1/2,2}([0, \infty) : \mathbb{C})$. Otherwise, we would have the existence of an arbitrarily large positive real number $\alpha > 0$ such that

$$\sup_{t\geqslant 0}\left|2^{-t}\frac{\ln(1 + (\alpha/(1 + t)))}{\ln(2 + t) \cdot \ln(2 + t + \alpha)}\right| \leqslant \varepsilon.$$

Taking $t = 0$, this simply leads us to a contradiction.

The class $UR_{\omega,c,1}([0, \infty) : X)$ is also extremely non-interesting due to the following characterization.

Proposition 4.1.22. *Suppose $c \in \mathbb{C} \setminus \{0\}$, $|c| < 1$ and $\omega > 0$. Then* $UR_{\omega,c,1}([0, \infty) : X) = C_0([0, \infty) : X)$.

Proof. If $f \in C_0([0, \infty) : X)$, then for each strictly increasing sequence (α_n) tending to plus infinity and for each real number $\varepsilon > 0$ we can always find an integer $n_0 \in \mathbb{N}$ such that $\|f(t + \alpha_n) - c^{\alpha_n/\omega} f(t)\| \leqslant (\varepsilon/2) + |c|^{\alpha_n/\omega}\|f(t)\| \leqslant (\varepsilon/2) + |c|^{\alpha_n/\omega}\|f\|_\infty \leqslant \varepsilon$, $t \geqslant 0$, $n \geqslant n_0$, which implies $f \in UR_{\omega,c,1}([0, \infty) : X)$. To prove the converse, let us first show that the assumption $f \in UR_{\omega,c,1}([0, \infty) : X)$ implies the boundedness of $f(\cdot)$. If (α_n) satisfies the requirements of the definition of the space $UR_{\omega,c,1}([0, \infty) : X)$, then we may assume without loss of generality that $\alpha_{n+1} - \alpha_n > 3$ for all $n \in \mathbb{N}$ and

$$\|f(t + \alpha_n)\| \leqslant 1 + |c|^{\alpha_n/\omega}\|f(t)\|, \quad t \geqslant 0, \ n \in \mathbb{N}. \tag{4.3}$$

Let $n \in \mathbb{N}$ be fixed and let $M_0 := \max_{t\in[0,\alpha_n]} \|f(t)\|$. Then (4.3) inductively implies that for arbitrary $T \in (0, \alpha_n]$ and for arbitrary $k \in \mathbb{N}$ we have

$$\|f(T + k\alpha_n)\| \leqslant \sum_{j=0}^{k-1} |c|^{\alpha_n j/\omega} + |c|^{k\alpha_n/\omega} M_0 \leqslant \sum_{j=0}^{\infty} |c|^{j/\omega} + M_0.$$

Therefore, $\|f(t)\| \leqslant \sum_{j=0}^{\infty} |c|^{j/\omega} + M_0$, $t \geqslant 0$, as claimed. The remainder of the proof is simple; since the function $f(\cdot)$ is bounded, we have the existence of an integer $n_1 \in \mathbb{N}$ such that

$$\|f(t + \alpha_n)\| \leqslant |c|^{\alpha_n/\omega}\|f\|_\infty + (\varepsilon/2) < \varepsilon, \quad t \geqslant 0, \ n \geqslant n_1,$$

and therefore $f \in C_0([0, \infty) : X)$. □

Now we will prove the following result.

Proposition 4.1.23. *Suppose that* $|c| < 1$ *and* $\omega > 0$. *Then* $f \in AP_{\omega,c,2}([0,\infty) : X)$ *if and only if the function* $f_{\omega,c}(\cdot)$ *is bounded continuous and for each* $\varepsilon > 0$ *and* $N > 0$ *the set of all positive real numbers* $\tau > 0$ *such that*

$$\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leqslant \varepsilon, \quad t \in [0,N] \tag{4.4}$$

is relatively dense in $[0,\infty)$.

Proof. Suppose first that $f \in AP_{\omega,c,2}([0,\infty) : X)$. Due to Proposition 4.1.17, the function $f_{\omega,c}(\cdot)$ is bounded. Let $\varepsilon > 0$ and $N > 0$ be fixed, and let $\varepsilon_0 > 0$ be such that $\varepsilon_0|c|^{-N/\omega} \leqslant \varepsilon$. By our assumption, the set of all positive reals $\tau > 0$ such that $\|f_{\omega,c}(t+\tau)-f_{\omega,c}(t)\| \leqslant \varepsilon_0|c|^{-t/\omega}$, $t \geqslant 0$ is relatively dense in $[0,\infty)$. If τ belongs to this set, then we have $\|f_{\omega,c}(t+\tau)-f_{\omega,c}(t)\| \leqslant \varepsilon_0|c|^{-t/\omega} \leqslant \varepsilon$, $t \in [0,N]$. For the converse, it suffices to assume $f_{\omega,c} \neq 0$. Fix a number $\varepsilon > 0$. In this case, we can find a number $N > 0$ such that $|c|^{t/\omega} \leqslant \varepsilon/(2(1+\|f_{\omega,c}\|_\infty))$ for all $t \geqslant N$. For this $\varepsilon > 0$ and $N > 0$ we can find a relatively dense set of positive reals τ satisfying (4.4). If τ belongs to this set, then there exist two possibilities: $t \geqslant N$ or $t \in [0,N)$. In the first case, we have $\|c^{t/\omega}[f_{\omega,c}(t+\tau)-f_{\omega,c}(t)]\| \leqslant \varepsilon|c|^{t/\omega} \leqslant \varepsilon$; in the second case, we have $\|c^{t/\omega}[f_{\omega,c}(t+\tau)-f_{\omega,c}(t)]\| \leqslant (2\varepsilon\|f_{\omega,c}\|_\infty)/(2(1+\|f_{\omega,c}\|_\infty)) < \varepsilon$. Hence, we have $\|f_{\omega,c}(t+\tau)-f_{\omega,c}(t)\| \leqslant \varepsilon_0|c|^{-t/\omega}$, $t \geqslant 0$ and the proof of the proposition is thereby complete. □

Remark 4.1.24.

(i) Let us recall that any Levitan N-almost periodic function $f_{\omega,c} : [0,\infty) \to X$ satisfies the requirement that for each $\varepsilon > 0$ and $N > 0$ the set of all positive reals $\tau > 0$ such that (4.4) holds is relatively dense in $[0,\infty)$ (cf. [697, Definition 2, p. 53]). In particular, the restriction of any almost automorphic function $f_{\omega,c} : \mathbb{R} \to X$ satisfies this condition. Denote by $AA_{[0,\infty)}(X)$ the vector space consisting of such functions; thus, $c^{\cdot/\omega}AA_{[0,\infty)}(X) \subseteq AP_{\omega,c,2}([0,\infty) : X)$. Recall also that the function $t \mapsto 1/(2+\cos t+\cos(\sqrt{2}t))$, $t \geqslant 0$ is Levitan N-almost periodic and unbounded.

(ii) According to [697, Definition 2, p. 80], a continuous function $f : I \to X$ is called recurrent if and only if for each $\varepsilon > 0$ and $N > 0$ the set of all positive reals $\tau > 0$ such that (4.4) holds is relatively dense in $[0,\infty)$ (the case $I = \mathbb{R}$ has been considered in [697], only). The Stepanov generalizations of recurrent functions can be also introduced but then it is not clear how one can consider the invariance of recurrence under the action of infinite convolution product given by (2.46) since the methods proposed in the proof of [631, Proposition 2.6.11] and related results do not work in this framework. Note also that we can extend the notion of (ω,c)-almost automorphicity by requiring that the function $f_{\omega,c}(\cdot)$ is recurrent.

(iii) Due to Corollary 4.1.19, $AP_{\omega,c,1}([0,\infty) : X)$ is the vector space together with the usual operations. This is no longer true for the space $AP_{\omega,c,2}([0,\infty) : X)$, which can be deduced from Proposition 4.1.23 and a counterexample constructed by W. A. Veech (see, e. g., [126, Example 2.8], and the corresponding example given in the pioneering paper [694] by B. Ya. Levin, as well as the articles [407] by J. Egawa, [766] by A. Michalowicz, S. Stoínski and [272] by D. N. Cheban). In particular, the space $AP_{\omega,c,2}([0,\infty) : X) \subseteq UR_{\omega,c,2}([0,\infty) : X)$ strictly contains $c^{\cdot/\omega}AA_{[0,\infty)}(X)$. On the other hand, the compactly almost automorphic function constructed by A. M. Fink in [443] is not asymptotically uniformly recurrent, as shown earlier. This implies that there exists a function $f \in c^{\cdot/\omega}AA_{[0,\infty)}(X)$ such that $f_{\omega,c}(\cdot)$ is not uniformly recurrent; in particular, $UR_{\omega,c,2}([0,\infty) : X)$ strictly contains $UR_{\omega,c}([0,\infty) : X)$.

(iv) As already seen, there exist two bounded, even, uniformly continuous, uniformly recurrent functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that its sum is not uniformly recurrent. Furthermore, we can choose $f(\cdot)$ and $g(\cdot)$ such that $f(0) = g(0) = 1$ and $|f(t) + g(t)| \leqslant 1$ for $|t| \geqslant 1$. Denote the restrictions of such functions to the non-negative real axis by the same symbols, and define after that $F(t) := 2^{-t}f(t)$, $t \geqslant 0$ and $G(t) := 2^{-t}g(t)$, $t \geqslant 0$. Then $F,\ G \in UR_{1,1/2}([0,\infty) : \mathbb{C}) \subseteq UR_{1,1/2,2}([0,\infty) : \mathbb{C})$ but $F + G \notin UR_{1,1/2,2}([0,\infty) : \mathbb{C})$. If we suppose the contrary, then we would have the existence of a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \left|2^{-t}[f(t+\alpha_n) + g(t+\alpha_n)] - 2^{-t}[f(t) + g(t)]\right| = 0,$$

which is impossible because for each $n \in \mathbb{N}$ such that $\alpha_n \geqslant 1$ we have

$$\sup_{t\geqslant 0} \left|2^{-t}[f(t+\alpha_n) + g(t+\alpha_n)] - 2^{-t}[f(t) + g(t)]\right|$$
$$\geqslant \left|f(0) + g(0) - [f(\alpha_n) + g(\alpha_n)]\right| = \left|2 - [f(\alpha_n) + g(\alpha_n)]\right| \geqslant 1.$$

In particular, this example can be used to show that the set $UR_{\omega,c,2}([0,\infty) : \mathbb{C})$ does not form a vector space together with the usual operations.

(v) Using Proposition 4.1.23, as well as the arguments contained in the proofs of Proposition 4.1.11 and [166, Theorem 8°, pp. 3-4], it follows that $AP_{\omega,c,2}([0,\infty) : X)$ is a complete metric space equipped with the distance $d(\cdot,\cdot) := \|\cdot - \cdot\|_{\omega,c}$.

Keeping in mind the proved results, it seems logical to consider the following notion for Stepanov classes, only.

Definition 4.1.25. Let $p \in \mathcal{P}([0,1])$, $c \in \mathbb{C}\setminus\{0\}$, $|c| \leqslant 1$ and $\omega > 0$. Then it is said that a function $f \in L^{p(x)}_{loc}([0,\infty) : X)$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent of type 2, resp. Stepanov $(p(x), \omega, c)$-almost periodic of type 2 if and only if

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \left\|c^{\frac{-\alpha_n}{\omega}} f(s+t+\alpha_n) - f(s+t)\right\|_{L^{p(s)}[0,1]} = 0,$$

resp. for each $\varepsilon > 0$ the set

$$\left\{\tau > 0 : \sup_{t\geqslant 0} \left\|c^{\frac{-\alpha_n}{\omega}} f(s+t+\alpha_n) - f(s+t)\right\|_{L^{p(s)}[0,1]} < \varepsilon\right\}$$

is relatively dense in $[0,\infty)$.

By $S^{p(x)}\mathrm{UR}_{\omega,c,2}([0,\infty) : X)$ and $S^{p(x)}\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$ we denote the space of all Stepanov $(p(x), \omega, c)$-uniformly recurrent functions of type 2 and the space of all Stepanov $(p(x), \omega, c)$-almost periodic functions of type 2, respectively. If $p(x) \equiv p \in [1,\infty)$, then the above classes are also denoted by $S^{p}\mathrm{UR}_{\omega,c,2}([0,\infty) : X)$ and $S^{p}\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$, respectively.

If $1 \leqslant p(x) \leqslant q(x) < \infty$ and $f \in S^{q(x)}\mathrm{UR}_{\omega,c,2}([0,\infty) : X)$, resp. $f \in S^{q(x)}\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$, then $f \in S^{p(x)}\mathrm{UR}_{\omega,c,2}([0,\infty) : X)$, resp. $f \in S^{p(x)}\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$; furthermore, the space $S^{p(x)}\mathrm{UR}_{\omega,c,2}([0,\infty) : X)$, resp. $S^{p(x)}\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$, contains the space $\mathrm{UR}_{\omega,c,2}([0,\infty) : X)$, resp. $\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$. It is simply verified that the space $S^{p(x)}\mathrm{UR}_{\omega,c,2}([0,\infty) : X)$, resp. $S^{p(x)}\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$, consists of those locally $p(x)$-integrable functions $f : I \to X$ for which $\hat{f}(\cdot)$ belongs to the space $\mathrm{UR}_{\omega,c,2}([0,\infty) : L^{p(x)}([0,1] : X))$, resp. $\mathrm{AP}_{\omega,c,2}([0,\infty) : L^{p(x)}([0,1] : X))$. Keeping in mind this observation, it is straightforward to transfer the previously proved results and the points [4′.–7′.] for the introduced Stepanov classes; details can be omitted. Note, finally, that $S^{p(x)}\mathrm{AP}_{\omega,c,2}([0,\infty) : X)$ is a complete metric space equipped with the distance $d(\cdot,\cdot) := \|\cdot - \cdot\|_{p,\omega,c}$.

4.1.2 Composition principles for (ω, c)-almost periodic type functions

The methods established in [650] enable one to formulate a great number of composition principles for (ω, c)-almost periodic type functions. We will explain this fact only in the case of consideration of [650, Theorem 2.9] for Stepanov uniformly recurrent functions. So, let us assume that the function $F : I \times Y \to X$ is continuous and the function $f_{\omega,c}(\cdot)$ is Stepanov p-uniformly recurrent, i. e., the function $f(\cdot)$ is Stepanov (p, ω, c)-almost periodic ($p > 1$, $\omega > 0$, $c \in \mathbb{C} \smallsetminus \{0\}$). Define the function $G : I \times Y \to X$ by

$$G(t, y) := c_1^{-\frac{t}{\omega_1}} F(t, c^{t/\omega} y), \quad t \in I,\ y \in Y,$$

where $c_1 \in \mathbb{C} \smallsetminus \{0\}$ and $\omega_1 > 0$. If the requirements of the above-mentioned theorem hold with the functions $f(\cdot)$ and $F(\cdot,\cdot)$ replaced, respectively, with the functions $f_{\omega,c}(\cdot)$ and $G(\cdot,\cdot)$, then the resulting function

$$t \mapsto G(t, f_{\omega,c}(t)) = c_1^{-t_1/\omega_1} F(t, f(t)), \quad t \in I,$$

is Stepanov q-uniformly recurrent so that the function $t \mapsto F(t, f(t))$, $t \in I$ is Stepanov (q, ω_1, c_1)-uniformly recurrent. More precisely, we have the following.

Theorem 4.1.26. *Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $p \in \mathcal{P}([0,1])$. Suppose that the following conditions hold:*

(i) *The function $G : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent and there exist a function $r(x) \geqslant \max(p(x), p(x)/(p(x)-1))$ and a function $L_G \in L_S^{r(x)}(I)$ such that* (2.20) *holds with the functions* $F(\cdot,\cdot)$ *and* $L_F(\cdot)$ *replaced therein with the functions* $G(\cdot,\cdot)$ *and* $L_G(\cdot)$*, respectively.*

(ii) *The function* $f_{\omega,c} : I \to Y$ *is Stepanov* $p(x)$*-uniformly recurrent and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{f_{\omega,c}(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* Y*.*

(iii) *For every compact set* $K \subseteq Y$*, there exists a strictly increasing sequence* (α_n) *of positive real numbers tending to plus infinity such that* (2.52) *holds with the function* $F(\cdot,\cdot)$ *replaced therein with the function* $G(\cdot,\cdot)$*, and* (2.3) *holds with the function* $f_{\omega,c}(\cdot)$ *and the norm* $\|\cdot\|$ *replaced, respectively, by the function* $\hat{f_{\omega,c}}(\cdot)$ *and the norm* $\|\cdot\|_{L^{p(x)}([0,1]:Y)}$ *therein.*

Set $q(x) := p(x)r(x)/(p(x)+r(x)) \in [1, p(x))$ *provided* $x \in [0,1]$ *and* $r(x) < +\infty$ *and* $q(x) := p(x)$ *for* $x \in [0,1]$ *and* $r(x) = +\infty$*. Then* $F(\cdot, f(\cdot))$ *is Stepanov* $(q(x), \omega_1, c_1)$*-uniformly recurrent.*

In the remainder of this subsection, we will furnish some composition principles for (ω, c)-uniformly recurrent functions of type 2; see also Corollary 4.1.19 and Proposition 4.1.22 (we can simply reformulate these results for (ω, c)-almost periodic functions of type 2). Hence, in the continuation of this subsection, we will assume that $|c| \leqslant 1$, $I = [0, \infty)$ and $i = 2$.

Suppose that $F : I \times Y \to X$ is a continuous function and there exists a finite constant $L > 0$ such that (2.51) holds. Define $\mathcal{F}(t) := F(t, f(t))$, $t \in I$. We will use the following estimate ($\tau \geqslant 0$, $\omega > 0$, $c \in \mathbb{C} \setminus \{0\}$, $t \in I$):

$$\begin{aligned}
&\big\|c^{(-\tau)/\omega}F\big(t+\tau, f(t+\tau)\big) - F\big(t, f(t)\big)\big\| \\
&\quad\leqslant \big\|c^{(-\tau)/\omega}F\big(t+\tau, f(t+\tau)\big) - F\big(t, c^{(-\tau)/\omega}f(t+\tau)\big)\big\| \\
&\qquad + \big\|F\big(t, c^{(-\tau)/\omega}f(t+\tau)\big) - F\big(t, f(t)\big)\big\| \\
&\quad\leqslant \big\|c^{(-\tau)/\omega}F\big(t+\tau, f(t+\tau)\big) - F\big(t, c^{(-\tau)/\omega}f(t+\tau)\big)\big\| + L\big\|c^{(-\tau)/\omega}f(t+\tau) - f(t)\big\|.
\end{aligned} \tag{4.5}$$

Remark 4.1.27. Although we will not employ this estimate henceforth, it should be noticed that we also have

$$\begin{aligned}
&\big\|F\big(t+\tau, f(t+\tau)\big) - c^{\tau/\omega}F\big(t, f(t)\big)\big\| \\
&\quad\leqslant \big\|F\big(t+\tau, f(t+\tau)\big) - F\big(t+\tau, c^{\tau/\omega}f(t)\big)\big\| + \big\|F\big(t+\tau, c^{\tau/\omega}f(t)\big) - c^{\tau/\omega}F\big(t, f(t)\big)\big\| \\
&\quad\leqslant L\big\|f(t+\tau) - c^{\tau/\omega}f(t)\big\| + \big\|F\big(t+\tau, c^{\tau/\omega}f(t)\big) - c^{\tau/\omega}F\big(t, f(t)\big)\big\|.
\end{aligned}$$

Using the proof of [631, Theorem 3.29] and (4.5), we can simply deduce the following result.

Theorem 4.1.28. *Suppose that* $F : I \times Y \to X$ *is a continuous function and there exists a finite constant* $L > 0$ *such that* (2.51) *holds.*

(i) *Suppose that* $f : I \to Y$ *is* (ω, c)*-uniformly recurrent of type* 2. *If there exists a strictly increasing sequence* (α_n) *of positive reals tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \left\| c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t) \right\| = 0$$

and

$$\lim_{n\to+\infty} \sup_{t\in I} \left\| c^{(-\alpha_n)/\omega} F\big(t+\alpha_n, f(t+\alpha_n)\big) - F\big(t, c^{(-\alpha_n)/\omega} f(t+\alpha_n)\big) \right\| = 0,$$

then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ *is* (ω, c)*-uniformly recurrent of type* 2.

(ii) *Suppose that* $f : I \to Y$ *is* (ω, c)*-almost periodic of type* 2. *If for each* $\varepsilon > 0$ *the set of all positive real numbers* $\tau > 0$ *such that*

$$\sup_{t\in I} \left\| c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t) \right\| < \varepsilon$$

and

$$\sup_{t\in I} \left\| c^{(-\tau)/\omega} F\big(t+\tau, f(t+\tau)\big) - F\big(t, c^{(-\tau)/\omega} f(t+\tau)\big) \right\| < \varepsilon$$

is relatively dense in $[0, \infty)$, *then the mapping* $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ *is* (ω, c)*-almost periodic of type* 2.

We can similarly reformulate the statements of [631, Theorem 3.30, Theorem 3.31] in our context (cf. also [49, Theorem 2.11] and [442, Theorem 2.11]).

Now we will provide two results for Stepanov classes of (ω, c)-uniformly recurrent functions of type 2. We will first state the following.

Theorem 4.1.29. *Let* $I = [0, \infty)$, $|c| \leqslant 1$, $\omega > 0$, $p,\ q,\ r \in \mathcal{P}([0,1])$, $p(x),\ q(x) \in [1, \infty)$, $r(x) \in [1, \infty]$, $1/p(x) = 1/q(x) + 1/r(x)$ *and the following conditions hold:*

(i) *The function* $F : I \times Y \to X$ *is Stepanov* $p(x)$*-uniformly recurrent and there exists a function* $L_F \in L_S^{r(x)}(I)$ *such that* (2.20) *holds.*

(ii) *There exists a strictly increasing sequence* (α_n) *of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \Big\| \sup_{u\in R(f)} \big\| c^{-\alpha_n/\omega} F(s+t+\alpha_n, u) - F\big(s+t, c^{\alpha_n/\omega} u\big) \big\| \Big\|_{L^{p(s)}[0,1]} = 0 \tag{4.6}$$

and

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \left\| c^{\frac{-\alpha_n}{\omega}} f(s+t+\alpha_n) - f(s+t) \right\|_{L^{p(s)}[0,1]} = 0.$$

Then the function $F(\cdot, f(\cdot))$ *is Stepanov* $(p(x), \omega, c)$*-uniformly recurrent of type* 2. *Furthermore, the assumption that* $F(\cdot, 0)$ *is Stepanov* $p(x)$*-bounded implies that the function* $F(\cdot, f(\cdot))$ *is also Stepanov* $p(x)$*-bounded.*

Proof. We will only provide the main details of the proof since it is very similar to the proof of [729, Theorem 2.2]. Using the arguments contained for showing the estimate (4.5), we get ($t \geqslant 0$, $n \in \mathbb{N}$)

$$\begin{aligned}&\|c^{-\alpha_n/\omega}F(t+\alpha_n, f(t+\alpha_n)) - F(t, f(t))\| \\ &\quad \leqslant \|c^{(-\alpha_n)/\omega}F(t+\alpha_n, f(t+\alpha_n)) - F(t, c^{(-\alpha_n)/\omega}f(t+\alpha_n))\| \\ &\qquad + L_F(t)\|c^{(-\alpha_n)/\omega}f(t+\alpha_n) - f(t)\|.\end{aligned} \tag{4.7}$$

Keeping in mind (4.7), we can repeat almost verbatim the arguments given in the proof of [729, Theorem 2.2] so as to conclude that there exists a finite constant $c_p > 0$ such that ($n \in \mathbb{N}$)

$$\begin{aligned}&\sup_{t\geqslant 0}\|c^{-\alpha_n/\omega}F(s+t+\alpha_n, f(s+t+\alpha_n)) - F(s, f(s))\|_{L^{p(s)}[0,1]} \\ &\quad \leqslant c_p\|L_F(\cdot)\|_{S^{r(x)}}^p \cdot \sup_{t\geqslant 0}\big\| \|c^{-\alpha_n/\omega}f(s+t+\alpha_n) - f(s+t)\|\big\|_{L^{p(s)}[0,1]} \\ &\qquad + c_p \sup_{t\geqslant 0}\Big\|\sup_{u\in R(f)}\|c^{-\alpha_n/\omega}F(s+t+\alpha_n, u) - F(s+t, u)\|\Big\|_{L^{p(s)}[0,1]}.\end{aligned}$$

By (4.6), this shows that the function $F(\cdot, f(\cdot))$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent of type 2. If the function $F(\cdot, 0)$ is Stepanov $p(x)$-bounded, then the arguments given on [729, p. 6, l.-1–l.-5] enable one to see that the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-bounded, as claimed. □

We can simply formulate a consequence of this result with the usual Lipshitzian condition used. Similarly, we can prove the following result.

Theorem 4.1.30. *Let $I = [0, \infty)$, $|c| \leqslant 1$, $\omega > 0$, $p \in \mathcal{P}([0,1])$, and the following conditions hold:*

(i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent and there exist a function $r(x) \geqslant \max(p(x), p(x)/(p(x)-1))$ and a function $L_F \in L_S^{r(x)}(I)$ such that (2.20) holds.*
(ii) *There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty}\sup_{t\geqslant 0}\Big\|\sup_{u\in R(f)}\|c^{-\alpha_n/\omega}F(s+t+\alpha_n, u) - F(s+t, c^{\alpha_n/\omega}u)\|\Big\|_{L^{p(s)}[0,1]} = 0$$

and

$$\lim_{n\to+\infty}\sup_{t\geqslant 0}\|c^{\frac{-\alpha_n}{\omega}}f(s+t+\alpha_n) - f(s+t)\|_{L^{p(s)}[0,1]} = 0.$$

Then $q(x) := p(x)r(x)/(p(x)+r(x))$ for $x \in [0,1]$ and $r(x) < +\infty$ and $q(x) := p(x)$ for $x \in [0,1]$ and $r(x) = +\infty$. Then the function $F(\cdot, f(\cdot))$ is Stepanov $(q(x), \omega, c)$-uniformly recurrent of type 2. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $q(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is also Stepanov $q(x)$-bounded.

Remark 4.1.31. Concerning Theorem 4.1.29 and Theorem 4.1.30, it should be noticed that we do not require that there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that the set $K := \{f(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact. For Stepanov (p,ω,c)-uniformly recurrent functions of type 2, we cannot assume, in (4.6), a slightly weaker condition (see [729, Lemma 2.1]):

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \sup_{u\in R(f)} \|c^{-\alpha_n/\omega}F(s+t+\alpha_n,u) - F(s+t,c^{\alpha_n/\omega}u)\|_{L^{p(s)}[0,1]} = 0.$$

4.1.3 (ω,c)-Almost periodic properties of convolution products and applications to integro-differential equations

In the first part of this subsection, we will examine the invariance of (ω,c)-almost periodic properties of the infinite convolution product (2.46), where a strongly continuous operator family $(R(t))_{t>0} \subseteq L(X,Y)$ satisfies certain assumptions. As already mentioned, the consideration is simple for the (ω,c)-uniformly recurrent functions, (ω,c)-almost periodic functions and (compactly) (ω,c)-almost automorphic functions because we then need to examine when the function $t \mapsto c^{-(t/\omega)}F(t)$, $t \in \mathbb{R}$ is uniformly recurrent, almost periodic or (compactly) almost automorphic, respectively. But we have

$$c^{-\frac{t}{\omega}}F(t) = \int_{-\infty}^{t} [c^{-\frac{t-s}{\omega}}R(t-s)][c^{-\frac{s}{\omega}}f(s)]\,ds, \quad t \in \mathbb{R},$$

so that the statements of [648, Proposition 3.1, 3.2] (uniform recurrence), [631, Proposition 2.6.11] (almost periodicity) and [631, Proposition 3.5.3] (almost automorphicity), for instance, can be simply reformulated in our context by replacing, respectively, the operator family $(R(t))_{t>0}$ and the function $f(\cdot)$ by the operator family $(c^{-\frac{t}{\omega}}R(t))_{t>0}$ and the function $c^{-\frac{\cdot}{\omega}}f(\cdot)$. We will do this only in the case of the last mentioned result (see [373] for the notion).

Proposition 4.1.32. *Suppose that* $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ *and* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying*

$$M := \sum_{k=0}^{\infty} \|c^{-\frac{\cdot+k}{\omega}}R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty.$$

If $c^{-\frac{\cdot}{\omega}}f : \mathbb{R} \to X$ *is* $S^{p(x)}$*-almost automorphic, then the function* $F : \mathbb{R} \to Y$*, given by* (2.46)*, is well defined and* (ω,c)*-almost automorphic.*

It is worth noting that this result can be applied in both cases $|c| > 1$ and $|c| < 1$, under suitable conditions. It is straightforward to incorporate the above results in the study of the existence and uniqueness of (ω,c)-almost periodic type solutions for the various classes of abstract inhomogeneous integro-differential equations. Keeping in mind Theorem 4.1.13, we will skip all related details with regard to the invariance of (ω, c)-uniform recurrence of type 1 (type 2) and (ω, c)-almost periodicity of type 1 (type 2) under the actions of infinite convolution products.

Due to the fact that

$$c^{-\frac{t}{\omega}} \int_0^t R(t-s)f(s)\,ds = \int_0^t \big[c^{-\frac{t-s}{\omega}} R(t-s)\big]\big[c^{-\frac{s}{\omega}} f(s)\big]\,ds, \quad t \geqslant 0, \tag{4.8}$$

we can similarly analyze the invariance of asymptotical (ω, c)-uniform recurrence, asymptotical (ω, c)-almost periodicity and asymptotical (compact) (ω, c)-almost automorphicity under the actions of finite convolution products; the interested reader may try to reformulate the statements of [631, Proposition 2.6.13, Theorem 2.9.5, Theorem 2.9.7, Theorem 2.9.15] in our new context.

If $|c| < 1$ and $\omega > 0$, then it is worth noting that the (ω, c)-uniform recurrence of type 2 and the (ω, c)-almost periodicity of type 2 cannot be so simply retained after the actions of finite convolution products. The situation is much simpler for the classes $\mathrm{AP}_{\omega,c,1}([0,\infty) : X)$ and $\mathrm{UR}_{\omega,c,1}([0,\infty) : X)$ ($S^{p(x)}\mathrm{AP}_{\omega,c,1}([0,\infty) : X)$ and $S^{p(x)}\mathrm{UR}_{\omega,c,1}([0,\infty) : X)$) because in this case we can apply Corollary 4.1.19, Proposition 4.1.22 and (4.8).

In the remainder of this subsection, we will provide a few applications to the abstract integro-differential equations and inclusions in Banach spaces.

1. In the concrete situation of [1067, Example 4], we know that the unique solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial condition $u(x,0) = f(x)$, is given by

$$u(x,t) := \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\,ds, \quad x \in \mathbb{R},\ t \geqslant 0. \tag{4.9}$$

Let the number $t_0 > 0$ be fixed, let $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and let the function $c^{-\cdot/\omega} f(\cdot)$ be bounded uniformly recurrent (almost periodic, (compactly) almost automorphic). Since $c^{-\frac{\cdot}{\omega}} e^{-\cdot^2/4t_0} \in L^1(\mathbb{R})$, we can apply Proposition 4.1.7 in order to see that the solution $x \mapsto u(x,t_0)$, $x \in \mathbb{R}$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(compactly) (ω, c)-almost automorphic). See also [49, Example 2.9].

2. It is worth noting that the notion from Definition 4.1.12 and Definition 4.1.25 can be introduced with the intervals $I = [-a, \infty)$, where $a > 0$ is an arbitrary real number. To explain the importance of this observation, we will reexamine [1067, Example 5]. It is well known that the unique regular solution of the wave equation $u_{xx}(x,t) = u_{tt}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial conditions $u(x,0) = f(x)$, $x \in \mathbb{R}$, $u_t(x,0) = g(x)$, $x \in \mathbb{R}$, is given by the d'Alembert formula (3.65). Here

we would like to note the following fact about the term

$$H_{t_0}(x) := \frac{1}{2}\int_{x-t_0}^{x+t_0} g(s)\,ds, \quad x \in \mathbb{R},$$

where $t_0 > 0$ is a fixed real number. Suppose that the function $g : [-t_0, \infty) \to \mathbb{C}$ is (ω, c)-uniformly recurrent of type 2, for example (the same comment applies to all other classes of functions introduced in Definition 4.1.12). Then there exists a strictly increasing sequence (α_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{t\geqslant -t_0} \left|c^{-\alpha_n/\omega} g(t+\alpha_n) - g(t)\right| = 0.$$

If $\varepsilon > 0$ is given, this implies the existence of an integer $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$,

$$\left|c^{-\alpha_n/\omega} H_{t_0}(x+\alpha_n) - H_{t_0}(x)\right| \leqslant \int_{-t_0}^{t_0} \left|c^{-\alpha_n/\omega} g(x+s+\alpha_n) - g(x+s)\right| ds \leqslant 2t_0\varepsilon, \quad x \geqslant 0.$$

Hence, the function $H_{t_0} : [0, \infty) \to \mathbb{C}$ is (ω, c)-uniformly recurrent of type 2.

It would be very enticing to provide certain applications of composition principles established in Subsection 4.1.2 in the qualitative analysis of solutions to the abstract semilinear Cauchy inclusions which belongs to the classes $AP_{\omega,c,2}([0,\infty))$ and $UR_{\omega,c,2}([0,\infty))$.

The case $|c| \neq 1$ is still unexplored in the theory of almost periodic functions and we are obliged to say that the classes of (ω, c)-almost periodic type functions with $|c| \neq 1$ have some very unusual and unpleasant features.

4.1.4 (ω, c)-Pseudo-almost periodic functions, (ω, c)-pseudo-almost automorphic functions and applications

In this subsection, we deal with the interval $I = \mathbb{R}$, only. Let us recall the (ω, c)-mean of a function $h : \mathbb{R} \to X$ is introduced in [48] by

$$\mathcal{M}_{\omega,c}(h) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} c^{-\sigma/\omega} h(\sigma)\,d\sigma,$$

whenever the limit exists. For example, for $h_1(t) = c^{t/\omega}$ and $h_2(t) = c^{t/\omega}e^{it}$, we see that $\mathcal{M}_{\omega,c}(h_1) = 1$ and $\mathcal{M}_{\omega,c}(h_2) = 0$. Furthermore, $\mathcal{M}_{\omega,c}$ is a linear and continuous operator. Indeed, if $c^{-t/\omega}h_n(t) \to c^{-t/\omega}h(t)$ uniformly as $n \to \infty$, then $\mathcal{M}_{\omega,c}(h_n) \to \mathcal{M}_{\omega,c}(h)$ as $n \to \infty$.

Remark 4.1.33. If $h(\cdot)$ is (ω, c)-almost periodic, then the mean $\mathcal{M}_{\omega,c}(h)$ always exists, because the function $c^{-(\cdot/\omega)}f(\cdot)$ is almost periodic and the usual mean value of any almost periodic function exists.

We will use the space

$$\mathrm{PAP}_{0;\omega,c}(\mathbb{R} : X) := \{h \in C(\mathbb{R} : X);\ c^{-\cdot/\omega}h(\cdot) \in \mathrm{PAP}_0(\mathbb{R} : X)\}.$$

A function $h(\cdot)$ is said to be c-ergodic if and only if $h(\cdot)$ belongs to this space.

Furthermore, we will use the following two types of (ω, c)-pseudo-ergodic components.

Definition 4.1.34. Let $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$.
(i) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 1)$-pseudo-ergodic vanishing if and only if $c^{-t/\omega}f(t, \cdot) \in \mathrm{PAP}_0(\mathbb{R}\times Y : X)$. The space of all such functions will be denoted by $\mathrm{PAP}_{0;\omega,c,1}(\mathbb{R} \times Y : X)$.
(ii) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 2)$-pseudo-ergodic vanishing if and only if $c^{-t/\omega}f(t, c^{t/\omega}\cdot) \in \mathrm{PAP}_0(\mathbb{R} \times Y : X)$. The space of all such functions will be denoted by $\mathrm{PAP}_{0;\omega,c,2}(\mathbb{R} \times Y : X)$.

Similarly, we will use two different types of (ω, c)-almost periodic functions, resp. (ω, c)-almost automorphic functions, depending on two variables. Now we would like to introduce the following definitions.

Definition 4.1.35. Let $c \in \mathbb{C}\setminus\{0\}$, $\omega > 0$ and $i = 1, 2$.
(i) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 1)$-almost periodic, resp. $(\omega, c, 1)$-almost automorphic, if and only if $c^{-t/\omega}f(t, \cdot) \in \mathrm{AP}(\mathbb{R} \times Y : X)$, resp. $c^{-t/\omega}f(t, \cdot) \in \mathrm{AA}(\mathbb{R}\times Y : X)$. The space of all such functions will be denoted by $\mathrm{AP}_{\omega,c,1}(\mathbb{R}\times Y : X)$, resp. $\mathrm{AA}_{\omega,c,1}(\mathbb{R} \times Y : X)$.
(ii) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 2)$-almost periodic, resp. $(\omega, c, 2)$-almost automorphic, if and only if $c^{-t/\omega}f(t, c^{t/\omega}\cdot) \in \mathrm{AP}(\mathbb{R} \times Y : X)$, resp. $c^{-t/\omega}f(t, c^{t/\omega}\cdot) \in \mathrm{AA}(\mathbb{R}\times Y : X)$. The space of all such functions will be denoted by $\mathrm{AP}_{\omega,c,2}(\mathbb{R}\times Y : X)$, resp. $\mathrm{AA}_{\omega,c,2}(\mathbb{R} \times Y : X)$.

Definition 4.1.36. Let $c \in \mathbb{C}\setminus\{0\}$, $\omega > 0$ and $i = 1, 2$.
(i) A function $f \in C(\mathbb{R} : X)$ is said to be (ω, c)-pseudo-almost periodic, resp. (ω, c)-pseudo-almost automorphic, if and only if it admits a decomposition $f(t) = g(t) + h(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is (ω, c)-almost periodic, resp. (ω, c)-almost automorphic, and $h \in \mathrm{PAP}_{0;\omega,c}(\mathbb{R} : X)$. The space of all such functions will be denoted by $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$, resp. $\mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$.
(ii) A function $f(\cdot, \cdot) \in C(\mathbb{R} \times Y : X)$ is said to be (ω, c, i)-pseudo-almost periodic, resp. (ω, c, i)-pseudo-almost automorphic, if and only if it admits a decomposition $f(t, x) = g(t, x) + h(t, x)$, $t \in \mathbb{R}$, $x \in X$, where $g(\cdot, \cdot)$ is (ω, c, i)-almost periodic,

resp. (ω, c, i)-almost automorphic, and $h(\cdot,\cdot) \in PAP_{0;\omega,i}(\mathbb{R} \times Y : X)$. The space of all such functions will be denoted by $PAP_{\omega,c,i}(\mathbb{R} \times Y : X)$, resp. $PAA_{\omega,c,i}(\mathbb{R} \times Y : X)$.

Theorem 4.1.37. *Let $f \in C(\mathbb{R} : X)$. Then $f(\cdot)$ is (ω, c)-pseudo-almost periodic, resp. (ω, c)-pseudo-almost automorphic, if and only if*

$$f(t) \equiv c^{\wedge}(t)u(t), \quad \textit{with } c^{\wedge}(t) \equiv c^{t/\omega},\ u \in PAP(\mathbb{R} : X), \tag{4.10}$$

resp.

$$f(t) \equiv c^{\wedge}(t)u(t), \quad \textit{with } c^{\wedge}(t) \equiv c^{t/\omega},\ u \in PAA(\mathbb{R} : X).$$

Proof. We will consider only (ω, c)-pseudo-almost periodic functions for simplicity. It is clear that if $f(\cdot)$ satisfies (4.10), then $f(\cdot)$ is an (ω, c)-pseudo-almost periodic function. In order to show the converse statement, let $f \in PAP_{\omega,c}(\mathbb{R} : X)$. Then there exist $g \in AP_{\omega,c}(\mathbb{R} : X)$ and $PAP_{0;\omega,c}(\mathbb{R} : X)$ such that $f = g + h$. Therefore,

$$u(t) = c^{-t/\omega}g(t) + c^{-t/\omega}h(t) = F_1(t) + F_2(t), \quad t \in \mathbb{R}.$$

So, $u(t)$ is written as a sum of $F_1(\cdot)$ which is almost periodic and $F_2(\cdot)$ which belongs to $PAP_{0;\omega,c}(\mathbb{R} : X)$. ☐

Remark 4.1.38. Let us note that the decompositions given in Definition 4.1.36 are unique; see also [48, Remark 2.9]. The proof of this fact can be left to the interested reader.

It can be simply shown that:

(i) We have $f + g \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f + g \in PAA_{\omega,c}(\mathbb{R} : X)$, and $\alpha h \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $\alpha h \in PAA_{\omega,c}(\mathbb{R} : X)$, provided $f, g, h \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f, g, h \in PAA_{\omega,c}(\mathbb{R} : X)$, and $\alpha \in \mathbb{C}$.

(ii) If $\tau \in \mathbb{R}$ and $f \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f \in PAA_{\omega,c}(\mathbb{R} : X)$, then $f_\tau(\cdot) \equiv f(\cdot + \tau) \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f_\tau(\cdot) \in PAA_{\omega,c}(\mathbb{R} : X)$.

Now we would like to endow the introduced space of (ω, c)-pseudo-almost periodic functions, resp. (ω, c)-pseudo-almost automorphic functions, with a certain norm.

Proposition 4.1.39. *The space $PAP_{\omega,c}(\mathbb{R} : X)$, resp. $PAA_{\omega,c}(\mathbb{R} : X)$, equipped with the norm $\|\cdot\|_{\omega,c}$ is a Banach space.*

Proof. We will consider the space $PAP_{\omega,c}(\mathbb{R} : X)$, only. Let (f_n) be a Cauchy sequence in $PAP_{\omega,c}(\mathbb{R} : X)$. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m, n \geqslant N$, we have

$$\|f_n - f_m\|_{\omega,c} < \varepsilon.$$

Since $f_m, f_n \in PAP_{\omega,c}(\mathbb{R} : X)$, there exist $u_m, u_n \in PAP(\mathbb{R} : X)$ such that $f_m(t) \equiv c^{\wedge}(t)u_m(t)$ and $f_n(t) \equiv c^{\wedge}(t)u_n(t)$ for all $t \in \mathbb{R}$. Now, for $m, n \geqslant N$ we have $\|u_m - u_n\|_\infty \leqslant$

$\|f_n - f_m\|_{\omega,c} < \varepsilon$. It follows that (u_n) is a Cauchy sequence in $\mathrm{PAP}(\mathbb{R} : X)$. Since $\mathrm{PAP}(\mathbb{R} : X)$ is complete, there exists $u \in \mathrm{PAP}(\mathbb{R} : X)$ such that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Let us define $f(t) := c^\wedge(t)u(t)$, $t \in \mathbb{R}$. We claim that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Indeed, $\|f_n - f\|_{\omega,c} = \sup_{t\in\mathbb{R}} \|u_n(t) - u(t)\| \to 0 \quad (n \to \infty)$. Hence, $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$ is a Banach space with the norm $\|\cdot\|_{\omega,c}$. □

Lemma 4.1.40 ([48]). *Assume that $k^\sim(\cdot) := c^\wedge(-\cdot)k(\cdot) \in L^1(\mathbb{R})$. Then $h \in \mathrm{PAP}_{0;\omega,c}(\mathbb{R} : X)$ implies that $k * h \in \mathrm{PAP}_{0;\omega,c}(\mathbb{R} : X)$.*

Theorem 4.1.41. *Let $f \in \mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$, resp. $f \in \mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$, with $f(\cdot) = c^\wedge(\cdot)p(\cdot)$, $p \in \mathrm{PAP}(\mathbb{R} : X)$, resp. $p \in \mathrm{PAA}(\mathbb{R} : X)$. If for some $k(\cdot)$ we have $k^\sim(\cdot) := c^\wedge(-\cdot)k(\cdot) \in L^1(\mathbb{R})$, then*

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds = c^\wedge(t)\big(k^\sim * p\big)(t), \quad t \in \mathbb{R}.$$

In particular, $k * f \in \mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$, resp. $k * f \in \mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$.

Proof. As before, we will consider the space $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$ only, because the proof is quite analogous for the space $\mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$. Since $p \in \mathrm{PAP}(\mathbb{R} : X)$, we see that there exist $p_1 \in \mathrm{AP}(\mathbb{R} : X)$ and $p_2 \in \mathrm{PAP}_0(\mathbb{R} : X)$ such that $p = p_1 + p_2$. Then $f = f_1 + f_2$, where $f_1(\cdot) = c^\wedge(\cdot)p_1(\cdot) \in \mathrm{AP}_{\omega,c}(\mathbb{R} : X)$ and $f_2(\cdot) = c^\wedge(\cdot)p_1(\cdot) \in \mathrm{PAP}_{0;\omega,c}(\mathbb{R} : X)$. For every $t \in \mathbb{R}$, we have

$$\begin{aligned}(k * f)(t) &= \int_{-\infty}^{\infty} k(t-s)f(s)\,ds \\ &= \int_{-\infty}^{\infty} k(t-s)f_1(s)\,ds + \int_{-\infty}^{\infty} k(t-s)f_2(s)\,ds \\ &= (k * f_1)(t) + (k * f_2)(t) =: I_1(t) + I_2(t).\end{aligned}$$

We see that $I_1 \in \mathrm{AP}_{\omega,c}(\mathbb{R} : X)$ and $I_2 \in \mathrm{PAP}_{0;\omega,c}(\mathbb{R} : X)$. Moreover, by definition of $f(\cdot)$, we have $(k * f)(\cdot) = c^\wedge(\cdot)(k^\sim * p)(\cdot)$ so that $k * f \in \mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$. □

Example 4.1.42. Let us consider the heat equation $u_t(x,t) = u_{xx}(x,t)$, $t > 0$, $x \in \mathbb{R}$, with the initial value condition $u(x,0) = f(x)$. Let $u(x,t)$ be a regular solution of this equation; see (4.9). Fix $t_0 > 0$ and assume that $f(\cdot)$ is an (ω, c)-pseudo-almost periodic function. Then, by Theorem 4.1.41, the solution $u(x,t_0)$ is (ω, c)-pseudo-almost periodic with respect to x.

To formulate the related composition principles, we will use two lemmas.

Lemma 4.1.43 (see [631, Lemma 2.12.2]). *Let $f \in \mathrm{PAP}(\mathbb{R} \times Y : X)$ and $u \in \mathrm{PAP}(\mathbb{R} : Y)$. Then the mapping $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ belongs to the space $\mathrm{PAP}(\mathbb{R} : X)$ provided that the following conditions hold:*

(i) *The set* $\{f(t,y) : t \in \mathbb{R},\ y \in B\}$ *is bounded for every bounded subset* $B \subseteq Y$.
(ii) $f(t,y)$ *is uniformly continuous in each bounded subset of* Y *uniformly in* $t \in \mathbb{R}$. *That is, for any* $\varepsilon > 0$ *and* $B \subseteq X$ *bounded, there exists* $\delta > 0$ *such that* $x,\ y \in B$ *and* $\|x - y\| \leqslant \delta$ *imply* $\|f(t,x) - f(t,y)\| \leqslant \varepsilon$ *for all* $t \in \mathbb{R}$.

Lemma 4.1.44 (see [631, Theorem 3.2.4]). *Suppose that* $f = g + \phi \in \mathrm{PAA}(\mathbb{R} \times Y : X)$ *with* $g \in \mathrm{AA}(\mathbb{R} \times Y : X)$, $\phi \in \mathrm{PAP}_0(\mathbb{R} \times Y : X)$ *and the following holds:*
(i) *the mapping* $(t,y) \mapsto g(t,y)$ *is uniformly continuous in any bounded subset* $B \subseteq Y$ *uniformly for* $t \in \mathbb{R}$;
(ii) *the mapping* $(t,y) \mapsto \phi(t,y)$ *is uniformly continuous in any bounded subset* $B \subseteq Y$ *uniformly for* $t \in \mathbb{R}$.

Then for each $u \in \mathrm{PAA}(\mathbb{R} : Y)$ *one has* $f(\cdot, u(\cdot)) \in \mathrm{PAA}(\mathbb{R} : X)$.

For simplicity, we will not consider Stepanov p-almost periodic functions and Stepanov p-almost automorphic functions depending on two variables here.

Suppose now that a continuous function $g : \mathbb{R}\times Y \to X$ satisfies $g(t+\omega, y) = cg(t,y)$ for all $t \in \mathbb{R}$ and $y \in Y$, resp. $g(t + \omega, cy) = cg(t,y)$ for all $t \in \mathbb{R}$ and $y \in Y$. Define the functions

$$G_1(t,y) := c^{-\frac{t}{\omega}} g(t,y), \quad t \in \mathbb{R},\ y \in Y \tag{4.11}$$

and

$$G_2(t,y) := c^{-\frac{t}{\omega}} g\big(t, c^{t/\omega} y\big), \quad t \in \mathbb{R},\ y \in Y.$$

Then, for every $t \in \mathbb{R}$ and $y \in Y$, we have

$$G_1(t+\omega, y) = c^{-\frac{t+\omega}{\omega}} g(t+\omega, y) = c^{-\frac{t+\omega}{\omega}} cg(t+\omega, y) = c^{-\frac{t}{\omega}} g(t,y) = G_1(t,y)$$

and

$$\begin{aligned} G_2(t+\omega, y) &= c^{-\frac{t+\omega}{\omega}} g\big(t+\omega, c^{\frac{t+\omega}{\omega}} y\big) = c^{-\frac{t+\omega}{\omega}} cg\big(t, c^{t/\omega} y\big) \\ &= c^{-t/\omega} g\big(t, c^{t/\omega} y\big) = G_2(t,y). \end{aligned}$$

In both cases, the function $G_i(\cdot,\cdot)$ is ω-periodic in time variable ($i = 1,2$). Furthermore, if the requirements of [48, Theorem 2.24] hold (case $i = 2$), then condition (i) of Lemma 4.1.44 holds with the function $g(\cdot,\cdot)$ replaced therein with the function $G_2(\cdot,\cdot)$, and condition (ii) of Lemma 4.1.44 holds with the function $\phi(\cdot,\cdot)$ replaced therein with the function $h_2(t,\cdot) \equiv c^{-t/\omega} h(t, c^{t/\omega}\cdot)$, $t \in \mathbb{R}$. Furthermore, $G_2 \in \mathrm{AA}(\mathbb{R} \times Y : X)$ and $h_2 \in \mathrm{PAP}_0(\mathbb{R} \times Y : X)$ so that repeating verbatim the arguments used in the proof of [711, Theorem 2.4] with appealing to [49, Theorem 2.11] in place of [711, Lemma 2.2] immediately yields a much simpler proof of [48, Theorem 2.24]. Furthermore, the statement of [49, Theorem 2.11] can be formulated for continuous functions which maps the space $\mathbb{R} \times Y$ into X; in other words, we can use two different pivot spaces X and Y. Keeping in mind this observation, we can immediately clarify an extension of [48, Theorem 2.24] in this context (the interested reader should try to reexamine [48, Theorem 2.25] for (ω, c)-pseudo-almost periodic functions and (ω, c)-pseudo-almost automorphic functions). Furthermore, we have the following result.

Proposition 4.1.45.
(i) *Suppose that* $f = g + \phi$ *with* $g \in \mathrm{AA}_{\omega,c,1}(\mathbb{R} \times Y : X)$, $\phi \in \mathrm{PAP}_{0;\omega,c,1}(\mathbb{R} \times Y : X)$ *and the following holds:*
  (a) *the mapping* $(t, y) \mapsto G_1(t, y)$ *given by* (4.11) *is uniformly continuous in any bounded subset* $B \subseteq Y$ *uniformly for* $t \in \mathbb{R}$;
  (b) *the mapping* $(t, y) \mapsto \phi_1(t, y)$ *given by* (4.11) *with the function* $g(\cdot, \cdot)$ *replaced therein with the function* $\phi(\cdot, \cdot)$, *is uniformly continuous in any bounded subset* $B \subseteq Y$ *uniformly for* $t \in \mathbb{R}$.

  Then for each $u \in \mathrm{PAA}(\mathbb{R} : Y)$ *one has* $f(\cdot, u(\cdot)) \in \mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$.

(ii) *Suppose that* $f = g + \phi$ *with* $g \in \mathrm{AA}_{\omega,c,2}(\mathbb{R} \times Y : X)$, $\phi \in \mathrm{PAP}_{0;\omega,c,2}(\mathbb{R} \times Y : X)$ *and the following holds:*
  (c) *the mapping* $(t, y) \mapsto G_2(t, y)$ *given by* (4.11) *is uniformly continuous in any bounded subset* $B \subseteq Y$ *uniformly for* $t \in \mathbb{R}$;
  (d) *the mapping* $(t, y) \mapsto \phi_2(t, y)$ *given by* (4.11), *with the function* $g(\cdot, \cdot)$ *replaced therein with the function* $\phi(\cdot, \cdot)$, *is uniformly continuous in any bounded subset* $B \subseteq Y$ *uniformly for* $t \in \mathbb{R}$.

  Then for each $u \in \mathrm{PAA}_{\omega,c}(\mathbb{R} : Y)$ *one has* $f(\cdot, u(\cdot)) \in \mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$.

We can also clarify the following result.

Proposition 4.1.46.
(i) *Let* $f \in \mathrm{PAP}_{\omega,c,1}(\mathbb{R} \times Y : X)$ *and* $u \in \mathrm{PAP}(\mathbb{R} : Y)$. *Then the mapping* $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ *belongs to the space* $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$ *provided that the following conditions hold:*
  (a) *The set* $\{c^{-t/\omega} f(t, y) : t \in \mathbb{R},\ y \in B\}$ *is bounded for every bounded subset* $B \subseteq Y$.
  (b) $c^{-t/\omega} f(t, y)$ *is uniformly continuous in each bounded subset of* Y *uniformly in* $t \in \mathbb{R}$.

(ii) *Let* $f \in \mathrm{PAP}_{\omega,c,2}(\mathbb{R} \times Y : X)$ *and* $u \in \mathrm{PAP}_{\omega,c}(\mathbb{R} : Y)$. *Then the mapping* $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ *belongs to the space* $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$ *provided that the following conditions hold:*
  (a) *The set* $\{c^{-t/\omega} f(t, c^{t/\omega} y) : t \in \mathbb{R},\ y \in B\}$ *is bounded for every bounded subset* $B \subseteq Y$.
  (b) $c^{-t/\omega} f(t, c^{t/\omega} y)$ *is uniformly continuous in each bounded subset of* Y *uniformly in* $t \in \mathbb{R}$.

Consider the semilinear fractional Cauchy inclusion (2.11), where $\gamma \in (0, 1]$, $f : \mathbb{R} \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator in

X satisfying condition (P). Then there exists a finite constant $M_0 > 0$ such that the degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ satisfies the estimate $\|T(t)\| \leqslant M_0 e^{-at} t^{\beta-1}$, $t > 0$. Let us recall that by a mild solution of problem (2.11), we mean any continuous function $t \mapsto u(t)$, $t \in \mathbb{R}$, satisfying

$$u(t) = \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

We will use the following auxiliary result.

Lemma 4.1.47 (see the proof of [631, Lemma 2.12.3]). *Suppose that $f : \mathbb{R} \to X$ is pseudo-almost periodic (pseudo-almost automorphic) and $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying that $\|R(t)\| \leqslant Me^{-bt} t^{\beta-1}$, $t > 0$ for some finite numbers $M \geqslant 1$, $b > 0$ and $\beta \in (0, 1]$. Then the function $F(t) := \int_{-\infty}^{t} R(t-s) f(s)\, ds$, $t \in \mathbb{R}$ is well defined and pseudo-almost periodic (pseudo-almost automorphic).*

Suppose now that

$$0 < M_0 / \big(a + (\ln|c|/\omega)\big) < 1 \tag{4.12}$$

and define the mapping

$$Pu : \mathrm{PAP}_{\omega,c}(\mathbb{R} : X) \to \mathrm{PAP}_{\omega,c}(\mathbb{R} : X), \quad \text{resp. } Pu : \mathrm{PAA}_{\omega,c}(\mathbb{R} : X) \to \mathrm{PAA}_{\omega,c}(\mathbb{R} : X),$$

by

$$(Pu)(t) := \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Under certain assumptions, the mapping $f(\cdot, u(\cdot))$ belongs to the class $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$, resp. $\mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$. Using the decomposition

$$\int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds = \int_{-\infty}^{t} \big[c^{-\frac{t-s}{\omega}} T(t-s)\big]\big[c^{-\frac{s}{\omega}} f(s, u(s))\big]\, ds, \quad t \in \mathbb{R},$$

the estimate (4.12) shows that the mapping $t \mapsto \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds$, $t \in \mathbb{R}$ belongs to the class $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$, resp. $\mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$. Hence, the mapping $P(\cdot)$ is well defined. By simple calculation, we get

$$\|Pu\|_{\omega,c} \leqslant \frac{M_0}{a + (\ln|c|/\omega)} \|Pu\|_{\omega,c}, \quad u \in \mathrm{PAP}_{\omega,c}(\mathbb{R} : X) \quad \big[u \in \mathrm{PAA}_{\omega,c}(\mathbb{R} : X)\big].$$

Applying the Banach contraction principle, the mapping $P(\cdot)$ has a unique fixed point, so that there exists a unique solution of the abstract semilinear Cauchy inclusion (2.11) which belongs to the class $\mathrm{PAP}_{\omega,c}(\mathbb{R} : X)$, resp. $\mathrm{PAA}_{\omega,c}(\mathbb{R} : X)$.

4.1.5 (ω,c)-Almost periodic distributions

Almost periodic distributions extending the classical Bohr and Stepanov almost periodic functions are introduced by L. Schwartz; see [913]. Asymptotical almost periodicity of Schwartz distributions was introduced by I. Cioransescu [299] (see also [208, 209, 631, 764, 958, 959], Subsection 4.2.8 and the list of references quoted therein).

This subsection introduces and investigates (ω,c)-almost periodicity (resp. asymptotic (ω,c)-almost periodicity) in the setting of Schwartz–Sobolev distributions. For simplicity, we will consider only scalar-valued distributions because the extensions to the vector-valued case are straightforward.

By $\mathcal{D} = C_0^{\infty}(\mathbb{R})$, $\mathcal{E} = C^{\infty}(\mathbb{R})$ and $\mathcal{S} = \mathcal{S}(\mathbb{R})$ we denote the Schwartz spaces of test functions, endowed with the usual topologies. If $\emptyset \neq \Omega \subseteq \mathbb{R}$, then by $\mathcal{D}_{\Omega}$ we denote the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\operatorname{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$ and $\mathcal{D}' := L(\mathcal{D} : \mathbb{C})$ stands for the space consisting of all scalar-valued distributions.

We will first introduce the space of smooth (ω,c)-almost periodic functions and investigate some of their basic properties. We will use the following notations:

$$\varphi_{\omega,c}(\cdot) := c^{-\frac{(\cdot)}{\omega}}\varphi(\cdot), \quad \varphi \in C^{\infty} \text{ or } L^p,\ 1 \leqslant p \leqslant +\infty \text{ and } T_{\omega,c} := c^{-\frac{(\cdot)}{\omega}}T,\ T \in \mathcal{D}', \tag{4.13}$$

where the equality is taken in the usual (resp. Lebesgue, distributional) sense.

To construct the (ω,c)-smooth almost periodic functions, we need to introduce some new functional spaces. Let $p \in [1,+\infty]$ and $f(\cdot)$ be a complex-valued measurable function on $\mathbb{R}$.

We say that $f(\cdot)$ is a (ω,c)-Lebesgue function with exponent p, if

$$\Big(\int_{\mathbb{R}} |f_{\omega,c}(t)|^p\,dt\Big)^{\frac{1}{p}} < \infty, \quad \text{for } 1 \leqslant p < +\infty,$$

and

$$\sup_{t\in\mathbb{R}} |f_{\omega,c}(t)| < \infty, \quad \text{for } p = +\infty.$$

We denote by $L^p_{\omega,c}$ the set of (ω,c)-Lebesgue functions with exponent p, i. e.,

$$L^p_{\omega,c} := \{f : \mathbb{R} \to \mathbb{C} \text{ measurable};\ f_{\omega,c} \in L^p\}.$$

When $c = 1$, $L^p_{\omega,c} := L^p$ is the classical Lebesgue space over $\mathbb{R}$.

Proposition 4.1.48. *The space $L^p_{\omega,c}$ endowed with the (ω,c)-norm*

$$\|f\|_{L^p_{\omega,c}} := \|f_{\omega,c}\|_{L^p}, \quad \textit{for } 1 \leqslant p < +\infty,$$

and

$$\|f\|_{L^{\infty}_{\omega,c}} := \|f\|_{\omega,c}, \quad \textit{for } p = +\infty,$$

is a Banach space.

Proposition 4.1.49. *$\mathcal{D}$ is dense in $L^p_{\omega,c}$*; $1 \leqslant p < \infty$.

Proof. Since $\mathcal{D}$ is dense in the space $\mathcal{C}_c$ of continuous functions with compact support, it suffices to show that $\mathcal{C}_c$ is dense in $L^p_{\omega,c}$ for $1 \leqslant p < \infty$.

Let S be the set of all simple measurable functions $s(\cdot)$, with complex values, defined on $\mathbb{R}$ and such that

$$m(\{t : s(t) \neq 0\}) < \infty.$$

First, it is clear that S is dense in $L^p_{\omega,c}$ for $1 \leqslant p < \infty$. Indeed, as $c^{-\frac{t}{\omega}} s \in L^p$, then $S \subseteq L^p_{\omega,c}$. Suppose $f \in L^p_{\omega,c}$ is positive and define the sequence $(s_n)_n$ such that

$$0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant f, \quad \text{and for each } t \in \mathbb{R},\ s_n(t) \to f(t) \text{ when } n \to +\infty.$$

Then $(f - s_n)_{\omega,c} = c^{-\frac{t}{\omega}}(f - s_n) \in L^p$, hence $s_n \in S$. Furthermore, since

$$\left|c^{-\frac{t}{\omega}}(f - s_n)\right|^p \leqslant f^p,$$

Lebesgue's dominated convergence theorem shows that

$$\|(f - s_n)_{\omega,c}\|_{L^p} = \|c^{-\frac{t}{\omega}}(f - s_n)\|_{L^p} \to 0$$

when $n \to +\infty$. Hence, $\|f - s_n\|_{L^p_{\omega,c}} \to 0$ when $n \to +\infty$. On the other hand, by Lusin's theorem, for $s \in S$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}_c$ such that $g(t) = s(t)$, except on a set of measure less than ε, and $|g| \leqslant \|s\|_\infty$, and since $s(\cdot)$ takes only a finite number of values, there exists a constant $C > 0$ which depends on c and ω such that

$$\|(g - s)_{\omega,c}\|_{L^p} = \left(\int_{\mathbb{R}} \left|c^{-\frac{t}{\omega}}(g(t) - s(t))\right|^p dt\right)^{\frac{1}{p}} \leqslant 2C\varepsilon^{\frac{1}{p}}\|s\|_\infty.$$

The density of S in $L^p_{\omega,c}$ completes the proof. □

We define

$$\mathcal{D}_{L^p_{\omega,c}} := \{\varphi \in \mathcal{C}^\infty : \varphi^{(j)}_{\omega,c} \in \mathcal{D}_{L^p},\ j \in \mathbb{N}\}.$$

When $c = 1$, we get $\mathcal{D}_{L^p_{\omega,c}} := \mathcal{D}_{L^p}$. Moreover, it is easy to show that the space $\mathcal{D}_{L^p_{\omega,c}}$, $1 \leqslant p \leqslant \infty$, endowed with the topology defined by the following countable family of norms

$$|\varphi|_{k,p;\omega,c} := \sum_{j \leqslant k} \|(\varphi_{\omega,c})^{(j)}\|_{L^p}, \quad k \in \mathbb{N},$$

is a Fréchet subspace of $\mathcal{C}^\infty$.

Proposition 4.1.50. *Let* $1 \leqslant p \leqslant \infty$. *If* $\varphi,\ \psi \in \mathcal{D}_{L^p_{2\omega,c}}$, *then* $\varphi\psi \in \mathcal{D}_{L^p_{\omega,c}}$.

Proof. Let $\varphi,\ \psi \in \mathcal{D}_{L^p_{2\omega,c}}$, then $\varphi_{2\omega,c} \in \mathcal{D}_{L^p}$ and $\psi_{2\omega,c} \in \mathcal{D}_{L^p}$, $j \in \mathbb{N}$. So $\varphi^{(j)}_{2\omega,c} \in L^p$ and $\psi^{(j)}_{2\omega,c} \in L^p$. By Leibniz's rule, we obtain

$$\big((\varphi\psi)_{\omega,c}\big)^{(j)} = \big(c^{-\frac{t}{2\omega}}\varphi c^{-\frac{t}{2\omega}}\psi\big)^{(j)} = (\varphi_{2\omega,c}\psi_{2\omega,c})^{(j)} = \sum_{i=0}^{j}\binom{i}{j}\varphi^{(i)}_{2\omega,c}\psi^{(j-i)}_{2\omega,c} \in L^p.$$

This shows that $(\varphi\psi)_{\omega,c} \in \mathcal{D}_{L^p}$. Hence, $\varphi\psi \in \mathcal{D}_{L^p_{\omega,c}}$. □

The following result shows that the family of norms $|\cdot|_{k,p;\omega,c}$ is submultiplicative.

Proposition 4.1.51. *Let $1 \leqslant p \leqslant \infty$. If $\varphi,\ \psi \in \mathcal{D}_{L^p_{2\omega,c}}$, then for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that*

$$|\varphi\psi|_{k,p;\omega,c} \leqslant C_k|\varphi|_{k,p;2\omega,c}.|\psi|_{k,p;2\omega,c}.$$

Proof. Let $\varphi,\ \psi \in \mathcal{D}_{L^p_{2\omega,c}}$. We have

$$\begin{aligned}\sum_{j\leqslant k}\big\|\big((\varphi\psi)_{\omega,c}\big)^{(j)}\big\|_{L^p} &= \left\|\binom{i}{j}(\varphi_{2\omega,c})^{(i)}(\psi_{2\omega,c})^{(j-i)}\right\|_{L^p}\\ &\leqslant \sum_{j\leqslant k}\sum_{i=1}^{j}\binom{i}{j}\big\|(\varphi_{2\omega,c})^{(i)}(\psi_{2\omega,c})^{(j-i)}\big\|_{L^p}\\ &\leqslant \sum_{j\leqslant k}\sum_{i=1}^{j}\binom{i}{j}\big\|(\varphi_{2\omega,c})^{(i)}\big\|_{L^p}\sum_{j\leqslant k}\sum_{i=1}^{j}\binom{i}{j}\big\|(\psi_{2\omega,c})^{(j-i)}\big\|_{L^p}.\end{aligned}$$

Set

$$C_k := \Bigg(\sum_{j\leqslant k}\sum_{i=1}^{j}\binom{i}{j}\Bigg)^2 > 0.$$

Then

$$|\varphi\psi|_{k,p;\omega,c} \leqslant C_k|\varphi|_{k,p;2\omega,c}\cdot|\psi|_{k,p;2\omega,c}.$$ □

For $1 \leqslant p < \infty$, we have $\mathcal{D} \subseteq \mathcal{D}_{L^p_{\omega,c}} \subseteq \mathcal{D}_{L^\infty_{\omega,c}}$. Moreover, it can be simply shown that, for $1 \leqslant p < \infty$, the space $\mathcal{D}$ is dense in $\mathcal{D}_{L^p_{\omega,c}}$.

The space $\mathcal{D}$ is not dense in $\mathcal{D}_{L^\infty_{\omega,c}}$. We then define $\dot{\mathcal{D}}_{L^\infty_{\omega,c}}$ as the subspace of all functions in $\mathcal{D}_{L^\infty_{\omega,c}}$ which vanish at infinity with all their derivatives. This space is the closure of the space $\mathcal{D}_{L^\infty_{\omega,c}}$ in $\mathcal{D}$. It is clear that $\dot{\mathcal{D}}_{L^\infty_{\omega,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{\omega,c}}$; hence, it is a Fréchet space. Moreover, it is easy to prove the following result:

Proposition 4.1.52. *For $1 \leqslant p < \infty$, we have*

$$\mathcal{D}_{L^p_{\omega,c}} \hookrightarrow \dot{\mathcal{D}}_{L^\infty_{\omega,c}} \hookrightarrow \mathcal{D}_{L^\infty_{\omega,c}},$$

with continuous embeddings.

Recall also the following space of smooth almost periodic functions introduced by L. Schwartz

$$\mathcal{B}_{\mathrm{ap}} := \{\varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in \mathrm{AP},\ j \in \mathbb{N}\}.$$

We have the following properties of $\mathcal{B}_{\mathrm{ap}}$.

Proposition 4.1.53.
(i) $\mathcal{B}_{\mathrm{ap}} = \mathrm{AP} \cap \mathcal{D}_{L^\infty}$.
(ii) $\mathcal{B}_{\mathrm{ap}}$ *is a closed differential subalgebra of* $\mathcal{D}_{L^\infty}$.
(iii) *If* $f \in L^1$ *and* $\varphi \in \mathcal{B}_{\mathrm{ap}}$, *then* $f * \varphi \in \mathcal{B}_{\mathrm{ap}}$.

Proof. See [913]. □

Now, we can introduce the space of smooth (ω, c)-almost periodic functions.

Definition 4.1.54. The space of smooth (ω, c)-almost periodic functions on $\mathbb{R}$ is defined by

$$\mathcal{B}_{\mathrm{AP}_{\omega,c}} := \{\varphi \in \mathcal{D}_{L^\infty_{\omega,c}} : \varphi^{(j)}_{\omega,c} \in \mathcal{B}_{\mathrm{ap}},\ j \in \mathbb{N}\}.$$

We endow $\mathcal{B}_{\mathrm{AP}_{\omega,c}}$ with the topology induced by $\mathcal{D}_{L^\infty_{\omega,c}}$. Some properties of $\mathcal{B}_{\mathrm{AP}_{\omega,c}}$ are given in the following.

Proposition 4.1.55.
(i) $\mathcal{B}_{\mathrm{AP}_{\omega,c}} = \mathrm{AP}_{\omega,c} \cap \mathcal{D}_{L^\infty_{\omega,c}}$.
(ii) $\mathcal{B}_{\mathrm{AP}_{\omega,c}}$ *is a closed subspace of* $\mathcal{D}_{L^\infty_{\omega,c}}$.
(iii) *If* $f \in L^1_{\omega,c}$ *and* $\varphi \in \mathcal{B}_{\mathrm{AP}_{\omega,c}}$, *then* $c^{\frac{t}{\omega}}(f_{\omega,c} * \varphi_{\omega,c}) \in \mathcal{B}_{\mathrm{AP}_{\omega,c}}$.

Proof. (i): Obvious.

(ii): It follows from (i) and the completeness of the space of almost periodic functions.

(iii): If $f \in L^1_{\omega,c}$ and $\varphi \in \mathcal{B}_{\mathrm{AP}_{\omega,c}}$, then $f_{\omega,c} \in L^1$ and $\varphi_{\omega,c} \in \mathcal{B}_{\mathrm{ap}}$. From Proposition 4.1.53, we have $f_{\omega,c} * \varphi_{\omega,c} \in \mathcal{B}_{\mathrm{ap}}$; hence,

$$c^{-\frac{t}{\omega}}\big(c^{\frac{t}{\omega}}(f_{\omega,c} * \varphi_{\omega,c})\big) \in \mathcal{B}_{\mathrm{ap}},$$

which shows that $c^{\frac{t}{\omega}}(f_{\omega,c} * \varphi_{\omega,c}) \in \mathcal{B}_{\mathrm{AP}_{\omega,c}}$. □

Corollary 4.1.56. *If* $f \in \mathcal{D}_{L^\infty_{\omega,c}}$ *and* $c^{\frac{t}{\omega}}(f_{\omega,c} * \varphi_{\omega,c}) \in \mathrm{AP}_{\omega,c}$, $\varphi \in \mathcal{D}$, *then* $f \in \mathcal{B}_{\mathrm{AP}_{\omega,c}}$.

Remark 4.1.57. It is clear that $\mathcal{B}_{\mathrm{AP}_{\omega,c}} \subseteq \mathrm{AP}_{\omega,c} \cap \mathcal{C}^\infty$, whereas the converse inclusion is not true. Indeed, the function

$$f(t) = 2^{-t}\sqrt{2 + \cos t + \cos\sqrt{2}t}, \quad t \in \mathbb{R},$$

is an element of $\mathrm{AP}_{\omega,c} \cap C^{\infty}$ with $c = 2$ and $\omega = 1$. However,

$$f'(t) = 2^{-t}\left(\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2+\cos t+\cos\sqrt{2}t}} - \ln 2\sqrt{2+\cos t+\cos\sqrt{2}t}\right), \quad t \in \mathbb{R},$$

is not bounded, because $\inf_{t\in\mathbb{R}}(2+\cos t+\cos\sqrt{2}t) = 0$ and therefore

$$\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2+\cos t+\cos\sqrt{2}t}} \notin \mathrm{AP}.$$

Hence, $f \notin \mathcal{B}_{\mathrm{AP}_{w,c}}$.

Now we would like to introduce the concept of (ω, c)-almost periodicity in the setting of Sobolev–Schwartz distributions. For this we need to introduce the so-called space of $L^p_{\omega,c}$-distributions, $1 \leqslant p \leqslant \infty$. We first recall the space of L^p-distributions, $1 \leqslant p \leqslant \infty$, which has been introduced by L. Schwartz in [913]. L. Schwartz has introduced the space $\mathcal{D}'_{L^p}$ as a topological dual of $\mathcal{D}_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$. These spaces is related to Sobolev spaces; for more details, see [118] and [913].

Definition 4.1.58. Let $1 < p \leqslant \infty$, the space $\mathcal{D}'_{L^p}$ is the topological dual of $\mathcal{D}_{L^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. An element of $\mathcal{D}'_{L^\infty}$ is called a bounded distribution.

Before we go any further, let us recall that the space of bounded distributions will be denoted slightly different in Subsection 4.2.8, where we will use the notation $\mathcal{D}'_{L^1}$. Now we will state the following result.

Theorem 4.1.59. *Let $T \in \mathcal{D}'$. Then the following statements are equivalent:*
(i) $T \in \mathcal{D}'_{L^p}$.
(ii) $T * \varphi \in L^p$, $\varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{N}, \exists (f_j)_{0\leqslant j\leqslant k} \subseteq L^p : T = \sum_{j=0}^{k} f_j^{(j)}$.

Proof. See [118] or [913]. □

Owing to the density of the space $\mathcal{D}$ in $\mathcal{D}_{L^p_{\omega,c}}$, $1 \leqslant p < \infty$, (resp. $\dot{\mathcal{D}}_{L^\infty_{\omega,c}}$), we see that the space $\mathcal{D}_{L^p_{\omega,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{\omega,c}}$) is a normal space of distributions, i. e., the elements of topological dual of $\mathcal{D}_{L^p_{\omega,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{\omega,c}}$) can be identified with continuous linear forms on $\mathcal{D}$.

Definition 4.1.60. For $1 < p \leqslant \infty$, we denote by $\mathcal{D}'_{L^p_{\omega,c}}$ the topological dual of $\mathcal{D}_{L^q_{\omega,c}}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

The following spaces of $L^p_{\omega,c}$-distributions are needed to define and study the (ω, c)-almost periodicity of distributions.

Definition 4.1.61.
(i) The topological dual of $\mathcal{D}_{L^1_{\omega,c}}$, denoted by $\mathcal{B}'_{\omega,c}$, is called the space of (ω, c)-bounded distributions.
(ii) The topological dual of $\dot{\mathcal{D}}_{L^\infty_{\omega,c}}$, denoted by $\mathcal{D}'_{L^1_{\omega,c}}$, is called the space of (ω, c)-integrable distributions.

By applying Theorem 4.1.59, we can easily show the following characterizations of $L^p_{\omega,c}$-distributions.

Theorem 4.1.62. *Let $T \in \mathcal{D}'$. Then the following statements are equivalent:*
(i) $T \in \mathcal{D}'_{L^p_{\omega,c}}$.
(ii) $c^{\frac{t}{\omega}}(T_{\omega,c} * \varphi) \in L^p_{\omega,c}$, $\varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{N}, \exists (f_j)_{0\leqslant j\leqslant k} \subseteq L^p_{\omega,c} : T = c^{\frac{t}{w}} \sum_{j=0}^k (f_{\omega,c})_j^{(j)}$, *where*

$$((f_{\omega,c})_j)_{0\leqslant j\leqslant k} = (c^{-\frac{t}{\omega}} f_j)_{0\leqslant j\leqslant k}.$$

Remark 4.1.63. As a consequence of Theorem 4.1.62, we see that $T \in \mathcal{D}'_{L^p_{\omega,c}}$ if and only if $T_{\omega,c} \in \mathcal{D}'_{L^p}$.

Returning to the notation (4.13), we recall that a distribution $T \in \mathcal{D}'$ is zero on an open subset V of $\mathbb{R}$ if

$$\langle T, \varphi\rangle = 0, \quad \varphi \in \mathcal{D}(V),$$

and that two distributions T, $S \in \mathcal{D}'$ coincide on V if $T - S = 0$ on V.

Lemma 4.1.64. *Let $f \in C^\infty$ and $T \in \mathcal{D}'$. If $fT = 0$, then $T = 0$ on the set $G = \{x \in \mathbb{R} : f(x) \neq 0\}$.*

Proof. Let $\varphi \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subseteq G$. Then we have

$$\langle T, \varphi\rangle = \left\langle T, f\frac{\varphi}{f}\right\rangle = \left\langle fT, \frac{\varphi}{f}\right\rangle = 0,$$

because $\frac{\varphi}{f} \in \mathcal{D}$ and by hypothesis $fT = 0$. □

Proposition 4.1.65. *Let $T \in \mathcal{D}'$. Then $T \in \mathcal{D}'_{L^p_{\omega,c}}, 1 \leqslant p \leqslant \infty$, if and only if, there exists $S \in \mathcal{D}'_{L^p}, 1 \leqslant p \leqslant \infty$, such that $T = c^{\frac{t}{\omega}} S$ in $\mathcal{D}'$.*

Proof. $(\to)$: If $T \in \mathcal{D}'_{L^p_{\omega,c}}$, then we have (see Remark 4.1.63) $T_{\omega,c} = c^{-\frac{t}{\omega}} T \in \mathcal{D}'_{L^p}$, so there exists $S \in \mathcal{D}'_{L^p}$ such that $c^{-\frac{t}{\omega}} T - S = 0$ in $\mathcal{D}'_{L^p}$, i. e., $c^{-\frac{t}{\omega}}(T - c^{\frac{t}{\omega}} S) = 0$ in $\mathcal{D}'_{L^p}$. By applying Lemma 4.1.64, it follows that

$$T = c^{\frac{t}{\omega}} S \quad \text{in } \mathcal{D}'.$$

($\leftarrow$): Suppose that $T \in \mathcal{D}'$ and there exists $S \in \mathcal{D}'_{L^p}, 1 \leqslant p \leqslant \infty$, such that $T = c^{\frac{t}{\omega}} S$ in $\mathcal{D}'$. Then $c^{-\frac{t}{\omega}} T = S \in \mathcal{D}'_{L^p}$ and hence $T \in \mathcal{D}'_{L^p_{\omega,c}}$. □

Recall that the space $\mathcal{B}'_{\mathrm{ap}}$ of almost periodic distributions was studied by L. Schwartz using the topological definition of Bochner's almost periodic functions. Let $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translated of T by h, denoted by $\tau_h T$, is defined by

$$\langle \tau_h T, \varphi \rangle := \langle T, \tau_{-h}\varphi \rangle, \quad \varphi \in \mathcal{D},$$

where $\tau_{-h}\varphi(x) := \varphi(x + h)$.

The following result gives the basic characterizations of Schwartz almost periodic distributions.

Theorem 4.1.66. *For any bounded distribution $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent:*
(i) *The set $\{\tau_h T : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$.*
(ii) $T * \varphi \in \mathrm{AP}, \varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{N}, \exists (f_j)_{0 \leqslant j \leqslant k} \subseteq \mathrm{AP} : T = \sum_{j=0}^{k} f_j^{(j)}$.

Proof. See [913]. □

Now we will introduce the following concept.

Definition 4.1.67. A distribution $T \in \mathcal{B}'_{\omega,c}$ is said to be (ω, c)-almost periodic, if and only if, $T_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$, i. e., the set $\{\tau_h T_{\omega,c} : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$. The set of (ω, c)-almost periodic distributions is denoted by $\mathcal{B}'_{\mathrm{AP}_{\omega,c}}$.

Example 4.1.68.
(i) The associated distribution of an (ω, c)-almost periodic function (resp. Stepanov (p, ω, c)-almost periodic function) is an (ω, c)-almost periodic distribution, i. e.

$$\mathrm{AP}_{\omega,c} \hookrightarrow \mathcal{B}'_{\mathrm{AP}_{\omega,c}} \quad (\text{resp. } S^p\mathrm{AP}_{\omega,c} \hookrightarrow \mathcal{B}'_{\mathrm{AP}_{\omega,c}}).$$

(ii) When $c = 1$ it follows that $\mathcal{B}'_{\mathrm{AP}_{\omega,c}} := \mathcal{B}'_{\mathrm{ap}}$.

The main characterizations of (ω, c)-almost periodic distributions are given in the following result.

Theorem 4.1.69. *Let $T \in \mathcal{B}'_{\omega,c}$. Then the following statements are equivalent:*
(i) $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$.
(ii) $c^{\frac{t}{w}}(T_{\omega,c} * \varphi) \in \mathrm{AP}_{\omega,c}, \varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{N}, \exists (f_j)_{0 \leqslant j \leqslant k} \subseteq \mathrm{AP}_{\omega,c} : T = c^{\frac{t}{\omega}} \sum_{j=0}^{k} (f_{\omega,c})_j^{(j)}$, *where*

$$((f_{\omega,c})_j)_{0 \leqslant j \leqslant k} = (c^{-\frac{t}{\omega}} f_j)_{0 \leqslant j \leqslant k}.$$

Proof. Since for every $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$, we have $T_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$, the result follows immediately from Theorem 4.1.66. □

The main properties of $\mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ are given in the following proposition.

Proposition 4.1.70.
(i) *If $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$, then $c^{\frac{t}{\omega}}(T_{w,c})^{(j)} \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$, $j \in \mathbb{N}$.*
(ii) *If $\varphi \in \mathcal{B}_{\mathrm{AP}_{\omega,c}}$ and $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$, then $\varphi_{\omega,c}T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$.*
(iii) *If $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ and $S \in \mathcal{D}'_{L^1_{\omega,c}}$, then $c^{\frac{t}{\omega}}(T_{w,c} * S_{\omega,c}) \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$.*

Proof. (i) Obvious.

(ii) If $\varphi \in \mathcal{B}_{\mathrm{AP}_{\omega,c}}$ and $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$, then $\varphi_{\omega,c} \in \mathcal{B}_{\mathrm{ap}}$ and $T_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$. Then we simply get $\varphi_{\omega,c}T_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$ and therefore

$$c^{-\frac{t}{\omega}}\big(c^{\frac{t}{\omega}}(\varphi_{\omega,c}T_{\omega,c})\big) \in \mathcal{B}'_{\mathrm{ap}},$$

which gives

$$c^{\frac{t}{\omega}}(\varphi_{\omega,c}T_{\omega,c}) \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}.$$

Hence, $\varphi_{\omega,c}T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$.

(iii) Let $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ and $S \in \mathcal{D}'_{L^1_{\omega,c}}$. Then $T_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$ and $S_{\omega,c} \in \mathcal{D}'_{L^1}$. Similarly as above, $T_{\omega,c} * S_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$, and

$$c^{-\frac{t}{\omega}}\big(c^{\frac{t}{\omega}}(T_{\omega,c} * S_{\omega,c})\big) \in \mathcal{B}'_{\mathrm{ap}}.$$

Hence, $c^{\frac{t}{\omega}}(T_{\omega,c} * S_{\omega,c}) \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$. □

The following result shows that $\mathcal{B}_{\mathrm{AP}_{\omega,c}}$ is dense in $\mathcal{B}'_{\mathrm{AP}_{\omega,c}}$.

Proposition 4.1.71. *Let $T \in \mathcal{B}'_{\omega,c}$. Then $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ if and only if there exists $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathrm{AP}_{\omega,c}}$ such that $\lim_{n\to+\infty} \varphi_n = T$ in $\mathcal{B}'_{\omega,c}$.*

Proof. If $T \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$, then $T_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$ and from the density of $\mathcal{B}_{\mathrm{ap}}$ in $\mathcal{B}'_{\mathrm{ap}}$ there exists $(\psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathrm{ap}}$ such that

$$\lim_{n\to+\infty} \psi_n = T_{\omega,c} \quad \text{in } \mathcal{D}'_{L^\infty};$$

this is equivalent to

$$c^{\frac{t}{\omega}} \lim_{n\to+\infty} \psi_n = \lim_{n\to+\infty}\big(c^{\frac{t}{\omega}}\psi_n\big) = c^{\frac{t}{\omega}}T_{\omega,c} = T \quad \text{in } \mathcal{B}'_{\omega,c}.$$

Hence, there exists $(\varphi_n)_{n\in\mathbb{N}} = (c^{\frac{t}{\omega}}\psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathrm{AP}_{\omega,c}}$ such that

$$\lim_{n\to+\infty} \varphi_n = T \quad \text{in } \mathcal{B}'_{\omega,c}.$$ □

Now we will introduce the concept of asymptotic (ω, c)-almost periodicity of distributions. Asymptotically almost periodic Schwartz distributions have been introduced and studied by I. Cioranescu in [299]. We recall the definition and some properties of asymptotically almost periodic Schwartz distributions ($\mathbb{R}_+ \equiv [0,\infty)$).

Definition 4.1.72. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called vanishing at infinity if

$$\lim_{h\to+\infty} \langle \tau_{-h}T, \varphi\rangle = 0 \quad \text{in } \mathbb{C},\ \varphi \in \mathcal{D}.$$

Denote by $\mathcal{B}'_{0+}$ the space of bounded distributions vanishing at infinity.

Definition 4.1.73. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called asymptotically almost periodic if there exist $R \in \mathcal{B}'_{\mathrm{ap}}$ and $S \in \mathcal{B}'_{0+}$ such that $T = R + S$ on $[0,\infty)$. The space of asymptotically almost periodic Schwartz distributions is denoted by $\mathcal{B}'_{\mathrm{aap}}(\mathbb{R}_+)$.

Proposition 4.1.74. *If $T \in \mathcal{B}'_{\mathrm{aap}}([0,\infty))$, the decomposition $T = R+S$ on $[0,\infty)$, is unique in $\mathcal{D}'_{L^\infty}$.*

Proof. See [299]. □

Set $\mathcal{D}_+ := \{\varphi \in \mathcal{D} : \operatorname{supp}(\varphi) \subseteq [0,\infty)\}$. Then we have the following characterization of the space $\mathcal{B}'_{\mathrm{aap}}([0,\infty))$.

Theorem 4.1.75. *Let $T \in \mathcal{D}'_{L^\infty}$. Then the following assertions are equivalent:*
(i) $T \in \mathcal{B}'_{\mathrm{aap}}(\mathbb{R}_+)$.
(ii) $T * \check{\varphi} \in \mathrm{AAP}([0,\infty))$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) := \varphi(-x)$.
(iii) $\exists k \in \mathbb{N}, \exists (f_j)_{0\leqslant j\leqslant k} \subseteq \mathrm{AAP}([0,\infty)) : T = \sum_{j=0}^{k} f_j^{(j)}$ *on* $\mathbb{R}_+$.

Proof. See [299]. □

Asymptotic (ω, c)-almost periodicity of distributions is introduced in the following definition.

Definition 4.1.76. Let $c \in \mathbb{C}$, $|c| \geqslant 1$ and $\omega > 0$. Then a distribution $T \in \mathcal{B}'_{\omega,c}$ is said to be asymptotically (ω, c)-almost periodic, if and only if, $T_{\omega,c} \in \mathcal{B}'_{\mathrm{aap}}([0,\infty))$. The space of asymptotically (ω, c)-almost periodic distributions is denoted by $\mathcal{B}'_{\mathrm{AAP}_{\omega,c}}([0,\infty))$.

Remark 4.1.77.
(i) When $c = 1$ it follows that $\mathcal{B}'_{\mathrm{AAP}_{\omega,c}}([0,\infty)) := \mathcal{B}'_{\mathrm{aap}}([0,\infty))$.
(ii) The associated distribution of an asymptotically (ω, c)-almost periodic function (resp. asymptotically Stepanov (p, ω, c)-almost periodic function) is an asymptotically (ω, c)-almost periodic distribution.

Let us define now the space $(\mathcal{B}'_{\omega,c})_{0+}$ of (ω, c)-bounded distributions vanishing at infinity as follows.

Definition 4.1.78. Let $c \in \mathbb{C}$, $|c| \geqslant 1$ and $\omega > 0$. A distribution $T \in \mathcal{B}'_{\omega,c}$ is said to be (ω, c)-bounded distribution vanishing at infinity, if and only if, $T_{\omega,c} \in \mathcal{B}'_{0+}$.

We have the following result.

Theorem 4.1.79. *Let $c \in \mathbb{C}$, $|c| \geqslant 1$, $\omega > 0$ and $T \in \mathcal{B}'_{\omega,c}$. Then $T \in \mathcal{B}'_{\mathrm{AAP}_{\omega,c}}([0,\infty))$, if and only if, there exist $R \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ and $S \in (\mathcal{B}'_{\omega,c})_{0+}$ such that*

$$T = R + S \quad \text{on } [0,\infty). \tag{4.14}$$

Proof. $(\rightarrow)$: Let $T \in \mathcal{B}'_{\mathrm{AAP}_{\omega,c}}([0,\infty))$. Then $T_{\omega,c} \in \mathcal{B}'_{\mathrm{aap}}([0,\infty))$ and by Definition 4.1.73, there exist $P \in \mathcal{B}'_{\mathrm{ap}}$ and $Q \in \mathcal{B}'_{0+}$ such that $T_{\omega,c} = P + Q$ on $[0,\infty)$. On the other hand, we have

$$\begin{aligned} T_{\omega,c} &= c^{-\frac{t}{\omega}} T = P + Q \rightarrow \langle c^{-\frac{t}{\omega}} T, \varphi \rangle = \langle P, \varphi \rangle + \langle Q, \varphi \rangle, \quad \varphi \in \mathcal{D} \\ &\rightarrow \langle T, \psi \rangle = \langle c^{\frac{t}{\omega}} P, \psi \rangle + \langle c^{\frac{t}{\omega}} Q, \psi \rangle, \quad \psi = c^{-\frac{t}{\omega}} \varphi \in \mathcal{D}. \end{aligned}$$

Thus there exist $R = c^{\frac{t}{\omega}} P \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ and $S = c^{\frac{t}{\omega}} Q \in (\mathcal{B}'_{\omega,c})_{0+}$ such that $T = R + S$ on $[0,\infty)$.

$(\leftarrow)$: If there exist $R \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ and $S \in (\mathcal{B}'_{\omega,c})_{0+}$ such that $T = R + S$ on $[0,\infty)$, then $c^{-\frac{t}{\omega}} T = c^{-\frac{t}{\omega}} R + c^{-\frac{t}{\omega}} S$ on $[0,\infty)$, i. e. $T_{\omega,c} = R_{\omega,c} + S_{\omega,c}$ on $[0,\infty)$, where $R_{\omega,c} \in \mathcal{B}'_{\mathrm{ap}}$ and $S_{\omega,c} \in \mathcal{B}'_{0+}$; hence $T_{\omega,c} \in \mathcal{B}'_{\mathrm{aap}}([0,\infty))$, which shows that $T \in \mathcal{B}'_{\mathrm{AAP}_{\omega,c}}(\mathbb{R}_+)$. □

Proposition 4.1.80. *The decomposition* (4.14) *is unique in $\mathcal{B}'_{\omega,c}$.*

Proof. Suppose that $T \in \mathcal{B}'_{\mathrm{AAP}_{\omega,c}}(\mathbb{R}_+)$ is such that $T = R + S$ on $[0,\infty)$, where $R \in \mathcal{B}'_{\mathrm{AP}_{\omega,c}}$ and $S \in (\mathcal{B}'_{\omega,c})_{0+}$. Then the result follows from the proof of the implication $(\leftarrow)$ of Theorem 4.1.79 and the uniqueness of the decomposition of asymptotically almost periodic distributions. □

Some characterizations of asymptotically (ω, c)-almost periodic distributions are given in the following result.

Theorem 4.1.81. *Let $c \in \mathbb{C}$, $|c| \geqslant 1$, $\omega > 0$ and $T \in \mathcal{B}'_{\omega,c}$. The following assertions are equivalent:*

(i) $T \in \mathcal{B}'_{\mathrm{AAP}_{\omega,c}}(\mathbb{R}_+)$.

(ii) $c^{\frac{t}{\omega}}(T_{\omega,c} * \check{\varphi}) \in \mathrm{AAP}_{\omega,c}([0,\infty))$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) := \varphi(-x)$.

(iii) $\exists k \in \mathbb{N}$, $\exists (f_j)_{0 \leqslant j \leqslant k} \subseteq \mathrm{AAP}_{\omega,c}([0,\infty)) : T = c^{\frac{t}{\omega}} \sum_{j=0}^{k} (f_{\omega,c})_j^{(j)}$ *on* $[0,\infty)$, *where* $((f_{\omega,c})_j)_{0 \leqslant j \leqslant k} = (c^{-\frac{t}{\omega}} f_j)_{0 \leqslant j \leqslant k}$.

Proof. It is clear that if $T \in \mathcal{B}'_{\mathrm{AAP}_{\omega,c}}(\mathbb{R}_+)$ then $T_{\omega,c} \in \mathcal{B}'_{\mathrm{aap}}(\mathbb{R}_+)$. Applying Theorem 4.1.75, we obtain the result. □

4.1.6 Linear differential equations in $\mathcal{B}'_{\mathrm{AP}_{\omega,c}}$

In this subsection, we will study the existence of distributional (ω, c)-almost periodic solutions of the following system of linear ordinary differential equations

$$T' = AT + S, \tag{4.15}$$

where $A = (a_{ij})_{1 \leqslant i,j \leqslant k}$, $k \in \mathbb{N}$, is a given square matrix of complex numbers, $S = (S_i)_{1 \leqslant i \leqslant k} \in (\mathcal{D}')^k$ is a vector distribution and $T = (T_i)_{1 \leqslant i \leqslant k}$ is the unknown vector distribution.

First, consider the system (4.15) with $S \in (\mathrm{AP})^k$ and recall the following result.

Theorem 4.1.82. *If the matrix A has no eigenvalues with real part zero, then for any $S \in (\mathrm{AP})^k$, there exists a unique solution $T \in (\mathrm{AP})^k$ of the system* (4.15).

Proof. See [299]. □

Let I_k be the unit matrix of order k. The following result gives a sufficient condition for the (ω, c)-almost periodicity of the solution (if it exists) of the system (4.15).

Theorem 4.1.83. *Let $S \in (\mathcal{B}'_{\mathrm{AP}_{\omega,c}})^k$. If the matrix $A - \frac{\log c}{\omega} I_k$ has no eigenvalues with real part zero, then the system* (4.15) *admits a unique solution $T \in (\mathcal{D}'_{L^\infty_{\omega,c}})^k$ which is an (ω, c)-almost periodic vector distribution.*

Proof. Let $\varphi \in \mathcal{D}$. We have

$$c^{-\frac{t}{\omega}} T' * \varphi = (c^{-\frac{t}{\omega}} T * \varphi)' + \frac{\log c}{\omega} c^{-\frac{t}{\omega}} T * \varphi. \tag{4.16}$$

On the other hand, if $T \in (\mathcal{D}'_{L^\infty_{\omega,c}})^k$ satisfies (4.15), then

$$c^{-\frac{t}{\omega}} T' * \varphi = A c^{-\frac{t}{\omega}} T * \varphi + c^{-\frac{t}{\omega}} S * \varphi.$$

So from (4.16), we have

$$(c^{-\frac{t}{\omega}} T * \varphi)' = \left(A - \frac{\log c}{\omega} I_k\right) c^{-\frac{t}{\omega}} T * \varphi + c^{-\frac{t}{\omega}} S * \varphi,$$

i. e.

$$(T_{\omega,c} * \varphi)' = \left(A - \frac{\log c}{\omega} I_k\right)(T_{\omega,c} * \varphi) + S_{\omega,c} * \varphi, \tag{4.17}$$

where

$$T_{\omega,c} * \varphi = ((T_{\omega,c})_i * \varphi)_{1 \leqslant i \leqslant k} = ((c^{-\frac{t}{\omega}} T_i) * \varphi)_{1 \leqslant i \leqslant k}$$

and

$$S_{\omega,c} * \varphi = \big((S_{\omega,c})_i * \varphi\big)_{1\leqslant i\leqslant k} = \big((c^{-\frac{t}{\omega}} S_i) * \varphi\big)_{1\leqslant i\leqslant k}.$$

Then the system (4.17) is equivalent in $(\mathcal{C}^\infty)^k$ to the following system of differential equations

$$P' = BP + Q,$$

with $B = A - \frac{\log c}{\omega} I_k$, $P = T_{\omega,c} * \varphi \in (\mathcal{C}^\infty)^k$ and $Q = S_{\omega,c} * \varphi \in (\mathrm{AP})^k$. According to Theorem 4.1.82, it follows that there exists a unique bounded solution P which is almost periodic; therefore, $(T_{\omega,c})_i * \varphi \in \mathrm{AP}$, $1 \leqslant i \leqslant k$, $\varphi \in \mathcal{D}$ and $c^{\frac{t}{\omega}}((T_{\omega,c})_i * \varphi) \in \mathrm{AP}_{\omega,c}$, $1 \leqslant i \leqslant k$, $\varphi \in \mathcal{D}$. Thus, according to Theorem 4.1.69, we get $(T_i)_{1\leqslant i\leqslant k} \in (\mathcal{B}'_{\mathrm{AP}_{\omega,c}})^k$. □

4.1.7 Asymptotically (ω, c)-almost periodic type solutions of abstract degenerate non-scalar Volterra equations

There are by now only a few relevant references concerning abstract non-scalar Volterra equations, degenerate or non-degenerate in the time variable. Concerning non-degenerate abstract Volterra equations of non-scalar type, mention should be made of the research monograph [857] by J. Prüss, the article [569] by M. Jung and the article [634] by M. Kostić. In [635], we have explained how the methods proposed in [857] and [634] can be helpful in the analysis of abstract degenerate Volterra equations of non-scalar type. In this subsection, we initiate the study of the existence and uniqueness of asymptotically almost periodic type solutions of the abstract degenerate non-scalar Volterra equations. In actual fact, we investigate asymptotically (ω, c)-almost periodic type solutions of the abstract degenerate non-scalar Volterra equations in Banach spaces (we can similarly analyze (ω, c)-asymptotically periodic solutions; the Stepanov, Weyl and Besicovitch generalizations of asymptotically (ω, c)-almost periodic functions will not be considered, as well). The material of this subsection is taken form our recent joint research article with V. E. Fedorov [659].

We will first recall the various notions of (A, k, B)-regularized C-pseudoresolvent families introduced in [635]; after that, we will analyze the existence and uniqueness of asymptotically (ω, c)-almost periodic type solutions of the abstract degenerate Cauchy problem

$$Bu(t) = f(t) + \int_0^t A(t-s)u(s)\,ds, \ t \in [0, \tau). \tag{4.18}$$

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)_Y$ be two non-trivial complex Banach spaces such that Y is continuously embedded in X. Let the operator $C \in L(X)$ be injective, and let $\tau \in (0, \infty]$.

The norm in X, resp. Y, will be denoted by $\|\cdot\|_X$, resp. $\|\cdot\|_Y$. We use the symbol B to denote a closed linear operator with domain and range contained in X; by $\|\cdot\|_{[D(B)]} := \|\cdot\| + \|B\cdot\|$ we denote the corresponding graph norm and by $[D(B)] = (D(B), \|\cdot\|_{[D(B)]})$ we denote the corresponding Banach space. If Z is a general topological space and $Z_0 \subseteq Z$, then by $\overline{Z_0}^Z$ we denote the adherence of Z_0 in Z. We will basically follow the notation employed in the monograph of J. Prüss [857] and our paper [635].

We start by recalling the following notion introduced in [635] (see also [633, Section 2.9]).

Definition 4.1.84. Let $k \in C([0,\tau))$ and $k \neq 0$, let $\tau \in (0,\infty]$, $f \in C([0,\tau) : X)$, and let $A \in L^1_{loc}([0,\tau) : L(Y,X))$. Then we say that a function $u \in C([0,\tau) : [D(B)])$ is:

(i) a strong solution of (4.18) if and only if $u \in L^\infty_{loc}([0,\tau) : Y)$ and (4.18) holds on $[0,\tau)$,

(ii) a mild solution of (4.18) if and only if there exist a sequence (f_n) in $C([0,\tau) : X)$ and a sequence (u_n) in $C([0,\tau) : [D(B)])$ such that $u_n(t)$ is a strong solution of (4.18) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ as well as $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $[0,\tau)$.

The following definition will be invaluably important in our further work [635].

Definition 4.1.85. Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. A family $(S(t))_{t\in[0,\tau)}$ in $L(X,[D(B)])$ is called an (A,k,B)-regularized C-pseudoresolvent family if and only if the following hold:

(S1) The mappings $t \mapsto S(t)x$, $t \in [0,\tau)$ and $t \mapsto BS(t)x$, $t \in [0,\tau)$ are continuous in X for every fixed $x \in X$, $BS(0) = k(0)C$ and $S(t)C = CS(t)$, $t \in [0,\tau)$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0,\tau)$. Then (S2) means $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0,\tau)$ and $(U(t)_{|Y})_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$.

(S3) The resolvent equations

$$BS(t)y = k(t)Cy + \int_0^t A(t-s)\,dU(s)y, \quad t \in [0,\tau),\ y \in Y, \tag{4.19}$$

$$BS(t)y = k(t)Cy + \int_0^t S(t-s)A(s)y\,ds, \quad t \in [0,\tau),\ y \in Y, \tag{4.20}$$

hold; (4.19), resp. (4.20), is called the first resolvent equation, resp. the second resolvent equation.

An (A,k,B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be an (A,k,B)-regularized C-resolvent family if additionally:

(S4) For every $y \in Y$, we have $S(\cdot)y \in L^\infty_{loc}([0,\tau) : Y)$.

An operator family $(S(t))_{t\in[0,\tau)}$ in $L(X,[D(B)])$ is called a weak (A,k,B)-regularized C-pseudoresolvent family if and only if (S1) and (4.20) hold. Finally, a weak (A,k,B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be a-regular ($a \in L^1_{\mathrm{loc}}([0,\tau))$) if and only if $a * S(\cdot)x \in C([0,\tau) : Y)$, $x \in \overline{Y}^X$.

As is well known, condition (S3) can be rewritten in the following equivalent form: (S3)$'$

$$BU(t)y = \Theta(t)Cy + \int_0^t A(t-s)U(s)y\,ds, \quad t \in [0,\tau),\ y \in Y,$$

$$BU(t)y = \Theta(t)Cy + \int_0^t U(t-s)A(s)y\,ds, \quad t \in [0,\tau),\ y \in Y.$$

We also need the following definition from [635].

Definition 4.1.86. Let $k \in C([0,\infty))$, $k \neq 0$, $A \in L^1_{\mathrm{loc}}([0,\infty) : L(Y,X))$, $\alpha \in (0,\pi]$, and let $(S(t))_{t\geqslant 0} \subseteq L(X,[D(B)])$ be a (weak) (A,k,B)-regularized C-(pseudo)resolvent family. Then it is said that $(S(t))_{t\geqslant 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)-resolvent family of angle α, if there exists an analytic function $\mathbf{S} : \Sigma_\alpha \to L(X,[D(B)])$ satisfying $\mathbf{S}(t) = S(t)$, $t > 0$, $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{S}(z)x = S(0)x$ and $\lim_{z\to 0, z\in\Sigma_\gamma} B\mathbf{S}(z)x = BS(0)x$ for all $\gamma \in (0,\alpha)$ and $x \in X$. We say that $(S(t))_{t\geqslant 0}$ is an exponentially bounded, analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, resp. bounded analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, of angle α, if $(S(t))_{t\geqslant 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family of angle α and for each $\gamma \in (0,\alpha)$ there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that $\|\mathbf{S}(z)\|_{L(X)} + \|B\mathbf{S}(z)\|_{L(X)} \leqslant M_\gamma e^{\omega_\gamma |z|}$, $z \in \Sigma_\gamma$. Since no confusion seems likely to arise, we shall identify $S(\cdot)$ and $\mathbf{S}(\cdot)$ in the sequel.

In [635], we have also introduced the notion of an (A,k,B)-regularized C-uniqueness family with a view to analyzing the uniqueness of solutions of the abstract Cauchy problem (4.18).

Definition 4.1.87. Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{\mathrm{loc}}([0,\tau) : L(Y,X))$. A strongly continuous operator family $(V(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be an (A,k,B)-regularized C-uniqueness family if and only if

$$V(t)By = k(t)Cy + \int_0^t V(t-s)A(s)y\,ds, \quad t \in [0,\tau),\ y \in Y \cap D(B).$$

We will use the following statements proved in [635, Proposition 2]:

[P] Assume that $(V(t))_{t\in[0,\tau)}$ is an (A,k,B)-regularized C-uniqueness family, $f \in C([0,\tau) : X)$ and $u(t)$ is a mild solution of (4.18). Then we have $(kC * u)(t) = (V * f)(t)$, $t \in [0,\tau)$.

[Q] Assume that $(S(t))_{t\in[0,\tau)}$ is an $(A,1,B)$-regularized C-pseudoresolvent family, $C^{-1}f \in C([0,\tau) : X)$ and $f(0) = 0$. Then we know that the following statements hold:

(a) Let $C^{-1}f \in AC_{\mathrm{loc}}([0,\tau) : Y)$ and $(C^{-1}f)' \in L^1_{\mathrm{loc}}([0,\tau) : Y)$. Then the function $t \mapsto u(t)$, $t \in [0,\tau)$ given by

$$u(t) = \int_0^t S(t-s)(C^{-1}f)'(s)\,ds = \int_0^t dU(s)(C^{-1}f)'(t-s)$$

is a strong solution of (4.18). Moreover, $u \in C([0,\tau) : Y)$.

(b) Let $(C^{-1}f)' \in L^1_{\mathrm{loc}}([0,\tau) : X)$ and $\overline{Y}^X = X$. Then the function $u(t) = \int_0^t S(t-s)(C^{-1}f)'(s)\,ds$, $t \in [0,\tau)$ is a mild solution of (4.18).

(c) Let $C^{-1}g \in W^{1,1}_{\mathrm{loc}}([0,\tau) : \overline{Y}^X)$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $f(t) = (a * g)(t)$, $t \in [0,\tau)$ and let $(S(t))_{t\in[0,\tau)}$ be a-regular. Then the function $u(t) = \int_0^t S(t-s)(a * (C^{-1}g)')(s)\,ds$, $t \in [0,\tau)$ is a strong solution of (4.18).

The uniqueness of solutions in (a), (b) or (c) holds provided that for each $y \in Y \cap D(B)$ we have $S(t)By = BS(t)y$, $t \in [0,\tau)$.

Even in the case that $B = C = I$ and $k(t) \equiv 1$, there exist examples of global not exponentially bounded (A,k,B)-regularized C-pseudoresolvent families (see, e. g., [857, Example 6.2, pp. 165–166]). For our purposes, it will be crucial to examine whether the operator family $(S(t))_{t\geqslant 0}$ is exponentially decaying as the time variable goes to plus infinity. The existence of a number $\varepsilon_0 \geqslant 0$ such that

$$\int_0^\infty e^{-\varepsilon t}\|A(t)\|_{L(Y,X)}\,dt < \infty, \quad \varepsilon > \varepsilon_0, \tag{4.21}$$

which has been used in [857] and [634, 635], is not sufficient to ensure the exponential decaying of $(S(t))_{t\geqslant 0}$ as $t \to +\infty$. Therefore, we must impose some extra conditions ensuring this property of $(S(t))_{t\geqslant 0}$, which will be extremely important for us.

Now we will state two simple results concerning this problematic. The two of them are basically deduced in [635].

Theorem 4.1.88. *Assume $\varepsilon_0 \geqslant 0$, $k(t)$ satisfies* (P1), $\omega \geqslant \max(\mathrm{abs}(k), \varepsilon_0)$, (4.21) *holds, $\alpha \in (0,\pi/2]$, there exists an analytic mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X,[D(B)])$ such that*

(i) $BH(\lambda)y - H(\lambda)\tilde{A}(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$; $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega$,

(ii) $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}[\|(\lambda-\omega)H(\lambda)\|_{L(X)} + \|(\lambda-\omega)BH(\lambda)\|_{L(X)}] < \infty$ *for all* $\gamma \in (0,\alpha)$,

(iii) *there exists an operator* $F \in L(X,[D(B)])$ *such that* $BFx = k(0)Cx$, $x \in X$ *and* $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda H(\lambda)x = Fx$, $x \in X$, *and*

(iv) $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda BH(\lambda)x = k(0)Cx$, $x \in X$, *provided that* $\overline{Y}^X \neq X$.

If there exists a real number $\omega_0 < 0$ such that the mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ can be analytically extended to the sector $\omega_0 + \Sigma_{\frac{\pi}{2}+\alpha}$ and condition (ii) *holds with the number ω replaced by the number ω_0 therein, then there exists a weak analytic (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t\geqslant 0}$ of angle α such that*

$$\sup_{z\in\Sigma_\gamma}\big[\|e^{-\omega_0 z}S(z)\|_{L(X)} + \|e^{-\omega_0 z}BS(z)\|_{L(X)}\big] < \infty \quad \textit{for all } \gamma \in (0, \alpha). \tag{4.22}$$

Proof. By [635, Theorem 3], we know that there exists a weak analytic (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t\geqslant 0}$ of angle α, satisfying that the estimate (4.22) holds with the number ω_0 replaced by the number ω. The final statement follows easily from this fact, [82, Theorem 2.6.1], the uniqueness theorem for the Laplace transform and the assumption we have made after the formulation of conditions (i)–(iv). □

We can similarly deduce the validity of the following result, which corresponds to [635, Theorem 4].

Theorem 4.1.89. *Assume $\alpha \in (0, \pi/2]$, $\varepsilon_0 \geqslant 0$, $k(t)$ satisfies* (P1) *and* (4.21) *holds. Let $\omega \geqslant \max(\mathrm{abs}(k), \varepsilon_0)$, and let there exist an analytic mapping $H : \omega+\Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ such that $H_{|Y} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(Y)$ is an analytic mapping, as well as that:*

(i) *One has*

$$\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}\big[\|(\lambda-\omega)H(\lambda)\|_{L(X)} + \|(\lambda-\omega)BH(\lambda)\|_{L(X)} + \|(\lambda-\omega)H(\lambda)\|_{L(Y)}\big] < \infty$$

for all $\gamma \in (0, \alpha)$,

(ii) *$BH(\lambda)y - H(\lambda)\tilde{A}(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$; $BH(\lambda)y - \tilde{A}(\lambda)H(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$; $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega_0$,*

(iii) *there exists an operator $F \in L(X, [D(B)])$ such that $BFx = k(0)Cx$, $x \in X$, $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0} \lambda H(\lambda)x = Fx$, $x \in X$, and*

(iv) *$\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0} \lambda BH(\lambda)x = k(0)Cx$, $x \in X$, provided that $\overline{Y}^X \neq X$.*

If there exists a real number $\omega_0 < 0$ such that the mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ can be analytically extended to the sector $\omega_0+\Sigma_{\frac{\pi}{2}+\alpha}$, the mapping $H_{|Y} : \omega+\Sigma_{\frac{\pi}{2}+\alpha} \to L(Y)$ can be analytically extended to the sector $\omega_0 + \Sigma_{\frac{\pi}{2}+\alpha}$, and condition (i) *holds with the number ω replaced by the number ω_0 therein, then there exists an analytic (A, k, B)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$ of angle α such that*

$$\sup_{z\in\Sigma_\gamma}\Big[\|e^{-\omega_0 z}S(z)\|_{L(X)} + \|e^{-\omega_0 z}BS(z)\|_{L(X)} + \sup_{z\in\Sigma_\gamma}\|e^{-\omega_0 z}S(z)\|_{L(Y)}\Big] < \infty$$

and the mapping $t \mapsto U(t) \in L(Y)$, $t > 0$ can be analytically extended to the sector Σ_α.

Remark 4.1.90. Concerning Theorem 4.1.88, it should be noted that we can also impose the condition that there exist a negative real number $\omega < 0$, a real number $\beta \in (0,1]$ and a number $\alpha_0 \in (0,\pi/2)$ such that $H(\cdot)$ is analytic on the region $\Omega \equiv \omega_0 + \Sigma_{(\pi/2)+\alpha}$, continuous on $\overline{\Omega}$ and satisfies the estimate

$$\sup_{\lambda\in\overline{\Omega}}\big[\big\|(1+|\lambda|)^{-\beta}H(\lambda)\big\|_{L(X)}+\big\|(1+|\lambda|)^{-\beta}BH(\lambda)\big\|_{L(X)}\big]<\infty.$$

Then the integral computation carried out in the proof of [82, Theorem 2.6.1] shows that there exists a weak analytic (A,k,B)-regularized C-pseudoresolvent family $(S(t))_{t\geqslant 0}$ of angle α such that

$$\sup_{z\in\Sigma_\gamma}\big[\big\|e^{-\omega_0 z}|z|^{\beta-1}S(z)\big\|_{L(X)}+\big\|e^{-\omega_0 z}|z|^{\beta-1}BS(z)\big\|_{L(X)}\big]<\infty \quad \text{for all } \gamma\in(0,\alpha).$$

A similar comment can be given in the case of Theorem 4.1.89.

Clearly, it is not trivial to practically verify the requirements of Theorem 4.1.88–Theorem 4.1.89 as well as that these theorems are not suitable for applications to the abstract fractional differential equations of non-scalar type. But, in many concrete situations, the requirements of these theorems can be very simply verified.

Example 4.1.91. Suppose that $X = Y$, $B = C = I$, $k(t) \equiv 1$, $\omega_0 < 0$, $0 < \alpha \leqslant \pi/2$ and D is a closed linear operator in X such that for each number $\gamma \in (0,\alpha)$ there exists a finite real number $M_\gamma > 0$ such that

$$\sup_{\lambda\in\omega_0+\Sigma_{(\pi/2)+\gamma}}\big\|\lambda(\lambda-D)^{-1}\big\|\times\big\|(\lambda-\omega_0)(\lambda-D)^{-1}\big\|<\infty.$$

Define $A(\cdot)$ through $\tilde{A}(\lambda) := (2D)/(\lambda) - (D^2)/(\lambda^2)$, $\lambda \neq 0$. Then the assumptions of Theorem 4.1.89 hold true because for each $\gamma \in (0,\pi/2)$ we have

$$\begin{aligned}
&\sup_{\lambda\in\omega_0+\Sigma_{(\pi/2)+\gamma}}\left|\frac{\lambda-\omega_0}{\lambda}\right|\times\big\|(I-\tilde{A}(\lambda))^{-1}\big\| \\
&\quad=\sup_{\lambda\in\omega_0+\Sigma_{(\pi/2)+\gamma}}\left|\frac{\lambda-\omega_0}{\lambda}\right|\times\left\|\left(I-\frac{2D}{\lambda}+\frac{D^2}{\lambda^2}\right)^{-1}\right\| \\
&\quad=\sup_{\lambda\in\omega_0+\Sigma_{(\pi/2)+\gamma}}\big\|\lambda(\lambda-D)^{-1}\big\|\times\big\|(\lambda-\omega_0)(\lambda-D)^{-1}\big\|<\infty.
\end{aligned}$$

Further possibilities to apply Theorem 4.1.88–Theorem 4.1.89 will be considered somewhere else. In [634, Theorem 3] and [635, Theorem 2], we have considered the hyperbolic perturbation results for the abstract non-scalar Volterra equations. Before proceeding further, we want also to observe that it is very difficult to say whether the perturbed resolvent solution family will be exponentially decaying if the initial resolvent solution family is exponentially decaying as time marches to plus infinity.

Concerning the exponential decaying rate at infinity of an (A,k,B)-regularized C-pseudoresolvent family $(S(t))_{t\geqslant 0}$, we would like to stress that, in [631, Remark 2.6.15], we have presented a simple idea which can be also applied in the qualitative analysis of asymptotically almost periodic type solutions of the abstract degenerate non-scalar Volterra integral equations. This will be the starting point for our investigations carried out in the remainder of subsection. First of all, we will clarify the following result which can be also formulated for analytic (A,k,B)-regularized C-pseudoresolvent families.

Proposition 4.1.92. *Suppose that* $z \in \mathbb{C}$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $k \neq 0$, $A \in L^1_{\mathrm{loc}}([0,\tau) : L(Y,X))$ *and* $(S(t))_{t\in[0,\tau)}$ *is an* (A,k,B)*-regularized* C*-pseudoresolvent family (weak* (A,k,B)*-regularized* C*-pseudoresolvent family). Define*

$$k_z(t) := e^{-zt}k(t), \quad A_z(t) := e^{-zt}A(t), \quad \text{and} \quad S_z(t) := e^{-zt}S(t), \quad t \in [0,\tau).$$

Then $(S_z(t))_{t\in[0,\tau)}$ *is an* (A_z,k_z,B)*-regularized* C*-pseudoresolvent family (weak* (A_z,k_z,B)*-regularized* C*-pseudoresolvent family). Furthermore,* $(S(t))_{t\in[0,\tau)}$ *is* a*-regular if and only if* $(S_z(t))_{t\in[0,\tau)}$ *is* a_z*-regular, where* $a_z(t) := e^{-zt}a(t)$, $t \in [0,\tau)$, *and* $(S_z(t))_{t\in[0,\tau)}$ *is an* (A_z,k_z,B)*-regularized* C*-resolvent family if* $(S(t))_{t\in[0,\tau)}$ *is an* (A,k,B)*-regularized* C*-resolvent family and* $\operatorname{Re} z \leqslant 0$.

Proof. We will provide the main details of the proof for (A,k,B)-regularized C-pseudoresolvent families, only. It is clear that condition (S1) holds true. In order to show (S2), define $U_z(t) := \int_0^t S_z(s)x\,ds$, $x \in X$, $t \in [0,\tau)$ and observe that the partial integration implies

$$U_z(t)x = e^{-zt}U(t)x + z\int_0^t e^{-zs}U(s)x\,ds, \quad x \in X,\ t \in [0,\tau). \tag{4.23}$$

This simply shows that $U_z(t)Y \subseteq Y$, $U_z(t)_{|Y} \in L(Y)$, $t \in [0,\tau)$ and $(U_z(t)_{|Y})_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$. We will prove only the first resolvent equation in (S3)$'$ because the second resolvent equation in (S3)$'$ [or (S3)] can be deduced almost trivially. So, let $y \in Y$ and $t \in [0,\tau)$ be fixed. Applying (4.23) twice and using the first resolvent equation in (S3)' for $(S(t))_{t\in[0,\tau)}$, we get

$$\begin{aligned} BU_z(t)y &= e^{-zt}\Bigg[\int_0^t e^{-zs}k(s)Cy\,ds + \int_0^t A(t-s)U(s)y\,ds\Bigg] \\ &\quad + z\int_0^t e^{-zs}\Bigg[\int_0^s e^{-zr}k(r)Cy\,dr + \int_0^s A(s-r)U(r)y\,dr\Bigg]ds \\ &= e^{-zt}\Bigg[\int_0^t e^{-zs}k(s)Cy\,ds + ze^{zt}\int_0^t e^{-zs}\int_0^s e^{-zr}k(r)Cy\,dr\,ds\Bigg] \\ &\quad + \int_0^t A(t-s)e^{-zt}U(s)y\,ds + z\big[e^{-z\cdot}A(\cdot) * 1 * e^{-z\cdot}U(\cdot)y\big](t). \end{aligned}$$

Partial integration shows that

$$e^{-zt}\left[\int_0^t e^{-zs}k(s)Cy\,ds + ze^{zt}\int_0^t e^{-zs}\int_0^s e^{-zr}k(r)Cy\,dr\,ds\right] = \int_0^t e^{-zs}k(s)Cy\,ds$$

and the required statement simply follows because the above equality yields

$$\begin{aligned} BU_z(t)y = &\int_0^t e^{-zs}k(s)Cy\,ds \\ &+ \int_0^t e^{-z(t-s)}A(t-s)\left[e^{-zs}U(s)y + z\int_0^s e^{-zr}U(r)y\,dr\right]ds. \end{aligned}$$

In order to see that $(S(t))_{t\in[0,\tau)}$ is a-regular if and only if $(S_z(t))_{t\in[0,\tau)}$ is a_z-regular, it suffices to observe that

$$\big(a_z * S_z(\cdot)x\big)(t) = e^{-zt}\big(a * S(\cdot)x\big)(t), \quad t \in [0,\tau),\ x \in \overline{Y}^X.$$

The remainder of the proof for (A, k, B)-regularized C-resolvent families is trivial. □

Now we will analyze the existence and uniqueness of asymptotically (ω, c)-almost periodic type solutions of the abstract Cauchy problem (4.18). First of all, we will state the following lemma whose proof is very simple and therefore is omitted.

Lemma 4.1.93. *Let $k \in C([0,\tau))$ and $k \neq 0$, let $\tau \in (0,\infty]$, $z \in \mathbb{C}$, $f \in C([0,\tau) : X)$, and let $A \in L^1_{\mathrm{loc}}([0,\tau) : L(Y,X))$. Suppose that $(V(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (A,k,B)-regularized C-uniqueness family. Define $f_z(t) := e^{-zt}f(t)$, $V_z(t) := e^{-zt}V(t)$ and $A_z(t) := e^{-zt}A(t)$ for all $t \in [0,\tau)$. Then we have:*

(i) *If $u(\cdot)$ is a strong (mild) solution of problem (4.18), then $u_z(\cdot) \equiv e^{-z\cdot}u(\cdot)$ is a strong (mild) solution of problem obtained by replacing, respectively, $f(\cdot)$ and $A(\cdot)$ in (4.18) by $f_z(\cdot)$ and $A_z(\cdot)$.*

(ii) *$(V_z(t))_{t\geqslant 0} \subseteq L(X)$ is an (A_z, k_z, B)-regularized C-uniqueness family.*

Now we will prove the following proposition.

Proposition 4.1.94. *Let $k \in C([0,\infty))$, $k \neq 0$, $\omega_0 \geqslant 0$, $\omega > 0$, $1 > \omega\omega_0$, $A \in L^1_{\mathrm{loc}}([0,\infty) : L(Y,X))$ and $(V(t))_{t\geqslant 0} \subseteq L(X)$ is an (A,k,B)-regularized C-uniqueness family such that $\|V(t)\| \leqslant Me^{\omega_0 t}$, $t \geqslant 0$. If $u(\cdot)$ is a mild solution of (4.18) and $f(\cdot)$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic), then the function $(kC * u)(\cdot)$ is likewise asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).*

Proof. Let $z = 1/\omega$. Due to our assumptions, we see that the operator family $(V_z(t) \equiv e^{-zt}V(t))_{t\geqslant 0}$ is exponentially decaying. By Lemma 4.1.93(i), $u_z(\cdot)$ is a strong (mild) solution of problem obtained by replacing, respectively, $f(\cdot)$ and $A(\cdot)$ in (4.18) by $f_z(\cdot)$ and $A_z(\cdot)$. Due to Lemma 4.1.93(ii), we see that $(V_z(t))_{t\geqslant 0} \subseteq L(X)$ is an (A_z, k_z, B)-regularized C-uniqueness family. Applying now [P], we get

$$(k_z C * u_z)(t) = (V_z * f_z)(t), \quad t \geqslant 0,$$

i. e.,

$$e^{-z\cdot}(kC * u)(t) = (V_z * f_z)(t), \quad t \geqslant 0.$$

We see that $f_z(\cdot)$ is asymptotically almost periodic (asymptotically almost automorphic, asymptotically compactly almost automorphic), so that the function $t \mapsto (V_z * f_z)(t)$, $t \geqslant 0$ has the same property [631]. This implies the required statement. □

It is clear that, if $(S(t))_{t\in[0,\tau)} \subseteq L(X, [D(B)])$ is a weak (A, k, B)-regularized C-pseudoresolvent family and $BS(t)y = S(t)By$, $t \in [0, \tau)$, $y \in Y \cap D(B)$, then $(S(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (A, k, B)-regularized C-uniqueness family. Using this observation, [P]–[Q] and Proposition 4.1.94, we may deduce the following.

Proposition 4.1.95. *Suppose that $(S(t))_{t\geqslant 0} \subseteq L(X, [D(B)])$ is an $(A, 1, B)$-regularized C-pseudoresolvent family, $BS(t)y = S(t)By$, $t \in [0, \tau)$, $y \in Y \cap D(B)$, $\omega_0 \geqslant 0$, $\omega > 0$, $1 > \omega\omega_0$, $\|S(t)\| \leqslant Me^{\omega_0 t}$, $t \geqslant 0$ and $f(\cdot)$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic). Then we have the following:*

(i) *Let $C^{-1}f \in AC_{\mathrm{loc}}([0, \infty) : Y)$, $(C^{-1}f)' \in L^1_{\mathrm{loc}}([0, \infty) : Y)$ and $f(0) = 0$. Then there exists a unique strong solution $u(\cdot)$ of (4.18); moreover, $u \in C([0, \tau) : Y)$ and the mapping $t \mapsto C\int_0^t u(s)\,ds$, $t \geqslant 0$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).*

(ii) *Let $(C^{-1}f)' \in L^1_{\mathrm{loc}}([0, \infty) : X)$ and $\overline{Y}^X = X$. Then there exists a unique mild solution $u(\cdot)$ of (4.18); moreover, the mapping $t \mapsto C\int_0^t u(s)\,ds$, $t \geqslant 0$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).*

(iii) *Let $C^{-1}g \in W^{1,1}_{\mathrm{loc}}([0, \infty) : \overline{Y}^X)$, $a \in L^1_{\mathrm{loc}}([0, \infty))$, $f(t) = (a * g)(t)$, $t \geqslant 0$ and let $(S(t))_{t\geqslant 0}$ be a-regular. Then there exists a unique strong solution $u(\cdot)$ of (4.18); moreover, the mapping $t \mapsto C\int_0^t u(s)\,ds$, $t \geqslant 0$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).*

It is worth noting that Proposition 4.1.95 can be deduced directly, as well as that some sufficient conditions ensuring the above features of the mapping $t \mapsto u(t)$, $t \geqslant 0$ can be also achieved. We will explain this only in the case of consideration of part (i). So, let us assume that $(S(t))_{t\geqslant 0} \subseteq L(X, [D(B)])$ is an $(A, 1, B)$-regularized C-pseudoresolvent family as well as that $C^{-1}f \in AC_{\mathrm{loc}}([0, \infty) : Y)$, $(C^{-1}f)' \in L^1_{\mathrm{loc}}([0, \infty) : Y)$ and $f(0) = 0$. Then the function $t \mapsto u(t)$, $t \geqslant 0$ given by $u(t) = \int_0^t S(t-s)(C^{-1}f)'(s)\, ds$ is a strong solution of (4.18). Let $\omega_0 \geqslant 0$, $\omega > 0$, $1 > \omega\omega_0$, let $\|S(t)\| \leqslant Me^{\omega_0 t}$, $t \geqslant 0$, and let the mapping $(C^{-1}f)'(\cdot)$ be asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic). Then we have

$$e^{-t/\omega}u(t) = e^{-t/\omega} \int_0^t S(t-s)(C^{-1}f)'(s)\, ds$$
$$= \int_0^t [e^{-(t-s)/\omega} S(t-s)][e^{-s/\omega}(C^{-1}f)'(s)]\, ds, \quad t \geqslant 0.$$

Since the operator family $(e^{-t/\omega}S(t))_{t\geqslant 0}$ is exponentially decaying, it follows that the function $t \mapsto e^{-t/\omega}u(t)$, $t \geqslant 0$ is asymptotically almost periodic (asymptotically almost automorphic, asymptotically compactly almost automorphic). Hence, the mapping $t \mapsto u(t)$, $t \geqslant 0$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).

Concerning the abstract non-degenerate Volterra equations of non-scalar type, it is clear that the above results can be applied to numerous problems in linear (thermo-)viscoelasticity and electrodynamics with memory (cf. [857, Chapter 9, Chapter 13] for more details); for example, in the analysis of viscoelastic Timoshenko beam in the case of non-synchronous materials. In both cases, degenerate and non-degenerate, we can make many applications of our results with the regularizing operator $C \neq I$; see, e. g., [634, Corollary 1, Example 1, Example 2] and the paragraph following [635, Theorem 2].

Finally, we would like to say a few words about the following special class of the abstract non-degenerate Volterra equations of non-scalar type (cf. also [278]):

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s) + f(t), \quad t \geqslant 0; \quad x(0) = x_0, \tag{4.24}$$

where A generates a strongly continuous semigroup on X and $(B(t))_{t\geqslant 0}$ is a family of linear operators on X such that, for almost every $t \geqslant 0$, the operator $B(t)$ maps continuously the space $Y = [D(A)]$ into X and there exists a locally integrable function $b : [0, \infty) \to [0, \infty)$ such that $\|B(t)y\|_{L(Y,X)} \leqslant b(t)\|y\|_Y$ for all $y \in Y$ and $t \geqslant 0$; see, e. g., [277, 278, 484, 485, 769] for more details about the subject. By a solution of (4.24), we mean any function $x \in C([0, \infty) : Y) \cap C^1([0, \infty) : X)$ satisfying the initial condition $x(0) = x_0$ and the first equality in (4.24) identically for $t \geqslant 0$. In the analysis of (4.24), the following notion of resolvent family (which is a very special case of the notion introduced in Definition 4.1.85) plays an important role.

Definition 4.1.96 (W. Desch, R. Grimmer, W. Schappacher [360, Definition, pp. 220–221]). A strongly continuous operator family $(R(t))_{t\geqslant 0} \subseteq L(X)$ is said to be a resolvent family for (4.24) if and only if $R(0) = I$, the mapping $y \mapsto R(t)y \in Y$, $t \geqslant 0$ belongs to the class $C([0,\infty) : Y) \cap C^1([0,\infty) : X)$ and the following resolvent equations hold:

$$R'(t)y = AR(t)y + \int_0^t B(t-s)R(s)y\,ds, \quad t \geqslant 0,$$

and

$$R'(t)y = R(t)Ay + \int_0^t R(t-s)B(s)y\,ds, \quad t \geqslant 0.$$

In [360, Proposition 2(c)], it has been proved that any solution of (4.24) has the form

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0.$$

The notion of a resolvent family for (4.24) has been extended by R. Grimmer in [483], where the author has analyzed the well-posedness of the following abstract differential first-order equation of non-convolution type:

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s) + f(t), \quad x(0) = x_0;$$

here, $A(t)$ and $B(t,s)$ are closed linear operators with fixed domain and the function $f : [0,\infty) \to X$ is continuous. In this paper, some particular results are given for the convolution case $B(t,s) \equiv B(t-s)$ and the usually considered autonomous case $A(t) \equiv A$, which turns the above equation in (4.24). We would like to especially emphasize that the author has shown, in [483, Theorem 4.1], that there exists an exponentially decaying resolvent family $(R(t))_{t\geqslant 0} \subseteq L(X)$ for (4.24) which decays exponentially in time. Hence, we can simply apply many structural results obtained so far in the analysis of the existence and uniqueness of asymptotically almost periodic type solutions of (4.24). As an application, we can consider the existence and uniqueness of asymptotically almost periodic type solutions of the equation

$$\begin{aligned} &c\Delta_t\theta(x,t) + \beta(0)\frac{\partial}{\partial t}\theta(x,t) \\ &\quad = \alpha_0\Delta_x\theta(x,t) - \int_{-\infty}^t \beta'(t-s)\frac{\partial}{\partial s}\theta(x,s)\,ds + \int_{-\infty}^t \alpha'(t-s)\Delta_x\theta(x,s)\,ds + \frac{\partial}{\partial t}r(x,t), \end{aligned}$$

which arises in the study of heat conduction in materials with memory; see [483] for further information.

We close this section by recalling that the following special class of second-order abstract Volterra equations of non-scalar type

$$u''(t) = Au(t) + \int_0^t B(t-s)u(s)\,ds + f(t), \quad t \geqslant 0,\ u(0) = x,\ u'(0) = y, \tag{4.25}$$

where A generates a strongly continuous cosine function and $B \in BV_{\mathrm{loc}}([0,\infty) : L([D(A)], X))$, has been systematically investigated starting from the 1970s; see, e. g., [241, 242, 322, 323, 359], and the references cited therein for more details on the subject. Almost periodic solutions of the abstract second-order differential equations of (4.25) and their generalizations with the added delay or nonlinear dissipative terms have been investigated in [41, 100, 445, 528, 809, 843, 1020]; see also [175–177, 615, 796, 943, 1049] and the reference lists of [631] and [859]. We want also to mention the article [87] by M. Arienmughare and T. Diagana, where the authors have employed the Drazin inverses to investigate the existence of almost periodic solutions to some singular systems of first-and second-order differential equations with complex coefficients (cf. also [89] and [437]).

4.2 *c*-Uniformly recurrent functions, *c*-almost periodic functions and semi-*c*-periodic functions

Besides the notion depending on two parameters ω and c, it is meaningful to consider the notion depending only on the parameter c. The main aim of this section is to introduce and analyze the classes of c-almost periodic functions, c-uniformly recurrent functions, semi-c-periodic functions and their Stepanov generalizations, where $c \in \mathbb{C} \setminus \{0\}$. We also introduce and investigate the corresponding classes of c-almost periodic type functions depending on two variables; several composition principles for c-almost periodic type functions are established in this direction. We provide some illustrative examples and applications to the abstract fractional semilinear integro-differential inclusions [before proceeding further, we would like to note that it is not clear how we can introduce and analyze the notion of (compact) c-almost automorphicity in a satisfactory way].

We will use the following auxiliary result, whose proof follows from the argumentation used in the proof that every orbit under an irrational rotation is dense in $S_1 \equiv \{z \in \mathbb{C} : |z| = 1\}$; see e. g. the solution given by C. Blatter in [179].

Lemma 4.2.1. *Suppose that $c = e^{i\pi\varphi}$, where $\varphi \in (-\pi, \pi] \setminus \{0\}$ is not rational. Then for each $c' \in S_1$ there exists a strictly increasing sequence (l_k) of positive integers such that $\sup_{k\in\mathbb{N}}(l_{k+1} - l_k) < \infty$ and $|c^{l_k} - c'| < \varepsilon$.*

Unless stated otherwise, we will always assume here that $c \in \mathbb{C}$ and $|c| = 1$. Let $f : I \to X$ be a continuous function and let a number $\varepsilon > 0$ be given. We call a number $\tau > 0$ an (ε, c)-period for $f(\cdot)$ if $\|f(t+\tau) - cf(t)\| \leqslant \varepsilon$ for all $t \in I$. By $\vartheta_c(f, \varepsilon)$ we denote the set consisting of all (ε, c)-periods for $f(\cdot)$.

We are concerned with the following notion.

Definition 4.2.2. It is said that $f(\cdot)$ is c-almost periodic if and only if for each $\varepsilon > 0$ the set $\vartheta_c(f, \varepsilon)$ is relatively dense in $[0, \infty)$. The space consisting of all c-almost periodic functions from the interval I into X will be denoted by $\mathrm{AP}_c(I : X)$.

If $c = -1$, then we recover the notion of almost anti-periodicity [666].

In general case, it is very simple to prove that the following holds (see, e. g., the proof of [166, Theorem 4°, p. 2]).

Proposition 4.2.3. *Suppose that $f : I \to X$ is c-almost periodic. Then $f(\cdot)$ is bounded.*

The following generalization of c-almost periodicity is also meaningful.

Definition 4.2.4. Let $c \in \mathbb{C} \setminus \{0\}$. Then a continuous function $f : I \to X$ is said to be c-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \|f(\cdot + \alpha_n) - cf(\cdot)\|_\infty = 0. \tag{4.26}$$

If $c = -1$, then we also say that the function $f(\cdot)$ is uniformly anti-recurrent. The space consisting of all c-uniformly recurrent functions from the interval I into X will be denoted by $\mathrm{UR}_c(I : X)$.

Define now $\mathbb{S} := \mathbb{N}$ if $I = [0, \infty)$, and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$. We will also consider the following notion.

Definition 4.2.5. Let $f \in C(I : X)$. It is said that $f(\cdot)$ is semi-c-periodic if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \|f(x + mp) - c^m f(x)\| \leqslant \varepsilon.$$

The space of all semi-c-periodic functions will be denoted by $\mathcal{SP}_c(I : X)$.

Suppose that $I = \mathbb{R}$, $f \in C(\mathbb{R} : X)$, $p > 0$ and $m \in \mathbb{N}$. Then we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \|f(x+mp) - c^m f(x)\| &= \sup_{x\in\mathbb{R}} \|f(x) - c^m f(x - mp)\| \\
&= \sup_{x\in\mathbb{R}} \|c^m [c^{-m} f(x) - f(x - mp)]\| \\
&= |c|^m \sup_{x\in\mathbb{R}} \|f(x - mp) - c^{-m} f(x)\| \\
&= \sup_{x\in\mathbb{R}} \|f(x - mp) - c^{-m} f(x)\| \in [0, \infty].
\end{aligned}$$

Therefore, we have the following.

Proposition 4.2.6. *Suppose that $f \in C(I : X)$. Then $f(\cdot)$ is semi-c-periodic if and only if*

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \|f(x + mp) - c^m f(x)\| \leqslant \varepsilon.$$

Furthermore, if $I = \mathbb{R}$, then the above is also equivalent with

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in -\mathbb{N} \quad \forall x \in I \quad \|f(x + mp) - c^m f(x)\| \leqslant \varepsilon.$$

It can be very simply shown that any semi-c-periodic function is bounded. Keeping in mind Proposition 4.2.6 and this observation, we may conclude that the notion introduced in Definition 4.2.5 is equivalent and extends the notion of semi-periodicity for case $c = 1$, introduced by J. Andres and D. Pennequin in [69], and the notion of semi-anti-periodicity for case $c = -1$, introduced by B. Chaouchi et al. in [262] (concerning the papers of J. Andres and his coauthors, mention should be made of [62–66], as well).

We continue by providing several illustrative examples.

Example 4.2.7. Let $f \equiv c \neq 0$. Due to (2.2), $f \notin \mathrm{ANP}(\mathbb{R} : X)$ and clearly $f(\cdot)$ is not semi-anti-periodic. On the other hand, $f(\cdot)$ is periodic and therefore semi-periodic.

Example 4.2.8. It can be simply verified that the function $f(x) := \sin x + \sin(\pi x\sqrt{2})$, $x \in \mathbb{R}$ is almost anti-periodic but not semi-periodic (see, e. g., [69, Remark 3] and [666, Example 2.1]).

Example 4.2.9 (a slight modification of [69, Example 1]). The function

$$f(x) := \sum_{n=1}^{\infty} \frac{e^{ix/(2n+1)}}{n^2}, \quad x \in \mathbb{R}$$

is semi-anti-periodic because it is a uniform limit of $[\pi \cdot (2n + 1)!!]$-anti-periodic functions

$$f_N(x) := \sum_{n=1}^{\infty} \frac{e^{ix/(2n+1)}}{n^2}, \quad x \in \mathbb{R} \quad (N \in \mathbb{N}).$$

On the other hand, the function $f(\cdot)$ cannot be periodic.

Example 4.2.10. Set $\mathbb{Q}_n := \{(2n + 1)/(2m + 1) : m, n \in \mathbb{Z}\}$. If $\theta > 0$ and $\sum_{\lambda\in\theta\cdot\mathbb{Q}_n} \|a_\lambda(f)\| < \infty$, then the function

$$f(t) := \sum_{\lambda\in\theta\cdot\mathbb{Q}_n} a_\lambda(f)e^{i\lambda t}, \quad t \in \mathbb{R},$$

is semi-anti-periodic. This can be inspected as in the proof of [69, Proposition 2] since the function $f_N(\cdot)$ used therein is anti-periodic with the anti-period $\pi q_1 \cdots q_N/\theta$.

The following important result holds true.

Proposition 4.2.11. *Suppose that* $f \in \mathrm{UR}_c(I : X)$ *and* $c \in \mathbb{C} \smallsetminus \{0\}$ *satisfies* $|c| \neq 1$. *Then* $f \equiv 0$.

Proof. Without loss of generality, we may assume that $I = [0, \infty)$. Suppose to the contrary that there exists $t_0 \geqslant 0$ such that $f(t_0) \neq 0$. Inductively, (4.26) implies

$$|c|^k m - \frac{|c|^k - 1}{n(|c| - 1)} \leqslant \|f(t)\| \leqslant |c|^k M - \frac{|c|^k - 1}{n(|c| - 1)}, \quad k \in \mathbb{N},\ t \in [k\alpha_n, (k+1)\alpha_n]. \tag{4.27}$$

Consider now case $|c| < 1$. Let $0 < \varepsilon < c\|f(t_0)\|$. Then (4.27) shows that there exist integers $k_0 \in \mathbb{N}$ and $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ with $k \geqslant k_0$ we have $\|f(t)\| \leqslant \varepsilon/2$, $t \in [k\alpha_n, (k+1)\alpha_n]$. Then the contradiction is obvious because for each $m \in \mathbb{N}$ with $m > n$ there exists $k \in \mathbb{N}$ such that $t_0 + \alpha_m \in [k\alpha_n, (k+1)\alpha_n]$ and therefore $\|f(t_0 + \alpha_m)\| \geqslant c\|f(t_0)\| - (1/m) \to c\|f(t_0)\| > \varepsilon$, $m \to +\infty$. Consider now case $|c| > 1$; let $n \in \mathbb{N}$ be such that $\|f(t_0)\| > 1/(n(|c| - 1))$ and $M := \max_{t \in [0, 2\alpha_n]} \|f(t)\| > 0$. Then for each $m \in \mathbb{N}$ with $m > n$ there exists $k \in \mathbb{N}$ such that $\alpha_m \in [(k-1)\alpha_n, k\alpha_n]$ and therefore $\|f(t + \alpha_m)\| \leqslant 1 + |c|M$, $t \in [0, 2\alpha_n]$. On the other hand, we obtain inductively from (4.26) that

$$\|f(t_0 + k\alpha_n)\| \geqslant |c|^k \left[\|f(t_0)\| - \frac{1}{n(|c| - 1)}\right] + \frac{1}{n(|c| - 1)} \to +\infty \quad \text{as } k \in \mathbb{N},$$

which immediately yields a contradiction. □

In accordance with the established result, it is reasonable to assume $|c| = 1$. This will be our standing assumption till the end of Subsection 4.2.2.

Proposition 4.2.12. *Suppose that* $I = \mathbb{R}$ *and* $f : \mathbb{R} \to X$. *Then the function* $f(\cdot)$ *is* c*-almost periodic (*c*-uniformly recurrent, semi-*c*-periodic) if and only if the function* $\check{f}(\cdot)$ *is* $1/c$*-almost periodic (*$1/c$*-uniformly recurrent, semi-*$1/c$*-periodic).*

Since for each number t, $\tau \in I$ and $m \in \mathbb{N}$ we have

$$\big|\|f(t+\tau)\| - \|f(t)\|\big| = \big|\|f(t+\tau)\| - \|c^m f(t)\|\big| \leqslant \|f(t+\tau) - c^m f(t)\|,$$

the subsequent result simply follows.

Proposition 4.2.13. *Suppose that* $f : I \to X$ *is* c*-almost periodic (*c*-uniformly recurrent, semi-*c*-periodic). Then* $\|f\| : I \to [0, \infty)$ *is almost periodic (uniformly recurrent, semi-periodic).*

Furthermore, we have ($x \in I$, $\tau > 0$, $l \in \mathbb{N}$)

$$f(x + l\tau) - c^l f(x) = \sum_{j=0}^{l-1} c^j [f(x + (l-j)\tau) - cf(x + (l-j-1)\tau)]. \tag{4.28}$$

Hence,

$$\|f(\cdot + l\tau) - c^l f(\cdot)\|_\infty \leqslant l\|f(\cdot + \tau) - cf(\cdot)\|_\infty.$$

The above estimate can be used to prove the following.

Proposition 4.2.14. *Let $f : I \to X$ be a c-almost periodic function (c-uniformly recurrent function, semi-c-periodic function), and let $l \in \mathbb{N}$. Then $f(\cdot)$ is c^l-almost periodic (c^l-uniformly recurrent, semi-c^l-periodic).*

Let us take into account the following condition:

$$p \in \mathbb{Z} \setminus \{0\}, \quad q \in \mathbb{N}, \ (p, q) = 1 \text{ and } \arg(c) = \pi p/q. \tag{4.29}$$

The next corollary of Proposition 4.2.14 follows immediately by plugging $l = q$.

Corollary 4.2.15. *Let $f : I \to X$ be a continuous function, and let* (4.29) *hold.*
(i) *If p is even and $f(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic), then $f(\cdot)$ is almost periodic (uniformly recurrent, semi-periodic).*
(ii) *If p is odd and $f(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic), then $f(\cdot)$ is almost anti-periodic (uniformly anti-recurrent, semi-anti-periodic).*

Therefore, if $\arg(c)/\pi \in \mathbb{Q}$, then the class of c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) is always contained in the class of almost periodic functions (uniformly recurrent functions, semi-periodic functions); in particular, we see that any almost anti-periodic function (uniformly anti-recurrent function, semi-anti-periodic function) is almost periodic (uniformly recurrent, semi-periodic).

Now we will prove the following.

Proposition 4.2.16. *Let $f : I \to X$ be a continuous function, and let* $\arg(c)/\pi \notin \mathbb{Q}$.
(i) *If $f(\cdot)$ is c-almost periodic, then $f(\cdot)$ is c'-almost periodic for all $c' \in S_1$.*
(ii) *If $f(\cdot)$ is bounded and c-uniformly recurrent, then $f(\cdot)$ is c'-uniformly recurrent for all $c' \in S_1$.*

Proof. We will prove only (i). Clearly, it suffices to consider the case in which the function $f(\cdot)$ is not identical to zero. Let $c' \in S_1$ and $\varepsilon > 0$ be fixed; then the prescribed assumption implies that the set $\{c^l : l \in \mathbb{N}\}$ is dense in S_1 and therefore there exists an increasing sequence (l_k) of positive integers such that $\lim_{k\to+\infty} c^{l_k} = c'$. By Proposition 4.2.3, the function $f(\cdot)$ is bounded; let $k \in \mathbb{N}$ be such that $|c^{l_k} - c'| < \varepsilon/(2\|f\|_\infty)$, and let $\tau > 0$ be any $(\varepsilon/2, c^{l_k})$-period for $f(\cdot)$. Then we have

$$\|f(x+\tau) - c' f(x)\| \leqslant \|f(x+\tau) - c^{l_k} f(x)\| + |c^{l_k} - c'| \cdot \|f\|_\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for any $x \in I$. This simply completes the proof. □

Proposition 4.2.17. *Let $f : I \to X$ be a continuous function. Then we have the following:*
(i) *If $f(\cdot)$ is semi-c-periodic and* $\arg(c)/\pi \in \mathbb{Q}$, *then $f(\cdot)$ is c'-almost periodic for all $c' \in \{c^l : l \in \mathbb{N}\}$.*
(ii) *If $f(\cdot)$ is semi-c-periodic and* $\arg(c)/\pi \notin \mathbb{Q}$, *then $f(\cdot)$ is c'-almost periodic for all $c' \in S_1$.*

Proof. Let $\varepsilon > 0$ be fixed. To prove (i), it suffices to show that $f(\cdot)$ is c-almost periodic (see Proposition 4.2.14). Since $\arg(c)/\pi \in \mathbb{Q}$ and (4.29) holds, then we have $c^{1+2lq} = c$ for all $l \in \mathbb{N}$. Then there exists $p > 0$ such that, for every $m \in \mathbb{N}$ and $x \in I$, we have $\|f(x+mp) - c^m f(x)\| \leqslant \varepsilon$. With $m = 1+2lq$, we have $\|f(x+(1+2lq)p) - c^{1+2lq}f(x)\| = \|f(x+(1+2lq)p) - cf(x)\| \leqslant \varepsilon$ so that the conclusion follows from the fact that the set $\{(1+2lq)p : l \in \mathbb{N}\}$ is relatively dense in $[0,\infty)$. Assume now that $\arg(c)/\pi \notin \mathbb{Q}$. To prove (ii), it suffices to consider case $f \neq 0$. Observe first that Lemma 4.2.1 shows that there exists a strictly increasing sequence (l_k) of positive integers such that $\sup_{k\in\mathbb{N}}(l_{k+1}-l_k) < \infty$ and $|c^{l_k} - c'| < \varepsilon/\|f\|_\infty$ for all $k \in \mathbb{N}$. With this sequence and the number $p > 0$ chosen as above, we have

$$\begin{aligned}\|f(x+pl_k) - c'f(x)\| &\leqslant \|f(x+pl_k) - c^{l_k}f(x)\| + |c^{l_k} - c'|\|f\|_\infty \\ &\leqslant \varepsilon + \varepsilon\|f\|_\infty/\|f\|_\infty = 2\varepsilon, \quad x \in I,\ k \in \mathbb{N}.\end{aligned}$$

Since the set $\{pl_k : k \in \mathbb{N}\}$ is relatively dense in $[0,\infty)$, the proof is completed. □

In connection with Proposition 4.2.17(ii), it is natural to ask whether the assumptions that the function $f(\cdot)$ is semi-c-periodic and $\arg(c)/\pi \notin \mathbb{Q}$ imply that $f(\cdot)$ is semi-c'-almost periodic for all $c' \in S_1$?

We continue by providing the following extension of [666, Theorem 2.2] (see also [166, pp. 3–4]).

Theorem 4.2.18. *Let $f : I \to X$ be c-almost periodic (c-uniformly recurrent, semi-c-periodic), and let $\alpha \in \mathbb{C}$. Then we have:*
(i) *$\alpha f(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic).*
(ii) *If $X = \mathbb{C}$ and $\inf_{x\in\mathbb{R}} |f(x)| = m > 0$, then $1/f(\cdot)$ is $1/c$-almost periodic ($1/c$-uniformly recurrent, semi-$1/c$-periodic).*
(iii) *If $(g_n : I \to X)_{n\in\mathbb{N}}$ is a sequence of c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g : I \to X$, then $g(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic).*
(iv) *If $a \in I$ and $b \in I \setminus \{0\}$, then the functions $f(\cdot + a)$ and $f(b\,\cdot)$ are likewise c-almost periodic (c-uniformly recurrent, semi-c-periodic).*

Let us recall that a continuous function $f : I \to X$ is called (p,c)-periodic if and only if $f(x+p) = cf(x)$, $x \in I$ ($p > 0$, $c \in \mathbb{C} \setminus \{0\}$). We say that a function $f : I \to X$ is c-periodic if and only if there exists $p > 0$ such that the function $f(\cdot)$ is (p,c)-periodic.

Keeping in mind Theorem 4.2.18(iii) and the proofs of [69, Lemma 1, Theorem 1], we can clarify the following extension of [262, Proposition 3].

Theorem 4.2.19. *Let $f \in C_b(I : X)$. Then $f(\cdot)$ is semi-c-periodic if and only if there exists a sequence (f_n) of c-periodic functions in $C_b(I : X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in I.*

We continue by providing two examples.

Example 4.2.20 (see also [666, Example 2.2]). The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) := \cos t$, $t \in \mathbb{R}$ is c-almost periodic (c-uniformly recurrent) if and only if $c = \pm 1$, while $f(\cdot)$ is semi-c-periodic if and only if $c = 1$; the function $f_\varphi : \mathbb{R} \to \mathbb{R}$ given by $f_\varphi(t) := e^{it\varphi}$, $t \in \mathbb{R}$ ($\varphi \in (-\pi, \pi] \smallsetminus \{0\}$) is c-almost periodic (semi-c-periodic) for any $c \in S_1$, while the function $f_0(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic) if and only if $c = 1$. Consider now the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) := 2^{-1} \cos 4t + 2\cos 2t$, $t \in \mathbb{R}$. Then we know that the function $g(\cdot)$ is (almost) periodic and not almost anti-periodic. Now we will prove that $g(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic) if and only if $c = 1$. Suppose that (α_n) is a strictly increasing sequence tending to plus infinity such that ($c = e^{i\alpha}$, $\alpha \in (-\pi, \pi]$):

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} \left|2^{-1}\cos(4t+\alpha_n)2\cos(2t+\alpha_n) - e^{i\alpha}\left[2^{-1}\cos 4t + 2\cos 2t\right]\right| = 0.$$

With $t = \pi$, the above implies

$$\lim_{n\to+\infty} [\cos 4\alpha_n + 4\cos 2\alpha_n - 5\cos\alpha] = 0 \quad \text{and} \quad \lim_{n\to+\infty} 5\sin\alpha = 0, \tag{4.30}$$

which immediately yields $\alpha = 0$ or $\alpha = \pi$. In the second case, the contradiction is obvious since the first limit equation in (4.30) cannot be fulfilled, while the case $\alpha = 0$ is possible and equivalent with the usual almost periodicity of $g(\cdot)$.

Example 4.2.21 (see also [69, Example 1] and [262, Example 4, Example 5]). Let p and q be odd natural numbers such that $p - 1 \equiv 0 \pmod q$, and let $c = e^{i\pi p/q}$. The function

$$f(x) := \sum_{n=1}^{\infty} \frac{e^{ix/(2nq+1)}}{n^2}, \quad x \in \mathbb{R},$$

is semi-c-periodic because it is a uniform limit of $[\pi \cdot (1+2q)\cdots(1+2Nq)]$-periodic functions

$$f_N(x) := \sum_{n=1}^{N} \frac{e^{ix/(2nq+1)}}{n^2}, \quad x \in \mathbb{R} \quad (N \in \mathbb{N}).$$

Now we will state and prove the following.

Proposition 4.2.22. *Suppose that $f : I \to \mathbb{R}$ is c-uniformly recurrent (semi-c-periodic) and $f \neq 0$. Then $c = \pm 1$ and moreover, if $f(t) \geqslant 0$ for all $t \in I$, then $c = 1$.*

Proof. We will consider the class of c-uniformly recurrent functions, only, when we may assume without loss of generality that $I = [0, \infty)$. Then $f \notin C_0([0,\infty) : \mathbb{R})$; namely, if we suppose the contrary, then there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and (4.26) holds. In particular, for every fixed number $t_0 \geqslant 0$ we have $\lim_{n\to+\infty} |f(t_0 + \alpha_n) - cf(t_0)| = 0$. This automatically yields $f(t_0) = 0$ and, since $t_0 \geqslant 0$ was arbitrary, we get $f = 0$ identically, which is a contradiction. Therefore, there exist a strictly increasing sequence $(t_l)_{l\in\mathbb{N}}$ of positive real numbers tending to plus infinity and a positive real number $a \geqslant \limsup_{t\to+\infty} |f(t)| > 0$ such that $|f(t_l)| \geqslant a/2$ for all $l \in \mathbb{N}$. Let $\varepsilon > 0$ be fixed. Then there exist two real numbers $t_0 > 0$ and $n_0 \in \mathbb{N}$ such that $|f(t+\alpha_n) - f(t)| \leqslant \varepsilon$ for all $t \geqslant t_0$ and $n \geqslant n_0$. If $\arg(c) = \varphi \in (-\pi, \pi]$, then we particularly get for each $t \geqslant t_0$ and $n \geqslant n_0$

$$\left|f(t+\alpha_n) - \cos\varphi \cdot f(t)\right| \leqslant \varepsilon \quad \text{and} \quad \left|\sin\varphi \cdot f(t)\right| \leqslant \varepsilon.$$

Plugging this in the second estimate $t = t_l$ for a sufficiently large $l \in \mathbb{N}$ we get $|\sin\varphi| \leqslant 2\varepsilon/a$. Since $\varepsilon > 0$ was arbitrary, we get $\sin\varphi = 0$ and $c = \pm 1$. Suppose, finally, that $f(t) \geqslant 0$ for all $t \geqslant 0$ and $c = -1$. Then we have $f(t+\alpha_n) + f(t) \leqslant 2\varepsilon$ for all $t \geqslant t_0$ and $n \geqslant n_0$. Plugging in again $t = t_l$ for a sufficiently large $l \in \mathbb{N}$ we get $a \leqslant \varepsilon$ for all $\varepsilon > 0$ and therefore $a = 0$, which is a contradiction. □

By the proof of Proposition 4.2.22, we have the following.

Proposition 4.2.23. *Suppose that $f : I \to X$ is c-uniformly recurrent (semi-c-periodic) and $f \neq 0$. Then $f \notin C_0(I : X)$.*

We continue by providing some illustrative applications of Proposition 4.2.22.

Example 4.2.24.
(i) The function $f : \mathbb{R} \to \mathbb{R}$, given by (2.28), is unbounded, uniformly continuous and uniformly recurrent. By the foregoing, $f(\cdot)$ is c-uniformly recurrent if and only if $c = 1$.
(ii) The function $g : \mathbb{R} \to \mathbb{R}$, given by

$$g(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\left(\frac{t}{3^n}\right) dt, \quad t \in \mathbb{R},$$

is unbounded, Lipschitz continuous and uniformly recurrent; furthermore, we have the existence of a positive integer $k_0 \in \mathbb{N}$ such that

$$\frac{1}{3^k\pi} \int_0^{3^k\pi} g(s)\, ds \geqslant \frac{1}{2}(\ln k - 1), \quad k \geqslant k_0, \tag{4.31}$$

and

$$\sup_{t\in\mathbb{R}} \left|g(t+3^n\pi) - g(t)\right| \leqslant \frac{\pi}{n+1} \sum_{j=1}^{\infty} 3^{-j}, \quad n \in \mathbb{N}. \tag{4.32}$$

This can be proved in exactly the same way as in the proof of [511, Theorem 1.1]. Define now $f(t) := \sin t \cdot g(t)$, $t \in \mathbb{R}$. Then (4.32) easily implies

$$\sup_{t\in\mathbb{R}} \left|f(t+3^n\pi) + f(t)\right| \leqslant \frac{\pi}{n+1} \sum_{j=1}^{\infty} 3^{-j}, \quad n \in \mathbb{N}.$$

Therefore, $f(\cdot)$ is uniformly anti-recurrent and Proposition 4.2.22 shows that the function $f(\cdot)$ is c-uniformly recurrent if and only if $c = \pm 1$. To prove that $f(\cdot)$ is Stepanov unbounded, observe that (4.31) implies the existence of a sequence $(t_k)_{k\in\mathbb{N}}$ of positive real numbers such that $g(t_k) \geqslant (1/2)(\ln k - 1)$ for all $k \geqslant k_0$. If we denote by $L \geqslant 1$ the Lipschitzian constant of the mapping $g(\cdot)$, then the above implies

$$g(x) \geqslant (1/2)(\ln k - 1) - 8L\pi, \quad x \in [t_k, t_k + 8\pi],\ k \geqslant k_0. \tag{4.33}$$

The existence of a constant $M > 0$ such that $\int_t^{t+1} |\sin s| \cdot g(s)\, ds < M$ for all $t \in \mathbb{R}$ would imply by (4.33) the existence of a sequence (a_k) of positive integers such that $[2a_k\pi + (\pi/2), 2a_k\pi + (\pi/2) + 1] \subseteq [t_k, t_k + 8\pi]$ and therefore (take $t = 2a_k\pi + (\pi/2)$)

$$\sin\bigl((\pi/2) + 1\bigr) \cdot \bigl[(1/2)(\ln k - 1) - 8L\pi\bigr] \leqslant M, \quad k \geqslant k_0,$$

which is a contradiction.

In connection with Proposition 4.2.22 and Proposition 4.2.23, we would like to present an illustrative example with the complex-valued functions.

Example 4.2.25. Let $h : I \to \mathbb{R}$, $q : I \to \mathbb{R}$ and $f(t) := h(t) + iq(t)$, $t \in I$. Suppose that $f : I \to \mathbb{C}$ is c-uniformly recurrent, where $c = e^{i\varphi}$ and $\sin\varphi \neq 0$. Then $h \in C_0(I : \mathbb{R})$ or $q \in C_0(I : \mathbb{R})$ implies $f \equiv 0$. To show this, observe that the c-uniform recurrence of $f(\cdot)$ implies the existence of a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\in I} \bigl|h(t + \alpha_n) - \cos\varphi \cdot h(t) + \sin\varphi \cdot q(t)\bigr| = 0, \quad \text{and}$$
$$\lim_{n\to+\infty} \sup_{t\in I} \bigl|q(t + \alpha_n) - \cos\varphi \cdot q(t) - \sin\varphi \cdot h(t)\bigr| = 0.$$

Since we have assumed that $\sin\varphi \neq 0$, the assumption $h \in C_0(I : \mathbb{R})$ ($q \in C_0(I : \mathbb{R})$) implies by the first (second) of the above equalities that $q \in C_0(I : \mathbb{R})$ ($h \in C_0(I : \mathbb{R})$). Hence, $f \in C_0(I : \mathbb{C})$ and the claimed statement follows by Proposition 4.2.23.

The space consisting of all almost periodic functions ($c = 1$) is the only function space from those introduced in Definition 4.2.2, Definition 4.2.4 and Definition 4.2.5 which has a linear vector structure.

Example 4.2.26.
(i) Suppose that $c = 1$. Then the set of all c-almost periodic functions is a vector space together with the usual operations, while the set of c-uniformly recurrent functions and the set of semi-c-periodic functions are not vector spaces together with the usual operations.

(ii) Suppose that $c = -1$. Then the set of all c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) is not a vector space together with the usual operations [666].

(iii) Suppose that $c \neq \pm 1$. Then the set of all c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) is not a vector space together with the usual operations. In actual fact, the functions $f_{\varphi,\pm} : \mathbb{R} \to \mathbb{R}$ given by $f_{\varphi,\pm}(t) := e^{\pm it\varphi}$, $t \in \mathbb{R}$ ($\varphi \in (-\pi, \pi] \setminus \{0\}$) are c-almost periodic (semi-c-periodic); see Example 4.2.20. The sum $f_{\varphi,+}(\cdot) + f_{\varphi,-}(\cdot) = 2\cos\varphi\cdot$ is not c-uniformly recurrent due to Proposition 4.2.22.

Similarly, we have the following.

Example 4.2.27. Let $f : I \to \mathbb{C}$ and $g : I \to X$.

(i) Suppose that $c = 1$. If $f \in AP(I : \mathbb{C})$ and $g \in AP(I : X)$, then $f \cdot g \in AP(I : X)$; furthermore, there exist $f \in UR(I : \mathbb{C})$ and $g \in UR(I : X)$ such that $f \cdot g \notin UR(I : X)$ [648]. It can be simply proved that the pointwise product of anti-periodic functions $f(t) := \cos t$, $t \in \mathbb{R}$ and $g(t) := \cos \sqrt{2}t$, $t \in \mathbb{R}$ is not a semi-periodic function (see, e. g., [69, Lemma 2]).

(ii) Suppose that $c = -1$. Then there exist an anti-periodic function $f(\cdot)$ and an anti-periodic function $g(\cdot)$ such that $f \cdot g(\cdot)$ is not anti-uniformly recurrent. We can simply take $X = \mathbb{C}$ and $f(t) := g(t) := \cos t$, $t \in I$.

(iii) Suppose that $c \neq \pm 1$. Then there exist a semi-c-periodic function $f(\cdot)$ and a semi-c-periodic function $g(\cdot)$ such that $f \cdot g(\cdot)$ is not c-uniformly recurrent. Consider again the functions $f_{\varphi,\pm} : \mathbb{R} \to \mathbb{R}$ given by $f_{\varphi,\pm}(t) := e^{\pm it\varphi}$, $t \in \mathbb{R}$ ($\varphi \in (-\pi, \pi] \setminus \{0\}$). They are semi-$c$-periodic but their pointwise product $f_{\varphi,+}(\cdot) \cdot f_{\varphi,-}(\cdot) = 1$ is not c-uniformly recurrent due to Proposition 4.2.22.

Let us recall that $ANP_0(I : X)$ and $ANP(I : X)$ stand for the linear span of almost anti-periodic functions $f : I \to E$ and its closure in $AP(I : X)$, respectively; by (2.2), we have $ANP(I : X) = AP_{\mathbb{R}\setminus\{0\}}(I : X)$. Now we will prove the following extension of this equality.

Theorem 4.2.28. *Denote by* $AP_{c,0}(I : X)$ *and* $AP_{c,0}(I : X)$ *the linear span of c-almost periodic functions* $f : I \to X$ *and its closure in* $AP(I : X)$*, respectively. Then the following hold:*

(i) *Let* $\arg(c) \in \pi \cdot \mathbb{Q}$. *Then we have* $AP_{c,0}(I : X) = AP_{\mathbb{R}\setminus\{0\}}(I : X)$.

(ii) *Let* $\arg(c) \notin \pi \cdot \mathbb{Q}$. *Then we have* $AP_{c,0}(I : X) \supseteq AP_{\mathbb{R}\setminus\{0\}}(I : X)$.

Proof. Assume first that $f \in AP_{\mathbb{R}\setminus\{0\}}(I : X)$. By spectral synthesis, we have

$$f \in \overline{\operatorname{span}\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\}},$$

where the closure is taken in the space $C_b(I : X)$. Since $\sigma(f) \subseteq \mathbb{R} \smallsetminus \{0\}$ and the function $t \mapsto e^{i\mu t}$, $t \in I$ ($\mu \in \mathbb{R} \smallsetminus \{0\}$) is c-almost periodic for all $c \in S_1$, we see that $\operatorname{span}\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\} \subseteq \mathrm{AP}_{c,0}(I : X)$. Hence, $f \in \mathrm{AP}_{c,0}(I : X)$. To complete the proof, it remains to consider case $\arg(c) \in \pi \cdot \mathbb{Q}$ and show that any function $f \in \mathrm{AP}_{c,0}(I : X)$ belongs to the space $\mathrm{AP}_{\mathbb{R}\smallsetminus\{0\}}(I : X)$. Furthermore, it suffices to consider case in which (4.29) holds with the number p even because otherwise we can apply Corollary 4.2.15(ii) and Proposition 4.2.16(i) to see that $\mathrm{AP}_{c,0}(I : X) \subseteq \mathrm{ANP}_0(I : X)$ and therefore $\mathrm{AP}_{c,0}(I : X) \subseteq \mathrm{ANP}(I : X)$, so that the statement directly follows from [666, Theorem 2.3]. We will prove that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds = 0; \tag{4.34}$$

clearly, by almost periodicity of $f(\cdot)$, the limit in (4.34) exists. Let $\varepsilon > 0$ be fixed, and let $l > 0$ satisfy the requirement that every interval of $[0, \infty)$ of length l contains a point τ such that $\|f(t + \tau) - cf(t)\| \leqslant \varepsilon$, $t \geqslant 0$. We have $c^q = 1$ and therefore $1 + c + \cdots + c^{q-1} = 0$; using this equality and the decomposition ($s \geqslant 0,\ n \in \mathbb{N}$)

$$\begin{aligned}
&\big\|f(s + (n-1)\tau) + f(s + (n-2)\tau) + \cdots + f(s)\big\| \\
&\quad\leqslant \varepsilon + \big\|(1 + c)f(s + (n-2)\tau) + f(s + (n-3)\tau) + \cdots + f(s)\big\| \\
&\quad\leqslant \varepsilon + \big\|(1 + c)f(s + (n-2)\tau) - (1 + c)cf(s + (n-3)\tau)\big\| \\
&\qquad + \big\|[1 + (1 + c)c]f(s + (n-3)\tau) + f(s + (n-4)\tau) \cdots + f(s)\big\| \\
&\quad\leqslant \varepsilon + |1 + c|\varepsilon + \big\|[1 + c + c^2]f(s + (n-3)\tau) + f(s + (n-4)\tau) + \cdots + f(s)\big\| \\
&\quad\leqslant \varepsilon + |1 + c|\varepsilon + \big|1 + c + c^2\big|\varepsilon + \cdots \\
&\quad\leqslant \varepsilon + |1 + c|\varepsilon + \big|1 + c + c^2\big|\varepsilon + \cdots + \big|1 + c + c^2 + \cdots + c^{q-2}\big|\varepsilon \\
&\qquad + \big\|f(s) + f(s + \tau) + \cdots + f(s + (n-1-q)\tau)\big\|,
\end{aligned}$$

we immediately see that there exists a finite constant $A \geqslant 1$ such that, for every $s \geqslant 0$ and $n \in \mathbb{N}$,

$$\big\|f(s + (n-1)\tau) + f(s + (n-2)\tau) + \cdots + f(s)\big\| \leqslant A\varepsilon\lceil n/q\rceil + A\|f\|_\infty.$$

Integrating this estimate over the segment $[0, n\tau]$, we get, for every $s \geqslant 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\left\|\int_0^{n\tau} f(s)\, ds\right\| &= \left\|\int_0^{\tau} \big[f(s + (n-1)\tau) + f(s + (n-2)\tau) + \cdots + f(s)\big]\, ds\right\| \\
&\leqslant A\tau\varepsilon\lceil n/q\rceil + A\tau\|f\|_\infty.
\end{aligned}$$

Dividing both sides of the above inequality by $n\tau$, we get

$$\lim_{n\to+\infty}\left\|\frac{1}{n\tau}\int_0^{n\tau} f(s)\, ds\right\| \leqslant A\varepsilon/q.$$

Since $\varepsilon > 0$ was arbitrary, this immediately yields (4.34). □

Now we will clarify the following result.

Proposition 4.2.29. *Suppose that $f : [0,\infty) \to X$ is c-almost periodic (semi-c-periodic). Then $\mathbb{E}f : \mathbb{R} \to X$ is a unique c-almost periodic extension (semi-c-periodic extension) of $f(\cdot)$ to the whole real axis.*

Proof. The proof for the class of c-almost periodic functions is very similar to the proof of [666, Proposition 2.2] and therefore is omitted. For the class of semi-c-periodic functions, the proof can be deduced as follows. Due to Proposition 4.2.17, we see that the function $f : [0,\infty) \to X$ is almost periodic, so that the function $\mathbb{E}f : \mathbb{R} \to X$ is a unique almost periodic extension of $f(\cdot)$ to the whole real axis. Therefore, it remains to be proved that $\mathbb{E}f(\cdot)$ is semi-c-periodic. Let $\varepsilon > 0$ be fixed. Then there exists $p > 0$ such that for all $m \in \mathbb{N}$ and $x \geqslant 0$ we have $\|f(x+mp)-c^m f(x)\| \leqslant \varepsilon$. For every fixed number $m \in \mathbb{N}$, the function $\mathbb{E}f(\cdot + mp) - c^m \mathbb{E}f(\cdot)$ is almost periodic so that the supremum formula implies

$$\begin{aligned}\sup_{x\in\mathbb{R}}\|\mathbb{E}f(x+mp)-c^m\mathbb{E}f(x)\| &= \sup_{x\geqslant 0}\|\mathbb{E}f(x+mp)-c^m\mathbb{E}f(x)\| \\ &= \sup_{x\geqslant 0}\|f(x+mp)-c^m f(x)\| \leqslant \varepsilon.\end{aligned}$$

This completes the proof. □

We continue by introducing the following notion.

Definition 4.2.30. A continuous function $f : I \to X$ is called asymptotically c-uniformly recurrent (asymptotically c-almost periodic, asymptotically semi-c-periodic) if and only if there are a c-uniformly recurrent (c-almost periodic, semi-c-periodic) function $g : \mathbb{R} \to X$ and a function $h \in C_0(I : X)$ such that $f(x) = g(x) + h(x)$, $x \in I$.

Definition 4.2.31. Let $p \in \mathcal{P}([0,1])$, and let $f \in L^{p(x)}_{\mathrm{loc}}(I : X)$.

(i) It is said that $f(\cdot)$ is Stepanov $(p(x),c)$-uniformly recurrent (Stepanov $(p(x),c)$-almost periodic, Stepanov semi-$(p(x),c)$-periodic) if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (2.5), is c-uniformly recurrent (c-almost periodic, semi-c-periodic).

(ii) It is said that $f(\cdot)$ is asymptotically Stepanov $(p(x),c)$-uniformly recurrent (asymptotically Stepanov $(p(x),c)$-almost periodic, asymptotically Stepanov semi-$(p(x), c)$-periodic) if and only if there are a Stepanov $(p(x),c)$-uniformly recurrent (Stepanov $(p(x),c)$-almost periodic, Stepanov semi-$(p(x),c)$-periodic) function $h(\cdot)$ and $q \in C_0(I : L^{p(x)}([0,1] : X))$ such that $f(t) = h(t) + q(t)$ for a. e. $t \in I$.

If $p(x) \equiv p \in [1,\infty)$, then we also say that the function $f(\cdot)$ is Stepanov (p,c)-uniformly recurrent (Stepanov (p,c)-almost periodic, Stepanov semi-(p,c)-periodic) and so on.

In the case $c = 1$, resp. $c = -1$, we also say that an (asymptotically) Stepanov $(p(x),c)$-uniformly recurrent ((asymptotically) Stepanov $(p(x),c)$-almost periodic/(asymptotically) Stepanov semi-$(p(x),c)$-periodic) function is (asymptotically) Stepanov

$p(x)$-uniformly recurrent, resp. (asymptotically) Stepanov $p(x)$-uniformly anti-recurrent ((asymptotically) Stepanov $p(x)$-almost periodic, resp. (asymptotically) Stepanov $p(x)$-almost anti-periodic/(asymptotically) Stepanov semi-$p(x)$-periodic, resp. (asymptotically) Stepanov semi-$p(x)$-anti-periodic).

Question 4.2.32. Assume that α, $\beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well-defined irrational number. We would like to raise the question whether the functions $f_{\alpha,\beta}(\cdot)$ and $g_{\alpha,\beta}(\cdot)$, given by (2.6) and (2.7), respectively, are Stepanov q-semi-periodic for any $1 \leqslant q < \infty$.

Example 4.2.33. Let us consider the function $f(x) := \sin x + \sin(\pi x\sqrt{2})$, $x \in \mathbb{R}$. Then a simple analysis involving the identity

$$f(x) = 2\sin\left(x\frac{1+\pi\sqrt{2}}{2}\right)\cdot\cos\left(x\frac{\pi\sqrt{2}-1}{2}\right), \quad x \in \mathbb{R},$$

shows that the function $\operatorname{sign}(f(\cdot))$ is identically equal to a function $F(\cdot)$ of the following, much more general form: Let $(a_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers satisfying $\lim_{n\to+\infty}(a_{n+1}-a_n) = +\infty$, $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to-\infty} a_n = -\infty$. Suppose that $(b_n)_{n\in\mathbb{Z}}$ is a sequence of non-zero real numbers satisfying that the sets $\{n \in \mathbb{Z} : b_n < 0\}$ and $\{n \in \mathbb{Z} : b_n > 0\}$ are infinite, as well as that there exists a finite positive constant $c > 0$ such that $c \leqslant |b_n - b_l|$ for any n, $l \in \mathbb{Z}$ such that $b_n b_l < 0$. Define the function $F : \mathbb{R} \to \mathbb{R}$ by $F(x) := b_n$ if $x \in [a_n, a_{n+1})$, for any $n \in \mathbb{Z}$. Then $F(\cdot)$ cannot be Stepanov q-semi-periodic for any finite real number $q \geqslant 1$. Otherwise, for $\varepsilon \in (0, c^q)$ we would be able to find a number $p > 0$ such that for each $m \in \mathbb{Z}$ and $x \in \mathbb{R}$ one has:

$$\int_0^1 \left|F(x+mp+s) - F(x+s)\right|^q ds < (1/2)^q.$$

Let $n \in \mathbb{Z}$ be such that $[x, x+1] \subseteq [a_n, a_{n+1})$ and $b_n < 0$, say. Without loss of generality, we may assume that the set $\{n \in \mathbb{N} : b_n > 0\}$ is infinite. Then the contradiction is obvious because, for every sufficiently large number $l \in \mathbb{N}$ with $b_l > 0$, we can find $m \in \mathbb{N}$ such that $[x+mp, x+mp+1] \subseteq [a_l, a_{l+1})$ so that

$$\int_0^1 \left|F(x+mp+s) - F(x+s)\right|^q ds \geqslant |b_n - b_l|^q \geqslant c^q.$$

In the remainder of this subsection, we will present two statements concerning the invariance of c-almost periodicity, c-uniform recurrence and semi-c-periodicity under the actions of infinite convolution products. We first state the following slight generalization of [666, Proposition 3.1], which can be deduced by using almost the same arguments as in the proof of Proposition 2.4.39 (similarly we can generalize [666, Proposition 3.2] for asymptotical c-almost type periodicity).

Proposition 4.2.34. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is Stepanov $(p(x),c)$-almost periodic (Stepanov $p(x)$-bounded and Stepanov $(p(x),c)$-uniformly recurrent/Stepanov $p(x)$-bounded and Stepanov semi-$(p(x),c)$-periodic), then the function $F(\cdot)$, given by* (2.46)*, is well defined and c-almost periodic (bounded c-uniformly recurrent/bounded and semi-c-periodic).*

We can also consider the situation in which the forcing term $f(\cdot)$ is not Stepanov $p(x)$-bounded (see Proposition 2.4.41).

Proposition 4.2.35. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$, $\check{f} : \mathbb{R} \to X$ is Stepanov $(p(x),c)$-almost periodic (Stepanov $(p(x),c)$-uniformly recurrent/Stepanov semi-$(p(x),c)$-periodic), $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and there exists a continuous function $P : \mathbb{R} \to [1,\infty)$ such that* (2.47)–(2.48) *hold. If the function $\hat{f} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ is uniformly continuous, then the function $F : \mathbb{R} \to Y$, given by* (2.46)*, is well defined and c-almost periodic (c-uniformly recurrent/semi-c-periodic).*

4.2.1 Composition principles for c-almost periodic type functions

In this subsection, we will clarify and prove several composition principles for c-almost periodic functions and c-uniformly recurrent functions.

Suppose that $F : I \times Y \to X$ is a continuous function and there exists a finite constant $L > 0$ such that (2.51) holds. Define $\mathcal{F}(t) := F(t,f(t))$, $t \in I$. We need the following estimates ($\tau \geqslant 0$, $c \in \mathbb{C} \setminus \{0\}$, $t \in I$):

$$\begin{aligned}&\|F(t+\tau,f(t+\tau)) - cF(t,f(t))\|\\&\quad\leqslant \|F(t+\tau,f(t+\tau)) - F(t+\tau,cf(t))\| + \|F(t+\tau,cf(t)) - cF(t,f(t))\|\\&\quad\leqslant L\|f(t+\tau) - cf(t)\| + \|F(t+\tau,cf(t)) - cF(t,f(t))\|.\end{aligned} \tag{4.35}$$

Using (4.35), we can simply deduce the following result.

Theorem 4.2.36. *Suppose that $F : I \times Y \to X$ is a continuous function and there exists a finite constant $L > 0$ such that* (2.51) *holds.*

(i) *Suppose that $f : I \to Y$ is c-uniformly recurrent. If there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \|f(t+\alpha_n) - cf(t)\| = 0 \tag{4.36}$$

and

$$\lim_{n\to+\infty} \sup_{t\in I} \|F(t+\alpha_n, cf(t)) - cF(t,f(t))\| = 0, \tag{4.37}$$

then the mapping $\mathcal{F}(t) := F(t,f(t))$, $t \in I$ is c-uniformly recurrent.

(ii) *Suppose that $f : I \to Y$ is c-almost periodic. If for each $\varepsilon > 0$ the set of all positive real numbers $\tau > 0$ such that*

$$\sup_{t\in I}\|f(t+\tau) - cf(t)\| < \varepsilon \tag{4.38}$$

and

$$\sup_{t\in I}\|F(t+\tau, cf(t)) - cF(t, f(t))\| < \varepsilon \tag{4.39}$$

is relatively dense in $[0,\infty)$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is c-almost periodic.

For the class of asymptotically c-almost periodic functions, the subsequent result simply follows from the previous theorem and the argumentation used in the proof of [364, Theorem 3.49].

Theorem 4.2.37. *Suppose that $F : I\times Y \to X$ and $Q : I\times Y \to X$ are continuous functions and there exists a finite constant $L > 0$ such that* (2.51) *holds and* (2.51) *holds with the function $F(\cdot,\cdot)$ replaced therein with the function $Q(\cdot,\cdot)$.*

(i) *Suppose that $g : I \to E$ is a c-uniformly recurrent function, $h \in C_0(I : Y)$ and $f(x) = g(x) + h(x)$, $x \in I$. If there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that* (4.36) *and* (4.37) *hold with the function $f(\cdot)$ replaced therein with the function $g(\cdot)$, $\lim_{|t|\to+\infty} Q(t, y) = 0$ uniformly for $y \in R(f)$, then the mapping $\mathcal{H}(t) := (F + Q)(t, f(t))$, $t \in I$ is asymptotically c-uniformly recurrent.*
(ii) *Suppose that $g : I \to Y$ is a c-almost periodic function, $h \in C_0(I : Y)$ and $f(x) = g(x) + h(x)$, $x \in I$. If for each $\varepsilon > 0$ the set of all positive real numbers $\tau > 0$ such that* (4.38) *and* (4.39) *hold with the function $f(\cdot)$ replaced therein with the function $g(\cdot)$, $\lim_{|t|\to+\infty} Q(t, y) = 0$ uniformly for $y \in R(f)$, then the mapping $\mathcal{H}(t) := (F + Q)(t, f(t))$, $t \in I$ is asymptotically c-almost periodic.*

For the Stepanov classes, we can also clarify certain results.

Theorem 4.2.38. *Let $p, q, r \in \mathcal{P}([0,1])$, let $p(x), q(x) \in [1,\infty)$, $r(x) \in [1,\infty]$, $1/p(x) = 1/q(x) + 1/r(x)$ and the following conditions hold:*

(i) *Let $F : I \times Y \to X$ and let there exist a function $L_F \in L_S^{r(x)}(I)$ such that* (2.20) *holds.*
(ii) *There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty}\sup_{t\in I}\sup_{u\in R(f)}\|F(s+t+\alpha_n, cu) - cF(s+t, u)\|_{L^{p(s)}[0,1]} = 0 \quad \textit{and}$$

$$\lim_{n\to+\infty}\sup_{t\in I}\|f(s+t+\alpha_n) - cf(s+t)\|_{L^{q(s)}[0,1]} = 0. \tag{4.40}$$

Then the function $F(\cdot, f(\cdot))$ is Stepanov $(p(x), c)$-uniformly recurrent. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $p(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-bounded, as well.

Similarly, we can prove the following.

Theorem 4.2.39. *Suppose that $p \in \mathcal{P}([0,1])$ and the following conditions hold:*
(i) *Let $F : I \times Y \to X$ and there exist a function $r(x) \geqslant \max(p(x), p(x)/(p(x)-1))$ and a function $L_F \in L_S^{r(x)}(I)$ such that (2.20) holds.*
(ii) *There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (4.40) holds and (4.40) holds with the function $q(\cdot)$ replaced by the function $p(\cdot)$ therein.*

Then $q(x) := p(x)r(x)/(p(x)+r(x))$ for $x \in [0,1]$, $r(x) < +\infty$ and $q(x) := p(x)$ for $x \in [0,1]$, $r(x) = +\infty$. Then the function $F(\cdot, f(\cdot))$ is Stepanov $(q(x), c)$-uniformly recurrent. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $q(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-bounded, as well.

The above results can be simply reformulated for the class of Stepanov $(p(x), c)$-almost periodic functions. For the classes of asymptotically Stepanov $(p(x), c)$-uniformly recurrent (asymptotically Stepanov $(p(x), c)$-almost periodic) functions, we can simply extend the assertions of [631, Proposition 2.7.3, Proposition 2.7.4]. Details will be left to the interested reader.

4.2.2 Applications to the abstract Volterra integro-differential inclusions

In this subsection, we will present some illustrative applications of our abstract results in the analysis of the existence and uniqueness of c-almost periodic type solutions to the abstract (semilinear) Volterra integro-differential inclusions.

Regarding semilinear problems, we can apply our results in the study of the existence and uniqueness of c-almost periodic solutions and c-uniformly recurrent solutions of the fractional semilinear Cauchy inclusion (2.11), where $D_{t,+}^{\gamma}$ denotes the Riemann–Liouville fractional derivative of order $\gamma \in (0,1)$, $F : \mathbb{R} \times Y \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator satisfying condition (P). To explain this in more detail, fix a strictly increasing sequence (α_n) of positive reals tending to plus infinity and define

$$\mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X) := \Big\{ f \in \mathrm{UR}_c(\mathbb{R} : X);\ f(\cdot) \text{ is bounded and } \lim_{n \to +\infty} \sup_{t \in \mathbb{R}} \| f(t+\alpha_n) - cf(t) \|_{\infty} = 0 \Big\}.$$

Equipped with the metric$d(\cdot, \cdot) := \| \cdot - \cdot \|_{\infty}$, $\mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X)$ becomes a complete metric space. Let $(R_\gamma(t))_{t>0}$ be the operator family considered in [631]. It is said that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (2.11) if and only if

$$u(t) = \int_{-\infty}^{t} R_\gamma(t-s)F(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Now we are able to state the following result, which is very similar to [631, Theorem 3.1] (for simplicity, we will consider the constant coefficient $p(x) \equiv p > 1$ here).

Theorem 4.2.40. *Suppose that the function $F : \mathbb{R} \times X \to X$ satisfies the requirement that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\in\mathbb{R}} \sup_{y\in B} \|F(t,y)\| \leqslant M_B$. Suppose, further, that the function $F : \mathbb{R} \times X \to X$ is Stepanov (p,c)-uniformly recurrent with $p > 1$, and there exist a number $r \geqslant \max(p, p/(p-1))$ and a function $L_F \in L_S^r(I)$ such that $q := pr/(p+r) > 1$ and* (2.20) *holds with $I = \mathbb{R}$. If* (2.58) *holds and there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, where*

$$M_n := \sup_{t\geqslant 0} \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \|R_\gamma(t-x_n)\| \\ \times \prod_{i=2}^{n} \|R_\gamma(x_i - x_{i-1})\| \prod_{i=1}^{n} L_F(x_i)\, dx_1\, dx_2 \cdots dx_n,$$

and (4.40) *holds with the set $R(f)$ replaced therein with an arbitrary bounded set $B \subseteq X$, then the abstract semilinear fractional Cauchy inclusion* (2.11) *has a unique bounded uniformly recurrent solution which belongs to the space* $\mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X)$.

Proof. Define $\Upsilon : \mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X) \to \mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X)$ by

$$(\Upsilon u)(t) := \int_{-\infty}^{t} R_\gamma(t-s)F(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Suppose that $u \in \mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X)$. Then $R(u) = B$ is a bounded set and the mapping $t \mapsto F(t, u(t))$, $t \in \mathbb{R}$ is bounded due to the prescribed assumption. Applying Theorem 4.2.39, we see that the function $F(\cdot, u(\cdot))$ is Stepanov (q,c)-uniformly recurrent. Define $q' := q/(q-1)$. By (2.56) and (2.58), we have $\|R_\gamma(\cdot)\| \in L^{q'}[0,1]$ and $\sum_{k=0}^{\infty} \|R_\gamma(\cdot)\|_{L^{q'}[k,k+1]} < \infty$. Applying Proposition 4.2.34, we see that the function

$$t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)F(s, u(s))\, ds, \quad t \in \mathbb{R}$$

is bounded and c-uniformly recurrent, implying that $\Upsilon u \in \mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X)$, as claimed. Furthermore, a simple calculation shows that

$$\|(\Upsilon^n u_1) - (\Upsilon^n u_2)\|_\infty \leqslant M_n \|u_1 - u_2\|_\infty, \quad u_1,\, u_2 \in \mathrm{BUR}_{(\alpha_n);c}(\mathbb{R} : X),\ n \in \mathbb{N}.$$

Since there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, the Bryant fixed point theorem shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point, finishing the proof of the theorem. □

Similarly we can analyze the existence and uniqueness of asymptotically Stepanov (p, c)-almost periodic solutions and Stepanov (p, c)-uniformly recurrent solutions of the fractional semilinear Cauchy inclusion $(\mathrm{DFP})_{f,\gamma,s}$.

As mentioned earlier, the unique regular solution of the heat equation $u_t(x, t) = u_{xx}(x, t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial condition $u(x, 0) = f(x)$, is given by (4.9). Let the number $t_0 > 0$ be fixed, and let the function $f(\cdot)$ be bounded c-uniformly recurrent (c-almost periodic, semi-c-periodic). Since $e^{-\cdot^2/4t_0} \in L^1(\mathbb{R})$, we can use the fact that the space of bounded c-uniformly recurrent functions (c-almost periodic functions, semi-c-periodic functions) is convolution invariant in order to see that the solution $x \mapsto u(x, t_0)$, $x \in \mathbb{R}$ is bounded and c-uniformly recurrent (c-almost periodic, semi-c-periodic).

4.2.3 Semi-c-periodic functions

Let us recall that $\mathbb{S} := \mathbb{N}$ if $I = [0, \infty)$, and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$. In this subsection, we will first extend the notion introduced in Definition 4.2.5 with general parameter $c \in \mathbb{C} \setminus \{0\}$.

Definition 4.2.41. Let $f \in C(I : X)$.
(i) It is said that $f(\cdot)$ is semi-c-periodic of type 1 if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \|f(x + mp) - c^m f(x)\| \leqslant \varepsilon. \tag{4.41}$$

(ii) It is said that $f(\cdot)$ is semi-c-periodic of type 2 if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \|c^{-m} f(x + mp) - f(x)\| \leqslant \varepsilon.$$

The space of all semi-c-periodic functions of type i will be denoted by $\mathcal{SP}_{c,i}(I : X)$, $i = 1, 2$.

Definition 4.2.42. Let $f \in C(I : X)$.
(i) It is said that $f(\cdot)$ is semi-c-periodic of type 1_+ if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \|f(x + mp) - c^m f(x)\| \leqslant \varepsilon.$$

(ii) It is said that $f(\cdot)$ is semi-c-periodic of type 2_+ if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \|c^{-m} f(x + mp) - f(x)\| \leqslant \varepsilon.$$

The space of all semi-c-periodic functions of type i_+ will be denoted by $\mathcal{SP}_{c,i,+}(I : X)$, $i = 1, 2$.

We have already seen that the notion of a semi-c-periodicity of type i (i_+), where $i = 1, 2$, is equivalent with the notion of semi-c-periodicity introduced in Definition 4.2.5, provided that $|c| = 1$.

Now we will focus our attention to the general case $c \in \mathbb{C} \setminus \{0\}$. We will first state the following.

Lemma 4.2.43.
(i) *If $|c| \geqslant 1$ and $f : I \to X$ is semi-c-periodic of type 1_+, then $f(\cdot)$ is semi-c-periodic of type 2_+.*
(ii) *If $|c| \leqslant 1$ and $f : I \to X$ is semi-c-periodic of type 2_+, then $f(\cdot)$ is semi-c-periodic of type 1_+.*

Proof. If $x \in I$, $p > 0$, $m \in \mathbb{N}$ and $|c| \geqslant 1$, then we have

$$\|f(x+mp) - c^m f(x)\| \leqslant \varepsilon \Rightarrow \|c^{-m} f(x+mp) - f(x)\| \leqslant \varepsilon,$$

which implies (i); the proof of (ii) is similar. □

Using the proofs of [69, Lemma 1, Theorem 1], we can clarify the following important lemma.

Lemma 4.2.44. *Suppose that $|c| \leqslant 1$, resp. $|c| \geqslant 1$, and $f : [0, \infty) \to X$ is semi-c-periodic of type 1_+, resp. 2_+. Then there exists a sequence $(f_n : [0, \infty) \to X)_{n \in \mathbb{N}}$ of c-periodic functions which converges uniformly to $f(\cdot)$.*

Now we are able to prove the following result.

Theorem 4.2.45. *Let $|c| \neq 1$, $i = 1, 2$ and $f : I \to X$. Then $f(\cdot)$ is c-periodic if and only if $f(\cdot)$ is semi-c-periodic of type i (i_+).*

Proof. Suppose that the function $f(\cdot)$ is (p, c)-periodic. Then we have $f(x + mp) = c^m f(x)$, $x \in I$, $m \in \mathbb{S}$, so that $f(\cdot)$ is automatically semi-c-periodic of type i (i_+). To prove the converse statement, let us observe that any semi-c-periodic function of type i is clearly semi-c-periodic of type i_+. Suppose first that $|c| > 1$. Due to Lemma 4.2.43(i), it suffices to show that, if $f(\cdot)$ is semi-c-periodic of type 2_+, then $f(\cdot)$ is c-periodic. Assume first $I = [0, \infty)$. Using Lemma 4.2.44, we get the existence of a sequence $(f_n : [0, \infty) \to X)_{n \in \mathbb{N}}$ of c-periodic functions which converges uniformly to $f(\cdot)$. Let $f_n(t + p_n) = c f_n(t)$, $t \geqslant 0$ for some sequence (p_n) of positive real numbers. Suppose first that (p_n) is bounded. Then there exist a strictly increasing sequence (n_k) of positive integers and a number $p \geqslant 0$ such that $\lim_{k \to +\infty} p_{n_k} = p$. Let $\varepsilon > 0$ be given. Then there exists an integer $k_0 \in \mathbb{N}$ such that $\|f(t) - f_{n_k}(t)\| \leqslant \varepsilon/(2 + 2|c|^{-1})$ for all real numbers $t \geqslant 0$ and all integers $k \geqslant k_0$. Furthermore, we have

$$\begin{aligned}&\|c^{-1} f(t + p_{n_k}) - f(t)\|\\ &\quad \leqslant \|c^{-1} f(t + p_{n_k}) - c^{-1} f_{n_k}(t + p_{n_k})\|\end{aligned}$$

$$
\begin{aligned}
&+ \|c^{-1}f_{n_k}(t+p_{n_k}) - f_{n_k}(t)\| + \|f_{n_k}(t)-f(t)\| \\
&= \|c^{-1}f(t+p_{n_k}) - c^{-1}f_{n_k}(t+p_{nk})\| + \|f_{n_k}(t)-f(t)\| \\
&\leqslant 2(1+|c|^{-1})\varepsilon/(2+2|c|^{-1}) = \varepsilon,
\end{aligned}
$$

for all real numbers $t \geqslant 0$ and all integers $k \geqslant k_0$. Letting $k \to +\infty$ we get $f(t+p) = cf(t)$ for all $t \geqslant 0$. If $p > 0$ the above shows that $f(\cdot)$ is (p, c)-periodic, while the assumption $p = 0$ yields $f \equiv 0$ or $c = 1$, i. e., $f(\cdot) \equiv 0$; in any case, $f(\cdot)$ is (p, c)-periodic. Suppose now that (p_n) is unbounded. Then, with the same notation as above, we may assume that $\lim_{k\to+\infty} p_{n_k} = +\infty$. Using the same computation, it follows that $\lim_{k\to+\infty} \|c^{-1}f(\cdot + p_{n_k}) - f(\cdot)\|_\infty = 0$, so that $f \in UR_c([0,\infty) : X)$. Due to Proposition 4.2.11, we get $f(\cdot) \equiv 0$. Assume now $I = \mathbb{R}$. By the foregoing arguments, we know that there exists $p > 0$ such that $f(x+p) = cf(x)$ for all $x \geqslant 0$. Let $x < 0$ and $\varepsilon > 0$ be fixed. Since $f(\cdot)$ is semi-c-periodic, there exists $p_\varepsilon > 0$ such that $\|c^{-m}f(x+p+mp_\varepsilon) - f(x+p)\| \leqslant \varepsilon$ and $\|c^{1-m}f(x+mp_\varepsilon) - cf(x)\| \leqslant \varepsilon$ for all $m \in \mathbb{N}$. For all sufficiently large integers $m \in \mathbb{N}$ we have $x + mp_\varepsilon > 0$ so that $c^{-m}f(x+p+mp_\varepsilon) = c^{1-m}f(x+mp_\varepsilon)$ and therefore $\|f(x+p)-cf(x)\| \leqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $f(x+p) = cf(x)$, which completes the proof in the case $|c| > 1$. Suppose now that $|c| < 1$. Due to Lemma 4.2.43(ii), it suffices to show that, if $f(\cdot)$ is semi-c-periodic of type 1_+, then $f(\cdot)$ is c-periodic. But then we can apply Lemma 4.2.44 again and similar arguments to above to complete the whole proof. □

Corollary 4.2.46. *Let $c \in \mathbb{C} \setminus \{0\}$, let $i = 1, 2$, and let $f(\cdot)$ be semi-c-periodic of type i (i_+). Then there exist two finite real constants $M > 0$ and $p > 0$ such that $\|f(t)\| \leqslant M|c|^{t/p}$, $t \in I$.*

Using Theorem 4.2.19 and the proof of Theorem 4.2.45, we may deduce the following corollaries.

Corollary 4.2.47. *Let $f \in C(I : X)$ and $c \in \mathbb{C} \setminus \{0\}$. Then $f(\cdot)$ is semi-c-periodic if and only if there exists a sequence (f_n) of c-periodic functions in $C(I : X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in I.*

Corollary 4.2.48. *Let $f \in C(I : X)$ and $|c| \neq 1$. If (f_n) is a sequence of c-periodic functions and $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in I, then $f(\cdot)$ is c-periodic.*

For the Stepanov classes, we will use the following notion.

Definition 4.2.49. Let $p \in \mathcal{P}([0,1])$, and let $f \in L^{p(x)}_{\mathrm{loc}}(I : X)$.

(i) It is said that $f(\cdot)$ is Stepanov semi-$(p(x), c)$-periodic of type i (i_+) if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (2.5), is semi-c-periodic of type i (i_+).

(ii) It is said that $f(\cdot)$ is asymptotically Stepanov semi-$(p(x), c)$-periodic of type i (i_+) if and only if there are a Stepanov semi-$(p(x), c)$-periodic function of type i (i_+) $h(\cdot)$ and $q \in C_0(I : L^{p(x)}([0,1] : X))$ such that $f(t) = h(t) + q(t)$ for a. e. $t \in I$.

If $p(x) \equiv [1, \infty)$, then we also say that the function $f(\cdot)$ is Stepanov semi-(p, c)-periodic of type i (i_+) and so on.

Remark 4.2.50. Let us observe that we can also analyze the following notion in the case that the parameter c is not given in advance (compare with (4.41)):

$$\forall \varepsilon > 0 \quad \exists c > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \| f(x + mp) - c^m f(x) \| \leqslant \varepsilon.$$

More details will be given elsewhere.

Semi-periodic functions depending on a parameter have been introduced in [69, Definition 4], where the authors have considered case in which $I = \mathbb{R}$, $E = \mathbb{R}^k$ and $c = 1$. We will not introduce the related notion in the case $|c| = 1$, which will be standing till the end of the subsection.

The composition theorems for semi-c-periodic functions have not been considered elsewhere even in the case $c = 1$. In order to formulate the first result in this direction, suppose that $t \in I$, $p > 0$, $m \in \mathbb{S}$ and $c \in \mathbb{C} \setminus \{0\}$. Let $F : I \times Y \to X$ be a continuous function. If there exists a finite constant $L \geqslant 1$ such that (2.51) holds, then we have

$$\begin{aligned} &\| F(t + mp, f(t + mp)) - c^m F(t, f(t)) \| \\ &\quad \leqslant \| F(t + mp, f(t + mp)) - F(t + mp, c^m f(t)) \| \\ &\qquad + \| F(t + mp, c^m f(t)) - c^m F(t, f(t)) \| \\ &\quad \leqslant L \| f(t + mp) - c^m f(t) \| + \| F(t + mp, c^m f(t)) - c^m F(t, f(t)) \|. \end{aligned} \tag{4.42}$$

Therefore, it is natural to consider the following condition:

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall t \in I \quad \| F(t + mp, c^m f(t)) - c^m F(t, f(t)) \| \leqslant \varepsilon. \tag{4.43}$$

Using the estimate (4.42), we can immediately clarify the following result which can be simply formulated for semi-c-periodic functions.

Theorem 4.2.51. *Suppose that $F : I \times Y \to X$ is a continuous function satisfying that there exists a finite real constant $L > 0$ such that* (2.51) *holds, $f : I \to Y$ is a continuous function and for each $\varepsilon > 0$ there exists $p > 0$ such that* (4.41) *and* (4.43) *hold. Then the function $t \mapsto F(t, f(t))$, $t \in I$, is semi-c-periodic.*

In the following result, we reconsider [364, Theorem 3.31] for semi-c-periodic functions.

Theorem 4.2.52. *Suppose that $F : I \times Y \to X$ is a continuous function, $f : I \to Y$ is a continuous function and $F(\cdot, \cdot)$ is uniformly continuous on set $\{\eta f(t) : \eta \in \mathbb{C},\ t \in I\}$, uniformly in $t \in I$ (that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\| f(t, x) - f(t, y) \| \leqslant \varepsilon$ for all $t \in I$ and $x, y \in \{\eta f(t) : \eta \in \mathbb{C},\ t \in I\}$). Suppose that for each $\varepsilon > 0$ there exists $p > 0$ such that* (4.41) *and* (4.43) *hold. Then the function $t \mapsto F(t, f(t))$, $t \in I$ is semi-c-periodic.*

Proof. Since (4.43) holds, the statement easily follows from the estimate (4.42) and the prescribed assumptions. □

For the Stepanov classes, we will first clarify the following result.

Theorem 4.2.53. *Suppose that $p_1 \in \mathcal{P}([0,1])$, $r(x) \geqslant \max(p_1(x)/(p_1(x)-1))$, and there exists a function $L_F \in L_S^{r(x)}(I)$ such that (2.20) holds. Suppose, further, that for each $\varepsilon > 0$ there exists $p > 0$ such that*

$$\forall m \in \mathbb{S} \quad \forall t \in I \quad \|F(s+t+mp, c^m f(s+t)) - c^m F(s+t, f(s+t))\|_{L^{p_1(s)}[0,1]} \leqslant \varepsilon$$

holds, as well as (4.41) holds, with the function $f(\cdot)$ and the space Y replaced therein with the function $\hat{f}(\cdot)$ and the space $L^{p_1(x)}([0,1] : Y)$. Then the function $F(\cdot, f(\cdot))$ is Stepanov semi-$(q(x),c)$-periodic with $q(x) := p(x)r(x)/(p(x)+r(x))$ for $x \in [0,1]$, $r(x) < \infty$ and $q(x) := p(x)$ for $x \in [0,1]$, $r(x) = +\infty$.

Proof. We will prove the theorem with the constant coefficient $p_1(x) \equiv p_1 \in [1,\infty)$. Let $\varepsilon > 0$ be given and let the number $p > 0$ satisfy the above requirements. Fix numbers $t \in I$ and $m \in \mathbb{Z}$. Arguing as in the proof of estimate (4.42), we get

$$\begin{aligned}
&\|F(t+mp, f(t+mp)) - c^m F(t, f(t))\| \\
&\quad \leqslant L_F(t)\|f(t+mp) - c^m f(t)\| + \|F(t+mp, c^m f(t)) - c^m F(t, f(t))\|.
\end{aligned}$$

Using the Hölder inequality and the inequality $q < p_1$, we get

$$\begin{aligned}
&\left(\int_t^{t+1} \|F(s+mp, f(s+mp)) - c^m F(s, f(s))\|^q \, ds \right)^{1/q} \\
&\quad \leqslant 2^{(q-1)/q} \big[\|L_F(\cdot)\|_{L^r[t,t+1]} \|f(\cdot + mp) - c^m f(\cdot)\|_{L^{p_1}[t,t+1]} \\
&\qquad + \|F(\cdot + mp, c^m f(\cdot)) - c^m F(\cdot, f(\cdot))\|_{L^{p_1}[t,t+1]} \big].
\end{aligned}$$

This completes the proof of the theorem in a routine manner. □

Remark 4.2.54. We will not reconsider the statement of [729, Lemma 2.1] here.

We can similarly prove the result which naturally corresponds to [639, Theorem 2.1] and the consequence for the usual Lipschitz condition used. Finally, we would like to clarify an interesting result concerning the existence and uniqueness of semi-c-periodic solutions of the following abstract semilinear fractional Cauchy problem:

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\, ds + F(t, u(t)), \quad t \in \mathbb{R}, \tag{4.44}$$

where $D^\alpha u(t)$ denotes the Weyl–Liouville fractional derivative of order $\alpha > 0$, $a \in L^1_{\mathrm{loc}}([0,\infty))$ is a scalar-valued kernel, the function $F(\cdot,\cdot)$ enjoys some properties and A

generates a non-degenerate α-resolvent operator family $(S_\alpha(t))_{t\geqslant 0}$ on X satisfying that $\int_0^\infty \|S_\alpha(t)\|\,dt < \infty$ (see R. Ponce [854] for more details; equations of this kind arise in the study of heat flow in materials with memory as well as in certain equations of population dynamics). By a mild solution of (4.44), we mean any continuous function $u : \mathbb{R} \to X$ such that

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s)F(s,u(s))\,ds, \quad t \in \mathbb{R}.$$

Now we are able to formulate the following theorem.

Theorem 4.2.55. *Suppose that $F : \mathbb{R} \times X \to X$ is a continuous function satisfying that there exists a finite real constant $L > 0$ such that* (2.51) *holds. If $L \int_0^\infty \|S_\alpha(t)\|\,dt < 1$, then the abstract fractional semilinear Cauchy inclusion* (4.44) *has a unique semi-c-periodic solution.*

Proof. It can be easily shown that the set $\mathcal{SP}_{c,1}(\mathbb{R} : X)$, equipped with the distance $d(u,v) := \sup_{t\in\mathbb{R}} \|u(t) - v(t)\|$, $u, v \in \mathcal{SP}_{c,1}(\mathbb{R} : X)$, is a complete metric space. Define the mapping

$$(\Lambda u)(t) := \int_{-\infty}^{t} S_\alpha(t-s)F(s,u(s))\,ds, \quad t \in \mathbb{R} \quad (u \in \mathcal{SP}_{c,1}(\mathbb{R} : X)).$$

Applying Theorem 4.2.51 and the foregoing arguments, we see that the mapping $\Lambda(\cdot)$ is well defined. Moreover, our assumption $L \int_0^\infty \|S_\alpha(t)\|\,dt < 1$ easily implies that $\Lambda(\cdot)$ is a contraction. The proof completes an application of the Banach contraction principle. □

Before we switch to the next subsection, we note that J. Cao, A. Debbouche and Y. Zhou have applied, in [239], the Krasnoselskii fixed point theorem and a decomposition technique to obtain some sufficient conditions ensuring the existence of asymptotically almost periodic mild solutions for (4.44).

4.2.4 Semi-Bloch *k*-periodicity

In this subsection, we will always assume that $I = [0,\infty)$ or $I = \mathbb{R}$; as before, we set $\mathbb{S} := \mathbb{N}$ if $I = [0,\infty)$, and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$. For the convenience of the reader, we recall that a bounded continuous function $f : I \to X$ is said to be Bloch (p,k)-periodic, or Bloch periodic with period p and Bloch wave vector or Floquet exponent k if and only if $f(x+p) = e^{ikp}f(x)$, $x \in I$, with $p > 0$ and $k \in \mathbb{R}$. The space of all functions $f : I \to X$ that are Bloch (p,k)-periodic will be denoted by $\mathcal{B}_{p,k}(I : X)$.

If $f \in \mathcal{B}_{p,k}(I : X)$, then we have

$$f(x + mp) = e^{ikmp} f(x), \quad x \in I,\ m \in \mathbb{S},$$

so that the function $f(\cdot)$ must be periodic provided that $kp \in \mathbb{Q}$; but, if $kp \notin \mathbb{Q}$, then the function $f(\cdot)$ need not be periodic as the following simple counterexample shows: The function

$$f(x) := e^{ix} + e^{i(\sqrt{2}-1)x}, \quad x \in \mathbb{R},$$

is Bloch (p, k)-periodic with $p = 2\pi + \sqrt{2}\pi$ and $k = \sqrt{2} - 1$ but not periodic.

Given $k \in \mathbb{R}$, we set $\mathcal{B}_k(I : X) := \bigcup_{p>0} \mathcal{B}_{p,k}(I : X)$. Observing that $f \in P_c(I : X)$ satisfies $f(x + p) = f(x)$ for all $x \in I$ and some $p > 0$ if and only if the function $F(x) := e^{ikx} f(x)$, $x \in I$ satisfies $F(x + p) = e^{ikp} F(x)$, $x \in I$, we may conclude that

$$\mathcal{B}_k(I : X) := \{e^{ik\cdot} f(\cdot) : f \in P_c(I : X)\}. \tag{4.45}$$

For more details on the Bloch (p, k)-periodic functions, see [522] and the references cited therein.

Let us define the notion of a semi-Bloch k-periodic function as follows.

Definition 4.2.56. Let $k \in \mathbb{R}$. A function $f \in C_b(I : X)$ is said to be semi-Bloch k-periodic if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \|f(x + mp) - e^{ikmp} f(x)\| \leqslant \varepsilon. \tag{4.46}$$

The space of all semi-Bloch k-periodic functions will be denoted by $\mathcal{SB}_k(I : X)$.

It is clear that Definition 4.2.56 provides a generalization of [69, Definition 2 and Definition 3], given only in the case that $I = \mathbb{R}$. In actual fact, a function $f : \mathbb{R} \to X$ is semi-periodic in the sense of above-mentioned (equivalent) definitions if and only if $f : \mathbb{R} \to X$ is semi-Bloch 0-periodic. Furthermore, it can be easily shown that for each $k \in \mathbb{R}$ any constant function $f \equiv c$ belongs to the space $\mathcal{SB}_k(I : X)$; for this, it is only worth noticing that for each $\varepsilon > 0$ and $k \neq 0$ we can take $p = 2\pi/k$ and (4.46) will be satisfied.

Remark 4.2.57. It is not so easy to introduce the concept of almost Bloch k-periodicity, where $k \in \mathbb{R}$. In order to explain this in more detail, assume that a function $f \in C_b(I : X)$ and a number $\varepsilon > 0$ are given. Let us say that a real number $p > 0$ is an (ε, k)-Bloch period for $f(\cdot)$ if and only if

$$\|f(x + p) - e^{ikp} f(x)\| \leqslant \varepsilon, \quad x \in I, \tag{4.47}$$

and $f(\cdot)$ is almost Bloch k-periodic if and only if for each $\varepsilon > 0$ the set constituted of all (ε, k)-Bloch periods for $f(\cdot)$ is relatively dense in $[0, \infty)$. But then we see that $f(\cdot)$ is almost Bloch k-periodic if and only if $f(\cdot)$ is almost periodic. To see this, it suffices to observe that (4.47) is equivalent with

$$\left\| e^{-ik(x+p)} f(x+p) - e^{-ikx} f(x) \right\| \leqslant \varepsilon, \quad x \in I,$$

so that, actually, the function $f(\cdot)$ is almost Bloch k-periodic if and only if the function $e^{-ik\cdot} f(\cdot)$ is almost periodic, which is equivalent to saying that the function $f(\cdot)$ is almost periodic. Furthermore, let $f(\cdot) \in \mathcal{S}B_k(I : X)$. Then for each number $\varepsilon > 0$ we see that the set constituted of all (ε, k)-Bloch periods for $f(\cdot)$ is relatively dense in $[0, \infty)$ since it contains the set $\{mp : m \in \mathbb{N}\}$, where $p > 0$ is determined by (4.46). In view of our previous conclusions, $f(\cdot)$ is almost periodic. In particular, any Bloch (p, k)-periodic function needs to be almost periodic, which has not been observed in the researches of Bloch periodic functions carried out so far (see, e. g., [381] and [522]).

Now we will prove the following result.

Proposition 4.2.58. *Let $k \in \mathbb{R}$ and $f \in C_b(I : X)$. Then the following hold:*
(i) *$f(\cdot)$ is semi-Bloch k-periodic if and only if $e^{-ik\cdot} f(\cdot)$ is semi-periodic.*
(ii) *$f(\cdot)$ is semi-Bloch k-periodic if and only if there exists a sequence (f_n) in $P_c(I : X)$ such that* $\lim_{n\to\infty} e^{ikx} f_n(x) = f(x)$ *uniformly in I.*
(iii) *$f(\cdot)$ is semi-Bloch k-periodic if and only if there exists a sequence (f_n) in $\mathcal{B}_k(I : X)$ such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *uniformly in I.*

Proof. The proof of (i) follows similarly as above. Since [69, Lemma 1 and Theorem 1] hold for the functions defined on the interval $I = [0, \infty)$, we see that (i) implies that $f(\cdot)$ is semi-Bloch k-periodic if and only if there exists a sequence (f_n) in $P_c(I : X)$ such that $\lim_{n\to\infty} e^{ikx} f_n(x) = f(x)$ uniformly in I. This proves (ii). For the proof of (iii), it suffices to apply (ii), (4.45) and the conclusion preceding it. □

Let $k \in \mathbb{R}$. Using Proposition 4.2.58 and [69, Proposition 2], we may construct a substantially large class of semi-Bloch k-periodic functions, which do not form a vector space due to a simple example in the second part of [69, Remark 3]; [69, Lemma 2] can be straightforwardly reformulated for semi-Bloch k-periodic functions, while the function given in [69, Example 1] can be simply used to provide an example of a scalar-valued semi-Bloch k-periodic function which is not contained in the space $\mathcal{B}_k(I : \mathbb{C})$. If we define Bloch k-quasi-periodic function

$$\mathcal{B}_{k;q}(I : X) := \{e^{ik\cdot} f(\cdot) : f \in QP^0(I : X)\},$$

where $QP^0(I : X)$ denotes the space of all quasi-periodic functions from I into X (see [69, 185] and the references cited therein for the notion), then [69, Theorem 2] can be also reformulated in our context; this also holds for [69, Example 2, Example 3].

By the foregoing, we have

$$\mathcal{B}_k(I : X) \subseteq \mathcal{S}B_k(I : X) \subseteq \mathrm{AP}(I : X) \subseteq \mathrm{BUC}(I : X), \quad k \in \mathbb{R}.$$

Example 4.2.59. The function $f(x) := \cos x$, $x \in \mathbb{R}$ is anti-periodic. Now we will prove that $f \in \mathcal{S}B_k(I : X)$ if and only if $k \in \mathbb{Q}$. For $k \in \mathbb{Q}$, this is clear because we can take p in (4.46) as a certain multiple of 2π. Let us assume now that $k \notin \mathbb{Q}$. Then it suffices to show that the function $e^{-ik\cdot}f(\cdot)$ is not semi-periodic. To this aim, let us observe that $\sigma(e^{-ik\cdot}f(\cdot)) = \{1-k, -1-k\}$ so that there does not exist a positive real number $\theta > 0$ such that $\sigma(e^{-ik\cdot}f(\cdot)) \subseteq \theta \cdot \mathbb{Q}$, which can be simply proved and which contradicts [69, Lemma 2].

Remark 4.2.60. Let $a \in \mathrm{AP}(I : \mathbb{C})$. Then we can introduce and analyze the following notion: A function $f \in C_b(I : X)$ is said to be semi $_a$-periodic if and only if there exists a sequence (f_n) in $P_c(I : X)$ such that $\lim_{n\to\infty} a(x)f_n(x) = f(x)$ uniformly in I. Any such function needs to be almost periodic. We will analyze this notion somewhere else.

Example 4.2.61. Roughly speaking, it is well known that the unique solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial condition $u(x,0) = f(x)$, is given by (4.9). By the conclusion from [522, Example 2.1], we know that, if the function $f(\cdot)$ is Bloch (p,k)-periodic, then the solution $u(x,\cdot)$ is likewise Bloch (p,k)-periodic ($p > 0$, $k \in \mathbb{R}$). Using this fact, the dominated convergence theorem and Proposition 4.2.58, if $f(\cdot)$ is semi-Bloch k-periodic, then the solution $u(x,\cdot)$ will be likewise semi-Bloch k-periodic.

Proposition 4.2.62. *Let $k \in \mathbb{R}$, let $p > 0$, and let a function $f \in C_b([0,\infty) : X)$ be given. If $f(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic), then the function $\mathbb{E}f(\cdot)$ is likewise Bloch (p,k)-periodic (semi-Bloch k-periodic).*

Proof. Suppose first that $f(\cdot)$ is Bloch (p,k)-periodic. Then $f(x+p) = e^{ikp}f(x)$, $x \geqslant 0$; we need to show that $(\mathbb{E}f)(x+p) = e^{ikp}(\mathbb{E}f)(x)$, $x \in \mathbb{R}$, i. e., $[W(x+p)f](0) = e^{ikp}[W(x)f](0)$, $x \in \mathbb{R}$. Since $W(x+p) = W(x)W(p)$, $x \in \mathbb{R}$, it suffices to show that $[W(x)f(\cdot+p)](0) = e^{ikp}[W(x)f](0)$, $x \in \mathbb{R}$, i. e., $[W(x)e^{ikp}f(\cdot)](0) = e^{ikp}[W(x)f](0)$, $x \in \mathbb{R}$, which is true. If $f(\cdot)$ is semi-Bloch k-periodic, then Proposition 4.2.58(iii) shows that there exists a sequence (f_n) in $\mathcal{B}_k([0,\infty) : X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in $[0,\infty)$. Due to the supremum formula, we see that $\lim_{n\to\infty}(\mathbb{E}f_n)(x) = (\mathbb{E}f)(x)$ uniformly in $\mathbb{R}$. By the first part of the proof, we know that for each $n \in \mathbb{N}$ the function $(\mathbb{E}f_n)(\cdot)$ belongs to the space $\mathcal{B}_k(\mathbb{R} : X)$. Applying again Proposition 4.2.58(iii), we see that $\mathbb{E}f(\cdot)$ is likewise semi-Bloch k-periodic. □

The proof of the following simple proposition is left to the interested reader.

Proposition 4.2.63. *Let $k \in \mathbb{R}$, let $p > 0$, and let $f : I \to X$. Then we have:*
(i) *If $f(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic), then $cf(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic) for any $c \in \mathbb{C}$.*
(ii) *If $X = \mathbb{C}$, $\inf_{x\in\mathbb{R}} |f(x)| = m > 0$ and $f(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic), then $1/f(\cdot)$ is Bloch $(p,-k)$-periodic (semi-Bloch $(-k)$-periodic).*

Now we will introduce the following definition.

Definition 4.2.64. Let $f \in C_b(I : X)$ and $k \in \mathbb{R}$. Then we say that $f(\cdot)$ is asymptotically semi Bloch k-periodic if and only if there exist a function $\phi \in C_0(I : X)$ and a semi Bloch k-periodic function $g : \mathbb{R} \to X$ such that $f(t) = g(t) + \phi(t)$ for all $t \geqslant 0$.

As already mentioned, the notion of Stepanov semi-periodicity has not been analyzed in [69]. We will use the following definitions.

Definition 4.2.65. Let $k \in \mathbb{R}$ and $p \in \mathcal{P}([0,1])$. Then we say that a function $f \in L_S^{p(x)}(I : X)$ is Stepanov $p(x)$-semi-Bloch k-periodic if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (2.5), is semi-Bloch k-periodic.

If $p(x) \equiv p \in [1,\infty)$, then we also say that the function $f(\cdot)$ is Stepanov p-semi-Bloch k-periodic.

Definition 4.2.66. Let $k \in \mathbb{R}$ and $p \in \mathcal{P}([0,1])$. Then we say that a function $f \in L_S^{p(x)}(I : X)$ is asymptotically Stepanov $p(x)$-semi-Bloch k-periodic if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (2.5), is asymptotically semi-Bloch k-periodic.

If $p(x) \equiv p \in [1,\infty)$, then we also say that the function $f(\cdot)$ is asymptotically Stepanov p-semi-Bloch k-periodic.

Let $p > 0$ and $k \in \mathbb{R}$. It should be noted that, if $f : I \to X$ is Bloch (p,k)-periodic, then $\hat{f} : I \to L^q([0,1] : X)$ is likewise Bloch (p,k)-periodic. Furthermore, it immediately follows from the corresponding definitions that, if $f : I \to X$ is semi-Bloch k-periodic, then $f(\cdot)$ is Stepanov q-semi-Bloch k-periodic for every number $q \in [1,\infty)$; a large class of non-continuous periodic or Bloch (p,k)-periodic functions can be used to provide that the converse statement does not hold in general. If $1 \leqslant q < q' < \infty$ and $f : I \to X$ is (asymptotically) Stepanov q'-semi-Bloch k-periodic, then $f(\cdot)$ is (asymptotically) Stepanov q-semi-Bloch k-periodic. To see that the converse statement does not hold in general, we will provide only one illustrative example.

Example 4.2.67. Suppose that $1 < q < \infty$. Let us revisit the example of H. Bohr and E. Følner once more; they have constructed an example of a Stepanov 1-almost periodic function $F : \mathbb{R} \to \mathbb{R}$ that is not Stepanov q-almost periodic (see [199, p. 70]). Moreover, for each $n \in \mathbb{N}$ there exists a bounded periodic function $F_n : \mathbb{R} \to \mathbb{R}$ with at most countable points of discontinuity such that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} \left|F_n(s) - F(s)\right| ds = 0. \tag{4.48}$$

Therefore, $\hat{F_n} : \mathbb{R} \to L^1([0,1] : \mathbb{R})$ is a bounded periodic function and, in addition to the above, $\hat{F_n}(\cdot)$ is continuous ($n \in \mathbb{N}$). Due to (4.48), we see that $\lim_{n\to\infty} \hat{F_n}(t) = \hat{F}(t)$ uniformly in $t \in \mathbb{R}$. This implies that the function $F(\cdot)$ is Stepanov 1-semi-periodic but not Stepanov q-semi-periodic because it is not Stepanov q-almost periodic.

The above conclusions can also be clarified for Stepanov $p(x)$-semi-Bloch k-periodic functions. Concerning the invariance of semi-Bloch k-periodicity under the actions of infinite convolution products, we have the following result.

Proposition 4.2.68. *Suppose that $k \in \mathbb{R}$, p, $q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is Stepanov $p(x)$-semi-Bloch $(-k)$-periodic, then the function $F(\cdot)$, given by (2.46), is well defined and semi-Bloch k-periodic.*

Proof. Using the same arguments as in the proof of Proposition 3.1.18, we see that $F(\cdot)$ is well defined and continuous. It remains to be proved that $F(\cdot)$ is semi-Bloch k-periodic. Let a number $\varepsilon > 0$ be given in advance. Then we can find a finite number $p > 0$ such that, for every $m \in \mathbb{Z}$ and $t \in \mathbb{R}$, we have

$$\|\check{f}(t + mp) - e^{-ikmp}\check{f}(t)\|_{L^{p(x)}[0,1]} \leqslant \varepsilon, \quad t \in \mathbb{R}.$$

Applying the Hölder inequality and this estimate, we get

$$\begin{aligned}
&\|F(t + mp) - e^{ikmp}F(t)\| \\
&\quad \leqslant \int_0^{\infty} \|R(r)\| \cdot \|f(t + mp - r) - e^{ikmp}f(t - r)\|\,dr \\
&\quad = \sum_{k=0}^{\infty} \int_0^1 \|R(r + k)\| \cdot \|f(t + k + mp - r) - e^{ikmp}f(t + k - r)\|\,dr \\
&\quad \leqslant 2\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} \|e^{-ikmp}\check{f}(r - t - mp - k) - \check{f}(r - t - k)\|_{L^{p(r)}[0,1]} \\
&\quad \leqslant 2\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]}\varepsilon = 2M\varepsilon, \quad t \in \mathbb{R},
\end{aligned}$$

which clearly implies the required conclusion. □

The above result can be simply applied in the study of existence and uniqueness of semi-Bloch k-periodic solutions of the fractional Cauchy inclusion (2.49). We can also analyze the invariance of asymptotical semi-Bloch k-periodicity under the actions of finite convolution products, applying the obtained results in the qualitative analysis of asymptotically (Stepanov) semi-Bloch k-periodic solutions of the abstract fractional Cauchy inclusion $(\mathrm{DFP})_{f,\gamma}$.

Let $p > 0$ and $k \in \mathbb{R}$. If $f : \mathbb{R} \to X$ is Bloch (p,k)-periodic and $a \in L^1(\mathbb{R})$, then the function $a * f(\cdot)$ is likewise Bloch (p,k)-periodic. Using the Young inequality and our previous results, it can be simply shown that the space of semi-Bloch k-periodic functions is convolution invariant.

Finally, let B be a subset of $\mathbb{R}^s$ and $f : \mathbb{R} \times B \to \mathbb{R}^s$. Then we say that the function $f(\cdot)$ is uniformly semi-Bloch k-periodic function if and only if for any compact subset

K of B, we have

$$\forall \varepsilon > 0\ \exists p \geqslant 0\ \forall m \in \mathbb{Z}\ \forall x \in \mathbb{R}\ \forall \alpha \in K\ \|f(x + mp, \alpha) - e^{ikmp} f(x, \alpha)\|_{\mathbb{R}^s} \leqslant \varepsilon.$$

We close the subsection with the observation that we can simply reformulate [69, Proposition 3] for uniformly semi-Bloch k-periodic functions and provide certain applications to the matrix differential equations, as has been done in [69, Theorem 4] for semi-periodic functions.

4.2.5 Weyl (p, c)-almost periodic type functions

The material of the next three subsections is taken from a joint paper [588] with Prof. M. T. Khalladi, A. Rahmani, M. Pinto and D. Velinov.

We first introduce the notion of an (equi-)Weyl (p, c)-almost periodic function as follows.

Definition 4.2.69. Let $1 \leqslant p < \infty$ and $f \in L^p_{\mathrm{loc}}(I : X)$.

(i) We say that the function $f(\cdot)$ is equi-Weyl (p, c)-almost periodic, $f \in e - W^p_{\mathrm{ap};c}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x \in I} \left[\frac{1}{l} \int\limits_x^{x+l} \|f(t + \tau) - cf(t)\|^p \, dt \right]^{1/p} \leqslant \varepsilon. \tag{4.49}$$

(ii) We say that the function $f(\cdot)$ is Weyl (p, c)-almost periodic, $f \in W^p_{\mathrm{ap};c}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l \to +\infty} \sup_{x \in I} \left[\frac{1}{l} \int\limits_x^{x+l} \|f(t + \tau) - cf(t)\|^p \, dt \right]^{1/p} \leqslant \varepsilon.$$

If $c = 1$, resp. $c = -1$, then we also say that $f(\cdot)$ is (equi-)Weyl p-almost periodic, resp. (equi-)Weyl p-almost anti-periodic.

It is clear that any equi-Weyl (p, c)-almost periodic function is Weyl (p, c)-almost periodic. The proofs of following results are trivial and therefore are omitted.

Proposition 4.2.70. *Suppose that $f : I \to X$ is (equi-)Weyl (p, c)-almost periodic. Then $\|f\| : I \to [0, \infty)$ is (equi-)Weyl p-almost periodic.*

Proposition 4.2.71. *Let $1 \leqslant p < \infty$ and $f \in L^p_{\mathrm{loc}}(I : X)$. If the function $f(\cdot)$ is (equi-)-Weyl (p, c)-almost periodic and $I = \mathbb{R}$, then the function $\check{f} : \mathbb{R} \to X$ is (equi-)Weyl $(p, 1/c)$-almost periodic.*

We will include the proof of the following proposition for the sake of completeness.

Proposition 4.2.72. *Let $1 \leqslant p < \infty$ and $f \in L^p_{\mathrm{loc}}(I : X)$. If the function $f(\cdot)$ is (equi-)Weyl (p, c)-almost periodic and $m \in \mathbb{N}$, then the function $f(\cdot)$ is (equi-)Weyl (p, c^m)-almost periodic.*

Proof. We will give the proof for the class of equi-Weyl (p, c)-almost periodic functions. Let $\varepsilon > 0$ be fixed; then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (4.49) holds true. Clearly, integrating the estimate (4.28) (with the number l replaced by the number m therein) over the segment $[x, x + l]$, where $x \in I$, we obtain the existence of a finite constant $c_p > 0$ such that

$$\begin{aligned}
&\left[\frac{1}{l}\int_x^{x+l}\|f(t+m\tau)-c^m f(t)\|^p\,dt\right]^{1/p} \\
&\quad\leqslant c_p\left[\sum_{j=0}^{m-1}\frac{|c|^{jp}}{l}\int_x^{x+l}\|f(t+(m-j)\tau)-cf(t+(m-j-1)\tau)\|^p\,dt\right]^{1/p} \\
&\quad\leqslant c_p\left[\sum_{j=0}^{m-1}\frac{|c|^{jp}}{l}\int_{x+(m-j-1)\tau}^{x+l+(m-j-1)\tau}\|f(t+\tau)-cf(t)\|^p\,dt\right]^{1/p} \\
&\quad\leqslant c_p\varepsilon\left[\sum_{j=0}^{m-1}|c|^{jp}\right]^{1/p}.
\end{aligned}$$

Therefore, for this number $\varepsilon > 0$, we can take the numbers $l > 0$ and $mL > 0$ in the definition of equi-Weyl (p, c)-almost periodicity. This completes the proof. □

The next corollary of Proposition 4.2.72 follows immediately.

Corollary 4.2.73. *Let $1 \leqslant p < \infty$, $f \in L^p_{\mathrm{loc}}(I : X)$, and let (4.29) hold with the numbers p and q replaced therein with the numbers m and n, respectively.*
(i) *If m is even and $f(\cdot)$ is an (equi-)Weyl (p, c)-almost periodic function, then $f(\cdot)$ is (equi-)Weyl p-almost periodic.*
(ii) *If m is odd and $f(\cdot)$ is an (equi-)Weyl (p, c)-almost periodic function, then $f(\cdot)$ is (equi-)Weyl p-almost anti-periodic.*

Proposition 4.2.74. *Let $1 \leqslant p < \infty$, $f \in L^p_{\mathrm{loc}}(I : X)$, and let $|c| = 1$, $\arg(c)/\pi \notin \mathbb{Q}$. If $f(\cdot)$ is (equi-)Weyl (p, c)-almost periodic and Stepanov p-bounded, then $f(\cdot)$ is (equi-)Weyl (p, c')-almost periodic for all $c' \in S_1$.*

Proof. It suffices to consider case in which the function $f(\cdot)$ is not almost everywhere equal to zero. Let the numbers $c' \in S_1$ and $\varepsilon > 0$ be fixed; then the set $\{c^l : l \in \mathbb{N}\}$ is dense in S_1 and therefore there exists an increasing sequence (l_k) of positive integers such that $\lim_{k\to+\infty} c^{l_k} = c'$. Let $k \in \mathbb{N}$ be such that $|c^{l_k} - c'| < \varepsilon/(2\|f\|_{S^p})$, and let $\varepsilon > 0$ be given. Then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (4.49) holds. Then we have

$$\|f(x+\tau) - c'f(x)\| \leqslant \|f(x+\tau) - c^{l_k}f(x)\| + |c^{l_k} - c'| \cdot \|f(x)\|,$$

for any $x \in I$. Then the conclusion follows from Proposition 4.2.72, after integrating the above estimate over the segment $[x, x+l]$ and using the estimate

$$\frac{1}{l}\int_x^{x+l} \|f(t)\|^p \, dt \leqslant \frac{1}{l}(1 + \lfloor l \rfloor)\|f\|_{S^p}^p. \qquad \square$$

The main structural properties of (equi-)Weyl (p, c)-almost periodic functions are collected in the following theorem (see also [631, Proposition 2.3.5]).

Theorem 4.2.75. *Let $f : I \to X$ be (equi-)Weyl (p, c)-almost periodic, and let $\alpha \in \mathbb{C}$. Then we have:*

(i) *$\alpha f(\cdot)$ is (equi-)Weyl (p, c)-almost periodic.*
(ii) *If $X = \mathbb{C}$ and $\operatorname{ess\,inf}_{x\in\mathbb{R}} |f(x)| = m > 0$, then $1/f(\cdot)$ is (equi-)Weyl $(p, 1/c)$-almost periodic).*
(iii) *If $(g_n : I \to X)_{n\in\mathbb{N}}$ is a sequence of bounded, continuous, (equi-)Weyl (p, c)-almost periodic functions and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g : I \to X$, then $g(\cdot)$ is (equi-)Weyl (p, c)-almost periodic.*
(iv) *If $a \in I$ and $b \in I \setminus \{0\}$, then the functions $f(\cdot + a)$ and $f(b\,\cdot)$ are likewise (equi-)Weyl (p, c)-almost periodic.*

Now we will provide two simple examples.

Example 4.2.76. Set $f(t) := \chi_{[0,1/2]}(t)$, $t \in \mathbb{R}$. Then for each number $l > 0$ we have

$$\frac{1}{l}\int_x^{x+l} |f(t+\tau) - cf(t)|^p \, dt \leqslant \frac{1}{2l}(1 + |c|)^p, \quad x \in \mathbb{R}.$$

This implies that $f(\cdot)$ is equi-Weyl (p, c)-almost periodic for each complex number $c \in \mathbb{C} \setminus \{0\}$ and for each finite exponent $p \geqslant 1$.

Example 4.2.77. Set $f(t) := \chi_{[0,\infty)}(t)$, $t \in \mathbb{R}$. Then for each number $l > 0$ we have

$$\sup_{x\in\mathbb{R}} \frac{1}{l}\int_x^{x+l} |f(t+\tau) - cf(t)|^p \, dt \geqslant |1 - c|^p,$$

so that $f(\cdot)$ cannot be Weyl (p, c)-almost periodic for $c \neq 1$. On the other hand, it is well known that $f(\cdot)$ is Weyl $(p, 1)$-almost periodic for any finite exponent $p \geqslant 1$.

Concerning the invariance of (equi-)Weyl (p,c)-almost periodicity under the actions of convolution products, we will only note that the statements of [631, Proposition 2.11.1, Theorem 2.11.4, Proposition 2.11.6] can be simply reformulated in our framework. The interested reader can try to slightly generalize the notions and results of this subsection for variable exponents $p(x)$.

4.2.6 S-asymptotically (ω, c)-periodic functions

We start this subsection by introducing the following notion.

Definition 4.2.78. Let $\omega \in I$. Then we say that a continuous function $f : I \to X$ is S-asymptotically (ω, c)-periodic if and only if $\lim_{|t|\to\infty} \|f(t+\omega) - cf(t)\| = 0$; a continuous function $f : I \to X$ is said to be S_c-asymptotically periodic if and only if there exists $\omega > 0$ such that $f(\cdot)$ is S-asymptotically (ω, c)-periodic. By $\mathrm{SAP}_{\omega;c}(I : X)$ and $\mathrm{SAP}_c(I : X)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is S-asymptotically ω-anti-periodic, resp. S-asymptotically anti-periodic.

This definition extends the well-known definition of an S-asymptotically ω-periodic function, introduced by H. Henríquez et al. [531] for case $I = \mathbb{R}$ and M. Kostić [647] for case $I = [0, \infty)$. Furthermore, this notion extends the notion of Y.-K. Chang and Y. Wei [259], where the authors have analyzed S-asymptotically Bloch type periodic functions and some applications to the semilinear evolution equations in Banach spaces ($c = e^{ik\omega}$ for some $k \in \mathbb{R}$ and $I = \mathbb{R}$; see, especially, [259, Subsection 4.2], where the authors investigate semilinear fractional differential equation of Sobolev type).

Definition 4.2.79. Let $p \in \mathcal{P}([0,1])$. A $p(x)$-locally integrable function $f(\cdot)$ is said to be Stepanov $p(x)$-asymptotically (ω, c)-periodic if and only if

$$\lim_{|t|\to\infty} \left\| f(s+t+\omega) - cf(s+t) \right\|_{L^{p(s)}[0,1]} = 0;$$

a $p(x)$-locally integrable function $f : I \to X$ is called Stepanov $p_c(x)$-asymptotically periodic if and only if there exists $\omega > 0$ such that $f(\cdot)$ is Stepanov $p(x)$-asymptotically (ω, c)-periodic.

By $S^{p(x)}\mathrm{SAP}_{\omega;c}(I : X)$ and $S^{p(x)}\mathrm{SAP}_c(I : X)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov $p(x)$-asymptotically ω-anti-periodic, resp. Stepanov $p(x)$-asymptotically anti-periodic.

If $p(x) \equiv p \in [1, \infty)$, then by $S^p\mathrm{SAP}_{\omega;c}(I : X)$ and $S^p\mathrm{SAP}_c(I : X)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov p-asymptotically ω-anti-periodic, resp. Stepanov p-asymptotically anti-periodic.

Now we will introduce the class of quasi-asymptotically c-almost periodic functions.

Definition 4.2.80. It is said that a continuous function $f : I \to X$ is quasi-asymptotically c-almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|f(t+\tau) - cf(t)\| \leqslant \varepsilon, \quad \text{provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

Denote by $Q - \mathrm{AAP}_c(I : X)$ the set consisting of all quasi-asymptotically c-almost periodic functions from I into X; if $c = -1$, then we also say that the function $f(\cdot)$ is quasi-asymptotically almost anti-periodic.

Next, we introduce the following notion of Stepanov (p, c)-quasi-asymptotical almost periodicity.

Definition 4.2.81. Let $p \in \mathcal{P}([0,1])$. A $p(x)$-locally integrable function $f(\cdot)$ is said to be Stepanov $(p(x), c)$-quasi-asymptotically almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|f(s+t+\tau) - cf(s+t)\|_{L^{p(s)}[0,1]} \leqslant \varepsilon, \quad \text{provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

By $S^{p(x)}Q - \mathrm{AAP}_c(I : X)$ we denote the set consisting of all Stepanov $p(x)$-quasi-asymptotically c-almost periodic functions from I into X; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov $p(x)$-quasi-asymptotically almost anti-periodic.

If $p(x) \equiv p \in [1, \infty)$, then we accept the usual terminology and then we denote the above space by $S^p Q - \mathrm{AAP}_c(I : X)$.

Remark 4.2.82. A $p(x)$-locally integrable function $f(\cdot)$ is Stepanov $(p(x), c)$-quasi-asymptotically almost periodic if and only if the function $f : I \to L^{p(x)}([0,1] : X)$ is quasi-asymptotically c-almost periodic. Similar statements hold for the class of Stepanov $p(x)$-asymptotically (ω, c)-periodic functions. This observation enables one to see that many results clarified below, like Proposition 4.2.83, Corollary 4.2.84 and Theorem 4.2.86, continue to hold for the corresponding Stepanov classes of functions under our consideration.

It is very simple to prove that any asymptotically c-almost periodic function is quasi-asymptotically c-almost periodic. Furthermore, (4.28) easily implies the following.

Proposition 4.2.83. *Let $\omega > 0$, $f : I \to X$ be an S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), and let $m \in \mathbb{N}$.*

Then $f(\cdot)$ is S-asymptotically $(m\omega, c^m)$-periodic (S_{c^m}-asymptotically periodic, quasi-asymptotically c^m-almost periodic).

The next corollary of Proposition 4.2.83 follows immediately.

Corollary 4.2.84. *Let $f : I \to X$ be a continuous function, and let (4.29) hold with the numbers p and q replaced therein with the numbers m and n, respectively.*
(i) *If m is even and $f(\cdot)$ is S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), then the function $f(\cdot)$ is S-asymptotically ω-anti-periodic (S-asymptotically anti-periodic, quasi-asymptotically almost anti-periodic).*
(ii) *If m is odd and $f(\cdot)$ is S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), then the function $f(\cdot)$ is S-asymptotically ω-periodic (S-asymptotically periodic, quasi-asymptotically almost periodic).*

Therefore, if $\arg(c)/\pi \in \mathbb{Q}$, then the class of S-asymptotically (ω, c)-periodic functions (S_c-asymptotically periodic functions, quasi-asymptotically c-almost periodic functions) is always contained in the class of S-asymptotically ω-periodic functions (S-asymptotically periodic functions, quasi-asymptotically almost periodic functions).

The following result holds true.

Corollary 4.2.85. *Suppose that $|c| = 1$ and $\arg(c)/\pi \notin \mathbb{Q}$. If the function $f(\cdot)$ is bounded S-asymptotically (ω, c)-periodic (bounded S_c-asymptotically periodic, bounded quasi-asymptotically c-almost periodic), then $f(\cdot)$ is S-asymptotically ω-periodic (S-asymptotically periodic, quasi-asymptotically almost periodic).*

Furthermore, a slight modification of the proof of [647, Theorem 2.5] shows that the following statement holds.

Theorem 4.2.86. *Let $F(I : X)$ be any space consisting of continuous functions $h : I \to X$ such that $\sup_{t\in I} \|h(t + \tau) - ch(t)\| = \sup_{t\geqslant a} \|h(t + \tau) - ch(t)\|$, $a \in I$. Then the following hold:*
(i) $\mathrm{AAA}_c(I : X) \cap Q - \mathrm{AAP}_c(I : X) = \mathrm{AAP}_c(I : X)$.
(ii) $\mathrm{AA}_c(\mathbb{R} : X) \cap Q - \mathrm{AAP}_c(\mathbb{R} : X) = \mathrm{AP}_c(\mathbb{R} : X)$.

We will include the proof of the following proposition for the sake of completeness (see also the proof of [647, Proposition 2.7]).

Proposition 4.2.87. *Let $|c| \leqslant 1$. Then $\mathrm{SAP}_{\omega;c}(I : X) \subseteq Q - \mathrm{AAP}_c(I : X)$.*

Proof. Let $\varepsilon > 0$ be given. Then we can take $L(\varepsilon) = 2\omega$ in definition of the space $Q - \mathrm{AAP}_c(I : X)$. Then any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains a number $\tau = n\omega$ for some $n \in \mathbb{N}$. For this n and ε, there exists a finite number $M(\varepsilon, n) > 0$ such that $\|f(t + \omega) -$

$cf(t)\| \leqslant \varepsilon/n\omega$ for $|t| \geqslant M(\varepsilon, n)$. Then we have

$$\|f(t+n\omega) - cf(t)\| \leqslant \sum_{k=0}^{n-1} |c|^{n-k-1} \|f(t+(k+1)\omega) - cf(t+k\omega)\|$$
$$\leqslant \sum_{k=0}^{n-1} \|f(t+(k+1)\omega) - cf(t+k\omega)\| \leqslant \sum_{k=0}^{n-1} \frac{\varepsilon}{n\omega} = \varepsilon/\omega,$$

provided $|t| \geqslant M(\varepsilon, n) + n\omega$. This completes the proof. □

The following proposition can be deduced using the argumentation contained in the proof of [647, Proposition 2.12].

Proposition 4.2.88. *We have* $S^p Q - \mathrm{AAP}_c(I : X) \subseteq W^p_{\mathrm{ap};c}(I : X)$.

The structural properties of quasi-asymptotically almost periodic functions clarified in [647, Theorem 2.13] can be slightly generalized in the following manner.

Theorem 4.2.89. *Let $f : I \to X$ be a quasi-asymptotically c-almost periodic function (Stepanov (p, c)-quasi-asymptotically almost periodic function). Then we have:*
(i) *$\alpha f(\cdot)$ is quasi-asymptotically c-almost periodic (Stepanov (p, c)-quasi-asymptotically almost periodic) for any $\alpha \in \mathbb{C}$.*
(ii) *If $X = \mathbb{C}$ and $\inf_{x \in I} |f(x)| = m > 0$ ($\operatorname{ess\,inf}_{x \in I} |f(x)| = m > 0$), then $1/f(\cdot)$ is quasi-asymptotically $1/c$-almost periodic (Stepanov $(p, 1/c)$-quasi-asymptotically almost periodic).*
(iii) *If $(g_n : I \to X)_{n \in \mathbb{N}}$ is a sequence of quasi-asymptotically c-almost periodic functions and $(g_n)_{n \in \mathbb{N}}$ converges uniformly to a function $g : I \to X$, then $g(\cdot)$ is quasi-asymptotically c-almost periodic.*
(iv) *If $(g_n : I \to X)_{n \in \mathbb{N}}$ is a sequence of Stepanov (p, c)-quasi-asymptotically almost periodic functions and $(g_n)_{n \in \mathbb{N}}$ converges to a function $g : I \to X$ in the space $L^p_S(I : X)$, then $g(\cdot)$ is Stepanov (p, c)-quasi-asymptotically almost periodic.*
(v) *The functions $f(\cdot + a)$ and $f(b\,\cdot)$ are likewise quasi-asymptotically c-almost periodic (Stepanov (p, c)-quasi-asymptotically almost periodic), where $a \in I$ and $b \in I \setminus \{0\}$.*

The space of quasi-asymptotically c-almost periodic functions is not closed under pointwise addition and multiplication (see also [647, Proposition 2.15, Example 2.16–Example 2.18]).

Concerning the invariance of quasi-asymptotical c-almost periodicity under the actions of convolution products, the structural results clarified in [647, Section 3] continue to hold for (Stepanov p-) bounded forcing terms $f(\cdot)$.

Proposition 4.2.90.
(i) *Suppose that $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family and $\int_0^\infty \|R(s)\|\, ds < \infty$. If the function $f \in Q - \mathrm{AAP}_c([0, \infty) : X)$ is bounded, then the function $F(\cdot)$, defined through* (3.59)*, with the function* $\mathrm{F}(\cdot)$ *replaced therein with the function $f(\cdot)$, belongs to the class $Q - \mathrm{AAP}_c([0, \infty) : Y)$.*

(ii) *Suppose that* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\int_0^{\infty} \|R(s)\|\, ds < \infty$. *If* $f \in Q-\mathrm{AAP}_c(\mathbb{R}:X)$ *is bounded, then the function* $\mathbf{F}(t)$, *defined through* (2.46), *belongs to the class* $Q-\mathrm{AAP}_c(\mathbb{R}:Y)$.

Proposition 4.2.91.

(i) *Suppose that* $1/p+1/q=1$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. *If* $f \in S^p Q-\mathrm{AAP}_c([0,\infty):X)$ *is Stepanov p-bounded, then the function* $F(\cdot)$, *defined by* (3.59), *belongs to the class* $Q-\mathrm{AAP}_c([0,\infty):Y)$.

(ii) *Suppose that* $1/p+1/q=1$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. *If* $f \in S^p Q-\mathrm{AAP}_c(\mathbb{R}:X)$ *is Stepanov p-bounded, then the function* $\mathbf{F}(\cdot)$, *defined by* (2.46), *belongs to the class* $Q-\mathrm{AAP}_c(\mathbb{R}:Y)$.

Before we move to the next subsection, let us note the obvious fact that the various notions of Stepanov quasi-asymptotically almost periodic functions in Lebesgue spaces with variable exponent, among many other classes of generalized almost periodic functions, can be slightly generalized using the difference $f(\cdot+\tau)-cf(\cdot)$. Albeit that a fairly complete analysis is out of the scope of this book, we will consider some classes of multi-dimensional c-almost periodic type functions in Part II.

4.2.7 Composition principles for quasi-asymptotically c-almost periodic functions

The main aim of this subsection is to introduce the class of quasi-asymptotically c-almost periodic functions depending on two parameters, its Stepanov generalization and to formulate several composition principles for quasi-asymptotically c-almost periodic functions. First of all, we will introduce the following definition.

Definition 4.2.92. Suppose that $F: I \times Y \to X$ is a continuous function and $\mathcal{F}$ is a non-empty collection of subsets of Y. Then we say that $F(\cdot,\cdot)$ is quasi-asymptotically c-almost periodic, uniformly on $\mathcal{F}$ if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying the requirement that there exists a finite number holds with a number $M(\varepsilon,\tau) > 0$ such that for each subset $B \in \mathcal{F}$ we have

$$\|F(t+\tau,x)-cF(t,x)\|_Y \leqslant \varepsilon, \quad \text{provided } t \in I,\ x \in B \text{ and } |t| \geqslant M(\varepsilon,\tau).$$

Denote by $Q-\mathrm{AAP}_{c;\mathcal{F}}(I \times Y : X)$ the set consisting of all quasi-asymptotically c-almost periodic functions $F: I \times Y \to X$ on $\mathcal{F}$.

Suppose that $F: I \times Y \to X$ is a continuous function and there exists a finite constant $L > 0$ such that (2.51) holds. Define $\mathcal{F}(t) := F(t, f(t))$, $t \in I$. Using (4.35) and the proofs of [364, Theorem 3.30, Theorem 3.31], we may deduce the following composition principles.

Theorem 4.2.93. *Suppose that $F \in Q - \mathrm{AAP}_c(I \times Y : X)$ and $f \in Q - \mathrm{AAP}_c(I : Y)$. If there exists a finite number $L > 0$ such that (2.51) holds and for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying*

$$\|F(t+\tau, cf(t)) - cF(t, f(t))\| \leqslant \varepsilon, \quad t \in I, \tag{4.50}$$

then the function $t \mapsto F(t, f(t))$, $t \in I$ belongs to the class $Q - \mathrm{AAP}_c(I : X)$.

Theorem 4.2.94. *Suppose that $F \in Q - \mathrm{AAP}_c(I \times Y : X)$ and $f \in Q - \mathrm{AAP}_c(I : Y)$. If the function $x \mapsto F(t, x)$, $t \in I$ is uniformly continuous on $R(f)$ uniformly for $t \in I$ and for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that (4.50) holds, then the function $t \mapsto F(t, f(t))$, $t \in I$ belongs to the class $Q - \mathrm{AAP}_c(I : X)$.*

The notion of a Stepanov (p, c)-quasi-asymptotically almost periodic function depending on two parameters can be also introduced, and [647, Theorem 2.23, Theorem 2.24] can be slightly generalized in this framework.

In [647, Section 4], we have analyzed the qualitative solutions of the abstract non-autonomous differential equations (3.63)–(3.64) and their semilinear analogues. We close the subsection with the observation that the structural results established in [647, Theorem 4.1, Theorem 4.3] can be simply reformulated in our context; for example, in the formulation of [647, Theorem 4.1], we can assume that

$$\sum_{k=0}^{\infty} \|\Gamma(t+\tau, t+\tau-\cdot) - c\Gamma(t, t-\cdot)\|_{L^q[k,k+1]} \leqslant \varepsilon, \quad \text{provided } t \geqslant M(\varepsilon, \tau),$$

in place of condition [647, (4.1)]. Then the unique mild solution $u(\cdot)$ of the abstract Cauchy problem (3.64) will belong to the class $Q - \mathrm{AAP}_c([0,\infty) : X) + \mathcal{F}$; see [647] for the notation. The structural results established for the abstract non-autonomous semilinear differential equations [647, Theorem 4.6, Theorem 4.7] also can be slightly generalized in our framework.

4.2.8 *c*-Almost periodic type distributions

Let us recall that the classes of scalar-valued bounded distributions and scalar-valued almost periodic distributions have been introduced by L. Schwartz [913] and later extended to the vector-valued case by I. Cioranescu in [300]. On the other hand, the class of scalar-valued asymptotically almost periodic distributions has been introduced by

I. Cioranescu in [299], while the notion of a vector-valued asymptotically almost periodic distribution has been analyzed by D. N. Cheban [269] following a different approach (cf. also I. K. Dontvi [392] and A. Halanay, D. Wexler [506]). For more details about the subject, we refer the reader to [124, 207–210, 665, 958, 959] as well as the recent research studies [211] by C. Bouzar, F. Z. Tchouar, [646] by M. Kostić and [663] by M. Kostić, S. Pilipović, D. Velinov.

In this subsection, which presents some results from our recent joint research study [436] with V. Fedorov, S. Pilipović and D. Velinov, we introduce and investigate various classes of vector-valued c-almost periodic type distributions and vector-valued asymptotically c-almost periodic type distributions. In order to be consistent with the notion employed in [646], we will say that $f : \mathbb{R} \to X$ is half-asymptotically c-almost periodic (half-asymptotically c-uniformly recurrent, half-asymptotically semi-c-periodic) if and only if there are a c-almost periodic function (c-uniformly recurrent function, semi-c-periodic function) $g : \mathbb{R} \to X$ and a function $h \in C_0([0,\infty) : X)$ such that $f(t) = g(t) + h(t)$, $t \geqslant 0$.

If $c = 1$, then we also say that $f(\cdot)$ is ((half-)asymptotically) uniformly recurrent (((half-)asymptotically) semi-periodic, ((half-)asymptotically) almost periodic); if $c = -1$, then we also say that $f(\cdot)$ is ((half-)asymptotically) almost anti-periodic (((half-)-asymptotically) uniformly anti-recurrent, ((half-)asymptotically) semi-anti-periodic). Note, if $f(\cdot)$ is c-almost periodic, then $f(\cdot)$ is almost periodic and therefore bounded (see [586]).

We will use the following lemma, which can be deduced with the help of [1078, Theorem 2.6] and [364, Theorem 3.36, Theorem 3.47; pp. 97-98].

Lemma 4.2.95. *Suppose that the sequence* $(f_n : \mathbb{R} \to X)$ *of asymptotically almost periodic functions (half-asymptotically almost periodic functions) converges uniformly to a function* $f : \mathbb{R} \to X$. *Then* $f(\cdot)$ *is asymptotically almost periodic (half-asymptotically almost periodic).*

For the sake of better readability, we will recall the notion of Schwartz distribution spaces as well as the basic definitions and results about vector-valued (asymptotically) almost periodic distributions (see also Subsection 4.1.5). Denote by $\mathcal{D}(X) = \mathcal{D}(\mathbb{R} : X)$ the Schwartz space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ with compact support in X. By $\mathcal{S}(X) = \mathcal{S}(\mathbb{R} : X)$ we denote the Schwartz space of all rapidly decreasing functions with values in X, and by $\mathcal{E}(X) = \mathcal{E}(\mathbb{R} : X)$ we denote the space of all infinitely differentiable functions with values in X; $\mathcal{D} \equiv \mathcal{D}(\mathbb{C})$, $\mathcal{S} \equiv \mathcal{S}(\mathbb{C})$ and $\mathcal{E} \equiv \mathcal{E}(\mathbb{C})$. The spaces of all linear continuous mappings from $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{E}$ into X are denoted by $\mathcal{D}'(X)$, $\mathcal{S}'(X)$ and $\mathcal{E}'(X)$, respectively [913]; $\mathcal{D}_0$ stands for the subspace of $\mathcal{D}$ consisting of all functions with the support contained in $[0,\infty)$. If $T \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}$, then we define $T * \varphi \in \mathcal{E}(X)$ by $(T * \varphi)(x) := \langle T, \varphi(x - \cdot)\rangle$. If $f : \mathbb{R} \to X$, then we define $\check{f} : \mathbb{R} \to X$ by $\check{f}(t) := f(-t)$, $t \in \mathbb{R}$; for any $T \in \mathcal{D}'(X)$, we define $\check{T} \in \mathcal{D}'(X)$ by $\langle \check{T}, \varphi\rangle := \langle T, \check{\varphi}\rangle$, $\varphi \in \mathcal{D}$.

Let $1 \leqslant p \leqslant \infty$. By $\mathcal{D}_{L^p}(\mathbb{R} : X)$ we denote the vector space consisting of all infinitely differentiable functions $f : \mathbb{R} \to X$ such that $f^{(j)} \in L^p(\mathbb{R} : X)$ for all $j \in \mathbb{N}_0$. The Fréchet topology on $\mathcal{D}_{L^p}(\mathbb{R} : X)$ is induced by the following system of norms:

$$\|f\|_k := \sum_{j=0}^{k} \|f^{(j)}\|_{L^p(\mathbb{R})}, \quad k \in \mathbb{N}.$$

If $X = \mathbb{C}$, then the above space is simply denoted by $\mathcal{D}_{L^p}$. The space of all linear continuous mappings $f : \mathcal{D}_{L^1} \to X$ is denoted by $\mathcal{D}'_{L^1}(X)$. Endowed with the strong topology, $\mathcal{D}'_{L^1}(X)$ becomes a complete locally convex space; $\mathcal{D}'_{L^1}(X)$ is a well-known space of bounded X-valued distributions. In the sequel, we will use the fact that a vector-valued distribution $T \in \mathcal{D}'(X)$ is bounded if and only if the function $T * \varphi$ is bounded for all $\varphi \in \mathcal{D}$; see, e. g., [300, Theorem 1.1].

Let $T \in \mathcal{D}'_{L^1}(X)$. Then the following assertions are equivalent [300]:
(i) $T * \varphi \in \mathrm{AP}(\mathbb{R} : X)$, $\varphi \in \mathcal{D}$.
(ii) There exist an integer $k \in \mathbb{N}$ and almost periodic functions $f_j(\cdot) : \mathbb{R} \to X$ $(0 \leqslant j \leqslant k)$ such that $T = \sum_{j=0}^{k} f_j^{(j)}$ in the distributional sense.

We say that a bounded distribution $T \in \mathcal{D}'_{L^1}(X)$ is almost periodic if and only if T satisfies any of the above two equivalent conditions; if this is the case, then the restriction of T to the space $\mathcal{S}$ is an X-valued tempered distribution [631]. By $B'_{\mathrm{AP}}(X)$ we denote the space consisting of all almost periodic distributions.

Define the space of bounded distributions tending to zero at plus infinity as follows:

$$B'_{+,0}(X) := \Big\{T \in \mathcal{D}'_{L^1}(X);\ \lim_{h \to +\infty} \langle T_h, \varphi \rangle = 0,\ \varphi \in \mathcal{D}\Big\},$$

where $\langle T_h, \varphi \rangle := \langle T, \varphi(\cdot - h) \rangle$, $T \in \mathcal{D}'(X)$, $h > 0$. A bounded distribution $T \in \mathcal{D}'_{L^1}(X)$ is said to be asymptotically almost periodic if and only if there exist an almost periodic distribution $T_{\mathrm{ap}} \in B'_{\mathrm{AP}}(X)$ and a bounded distribution tending to zero at plus infinity $Q \in B'_{+,0}(X)$ such that $\langle T, \varphi \rangle = \langle T_{\mathrm{ap}}, \varphi \rangle + \langle Q, \varphi \rangle$, $\varphi \in \mathcal{D}_0$. By $B'_{\mathrm{AAP}}(X)$ we denote the vector space consisting of all asymptotically almost periodic distributions (see, e. g., [646, Definition 1]).

Let $T \in \mathcal{D}'_{L^1}(X)$. Then the following assertions are equivalent (see, e. g., [646, Theorem 1]):
(i) $T \in B'_{\mathrm{AAP}}(X)$.
(ii) The function $T * \varphi$ is half-asymptotically almost periodic for all $\varphi \in \mathcal{D}_0$.
(iii) The function $T * \varphi$ is half-asymptotically almost periodic for all $\varphi \in \mathcal{D}$.
(iv) There exist an integer $k \in \mathbb{N}$ and half-asymptotically almost periodic functions $f_j(\cdot) : \mathbb{R} \to X$ $(0 \leqslant j \leqslant k)$ such that $T = \sum_{j=0}^{k} f_j^{(j)}$ on $[0, \infty)$, i. e.,

$$\langle T, \varphi \rangle = \sum_{j=0}^{k} (-1)^j \int_0^{\infty} \varphi^{(j)}(t) f_j(t)\, dt, \quad \varphi \in \mathcal{D}_0. \tag{4.51}$$

(v) There exists a sequence (T_n) of half-asymptotically almost periodic functions from $\mathcal{E}(X)$ such that $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'_{L^1}(X)$.

For the first time in the existing literature, we consider here the space

$$B'_0(X) := \Big\{T \in \mathcal{D}'_{L^1}(X);\ \lim_{|h|\to+\infty} \langle T_h, \varphi\rangle = 0,\ \varphi \in \mathcal{D}\Big\},$$

which is slightly different from the space $B'_{+,0}(X)$ used above. For example, the regular distribution determined by the locally integrable function $f : \mathbb{R} \to \mathbb{R}$, given by $f(t) := 1$ for $t \leqslant 0$ and $f(t) := 0$ for $t > 0$, belongs to the space $B'_{+,0}(X)$ but not to the space $B'_0(X)$. Since for every fixed test function $\varphi \in \mathcal{D}$ and for every real number $h \in \mathbb{R}$ we have

$$\big\langle \check{T}, \varphi(\cdot - h)\big\rangle = \big\langle T, \varphi(-\cdot - h)\big\rangle = \big\langle T, \check{\varphi}(\cdot - h)\big\rangle,$$

it follows that $T \in B'_0(X)$ if and only if $T \in B'_{+,0}(X)$ and $\check{T} \in B'_{+,0}(X)$. Therefore, [299, Proposition 1] immediately implies the following result (see also [211, Proposition 10]).

Proposition 4.2.96. *Suppose that $T \in \mathcal{D}'_{L^1}(X)$. Then the following statements are equivalent:*
(i) $T \in B'_0(X)$.
(ii) *The restrictions of functions $T * \varphi$ and $\check{T} * \varphi$ to the non-negative real axis belong to the space $C_0([0,\infty) : X)$ for all $\varphi \in \mathcal{D}$.*
(iii) *There exist an integer $k \in \mathbb{N}$ and functions $f_j \in C_0(\mathbb{R} : X)$ $(0 \leqslant j \leqslant k)$ such that $T = \sum_{j=0}^{k} f_j^{(j)}$.*
(iv) *There exists a sequence (T_n) in $\mathcal{E}'(X)$ which converges to T for topology of $\mathcal{D}'_{L^1}(X)$.*

We continue by introducing the following notion.

Definition 4.2.97. Let $T \in \mathcal{D}'(X)$ and $c \in \mathbb{C} \setminus \{0\}$.
(i) T is said to be a c-almost periodic (c-uniformly recurrent, semi-c-periodic) distribution, (AP_c) ((UR_c), (SAP_c)) distribution in short, if and only if $T * \varphi \in \mathrm{AP}_c(\mathbb{R} : X)$ ($T * \varphi \in \mathrm{UR}_c(\mathbb{R} : X)$, $T * \varphi \in \mathrm{SAP}_c(\mathbb{R} : X)$) for all $\varphi \in \mathcal{D}$. By $B'_{\mathrm{AP}_c}(X)$ ($B'_{\mathrm{UR}_c}(X)$, $B'_{\mathrm{SAP}_c}(X)$) we denote the space of all c-almost periodic (c-uniformly recurrent, semi-c-periodic) distributions;
(ii) T is said to be a (half-)asymptotically c-almost periodic ((half-)asymptotically c-uniformly recurrent, (half-)asymptotically semi-c-periodic) distribution if and only if the function $T * \varphi$ is (half-)asymptotically c-almost periodic ((half-)asymptotically c-uniformly recurrent, (half-)asymptotically semi-c-periodic) for all $\varphi \in \mathcal{D}$;
(iii) T is said to be a (half-)asymptotically $(\mathcal{D}_0, c)$-almost periodic ((half-)asymptotically $(\mathcal{D}_0, c)$-uniformly recurrent, (half-)asymptotically semi-$(\mathcal{D}_0, c)$-periodic) distribution if and only if the function $T * \varphi$ is (half-)asymptotically c-almost periodic ((half-)asymptotically c-uniformly recurrent, (half-)asymptotically semi-c-periodic) for all $\varphi \in \mathcal{D}_0$.

Remark 4.2.98. We have already introduced the notion of a semi-Bloch k-periodic function ($k \in \mathbb{R}$). The class of semi-Bloch k-periodic distributions can be also introduced but we will skip all related details concerning this notion for simplicity.

All distribution spaces introduced in Definition 4.2.97 are closed under differentiation. It is also clear that, if $T \in \mathcal{D}'(X)$ belongs to any of the spaces introduced above, then the distribution αT belongs to the same space, where $\alpha \in \mathbb{C}$ and $\langle \alpha T, \varphi \rangle := \langle T, \alpha\varphi \rangle$, $\varphi \in \mathcal{D}$; but, if $c \neq 1$, then the spaces of c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) are not closed under pointwise addition, which continues to hold for corresponding distribution spaces. Furthermore, since every c-almost periodic (semi-c-periodic) function is almost periodic and therefore bounded continuous, an application of [300, Theorem 1.1] shows that any c-almost periodic (semi-c-periodic) distribution is a bounded distribution. This is no longer true for the c-uniformly recurrent distributions because there exists an unbounded uniformly recurrent function [511] and therefore the regular distribution determined by this function is a uniformly recurrent distribution which is not a bounded distribution ($c = 1$).

We continue by stating the following.

Proposition 4.2.99. *Suppose that T is a c-uniformly recurrent distribution and $c \in \mathbb{C} \setminus \{0\}$ satisfies $|c| \neq 1$. Then $T \equiv 0$.*

Proof. By definition, we have $T * \varphi \in \mathrm{UR}_c(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}$. Since $|c| \neq 1$, Proposition 4.2.11 shows that $T * \varphi \equiv 0$ for all $\varphi \in \mathcal{D}$. This immediately implies $T = 0$. □

A distribution $T \in \mathcal{D}'(X)$ is called c-periodic if and only if $T * \varphi \in P_c(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}$. Similarly, we have the following.

Proposition 4.2.100. *Suppose that T is a semi-c-periodic distribution and $c \in \mathbb{C} \setminus \{0\}$ satisfies $|c| \neq 1$. Then T is c-periodic.*

Keeping in mind Proposition 4.2.99 and Proposition 4.2.100, it seems reasonable to impose the following condition.

Blank Hypothesis. Unless stated otherwise, we will always assume in the sequel of this subsection that $c \in \mathbb{C}$ and $|c| = 1$.

We continue by stating the following proposition.

Proposition 4.2.101. *The following statements are equivalent:*
(i) $T \in B'_{\mathrm{AP}_c}(X)$ ($T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$);
(ii) $\check{T} \in \mathcal{D}'_{\mathrm{AP}_{1/c}}(X)$ ($\check{T} \in \mathcal{D}'_{\mathrm{UR}_{1/c}}(X)$, $\check{T} \in \mathcal{D}'_{\mathrm{SAP}_{1/c}}(X)$).

Proof. Clearly, it suffices to show that (i) implies (ii). We will do that only for c-almost periodicity. Let $\varphi \in \mathcal{D}$ be fixed; we need to show that $\check{T} * \varphi \in \mathrm{AP}_{1/c}(\mathbb{R} : X)$. Keeping in mind Proposition 4.1.15, it suffices to show that

$$\check{T} * \varphi = T \check{*} \check{\varphi}, \quad \varphi \in \mathcal{D}. \tag{4.52}$$

To prove this equality, fix a real number $t \in \mathbb{R}$. Then (4.52) follows from the next simple computations:

$$\begin{aligned}(\check{T} * \varphi)(t) &= \langle \check{T}, \varphi(t - \cdot)\rangle = \langle T, \varphi(t \check{-} \cdot)\rangle = \langle T, \varphi(t + \cdot)\rangle,\\ T \check{*} \check{\varphi}(t) &= (T * \check{\varphi})(-t) = \langle T, \check{\varphi}(-t - \cdot)\rangle = \langle T, \varphi(t - \cdot)\rangle = \langle T, \varphi(t + \cdot)\rangle. \qquad \square\end{aligned}$$

We continue by observing that Proposition 4.2.70 and Proposition 4.2.14 directly imply the following: if $T \in B'_{\mathrm{AP}_c}(X)$ ($T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$), then $\|T * \varphi\| : I \to [0, \infty)$ is almost periodic (uniformly recurrent, semi-periodic) for all $\varphi \in \mathcal{D}$ as well as $T \in B'_{\mathrm{AP}_{c^l}}(X)$ ($T \in B'_{\mathrm{UR}_{c^l}}(X)$, $T \in B'_{\mathrm{SAP}_{c^l}}(X)$) for any positive integer $l \in \mathbb{N}$. Furthermore, Corollary 4.2.73, Proposition 4.2.74 and Proposition 4.2.17 directly imply the following:

(i) Suppose that

$$p \in \mathbb{Z} \setminus \{0\}, \quad q \in \mathbb{N}, \quad (p, q) = 1 \quad \text{and} \quad \arg(c) = \frac{p}{q}\pi$$

and $T \in B'_{\mathrm{AP}_c}(X)$ ($T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$).
   (a) If p is even, then $T \in B'_{\mathrm{AP}}(X)$ ($T \in B'_{\mathrm{UR}}(X)$, $T \in B'_{\mathrm{SAP}}(X)$).
   (b) If p is odd, then T is almost anti-periodic (uniformly anti-recurrent, semi-anti-periodic) distribution.

(ii) Suppose that $\arg(c)/\pi \notin \mathbb{Q}$ and $T \in B'_{\mathrm{AP}_c}(X)$. Then $T \in B'_{\mathrm{AP}_{c'}}(X)$ for all $c' \in S_1 \equiv \{z \in \mathbb{C} : |z| = 1\}$.

(iii) Suppose that $\arg(c)/\pi \in \mathbb{Q}$ and $T \in B'_{\mathrm{SAP}_c}(X)$. Then $T \in B'_{\mathrm{AP}_{c'}}(X)$ for all $c' \in \{c^l : l \in \mathbb{N}\}$.

(iv) Suppose that $\arg(c)/\pi \notin \mathbb{Q}$ and $T \in B'_{\mathrm{SAP}_c}(X)$. Then $T \in B'_{\mathrm{AP}_{c'}}(X)$ for all $c' \in S_1$.

The following statements known for functions can also be simply deduced for distributions:

(i) Suppose that $c = 1$. Then the set consisting of all c-almost periodic distributions is a vector space together with the usual operations, while the set consisting of c-uniformly recurrent distributions and the set consisting of semi-c-periodic distributions are not vector spaces together with the usual operations.

(ii) Suppose that $c \neq 1$. Then the set consisting of all c-almost periodic (c-uniformly recurrent, semi-c-periodic) distributions is not a vector space together with the usual operations.

It is worthwhile to mention that all established statements concerning the pointwise products of c-almost periodic type functions with the scalar-valued functions can be reformulated for the pointwise products of c-almost periodic type distributions with the scalar-valued infinitely differentiable functions; concerning Stepanov classes of c-almost periodic type functions, it should be noticed that [646, Proposition 1] continues to hold in our new framework. Details can be left to the interested reader.

We proceed by stating the following simple result.

Proposition 4.2.102. *Let $h \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$ and $T \in B'_{\mathrm{AP}_c}(X)$ ($T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$). Then:*

(i) *Any translation T_h of $T \in B'_{\mathrm{AP}_c}(X)$ ($T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$) belongs to $B'_{\mathrm{AP}_c}(X)$ ($B'_{\mathrm{UR}_c}(X)$, $B'_{\mathrm{SAP}_c}(X)$).*

(ii) *Define the distribution Tb by $\langle Tb, \varphi\rangle := \langle T, \varphi(b\cdot)\rangle$, $\varphi \in \mathcal{D}$. Then $Tb \in B'_{\mathrm{AP}_c}(X)$ ($Tb \in B'_{\mathrm{UR}_c}(X)$, $Tb \in B'_{\mathrm{SAP}_c}(X)$).*

Proof. We will prove the proposition only for c-almost periodicity. To show (i), suppose that $T \in B'_{\mathrm{AP}_c}(X)$ and $\varphi \in \mathcal{D}$. Then we know that $T * \varphi \in \mathrm{AP}_c(I : X)$. Due to the first part of [586, Theorem 2.13(iv)], the above implies that the function $x \mapsto \langle T, \varphi(x+h-\cdot)\rangle$, $x \in \mathbb{R}$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic). Now the conclusion follows from the calculation

$$(T_h * \varphi)(x) = \langle T_h, \varphi(x-\cdot)\rangle = \langle T_h, \varphi(x-\cdot)\rangle = \langle T, \varphi(x+h-\cdot)\rangle, \quad x \in \mathbb{R}.$$

To show (ii), define the test function $\varphi^b(\cdot)$ by $\varphi^b(t) := \varphi(bt)$, $t \in \mathbb{R}$. Then $G^b := T * \varphi^b \in \mathrm{AP}_c(\mathbb{R} : X)$ and the required conclusion follows from the second part of Theorem 4.2.75(iv) and the calculation

$$\begin{aligned}(T^b * \varphi)(t) &= \langle T^b, \varphi(t-\cdot)\rangle = \langle T, \varphi(t-b\cdot)\rangle \\ &= \langle T, \varphi(b((t/b)-\cdot))\rangle = \langle T, \varphi^b((t/b)-\cdot)\rangle = G^b(t/b), \quad t \in \mathbb{R}.\end{aligned}$$ □

The following result is a distributional analogue of Proposition 4.2.23.

Proposition 4.2.103. *Let $T \in B'_{\mathrm{UR}_c}(X)$ ($T \in B'_{\mathrm{SAP}_c}(X)$) and $T \neq 0$. Then $T \notin B'_{+,0}(X)$.*

Proof. Since $T \neq 0$, there exists $\varphi \in \mathcal{D}$ such that $T * \check{\varphi} \neq 0$. Clearly, $T * \check{\varphi}$ is a c-uniformly recurrent function (semi-c-periodic function), so that [586, Proposition 2.18] implies $T * \check{\varphi} \notin C_0(\mathbb{R} : X)$. Assume to the contrary that $T \in B'_{+,0}(X)$. Then we have

$$(T * \check{\varphi})(t) = \langle T, \varphi(\cdot - t)\rangle \to 0 \quad \text{as } t \to +\infty,$$

which is a contradiction. □

The following result will be important in our further analyses.

Theorem 4.2.104. *Suppose that there exist an integer $k \in \mathbb{N}$ and c-almost periodic (c-uniformly recurrent, semi-c-periodic) functions $f_j : \mathbb{R} \to X$ ($0 \leqslant j \leqslant k$) such that the function*

$$t \mapsto f(t) \equiv (f_0(t), \ldots, f_k(t)), \quad t \in \mathbb{R} \tag{4.53}$$

is c-almost periodic (c-uniformly recurrent, semi-c-periodic). Define $T := \sum_{j=0}^{k} f_j^{(j)}$. Then $T \in B'_{\mathrm{AP}_c}(X)$ ($T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$).

Proof. We will prove the theorem only for c-almost periodicity because the proofs for c-uniform recurrence and semi-c-periodicity are quite analogous. It is clear that $T \in \mathcal{D}'(X)$, and that (4.51) shows that for each $\varphi \in \mathcal{D}$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}(T * \varphi)(t) = \langle T, \varphi(t - \cdot)\rangle &= \sum_{j=0}^{k} \int_{-\infty}^{+\infty} \varphi^{(j)}(t - v) f_j(v)\, dv \\ &= \sum_{j=0}^{k} \int_{-\infty}^{+\infty} \varphi^{(j)}(v) f_j(t - v)\, dv. \qquad (4.54)\end{aligned}$$

Let $\varepsilon > 0$ be given. Then the set $\theta_c(f, \varepsilon)$ is relatively dense in $[0, \infty)$; let $\tau \in \theta_c(f, \varepsilon)$ be arbitrary. Then the above computation shows that

$$\begin{aligned}\|(T * \varphi)(t + \tau) - c(T * \varphi)(t)\| &\leqslant \sum_{j=0}^{k} \int_{-\infty}^{+\infty} |\varphi^{(j)}(v)| \cdot \|f_j(t + \tau - v) - cf_j(t - v)\|\, dv \\ &\leqslant \varepsilon \sum_{j=0}^{k} \int_{-\infty}^{+\infty} |\varphi^{(j)}(v)|\, dv, \quad \varphi \in \mathcal{D},\ t \in \mathbb{R},\end{aligned}$$

which simply implies the required statement. □

The following counterexample demonstrates the fact that Theorem 4.2.104 does not generally hold if the function $f(\cdot)$, defined by (4.53), is not c-almost periodic (c-uniformly recurrent, semi-c-periodic); we will provide a direct non-trivial calculation showing this.

Example 4.2.105. Suppose that $c = -1$, $k = 1$, $f_0(t) = \cos t$ and $f_1(t) = \cos(2t)$ for all $t \in \mathbb{R}$. Then the function $f(\cdot)$, defined by (4.53), is not almost anti-periodic (see [666, Example 2.2(ii)]) and we have

$$\langle T, \varphi\rangle = \int_{-\infty}^{+\infty} \varphi(v) \cos v\, dv - \int_{-\infty}^{+\infty} \varphi'(v) \cos(2v)\, dv, \quad \varphi \in \mathcal{D}.$$

Due to (4.54), we have

$$\begin{aligned}&(T * \varphi)(t + \tau) + (T * \varphi)(t) \\ &\quad = \int_{-\infty}^{+\infty} \varphi(v)[\cos(t + \tau - v) + \cos(t - v)]\, dv \\ &\qquad + \int_{-\infty}^{+\infty} \varphi'(v)[\cos(2(t + \tau - v)) + \cos(2(t - v))]\, dv, \quad \varphi \in \mathcal{D},\ t \in \mathbb{R}.\end{aligned}$$

Applying the partial integration, the above implies

$$\begin{aligned}(T * \varphi)(t+\tau) + (T * \varphi)(t) \\ = \int_{-\infty}^{+\infty} \varphi(v)\big[\cos(t+\tau-v) + \cos(t-v) - 2\sin\big(2(t+\tau-v)\big) - 2\sin\big(2(t-v)\big)\big]\,dv,\end{aligned} \tag{4.55}$$

for any $\varphi \in \mathcal{D}$ and $t \in \mathbb{R}$. Suppose that $\varphi \in \mathcal{D}$ is non-negative and its support belongs to the interval $[1/6, 1/4] \subset [0, 1/4]$ and that

$$0 < \varepsilon < \int_0^{1/4} \varphi(v) \min\big((\sin v)/2, \cos v, 2\sin(2v)\big)\,dv \tag{4.56}$$

and

$$0 < \varepsilon < \frac{2}{3}\int_0^{1/4} \varphi(v)\,dv \cdot \sin\left(\frac{\pi}{4} - \frac{1}{8}\right), \quad 0 < \varepsilon < 2\sin\frac{1}{12} \cdot \cos\frac{1}{4}\int_0^{1/4} \varphi(v)\,dv. \tag{4.57}$$

We will prove that the set $\theta_{-1}(T * \varphi, \varepsilon)$ is empty in the following, a rather technical, way. Suppose to the contrary that $\tau \in \theta_{-1}(T * \varphi, \varepsilon)$. Then (4.55) implies

$$\left|\int_0^{1/4} \varphi(v)\big[\cos(t+\tau-v) + \cos(t-v) - 2\sin\big(2(t+\tau-v)\big) - 2\sin\big(2(t-v)\big)\big]\,dv\right| < \varepsilon \tag{4.58}$$

for all $t \in \mathbb{R}$. Plugging $t = -\tau$, $t = \pi - \tau$ and $t = (\pi/2) - \tau$ in (4.58), we get

$$\left|\int_0^{1/4} \varphi(v)\big[\cos(v) + \cos(\tau+v) + 2\sin(2v) + 2\sin\big(2(\tau+v)\big)\big]\,dv\right| < \varepsilon, \tag{4.59}$$

$$\left|\int_0^{1/4} \varphi(v)\big[-\cos(v) - \cos(\tau+v) + 2\sin(2v) + 2\sin\big(2(\tau+v)\big)\big]\,dv\right| < \varepsilon \tag{4.60}$$

and

$$\left|\int_0^{1/4} \varphi(v)\big[\sin(v) + \sin(\tau+v) - 2\sin(2v) - 2\sin\big(2(\tau+v)\big)\big]\,dv\right| < \varepsilon, \tag{4.61}$$

respectively. Adding and subtracting of (4.59) and (4.60), we get

$$\left|\int_0^{1/4} \varphi(v)\big[\cos(v) + \cos(\tau+v)\big]\,dv\right| < \varepsilon \tag{4.62}$$

and

$$\left| \int_0^{1/4} \varphi(v)\big[\sin(2v) + \sin\big(2(\tau + v)\big)\big]\, dv \right| < \varepsilon/2, \tag{4.63}$$

respectively. Inserting (4.63) in (4.61), we get

$$\left| \int_0^{1/4} \varphi(v)\big[\sin(v) + \sin(\tau + v)\big]\, dv \right| < 2\varepsilon. \tag{4.64}$$

Furthermore, there exist $k \in \mathbb{N} \cup \{-1, 0\}$ and $a \in [0, 2\pi)$ such that $\tau = (2k + 1)\pi + a$. Then (4.62)–(4.64) give

$$\left| \int_0^{1/4} \varphi(v)\big[\cos(v) - \cos(v + a)\big]\, dv \right| < \varepsilon, \tag{4.65}$$

$$\left| \int_0^{1/4} \varphi(v)\big[\sin(2v) + \sin(2a + 2v)\big]\, dv \right| < \varepsilon/2, \tag{4.66}$$

and

$$\left| \int_0^{1/4} \varphi(v)\big[\sin(v) - \sin(v + a)\big]\, dv \right| < 2\varepsilon. \tag{4.67}$$

If $a \in [0, (\pi/2) - (1/4)]$, then $2a + 2v \in [0, \pi/2]$ for all $v \in [0, 1/4]$ and the contradiction is obvious due to our choice of the value of ε in (4.56) and the estimate (4.66); if $a \in [\pi, 2\pi - (1/4)]$, then $a + v \in [\pi, 2\pi]$ for all $v \in [0, 1/4]$ and the contradiction is obvious due to our choice of number ε in (4.56) and the estimate (4.67). Furthermore, if $a \in [\pi/2, \pi]$, then $a + v \in [\pi/2, 3\pi/2]$ for all $v \in [0, 1/4]$ and the contradiction is obvious due to our choice of the value of ε in (4.56) and the estimate (4.65). If $a \in [(\pi/2) - (1/4), \pi/2]$, then the estimates (4.65) and (4.67) imply

$$\left| \int_0^{1/4} \varphi(v) \sin\big(v + (a/2)\big) \cdot \sin(a/2)\, dv \right| < \varepsilon/2$$

and

$$\left| \int_0^{1/4} \varphi(v) \cos\big(v + (a/2)\big) \cdot \sin(a/2)\, dv \right| < \varepsilon,$$

respectively. By adding, we get

$$\left| \int_0^{1/4} \varphi(v)\big[\sin\big(v + (a/2)\big) + \cos\big(v + (a/2)\big)\big]\, dv \right| < \frac{3}{2}\varepsilon \cdot \big[\sin(a/2)\big]^{-1} \leqslant \frac{3}{2\sin(\frac{\pi}{4} - \frac{1}{8})}\varepsilon,$$

which is a contradiction due to our choice of ε in (4.57) and the fact that $a+v \leqslant (\pi+1)/4$ for all $v \in [0,1/4]$ and therefore $\sin(v+(a/2)) + \cos(v+(a/2)) \geqslant 1$ for all $v \in [0,1/4]$. Finally, if $a \in [2\pi - (1/4), 2\pi)$, then

$$\begin{aligned}
&\int_0^{1/4} \varphi(v)\big[\sin(2v) + \sin(2a+2v)\big]\,dv \\
&= 2\int_0^{1/4} \varphi(v)\sin(2v+a)\cdot\cos(a)\,dv = 2\int_{1/6}^{1/4} \varphi(v)\sin(2v+a)\cdot\cos(a)\,dv \\
&\geqslant 2\sin\frac{1}{12}\cdot\cos\frac{1}{4}\int_0^{1/4}\varphi(v)\,dv,
\end{aligned}$$

which contradicts the second inequality in (4.57).

We continue by stating the following structural characterization of the space $B'_{\mathrm{AP}_c}(X)$ ($B'_{\mathrm{UR}_c}(X)$, $B'_{\mathrm{SAP}_c}(X)$).

Theorem 4.2.106. *Let $T \in B'_{\mathrm{AP}_c}(X)$ ($T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$) and let T be a bounded distribution. Then there exist an integer $p \in \mathbb{N}$ and a c-almost periodic (bounded c-uniformly recurrent, semi-c-periodic) function $F : \mathbb{R} \to X$ such that*

$$T = \sum_{j=0}^{p}(-1)^j\binom{p}{j}F^{(2j)} \tag{4.68}$$

in the distributional sense.

Proof. The proof essentially follows from the argumentation contained in the proof of [210, Theorem 1]; we will only outline the main details for c-almost periodicity because the proofs for c-uniform recurrence and semi-c-periodicity are quite analogous. Let us consider a fundamental solution G of the differential operator $(1 - d^2/dx^2)^p$ for a certain sufficiently large natural number $p \in \mathbb{N}$ depending on T. By the proof of the above-mentioned theorem, we see that the convolution $F := T * G$ exists as a continuous function and (4.68) holds in the distributional sense; furthermore, there exists a sequence (φ_k) in $\mathcal{D}$ such that $\lim_{k\to+\infty}(T * \varphi_k)(t) = F(t)$, uniformly in $t \in \mathbb{R}$. Since for each integer $k \in \mathbb{N}$ the function $(T * \varphi_k)(\cdot)$ is c-almost periodic (apply also [300, Theorem 1.1] for c-uniform recurrence), an application of Theorem 4.2.75(iii) shows that $F(\cdot)$ is c-almost periodic, as well. This completes the proof. □

Now we are able to formulate and prove the following result.

Theorem 4.2.107. *Suppose that $T \in \mathcal{D}'_{L^1}(X)$. Then the following statements are equivalent:*

(i) *We have* $T \in B'_{\mathrm{AP}_c}(X)$ *(*$T \in B'_{\mathrm{UR}_c}(X)$, $T \in B'_{\mathrm{SAP}_c}(X)$*).*
(ii) *There exist an integer* $p \in \mathbb{N}$ *and a c-almost periodic (bounded c-uniformly recurrent, semi-c-periodic) function* $F : \mathbb{R} \to X$ *such that* (4.68) *holds in the distributional sense.*
(iii) *There exist an integer* $k \in \mathbb{N}$ *and c-almost periodic (bounded c-uniformly recurrent, semi-c-periodic) functions* $f_j : \mathbb{R} \to X$ $(0 \leqslant j \leqslant k)$ *such that the function* $f(\cdot)$*, defined through* (4.53)*, is c-almost periodic (c-uniformly recurrent, semi-c-periodic) and* $T = \sum_{j=0}^{k} f_j^{(j)}$.
(iv) *There exists a sequence* (T_n) *of c-almost periodic functions (bounded c-uniformly recurrent functions, semi-c-periodic functions) from* $\mathcal{E}(X)$ *such that* $\lim_{n\to\infty} T_n = T$ *in* $\mathcal{D}'_{L^1}(X)$.

Proof. The implication (i) ⇒ (ii) is proved in Theorem 4.2.106, while the implication (ii) ⇒ (iii) is trivial. The implication (iii) ⇒ (i) follows from Theorem 4.2.104; therefore, we have proved the equivalence of statements (i), (ii) and (iii). Their equivalence with (iv) essentially follows from the argumentation contained in the proof of [210, Proposition 7]; see also the proof of Theorem 4.2.112 below. □

As a direct consequence of Theorem 4.2.107 (see also [646, Remark 1(ii)]), we have the following.

Corollary 4.2.108. *Let* (T_n) *be a sequence in* $B'_{\mathrm{AP}_c}(X)$ $(B'_{\mathrm{UR}_c}(X) \cap \mathcal{D}'_{L^1}(X), B'_{\mathrm{SAP}_c}(X))$*, and let* $\lim_{n\to\infty} T_n = T$ *in* $\mathcal{D}'_{L^1}(X)$*. Then* $T \in B'_{\mathrm{AP}_c}(X)$ $(T \in B'_{\mathrm{UR}_c}(X) \cap \mathcal{D}'_{L^1}(X), T \in B'_{\mathrm{SAP}_c}(X))$*.*

For the sequel, we need the following definition.

Definition 4.2.109. Suppose that $T \in \mathcal{D}'(X)$.
(i) We say that T is an asymptotically c-almost periodic distribution of type 1 (asymptotically c-uniformly recurrent distribution of type 1, asymptotically semi-c-periodic distribution of type 1) if and only if there exist a c-almost periodic (c-uniformly recurrent, semi-c-periodic) distribution $T_{\mathrm{apc}} \in B'_{\mathrm{AP}_c}(X)$, $(T_{\mathrm{urc}} \in B'_{\mathrm{UR}_c}(X)$, $T_{\mathrm{sapc}} \in B'_{\mathrm{SAP}_c}(X))$ and a distribution $Q \in B'_0(X)$ such that $\langle T, \varphi\rangle = \langle T_{\mathrm{apc}}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}$, $(\langle T, \varphi\rangle = \langle T_{\mathrm{urc}}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}$, $\langle T, \varphi\rangle = \langle T_{\mathrm{sapc}}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D})$.
(ii) We say that T is an asymptotically $(\mathcal{D}_0, c)$-almost periodic distribution of type 1 (asymptotically $(\mathcal{D}_0, c)$-uniformly recurrent distribution of type 1, asymptotically semi-$(\mathcal{D}_0, c)$-periodic distribution of type 1) if and only if there exist a c-almost periodic (c-uniformly recurrent, semi-c-periodic) distribution $T_{\mathrm{apc}} \in B'_{\mathrm{AP}_c}(X)$, $(T_{\mathrm{urc}} \in B'_{\mathrm{UR}_c}(X)$, $T_{\mathrm{sapc}} \in B'_{\mathrm{SAP}_c}(X))$ and a distribution $Q \in B'_0(X)$ such that $\langle T, \varphi\rangle = \langle T_{\mathrm{apc}}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$, $(\langle T, \varphi\rangle = \langle T_{\mathrm{urc}}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$, $\langle T, \varphi\rangle = \langle T_{\mathrm{sapc}}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0)$.

Remark 4.2.110. Concerning Definition 4.2.109(ii), it should be noted that it is completely irrelevant whether we will write $Q \in B'_0(X)$ or $Q \in B'_{+,0}(X)$ here because any element $Q \in B'_{+,0}(X)$ can be extended to an element $\tilde{Q} \in B'_0(X)$ by the formula $\tilde{Q} := F \cdot Q$, where $F \in C^{\infty}(\mathbb{R})$ is any fixed function satisfying $F(t) = 1$ for all $t \geqslant 0$ and $F(t) = 0$ for all $t \leqslant -1$.

Remark 4.2.111. We note that the decompositions in Definition 4.2.109 are unique in the case of consideration of c-almost periodicity (semi-c-periodicity) because they are unique for almost periodicity [646].

Now we will prove the following asymptotical analogue of Theorem 4.2.107, which gives some new insights at the assertion of [646, Theorem 1] and [211, Theorem 2] (in the last mentioned theorem, C. Bouzar and F. Z. Tchouar have recently established a structural characterization for the space of asymptotically almost automorphic distributions following the approach of I. Cioranescu from [299] (see also [646, Theorem 2]); our novelty here is the use of approach obeyed in the proof of [210, Proposition 7], with a direct proof of implication (i) $\Rightarrow$ (ii) and a new characterization (iii) for the class of vector-valued asymptotically almost automorphic distributions).

Theorem 4.2.112. *Suppose that $T \in \mathcal{D}'_{L^1}(X)$. Then the following statements are equivalent:*

(i) *T is (half-)asymptotically $(\mathcal{D}_0, c)$-almost periodic ((half-)asymptotically semi-$(\mathcal{D}_0, c)$-periodic).*

(ii) *T is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic).*

(iii) *There exist an integer $p \in \mathbb{N}$ and a bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) function $F : \mathbb{R} \to X$ such that* (4.68) *holds in the distributional sense.*

(iii)′ *There exist an integer $p \in \mathbb{N}$ and a bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) function $F : \mathbb{R} \to X$ such that* (4.68) *holds in the distributional sense (*(4.68) *holds in the distributional sense on* $[0, \infty)$*).*

(iv) *There exist an integer $k \in \mathbb{N}$ and bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) functions $f_j : \mathbb{R} \to X$ $(0 \leqslant j \leqslant k)$ such that the function $f(\cdot)$, defined through* (4.53)*, is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) and $T = \sum_{j=0}^{k} f_j^{(j)}$.*

(iv)′ *There exist an integer $k \in \mathbb{N}$ and bounded (half-)asymptotically c-almost periodic (bounded (half-)asymptotically semi-c-periodic) functions $f_j : \mathbb{R} \to X$ $(0 \leqslant j \leqslant k)$ such that the function $f(\cdot)$, defined through* (4.53)*, is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) and $T = \sum_{j=0}^{k} f_j^{(j)}$ ($T = \sum_{j=0}^{k} f_j^{(j)}$ on* $[0, \infty)$*).*

(v) *T is an asymptotically c-almost periodic distribution of type* 1 *(asymptotically semi-c-periodic distribution of type* 1*), in the case of consideration of asymptotical c-almost periodicity (asymptotical semi-c-periodicity), resp. T is an asymptotically $(\mathcal{D}_0, c)$-almost periodic distribution of type* 1 *(asymptotically semi-$(\mathcal{D}_0, c)$–*

periodic distribution of type 1*), in the case of consideration of half-asymptotical c-almost periodicity (half-asymptotical semi-c-periodicity).*

(vi) *There exists a sequence (T_n) of bounded (half-)asymptotically c-almost periodic functions (bounded (half-)asymptotically semi-c-periodic functions) from $\mathcal{E}(X)$ such that* $\lim_{n\to\infty} T_n = T$ *in* $\mathcal{D}'_{L^1}(X)$.

Proof. We will prove the implication (i) ⇒ (ii) only for half-asymptotical $(\mathcal{D}_0, c)$-almost periodicity. Let $\varphi \in \mathcal{D}$ be given and let $\operatorname{supp}(\varphi) \subseteq [a,b]$. If $a \geqslant 0$, then $\varphi \in \mathcal{D}_0$ and therefore the function $T * \varphi$ is half-asymptotically c-almost periodic, as required. If $a < 0$, then we consider the function $\varphi_a(\cdot) := \varphi(\cdot + a) \in \mathcal{D}_0$. Since the convolution mapping is translation invariant, we see that the function $(T * \varphi)_a(\cdot) = (T * \varphi_a)(\cdot)$ is half-asymptotically c-almost periodic, so that there exist a c-almost periodic function $g : \mathbb{R} \to X$ and a function $h \in C_0([0,\infty) : X)$ such that $(T*\varphi)_a(t) = (T*\varphi_a)(t) = g(t)+h(t)$ for all $t \geqslant 0$. This implies $(T * \varphi)(t) = g(t-a) + h(t-a) := g_a(t) + h_a(t)$, $t \geqslant a$. It is clear that the restriction of the function $h_a(\cdot)$ to the non-negative real axis belongs to the space $C_0([0,\infty) : X)$, so that the statement (ii) follows by applying Theorem 4.2.75(iv) with $I = [0,\infty)$ and the number a replaced therein with the number $-a > 0$. The implication (ii) ⇒ (iii) can be proved following the lines of the proof of Theorem 4.2.107; we will use the same notation. As in the proof of the above-mentioned result, we see that $\lim_{k\to+\infty}(T * \varphi_k)(t) = F(t)$, uniformly in $t \in \mathbb{R}$; due to [300, Theorem 1.1], the function $F(\cdot)$ is bounded. In the newly arisen situation, the function $(T * \varphi_k)(\cdot)$ is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) for all integers $k \in \mathbb{N}$. Therefore, there exist a c-almost periodic function (semi-c-periodic function) $g_k : \mathbb{R} \to X$ and a function $h_k \in C_0(\mathbb{R} : X)$, resp. $h_k \in C_0([0,\infty) : X)$, such that $(T * \varphi_k)(t) = g_k(t) + h_k(t)$, $t \in \mathbb{R}$, resp. $(T * \varphi_k)(t) = g_k(t) + h_k(t)$, $t \geqslant 0$ $(k \in \mathbb{N})$. Since for each integer $k \in \mathbb{N}$ the function $g_k(\cdot)$ is almost periodic, Lemma 4.2.95 shows that there exist an almost periodic function $g : \mathbb{R} \to X$ and a function $\phi \in C_0(\mathbb{R} : X)$, resp. $\phi \in C_0([0,\infty) : X)$, such that $F(t) = g(t)+\phi(t)$ for all $t \in \mathbb{R}$, resp. $F(t) = g(t)+\phi(t)$ for all $t \geqslant 0$. But the argumentation contained in the proofs of [364, Theorem 3.36, Theorem 3.47; pp. 97–98] also shows that the sequence of functions (g_k) converges to the function $g(\cdot)$, uniformly on $\mathbb{R}$. Since for each integer $k \in \mathbb{N}$ the function $g_k(\cdot)$ is c-almost periodic (semi-c-periodic), an application of Theorem 4.2.75(iii) shows that the function $g(\cdot)$ is also c-almost periodic (semi-c-periodic). This implies (iii). The implications (iii) ⇒ (iv) ⇒ (iv)$'$ are trivial. We will prove that (iv)$'$ implies (v) only for half-asymptotical c-almost periodicity. It simply follows that there exist c-almost periodic functions $g_j : \mathbb{R} \to X$ and functions $h_j \in C_0([0,\infty) : X)$ $(0 \leqslant j \leqslant k)$ such that the function $t \mapsto (g_0(t),\dots,g_k(t))$, $t \in \mathbb{R}$ is c-almost periodic as well as that $f_j(t) = g_j(t) + h_j(t)$ for all $t \geqslant 0$. Define $T_{\mathrm{apc}} \in B'_{\mathrm{AP}_c}(X)$ by

$$T_{\mathrm{apc}}(\varphi) := \sum_{j=0}^{k}(-1)^j \int_{-\infty}^{+\infty} \varphi^{(j)}(v) g_j(v)\, dv, \quad \varphi \in \mathcal{D},$$

(see Theorem 4.2.104) and

$$Q(\varphi) := \sum_{j=0}^{k} (-1)^j \int_{-\infty}^{+\infty} \varphi^{(j)}(v) h_j^e(v)\, dv, \quad \varphi \in \mathcal{D},$$

where $h_j^e(\cdot)$ denotes the even extension of the function $h_j(\cdot)$ to the whole real axis. It is clear that we have $\langle T, \varphi\rangle = \langle T_{\mathrm{apc}}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$. In order to see that $Q \in B_0'(X)$, it suffices to observe that, for every test function $\varphi \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subseteq [a, b]$, we see that

$$\langle Q, \varphi(\cdot - h)\rangle := \sum_{j=0}^{k} \int_{a}^{b} \varphi^{(j)}(v) h_j^e(v + h)\, dv, \quad \varphi \in \mathcal{D},\ h \in \mathbb{R},$$

and therefore $\lim_{|h|\to+\infty} \langle Q, \varphi(\cdot - h)\rangle = 0$, $\varphi \in \mathcal{D}$. In order to see that (v) implies (i), it suffices to repeat verbatim the argumentation given in [646, Remark 2]. We will prove that (vi) implies (i) only for half-asymptotical c-almost periodicity. Using the argumentation contained in the proof of [210, Proposition 7], it suffices to show that, for every fixed function $\varphi \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subseteq [0, b]$ and for every fixed bounded half-asymptotically c-almost periodic function $f : \mathbb{R} \to X$, the function $\varphi * f$ is bounded and half-asymptotically c-almost periodic. This is clear for boundedness; in order to see that the function $\varphi * f$ is half-asymptotically c-almost periodic, we can argue as follows. Let $g : \mathbb{R} \to X$ be a c-almost periodic function and let $h \in C_0([0, \infty) : X)$ such that $f(t) = g(t) + h(t)$ for all $t \geqslant 0$. Then we have

$$(\varphi * f)(t) = \int_{-\infty}^{+\infty} \varphi(s) g(t - s)\, ds + \int_{0}^{b} \varphi(s) h(t - s)\, ds, \quad t \geqslant b,$$

so that the final conclusion follows from the fact that the space consisting of all c-almost periodic functions is convolution invariant and that the function

$$t \mapsto (\varphi * f)(t) - \int_{-\infty}^{+\infty} \varphi(s) g(t - s)\, ds, \quad t \geqslant 0,$$

belongs to the class $C_0([0, \infty) : X)$, which a simple consequence of the last equality. The implication (i) $\Rightarrow$ (vi) follows directly from the corresponding part of the proof of [210, Proposition 7]. Therefore, we have proved the equivalence of all statements (i)–(vi). Since (iii)$'$ implies (iv)$'$ and (iv)$'$ implies (v), we see that (iii)$'$ or (iv)$'$ implies all other statements (i)–(vi). On the other hand, it is clear that (iii) implies (iii)$'$, finishing the proof. □

Corollary 4.2.113. *Let (T_n) be a sequence of bounded (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic) distributions, and let* $\lim_{n\to\infty} T_n = T$ *in $\mathcal{D}'_{L^1}(X)$. Then T is (half-)asymptotically c-almost periodic ((half-)asymptotically semi-c-periodic).*

Remark 4.2.114.
(i) It is worth noting that the implications (i) ⇒ (ii) and (iii) ⇒ (iv) ⇒ (v) ⇒ (i) and the equivalence (vi) ⇔ (i) can be formulated for bounded (half-)asymptotically c-uniformly recurrent functions, but it is not clear how one can prove that (ii) implies (iii) in this framework.
(ii) Using the idea from the proof of implication (i) ⇒ (ii) of Theorem 4.2.112, we may conclude that a distribution $T \in \mathcal{D}'(X)$ is c-periodic (c-almost periodic, c-uniformly recurrent, semi-c-periodic) if and only if the function $T*\varphi$ is c-periodic (c-almost periodic, c-uniformly recurrent, semi-c-periodic) for all $\varphi \in \mathcal{D}_0$.
(iii) If $c \neq 1$, then it is not clear how we can introduce and analyze the classes of c-almost automorphic functions and c-almost automorphic distributions.

Let $n \in \mathbb{N}$, and let $A = [a_{ij}]_{1\leqslant i,j\leqslant n}$ be a given complex matrix such that $\sigma(A) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. Following the analysis of C. Bouzar and M. T. Khalladi [207], we will provide here a small application in the analysis of the existence of half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) solutions of equation

$$T' = AT + G, \quad T \in \mathcal{D}'(X^n) \text{ on } [0,\infty), \tag{4.69}$$

where G is a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) X^n-valued distribution; applications can be also made to certain classes of functional-differential equations (see also the interesting research papers [747] by N. M. Man, N. V. Minh and [921, 922] by I. F. Shahpazova concerning this issue). By a solution of (4.69), we mean any element $T \in \mathcal{D}'(X^n)$ such that (4.69) holds in the distributional sense on $[0,\infty)$. Since the spaces of half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) distributions are not closed under the pointwise addition of functions, some obvious unpleasant difficulties occur in the case that $c \neq 1$. In the one-dimensional case, these difficulties can be overcome, fortunately.

Theorem 4.2.115. *Suppose that F is a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) distribution, $n = 1$ and $a_{11} = \lambda < 0$. Then there exists a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) distributional solution of* (4.69). *Furthermore, any distributional solution T of* (4.69) *is half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic).*

Proof. By Theorem 4.2.112, we know that there exist an integer $p \in \mathbb{N}$ and a bounded half-asymptotically c-almost periodic (bounded half-asymptotically semi-c-periodic) function $F : \mathbb{R} \to X$ such that (4.68) holds in the distributional sense, with T replaced with G therein. By the proof of [646, Theorem 4], given in the ultradistributional case, we get the existence of a positive integer $m \in \mathbb{N}$, continuous functions $F_j : [0,\infty) \to X$ $(0 \leqslant j \leqslant m)$ and a function $Q \in C_0([0,\infty) : X)$ such that any function $F_j(\cdot)$ has the form

$$F_j(t) = c_{1,j}(\lambda)F(t) + c_{2,j}(\lambda)\int_0^t e^{\lambda(t-s)}F(s)\,ds, \quad t \geqslant 0,$$

for certain complex numbers $c_{1,j}(\lambda)$ and $c_{2,j}(\lambda)$ $(0 \leqslant j \leqslant m)$ and $T = Q + \sum_{j=0}^{m} F_j^{(j)}$ on $[0,\infty)$. By the proofs of [631, Proposition 2.6.11] and [337, Lemma 4.1], we see that the function $t \mapsto (F_0(t),\ldots,F_m(t))$, $t \geqslant 0$ is half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) so that it can be uniquely extended to a half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) function $t \mapsto (\tilde{F_0}(t),\ldots,\tilde{F_m}(t))$, $t \in \mathbb{R}$ due to Proposition 4.2.29. Define $T_0 := \sum_{j=0}^{m} \tilde{F_j}^{(j)}$ and $T_1 := Q_e$, where Q_e denotes the even extension of the function $Q(\cdot)$ to the whole real axis. Then $T = T_0 + T_1$ on $[0,\infty)$, T is c-almost periodic (semi-c-periodic) and $T_1 \in B_0'(X)$, so that T is half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic). The existence of solutions is proved in a similar fashion. □

Unfortunately, the use of [207, Lemma 1] and the arguments contained in the proof of [207, Theorem 3, pp. 117-118] do not enable us to extend Theorem 4.2.115 to the multi-dimensional case. Keeping in mind the proofs of [646, Theorem 4] and Theorem 4.2.115, we can only prove the following.

Theorem 4.2.116. *Let there exist an integer $m \in \mathbb{N}$ and half-asymptotically c-almost periodic (half-asymptotically semi-c-periodic) X^n-valued functions $G_j(\cdot)$ $(0 \leqslant j \leqslant m)$ such that $G = \sum_{j=0}^{m} G_j^{(j)}$ on $[0,\infty)$. Then there exists a solution T of (4.69) which has the same form as G; furthermore, any distributional solution T of (4.69) has the same form as G (with the meaning being clear).*

5 Notes and appendices to Part I

In this chapter, we will briefly consider several important topics which have not been discussed in the previous part of this monograph.

Recurrent strongly continuous semigroups

The notion of a uniformly recurrent operator is closely connected with the notion of a recurrent operator in a complex Banach space X. Let us recall that a linear operator $T : X \to X$ is called recurrent if and only if for every non-empty open subset U of X there exists some $k \in \mathbb{N}$ such that $U \cap T^{-k}(U) \neq \emptyset$. A vector $x \in X$ is said to be recurrent for T if and only if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to +\infty$; the set consisting of all recurrent vectors of T will be denoted by $\mathrm{Rec}(T)$. A much stronger notion than the recurrence is the measure theoretic rigidity, introduced in the ergodic theoretic setting by H. Furstenberg and B. Weiss ([459]; see also [458]). This concept, in the context of topological dynamical systems, is known as (uniform) rigidity; it was introduced by S. Glasner and D. Maon [472]. We say that a bounded linear operator $T : X \to X$ is rigid if and only if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to +\infty$, for every $x \in X$. A bounded linear operator $T : X \to X$ is called uniformly rigid if and only if there exists an increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that

$$\|T^{k_n} - I\| = \sup_{\|x\|\leqslant 1} \|T^{k_n}x - x\| \to 0 \quad \text{as } n \to +\infty.$$

For more details about recurrent and rigid operators on Banach spaces, see the research articles [316] by G. Costakis, A. Manoussos, I. Parissis and [317] by G. Costakis, I. Parissis.

For families of bounded linear operators, we will use the following notion (the material is taken from [274], a joint research study with C.-C. Chen and D. Velinov).

Definition 5.0.1. Let $I = [0, \infty)$ or $I = \mathbb{R}$. We say that a family $(W(t))_{t\in I}$ of bounded linear operators on X is recurrent if and only if for every open non-empty set $U \subseteq X$ there exists some $t \in I$ such that $U \cap (W(t))^{-1}(U) \neq \emptyset$. A vector $x \in X$ is called a recurrent vector for $(W(t))_{t\in I}$ if and only if there exists an unbounded sequence of numbers (t_k) in I such that $W(t_k)x \to x$ as $k \to +\infty$. By $\mathrm{Rec}(W(t))$ we denote the set consisting of all recurrent vectors for $(W(t))_{t\in I}$.

Definition 5.0.2. We say that a family $(W(t))_{t\in I}$ of bounded linear operators on X is rigid if and only if there exists an unbounded sequence of numbers (t_k) in I such that $W(t_k)x \to x$ as $k \to +\infty$, for every $x \in X$, i. e., $W(t_k) \to I$ as $k \to +\infty$ in the strong operator topology, while $(W(t))_{t\in I}$ is called uniformly rigid if and only if there exists an unbounded sequence (t_k) in I such that $\|W(t_k) - I\| \to 0$ as $k \to \infty$.

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The following result is fundamental.

Theorem 5.0.3. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0,\infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. The following statements are equivalent:*
(i) $(T(t))_{t\in I}$ *is recurrent.*
(ii) $\overline{\mathrm{Rec}\,(T(t))} = X$.

If this is the case, the set of recurrent vectors for $(T(t))_{t\in I}$ is a G_δ-subset of X.

Proof. First we will show that (ii) $\Rightarrow$ (i). Let $\overline{\mathrm{Rec}\,(T(t))} = X$ and U be an arbitrary open non-empty subset in X. Let x be a recurrent vector and $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$, where $B(x,\varepsilon) = \{y \in X : \|x - y\| < \varepsilon\}$. Then there exists $t \in I$ such that $\|T(t)x - x\| < \varepsilon$. Thus $x \in U \cap T(t)(U) \neq \emptyset$, so $(T(t))_{t\in I}$ is recurrent. Now, we will show that (i) $\Rightarrow$ (ii). Let $(T(t))_{t\in I}$ be recurrent and let $B = B(x,\varepsilon)$ be an open ball in X, for fixed $x \in X$ and $\varepsilon < 1$. The proof will end if we show that there exists a recurrent vector in B. We use the recurrence property of $(T(t))_{t\in I}$. So, there exists $t_1 \in I$ such that $x_1 \in B \cap T(t_1)^{-1}(B)$, for some $x_1 \in E$. Since $(T(t))_{t\in I}$ is strongly continuous, we see that there exists $\varepsilon_1 < 1/2$ such that $B_2 = B(x_1,\varepsilon_1) \subseteq B \cap T(t_1)^{-1}(B)$. Since $(T(t))_{t\in I}$ is recurrent, there exists $t_2 \in I$ with $|t_2| > |t_1|$ and some $x_2 \in E$ such that $x_2 \in B_2 \cap T(t_2)^{-1}(B_2)$. Using the same argument with the strong continuity and recurrence of $(T(t))_{t\in I}$, we can inductively construct a sequence (x_n) in X, an unbounded sequence (t_n) in I and a decreasing sequence of positive real numbers (ε_n), such that for every integer $n \in \mathbb{N}$ one has $\varepsilon_n < 2^{-n}$,

$$B(x_n,\varepsilon_n) \subseteq B(x_{n-1},\varepsilon_{n-1}) \quad \text{and} \quad T(t_n)\big(B(x_n,\varepsilon_n)\big) \subseteq B(x_{n-1},\varepsilon_{n-1}).$$

By Cantor's theorem we have

$$\bigcap_{n=1}^{\infty} B(x_n,\varepsilon_n) = \{y\},$$

for some $y \in X$. It is clear that $T(t_n)y \to y$ as $n \to +\infty$. Hence $y \in B$ is a recurrent vector in the open ball B, so the proof of (ii) $\Rightarrow$ (i) is finished. Let us prove that

$$\mathrm{Rec}\big(T(t)\big) = \bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\left\{x \in X : \big\|T(q_n)x - x\big\| < \frac{1}{k}\right\} =: \mathrm{R}(T(t)), \tag{5.1}$$

where (q_n) denotes the sequence consisting of all rational numbers which do have the modulus strictly greater than 1. It simply follows that $\mathrm{Rec}(T(t))$ is contained in the set $\mathrm{R}(T(t))$. For the opposite inclusion, for each element $x \in \mathrm{R}(T(t))$ and for each integer $k \in \mathbb{N}$ we can pick up a rational number q_k which do have the module strictly greater than 1 and for which $\|T(q_k)x - x\| < 1/k$. If the sequence (q_k) is unbounded, we have done. If not, then there exists a convergent subsequence (q_{n_k}) of (q_n) such that $\lim_{k\to\infty} q_{n_k} = q$ for some real number $q \in I$ such that $|q| \geqslant 1$. In this case, the strong continuity of $(T(t))_{t\in I}$ shows that $x = T(q)x$ so that clearly $x \in \mathrm{Rec}(T(t))$ because, in this case, we have $T(nq)x = x$ for all $n \in \mathbb{N}$. Hence, (5.1) holds and $(T(t))_{t\in I}$ is a G_δ subset of X. ☐

Using the representation formula (5.1) and the proof of [316, Proposition 2.6], it can be easily shown that the following result holds good.

Theorem 5.0.4. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group on X. Then $(T(t))_{t\geqslant 0}$ is recurrent if and only if $(T(-t))_{t\geqslant 0}$ is recurrent.*

We continue by stating the following continuous analogue of [316, Proposition 2.3(i)].

Theorem 5.0.5. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0,\infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. Then, for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have* $\operatorname{Rec}(T(t)) = \operatorname{Rec}(\lambda T(t))$.

Proof. It is enough to show that $\operatorname{Rec}(T(t)) \subseteq \operatorname{Rec}(\lambda T(t))$. For $x \in \operatorname{Rec}(T(t))$, we define the set $L = \{|\mu| = 1 : \lambda^n T(t_n)x \to \mu x, \text{for some unbounded sequence } (t_n) \text{ in } I\}$. To finish the proof, we have to prove that $1 \in L$. First of all, let us note that $L \neq \emptyset$. Since $x \in \operatorname{Rec}(T(t))$, there exists an unbounded sequence (t_n) in I such that $T(t_n)x \to x$. There exists a subsequence of (t_n), denoted by (t_{n_k}), such that $\lambda^{t_{n_k}} \to \rho$ as $k \to \infty$, for some $|\rho| = 1$. Hence, we have $\lambda^{t_{n_k}} T(t_{n_k})x \to \rho x$ as $k \to \infty$, which means that $\rho \in L$. Let $\mu_1, \mu_2 \in L$ and $\varepsilon > 0$ be fixed. Since $\mu_1 \in L$, there exist a positive integer $n_1 \in \mathbb{N}$ and a real number $t_1 \in I$, with modulus sufficiently large, such that

$$\|\lambda^{n_1} T(t_1)x - \mu_1 x\| < \frac{\varepsilon}{2}.$$

Since $\mu_2 \in L$, there are a positive integer $n_2 \in \mathbb{N}$ and a real number $t_2 \in I$, with module sufficiently large, such that

$$\|\lambda^{n_2} T(t_2)x - \mu_2 x\| < \frac{\varepsilon}{2\|T(t_1)\|}.$$

Hence,

$$\begin{aligned}\|\lambda^{n_1+n_2} T(t_1+t_2)x - \mu_1\mu_2 x\| &\leqslant \|\lambda^{n_1} T(t_1)(\lambda^{n_2} T(t_2)x - \mu_2 x)\| + \|\mu_2(\lambda^{n_1} T(t_1)x - \mu_1 x)\| \\ &\leqslant \|T(t_1)\|\|(\lambda^{n_2} T(t_2))x - \mu_2 x\| + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

so that $\mu_1\mu_2 \in L$. Hence, $\mu^n \in L$ for $\mu \in L$. If μ is a rational rotation, this means that $1 \in L$ and we are done. If μ is an irrational rotation, there is a strictly increasing sequence of positive integers (s_k) such that $\mu^{s_k} \to 1$. Since L is closed, it follows that $1 \in L$. □

Theorem 5.0.6. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0,\infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. If $(T(t) \oplus T(t))_{t\in I}$ is recurrent, then $(T(t))_{t\in I}$ is likewise recurrent.*

Proof. Let $x_1 \oplus x_2$ be a recurrent vector for $(T(t) \oplus T(t))_{t\in I}$. Then it is clear that x_1 and x_2 are recurrent vectors for $(T(t))_{t\in I}$; hence, $(T(t))_{t\in I}$ is recurrent. □

The question whether the direct sum $(T(t) \oplus T(t))_{t\in I}$ of recurrent strongly continuous operator families $(T(t))_{t\in I}$ is recurrent is not simple. The answer is affirmative if $(T(t))_{t\in I}$ possesses some extra properties (see [316] for more details about the single-valued case).

The following continuous analogue of [316, Proposition 2.3(ii)] appears in this monograph for the first time.

Theorem 5.0.7. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group. Then the following assertions are equivalent:*
(i) $(T(t))_{t\geqslant 0}$ *is recurrent.*
(ii) *For every $t_0 > 0$, the operator $T(t_0)$ is recurrent.*
(iii) *There exists $t_0 > 0$ such that the operator $T(t_0)$ is recurrent.*

If this is the case, then, for every $t_0 \in I \smallsetminus \{0\}$, we have $\mathrm{Rec}(T(t)) = \mathrm{Rec}(T(t_0))$.

Proof. All non-trivial that we need to show is that (i) implies (ii), with the equality $\mathrm{Rec}(T(t)) = \mathrm{Rec}(T(t_0))$ for any fixed number $t_0 > 0$. To see this, assume that $(T(t))_{t\geqslant 0}$ is a recurrent C_0-semigroup. Then it is clear that $\mathrm{Rec}(T(t)) \supseteq \mathrm{Rec}(T(t_0))$ and, owing to Theorem 5.0.3, all that we need to prove is that the preassumption $x \in \mathrm{Rec}(T(t))$ implies $x \in \mathrm{Rec}(T(t_0))$. Without loss of generality, we can assume that $t_0 = 1$. Indeed, we can consider the semigroup $(\tilde{T}(t))_{t\geqslant 0}$, with $\tilde{T}(t) := T(tt_0)$, for every $t \geqslant 0$. It is clear that x is a recurrent vector for $(\tilde{T}(t))_{t\geqslant 0}$ and $\tilde{T}(1) = T(t_0)$. Denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$ and define the mapping $\phi : [0,\infty) \to \mathbb{T}$ by $\phi(t) := e^{2\pi it}$, $t \geqslant 0$. For every $u \in X$, we define the set

$$F_u := \Big\{\lambda \in \mathbb{T} : \exists (t_n)_n \in (0,\infty) \text{ s. t. } \lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} T(t_n)u = u \text{ and } \lim_{n\to\infty} \phi(t_n) = \lambda\Big\}.$$

Note that the set F_u is not empty by its definition and the recurrence property of the semigroup $(T(t))_{t\geqslant 0}$. The set F_u is closed for $u \in X$, as can be easily proved. Next, we will prove that, if $u \in X$ and $\lambda,\ \mu \in F_u$, then $\lambda\mu \in F_u$. Let U be an open balanced neighborhood of zero in X and $\varepsilon > 0$ arbitrary. Then we can find $t_1 > 0$ such that $\|T(t_1)u - \lambda u\| \leqslant \varepsilon/2$ and $|\phi(t_1) - \mu| < \varepsilon/2$. Choose an open balanced neighborhood of zero V in X and number $t_2 > 0$ such that $T(t_1)(V) \subseteq U$, $T(t_2)u - \mu u \in V$ and $|\phi(t_2) - \lambda| < \varepsilon/2$. Hence,

$$\begin{aligned} T(t_1 + t_2)u - \lambda\mu u &= T(t_1)\big(T(t_2)u - \mu u\big) + \mu\big(T(t_1)u - \lambda u\big) \\ &\in T(t_1)(V) + B(0,\varepsilon/2) \subseteq U + B(0,\varepsilon/2), \end{aligned}$$

so that

$$\big|\phi(t_1 + t_2) - \lambda\mu\big| = \big|\phi(t_1)\phi(t_2) - \lambda\mu\big| \leqslant \big|\phi(t_1) - \mu\big| \cdot \big|\phi(t_2)\big| + |\mu| \cdot \big|\phi(t_2) - \lambda\big| < \varepsilon.$$

This simply implies that $\lambda\mu \in F_u$ as claimed. Furthermore, it is clear that there exists $x \in (-\pi,\pi]$ such that $e^{ix} = \lambda \in F_u$. If x is rational, then using the fact that F_u is closed under multiplication immediately gives $1 \in F_u$. If x is not rational, then F_u is dense in $\mathbb{T}$ since it contains the set $\{e^{inx} : n \in \mathbb{N}\}$ so that $1 \in F_u$ again. Hence, $1 \in F_u$. Suppose now $u \in \operatorname{Rec}(T(t))$. Then we have the existence of a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers tending to infinity such that $\lim_{n\to\infty} T(t_n)u = u$ and $\lim_{\to\infty} \phi(t_n) = 1$. Let (k_n) be a sequence of positive integers and $\varepsilon_n \in [-1,1]$ such that $t_n = k_n + \varepsilon_n$ for all $n \in \mathbb{N}$. Obviously, $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, $\|T(k_n)u - u\| \leqslant \|T(-\varepsilon_n)[T(t_n)u - u] + [T(-\varepsilon_n)u - u]\| \leqslant \sup_{\xi\in[-1,1]} \|T(\xi)\| \cdot \|T(t_n)u - u\| + \|T(-\varepsilon_n)u - u\| \to 0$ as $n \to +\infty$. As a consequence, we have $u \in \operatorname{Rec}(T(1))$. □

Remark 5.0.8. Condition that $(T(t))_{t\geqslant 0}$ can be extended to a C_0-group seems to be slightly redundant. Due to [829, Theorem 6.5, p. 24], this is the case provided that there exists a finite number $t_0 > 0$ such that $[T(t_0)]^{-1} \in L(X)$.

Suppose that $\Delta = [0,\infty)$ or $\Delta = \mathbb{R}$. A measurable function $\rho : \Delta \to (0,\infty)$ is said to be an admissible weight function if and only if there exist constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\rho(t) \leqslant Me^{\omega|t'|}\rho(t+t')$ for all $t,\ t' \in \Delta$. Let us introduce the Banach spaces

$$L^p_\rho(\Delta,\mathbb{C}) := \{u : \Delta \to \mathbb{C}\,; u(\cdot) \text{ is measurable and } \|u\|_p < \infty\},$$

where $p \in [1,\infty)$ and $\|u\|_p := (\int_\Delta |u(t)|^p \rho(t)\, dt)^{1/p}$, and

$$C_{0,\rho}(\Delta,\mathbb{C}) := \Big\{u : \Delta \to \mathbb{K}\,; u(\cdot) \text{ is continuous and } \lim_{t\to\infty} u(t)\rho(t) = 0\Big\},$$

with $\|u\| := \sup_{t\in\Delta} |u(t)\rho(t)|$. For any function $f : \Delta \to \mathbb{C}$, we define $T(t)f := f(\cdot+t)$, $t \in \Delta$. If $\rho(\cdot)$ is an admissible weight function and $\Delta = [0,\infty)$, resp. $\Delta = \mathbb{R}$, then the translation semigroup, resp. group, $(T(t))_{t\in\Delta}$ is strongly continuous on $L^p_\rho(\Delta,\mathbb{C})$ and $C_{0,\rho}(\Delta,\mathbb{C})$. Recently, Z. Yin and Y. Wei have considered the weak recurrence of translation operators on weighted Lebesgue spaces and weighted continuous function spaces [1060]. They have shown that the existence of a function $f \in X$, where $X = L^p_\rho([0,\infty),\mathbb{C})$ or $X = C_{0,\rho}([0,\infty),\mathbb{C})$, satisfying that there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that (compare with (2.3))

$$\lim_{n\to+\infty} \big\|f(\cdot + \alpha_n) - f(\cdot)\big\|_X = 0$$

is equivalent to saying that $\liminf_{t\to+\infty} \rho(t) = 0$ (the hypercyclicity of $(T(t))_{t\geqslant 0}$); see also the preprint [219] by W. Brian and J. P. Kelly.

For more details about recurrent sets of operators, we refer the reader to the recent paper [57] by M. Amouch and O. Benchiheb.

Lower and upper densities

In Subsection 2.4.1, we have used various notions of lower and upper densities for a subset $A \subseteq [0,\infty)$ which can take, generally speaking, any value in the range $[0,\infty]$. Without any doubt, the most important densities are those ones with values in the

range $[0,1]$. As in the discrete case, the minimal conditions which should satisfy any lower or upper density $d : P([0,\infty)) \to [0,1]$ are: $d(\emptyset) = 0$, $d([0,\infty)) = 1$ and $d(A) \leqslant d(B)$, whenever $A, B \subseteq [0,\infty)$ and $A \subseteq B$. But some other axioms are needed for obtaining a good definition of density. For example, following A. R. Freedman and J. J. Sember [455] we can consider the upper density $\delta^\star(\cdot) : P([0,\infty)) \to [0,1]$ with the following properties:
(l1) $\delta^\star(A \cup B) \leqslant \delta^\star(A) + \delta^\star(B)$;
(l2) $\delta^\star(A) = \delta^\star(B)$, provided that $A\Delta B$ is bounded;
(l3) $\delta_\star(A) \leqslant \delta^\star(A)$.

It is also worth noting that we can consider the upper density $\nu^\star : P([0,\infty)) \to [0,1]$ with the following properties introduced recently by P. Leonetti and S. Tringali in the discrete case [693]:
(f1) $\nu^\star(A \cup B) \leqslant \nu^\star(A) + \nu^\star(B)$;
(f2) $\nu^\star(\alpha A) = \alpha^{-1}\nu^\star(A)$, provided that $\alpha > 0$;
(f3) $\nu^\star(A + \alpha) = \nu^\star(A)$, provided that $\alpha > 0$.

Besides that, it could be of some importance to analyze many other notions of lower and upper densities in the continuous setting, like the notions of upper logarithmic, upper Buck, upper Pólya or upper analytic densities (see also the classical studies by A. S. Besicovitch [168–170], the monograph [353] by C. De Lellis and the doctoral dissertation of N. F. G. Martin [751]). For further information, see also [481, 632] and the references cited therein.

Remotely almost periodic solutions of ordinary differential equations

In this section, we will briefly describe the notion and results obtained in our recent joint paper with C. Maulén, S. Castillo and M. Pinto [757]. For the sake of brevity, we will not include the proofs of structural results regarding remotely almost periodic solutions of ordinary differential equations (besides the already mentioned research articles and monographs concerning this issue, we want also to recommend to the reader [13, 17, 37, 60, 71, 145–148, 178, 238, 254, 329, 354–356, 398, 449, 452, 457, 475, 490, 576, 676, 734, 762] and [784–788, 820, 830, 852, 893, 924, 1050]).

To better understand the space of remotely almost periodic functions, denoted by $\mathrm{RAP}(\mathbb{R} : \mathbb{C}^n)$, we will recall the notion of a slowly oscillating function (the corresponding space is denoted by $\mathrm{SO}(\mathbb{R} : \mathbb{C}^n)$ henceforth): A function $f \in \mathrm{BUC}(\mathbb{R} : \mathbb{C}^n)$ is called slowly oscillating if and only if for every $a \in \mathbb{R}$ we have

$$\lim_{|t|\to+\infty} \left\| f(t+a) - f(t) \right\| = 0.$$

Now we recall the notion of a remotely almost periodic function.

Definition 5.0.9. A function $f \in BUC(\mathbb{R} : \mathbb{C}^n)$ is called remotely almost periodic if and only if $\varepsilon > 0$ we see that the set

$$T(f, \varepsilon) := \Big\{\tau \in \mathbb{R} : \limsup_{|t| \to +\infty} \|f(t + \tau) - f(t)\| < \varepsilon\Big\}$$

is relatively dense in $\mathbb{R}$.

Any number $\tau \in T(f, \varepsilon)$ is called an ε-remote-translation vector of $f(\cdot)$. It will be assumed here that any remotely almost periodic function $f : \mathbb{R} \to \mathbb{R}$ possesses the mean value

$$\mathcal{M}(f) := \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(s)\, ds.$$

Consider now the following systems of differential equations:

$$\frac{dx}{dt} = A(t)x(t) \tag{5.2}$$

and

$$\frac{dx}{dt} = A(t)x(t) + f(t), \tag{5.3}$$

where $A(t)$ is a complex-valued matrix of format $n \times n$ for all $t \in \mathbb{R}$.

In [757], we have used the following notion.

Definition 5.0.10. Let $\Phi(\cdot)$ be a fundamental matrix of Eq. (5.2). Then we say that Eq. (5.2) has an (α, K, P)-exponential dichotomy if and only if there exist positive constants α, $K > 0$ and a projection P ($P^2 = P$) such that

$$\|G(t, s)\| \leqslant Ke^{-\alpha|t-s|}, \quad t,\ s \in \mathbb{R},$$

where the Green function $G(t, s)$ of (5.2) is given by $G(t, s) := \Phi(t)P\Phi^{-1}(s)$ for $t \geqslant s$ and $G(t, s) := -\Phi(t)[I - P]\Phi^{-1}(s)$ for $t < s$.

The notion of bi-almost periodicity of the Green function, which has been omitted or less considered for a long time, plays a crucial role in our study.

Definition 5.0.11. We say that the Green function $G(t, s)$ of (5.2) is exponentially bi-almost periodic if and only if for all $\varepsilon > 0$ there exist positive real constants $\alpha' > 0$, $c > 0$ and a relatively dense set $T(G, \varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G, \varepsilon)$, we have

$$\|G(t + \tau, s + \tau) - G(t, s)\| \leqslant \varepsilon c e^{-\alpha'|t-s|}, \quad t,\ s \in \mathbb{R}.$$

We have also used the following notions.

Definition 5.0.12. We say that the Green function $G(t, s)$ of (5.2) is integro bi-almost periodic if and only if for all $\varepsilon > 0$ there exist a positive real constant $c > 0$ and a relatively dense set $T(G, \varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G, \varepsilon)$, we have

$$\int_{-\infty}^{+\infty} \|G(t+\tau, s+\tau) - G(t, s)\| \, ds \leqslant \varepsilon c, \quad t \in \mathbb{R}.$$

Definition 5.0.13. Let $\alpha > 0$. Then we say that the Green function $G(t, s)$ of (5.2) is α-exponentialy bi-remotely almost periodic if and only if for every $\varepsilon > 0$ there exist a positive real constant $c > 0$ and a relatively dense set $T(G, \varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G, \varepsilon)$, we have

$$\limsup_{|t|\to\infty} \|e^{\alpha(t-s)}[G(t+\tau, s+\tau) - G(t, s)]\| \leqslant \varepsilon c, \quad t,\ s \in \mathbb{R},\ t \geqslant s,$$

and

$$\limsup_{|t|\to\infty} \|e^{\alpha(s-t)}[G(t+\tau, s+\tau) - G(t, s)]\| \leqslant \varepsilon c, \quad t,\ s \in \mathbb{R},\ t < s.$$

Definition 5.0.14. Let $\alpha > 0$. Then we say that the Green function $G(t, s)$ of (5.2) is integro bi-remotely almost periodic if and only if for every $\varepsilon > 0$ there exist a positive real constant $c > 0$ and a relatively dense set $T(G, \varepsilon)$ in $\mathbb{R}$ such that, for every $\tau \in T(G, \varepsilon)$, we have

$$\limsup_{|t|\to\infty} \int_{-\infty}^{+\infty} \|G(t+\tau, s+\tau) - G(t, s)\| \, ds \leqslant \varepsilon c, \quad t \in \mathbb{R}.$$

The following results have been established.

Theorem 5.0.15. *If $a(\cdot)$ is a remotely almost periodic function with $\mathcal{M}(a) \neq 0$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $\tau \in T(a, \delta)$, we have*

$$\limsup_{|t|\to\infty} \int_{-\infty}^{t} \left|e^{\int_{s+\tau}^{t+\tau} a(r)\,dr - \int_s^t a(r)\,dr}\right| ds < \varepsilon, \quad \textit{provided } t < s \textit{ and } \mathcal{M}(a) < 0$$

and

$$\limsup_{|t|\to\infty} \int_{t}^{+\infty} \left|e^{\int_{s+\tau}^{t+\tau} a(r)\,dr - \int_s^t a(r)\,dr}\right| ds < \varepsilon, \quad \textit{provided } t \geqslant s \textit{ and } \mathcal{M}(a) > 0.$$

Theorem 5.0.16. *Suppose that $f \in \mathrm{RAP}(\mathbb{R} : \mathbb{C}^n)$ and the homogeneous system* (5.2) *has an (α, K, P)-exponential dichotomy and the associated Green function is integro bi-remotely almost periodic. Then the unique bounded solution of* (5.3) *is remotely almost periodic.*

Theorem 5.0.17. *Let* $f \in \mathrm{RAP}(\mathbb{R} : \mathbb{R}^n)$ *and let* $g(\cdot)$ *be remotely almost periodic in the first variable and locally Lipschitz in the second variable. Suppose, further, that the homogeneous system* (5.2) *has an* (α, K, P)*-exponential dichotomy and the associated Green function is integro bi-remotely almost periodic. Then there exists a positive constant* μ_0 *such that the assumption* $\mu \in [0, \mu_0)$ *implies that the differential equation*

$$z'(t) = A(t)z(t) + f(t) + \mu g(t, z(t)) \tag{5.4}$$

has a unique bounded solution which is remotely almost periodic.

We have also analyzed the Richard–Chapman equation with an external perturbation $f(\cdot)$:

$$x'(t) = x(t)[a(t) - b(t)x^{\theta}(t)] + f(t), \tag{5.5}$$

where $\theta \geqslant 0$. Consider the following hypotheses:
(H1) $a(t)$, $b(t)$ and $f(t)$ are remotely almost periodic functions;
(H2) $0 < \alpha \leqslant a(t) \leqslant A$, $0 < \beta \leqslant b(t) \leqslant B$, $0 < f(t) < F$;
(H3) With $\omega = A^{-1}[\beta - \gamma^{(1+\theta)/\theta}F]$ and $\gamma = B/\alpha$, we have $(1+\theta)F\gamma^{1/\theta}\theta^{-1}\alpha^{-1} < 1$ and $\beta(1+\theta)B\theta^{-1} < 1$.

We have proved the following result on the existence and uniqueness of positive remotely almost periodic solutions to (5.5).

Theorem 5.0.18. *Suppose that the hypotheses* (H1)–(H3) *hold. Then Eq.* (5.5) *has a unique remotely almost periodic solution* $\phi^*(t)$ *satisfying* $\gamma^{-1/\theta} \leqslant \phi^* \leqslant \omega^{-1/\theta}$ *for all* $t \in \mathbb{R}$.

Almost periodic functions of complex variables

The theory of almost periodic functions of one complex variable, initiated already by H. Bohr in the third part of [196], is still very popular and attracts the attention of numerous mathematicians (see, e. g., [426, 559, 917, 918]). Suppose that $-\infty \leqslant \alpha < \beta \leqslant +\infty$ and the function $f : \Omega \equiv \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\} \to X$ is analytic. Then we say that $f(\cdot)$ is almost periodic if and only if for any $\varepsilon > 0$ and every reduced strip $\{z \in \mathbb{C} : \alpha' < \operatorname{Re} z < \beta'\}$, where $\alpha < \alpha' < \beta' < \beta$, there exists a number $l > 0$ such that each subinterval of length l of $\mathbb{R}$ contains a number τ satisfying the inequality

$$\|f(z + i\tau) - f(z)\| \leqslant \varepsilon \quad \text{for } \alpha' < \operatorname{Re} z < \beta'.$$

In particular, this definition implies that, for any fixed $\sigma \in (\alpha, \beta)$, the function $f_\sigma(t) := f(\sigma + it)$, $t \in \mathbb{R}$ is almost periodic. Moreover, the definition implies that the almost periodicity should be uniform on the various straight lines, with the meaning clear. The Fourier series of these functions can be obtained from a certain exponential series with complex coefficients; the associated series is called the Dirichlet series of $f(\cdot)$.

As for the functions of one real variable, Bohr's notion of almost periodicity of $f(\cdot)$ in a vertical strip Ω is equivalent to the relative compactness of the set of its vertical translates, $\{f(\cdot + ih) : h \in \mathbb{R}\}$, with the topology of the uniform convergence on reduced strips. Mean motions and zeros of generalized almost periodic analytic functions have been analyzed by V. Borchsenius and B. Jessen in [200], where the reader can find several important applications to the Riemann zeta function (see also [783] and the references there for further information about applications of results from the theory of almost periodic analytic functions to the Riemann zeta function).

We would like to accent that the notions of uniform recurrence and $\odot_g$-almost periodicity for the functions of one real variable can be simply modified and introduced for the functions of one complex variable. For more details about almost periodic analytic functions of several complex variables, we refer the reader to [430, 877] and the references therein.

$C^{(n)}$-almost periodic functions

The notion of $C^{(n)}$-almost periodicity was introduced by M. Adamczak [16] in 1997 and later received a great attention of many other authors. In this monograph, we will not consider $C^{(n)}$-almost periodic type functions and solutions of integro-differential equations; we shall only say a few words about generalized $C^{(n)}$-almost periodic functions and possibilities for further expansions.

Several different classes of Stepanov-like $C^{(n)}$-pseudo almost automorphic functions have been analyzed by T. Diagana, V. Nelson and G. M. N'Guérékata in [374]. For example, let $1 \leqslant p < \infty$, let $n \in \mathbb{N}$, and let $f \in L^p_{\mathrm{loc}}(I : X)$.

(i) We say that the function $f(\cdot)$ is Stepanov-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - \mathrm{APS}^p(I : X)$ for short, if and only if for each $k = 0, 1, \ldots, n$, we have $f^{(k)} \in \mathrm{APS}^p(I : X)$.

(ii) We say that the function $f \in L^p_{\mathrm{loc}}([0, \infty) : X)$ is asymptotically Stepanov-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - \mathrm{AAPS}^p([0, \infty) : X)$ for short, if and only if for each $k = 0, 1, \ldots, n$, we have $f^{(k)} \in \mathrm{AAPS}^p([0, \infty) : X)$. The following definitions have been analyzed in [631]:

(iii) We say that the function $f(\cdot)$ is equi-Weyl p-$C^{(n)}$-almost periodic, $f \in e - C^{(n)} - W^p_{\mathrm{ap}}(I : X)$ for short, if and only if for each $k = 0, 1, \ldots, n$, we have $f^{(k)} \in e - W^p_{\mathrm{ap}}(I : X)$.

(iv) We say that the function $f(\cdot)$ is Weyl p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - W^p_{\mathrm{ap}}(I : X)$ for short, if and only if for each $k = 0, 1, \ldots, n$, we have $f^{(k)} \in W^p_{\mathrm{ap}}(I : X)$.

(v) We say that he function $f(\cdot)$ is Besicovitch–Doss p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - \mathrm{B}^p(I : X)$ for short, if and only if for each $k = 0, 1, \ldots, n$, we have $f^{(k)} \in \mathrm{B}^p(I : X)$.

Using the same idea, we can introduce and analyze a great number of $C^{(n)}$-almost automorphic function spaces [631]. For example, the function

$$f(t) = \sum_{n=1}^{\infty} \frac{\sin nt}{n^4}, \quad t \in \mathbb{R},$$

is $C^{(2)}$-almost periodic but not $C^{(3)}$-almost automorphic. Furthermore, for any real-valued function $g \in C^{(3)} - AA(\mathbb{R} : \mathbb{C})$ satisfying $\inf_{t\in\mathbb{R}} g'''(t) > 0$, we see that the function

$$f(t) = \sum_{n=1}^{\infty} \frac{g(nt)}{n^4}, \quad t \in \mathbb{R},$$

belongs to the space $C^{(2)}\text{-AAS}^1(\mathbb{R} : \mathbb{C}) \setminus C^{(3)}\text{-AAS}^1(\mathbb{R} : \mathbb{C})$; see e. g. [374, Example 2.23]. It is clear that we can slightly generalize the notion of all above-mentioned function spaces by using the definitions and results from the theory of $L^{p(x)}$-spaces.

Riemann–Stepanov almost periodicity, Riemann–Weyl almost periodicity and Riemann–Besicovitch almost periodicity

In [396], R. Doss has analyzed the classes of Riemann–Stepanov almost periodic functions, Riemann–Weyl almost periodic functions and Riemann–Besicovitch almost periodic functions. All considerations in this paper are carried out with the scalar-valued functions.

Following [396, Definition 1], we say that an essentially bounded function $f : I \to X$ is Riemann–Stepanov almost periodic if and only if for every $\varepsilon > 0$ there exist $\delta > 0$ and numbers $\pi_1 \in I, \ldots, \pi_m \in I$ such that

$$\sup_{x\in I} \overline{\int_x^{x+1}} \|f(t+\tau_t) - f(t)\|\,dt < \varepsilon \tag{5.6}$$

provided that $|\tau_t| < \delta \pmod{\pi_k}$, $k \in \mathbb{N}_m$; here, $\overline{\int}$ denotes the upper Lebesgue integral. If we replace the quantity in (5.6) with

$$\limsup_{l\to+\infty} \sup_{x\in I} \frac{1}{l} \overline{\int_x^{x+l}} \|f(t+\tau_t) - f(t)\|\,dt < \varepsilon$$

resp.,

$$\limsup_{l\to+\infty} \frac{1}{2l} \overline{\int_{-l}^{l}} \|f(t+\tau_t) - f(t)\|\,dt, \quad \text{if } I = \mathbb{R}, \quad \text{resp.}$$
$$\limsup_{l\to+\infty} \frac{1}{l} \overline{\int_{0}^{l}} \|f(t+\tau_t) - f(t)\|\,dt, \quad \text{if } I = [0,\infty),$$

then we say that $f(\cdot)$ is Riemann–Weyl almost periodic, resp. Riemann–Besicovitch almost periodic.

Following A. S. Kovanko [670], R. Doss has also introduced the classes of Kovanko–Stepanov almost periodic functions, Kovanko–Weyl almost periodic functions and Kovanko–Besicovitch almost periodic functions (see [396, Definition 2]). These classes can be simply introduced in the vector-valued case.

For any measurable set $E \subseteq I$, we introduce the quantities

$$S(E) := \sup_{x\in I} \int_x^{x+1} \chi_E(t)\,dt,$$

$$W(E) := \lim_{l\to+\infty} \sup_{x\in I} \frac{1}{l} \int_x^{x+l} \chi_E(t)\,dt,$$

and

$$B(E) := \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \chi_E(t)\,dt, \quad \text{if } I = \mathbb{R}, \quad \text{resp.}$$

$$B(E) := \limsup_{l\to+\infty} \frac{1}{l} \int_{0}^{l} \chi_E(t)\,dt, \quad \text{if } I = [0,\infty).$$

In [396, Theorem 1], it has been proved that an essentially bounded function $f : I \to X$ is Riemann–Stepanov almost periodic if and only if for every $\varepsilon > 0$ there exist a measurable set $E \subseteq I$ and numbers $\delta > 0$, $\pi_1 \in I, \ldots, \pi_m \in I$ such that $S(I \setminus E) < \varepsilon$ and $|f(x)-f(x')| < \varepsilon$ provided $x \in E$ and $|x-x'| < \delta \,(\mathrm{mod}\, \pi_k)$, $k \in \mathbb{N}_m$. For the Riemann–Weyl almost periodicity and the Riemann–Besicovitch almost periodicity, we have the same statement with the quantity $S(I \setminus E)$ replaced, respectively, by $W(I \setminus E)$ and $B(I \setminus E)$. We would like to note that the proof of necessity in this theorem works for the vector-valued functions, as it can be simply approved. But the proof of sufficiency in this theorem and the statement of [396, Theorem 2] are intended solely for the scalar-valued functions. Furthermore, in the scalar-valued case, we see that the concepts Riemann–Weyl almost periodicity and the Riemann–Besicovitch almost periodicity coincide.

Due to [396, Theorem 3], we see that an essentially bounded function $f : I \to X$ is Riemann–Stepanov almost periodic if and only if for every $\varepsilon > 0$ there exist a measurable set $E \subseteq I$ and a trigonometric polynomial $q(\cdot)$ such that $S(I \setminus E) < \varepsilon$ and $|f(x) - q(x)| < \varepsilon$ provided $x \in E$. For the Riemann–Weyl almost periodicity and the Riemann–Besicovitch almost periodicity, we have the same statement with the quantity $S(I \setminus E)$ replaced, respectively, by $W(I \setminus E)$ and $B(I \setminus E)$. We would like to note that the proof of sufficiency in this theorem works for the vector-valued functions.

Nemytskii operators between Stepanov almost periodic function spaces

Let p and q be two real numbers belonging to the interval $[1,\infty)$, and let $T > 0$. It is said that $f : (0,T) \times X \to Y$ is a Carathéodory function if and only if the following holds:

(i) the mapping $t \mapsto f(t,x)$, $t \in (0,T)$ is measurable for any fixed element $x \in X$;
(ii) for a. e. $t \in (0,T)$ the function $f(t,\cdot)$ is continuous from X and Y.

Consider now the Nemytskii operator $\mathcal{N}_f : L^p((0,T):X) \to L^q((0,T):Y)$ by

$$[\mathcal{N}_f(\omega)](t) := f\big(t,\omega(t)\big), \quad t \in (0,T),\ \omega \in L^p\big((0,T):X\big).$$

The well known result of R. Lucchetti and F. Patrone [735, Theorem 3.1] states that the Nemytskii operator is a well defined between these spaces if and only if there exist $a > 0$ and $b \in L^p((0,T))$ such that for all $x \in X$ and a. e. $t \in (0,T)$ we have

$$\|f(t,x)\| \leqslant a\|x\|^{p/q} + b(t).$$

In this case, the Nemytskii operator is continuous.

Concerning the Nemytskii operator between the spaces of almost periodic functions $AP(\mathbb{R} : X)$ and $AP(\mathbb{R} : Y)$, it should be noted that we have the equivalence of the following statements (see e. g. J. Blot, P. Cieutat, G. M. N'Guérékata and D. Pennequin [183]):

(i) The Nemytskii operator $\mathcal{N}_f : AP(\mathbb{R} : X) \to AP(\mathbb{R} : Y)$ is continuous.

(ii) For each compact set $K \subseteq X$ and for each $\varepsilon > 0$ the set

$$\Big\{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \sup_{x\in K} \|f(t+\tau,x) - f(t,x)\| \leqslant \varepsilon\Big\}$$

is relatively dense in $\mathbb{R}$.

(iii) For all $x \in X$, $f(\cdot,x) \in AP(\mathbb{R} : Y)$ and for each compact set $K \subseteq X$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x_1,\ x_2 \in K$ and for each $t \in \mathbb{R}$ we have the implication: $\|x_1 - x_2\| \leqslant \delta \Rightarrow \|f(t,x_1) - f(t,x_2)\| \leqslant \varepsilon$.

A similar statement holds for the continuity of Nemytskii operator between the spaces of almost automorphic functions $AA(\mathbb{R} : X)$ and $AA(\mathbb{R} : Y)$; see, e. g., the recent paper [295, Theorem 2.3] by P. Cieutat. Several necessary and sufficient conditions clarifying the continuity of Nemytskii operators between almost periodic and almost automorphic spaces in the sense of Stepanov approach can be found in [295, Section 4].

Geometric properties of generalized almost periodic function spaces of Orlicz type

In his fundamental paper [537], T. R. Hillmann has investigated the Besicovitch–Orlicz spaces of almost periodic functions. After that, numerous mathematicians working in the field of almost periodic functions have investigated the geometric properties of generalized almost periodic function spaces of Orlicz type.

We will inscribe here the results of M. Morsli, M. Smaali established in [793] and the results of F. Bedouhene, Y. Djabri, F. Boulahia established in [140], only; for more details on the subject, we refer the reader to [142, 279, 791, 792] and the references in these papers. Assume that the function $\varphi : \mathbb{R} \times [0,\infty) \to [0,\infty)$ satisfies the following conditions:

(i) For every $t \in \mathbb{R}$, we have $\varphi(t,0) = 0$.

(ii) For every $t \in \mathbb{R}$, the mapping $u \mapsto \varphi(t,u)$, $u \geqslant 0$ is convex.

(iii) $\varphi(t+1,u) = \varphi(t,u)$ for all $t \in \mathbb{R}$ and $u \geqslant 0$.
(iv) For every $u > 0$, we have $\inf_{t\in\mathbb{R}} \varphi(t,u) = \phi(u) > 0$.

If $f : \mathbb{R} \to [0,+\infty]$ is a measurable function, then it is well known that the functional

$$f \mapsto \rho_\varphi(f) := \limsup_{t\to+\infty} \frac{1}{2t} \int_{-t}^{t} \varphi(t,|f(t)|)\,dt, \quad f \in M(\mathbb{R}),$$

is convex and pseudomodular.

In [793], the authors have defined the Besicovitch–Musielak–Orlicz space associated to $\varphi(\cdot,\cdot)$ by

$$B^\varphi(\mathbb{R}) := \Big\{f \in M(\mathbb{R}) : \lim_{\alpha\to 0+} \rho_\varphi(\alpha f) = 0\Big\}.$$

We have

$$B^\varphi(\mathbb{R}) = \{f \in M(\mathbb{R}) : (\exists \alpha > 0)\, \rho_\varphi(\alpha f) < \infty\}.$$

The space $B^\varphi(\mathbb{R})$ is equipped with the pseudonorm

$$\|f\|_\varphi := \{k > 0 : \rho_\varphi(f/k) \leqslant 1\}.$$

The authors have introduced two different types of Besicovitch–Musielak–Orlicz spaces of almost periodic functions, $\tilde{B}^\varphi_{a.p.}(\mathbb{R})$ and $B^\varphi_{a.p.}(\mathbb{R})$, as follows: A function $f : \mathbb{R} \to \mathbb{C}$ is said to belong the space $B^\varphi_{a.p.}(\mathbb{R})$, resp. $\tilde{B}^\varphi_{a.p.}(\mathbb{R})$, if and only if there exists a sequence (f_n) of trigonometric polynomials such that for every $k > 0$, resp. there exists $k > 0$, such that $\lim_{n\to+\infty} \rho_\varphi(k(f_n - f)) = 0$. Then we clearly have

$$B^\varphi_{a.p.}(\mathbb{R}) \subseteq \tilde{B}^\varphi_{a.p.}(\mathbb{R}) \subseteq B^\varphi(\mathbb{R}).$$

If $\varphi(t,|x|) = |x|$, then by $B^1_{a.p.}(\mathbb{R})$, $\tilde{B}^1_{a.p.}(\mathbb{R})$ and $B^1(\mathbb{R})$ we denote the respective spaces.

Let us recall that a function $\varphi : \mathbb{R} \times [0,\infty) \to [0,\infty)$ is strictly convex if and only if $\varphi(t,\lambda u + (1-\lambda)v) < \lambda\varphi(t,u) + (1-\lambda)\varphi(t,v)$ for a. e. $t \in \mathbb{R}$ and for all $\lambda \in (0,1)$, $0 \leqslant u < v < \infty$. On the other hand, a normed linear space $(E, \|\cdot\|)$ is said to be strictly convex if and only if

$$\left\|\frac{x+y}{2}\right\| < 1, \quad \text{provided that } \|x\| = \|y\| = 1 \text{ and } x \neq y.$$

It is said that the function $\varphi(\cdot,\cdot)$ satisfies the Δ_2-condition if and only if there exist a number $k > 1$ and a measurable nonnegative function $h(\cdot)$ such that $\rho_\varphi(h) < \infty$ and $\varphi(t,2u) \leqslant k\varphi(t,u)$ for almost all $t \in \mathbb{R}$ and all $u \geqslant h(t)$.

Let $f \in B^{\varphi}_{a.p.}(\mathbb{R})$. Then, due to [793, Proposition 1], we have $\varphi(\cdot, |f(\cdot)|) \in B^{1}_{a.p.}(\mathbb{R})$ so that the limit

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(t, |f(t)|)\, dt$$

always exists and is finite. The main result of paper is [793, Theorem 1], which states that the space $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ is strictly convex if and only if $\varphi(\cdot,\cdot)$ is strictly convex and satisfies the Δ_2-condition.

Ergodicity in Stepanov–Orlicz spaces has been investigated in [140]. Let us recall that a convex function $\phi : \mathbb{R} \to [0,\infty)$ is said to be an Orlicz function if and only if it is non-decreasing, even and continuous on $\mathbb{R}$ and satisfies $\phi(0) = 0$, $\phi(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty} \phi(u) = +\infty$. In the newly arisen situation, we say that the function $\phi(\cdot)$ satisfies the Δ_2-condition if and only if there exist real numbers $k > 1$ and $u_0 > 0$ such that $\phi(2u) \leqslant k\varphi(u)$ for $|u| \geqslant u_0$. For any Orlicz function $\phi : \mathbb{R} \to [0,\infty)$, it can be simply proved that $f \in \mathrm{PAP}_0(\mathbb{R} : X)$ if and only if $\phi(\|f\|) \in \mathrm{PAP}_0(\mathbb{R} : X)$.

For any vector-valued measurable function $f : \mathbb{R} \to X$, we define the positive functional

$$\rho_{S^{\phi}}(f) := \sup_{x\in\mathbb{R}} \int_{x}^{x+1} \phi(\|f(s)\|)\, ds.$$

The Stepanov–Orlicz function space generated by ϕ is defined by

$$BS^{\phi}(\mathbb{R}, X) := \{f \in M(\mathbb{R} : X);\ (\exists\alpha > 0)\, \rho_{S^{\phi}}(\alpha f) < \infty\}.$$

We know that the vector space $BS^{\phi}(\mathbb{R}, X)$ equipped with the Luxemburg norm

$$\|f\|_{S^{\phi}} := \inf\left\{k > 0 : \sup_{x\in\mathbb{R}} \int_{x}^{x+1} \phi(\|f(s)\|/k)\, ds \leqslant 1\right\}$$

is a Banach space. It is also worth noting that the Morse–Transue space type

$$\widetilde{BS}^{\phi}(\mathbb{R}, X) := \{f \in M(\mathbb{R} : X);\ (\exists\alpha > 0)\, \rho_{S^{\phi}}(\alpha f) < \infty\}$$

equipped with the Luxemburg norm is a closed subspace of $BS^{\phi}(\mathbb{R}, X)$, which is commonly called the Besicovitch–Orlicz class. We know that $BS^{\phi}(\mathbb{R}, X) = \widetilde{BS}^{\phi}(\mathbb{R}, X)$ if and only if $\phi(\cdot)$ satisfies the Δ_2-condition.

Furthermore, for any $p \in C_+(\mathbb{R})$ we define the Musielak–Orlicz modular type space

$$BS^{p(\cdot)}(\mathbb{R}, X) := \left\{f \in M(\mathbb{R} : X);\ (\exists\alpha > 0)\ \sup_{x\in\mathbb{R}} \int_{x}^{x+1} (\|f(s)\|/k)^{p(s)}\, ds \leqslant 1\right\}.$$

For any function $f \in BS^{p(\cdot)}(\mathbb{R}, X)$, the notion of $BS^{p(\cdot)}(\mathbb{R}, X)$-ergodicity in norm sense and the notion of $BS^{p(\cdot)}(\mathbb{R}, X)$-ergodicity in modular sense are introduced in [140, Definition 3.1] and [140, Definition 3.2], respectively. Due to [140, Proposition 3.4], these concepts are equivalent.

Let $\phi : \mathbb{R} \to [0, \infty)$ be an Orlicz function. In [140, Definition 3.6], the authors introduced the notions of norm ergodicity in Stepanov Orlicz sense, modular ergodicity in Stepanov Orlicz sense and strongly modular ergodicity in Stepanov Orlicz sense for a given function $f \in BS^{\phi}(\mathbb{R}, X)$. After that, the authors further explored this notion in [140, Theorem 3.8, Theorem 3.10, Theorem 3.11] and provided several illustrative examples in [140, Section 4].

Density theorems for almost periodic functions in Hilbert spaces

In this section, we will inscribe a few relevant results obtained by A. Haraux and V. Komornik in [510]; these results have been obtained in their investigation of the oscillatory properties of the wave equation. Denote by X_T the vector space of all square-integrable functions with zero mean

$$X_T := \left\{ f \in L^2_{\text{loc}}(\mathbb{R} : \mathbb{C})\,;\, f(t+T) \equiv f(t),\ \int_0^T f(t)\,dt = 0 \right\},$$

where $T > 0$. If the set $A = \{T_1, \dots, T_N\}$ is a given set of positive real numbers, we define

$$X := X_{T_1} + \cdots + X_{T_N}.$$

If V is a certain collection of complex-valued functions and I is an interval in $\mathbb{R}$, then we set $V_I := \{f_I : f \in V\}$. In [510, Theorem 1], the authors have proved that there exists a positive real number $T(A)$ such that for any interval $I \subseteq \mathbb{R}$ we have

$$X_I \text{ is dense in } L^2(I) \quad \text{if and only if} \quad |I| < T(A),$$

where $|I|$ denotes the length of interval I; furthermore, the orthogonal complement of X_I in $L^2(I)$ is finite dimensional if $|I| = T(A)$ and infinite dimensional if $|I| > T(A)$. Suppose that $|I| = T(A)$ and the orthogonal complement of X_I in $L^2(I)$ is p-dimensional for some integer $p \in \mathbb{N}$. If P_{p-1} denotes the vector space consisting of all complex polynomials of degree $\leqslant p-1$ (including also the zero polynomial), then [510, Theorem 3(a)] states that Y_I is dense in $L^2(I)$, where $Y := P_{p-1} + X$; furthermore, $Y_I = L^2(I)$ if and only if $p = 1$, which is equivalent to saying that $P_i/P_j \in \mathbb{Q}$ for $1 \leqslant i \leqslant j \leqslant N$. Due to [510, Theorem 3(b)], there exists a real-valued function $h \in L^2(I)$ such that the functions $h, h', \dots, h^{p-1}$ span X_I; furthermore, if we extend the function $h(\cdot)$ by zero to the whole real line and denote the obtained function by $H(\cdot)$, then we know that the function $H(\cdot)$ is a nonzero finite linear combinations of Dirac measures.

Almost periodicity in chaos
In this part, we will only draw the attention of the reader to the results presented in the tenth chapter of the recent research monograph [34] by M. Akhmet. In [34, Section 10], the author has investigated the dynamical properties of the following system:

$$y' = Ay + G(t, y) + H\big(x(t)\big), \quad t \in \mathbb{R}, \tag{5.7}$$

where $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in both variables, almost periodic in variable t uniformly for $y \in \mathbb{R}^n$, the function $H : \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and all eigenvalues of the constant $n \times n$ real matrix A have negative real parts. Roughly speaking, if the perturbation part $H(x(t))$ is chaotic in a certain sense, then the system (5.7) has the interesting feature of chaos with infinitely many almost periodic motions. The obtained results are well illustrated with several numerical tests involving the coupled Duffing oscillators, for which it is well known that play an important role in modeling of the enhanced signal propagation (see also [35] and [36]). The most important notion used in [34, Section 10] is the notion of Li–Yorke chaotic set with infinitely many almost periodic motions, which is introduced in [34, Definition 10.1] for the equicontinuous families of uniformly bounded functions $x : \mathbb{R} \to \Lambda$, where Λ is a non-empty compact subset of $\mathbb{R}^m$. We would like to note here that this notion can be introduced in the infinite-dimensional setting, even for other types of chaos like distributional chaos or mean Li–Yorke chaos [632].

Almost periodicity in mathematical biology
There exist numerous research articles concerning almost periodic and almost automorphic type solutions for various classes of ordinary and partial differential equations (see, e. g., [8, 28, 72, 154, 384, 388, 401, 563, 719, 889]). In this part, we will present the main details of the investigation [388] carried out by H.-S. Ding, J. Liang, T.-J. Xiao and the investigation [1083] carried out by H. Zhang, M. Yang and L. Wang. The nonlinear functional differential equation

$$x'(t) = -ax(t) + \frac{p}{1 + x^n(t - \tau)}, \quad n > 0, \tag{5.8}$$

has been proposed by M. C. Mackey, L. Glass [743] for modeling of hematopoiesis describing the process of production of all types of blood cells generated by a remarkable self-regulated system that is responsive to the demands put upon it. The authors of [388] have studied the following modification of (5.8):

$$x'(t) = -a(t)x(t) + \frac{p(t)x^l(t - \tau(t))}{1 + x^l(t - \tau(t))}, \quad n > 0,$$

where $a, p, \tau : \mathbb{R} \to (0, \infty)$ are almost periodic functions, $0 < m \leqslant 1$ and $l > 0$. The almost periodic solutions of this equation have been previously studied in [390] and

[822] by using the Leggett–Williams fixed point theorem, which involves the compactness of operators. In contrast to this, the authors of [388] have employed a fixed point theorem in cones, which does not require such conditions. The authors of [1083] have considered the existence and global exponential convergence of positive almost periodic solutions for the generalized model of hematopoiesis, described by the following nonlinear functional differential equation:

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}, \quad n > 0, \tag{5.9}$$

where $a,\ b_i,\ \tau_i : \mathbb{R} \to (0, \infty)$ are continuous functions for $i = 1, 2, \ldots, m$; clearly, this equation is a generalization of (5.8). This model has been proposed by I. Gyori, G. Ladas [504] to describe the dynamics of hematopoiesis, i. e., blood cell production. In any reasonable biological interpretation of model (5.9), only positive functions $x(\cdot)$ can be accepted as solutions. In [51], J. O. Alzabut, J. J. Nieto and G. Tr. Stamov have analyzed the existence and exponential stability of a positive almost periodic solution for (5.9), provided that $m = 1$ and $\sup_{t\in\mathbb{R}} b_1(t) < \inf_{t\in\mathbb{R}} a(t)$. Furthermore, X. Wang and H. Zhang have proved a new fixed point theorem in [1023] in order to establish sufficient conditions for the existence, nonexistence and uniqueness of positive almost periodic solutions of (5.9) with $n > 1$. The main results of [1083] are Theorem 3.1 and Theorem 3.2, in which the authors have not used the requirements from [51] and [1023]. They have assumed that $a,\ b_i,\ \tau_i : \mathbb{R} \to (0, \infty)$ are almost periodic functions for $i = 1, 2, \ldots, m$. Set

$$a^- := \inf_{t\in\mathbb{R}} a(t), \quad a^+ := \sup_{t\in\mathbb{R}} a(t), \quad b_i^- := \inf_{t\in\mathbb{R}} b_i(t) > 0, \quad b_i^+ := \sup_{t\in\mathbb{R}} b_i(t),$$
$$r := \max_{1\leqslant q\leqslant n} \sup_{t\in\mathbb{R}} \tau_i(t) > 0, \quad M_1 := \frac{\sum_{i=1}^m b_i^+}{a^-}, \quad M_2 := \frac{\sum_{i=1}^m b_i^-}{a^+(1 + M_1^n)},$$

and suppose that

$$n \sum_{i=1}^{m} b_i^+ < a^-.$$

Then there exists a unique positive almost periodic solution of (5.9) in the closed set $B^* = \{f \in AP(\mathbb{R} : \mathbb{R}); M_2 \leqslant \|f\|_\infty \leqslant M_1\}$. If we denote by $x^*(\cdot)$ this solution, then any solution $x(t; t_0, \varphi)$ of Eq. (5.9) equipped with the initial condition

$$x_{t_0} = \varphi, \quad \varphi \in C_+,\ \varphi(0) > 0$$

converges exponentially to $x^*(t)$ as $t \to +\infty$; see [1083] for the notion and more details.

Almost periodic solutions and almost automorphic solutions of abstract difference equations

Let $l^{\infty}(\mathbb{N} : X)$ denote the Banach space of all bounded X-valued sequences equipped with the sup-norm. We say that an X-valued sequence $(x_n)_{n\in\mathbb{N}}$ is almost periodic if and only if for each $\varepsilon > 0$ there exists an integer $N_0(\varepsilon) \in \mathbb{N}$ such that among any $N_0(\varepsilon)$ consecutive natural numbers, there exists at least one natural number $\tau \in \mathbb{N}$ satisfying that

$$\|x_{n+\tau} - x_n\| \leqslant \varepsilon, \quad n \in \mathbb{N};$$

the number τ is said to be an ε-period of the sequence $(x_n)_{n\in\mathbb{N}}$. Any almost periodic X-valued sequence is bounded.

The class of almost automorphic sequences has been already analyzed in the old papers by S. Bochner and W. A. Veech. We say that an X-valued sequence $(x_n)_{n\in\mathbb{Z}}$ is almost automorphic if and only if for every sequence $(h'_k)_{k\in\mathbb{Z}}$ there exist a subsequence $(h_k)_{k\in\mathbb{Z}}$ of $(h'_k)_{k\in\mathbb{Z}}$ and an X-valued sequence $(y_n)_{n\in\mathbb{Z}}$ satisfying

$$\lim_{n\to\infty} x_{n+h_k} = y_n, \quad n \in \mathbb{Z} \quad \text{and} \quad \lim_{n\to\infty} y_{n-h_k} = x_n, \quad n \in \mathbb{Z}.$$

The notion asymptotical almost (automorphy) periodicity for an X-valued sequence $(x_n)_{n\in\mathbb{Z}}$ can be also introduced and analyzed.

Regarding almost (automorphic) periodic sequences and almost (automorphic) periodic type solutions of abstract differences equations, we have quoted some basic references in the corresponding part of [631, Section 3.11]. Besides these references, mention should be made of [2, 50, 134, 243, 244, 349, 390, 507, 728, 730, 947–949, 1003] and [1038]; the research monographs by R. Agarwal [20] and R. Agarwal, C. Cuevas, C. Lizama [21] are also of importance.

c-Almost periodic ultradistributions and *c*-almost periodic hyperfunctions

In this section, we analyze c-almost periodic ultradistributions and c-almost periodic hyperfunctions; we will skip all related details concerning c-uniformly recurrent ultradistributions (hyperfunctions) and semi-c-periodic ultradistributions (hyperfunctions). The material is taken from [655].

Assume that (M_p) is a sequence of positive real numbers satisfying $M_0 = 1$ and the following conditions:

(M.1) $M_p^2 \leqslant M_{p+1}M_{p-1}, p \in \mathbb{N}$,

(M.2) $M_p \leqslant AH^p \sup_{0\leqslant i\leqslant p} M_i M_{p-i}, p \in \mathbb{N}$, for some $A, H > 1$.

We will occasionally use the conditions:

(M.3′) $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$,

(M.3) $\sup_{p\in\mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1}M_{p+1}}{pM_pM_q} < \infty$,

(C) The sequence (M_p^2) satisfies (M.3).

Let us recall that conditions (M.3$'$) and (C) are substantially weaker than (M.3) and that condition (C) has been essentially employed in the analysis of almost periodic hyperfunctions [291] carried out by J. Chung, S.-Y. Chung, D. Kim, H. J. Kim and the analysis of representations of quasianalytic ultradistributions carried out by S.-Y. Chung, D. Kim [292] (it is well known that (M_p) satisfies (C) if and only if there exists a positive integer $k \in \mathbb{N}$ such that $\liminf_{p\to+\infty}(m_{kp}/m_p)^2 > k$, where $m_p := M_p/M_{p-1}$ for all $p \in \mathbb{N}$ and that H. Petzche has proved, in [837], that (M_p) satisfies (M.3) if and only if there exists a positive integer $k \in \mathbb{N}$ such that $\liminf_{p\to+\infty} m_{kp}/m_p > k$). If $s > 1$, then the Gevrey sequence $(p!^s)$ satisfies the above conditions, while the sequence $(p!^s)$ satisfies (M.1), (M.2) and (C) for $s > 1/2$.

The space of Beurling, resp., Roumieu ultradifferentiable functions, is defined by $\mathcal{D}^{(M_p)} := \operatorname{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{(M_p)}$, resp., $\mathcal{D}^{\{M_p\}} := \operatorname{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{\{M_p\}}$, where $\mathcal{D}_K^{(M_p)} := \operatorname{projlim}_{h\to\infty}\mathcal{D}_K^{M_p,h}$, resp., $\mathcal{D}_K^{\{M_p\}} := \operatorname{indlim}_{h\to 0}\mathcal{D}_K^{M_p,h}$,

$$\mathcal{D}_K^{M_p,h} := \{\phi \in C^\infty(\mathbb{R}) : \operatorname{supp}\phi \subseteq K, \|\phi\|_{M_p,h,K} < \infty\}$$

and

$$\|\phi\|_{M_p,h,K} := \sup\left\{\frac{h^p|\phi^{(p)}(t)|}{M_p} : t \in K, p \in \mathbb{N}_0\right\}.$$

The asterisk $*$ is used to designate both, the Beurling case (M_p) or the Roumieu case $\{M_p\}$. The space consisting of all linear continuous functions from $\mathcal{D}^*$ into X, denoted by $\mathcal{D}'^*(X) := L(\mathcal{D}^* : X)$, is said to be the space of all X-valued ultradistributions of $*$-class.

Let us recall (see [617–619] for the basic introduction to the theory of ultradistributions) that an entire function of the form $P(\lambda) = \sum_{p=0}^{\infty} a_p\lambda^p$, $\lambda \in \mathbb{C}$, is of class (M_p), resp., of class $\{M_p\}$, if there exist $l > 0$ and $C > 0$, resp., for every $l > 0$ there exists a constant $C > 0$, such that $|a_p| \leqslant Cl^p/M_p$, $p \in \mathbb{N}$. The corresponding ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_pD^p$ is of class (M_p), resp., of class $\{M_p\}$. For more details about convolution of scalar-valued ultradistributions (ultradifferentiable functions), see [617]. The convolution of Banach space valued ultradistributions and scalar-valued ultradifferentiable functions will be taken in the sense of the considerations given on page 685 of [619]. As in the distributional case, we define $\langle T_h, \phi\rangle := \langle T, \phi(\cdot - h)\rangle$, $T \in \mathcal{D}'^*(X)$, $h > 0$, $\phi \in \mathcal{D}^*$.

The Sato space $\mathcal{F}_H$ consists of all infinitely differentiable functions $\phi : \mathbb{R} \to \mathbb{C}$ satisfying that there exist $h > 0$ and $k > 0$ such that

$$\|\phi\|_{p,k} := \sup_{x\in\mathbb{R},p\in\mathbb{N}_0}\frac{h^p|\phi^{(p)}(x)|e^{k|x|}}{p!} < +\infty.$$

Let $\mathcal{F}_H$ be topologized by the corresponding inductive limit topology induced by these seminorms. The space of all X-valued Fourier hyperfunctions, denoted by $\mathcal{F}_H'(X)$, is defined as the space of all linear continuous mappings $T : \mathcal{F}_H \to X$, equipped with the strong topology.

Now we will consider bounded ultradistributions and bounded hyperfunctions with values in complex Banach spaces. First of all, for every $h > 0$, we define

$$\mathcal{D}_{L^1}\big((M_p), h\big) := \Big\{f \in \mathcal{D}_{L^1};\ \|f\|_{1,h} := \sup_{p\in\mathbb{N}_0} \frac{h^p\|f^{(p)}\|_1}{M_p} < \infty\Big\}.$$

Then $(\mathcal{D}_{L^1}((M_p), h), \|\cdot\|_{1,h})$ is a Banach space and the space of all X-valued bounded Beurling ultradistributions of class (M_p), resp., X-valued bounded Roumieu ultradistributions of class $\{M_p\}$, is defined as the space consisting of all linear continuous mappings from $\mathcal{D}_{L^1}((M_p))$, resp., $\mathcal{D}_{L^1}(\{M_p\})$, into X, where

$$\mathcal{D}_{L^1}\big((M_p)\big) := \operatorname{projlim}_{h\to+\infty} \mathcal{D}_{L^1}\big((M_p), h\big),$$

resp.,

$$\mathcal{D}_{L^1}\big(\{M_p\}\big) := \operatorname{indlim}_{h\to 0+} \mathcal{D}_{L^1}\big((M_p), h\big).$$

These spaces, carrying the strong topologies, will be shortly denoted by $\mathcal{D}'_{L^1}((M_p) : X)$, resp., $\mathcal{D}'_{L^1}(\{M_p\} : X)$. It is well known that $\mathcal{D}^{(M_p)}$, resp. $\mathcal{D}^{\{M_p\}}$, is a dense subspace of $\mathcal{D}_{L^1}((M_p))$, resp., $\mathcal{D}_{L^1}(\{M_p\})$, and that $\mathcal{D}_{L^1}((M_p)) \subseteq \mathcal{D}_{L^1}(\{M_p\})$.

In particular case $M_p := p!$, the space $\mathcal{D}'_{L^1}(\{p!\} : X)$ is said to be the space of bounded hyperfunctions. As in the scalar-valued case, this space is contained in the space $\mathcal{F}'_H(X)$ of all X-valued Fourier hyperfunctions (see also [293, Definition 3.1] for the multi-dimensional analogue).

Recall that the heat kernel $E(x,t)$ is defined by $E(x,t) := (4\pi t)^{-1/2}e^{-x^2/4t}$, $x \in \mathbb{R}$, $t > 0$ and $E(x,t) := 0$, $x \in \mathbb{R}$, $t \leqslant 0$. It can be simply shown that the function $E(\cdot,t)$ belongs to the Sato space for every fixed real number $t > 0$ and that for each $x \in \mathbb{R}$ and $t > 0$ the function $E(x-\cdot,t)$ belongs to the space $\mathcal{D}_{L^1}(\{p!\} : X)$. Hence, for each Fourier hyperfunction $T \in \mathcal{F}'_H(X)$, its Gauss transform $u(x,t) := \langle T, E(x-\cdot,t)\rangle$ is infinitely differentiable in $\mathbb{R} \times (0,\infty)$.

We would like to note that the statements of [293, Theorem 3.4, Theorem 3.5] continue to hold in the vector-valued case. In connection with this observation, it should be only observed that the existence of the functions $g(x)$ and $h(x)$, established on [293, p. 2425, l.-3] (see also [291, p. 735, l.-1; l.-5]), follows from the facts (see [82, Example 3.7.6, Example 3.7.8] for more details) that the Laplacian Δ with maximal distributional domain ($\equiv A$) generates a strongly continuous Gaussian semigroup on $L^p(\mathbb{R}^n : X)$, the operator A generates a polynomially bounded once integrated Gaussian semigroup on $L^\infty(\mathbb{R}^n : X)$, the basic results hold about the existence and uniqueness of mild solutions of the abstract (ill-posed) Cauchy problems of the first order and the conclusion established on [293, p. 2425, l.-4]. In particular, the statement of [291, Theorem 3.1] can be extended to the vector-valued case.

Theorem 5.0.19. *Suppose that $T \in \mathcal{F}'_H(X)$. Then the following statements are equivalent:*

(i) *We have $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$.*
(ii) *$T * \varphi \in L^\infty(\mathbb{R} : X)$ for all $\varphi \in \mathcal{F}_H$.*
(iii) *There exist two bounded continuous functions $f : \mathbb{R} \to X$, $g : \mathbb{R} \to X$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta)f + g$.*
(iv) *The Gauss transform $u(x,t)$ of T is infinitely differentiable in $(0,\infty)^2$ and solves the heat equation in $(0,\infty)^2$, and for every $\varepsilon > 0$ there exists a constant $c > 0$ such that*

$$\|u(x,t)\| \leqslant ce^{\varepsilon/t}, \quad x \in \mathbb{R},\ t > 0,$$

and

$$\langle T, \varphi \rangle = \lim_{t \to 0+} \int_{-\infty}^{+\infty} u(x,t)\varphi(x)\,dx, \quad \varphi \in \mathcal{D}_{L^1}(\{p!\} : X).$$

In connection with bounded quasianalytic ultradistributions, we would like to note that the statement of [291, Lemma 4.2] also holds in the vector-valued case.

Concerning c-almost periodic ultradistributions, we will use the function space

$$\mathcal{E}^*_{\mathrm{AP}_c}(X) := \{\phi \in \mathcal{E}^*(X) : \phi^{(i)} \in \mathrm{AP}_c(\mathbb{R} : X) \text{ for all } i \in \mathbb{N}_0\},$$

which is a slight generalization of the space $\mathcal{E}^*_{\mathrm{AP}}(X)$ used in [640], with $c = 1$.

In [301] and [640], a bounded X-valued ultradistribution $T \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $T \in \mathcal{D}'_{L^1}(\{M_p\} : X)$, is said to be almost periodic of Beurling class (M_p), resp., almost periodic of Roumeiu class $\{M_p\}$, if and only if there exists a sequence of X-valued trigonometric polynomials converging to T in $\mathcal{D}'_{L^1}((M_p) : X)$, resp., $\mathcal{D}'_{L^1}(\{M_p\} : X)$. If the sequence (M_p) satisfies (M.3), then $T \in \mathcal{D}'_{L^1}((M_p) : X)$ is almost periodic if and only if $T * \varphi \in \mathrm{AP}(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}^{(M_p)}$.

Concerning [640, Theorem 2], the following result should be stated for c-almost periodicity.

Theorem 5.0.20. *Let (M_p) satisfy the conditions* (M.1), (M.2) *and* (M.3′), *and let $T \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $T \in \mathcal{D}'_{L^1}(\{M_p\} : X)$. Consider the following assertions:*

(i) *There exists an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class (M_p), resp., of class $\{M_p\}$, and functions $f, g \in \mathrm{AP}_c(\mathbb{R} : X)$ such that the function $t \mapsto (f(t), g(t))$, $t \in \mathbb{R}$ is c-almost periodic and $T = P(D)f + g$ for all $\varphi \in \mathcal{D}_{L^1}((M_p))$, resp., $\varphi \in \mathcal{D}_{L^1}(\{M_p\})$.*
(ii) *For every $\varphi \in \mathcal{D}^*$, we have $T * \varphi \in \mathrm{AP}_c(\mathbb{R} : X)$.*
(iii) *$T \in \mathcal{D}'^*_{L^1}((M_p) : X)$, resp. $T \in \mathcal{D}'^*_{L^1}(\{M_p\} : X)$, and there exists a sequence (ϕ_n) in $\mathcal{E}^*_{\mathrm{AP}_c}(X)$ such that $\lim_{n\to\infty} \phi_n = T$ for the topology of $\mathcal{D}'_{L^1}((M_p) : X)$, resp. $\mathcal{D}'_{L^1}(\{M_p\} : X)$.*
(iv) *There exists $h > 0$ such that for each compact set $K \subseteq \mathbb{R}$, in the Beurling case, resp., for each compact set $K \subseteq \mathbb{R}$ and for each $h > 0$, in the Roumieu case, the following holds: $T * \varphi \in \mathrm{AP}_c(\mathbb{R} : X)$, $\varphi \in \mathcal{D}_K^{M_p,h}$.*

Then we have (i) ⇒ (ii) ⇔ (iii) ⇔ (iv).

Unfortunately, if (M_p) additionally satisfies (M.3), then the equivalence of the above assertions cannot be so simply clarified in the Beurling case (see, e. g., [301, Lemma 2] and the proofs of [301, Theorem 1, Theorem 2]); more precisely, it is not clear how one can prove that (iv) implies (i) for c-almost periodicity; we can only prove that (iv) implies that there exists an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class (M_p), resp., of class $\{M_p\}$, and functions f, $g \in AP_c(\mathbb{R} : X)$ such that $T = P(D)f + g$ for all $\varphi \in \mathcal{D}_{L^1}((M_p))$, resp., $\varphi \in \mathcal{D}_{L^1}(\{M_p\})$.

Concerning almost periodic quasianalytic ultradistributions, we would like to note that the statement of [291, Theorem 4.3] continues to hold in the vector-valued case. Concerning c-almost periodic quasianalytic ultradistributions and asymptotically c-almost periodic ultradistributions of $*$-class, let us only mention that the notion introduced in [640, Definition 1, Definition 2] and the notion of the space $B_0'(X)$ can be straightforwardly extended to the ultradistributional case (cf. also the recent article [342] by A. Debrouwere, L. Neyt and J. Vindas). Also, it could be very interesting to reconsider [640, Theorem 3] for asymptotical c-almost periodicity.

Now we would like to say something about the class of c-almost periodic hyperfunctions. We will follow the approach of J. Chung, S.-Y. Chung, D. Kim and H. J. Kim obeyed in [291]. In this paper, the authors use the operation calculus approach to hyperfunctions developed by T. Matsuzawa in [754–756], which is based on the use of Gauss kernels.

First of all, we introduce the vector-valued analogue of [291, Definition 3.2]:

Definition 5.0.21. A hyperfunction $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$ is said to be almost periodic if and only if there exists a sequence of trigonometric polynomials in X which converges to T in $\mathcal{D}'_{L^1}(\{p!\} : X)$.

Furthermore, we wish to emphasize that the statement of [291, Theorem 3.5] can be extended to the vector-valued case.

Theorem 5.0.22. *Suppose that $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$. Then the following statements are equivalent:*

(i) *T is almost periodic.*
(ii) *$T * \varphi \in AP(\mathbb{R} : X)$ for all $\varphi \in \mathcal{F}_H$.*
(iii) *There exist two almost periodic functions $f : \mathbb{R} \to X$, $g : \mathbb{R} \to X$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta)f + g$.*
(iv) *The Gauss transform $u(x, t)$ of T is almost periodic.*

Now we would like to introduce the notion of a c-almost periodic hyperfunction, which extends the notion of an almost periodic hyperfunction ($c = 1$) due to Theorem 5.0.22(ii).

Definition 5.0.23. Suppose that $c \in S_1$ and $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$. Then T is said to be c-almost periodic if and only if $T * \varphi \in AP_c(\mathbb{R} : X)$ for all $\varphi \in \mathcal{F}_H$.

Immediately from definition, it follows that any c-almost periodic hyperfunction is almost periodic, bounded and belongs to the Fourier class of hyperfunctions and that the space of c-almost periodic functions is closed under differentiation. Many structural properties of c-almost periodic hyperfunctions can be obtained by using the corresponding structural properties of space $AP_c(\mathbb{R} : X)$ given in [586]; for example, any almost anti-periodic hyperfunction (obtained by plugging $c = -1$ in the above definition) is almost periodic and any c-almost periodic hyperfunction is almost anti-periodic, provided that $|c| = 1$, $p \in \mathbb{Z}\backslash\{0\}$, $q \in \mathbb{N}$, $(p, q) = 1$ and $\arg(c) = (p/q)\pi$. Furthermore, many structural properties of c-almost periodic hyperfunctions can be obtained analogously as for c-almost periodic distributions; for example, the statements of [436, Proposition 2.5, Proposition 2.6] continue to hold for c-almost periodic hyperfunctions.

Concerning c-almost periodic hyperfunctions, we have the following analogue of Theorem 5.0.22.

Theorem 5.0.24. *Suppose that $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$. Consider the following statements:*
(i) *There exists a X^2-valued c-almost periodic function $x \mapsto (f(x), g(x))$, $x \in \mathbb{R}$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta)f + g$.*
(ii) *For every φ, $\psi \in \mathcal{F}_H$, the function $x \mapsto ((T * \varphi)(x), (T * \psi)(x))$, $x \in \mathbb{R}$ is c-almost periodic.*
(iii) *T is c-almost periodic.*
(iv) *The Gauss transform $u(x, t)$ of T is c-almost periodic.*

Then we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. The proofs of the equivalence (iii) $\Leftrightarrow$ (iv), the implication (ii) $\Rightarrow$ (i) and the implication (ii) $\Rightarrow$ (iii) can be given similarly as in the proof of [291, Theorem 3.5]. In order to see that (i) implies (ii), we can argue as in the proof of [436, Theorem 2.8]. In actual fact, let φ, $\psi \in \mathcal{F}_H$. Let $\varepsilon > 0$ be given, and let τ be a common (c, ε)-almost period of functions $f(\cdot)$ and $g(\cdot)$. If we assume that $\|\varphi\|_{p,k} < \infty$ for some h, $k > 0$, then for each $t \in \mathbb{R}$ we have

$$(T * \varphi)(t) = \langle T, \varphi(t - \cdot)\rangle = \sum_{p=0}^{\infty} (-1)^p a_p \int_{-\infty}^{+\infty} \varphi^{(2p)}(v) f(t - v)\, dv + \int_{-\infty}^{+\infty} \varphi(v) g(t - v)\, dv$$

and therefore

$$\begin{aligned}
&\|(T * \varphi)(t + \tau) - c(T * \varphi)(t)\| \\
&\quad \leqslant \Bigg\| \sum_{p=0}^{\infty} (-1)^p a_p \int_{-\infty}^{+\infty} \varphi^{(2p)}(v)[f(t + \tau - v) - cf(t - v)]\, dv \\
&\qquad + \int_{-\infty}^{+\infty} \varphi(v)[g(t + \tau - v) - cg(t - v)]\, dv \Bigg\| \\
&\quad \leqslant \varepsilon \Bigg[\sum_{p=0}^{\infty} |a_p| \int_{-\infty}^{+\infty} \big|\varphi^{(2p)}(v)\big|\, dv + \int_{-\infty}^{+\infty} \big|\varphi(v)\big|\, dv \Bigg].
\end{aligned}$$

We have the existence of a finite real number $M \geqslant 1$ such that $|\varphi^{(2p)}(v)| \leqslant Mh^{-p}e^{-k|v|}(2p)!$ for all $p \in \mathbb{N}_0$ and $v \in \mathbb{R}$. Moreover, for any $l \in (0, h/4)$, we have the existence of a finite real number $c > 0$ such that $|a_p| \leqslant cl^p p!^2$ for all $p \in \mathbb{N}_0$, so that we can continue the calculation as follows:

$$\leqslant \varepsilon\left[\sum_{p=0}^{\infty} cl^p p!^2 Mh^{-p}(2p)! \int_{-\infty}^{+\infty} e^{-k|v|}\,dv + \int_{-\infty}^{+\infty} e^{-k|v|}\,dv\right]$$

$$\leqslant \varepsilon\left[\sum_{p=0}^{\infty} cl^p 2^{2p} Mh^{-p} \int_{-\infty}^{+\infty} e^{-k|v|}\,dv + \int_{-\infty}^{+\infty} e^{-k|v|}\,dv\right].$$

A similar estimate holds with the function $\psi(\cdot)$ considered, with the same number τ. This simply completes the proof. □

Remark 5.0.25. Consider the following condition:

(i)′ There exist two c-almost periodic functions $x \mapsto f(x)$, $x \in \mathbb{R}$, $x \mapsto g(x)$, $x \in \mathbb{R}$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta)f + g$.

Then clearly (i) implies (i)′ but it is not clear whether (i)′ implies (ii).

Affine-periodic solutions and pseudo affine-periodic solutions for various classes of systems of ordinary differential equations

In a great number of recent research studies, the notions of affine-periodicity and pseudo affine-periodicity play an incredible role in the qualitative analysis of solutions for various classes of systems of ordinary differential equations, systems of functional differential equations and systems of Newtonian equations of motion with friction; see, e. g., [255, 282, 703, 707, 760, 1016, 1017, 1035, 1087] for some results obtained by Chinese mathematicians in this direction. In this section, we will only describe the main ideas of research studies carried out by X. Chang, Y. Li in [255] and Y. Li, H. Wang, X. Yang in [707]. By a (Q, T) affine-periodic function $x : \mathbb{R} \to \mathbb{R}^n$ we mean any continuous function $x(\cdot)$ for which

$$x(t + T) = Qx(t), \quad t \in \mathbb{R},$$

where Q is a regular matrix of format $n \times n$ and $T > 0$. Some qualitative properties of $x(\cdot)$ like periodicity, subharmonicity, or quasi-periodicity are induced by the corresponding qualitative properties of matrix Q; the most important subclasses of regular matrices for Q are power identity matrices, i. e., those matrices for which we have $Q^k = I$ for some integer $k \in \mathbb{Z}$, or orthogonal matrices belonging to the group $O(n)$; these subclasses are important for modeling certain real phenomena describing rotation motions in body from mechanics.

In [255], X. Chang and Y. Li have investigated the rotating periodic solutions of second-order dissipative dynamical systems. More precisely, the authors have considered the following dissipative dynamical system:

$$u'' + cu' + \nabla g(u) + h(u) = e(t), \quad t \in \mathbb{R},$$

where $c > 0$ is a constant, $g(u) = g(|u|)$, $h \in C(\mathbb{R}^n : \mathbb{R}^n)$, $h(u) = Qh(Q^{-1}u)$ for some orthogonal matrix $Q \in O(n)$ and $e \in C(\mathbb{R} : \mathbb{R}^n)$ satisfies $e(t + T) = Qe(t)$ for all $t \in \mathbb{R}$. It has been shown that the above equation admits a solution of the form $u(t+T) = Qu(t)$, $t \in \mathbb{R}$, which is usually called rotating periodic solution.

In [707], Y. Li, H. Wang and X. Yang have analyzed Fink's conjecture on affine-periodic solutions and Levinson's conjecture to Newtonian systems. The authors have analyzed the following system of ordinary differential equations:

$$x'(t) = f\big(t, x(t)\big), \quad t \in \mathbb{R},$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n : \mathbb{R}^n)$, $f(t,x) \equiv Qf(t, Q^{-1}x)$, the following system of functional differential equations:

$$x'(t) = F(t, x_t), \quad t \in \mathbb{R},$$

where $x_t(s) = x(t+s)$ for $s \in [-r, 0]$ and fixed $r > 0$, $F : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous with C being the Banach space of continuous functions $C([-r, 0] : \mathbb{R}^n)$ equipped with the sup-norm and $F(t, \varphi) \equiv QF(t, Q^{-1}\varphi)$, and the following system of Newtonian equations of motion with friction:

$$x'' + A(t,x)u' + \nabla V(x) + h(u) = e(t), \quad t \in \mathbb{R},$$

where $A : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^{m,m}$, $V : \mathbb{R}^m \to \mathbb{R}$ and $e : \mathbb{R} \to \mathbb{R}^m$ are continuous, $A(t,x)$ satisfies the local Lipschitz condition with respect to the variable x, $V(\cdot)$ is continuously differentiable and $A(t+T,x)y \equiv QA(t, Q^{-1}x)Q^{-1}y$, $\nabla V(x) \equiv Q\nabla V(Q^{-1}x)$, $e(t+T) \equiv Qe(t)$. Following the authors, such a vectorial equation is called a (Q,T) affine-periodic ordinary differential equation, a (Q,T) affine-periodic functional differential equation, or a (Q,T) affine-periodic Newtonian equation, respectively. Practically, the authors have essentially verified Levinson's conjecture for Newtonian systems with friction and proposed the problem of existence of a (Q,T) affine-periodic solution for a Newtonian system with friction.

$\mathbb{T}$-Almost periodic functions

Assume that $\mathbb{T} : X \to X$ is a linear isomorphism. The notion of (Q,T) affine-periodicity is a special case of the notion of $(w, \mathbb{T})$-periodicity, which has recently been introduced and analyzed in the infinite-dimensional setting by M. Fečkan, K. Liu and J.-R. Wang in [434]: A function $h : I \to X$ is called $(w, \mathbb{T})$-periodic if and only if there exists $w \in I \setminus \{0\}$

such that $h(t+w) = \mathbb{T}h(t)$ for all $t \in I$. In the same paper, the authors have investigated the existence and uniqueness of (w, T)-periodic solutions of the following semilinear impulsive differential equations:

$$y'(t) = Cy(t) + h(t, y(t)), \quad t \neq \tau_i,\ i \in \mathbb{N};$$
$$\Delta y_{|t=\tau_i} = y(\tau_i^+) - y(\tau_i^-) = Dy(\tau_i^-) + d_i,$$

where C is the generator a strongly continuous semigroup on X, $D \in L(X)$, $y(\tau_i^+)$ and $y(\tau_i^-)$ denote the right and left limits of function $y(t)$ at the point $t = \tau_i > 0$, respectively, and $y(\tau_i) \equiv y(\tau_i^-)$.

Now, we consider the following notion: For a given $\varepsilon > 0$, a real number $\tau > 0$ is called $(\varepsilon, \mathbb{T})$-almost period of a continuous function $f : I \to X$ if and only if

$$\|f(t + \tau) - \mathbb{T}f(t)\| < \varepsilon, \quad t \in I.$$

Denote by $\vartheta_{\mathbb{T}}(f, \varepsilon)$ the set of all $(\varepsilon, \mathbb{T})$-almost periods of $f(\cdot)$, i. e.,

$$\vartheta_{\mathbb{T}}(f, \varepsilon) := \{\tau \in I : \sup_{t\in I}\|f(t + \tau) - \mathbb{T}f(t)\| < \varepsilon\}.$$

A continuous function $f : I \to X$ is called $\mathbb{T}$-almost periodic if and only if for any $\varepsilon > 0$ the set $\vartheta_{\mathbb{T}}(f, \varepsilon)$ is relatively dense in $[0, \infty)$.

In the case that there exists an integer $k \in \mathbb{N}_0$ such that $\mathbb{T}^k = I$, the notion of $(w, \mathbb{T})$-periodicity is a special case of the notion of $\mathbb{T}$-almost periodicity; the converse statement does not true in general. In the case that $\mathbb{T} = cI$, where $c \in \mathbb{C} \setminus \{0\}$ and I denotes the identity operator on X, the notion of $\mathbb{T}$-almost periodicity reduces to the notion of c-almost periodicity.

For more details about the class of $\mathbb{T}$-almost periodic functions as well as the general class of multi-dimensional ρ-almost periodic functions, we refer the reader to the forthcoming paper [433] by M. Fečkan et al.

Interpolation by periodic and almost periodic functions

The problems of interpolation by periodic and almost periodic functions were intensively studied by a group of Polish mathematicians during the 1960s. Probably the first fundamental result in this direction was obtained in 1961 by J. Mycielski [804], who proved that there exists a sequence (t_n) of positive real numbers such that, for every sequence (ε_n) in $\{0, 1\}$, there exists a continuous periodic function $f : \mathbb{R} \to \mathbb{C}$ such that $f(t_n) = \varepsilon_n$ for all $n \in \mathbb{N}$, answering so a question proposed earlier by E. Marczewski and C. Ryll-Nardzewski. Two years later, this result was extended by J. S. Lipiński in [716], who proved that there exists a sequence (t_n) of positive real numbers such that, for every bounded real function $g(\cdot)$ defined on the set $\{t_n : n \in \mathbb{N}\}$, there exists a continuous periodic function $f : \mathbb{R} \to \mathbb{C}$ such that $f(t_n) = g(t_n)$ for all $n \in \mathbb{N}$. The essence of the above-mentioned results is the rapid increasing of the sequence (t_n) as $n \to +\infty$:

in [804], we concretely have $t_n = (3+\alpha)^n$, where $\alpha > 0$. In [887], C. Ryll-Nardzewski has shown that, for every sequence (ε_n) in $\{0,1\}$, there exists a continuous periodic function $f : \mathbb{R} \to \mathbb{C}$ such that $f(3^n) = \varepsilon_n$ for all $n \in \mathbb{N}$ and that there does not exist a sequence (t_n) of positive real numbers with $t_n = O(2^n)$, $n \in \mathbb{N}$ satisfying the above property.

Interpolation by almost periodic functions was investigated for the first time by S. Hartman [515] in 1961 and later reconsidered in a series of his joint research papers with C. Ryll-Nardzewski [517–519] during the period 1964–1967. In [517], the authors analyzed the following properties for the subset Λ of the real line $\mathbb{R}$ (and the Abelian topological groups):

I. Λ satisfies the property I if and only if any bounded, uniformly continuous function $g : \Lambda \to \mathbb{C}$ can be extended to an almost periodic function $f : \mathbb{R} \to \mathbb{C}$;

I_0. Λ satisfies the property I_0 if and only if any bounded function $g : \Lambda \to \mathbb{C}$ can be extended to an almost periodic function $f : \mathbb{R} \to \mathbb{C}$.

The authors first proved that there is no sequence (ε_n) in $\{0,1\}$ and there is no almost periodic function $f : \mathbb{R} \to \mathbb{C}$ such that $f(n^\alpha) = \varepsilon_n$ for all $n \in \mathbb{N}$, provided that $\alpha > 0$ is not an integer; this essentially follows from the equality

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n^\alpha) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt,$$

which is valid for these values of number $\alpha > 0$. The main results concerning the properties I and I_0 and extensions to uniformly continuous almost periodic functions were proved in [517, Theorem 1, Theorem 2], while the third main result of this paper, [517, Theorem 3], analyzes a similar problem for extensions to Stepanov almost periodic functions. In [963], E. Strzelecki proved that any sequence (t_n) of positive real numbers such that $t_{n+1}/t_n > 1+\delta$, $n \in \mathbb{N}$, where $\delta > 0$ is a fixed real number, has the property I_0; later, this result was extended in [518, Theorem 5]. Interpolation by Levitan almost periodic functions was considered by S. Hartman in [516] (1974).

We close Part I with the observation that we will not analyze the Bohr compactifications nor the interplays between the almost periodicity and the representation theory in this monograph. Concerning these important subjects, we refer the reader to the doctoral dissertation of L. Riggins [872], the research articles [43] by R. Alizade, A. Pancar, [330] by B. A. Davey, M. Haviar, H. Priestley, [462] by J. Galindo, S. Hernández, T.-S. Wu, [512–514] by J. E. Hart, K. Kunen, [1104] by P. Zlatoš, the articles [539, 885, 888] (Bohr compactifications) and the presentation [1022] by S. Wang, the research articles [110] by U. Bader, C. Rosendal, R. Sauer, [334] by M. Daws, [479] by A. Gorbis, A. Tempelman, [740] by M. Yu. Ljubich, Yu. I. Ljyubich, [573] by M. I. Kadets, the articles [451, 571, 741] (almost periodic representations) and the references cited therein.

Part II: **Multi-dimensional almost periodic type functions and applications**

The main aim of this part is to consider various types of multi-dimensional almost periodic functions and multi-dimensional almost automorphic functions with values in complex Banach spaces. Unless stated otherwise, we assume that $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are complex Banach spaces, $n \in \mathbb{N}$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X, R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$; usually, $\mathcal{B}$ denotes the collection of all bounded subsets of X or all compact subsets of X. Set $\mathcal{B}_X := \{y \in X : (\exists B \in \mathcal{B})\, y \in B\}$. Although it may seem slightly redundant, we will always assume henceforth that $\mathcal{B}_X = X$, i. e., that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

If $\mathbf{t_0} \in \mathbb{R}^n$ and $\epsilon > 0$, then we set $B(\mathbf{t}_0, \epsilon) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t_0}| \leq \epsilon\}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. By $(e_1, e_2, \ldots, e_n)$ we denote the standard basis of $\mathbb{R}^n$.

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6 Multi-dimensional almost periodic type functions and applications

This chapter consists of four sections, Section 6.1–Section 6.4.

6.1 Multi-dimensional almost periodic type functions

In this section, we provide deeper insight into multi-dimensional almost periodic type functions and their applications to abstract Volterra integro-differential equations.

Suppose that $F : \mathbb{R}^n \to X$ is a continuous function. Let us recall that $F(\cdot)$ is almost periodic if and only if, for every real number $\varepsilon > 0$, there exists a real number $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l)$ with

$$\|F(\mathbf{t} + \tau) - F(\mathbf{t})\| \leqslant \varepsilon, \quad \mathbf{t} \in \mathbb{R}^n;$$

equivalently, for every sequence $(\mathbf{b}_n)$ in $\mathbb{R}^n$, there exists a subsequence $(\mathbf{a}_n)$ of $(\mathbf{b}_n)$ such that $(F(\cdot + \mathbf{a}_n))$ converges in $C_b(\mathbb{R}^n : X)$, or there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$ which converges uniformly to $F(\cdot)$. If this is the case, then the mean value

$$M(F) := \lim_{T \to +\infty} \frac{1}{(2T)^n} \int\limits_{s+K_T} F(\mathbf{t})\, d\mathbf{t}$$

exists and is independent of $s \in \mathbb{R}^n$, where $K_T := \{\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : |t_i| \leqslant T \text{ for } 1 \leqslant i \leqslant n\}$.

The notion of an almost periodic function $f : G \to E$, where G is a topological group and E is a complete locally convex space, was introduced in the landmark paper by S. Bochner and J. von Neumann [191] (1935); see also the paper by J. von Neumann [811] (1934) for the scalar-valued case $E = \mathbb{C}$. Almost periodic functions on (semi-)topological (semi-)groups have been also analyzed in the research monographs [158] by J. F. Berglund, K. H. Hofmann, [233] by R. B. Burckel, [696] by B. M. Levitan and [824] by A. A. Pankov, the doctoral dissertations of A. B. Ferrentino [440] and X. Zhu [1103], the survey article [934] by A. I. Shtern as well as the articles [38, 143, 159, 294, 304, 351, 352, 473, 525, 582, 773, 778, 779, 1048]; for more details about almost automorphic functions on (semi-)topological groups, the reader may consult [772] and [870].

Working with general almost periodic functions on topological groups is rather non-trivial and, clearly, it is very difficult to provide certain applications to the abstract PDEs following this general approach. Because of that, we have decided to concretize the situation here by considering various notions of almost periodicity for the vector-valued functions defined on the domain of form $I \times X$, where $\emptyset \neq I \subseteq \mathbb{R}^n$ generally does not satisfy the semigroup property $I + I \subseteq I$ or contain the zero vector. Actually, the main aim of this section is to introduce and systematically analyze various

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classes of (asymptotically) $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions and (asymptotically) Bohr $\mathcal{B}$-almost periodic type functions as well as to provide several important applications to the abstract PDEs in Banach spaces. With the exception of the usual notion of Bohr almost periodicity only, the introduced notion is new even in the one-dimensional setting.

In Definition 6.1.1 and Definition 6.1.2, we introduce the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity and the notion of $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodicity for a continuous function $F : I \times X \to Y$. The convolution invariance of space consisting of all $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic functions is stated in Proposition 6.1.5, while the supremum formula for the class of $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions is stated in Proposition 6.1.6 (we also analyze the relative compactness of the range of the restrictions of an $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function $F : I \times X \to Y$ to the sets of form $I \times B$, $B \in \mathcal{B}$).

The notion of Bohr $\mathcal{B}$-almost periodicity and the notion of $\mathcal{B}$-uniform recurrence for a continuous function $F : I \times X \to Y$ are introduced in Definition 6.1.9, provided that the region I satisfies $I+I \subseteq I$. Numerous illustrative examples of Bohr $\mathcal{B}$-almost periodic functions and $\mathcal{B}$-uniformly recurrent functions are presented in Example 6.1.12 and Example 6.1.13. In Definition 6.1.14, we introduce the notion of Bohr $(\mathcal{B}, I')$-almost periodicity and $(\mathcal{B}, I')$-uniform recurrence, provided that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. After that, we provide several examples of Bohr $(\mathcal{B}, I')$-almost periodic functions and $(\mathcal{B}, I')$-uniformly recurrent functions in Example 6.1.16. The relative compactness of range $F(I \times B)$, where $B \in \mathcal{B}$, for a Bohr $\mathcal{B}$-almost periodic function $F : I \times X \to Y$ is analyzed in Proposition 6.1.17. The Bochner criterion for Bohr $\mathcal{B}$-almost periodic functions is stated in Theorem 6.1.18. Proposition 6.1.19 is crucial for clarifying the composition principles of Bohr $\mathcal{B}$-almost periodic functions; there we investigate the common ε-periods (see Definition 6.1.9(i)) for the finite families of Bohr $\mathcal{B}$-almost periodic functions defined on $\mathbb{R}^n \times X$ (see also Proposition 6.1.20 and Proposition 6.1.21, where we analyze the pointwise products of Bohr $\mathcal{B}$-functions and $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions with scalar-valued functions). The uniform continuity of a Bohr $\mathcal{B}$-almost periodic function $F : I \times X \to Y$ is analyzed in Proposition 6.1.22. The analysis carried out in this proposition indicates again that it is very unpleasant to work, in the case of consideration of Bohr $\mathcal{B}$-almost periodicity, with a general region $I \neq \mathbb{R}^n$.

Definition 6.1.26 introduces the notion of space $C_{0,\mathbb{D}}(I \times X : Y)$, which is crucial for introducing the notions of various types of $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity and $\mathbb{D}$-asymptotically Bohr $\mathcal{B}$-almost periodicity; see Definition 6.1.27. The main structural properties of introduced classes of almost periodic functions are established in Proposition 6.1.28 and Proposition 6.1.29–Proposition 6.1.32. Definition 6.1.33 introduces the notion of $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I')$-almost periodicity of type 1 and $\mathbb{D}$-asymptotical $(\mathcal{B}, I')$-uniform recurrence of type 1, which are further analyzed in terms of several results preceding Subsection 6.1.4, where we investigate the differentiation and integration of $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions. Our main results in this part are Theorem 6.1.35 and Theorem 6.1.40.

Concerning the proof of Theorem 6.1.35, we should recall that H. Bart and S. Goldberg have proved in [119] that, for every function $f \in AP([0,\infty) : X)$, there exists a unique almost periodic function $\mathbb{E}f : \mathbb{R} \to X$ such that $\mathbb{E}f(t) = f(t)$ for all $t \geqslant 0$. We will investigate the extensions of multi-dimensional almost periodic functions and multi-dimensional uniformly recurrent functions in Remark 4.2.98, Theorem 6.1.37 and Corollary 6.1.38 following the method proposed in the proof of Theorem 6.1.35, which is essentially based on the argumentation contained in the proof of [881, Theorem 3.4], the important theoretical result deduced by W. M. Ruess and W. H. Summers. Subsection 6.1.5 is devoted to the study of composition theorems for multi-dimensional almost periodic type functions. The final subsection is reserved for some applications of our theoretical results to the abstract Volterra integro-differential equations in Banach spaces.

6.1.1 $(\mathrm{R}, \mathcal{B})$-Multi-almost periodic type functions

The main aim of this subsection is to analyze $(\mathrm{R}, \mathcal{B})$-multi-almost periodic type functions. Let us recall that $\mathcal{B}$ denotes a non-empty collection of non-empty subsets of X, R denotes a non-empty collection of sequences in $\mathbb{R}^n$ and R_X denotes a non-empty collection of sequences in $\mathbb{R}^n \times X$.

In the following two definitions, we introduce the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity and one of its most important generalizations, the notion of $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost periodicity (the both notions can be introduced on general semitopological groups but the possibility for providing certain applications to the abstract PDEs is rather confined in this approach).

Definition 6.1.1. Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function, and the following condition holds:

$$\text{If } \mathbf{t} \in I,\ \mathbf{b} \in \mathrm{R} \text{ and } l \in \mathbb{N},\ \text{then we have } \mathbf{t} + \mathbf{b}(l) \in I. \tag{6.1}$$

Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : I \times X \to Y$ such that

$$\lim_{l \to +\infty} F\big(\mathbf{t} + \big(b_{k_l}^1, \ldots, b_{k_l}^n\big); x\big) = F^*(\mathbf{t}; x), \tag{6.2}$$

uniformly for all $x \in B$ and $\mathbf{t} \in I$. By $AP_{(\mathrm{R},\mathcal{B})}(I \times X : Y)$ we denote the space consisting of all $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions.

Definition 6.1.2. Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function, and the following condition holds:

$$\text{If } \mathbf{t} \in I,\ (\mathbf{b}; \mathbf{x}) \in \mathrm{R}_\mathrm{X} \text{ and } l \in \mathbb{N},\ \text{then we have } \mathbf{t} + \mathbf{b}(l) \in I. \tag{6.3}$$

Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost periodic if and only if for every $B\in\mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k=((b_k^1,b_k^2,\ldots,b_k^n);x_k)_k)\in\mathrm{R}_{\mathrm{X}}$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l}=((b_{k_l}^1,b_{k_l}^2,\ldots,b_{k_l}^n);x_{k_l})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^*:I\times X\to Y$ such that

$$\lim_{l\to+\infty}F\bigl(\mathbf{t}+\bigl(b_{k_l}^1,\ldots,b_{k_l}^n\bigr);x+x_{k_l}\bigr)=F^*(\mathbf{t};x), \tag{6.4}$$

uniformly for all $x\in B$ and $\mathbf{t}\in I$. By $AP_{(\mathrm{R}_{\mathrm{X}},\mathcal{B})}(I\times X:Y)$ we denote the space consisting of all $(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost periodic functions.

In our further investigations of $(\mathrm{R},\mathcal{B})$-multi-almost periodicity ($(\mathrm{R}_X,\mathcal{B})$-multi-almost periodicity), we will always assume that (6.1) ((6.3)) holds for I and R (I and R_X). Before we proceed, we would like to provide several useful observations about the notion introduced above.

Remark 6.1.3.
(i) The notion introduced in Definition 6.1.1 is a special case of the notion introduced in Definition 6.1.2. In order to see this, suppose that the function $F:I\times X\to Y$ is continuous. Set

$$\mathrm{R}_{\mathrm{X}}:=\bigl\{b:\mathbb{N}\to I\times X;\ (\exists a\in\mathrm{R})b(l)=\bigl(a(l);0\bigr)\text{ for all }l\in\mathbb{N}\bigr\}.$$

Then it is clear that (6.1) holds for I and R if and only if (6.3) holds for I and R_X; furthermore, with this collection of sequences, we find that $F(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic if and only if $F(\cdot;\cdot)$ is $(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost periodic. It is also clear that, if the function $F(\cdot;\cdot)$ is $(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost periodic, then we have $F^*(\mathbf{t};x)\in\overline{R(F)}$ for all $x\in X$ and $\mathbf{t}\in I$.

(ii) The domain I from the above two definitions is rather general. For example, if $n=1$, $I=[0,\infty)$, $X=\{0\}$, $\mathcal{B}=\{X\}$ and R is the collection of all sequences in $[0,\infty)$, then the notion of $(\mathrm{R},\mathcal{B})$-multi-almost periodicity is equivalent with the notion of asymptotical almost periodicity considered usually since a function $f:[0,\infty)\to Y$ is asymptotically almost periodic if and only if the set $H(f):=\{f(\cdot+s):s\geqslant 0\}$ is relatively compact in $C_b([0,\infty):X)$, which means that for any sequence (b_n) of non-negative real numbers there exists a subsequence (a_n) of (b_n) such that $(f(\cdot+a_n))$ converges in $C_b([0,\infty):X)$. Moreover, if I is a cone in $\mathbb{R}^n$, $X=\{0\}$, $\mathcal{B}=\{X\}$, $Y=\mathbb{C}$ and R is a collection of all sequences in I, then a well-known result of K. deLeeuw and I. Glicksberg [352, Theorem 9.1] says that any $(\mathrm{R},\mathcal{B})$-multi-almost periodic function $F:I\to\mathbb{C}$ can be uniformly approximated by linear combinations of semicharacters of I, which will be exponential functions in this case. If $X=\{0\}$, then we also say that the function $F:I\to Y$ is R-multi-almost periodic, resp. R_{X}-multi-almost periodic.

(iii) It is clear that an R-multi-almost periodic function need not be bounded in general; for example, if R is the collection of all bounded sequences in $\mathbb{R}^n$, then an application of the Bolzano–Weierstrass theorem shows that the identical mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$ is R-multi-almost periodic. The existence of an unbounded sequence $(\mathbf{b}_k) \in \mathrm{R}$ does not imply the boundedness of $F(\cdot)$, as well; for example, any unbounded uniformly recurrent function $F : \mathbb{R}^n \to Y$ satisfying the estimate (6.9) below with the sequence (τ_k) in $\mathbb{R}^n$ satisfying $\lim_{k\to+\infty} |\tau_k| = +\infty$ is R-multi-almost periodic with R being the collection consisting of the sequence (τ_k) and all its subsequences.

(iv) Suppose $0 \in I$, $I + I \subseteq I$, $\mathrm{R_X}$ denotes the collection of all sequences in $I \times X$ and $\mathcal{B} = \{X\}$. Let us recall that two sufficient conditions for a continuous function $F : I \times X \to Y$ to be $(\mathrm{R_X}, \mathcal{B})$-multi-almost periodic were obtained by P. Milnes in [773, Theorem 2] and T. Kayano in [582, Theorem 3]; some equivalent conditions for $F(\cdot;\cdot)$ to be $(\mathrm{R_X}, \mathcal{B})$-multi-almost periodic can be found in [773, Theorem 1(i)] and [582, Theorem 4(d)].

Let $k \in \mathbb{N}$ and $F_i : I \times X \to Y_i$ $(1 \leqslant i \leqslant k)$. Then we define the function $(F_1, \ldots, F_k) : I \times X \to Y_1 \times \cdots \times Y_k$ by

$$(F_1, \ldots, F_k)(\mathbf{t}; x) := \big(F_1(\mathbf{t}; x), \ldots, F_k(\mathbf{t}; x)\big), \quad \mathbf{t} \in I,\ x \in X.$$

Using an induction argument and an elementary argumentation, we may deduce the following.

Proposition 6.1.4.

(i) *Suppose that $k \in \mathbb{N}$, $\emptyset \neq I \subseteq \mathbb{R}^n$,* (6.1) *holds and for any sequence which belongs to* R *we find that any its subsequence also belongs to* R. *If the function $F_i(\cdot;\cdot)$ is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic for* $1 \leqslant i \leqslant k$*, then the function* $(F_1, \ldots, F_k)(\cdot;\cdot)$ *is also* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic.*

(ii) *Suppose that $k \in \mathbb{N}$, $\emptyset \neq I \subseteq \mathbb{R}^n$,* (6.1) *holds and for any sequence which belongs to* $\mathrm{R_X}$ *we find that any its subsequence also belongs to* $\mathrm{R_X}$. *If the function $F_i(\cdot;\cdot)$ is* $(\mathrm{R_X}, \mathcal{B})$*-multi-almost periodic for* $1 \leqslant i \leqslant k$*, then the function* $(F_1, \ldots, F_k)(\cdot;\cdot)$ *is also* $(\mathrm{R_X}, \mathcal{B})$*-multi-almost periodic.*

The convolution invariance of $(\mathrm{R_X}, \mathcal{B})$-multi-almost periodicity is analyzed in the following proposition (see also Theorem 7.3.6 below).

Proposition 6.1.5. *Suppose that $h \in L^1(\mathbb{R}^n)$, the function $F(\cdot;\cdot)$ is* $(\mathrm{R_X}, \mathcal{B})$*-multi-almost periodic and for each bounded subset D of X there exists a constant $c_D > 0$ such that $\|F(\mathbf{t}; x)\|_Y \leqslant c_D$ for all $\mathbf{t} \in \mathbb{R}^n$, $x \in D$. Suppose, further, that for each sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)_k) \in \mathrm{R_X}$ and for each set $B \in \mathcal{B}$ we find that $B + \{x_k : k \in \mathbb{N}\}$ is a bounded set in X. Then the function*

$$(h * F)(\mathbf{t}; x) := \int\limits_{\mathbb{R}^n} h(\sigma) F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X,$$

is $(\mathrm{R_X},\mathcal{B})$*-multi-almost periodic and satisfies the requirement that for each bounded subset D of X there exists a constant* $c_D' > 0$ *such that* $\|(h * F)(\mathbf{t};x)\|_Y \leqslant c_D'$ *for all* $\mathbf{t} \in \mathbb{R}^n$, $x \in D$.

Proof. Since $h \in L^1(\mathbb{R}^n)$, the prescribed assumptions imply that the function $(h*F)(\cdot;\cdot)$ is well defined as well as that for each bounded subset D of X there exists a constant $c_D'' > 0$ such that $\|(h*F)(\mathbf{t};x)\|_Y \leqslant c_D''$ for all $\mathbf{t} \in \mathbb{R}^n$, $x \in D$. The continuity of the function $(h*F)(\cdot;\cdot)$ follows from the dominated convergence theorem and the same assumption on the function $F(\cdot;\cdot)$. Let the set $B \in \mathcal{B}$ be fixed. Then for each sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k) \in \mathrm{R_X}$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (6.4) holds. By our assumption, $B + \{x_k : k \in \mathbb{N}\}$ is a bounded set in X so that there exists a finite real constant $c_B''' > 0$ such that $\|F^*(\mathbf{t};x)\|_Y \leqslant c_B'''$ for all $\mathbf{t} \in \mathbb{R}^n$, $x \in B$. Keeping this in mind and our standing hypothesis $X_{\mathcal{B}} = X$, we see that the function $(h*F^*)(\cdot;\cdot)$ is well defined. The remainder of the proof can be deduced by using the estimate

$$\begin{aligned}&\big\|(h * F)(\mathbf{t} + \mathbf{b}_{k_l}; x + x_{k_l}) - (h * F^*)(\mathbf{t}; x)\big\|_Y \\ &\quad \leqslant \int_{\mathbb{R}^n} \big|h(\sigma)\big| \big\|F(\mathbf{t} + \mathbf{b}_{k_l} - \sigma; x + x_{k_l}) - F^*(\mathbf{t} - \sigma; x)\big\|_Y \, d\sigma,\end{aligned}$$

which holds for any $\mathbf{t} \in \mathbb{R}^n$, $l \in \mathbb{N}$ and $x \in X$; see Definition 6.1.2. □

Almost directly from the above definitions we may conclude the following: If $X \in \mathcal{B}$, $I = \mathbb{R}^n$ and $\mathrm{R_X}$ is a collection of all sequences in $\mathbb{R}^n \times X$, then the notion of $(\mathrm{R_X},\mathcal{B})$-multi-almost periodicity is equivalent with the usual notion of almost periodicity (see, e. g., [696, p. 255]).

For the notion introduced in Definition 6.1.1, the supremum formula can be proved under the following conditions.

Proposition 6.1.6. *Suppose that* $F : I \times X \to Y$ *is* $(\mathrm{R},\mathcal{B})$*-multi-almost periodic,* $a \geqslant 0$ *and* $x \in X$. *If there exists a sequence* $b(\cdot)$ *in* R *whose any subsequence is unbounded and for which we have* $\mathbf{T} - \mathbf{b}(l) \in I$ *whenever* $\mathbf{T} \in I$ *and* $l \in \mathbb{N}$, *then*

$$\sup_{\mathbf{t}\in I} \big\|F(\mathbf{t};x)\big\|_Y = \sup_{\mathbf{t}\in I, |t|\geqslant a} \big\|F(\mathbf{t};x)\big\|_Y. \tag{6.5}$$

Proof. We will include all relevant details of the proof for the sake of completeness. Let $\varepsilon > 0$, $a \geqslant 0$ and $x \in X$ be given. Then (6.5) will be satisfied if we prove that

$$\big\|F(\mathbf{t};x)\big\|_Y \leqslant \varepsilon + \sup_{\mathbf{t}\in I, |t|\geqslant a} \big\|F(\mathbf{t};x)\big\|_Y. \tag{6.6}$$

Let $B \in \mathcal{B}$ be such that $x \in B$, and let $b(\cdot)$ be any sequence in R with the prescribed assumptions. Then there exists an integer $l_0 \in \mathbb{N}$ such that

$$\big\|F(\mathbf{T} - (b_{k_{l_0}}^1, \ldots, b_{k_{l_0}}^n); x) - F(\mathbf{T} - (b_{k_l}^1, \ldots, b_{k_l}^n); x)\big\|_Y \leqslant \varepsilon, \quad l \geqslant l_0,\ \mathbf{T} \in I,\ x \in B.$$

Plugging $\mathbf{t} = \mathbf{T} - (b_{k_{l_0}}^1, \ldots, b_{k_{l_0}}^n)$, we simply obtain (6.6). □

Now we will prove the following result.

Proposition 6.1.7. *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is $(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodic and, for every sequence which belongs to R_X, any its subsequence also belongs to R_X. If the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on X, then the function $F(\cdot;\cdot)$ is $(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodic.*

Proof. The proof is very similar to the proof of [492, Theorem 2.1.10] but we will provide all relevant details. Let $\mathbf{t} \in I$ and $x \in X$ be given. In order to prove that the function $F(\cdot;\cdot)$ is continuous at $(\mathbf{t};x)$, observe first that our standing assumption $\mathcal{B}_X = X$ gives the existence of a set $B \in \mathcal{B}$ such that $x \in B$. Since the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on X, we have the existence of a positive integer $n_0 \in \mathbb{N}$ such that $\|F_{n_0}(\mathbf{t}';x') - F(\mathbf{t}';x')\|_Y \leqslant \varepsilon/3$ for all $\mathbf{t}' \in I$ and $x' \in X$. After that, it suffices to observe that

$$\begin{aligned}\|F(\mathbf{t};x) - F(\mathbf{t}';x')\|_Y \leqslant \|F(\mathbf{t}';x') - F_{n_0}(\mathbf{t}';x')\|_Y + \|F_{n_0}(\mathbf{t}';x') - F_{n_0}(\mathbf{t};x)\|_Y \\ + \|F_{n_0}(\mathbf{t};x) - F(\mathbf{t};x)\|_Y, \quad \mathbf{t}' \in I,\ x' \in X,\end{aligned} \tag{6.7}$$

as well as to employ the continuity of $F_{n_0}(\cdot;\cdot)$ at $(\mathbf{t};x)$. Furthermore, let the set $B \in \mathcal{B}$ and the sequence $((\mathbf{b}_k;x_k) = ((b_k^1,b_k^2,\ldots,b_k^n);x_k)) \in \mathrm{R}_\mathrm{X}$ be given. Since we have assumed that, for every sequence which belongs to R_X, any of its subsequences also belongs to R_X, using the diagonal procedure we get the existence of a subsequence $((\mathbf{b}_{k_l};x_{k_l}) = ((b_{k_l}^1,b_{k_l}^2,\ldots,b_{k_l}^n);x_{k_l}))$ of $((\mathbf{b}_k;x_k))$ such that for each integer $j \in \mathbb{N}$ there exists a function $F_j^* : I \times X \to Y$ such that

$$\lim_{l\to+\infty} \|F_j(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l}) - F_j^*(\mathbf{t};x)\|_Y = 0, \tag{6.8}$$

uniformly for $x \in B$ and $\mathbf{t} \in I$. Fix now a positive real number $\varepsilon > 0$. Since

$$\begin{aligned}\|F_i^*(\mathbf{t};x) - F_j^*(\mathbf{t};x)\|_Y &\leqslant \|F_i^*(\mathbf{t};x) - F_i(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l})\|_Y \\ &\quad + \|F_i(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l}) - F_j(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l})\|_Y \\ &\quad + \|F_j(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l}) - F_j^*(\mathbf{t};x)\|_Y,\end{aligned}$$

and (6.8) holds, we can find a number $l_0 \in \mathbb{N}$ such that for all integers $l \geqslant l_0$ we have

$$\begin{aligned}&\|F_i^*(\mathbf{t};x) - F_i(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l})\|_Y \\ &\quad + \|F_j(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l}) - F_j^*(\mathbf{t};x)\|_Y < 2\varepsilon/3,\end{aligned}$$

uniformly for $x \in B$ and $\mathbf{t} \in I$. Since the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for all integers $i, j \in \mathbb{N}$ with $\min(i,j) \geqslant N(\varepsilon)$ we have

$$\|F_i(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l}) - F_j(\mathbf{t} + (b_{k_l}^1,\ldots,b_{k_l}^n);x + x_{k_l})\|_Y < \varepsilon/3,$$

uniformly for $x \in B$ and $\mathbf{t} \in I$. This implies that $(F_j^*(\mathbf{t};x))$ is a Cauchy sequence in Y and therefore convergent to an element $F^*(\mathbf{t};x)$, say. The above arguments simply yield that $\lim_{j\to+\infty} F_j^*(\mathbf{t};x) = F^*(\mathbf{t};x)$ uniformly for $\mathbf{t} \in I$ and $x \in B$. Furthermore, observe that for each $j \in \mathbb{N}$ we have

$$\begin{aligned}
&\|F(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x + x_{k_l}) - F^*(\mathbf{t};x)\|_Y \\
&\quad \leqslant \|F(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x + x_{k_l}) - F_j(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x + x_{k_l})\|_Y \\
&\qquad + \|F_j(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x + x_{k_l}) - F_j^*(\mathbf{t};x)\|_Y + \|F_j^*(\mathbf{t};x) - F^*(\mathbf{t};x)\|_Y.
\end{aligned}$$

It can be simply shown that there exists a number $j_0(\varepsilon) \in \mathbb{N}$ such that for all integers $j \geqslant j_0$ the first addend and the third addend in the above estimate are less or greater than $\varepsilon/3$, uniformly for $x \in B$ and $\mathbf{t} \in I$. For the second addend, take any integer $l \in \mathbb{N}$ such that

$$\|F_j(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x + x_{k_l}) - F_j^*(\mathbf{t};x)\|_Y < \varepsilon/3, \quad x \in B,\ \mathbf{t} \in I.$$

This completes the proof in a routine manner. □

We can similarly deduce the following.

Corollary 6.1.8. *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic and, for every sequence which belongs to R, any its subsequence also belongs to R. If for each $B \in \mathcal{B}$ there exists $\varepsilon_B > 0$ such that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$, then the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.*

Proof. The proof is almost completely the same as the proof of the previous proposition and we will only emphasize the main differences. The first difference is with regards to the continuity of the function $F(\cdot;\cdot)$ at $(\mathbf{t};x)$, where $\mathbf{t} \in I$ and $x \in X$ are given in advance. As above, we have the existence of a set $B \in \mathcal{B}$ such that $x \in B$. Since the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B' \equiv B^\circ \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$, we have the existence of a positive integer $n_0 \in \mathbb{N}$ such that $\|F_{n_0}(\mathbf{t}';x') - F(\mathbf{t}';x')\|_Y \leqslant \varepsilon/3$ for all $\mathbf{t}' \in I$ and $x' \in B'$. After that, it suffices to apply (6.7) and the continuity of $F_{n_0}(\cdot;\cdot)$ at $(\mathbf{t};x)$ (it should be noted that this part can be applied for proving the continuity of the function $F(\cdot;\cdot)$ at $(\mathbf{t};x)$ in the previous proposition under this weaker condition). The second difference is with regards to the uniform continuity; in Proposition 6.1.7, it is necessary to assume that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the whole space X. In the newly arisen situation, it suffices to assume that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set B, only. □

The following special cases will be very important for us in the sequel:

L1. $\mathrm{R} = \{b : \mathbb{N} \to \mathbb{R}^n;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a, a, a, \ldots, a) \in \mathbb{R}^n : a \in \mathbb{R}\}\}$. If $n = 2$ and $\mathcal{B}$ denotes the collection of all bounded subsets of X, then we also say that the function $F(\cdot;\cdot)$ is bi-almost periodic.
The notion of bi-almost periodicity plays an incredible role in the research study [674] by H. C. Koyuncuoğlu and M. Adıvar, where the authors have analyzed the existence of almost periodic solutions for a class of discrete Volterra systems and the research study [851] by M. Pinto and C. Vidal, where the authors have used the notion of integrable bi-almost periodic Green functions of linear homogeneous differential equations and the Banach contraction principle to show the existence of almost and pseudo-almost periodic mild solutions for a class of the abstract differential equations with constant delay (see also the research article [264], where A. Chávez, S. Castillo and M. Pinto have used the notion of bi-almost-automorphicity in their investigation of almost automorphic solutions of abstract differential equations with piecewise constant arguments, as well as [148, 717, 745]). The notion of k-bi-almost periodicity was introduced by M. Pinto in [848] and further analyzed in [315, Section 4], where the authors have analyzed the existence and uniqueness of weighted pseudo-almost periodic solutions for a class of abstract integro-differential equations.

L2. R is the collection of all sequences $b(\cdot)$ in $\mathbb{R}^n$, resp. R_{X} is the collection of all sequences in $\mathbb{R}^n \times X$. This is the limit case in our analysis because, in this case, any $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function, resp. $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic function, is automatically $(\mathrm{R}_1, \mathcal{B})$-multi-almost periodic, resp. $(\mathrm{R}_{1\mathrm{X}}, \mathcal{B})$-multi-almost periodic, for any other collection R_1 of sequences $b(\cdot)$ in $\mathbb{R}^n$, resp. any other collection $\mathrm{R}_{1\mathrm{X}}$ is the collection of sequences in $\mathbb{R}^n \times X$.

Concerning Bohr type definitions, we will consider the following notion (see also the paper [836] by A. I. Perov and T. K. Kacaran).

Definition 6.1.9. Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that

$$\|F(\mathbf{t} + \tau; x) - F(\mathbf{t}; x)\|_Y \leqslant \varepsilon, \quad \mathbf{t} \in I,\ x \in B.$$

(ii) $F(\cdot;\cdot)$ is $\mathcal{B}$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \|F(\mathbf{t} + \tau_k; x) - F(\mathbf{t}; x)\|_Y = 0. \tag{6.9}$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Bohr almost periodic (uniformly recurrent).

Remark 6.1.10. Suppose that $F : I \times X \to Y$ is a continuous function. If $\mathcal{B}'$ is a certain collection of subsets of X which contains $\mathcal{B}$, R' is a certain collection of sequences in $\mathbb{R}^n$ which contains R and Eq. (6.1) holds with the family R replaced with the family R' therein, resp. R'_{X} is a certain collection of sequences in $\mathbb{R}^n \times X$ which contains R_{X} and Eq. (6.3) holds with the family R_{X} replaced with the family R'_{X} therein. If $F(\cdot;\cdot)$ is $(\mathrm{R}', \mathcal{B}')$-multi-almost periodic, resp. $(\mathrm{R}'_{\mathrm{X}}, \mathcal{B}')$-multi-almost periodic, then $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic. Similarly, if $F(\cdot;\cdot)$ is Bohr $\mathcal{B}'$-almost periodic ($\mathcal{B}$-uniformly recurrent) for some family $\mathcal{B}'$ which contains $\mathcal{B}$, then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic ($\mathcal{B}$-uniformly recurrent). Therefore, it is important to know the maximal collections $\mathcal{B}$, R and R_X, with the meaning clear, for which the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic, Bohr $\mathcal{B}$-almost periodic or $\mathcal{B}$-uniformly recurrent.

It is clear that any Bohr ($\mathcal{B}$-)almost periodic function is ($\mathcal{B}$-)uniformly recurrent; in general, the converse statement does not hold. It is also clear that, if $F(\cdot;\cdot)$ is $\mathcal{B}$-uniformly recurrent and $x \in X$, then we have the following supremum formula:

$$\sup_{\mathbf{t}\in I} \|F(\mathbf{t}; x)\|_Y = \sup_{\mathbf{t}\in I, |t|\geqslant a} \|F(\mathbf{t}; x)\|_Y,$$

which in particular shows that for each $x \in X$ the function $F(\cdot; x)$ is identically equal to zero provided that the function $F(\cdot;\cdot)$ is $\mathcal{B}$-uniformly recurrent and $\lim_{t\in I, |t|\to+\infty} F(\mathbf{t}; x) = 0$. The statements of [166, Theorem 7, p. 3] and Proposition 6.1.5 can be reformulated in this framework, as well.

Keeping in mind the proof of [697, Property 4, p. 3], the following result can be proved as in the one-dimensional case.

Proposition 6.1.11. *Suppose that $F : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent), and $\phi : Y \to Z$ is uniformly continuous on $\overline{R(F)}$. Then $\phi \circ F : I \times X \to Z$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent).*

We continue by providing several illustrative examples and useful observations.

Example 6.1.12. In contrast with the class of Bohr $\mathcal{B}$-almost periodic functions, we can simply construct a great number of multi-dimensional $\mathcal{B}$-uniformly recurrent functions by using Proposition 6.1.11 and the fact that, for any given tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \neq 0$, the linear function

$$g(\mathbf{t}) := a_1 t_1 + \cdots + a_n t_n, \quad \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$$

is uniformly recurrent provided that $n > 1$. To verify this, it suffices to observe that the set $W := \{(t_1, \ldots, t_n) \in \mathbb{R}^n : a_1 t_1 + \cdots + a_n t_n = 0\}$ is a non-trivial linear submanifold of $\mathbb{R}^n$ as well as that $g(\mathbf{t} + \mathbf{t}') = g(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{t}' \in W$. Therefore, for any uniformly continuous function $\phi : \mathbb{R} \to X$, we find that the function $\phi \circ g : \mathbb{R}^n \to X$ is uniformly recurrent.

Example 6.1.13.

(i) Suppose that $F_j : X \to Y$ is a continuous function, for each $B \in \mathcal{B}$ we have $\sup_{x \in B} \|F_j(x)\|_Y < \infty$ and the complex-valued mapping $t \mapsto \int_0^t f_j(s)\,ds$, $t \geqslant 0$ is almost periodic $(1 \leqslant j \leqslant n)$. Set

$$F(t_1, \ldots, t_{n+1}; x) := \sum_{j=1}^{n} \int_{t_j}^{t_{j+1}} f_j(s)\,ds \cdot F_j(x) \quad \text{for all } x \in X \text{ and } t_j \geqslant 0,\ 1 \leqslant j \leqslant n.$$

Then the mapping $F : [0,\infty)^{n+1} \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic. In actual fact, for every $B \in \mathcal{B}$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&\|F(t_1 + \tau_1, \ldots, t_{n+1} + \tau_{n+1}; x) - F(t_1, \ldots, t_{n+1}; x)\|_Y \\
&\quad \leqslant \sum_{j=1}^{n} \left| \int_{t_j+\tau_j}^{t_{j+1}+\tau_{j+1}} f_j(s)\,ds - \int_{t_j}^{t_{j+1}} f_j(s)\,ds \right| \cdot \|F_j(x)\|_Y \\
&\quad \leqslant \sum_{j=1}^{n} \left\{ \left| \int_0^{t_{j+1}+\tau_{j+1}} f_j(s)\,ds - \int_0^{t_{j+1}} f_j(s)\,ds \right| \right. \\
&\qquad \left. + \left| \int_0^{t_j+\tau_j} f_j(s)\,ds - \int_0^{t_j} f_j(s)\,ds \right| \right\} \cdot \|F_j(x)\|_Y \\
&\quad \leqslant M \sum_{j=1}^{n} \left\{ \left| \int_0^{t_{j+1}+\tau_{j+1}} f_j(s)\,ds - \int_0^{t_{j+1}} f_j(s)\,ds \right| \right. \\
&\qquad \left. + \left| \int_0^{t_j+\tau_j} f_j(s)\,ds - \int_0^{t_j} f_j(s)\,ds \right| \right\},
\end{aligned}$$

where $M = \sup_{x \in B, 1 \leqslant j \leqslant n} \|F_j(x)\|_Y$. The corresponding statement follows by considering the common $\varepsilon/(2nM)$-periods τ_j of the functions $\int_0^{\cdot} f_{j-1}(s)\,ds$ and $\int_0^{\cdot} f_j(s)\,ds$ for $2 \leqslant j \leqslant n$, the $\varepsilon/(2nM)$-periods τ_1 of the function $\int_0^{\cdot} f_1(s)\,ds$ and the $\varepsilon/(2nM)$-periods τ_{n+1} of the function $\int_0^{\cdot} f_{n+1}(s)\,ds$. Furthermore, let us denote by $G_j(\cdot)$ the unique almost periodic extension of the function $t \mapsto \int_0^t f_j(s)\,ds$, $t \geqslant 0$ to the whole real line $(1 \leqslant j \leqslant n)$. Let $(\mathbf{b}_k)$ be any sequence in $\mathbb{R}^{n+1}$. Then we can use Theorem 6.1.18 below to conclude that there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that, for every $j \in \mathbb{N}_n$, $F_j(t_j + b^j_{k_l}, t_{j+1} + b^{j+1}_{k_l}) = G_j(t_{j+1} + b^{j+1}_{k_l}) - G_j(t_j + b^j_{k_l})$ converges to a function $F_j^*(t_j, t_{j+1})$ as $l \to +\infty$, uniformly for $(t_j, t_{j+1}) \in \mathbb{R}^2$. Define

$$F(t_1, \ldots, t_{n+1}; x) := \sum_{j=1}^{n} F_j^*(t_j, t_{j+1}) F_j(x), \quad \text{for all } x \in X \text{ and } t_j \geqslant 0,\ 1 \leqslant j \leqslant n.$$

Let $B \in \mathcal{B}$ be fixed. Then it can be simply shown that

$$\lim_{l\to+\infty} F(t_1 + b^1_{k_l}, \dots, t_{n+1} + b^{n+1}_{k_l}; x) = F(t_1, \dots, t_{n+1}; x)$$

uniformly for $x \in B$ and $\mathbf{t} = (t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1}$. Hence, the function $F_1(\cdot)$ is R-multi-almost periodic with R being the collection of all sequences in $\mathbb{R}^{n+1}$.

(ii) Suppose that $F : X \to Y$ is a continuous function, for each $B \in \mathcal{B}$ we have $\sup_{x\in B} \|F(x)\|_Y < \infty$ and the complex-valued mapping $t \mapsto f_j(t)$, $t \geqslant 0$ is almost periodic, resp. bounded and uniformly recurrent ($1 \leqslant j \leqslant n$). Set

$$F(t_1, \dots, t_n; x) := \prod_{j=1}^{n} f_j(t_j) \cdot F(x) \quad \text{for all } x \in X \text{ and } t_j \geqslant 0,\ 1 \leqslant j \leqslant n.$$

Then the mapping $F : [0,\infty)^n \times X \to X$ is Bohr $\mathcal{B}$-almost periodic, resp. $\mathcal{B}$-uniformly recurrent. In actual fact, for every $B \in \mathcal{B}$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&\|F(t_1 + \tau_1, \dots, t_n + \tau_n; x) - F(t_1, \dots, t_n; x)\|_Y \\
&\quad \leqslant M\Biggl\{ |f_1(t_1 + \tau_1) - f_1(t_1)| \cdot \prod_{j=2}^{n} |f_j(t_j + \tau_j)| + |f_1(t_1)| \cdot \prod_{j=2}^{n} |f_j(t_j + \tau_j) - f_j(t_j)| \Biggr\} \\
&\quad \leqslant M|f_1(t_1 + \tau_1) - f_1(t_1)| \cdot \prod_{j=2}^{n} \|f_j\|_\infty + \|f_1\|_\infty \cdot \prod_{j=2}^{n} |f_j(t_j + \tau_j) - f_j(t_j)|,
\end{aligned}$$

where $M = \sup_{x\in B} \|F(x)\|_Y$. Repeating this procedure, we simply get the required statement; furthermore, we can use the usual Bochner criterion and repeat the above calculus in order to see that the function $F_2(\cdot)$ is R-multi-almost periodic with R being the collection of all sequences in $\mathbb{R}^n$.

(iii) Suppose that $G : [0,\infty)^n \to \mathbb{C}$ is almost periodic, resp. bounded and uniformly recurrent, $F : [0,\infty) \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, resp. $\mathcal{B}$-uniformly recurrent, and for each set $B \in \mathcal{B}$ we have $\sup_{t\geqslant 0; x\in B} \|F(t;x)\|_Y < \infty$. Set

$$\begin{gathered}
F(t_1, \dots, t_{n+1}; x) := G(t_1, \dots, t_n) \cdot F(t_{n+1}; x) \\
\text{for all } x \in X \text{ and } t_j \geqslant 0,\ 1 \leqslant j \leqslant n+1.
\end{gathered}$$

Then the mapping $F : [0,\infty)^{n+1} \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, resp. $\mathcal{B}$-uniformly recurrent, which can be simply shown by using the estimate ($t_i, \tau_i \geqslant 0$ for $1 \leqslant i \leqslant n+1$; $x \in X$):

$$\begin{aligned}
&\|F(t_1 + \tau_1, \dots, t_{n+1} + \tau_{n+1}; x) - F(t_1, \dots, t_{n+1}; x)\|_Y \\
&\quad \leqslant |G(t_1 + \tau_1, \dots, t_n + \tau_n) - G(t_1, \dots, t_n)| \cdot \|F(t_{n+1} + \tau_{n+1}; x)\|_Y \\
&\qquad + |G(t_1, \dots, t_n)| \cdot \|F(t_{n+1} + \tau_{n+1}; x) - F(t_{n+1}; x)\|_Y,
\end{aligned}$$

the boundedness of the function $G(\cdot, \dots, \cdot)$ and the assumption that for each set $B \in \mathcal{B}$ we have $\sup_{t\geqslant 0; x\in B} \|F(t;x)\|_Y < \infty$ (see also Proposition 6.1.17 below).

It is worth noting that we can extend the notion introduced in Definition 6.1.9 as follows.

Definition 6.1.14. Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I')$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that

$$\|F(\mathbf{t} + \tau; x) - F(\mathbf{t}; x)\|_Y \leqslant \varepsilon, \quad \mathbf{t} \in I,\ x \in B. \tag{6.10}$$

(ii) $F(\cdot;\cdot)$ is $(\mathcal{B}, I')$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_n) in I' such that $\lim_{n\to+\infty} |\tau_n| = +\infty$ and

$$\lim_{n\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \|F(\mathbf{t} + \tau_n; x) - F(\mathbf{t}; x)\|_Y = 0.$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Bohr I'-almost periodic (I'-uniformly recurrent).

Clearly, the notion from Definition 6.1.9 is recovered by plugging $I' = I$ and any $(\mathcal{B}, I')$-uniformly recurrent function is $(\mathcal{B}, I)$-uniformly recurrent provided that $I+I \subseteq I$. This is not true for almost periodicity: we can simply construct a great number of corresponding examples showing that the notion of $(\mathcal{B}, I')$-almost periodicity is neither stronger nor weaker than the notion of $(\mathcal{B}, I)$-almost periodicity, provided that $I+I \subseteq I$.

Before proceeding any further, we would like to notice that the assumption $I+I' \subseteq I$ is crucial as well as that the notion introduced above can be understood even if the assumption $I' \subseteq I$ is neglected (a similar comment can be made for the corresponding Stepanov and Weyl classes of multi-dimensional almost periodic functions analyzed below). For example, we have the following.

Example 6.1.15. Suppose that $L > 0$ is a fixed real number as well as that the functions $t \mapsto f(t)$, $t \in \mathbb{R}$ and $t \mapsto g(t)$, $t \in \mathbb{R}$ are almost periodic. Set $I := \{(x, y) \in \mathbb{R}^2 : |x-y| \geqslant L\}$, $I' := \{(\tau, \tau) : \tau \in \mathbb{R}\}$ and

$$u(x, y) := \frac{f(x) + g(y)}{x - y}, \quad (x, y) \in I.$$

Then $I + I' \subseteq I$ but I' is not a subset of I. Furthermore, if $\varepsilon > 0$ is given and $\tau > 0$ is a common ε-period of the functions $f(\cdot)$ and $g(\cdot)$, then we have

$$\begin{aligned} \|u(x + \tau, y + \tau) - u(x, y)\| &\leqslant \frac{\|f(x + \tau) - f(x)\| + \|g(y + \tau) - g(y)\|}{|x - y|} \\ &\leqslant 2\varepsilon/L, \quad (x, y) \in I. \end{aligned}$$

This implies that the function $u(\cdot,\cdot)$ is Bohr I'-almost periodic. Observe, finally, that under some regularity conditions on the functions $f(\cdot)$ and $g(\cdot)$, the function $u(\cdot,\cdot)$ is a solution of the partial differential equation

$$u_{xy} - \frac{u_x}{x-y} + \frac{u_y}{x-y} = 0.$$

In many concrete situations, the situation in which $I' \neq I$ can occur; for example, we have the following.

Example 6.1.16.

(i) Suppose that the complex-valued mapping $t \mapsto \int_0^t f_j(s)\,ds$, $t \geqslant 0$ is almost periodic, resp. bounded and uniformly recurrent $(1 \leqslant j \leqslant n)$. Set

$$F_1(t_1,\ldots,t_{2n}) := \prod_{j=1}^{n} \int\limits_{t_j}^{t_{j+n}} f_j(s)\,ds \quad \text{and} \quad t_j \in \mathbb{R}, \quad 1 \leqslant j \leqslant 2n.$$

Then the argumentation used in Example 6.1.13(i)–(ii) shows that the mapping $F_1 : \mathbb{R}^{2n} \to \mathbb{C}$ is both Bohr I'-almost periodic, resp. I'-uniformly recurrent, where $I' = \{(\tau,\tau) : \tau \in \mathbb{R}^n\}$; in the case of consideration of almost periodicity, we can use [631, Theorem 2.1.1(xiv)] in order to see that $F_1(\cdot)$ is also Bohr I''-almost periodic, where $I'' = \{(a,a,\ldots,a) \in \mathbb{R}^{2n} : a \in \mathbb{R}\}$. Furthermore, in the same case, we can use Theorem 6.1.18 below and the usual Bochner criterion for the functions of one real variable to see that the function $F_1(\cdot)$ is Bohr almost periodic because it is R-almost periodic with R being the collection of all sequences in $\mathbb{R}^{2n}$.

(ii) Suppose that an X-valued mapping $t \mapsto \int_0^t f_j(s)\,ds$, $t \in \mathbb{R}$, is almost periodic, resp. bounded and uniformly recurrent, as well as that a strongly continuous operator family $(T_j(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ is uniformly bounded $(1 \leqslant j \leqslant n)$. Set

$$F_2(t_1,\ldots,t_{2n}) := \sum_{j=1}^{n} T_j(t_j - t_{j+n}) \int\limits_{t_j}^{t_{j+n}} f_j(s)\,ds$$

for all $t_j \in \mathbb{R}$, $1 \leqslant j \leqslant 2n$.

Since, for every t_i, $\tau_i \in \mathbb{R}$ $(1 \leqslant j \leqslant 2n)$ with $\tau_j = \tau_{j+n}$ $(1 \leqslant j \leqslant n)$, we have

$$\begin{aligned}
&\|F_2(t_1+\tau_1,\ldots,t_{2n}+\tau_{2n}) - F_2(t_1,\ldots,t_{2n})\|_Y \\
&\quad \leqslant M \sum_{j=1}^{n} \left\{ \left\| \int\limits_0^{t_j+\tau_j} f_j(s)\,ds - \int\limits_0^{t_j} f_j(s)\,ds \right\| + \left\| \int\limits_0^{t_{j+n}+\tau_j} f_j(s)\,ds - \int\limits_0^{t_{j+n}} f_j(s)\,ds \right\| \right\},
\end{aligned}$$

where $M = \sup_{1\leqslant j\leqslant n} \sup_{t\in\mathbb{R}} \|T_j(t)\|$, we may conclude as above that the mapping $F_2 : \mathbb{R}^{2n} \to Y$ is Bohr I'-almost periodic, resp. I'-uniformly recurrent, where $I' = \{(\tau,\tau) : \tau \in \mathbb{R}^n\}$, but not generally almost periodic, in the case of consideration of almost periodicity; in this case, we also find that $F_2(\cdot)$ is Bohr I''-almost periodic, where $I'' = \{(a,a,\ldots,a) \in \mathbb{R}^{2n} : a \in \mathbb{R}\}$, and that the function $F_2(\cdot)$ is R-multi-almost periodic with R being the collection of all sequences in I'.

(iii) Suppose that an X-valued mapping $t \mapsto f_j(t)$, $t \in \mathbb{R}$ is almost periodic, resp. bounded and uniformly recurrent, $F_j : X \to X$ is continuous as well as that a strongly continuous operator family $(T_j(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ satisfies the requirement that $\|T_j(t)\| \leqslant M_j e^{-\omega_j|t|}$, $t \in \mathbb{R}$ with $\omega_j > \|f_j\|_\infty$ $(1 \leqslant j \leqslant n)$. Set

$$\mathbb{F}_3(t_1,\ldots,t_{2n};x) := \sum_{j=1}^{n} e^{\int_{t_j}^{t_{j+n}} f_j(s)\,ds} T_j(t_j - t_{j+n})F_j(x)$$

$$\text{for all } x \in X \text{ and } t_j \in \mathbb{R},\ 1 \leqslant j \leqslant 2n.$$

Suppose, additionally, that for each set $B \in \mathcal{B}$ we have

$$\sup_{1\leqslant j\leqslant n; x\in B} \|F_j(x)\| < \infty.$$

Arguing as before, we may conclude with the help of the elementary inequality $|e^z - 1| \leqslant |z| \cdot e^{|z|}$, $z \in \mathbb{C}$ that the mapping $\mathbb{F}_3 : \mathbb{R}^{2n} \times X \to Y$ is Bohr $(\mathcal{B}, I')$-almost periodic, resp. $(\mathcal{B}, I')$-uniformly recurrent, where $I' = \{(\tau,\tau) : \tau \in \mathbb{R}^n\}$, but not generally Bohr $\mathcal{B}$-almost periodic, in the case of consideration of almost periodicity; in this case, we also find that $\mathbb{F}_3(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I'')$-almost periodic, where $I'' = \{(a,a,\ldots,a) \in \mathbb{R}^{2n} : a \in \mathbb{R}\}$, and that the function $F_2(\cdot)$ is R-multi-almost periodic with R being the collection of all sequences in I'.

(iv) Suppose that $-\infty \leqslant \alpha < \beta \leqslant +\infty$ and $f : \Omega \equiv \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\} \to X$ is an analytic almost periodic function (see also [824, Appendix 3]). Set, for $\alpha < \alpha' < \beta' < \beta$, $I_{\alpha',\beta'} := [\alpha',\beta'] \times \mathbb{R}$, $I'_{\alpha',\beta'} := \{0\} \times \mathbb{R}$ and $F(x,y) := f(x+iy)$, $(x,y) \in I_{\alpha',\beta'}$. Then $F(\cdot,\cdot)$ is Bohr $I'_{\alpha',\beta'}$-almost periodic.

(v) In connection with Example 6.1.12 and the notion introduced in Definition 6.1.14, the following should be stated: Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I_0 = [0,\infty)$ or $I_0 = \mathbb{R}$, $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{R}^n \neq 0$ and the linear function $g(\mathbf{t}) := a_1t_1 + \cdots + a_nt_n$, $\mathbf{t} = (t_1,\ldots,t_n) \in I$ maps surjectively the region I onto I_0. Suppose, further, that $f : I_0 \to X$ is a uniformly recurrent function as well as that a sequence (α_k) in I_0 satisfies the requirement that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and $\lim_{k\to+\infty} \sup_{t\in I_0} \|f(t+\alpha_k) - f(t)\| = 0$. Define $I' := g^{-1}(\{\alpha_k : k \in \mathbb{N}\})$ and $F : I \to X$ by $F(\mathbf{t}) := f(g(\mathbf{t}))$, $\mathbf{t} \in I$. Then $F(\cdot)$ is I'-uniformly recurrent, and $F(\cdot)$ is not almost periodic provided that $f(\cdot)$ is not almost periodic. In order to see this, observe that the surjectivity of the mapping $g : I \to I_0$ implies the existence of a sequence (τ_k) in I' such that $g(\tau_k) = \alpha_k$ for all $k \in \mathbb{N}$. Due to the Cauchy–Schwarz inequality, we have $|\tau_k| \geqslant |\alpha_k|/|a| \to +\infty$ as $k \to +\infty$. Furthermore, for every $\mathbf{t} \in I$, we have

$$\begin{aligned}
&\|F(\mathbf{t}+\tau_k) - F(\mathbf{t})\| \\
&\quad \leqslant \|f(g(\mathbf{t}+\tau_k)) - f(g(\mathbf{t})+\alpha_k)\| + \|f(g(\mathbf{t})+\alpha_k) - f(g(\mathbf{t}))\| \\
&\quad = \|f(g(\mathbf{t}+\tau_k)) - f(g(\mathbf{t}+\tau_k))\| + \|f(g(\mathbf{t})+\alpha_k) - f(g(\mathbf{t}))\| \\
&\quad = \|f(g(\mathbf{t})+\alpha_k) - f(g(\mathbf{t}))\| \leqslant \sup_{t\in I_0}\|f(t+\alpha_k) - f(t)\| \to 0,
\end{aligned}$$

as $k \to +\infty$. Suppose now that $f(\cdot)$ is not almost periodic. We will prove that $F(\cdot)$ is not almost periodic, as well. Let $l > 0$ be arbitrary. Due to our assumption, there exists $\varepsilon > 0$ such that there exists a subinterval $I'' \subseteq I_0$ of length $2|a|l$ such that for each $\tau \in I''$ there exists $t \in I_0$ such that $\|f(t+\tau) - f(t)\| > \varepsilon$. Let i'' be the center of I''. Then there exists $\mathbf{t}_0 \in I$ such that $g(\mathbf{t}_0) = i''$ and this simply implies that for each $\alpha \in B(\mathbf{t}_0, l) \cap I$ we have $g(\alpha) \in I''$. Therefore, for fixed α from this range, we can find $t \in I_0$ such that $\|f(t+\tau) - f(t)\| > \varepsilon$, where $\tau = g(\alpha)$. By surjectivity of $g(\cdot)$, we have the existence of a tuple $\mathbf{t} \in I$ such that $g(\mathbf{t}) = t$, which gives the required result.

(vi) Let $f : \mathbb{R} \to \mathbb{R}$ be given by (2.28). Then $f(\cdot)$ is uniformly continuous, uniformly recurrent (the sequence $(\alpha_k \equiv 2^k \pi)_{k \in \mathbb{N}}$ can be chosen in definition of uniform recurrence) and unbounded. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \neq 0$, let $I' = g^{-1}(\{2^k \pi : k \in \mathbb{N}\})$ and let $F : \mathbb{R}^n \to \mathbb{R}$ be given by $F(\mathbf{t}) := f(a_1 t_1 + \cdots + a_n t_n)$, $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$. Then the function $F(\cdot)$ is I'-uniformly recurrent and not almost periodic; furthermore, $F(\cdot)$ is uniformly continuous and unbounded.

(vii) Suppose that K is a bounded Lebesgue measurable set and $I+K \subseteq I$. Then we can simply prove that the Bohr $(\mathcal{B}, I')$-almost periodicity, resp. $(\mathcal{B}, I')$-uniform recurrence, of the function $F : I \times X \to Y$ implies the Bohr $(\mathcal{B}, I')$-almost periodicity, resp. $(\mathcal{B}, I')$-uniform recurrence, of the function $G : I \times X \to Y$ defined by

$$G(\mathbf{t}; x) := \int\limits_{\mathbf{t}}^{\mathbf{t}+K} F(\sigma; x)\, d\sigma = \int\limits_{K} F(\sigma + \mathbf{t}; x)\, d\sigma, \quad \mathbf{t} \in I,\ x \in X,$$

which extends the conclusions established in [1067, Example 7, p. 33] to the multi-dimensional case; furthermore, if $F : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic and for each $x \in X$ we have $\sup_{\mathbf{t} \in I} \|F(\mathbf{t}; x)\|_Y < \infty$, resp. $F : I \times X \to Y$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic and for each $B \in \mathcal{B}$, $x \in B$ and each sequence (x_k) in X for which there exists a sequence $(\mathbf{b}_k)$ in I such that $(\mathbf{b}_k; x_k) \in \mathrm{R}_{\mathrm{X}}$ we have $\sup_{\mathbf{t} \in I} \|F(\mathbf{t} + \mathbf{b}_k; x + x_k)\|_Y < \infty$, then the use of dominated convergence theorem shows that the function $G(\cdot; \cdot)$ is likewise $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic.

(viii) The notion of Bloch (p, k)-periodicity can be simply extended to the functions of several real variables as follows [521, 522]: a bounded continuous function $F : I \to X$ is said to be Bloch $(\mathbf{p}, \mathbf{k})$-periodic, or Bloch periodic with period $\mathbf{p}$ and Bloch wave vector or Floquet exponent $\mathbf{k}$, where $\mathbf{p} \in I$ and $\mathbf{k} \in \mathbb{R}^n$ if and only if $F(\mathbf{x}+\mathbf{p}) = e^{i\langle \mathbf{k}, \mathbf{p} \rangle} F(\mathbf{x})$, $\mathbf{x} \in I$ (of course, we assume here that $\mathbf{p} + I \subseteq I$). Arguing as in Remark 4.2.57, we may conclude that the Bloch $(\mathbf{p}, \mathbf{k})$-periodicity of the function $F(\cdot)$ implies the Bohr $(\mathcal{B}, I')$-almost periodicity of function $e^{-i\langle \mathbf{k}, \cdot \rangle} F(\cdot)$ with I' being the intersection of I and the one-dimensional submanifold generated by the vector $\mathbf{p}$; furthermore, if $\mathbf{k}$ is orthogonal to $\mathbf{p}$, then the function $F(\cdot)$ will be Bohr $(\mathcal{B}, I')$-almost periodic.

The previous examples show that the notions of Bohr I'-almost periodicity and Bohr I'-uniform recurrence are extremely important in the case that $I' \neq I$. But we feel it is our duty to say that the fundamental properties of Bohr I'-almost periodic functions and Bohr I'-uniformly recurrent functions cannot be so simply clarified in the case that $I' \neq I$. Because of that, we will basically assume henceforth that $I' = I$; our research has thrown up many questions in need of further analyses of Bohr I'-almost periodicity and Bohr I'-uniform recurrence in the case $I' \neq I$.

It can be simply shown that the subsequent proposition is applicable if $I = [0, \infty)^n$ or $I = \mathbb{R}^n$.

Proposition 6.1.17. *Suppose that* $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, *I is closed,* $F : I \times X \to Y$ *is Bohr $\mathcal{B}$-almost periodic and $\mathcal{B}$ is any family of compact subsets of X. If*

$$\begin{aligned} &(\forall l > 0)\,(\exists \mathbf{t_0} \in I)\,(\exists k > 0)\,(\forall \mathbf{t} \in I)\big(\exists \mathbf{t}_0' \in I\big) \\ &(\forall \mathbf{t}_0'' \in B(\mathbf{t}_0', l) \cap I)\; \mathbf{t} - \mathbf{t}_0'' \in B(\mathbf{t_0}, kl) \cap I, \end{aligned} \tag{6.11}$$

then for each $B \in \mathcal{B}$ the set $\{F(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in B\}$ is relatively compact in Y; in particular, $\sup_{\mathbf{t} \in I; x \in B} \|F(\mathbf{t}; x)\|_Y < \infty$.

Proof. Let $B \in \mathcal{B}$ and $\varepsilon > 0$ be given. Then we can find a finite number $l > 0$ such that for each $\mathbf{s} \in I$ there exists $\tau \in B(\mathbf{s}, l) \cap I$ such that (6.10) holds with $\mathbf{t}_0 = \mathbf{s}$. Let $\mathbf{t}_0 \in I$ and $k > 0$ be such that (6.11) holds. Since I is closed and B is compact, we find that the set $\{F(\mathbf{s}; x) : \mathbf{s} \in B(\mathbf{t_0}, kl) \cap I,\ x \in B\}$ is compact in Y. Let $\mathbf{t} \in I$ be fixed. We will show that $\|F(\mathbf{t}; x)\|_Y \leqslant M + \varepsilon$ for all $x \in B$. By our assumption, there exists $\mathbf{t}_0' \in I$ such that, for every $\mathbf{t}_0'' \in B(\mathbf{t}_0', l) \cap I$, we have $\mathbf{t} \in \mathbf{t}_0'' + [B(\mathbf{t_0}, kl) \cap I]$. On the other hand, there exists $\tau = \mathbf{t}_0'' \in B(\mathbf{t}_0', l) \cap I$ such that $\|F(\mathbf{s} + \tau; x) - F(\mathbf{s}; x)\|_Y \leqslant \varepsilon$, $\mathbf{s} \in I$, $x \in B$. Clearly, $\mathbf{s} = \mathbf{t} - \tau \in B(\mathbf{t_0}, kl) \cap I$, which simply implies from the last estimate that $\|F(\mathbf{t}; x)\|_Y \leqslant M + \varepsilon$ for all $x \in B$. This implies that $\{F(\mathbf{s}; x) : \mathbf{s} \in B(\mathbf{t_0}, kl) \cap I,\ x \in B\}$ is an ε-net for $\{F(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in B\}$, which completes the proof in a routine manner. □

Suppose now that $F : \mathbb{R}^n \times X \to Y$ is a Bohr $\mathcal{B}$-almost periodic function, where $\mathcal{B}$ is any family of compact subsets of X. Let $B \in \mathcal{B}$ be fixed. We will consider the Banach space $l_\infty(B : Y)$ consisting of all bounded functions $f : B \to Y$, equipped with the sup-norm. Define the function $F_B : \mathbb{R}^n \to l_\infty(B : Y)$ by

$$[F_B(\mathbf{t})](x) := F(\mathbf{t}; x), \quad \mathbf{t} \in \mathbb{R}^n,\ x \in B. \tag{6.12}$$

By Proposition 6.1.17, this mapping is well defined. Furthermore, this mapping satisfies the requirement that for each $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \mathbb{R}^n$ such that

$$d\big(F_B(\mathbf{t} + \tau), F_B(\mathbf{t})\big) \leqslant \varepsilon, \quad \mathbf{t} \in I.$$

Hence, $F_B(\cdot)$ is Bohr almost periodic in the sense of definition given in [824, Subsection 1.2, p. 7]. By [824, Theorem 1.2, p. 7], it follows that $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic with R being the collection of all sequences in $\mathbb{R}^n$ (case [L2]). The converse statement can be deduced similarly, and therefore, the following Bochner criterion holds good.

Theorem 6.1.18. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is continuous, $\mathcal{B}$ is any family of compact subsets of X and* R *is the collection of all sequences in $\mathbb{R}^n$. Then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if $F(\cdot;\cdot)$ is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic.*

As a direct consequence of Proposition 6.1.4(i) and Theorem 6.1.18, we have the following important result for our further investigations (see [442, pp. 17–24] for the notion of a uniformly almost periodic family, where the corresponding problematic has been considered for infinite number of almost periodic functions; for the sake of brevity, we will not consider this topic here).

Proposition 6.1.19. *Suppose that $k \in \mathbb{N}$ and $\mathcal{B}$ is any family of compact subsets of X. If the function $F_i : \mathbb{R}^n \times X \to Y_i$ is Bohr $\mathcal{B}$-almost periodic for $1 \leqslant i \leqslant k$, then the function $(F_1, \ldots, F_k)(\cdot;\cdot)$ is also Bohr $\mathcal{B}$-almost periodic.*

As a consequence, we find that the Bohr $\mathcal{B}$-almost periodic functions $F_i(\cdot;\cdot)$ can share the same ε-periods in Definition 6.1.9(i), i. e., for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \mathbb{R}^n$ such that (6.10) holds for all $F = F_i$ and $Y = Y_i$, $1 \leqslant i \leqslant k$ (observe that the original proof of H. Bohr, see e. g. [198, pp. 36–38], does not work in the multi-dimensional case $n > 1$).

Now we can simply prove the following.

Proposition 6.1.20. *Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ is Bohr almost periodic and $F : \mathbb{R}^n \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, where $\mathcal{B}$ is any family of compact subsets of X. Define $F_1(\mathbf{t}; x) := f(\mathbf{t})F(\mathbf{t}; x)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$. Then $F_1(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic.*

Proof. Let $B \in \mathcal{B}$ and $\varepsilon > 0$ be fixed. Due to Proposition 6.1.17, there exists a finite real constant $M \geqslant 1$ such that $|f(\mathbf{t})| + \|F(\mathbf{t}; x)\|_Y \leqslant M$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$. Let $\tau \in \mathbb{R}^n$ be a common $(\varepsilon/2M)$-period for the functions $f(\cdot)$ and $F(\cdot;\cdot)$. Then the required statement simply follows from the next estimates:

$$
\begin{aligned}
&\big\|f(\mathbf{t}+\tau)F(\mathbf{t}+\tau; x) - f(\mathbf{t})F(\mathbf{t}; x)\big\|_Y \\
&\quad \leqslant |f(\mathbf{t}+\tau) - f(\mathbf{t})| \cdot \big\|F(\mathbf{t}+\tau; x)\big\|_Y + |f(\mathbf{t}+\tau)| \cdot \big\|F(\mathbf{t}+\tau; x) - F(\mathbf{t}; x)\big\|_Y \\
&\quad \leqslant 2M\big[\big|f(\mathbf{t}+\tau) - f(\mathbf{t})\big| + \big\|F(\mathbf{t}+\tau; x) - F(\mathbf{t}; x)\big\|_Y\big] \leqslant 2M\varepsilon/2M = \varepsilon. \qquad \square
\end{aligned}
$$

We can similarly prove the following analogue of Proposition 6.1.20 for $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions.

Proposition 6.1.21. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $f : I \to \mathbb{C}$ is bounded* R*-multi-almost periodic and $F : I \times X \to Y$ is a* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic function whose restriction to any set $I \times B$, where $B \in \mathcal{B}$ is arbitrary, is bounded. Define $F_1(\mathbf{t}; x) := f(\mathbf{t})F(\mathbf{t}; x)$, $\mathbf{t} \in I$,*

$x \in X$. Then $F_1(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, provided that for each sequence $(\mathbf{b}_k)$ in R we find that any its subsequence also belongs to R.

Using the decomposition

$$\|F(\mathbf{t}';x) - F(\mathbf{t}'';y)\|_Y \leqslant \|F(\mathbf{t}';x) - F(\mathbf{t}' + \tau;x)\|_Y + \|F(\mathbf{t}' + \tau;x) - F(\mathbf{t}'' + \tau;y)\|_Y \\ + \|F(\mathbf{t}'' + \tau;y) - F(\mathbf{t}'';y)\|_Y, \quad \mathbf{t}',\ \mathbf{t}'' \in I,\ x,\ y \in X,$$

and the assumptions clarified below, we can repeat almost literally the argumentation contained in the proof of [166, Theorem 5, p. 2] in order to see that the following result holds (unfortunately, the situation in which $I = [0,\infty)^n$ is not covered by this result in contrast with the usually considered case $I = \mathbb{R}^n$).

Proposition 6.1.22. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, I is closed and $F : I \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, where $\mathcal{B}$ is a family consisting of some compact subsets of X. If the condition holds that*

$$(\exists \mathbf{t_0} \in I)\,(\forall \varepsilon > 0)(\forall l > 0)\,(\exists l' > 0)\,(\forall \mathbf{t}',\ \mathbf{t}'' \in I) \\ B(\mathbf{t_0}, l) \cap I \subseteq B(\mathbf{t_0} - \mathbf{t}', l') \cap B(\mathbf{t_0} - \mathbf{t}'', l'),$$

then for each $B \in \mathcal{B}$ the function $F(\cdot;\cdot)$ is uniformly continuous on $I \times B$.

The following is a multi-dimensional extension of [388, Lemma 1.3(f)].

Example 6.1.23. Suppose that $f : \mathbb{R}^n \to X$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are Bohr almost periodic functions. Define the function

$$F(\mathbf{t}) := f(\mathbf{t} - g(\mathbf{t})), \quad \mathbf{t} \in \mathbb{R}^n.$$

Then the function $F(\cdot)$ is Bohr almost periodic, as well. We can show this similarly as in the proof of the above-mentioned lemma, with appealing to Proposition 6.1.19 and Proposition 6.1.22.

6.1.2 Strongly $\mathcal{B}$-almost periodic functions

This subsection analyzes strongly $\mathcal{B}$-almost periodic functions and their relations with Bohr $\mathcal{B}$-almost periodic functions and Bohr $(\mathcal{B}, I')$-almost periodic functions. First of all, we will introduce the following definition.

Definition 6.1.24. Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$ and $F : I \times X \to Y$ is a continuous function. Then we say that $F(\cdot;\cdot)$ is strongly $\mathcal{B}$-almost periodic if and only if for each $B \in \mathcal{B}$ there exists a sequence $(P_k^B(\mathbf{t};x))$ of trigonometric polynomials such that $\lim_{k\to+\infty} P_k^B(\mathbf{t};x) = F(\mathbf{t};x)$, uniformly for $\mathbf{t} \in I$ and $x \in B$. Here, by a trigonometric polynomial $P : I \times X \to Y$ we mean any linear combination of functions like

$$e^{i[\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n]} c(x), \tag{6.13}$$

where λ_i are real numbers $(1 \leqslant i \leqslant n)$ and $c : X \to Y$ is a continuous mapping.

The following proposition is of fundamental importance.

Proposition 6.1.25. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$ and $F : I \times X \to Y$ is a strongly $\mathcal{B}$-almost periodic function, where $\mathcal{B}$ is any collection of bounded subsets of X. Then we have the following:*

(i) *for every $j \in \mathbb{N}_n$ and $\varepsilon > 0$, there exists a finite real number $l > 0$ such that every interval $S \subseteq \mathbb{R}$ of length l contains a point $\tau_j \in I$ such that*

$$\|F(t_1, t_2, \ldots, t_j + \tau_j, \ldots, t_n; x) - F(t_1, t_2, \ldots, t_j, \ldots, t_n; x)\| \leqslant \varepsilon, \tag{6.14}$$

provided $(t_1, t_2, \ldots, t_j + \tau_j, \ldots, t_n) \in I$, $(t_1, \ldots, t_n) \in I$ and $x \in B$;

(ii) *for every $\varepsilon > 0$, there exists a finite real number $l > 0$ such that, for every $j \in \mathbb{N}_n$ and every interval $S \subseteq \mathbb{R}$ of length l, there exists a point $\tau_j \in I$ such that* (6.14) *holds provided that $(t_1, t_2, \ldots, t_j + \tau_j, \ldots, t_n) \in I$ for all $j \in \mathbb{N}_n$, $(t_1, \ldots, t_n) \in I$ and $x \in B$;*

(iii) *for every $\varepsilon > 0$, there exists a finite real number $l > 0$ such that every interval $S \subseteq \mathbb{R}$ of length l contains a point $\tau \in I$ such that, for every $j \in \mathbb{N}_n$,* (7) *holds with the number τ_j replaced by the number τ therein;*

(iv) *$F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic provided that $I + I \subseteq I$ and that, for every points $(t_1, \ldots, t_n) \in I$ and $(\tau_1, \ldots, \tau_n) \in I$, the points $(t_1, t_2 + \tau_2, \ldots, t_n + \tau_n)$, $(t_1, t_2, t_3 + \tau_3, \ldots, t_n + \tau_n), \ldots, (t_1, t_2, \ldots, t_{n-1}, t_n + \tau_n)$, also belong to I;*

(v) *$F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I \cap \Delta_n)$-almost periodic provided that $I \cap \Delta_n \neq \emptyset$, $I + (I \cap \Delta_n) \subseteq I$ and that, for every points $(t_1, \ldots, t_n) \in I$ and $(\tau, \ldots, \tau) \in I \cap \Delta_n$, the points $(t_1, t_2 + \tau, \ldots, t_n + \tau)$, $(t_1, t_2, t_3 + \tau, \ldots, t_n + \tau), \ldots, (t_1, t_2, \ldots, t_{n-1}, t_n + \tau)$, also belong to $I \cap \Delta_n$.*

Proof. The proof is not difficult and we will present the most relevant details only. For the proof of (iii), we can verify first the validity of this statement for any trigonometric polynomial $P(\cdot;\cdot)$ by using the fact that for each set $B \in \mathcal{B}$, which is bounded due to our assumption, we find that the set $c(B)$ is bounded in Y for any addend of $P(\cdot;\cdot)$ of form (6.13) and the fact that any finite set of almost periodic functions of one real variable has a common set of joint ε-periods for each $\varepsilon > 0$. In general case, there exists a sequence $(P_k^B(\mathbf{t}; x))$ of trigonometric polynomials such that $\lim_{k \to +\infty} P_k^B(\mathbf{t}; x) = F(\mathbf{t}; x)$, uniformly for $\mathbf{t} \in I$ and $x \in B$. Then, for a real number $\varepsilon > 0$ given in advance, we can find an integer $k_0 \in \mathbb{N}$ such that $\|P_{k_0}^B(\mathbf{t}; x) - F(\mathbf{t}; x)\|_Y \leqslant \varepsilon/3$, for every $\mathbf{t} \in I$, $x \in B$ and $k \in \mathbb{N}$ with $k \geqslant k_0$. Using the well known estimate ($\mathbf{t}'$, $\mathbf{t}'' \in I$; $x \in B$):

$$\begin{aligned} &\|F(\mathbf{t}'; x) - F(\mathbf{t}''; x)\|_Y \\ &\quad \leqslant \|F(\mathbf{t}'; x) - P_{k_0}^B(\mathbf{t}'; x)\|_Y + \|P_{k_0}^B(\mathbf{t}'; x) - P_{k_0}^B(\mathbf{t}''; x)\|_Y + \|P_{k_0}^B(\mathbf{t}''; x) - F(\mathbf{t}''; x)\|_Y, \end{aligned}$$

the required statement readily follows; the proofs of (i) and (ii) are analogous. For the proof of (iv), we can use (i) and the estimates

$$\begin{aligned} &\|F(t_1 + \tau_1, t_2 + \tau_2, \ldots, t_j + \tau_j, \ldots, t_n + \tau_n) - F(t_1, t_2, \ldots, t_n)\| \\ &\quad \leqslant \|F(t_1 + \tau_1, t_2 + \tau_2, \ldots, t_j + \tau_j, \ldots, t_n + \tau_n) - F(t_1, t_2 + \tau_2, \ldots, t_j + \tau_j, \ldots, t_n + \tau_n)\| \end{aligned}$$

$$
\begin{aligned}
&+ \|F(t_1, t_2 + \tau_2, \ldots, t_j + \tau_j, \ldots, t_n + \tau_n) - F(t_1, t_2, \ldots, t_n)\| \\
&\leqslant \|F(t_1 + \tau_1, t_2 + \tau_2, \ldots, t_j + \tau_j, \ldots, t_n + \tau_n) - F(t_1, t_2 + \tau_2, \ldots, t_j + \tau_j, \ldots, t_n + \tau_n)\| \\
&+ \|F(t_1, t_2 + \tau_2, \ldots, t_j + \tau_j, \ldots, t_n + \tau_n) - F(t_1, t_2, t_3 + \tau_3, \ldots, t_n + \tau_n)\| \\
&+ \|F(t_1, t_2, t_3 + \tau_3, \ldots, t_n + \tau_n) - F(t_1, t_2, \ldots, t_n)\| \leqslant \cdots,
\end{aligned}
$$

while for the proof of (v), we can use (iii) and the above estimates with $\tau = \tau_1 = \tau_2 = \cdots = \tau_n$. □

Concerning Proposition 6.1.25(iv)–(v), it is natural to ask the following: Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$ and $F : I \times X \to Y$ is a Bohr $\mathcal{B}$-almost periodic (Bohr $(\mathcal{B}, I \cap \Delta_n)$-almost periodic) function. What conditions ensure the strong $\mathcal{B}$-almost periodicity of $F(\cdot;\cdot)$? After proving Theorem 6.1.37, it will be clear from a combination with Proposition 6.1.25(iv) that the notion of strong Bohr $\mathcal{B}$-almost periodicity for continuous functions $F : I \to Y$ coincides with the notion of Bohr $\mathcal{B}$-almost periodicity, provided that I is a convex polyhedral; as a simple consequence of the last mentioned theorem, we also find that the uniform convergence of a sequence of scalar-valued trigonometric polynomials on a convex polyhedral in $\mathbb{R}^n$ always implies the uniform convergence of this sequence on the whole space $\mathbb{R}^n$ (in the present state of our knowledge, we really do not know whether this result was known before stating the above).

It could be interesting to formulate some statements concerning the relationship between the strong $\mathcal{B}$-almost periodicity and the $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodicity.

6.1.3 $\mathbb{D}$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions

We start this subsection by introducing the following definition.

Definition 6.1.26. Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. By $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ we denote the vector space consisting of all continuous functions $Q : I \times X \to Y$ such that, for every $B \in \mathcal{B}$, we have $\lim_{t \in \mathbb{D}, |t| \to +\infty} Q(\mathbf{t}; x) = 0$, uniformly for $x \in B$.

Now we are ready to introduce the following notion.

Definition 6.1.27. Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, and $F : I \times X \to Y$ is a continuous function. Then we say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $\mathbb{D}$-asymptotically $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost periodic, if and only if there exist an $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function $G : I \times X \to Y$, resp. an $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost periodic function $G : I \times X \to Y$, and a function $Q \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$.

Let $I = \mathbb{R}^n$. Then it is said that $F(\cdot;\cdot)$ is asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. asymptotically $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost periodic, if and only if $F(\cdot;\cdot)$ is $\mathbb{R}^n$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $\mathbb{R}^n$-asymptotically $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost periodic.

We similarly introduce the notions of ($\mathbb{D}$-)asymptotical Bohr $\mathcal{B}$-almost periodicity, ($\mathbb{D}$-)asymptotical uniform recurrence, ($\mathbb{D}$-)-asymptotical Bohr $(\mathcal{B}, I')$-almost periodicity and ($\mathbb{D}$-)asymptotical $(\mathcal{B}, I')$-uniform recurrence. If $F(\cdot;\cdot)$ is I-asymptotically uniformly recurrent, $G : I \times X \to Y$, $Q \in C_{0,I,\mathcal{B}}(I \times X : Y)$ and $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$, then we can simply show that, for every $x \in X$, we have

$$\overline{\{G(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in X\}} \subseteq \overline{\{F(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in X\}}. \tag{6.15}$$

The first part of following proposition can be deduced as in the one-dimensional case; keeping in mind the inclusion (6.15) and the argumentation used in the proof of [364, Theorem 4.29], we can simply deduce the second part (see also Proposition 6.1.7, Corollary 6.1.8 and Proposition 6.1.32 for the corresponding results regarding the classes of $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions and $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic functions).

Proposition 6.1.28.

(i) *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic ($\mathcal{B}$-uniformly recurrent). If for each $B \in \mathcal{B}$ there exists $\varepsilon_B > 0$ such that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x \in \partial B} B(x, \varepsilon_B)$, then the function $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic ($\mathcal{B}$-uniformly recurrent).*

(ii) *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is I-asymptotically Bohr $\mathcal{B}$-almost periodic (I-asymptotically $\mathcal{B}$-uniformly recurrent). If for each $B \in \mathcal{B}$ there exists $\varepsilon_B > 0$ such that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x \in \partial B} B(x, \varepsilon_B)$, then the function $F(\cdot;\cdot)$ is I-asymptotically Bohr $\mathcal{B}$-almost periodic (I-asymptotically $\mathcal{B}$-uniformly recurrent).*

The proof of following proposition, which can be also clarified for the classes of $\mathbb{D}$-asymptotically almost periodic type functions introduced in Definition 6.1.27, is simple and therefore omitted.

Proposition 6.1.29.

(i) *Suppose that $c \in \mathbb{C}$ and $F(\cdot;\cdot)$ is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent). Then $cF(\cdot;\cdot)$ is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent).*

(ii) (a) *Suppose that $\tau \in \mathbb{R}^n$, $\tau + I \subseteq I$ and $F(\cdot;\cdot)$ is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent). Then $F(\cdot + \tau;\cdot)$ is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent).*

(b) *Suppose that $x_0 \in X$ and $F(\cdot;\cdot)$ is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent). Then $F(\cdot;\cdot + x_0)$ is* $(\mathrm{R}, \mathcal{B}_{x_0})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B}_{x_0})$*-multi-almost periodic (Bohr $\mathcal{B}_{x_0}$-almost periodic/$\mathcal{B}_{x_0}$-uniformly recurrent), where $\mathcal{B}_{x_0} \equiv \{-x_0 + B : B \in \mathcal{B}\}$.*

(c) *Suppose that* $\tau \in \mathbb{R}^n$, $\tau + I \subseteq I$, $x_0 \in X$ *and* $F(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr* $\mathcal{B}$*-almost periodic/*$\mathcal{B}$*-uniformly recurrent). Then* $F(\cdot+\tau;\cdot+x_0)$ *is* $(\mathrm{R}, \mathcal{B}_{x_0})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B}_{x_0})$*-multi-almost periodic (Bohr* $\mathcal{B}_{x_0}$*-almost periodic/*$\mathcal{B}_{x_0}$*-uniformly recurrent).*

(iii) (a) *Suppose that* $c \in \mathbb{C} \smallsetminus \{0\}$, $cI \subseteq I$ *and* $F(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr* $\mathcal{B}$*-almost periodic/*$\mathcal{B}$*-uniformly recurrent). Then* $F(c\cdot;\cdot)$ *is* $(\mathrm{R}_c, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X},c}, \mathcal{B})$*-multi-almost periodic (Bohr* $\mathcal{B}$*-almost periodic/*$\mathcal{B}$*-uniformly recurrent), where* $\mathrm{R}_c \equiv \{c^{-1}\mathbf{b} : \mathbf{b} \in \mathrm{R}\}$ *and* $\mathrm{R}_{\mathrm{X},c} \equiv \{c^{-1}\mathbf{b} : \mathbf{b} \in \mathrm{R}_{\mathrm{X}}\}$.

(b) *Suppose that* $c_2 \in \mathbb{C} \smallsetminus \{0\}$*, and* $F(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr* $\mathcal{B}$*-almost periodic/*$\mathcal{B}$*-uniformly recurrent). Then* $F(\cdot;c_2\cdot)$ *is* $(\mathrm{R}, \mathcal{B}_{c_2})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B}_{c_2})$*-multi-almost periodic (Bohr* $\mathcal{B}_{c_2}$*-almost periodic/*$\mathcal{B}_{c_2}$*-uniformly recurrent), where* $\mathcal{B}_{c_2} \equiv \{c_2^{-1}B : B \in \mathcal{B}\}$.

(c) *Suppose that* $c_1 \in \mathbb{C}\smallsetminus\{0\}$, $c_2 \in \mathbb{C}\smallsetminus\{0\}$, $c_1 I \subseteq I$ *and* $F(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic (Bohr* $\mathcal{B}$*-almost periodic/*$\mathcal{B}$*-uniformly recurrent). Then* $F(c_1\cdot;c_2\cdot)$ *is* $(\mathrm{R}_{c_1}, \mathcal{B}_{c_2})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X},c_1}, \mathcal{B}_{c_2})$*-multi-almost periodic (Bohr* $\mathcal{B}_{c_2}$*-almost periodic/*$\mathcal{B}_{c_2}$*-uniformly recurrent).*

(iv) *Suppose that* α, $\beta \in \mathbb{C}$ *and, for every sequence which belongs to* R *(*R_{X}*), we find that any its subsequence belongs to* R *(*R_{X}*). If* $F(\cdot;\cdot)$ *and* $G(\cdot;\cdot)$ *are* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic, then* $(\alpha F + \beta G)(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp.* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost periodic.*

(v) *Suppose that* α, $\beta \in \mathbb{C}$*. If* $F : \mathbb{R}^n \times X \to Y$ *and* $G : \mathbb{R}^n \times X \to Y$ *are Bohr* $\mathcal{B}$*-almost periodic, then* $(\alpha F + \beta G)(\cdot;\cdot)$ *is Bohr* $\mathcal{B}$*-almost periodic.*

Due to Proposition 6.1.8 and Proposition 6.1.29(ii), we may conclude that, in the case that $X = \{0\}$, the limit function $F^*(\cdot)$ in (6.1) is likewise R-multi-almost periodic. In such a way, we can extend the statements of [189, Theorem 1] and [915, Lemma 1] for vector-valued functions; the statement of [915, Lemma 3] also holds for vector-valued functions.

Using Proposition 6.1.29(iv) and the supremum formula clarified in Proposition 6.1.6, we can simply deduce that the decomposition in Definition 6.1.27 is unique.

Proposition 6.1.30.

(i) *Suppose that there exist a function* $G_i(\cdot;\cdot)$ *which is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic and a function* $Q_i \in C_{0,I,\mathcal{B}}(I \times X : Y)$ *such that* $F(\mathbf{t};x) = G_i(\mathbf{t};x) + Q_i(\mathbf{t};x)$ *for all* $\mathbf{t} \in I$ *and* $x \in X$ *(*$i = 1,2$*). Suppose that, for every sequence which belongs to* R*, any its subsequence belongs to* R*. If there exists a sequence* $b(\cdot)$ *in* R *whose any subsequence is unbounded and for which we have* $\mathbf{T} - \mathbf{b}(l) \in I$ *whenever* $\mathbf{T} \in I$ *and* $l \in \mathbb{N}$*, then* $G_1 \equiv G_2$ *and* $Q_1 \equiv Q_2$.

(ii) *Suppose that* $\mathcal{B}$ *is any collection of compact subsets of* X*, there exist a Bohr* $\mathcal{B}$*-almost periodic function* $G_i : \mathbb{R}^n \times X \to Y$ *and a function* $Q_i \in C_{0,I,\mathcal{B}}(I \times X : Y)$ *such that*

$F(\mathbf{t};x) = G_i(\mathbf{t};x) + Q_i(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$ ($i = 1,2$). Then $G_1 \equiv G_2$ and $Q_1 \equiv Q_2$.

For the sequel, we need the following auxiliary lemma.

Lemma 6.1.31. *Suppose that there exist an $(\mathrm{R},\mathcal{B})$-multi-almost periodic function $G(\cdot;\cdot)$ and a function $Q \in C_{0,I,\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$. Then (6.15) holds provided that for each sequence $\mathbf{b} \in \mathrm{R}$ we have $I \pm \mathbf{b}(l) \in I$, $l \in \mathbb{N}$ and there exists a sequence in R whose any subsequence is unbounded.*

Now we are in a position to clarify the following result.

Proposition 6.1.32. *Suppose that, for every sequence $\mathbf{b}(\cdot)$ which belongs to R, any its subsequence belongs to R and $\mathbf{T} - \mathbf{b}(l) \in I$ whenever $\mathbf{T} \in I$ and $l \in \mathbb{N}$. Suppose, further, that there exists a sequence in R any subsequence of which is unbounded. If for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is I-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic and for each $B \in \mathcal{B}$ the sequence $(F_j(\cdot;\cdot))$ converges to $F(\cdot;\cdot)$ uniformly on the set $B^\circ \cup \bigcup_{x\in\partial B} B(x,\varepsilon_B)$, then the function $F(\cdot;\cdot)$ is I-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic.*

Proof. Due to Proposition 6.1.30, we know that there exist a uniquely determined function $G(\cdot;\cdot)$ which is $(\mathrm{R},\mathcal{B})$-multi-almost periodic and a uniquely determined function $Q \in C_{0,I,\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$. Furthermore, we have

$$F_j(\mathbf{t};x) - F_m(\mathbf{t};x) = \left[G_j(\mathbf{t};x) - G_m(\mathbf{t};x)\right] + \left[Q_j(\mathbf{t};x) - Q_m(\mathbf{t};x)\right],$$

for all $\mathbf{t} \in I$, $x \in X$ and $j,\ m \in \mathbb{N}$. Due to Proposition 6.1.29(iv), we find that the function $G_j(\cdot;\cdot) - G_m(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic ($j,\ m \in \mathbb{N}$). Keeping in mind this fact and Lemma 6.1.31 and the argumentation used in the proof of [364, Theorem 4.29], we get

$$\begin{aligned} &3 \sup_{\mathbf{t}\in I, x\in X} \left\|F_j(\mathbf{t};x) - F_m(\mathbf{t};x)\right\|_Y \\ &\quad \geqslant \sup_{\mathbf{t}\in I, x\in X} \left\|G_j(\mathbf{t};x) - G_m(\mathbf{t};x)\right\|_Y + \sup_{\mathbf{t}\in I, x\in X} \left\|Q_j(\mathbf{t};x) - Q_m(\mathbf{t};x)\right\|_Y, \end{aligned}$$

for any $j,\ m \in \mathbb{N}$. This implies that the sequences $(G_j(\cdot;\cdot))$ and $(Q_j(\cdot;\cdot))$ converge uniformly to the functions $G(\cdot;\cdot)$ and $Q(\cdot;\cdot)$, respectively. Due to Proposition 6.1.8, we see that the function $G(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic. The final conclusion follows from the obvious equality $F = G + Q$ and the fact that $C_{0,I,\mathcal{B}}(I \times X : Y)$ is a Banach space. □

Before we move ourselves to the next subsection, we would like to introduce the following general definition in a Bohr like manner; for any set $\Lambda \subseteq \mathbb{R}^n$, we define $\Lambda_M := \{\lambda \in \Lambda;\ |\lambda| \geqslant M\}$:

Definition 6.1.33. Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B},I')$-almost periodic of type 1 if and only if for every $B\in\mathcal{B}$ and $\varepsilon>0$ there exist $l>0$ and $M>0$ such that for each $\mathbf{t}_0\in I'$ there exists $\tau\in B(\mathbf{t}_0,l)\cap I'$ such that

$$\|F(\mathbf{t}+\tau;x)-F(\mathbf{t};x)\|_Y\leqslant\varepsilon,\quad \text{provided } \mathbf{t},\ \mathbf{t}+\tau\in\mathbb{D}_M,\ x\in B. \tag{6.16}$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\mathcal{B},I')$-uniformly recurrent of type 1 if and only if for every $B\in\mathcal{B}$ there exist a sequence (τ_n) in I' and a sequence (M_n) in $(0,\infty)$ such that $\lim_{n\to+\infty}|\tau_n|=\lim_{n\to+\infty}M_n=+\infty$ and

$$\lim_{n\to+\infty}\sup_{\mathbf{t},\mathbf{t}+\tau_n\in\mathbb{D}_{M_n};x\in B}\|F(\mathbf{t}+\tau_n;x)-F(\mathbf{t};x)\|_Y=0.$$

If $I'=I$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $\mathcal{B}$-almost periodic of type 1 ($\mathbb{D}$-asymptotically $\mathcal{B}$-uniformly recurrent of type 1); furthermore, if $X\in\mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr I'-almost periodic of type 1 ($\mathbb{D}$-asymptotically I'-uniformly recurrent of type 1). If $I'=I$ and $X\in\mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr almost periodic of type 1 ($\mathbb{D}$-asymptotically uniformly recurrent of type 1).

The proof of following proposition is trivial and therefore omitted.

Proposition 6.1.34. *Suppose that $\mathbb{D}\subseteq I\subseteq\mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset\neq I'\subseteq I\subseteq\mathbb{R}^n$, $F: I\times X\to Y$ is a continuous function and $I+I'\subseteq I$. If $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B},I')$-almost periodic, resp. $\mathbb{D}$-asymptotically $(\mathcal{B},I')$-uniformly recurrent, then $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B},I')$-almost periodic of type 1, resp. $\mathbb{D}$-asymptotically $(\mathcal{B},I')$-uniformly recurrent of type 1.*

Suppose now that the general assumptions from the preamble of Definition 6.1.33 hold true. Keeping in mind Proposition 6.1.34 and Remark 6.1.3(i)–(ii), it is natural to ask the following:

(i) In which cases the $\mathbb{D}$-asymptotical Bohr $(\mathcal{B},I')$-almost periodicity of type 1, resp. $\mathbb{D}$-asymptotical $(\mathcal{B},I')$-uniform recurrence of type 1, implies the $\mathbb{D}$-asymptotical Bohr $(\mathcal{B},I')$-almost periodicity, resp. $\mathbb{D}$-asymptotical $(\mathcal{B},I')$-uniform recurrence of the function $F(\cdot;\cdot)$?
(ii) In which cases the asymptotical Bohr $\mathcal{B}$-almost periodicity (of type 1) implies the $(\mathrm{R},\mathcal{B})$-multi-almost periodicity of $F(\cdot;\cdot)$, where R denotes the collection of all sequences in I?
(iii) In which cases the asymptotical Bohr $\mathcal{B}$-almost periodicity (of type 1) is a consequence of the $(\mathrm{R},\mathcal{B})$-multi-almost periodicity of $F(\cdot;\cdot)$, where R denotes the collection of all sequences in I?

Concerning the item (ii), it is well known that the answer is negative provided that $X=\{0\}$, $\mathcal{B}=X$ and $I=\mathbb{R}$ because, in this case, the asymptotical Bohr $\mathcal{B}$-almost periodicity of $F : \mathbb{R} \to Y$ is equivalent with the asymptotical Bohr $\mathcal{B}$-almost periodicity of type 1 of $F(\cdot)$, i. e., the usual asymptotical almost periodicity of $F(\cdot)$, while the $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity of $F(\cdot)$ is equivalent in this case with the usual almost periodicity of $F(\cdot)$; cf. [1078, Definition 2.2, Definition 2.3; Theorem 2.6] for the notion used. But we have the following statement in the particular case $I = [0, \infty)^n$.

Proposition 6.1.35. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, I is closed and $F : I \times X \to Y$ is a uniformly continuous, I-asymptotically Bohr $\mathcal{B}$-almost periodic function of type* 1, *where $\mathcal{B}$ is any family of compact subsets of X. If*

$$(\forall l > 0)\,(\forall M > 0)\,(\exists \mathbf{t_0} \in I)\,(\exists k > 0)\,(\forall \mathbf{t} \in I_{M+l})\big(\exists \mathbf{t}_0' \in I\big)$$
$$(\forall \mathbf{t}_0'' \in B(\mathbf{t}_0', l) \cap I)\ \mathbf{t} - \mathbf{t}_0'' \in B(\mathbf{t_0}, kl) \cap I_M, \tag{6.17}$$

there exists $L > 0$ such that $I_{kL} \smallsetminus I_{(k+1)L} \neq \emptyset$ for all $k \in \mathbb{N}$ and $I_M + I \subseteq I_M$ for all $M > 0$, then the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, where R denotes the collection of all sequences in I. Furthermore, if $X = \{0\}$ and $\mathcal{B} = \{X\}$, then $F(\cdot)$ is I-asymptotically Bohr almost periodic function.

Proof. Let $B \in \mathcal{B}$ and $\varepsilon > 0$ be fixed. Since $F(\cdot;\cdot)$ is uniformly continuous, we find that the function $\mathbf{F}_B(\cdot)$, given by (6.12), is likewise uniformly continuous. Arguing as in the proof of Proposition 6.1.17, the assumption (6.17) enables one to deduce that the set $\{F(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in B\}$ is relatively compact in Y as well as that the set $\{F_B(\mathbf{t}) : \mathbf{t} \in I\}$ is relatively compact in the Banach space $BUC(B : Y)$, consisting of all bounded, uniformly continuous functions from B into Y, equipped with the sup-norm. We know that there exist $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (6.16) holds with $\mathbb{D} = I$. Using these facts, we can slightly modify the first part of the proof of [881, Theorem 3.3] (with the segment $[N, 3N]$ replaced therein with the set $I_N \smallsetminus I_{3N}$, where $N = \max(L, l, M)$, and the number $\tau_k \in [kN, (k+1)N]$ replaced therein by the number $\tau_k \in I_{kL} \smallsetminus I_{(k+1)L}$; we need the condition $I_M + I \subseteq I_M$, $M > 0$ in order to see that the estimate given on [881, l. 2, p. 23] holds in our framework) in order to see that the set of translations $\{F_B(\cdot + \tau) : \tau \in I\}$ is relatively compact in $BUC(B : Y)$, which simply implies that $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, where R denotes the collection of all sequences in I. Suppose now that $X = \{0\}$ and $\mathcal{B} = \{X\}$. Then for each integer $k \in \mathbb{N}$ there exist $l_k > 0$ and $M_k > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (6.16) holds with $\varepsilon = 1/k$ and $\mathbb{D} = I$. Let τ_k be any fixed element of I such that $|\tau_k| > M_k + k^2$ and (6.16) holds with $\varepsilon = 1/k$ and $\mathbb{D} = I$ ($k \in \mathbb{N}$). Then the first part of the proof yields the existence of a subsequence (τ_{k_l}) of (τ_k) and a function $F^* : I \to Y$ such that $\lim_{l \to +\infty} F(\mathbf{t} + \tau_{k_l}) = F^*(\mathbf{t})$, uniformly for $t \in I$. The mapping $F^*(\cdot)$ is clearly continuous and now we will prove that $F^*(\cdot)$ is Bohr almost periodic. Let $\varepsilon > 0$ be fixed, and let $l > 0$ and $M > 0$ be such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (6.16) holds with $\mathbb{D} = I$ and the number ε replaced therein by $\varepsilon/3$. Let $\mathbf{t} \in I$ be fixed, and let $l_0 \in \mathbb{N}$ be such that $|\mathbf{t} + \tau_{k_{l_0}}| \geqslant M$ and $|\mathbf{t} + \tau + \tau_{k_{l_0}}| \geqslant M$.

Then we have

$$\begin{aligned}&\|F^*(\mathbf{t}+\tau)-F^*(\mathbf{t})\|\\&\quad\leqslant\|F^*(\mathbf{t}+\tau)-F^*(\mathbf{t}+\tau+\tau_{k_{l_0}})\|+\|F^*(\mathbf{t}+\tau+\tau_{k_{l_0}})-F^*(\mathbf{t}+\tau_{k_{l_0}})\|\\&\qquad+\|F^*(\mathbf{t}+\tau_{k_{l_0}})-F^*(\mathbf{t})\|\leqslant 3\cdot(\varepsilon/3)=\varepsilon,\end{aligned}$$

as required. The fact that the function $\mathbf{t}\mapsto F(\mathbf{t})-F^*(\mathbf{t})$, $\mathbf{t}\in I$ belongs to the space $C_{0,I}(I:Y)$ follows trivially by definition of $F^*(\cdot)$. The proof of the proposition is thereby complete. □

Remark 6.1.36. Suppose that the requirements of Theorem 6.1.35 hold with $X=\{0\}$ and $\mathcal{B}=\{X\}$. Suppose further that, for every $\mathbf{t}'\in\mathbb{R}^n$, there exist $\delta>0$ and $l_0\in\mathbb{N}$ such that the sequence (τ_k) from the above proof satisfies the requirement that $\mathbf{t}''+\tau_{k_l}\in I$ for all $l\in\mathbb{N}$ with $l\geqslant l_0$ and $\mathbf{t}''\in B(\mathbf{t}',\delta)$. Then the limit $\lim_{l\to+\infty}F(\mathbf{t}'+\tau_{k_l}):=\tilde{F^*}(\mathbf{t}')$ exists for all $\mathbf{t}'\in\mathbb{R}^n$, which can be easily seen from the estimate

$$\begin{aligned}&\|F(\mathbf{t}'+\tau_{k_{l_1}})-F(\mathbf{t}'+\tau_{k_{l_2}})\|_Y\\&\quad\leqslant\|F(\mathbf{t}'+\tau_{k_{l_1}})-F(\mathbf{t}'+\tau_{k_{l_1}}+\tau)\|_Y+\|F(\mathbf{t}'+\tau_{k_{l_1}}+\tau)-F(\mathbf{t}'+\tau_{k_{l_2}}+\tau)\|_Y\\&\qquad+\|F(\mathbf{t}'+\tau_{k_{l_2}}+\tau)-F(\mathbf{t}'+\tau_{k_{l_2}})\|_Y\\&\quad\leqslant 3\cdot(\varepsilon/3)=\varepsilon,\end{aligned}\tag{6.18}$$

which is valid for all numbers τ such that there exist $l>0$ and $M>0$ such that for each $\mathbf{t}_0\in I$ there exists $\tau\in B(\mathbf{t}_0,l)\cap I$ such that (6.16) holds with the number ε replaced therein with the number $\varepsilon/3$ and $\mathbb{D}=I$, all sufficiently large natural numbers l_1 and l_2 depending on τ, where we have also applied the Cauchy criterion of convergence for the limit $\lim_{l\to+\infty}F(\mathbf{t}+\tau_{k_l})=F^*(\mathbf{t})$, uniform in $t\in I$ and our assumption $I+I\subseteq I$. The function $\tilde{F^*}(\cdot)$ is clearly continuous and it can be easily shown that it is Bohr I-almost periodic. Furthermore, if for every $\mathbf{t}'\in\mathbb{R}^n$ and M_1, $M_2>0$ there exists $l_0\in\mathbb{N}$ such that $\mathbf{t}'+\tau_{k_l}-\tau\in I_{M_2}$ for all $l\in\mathbb{N}$ with $l\geqslant l_0$, then $\tilde{F^*}(\cdot)$ is Bohr $(I\cup(-I))$-almost periodic. Using a simple translation argument, the above gives an extension of [881, Theorem 3.4] in Banach spaces.

Keeping in mind the proof of Theorem 6.1.35 and our analysis from Remark 4.2.98, we can also deduce the following result concerning the extensions of Bohr I'-almost periodic functions.

Theorem 6.1.37. *Suppose that $I'\subseteq I\subseteq\mathbb{R}^n$, $I+I'\subseteq I$, the set I' is unbounded, $F:I\to Y$ is a uniformly continuous, Bohr I'-almost periodic function, resp. a uniformly continuous, I'-uniformly recurrent function, $S\subseteq\mathbb{R}^n$ is bounded and the following condition holds:*
(AP-E) *For every $\mathbf{t}'\in\mathbb{R}^n$, there exists a finite real number $M>0$ such that $\mathbf{t}'+I'_M\subseteq I$.*

Define $\Omega_S:=[(I'\cup(-I'))+(I'\cup(-I'))]\cup S$. Then there exists a uniformly continuous, Bohr Ω_S-almost periodic, resp. a uniformly continuous, Ω_S-uniformly recurrent, function

$\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$; furthermore, in the almost periodic case, the uniqueness of such a function $\tilde{F}(\cdot)$ holds provided that $\mathbb{R}^n \setminus \Omega_S$ is a bounded set.

Proof. We will prove the theorem only for uniformly continuous, Bohr I'-almost periodic functions. In this case, for each natural number $k \in \mathbb{N}$ there exists a point $\tau_k \in I'$ such that $\|F(\mathbf{t} + \tau_k) - F(\mathbf{t})\|_Y \leqslant 1/k$ for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$; furthermore, since the set I' is unbounded, we may assume without loss of generality that $\lim_{k\to+\infty} |\tau_k| = +\infty$. Hence, one has $\lim_{k\to+\infty} F(\mathbf{t}+\tau_k) = F(\mathbf{t})$, uniformly for $t \in I$. If $\mathbf{t}' \in \mathbb{R}^n$, then we can use our assumption on the existence of a finite real number $M > 0$ such that $\mathbf{t}' + I'_M \subseteq I$, and the corresponding argumentation from Remark 4.2.98 (see (6.18)), in order to conclude that $\lim_{k\to+\infty} F(\mathbf{t}' + \tau_k) := \tilde{F}(\mathbf{t}')$ exists. The function $\tilde{F}(\cdot)$ is clearly uniformly continuous because $F(\cdot)$ is uniformly continuous; furthermore, by construction, we find that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$. Now we will prove that the function $\tilde{F}(\cdot)$ is Bohr Ω_S-almost periodic. Suppose that a number $\varepsilon > 0$ is given. Then we know that there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that $\|F(\mathbf{t} + \tau) - F(\mathbf{t})\|_Y \leqslant \varepsilon/2$ for all $\mathbf{t} \in I$. Let $\mathbf{t}' \in \mathbb{R}^n$ be fixed. For any such numbers $\mathbf{t}_0 \in I'$ and $\tau \in B(\mathbf{t}_0, l) \cap I'$, we have

$$\begin{aligned}\big\|\tilde{F}(\mathbf{t}' + \tau) - \tilde{F}(\mathbf{t}')\big\|_Y &= \Big\|\lim_{k\to+\infty}\big[F(\mathbf{t}' + \tau + \tau_k) - F(\mathbf{t}' + \tau_k)\big]\Big\|_Y \\ &\leqslant \limsup_{k\to+\infty}\big\|F(\mathbf{t}' + \tau + \tau_k) - F(\mathbf{t}' + \tau_k)\big\|_Y \leqslant \varepsilon/2, \quad \mathbf{t}' \in \mathbb{R}^n.\end{aligned}$$

This clearly implies

$$\big\|\tilde{F}(\mathbf{t}' - \tau) - \tilde{F}(\mathbf{t}')\big\|_Y \leqslant \varepsilon/2, \quad \mathbf{t}' \in \mathbb{R}^n,$$

which further implies that $F(\cdot)$ is Bohr $(I' \cup (-I'))$-almost periodic since $-\mathbf{t}_0 \in I'$ and $-\tau \in B(-\mathbf{t}_0, l) \cap (-I')$. Take now any number $\tau \in \Omega$; then τ can be written as a sum of two elements τ_1 and τ_2 from the set $(I' \cup (-I'))$ and, as a such, it will satisfy

$$\begin{aligned}\big\|\tilde{F}(\mathbf{t}' + \tau) - \tilde{F}(\mathbf{t}')\big\|_Y &= \big\|F(\mathbf{t}' + \tau_1 + \tau_2) - F(\mathbf{t}')\big\|_Y \\ &\leqslant \big\|F(\mathbf{t}' + \tau_1 + \tau_2) - F(\mathbf{t}' + \tau_1)\big\|_Y + \big\|F(\mathbf{t}' + \tau_1) - F(\mathbf{t}')\big\|_Y \\ &\leqslant 2 \cdot (\varepsilon/2) = \varepsilon,\end{aligned}$$

for any $\mathbf{t}' \in \mathbb{R}^n$. Therefore, $F(\cdot)$ is Bohr Ω-almost periodic, which clearly implies that $F(\cdot)$ is Bohr Ω_S-almost periodic, as well.

Assume, finally, that the set $\mathbb{R}^n \setminus \Omega$ is bounded. Then the function $\tilde{F}(\cdot)$ is Bohr almost periodic and any function $\tilde{G} : \mathbb{R}^n \to Y$ which extends the function $F(\cdot)$ to the whole space and satisfies the above requirements must be Bohr almost periodic. Therefore, $\tilde{G}(\cdot)$ is compactly almost automorphic (cf. Section 8.1 for the notion) so that the sequence (τ_k) has a subsequence (τ_{k_l}) such that

$$\lim_{l\to+\infty}\lim_{m\to+\infty} \tilde{G}(\mathbf{t}' + \tau_{k_m} - \tau_{k_l}) = \tilde{G}(\mathbf{t}'), \quad \mathbf{t}' \in \mathbb{R}^n.$$

But, for every $l \in \mathbb{N}$ and $\mathbf{t}' \in \mathbb{R}^n$, we have $\lim_{m\to+\infty} \tilde{G}(\mathbf{t}'+\tau_{k_m}-\tau_{k_l}) = \lim_{m\to+\infty} F(\mathbf{t}'+\tau_{k_m}-\tau_{k_l})$, so that the final conclusion follows from the almost automorphicity of function $\tilde{F}(\cdot)$ and the equality

$$\lim_{l\to+\infty}\lim_{m\to+\infty} \tilde{F}(\mathbf{t}' + \tau_{k_m} - \tau_{k_l}) = \tilde{F}(\mathbf{t}'),$$

which holds pointwise on $\mathbb{R}^n$. □

In particular, if $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis of $\mathbb{R}^n$, and

$$I' = I = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n : \alpha_i \geqslant 0 \text{ for all } i \in \mathbb{N}_n\}$$

is a convex polyhedral in $\mathbb{R}^n$, then we have $\Omega = \mathbb{R}^n$ and, in this case, there exists a unique Bohr almost periodic extension of the function $F : I \to Y$ to the whole Euclidean space, so that Proposition 6.1.19, Proposition 6.1.20, Proposition 6.1.21 and Proposition 6.1.29(v) continue to hold in this framework (it is worth noting that the almost periodic functions of one real variable have been used in the recent investigation of M. Cekić, B. Georgiev and M. Mukherjee [253] regarding dynamical properties of the billiard flow on convex polyhedra; see also J. P. Gaivao [460] for a related problematic).

We also state the following important corollary of Theorem 6.1.37.

Corollary 6.1.38 (The uniqueness theorem for almost periodic functions). *Suppose that $I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, condition* (AP-E) *holds with $I' = I$, and $\mathbb{R}^n \setminus [(I \cup (-I)) + (I \cup (-I))]$ is a bounded set. If $F : \mathbb{R}^n \to Y$ and $G : \mathbb{R}^n \to Y$ are two Bohr almost periodic functions and $F(\mathbf{t}) = G(\mathbf{t})$ for all $\mathbf{t} \in I$, then $F(\mathbf{t}) = G(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$.*

Now we would like to propose the following definition.

Definition 6.1.39. Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$ and $I + I \subseteq I$. Then we say that I is admissible with respect to the almost periodic extensions if and only if for any complex Banach space Y and for any uniformly continuous, Bohr almost periodic function $F : I \to Y$ there exists a unique Bohr almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$.

By the foregoing, it is clear that any non-empty subset I of $\mathbb{R}^n$ which is closed under addition and satisfies the requirement that condition (AP-E) holds with $I' = I$ as well as the set $\mathbb{R}^n \setminus [(I \cup (-I)) + (I \cup (-I))]$ is bounded (in particular, this holds for convex polyhedrals) has to be admissible with respect to the almost periodic extensions. But it is clear that the set $I = [0, \infty) \times \{0\} \subseteq \mathbb{R}^2$ is not admissible with respect to the almost periodic extensions since there is no almost periodic extension of the function $F(x, y) = y$, $(x, y) \in I$ to the whole plane. Further analysis of the notion introduced in Definition 6.1.39 is out of scope of this book.

Concerning item (iii), we will clarify the following result.

Theorem 6.1.40. *Suppose that $0 \in I \subseteq \mathbb{R}^n$, I is closed, $I + I \subseteq I$ and $\emptyset \neq I' \subseteq I$. Suppose, further, that the set $\mathbb{D} \subseteq I$ is unbounded and condition* (MD) *holds, where:*

(MD) *For each* $M_0 > 0$ *there exists a finite real number* $M_1 > M_0$ *such that* $\mathbb{D}_{M_1} - \mathbf{t} \in I$ *and* $I'_{M_1} - \mathbf{t} \in I'$ *for all* $\mathbf{t} \in I \setminus I_{M_0}$.

Let R *denote the collection of all sequences in* I, *and let* $\mathcal{B}$ *denote any family of compact subsets of* X. *Then any* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic function* $F : I \times X \to Y$ *is* $\mathbb{D}$*-asymptotically Bohr* $(\mathcal{B}, I')$*-almost periodic of type* 1.

Proof. Let $B \in \mathcal{B}$ and $\varepsilon > 0$ be fixed. Since I is closed, we find that the restriction of function $F(\cdot;\cdot)$ to the set $I \times B$ is uniformly continuous, which easily implies that the function $F_B : I \to \mathrm{BUC}(B : Y)$, given by (6.12), is well defined and uniformly continuous. Now we will prove that the function $F_B(\cdot)$ has a relatively compact range. Denote $K_k = [-k, k]^n$ for all integers $k \in \mathbb{N}$. Since the set $F_B(K_k \cap I)$ is relatively compact in $\mathrm{BUC}(B : Y)$ for all integers $k \in \mathbb{N}$, it suffices to show that there exists $k \in \mathbb{N}$ such that, for every $\mathbf{t} \in I$, there exists a point $\mathbf{s} \in I \cap K_k$ such that $\|F(\mathbf{t}; x) - F(\mathbf{s}; x)\|_Y \leqslant \varepsilon$ for all $x \in B$. Suppose the contrary holds. Then for each $k \in \mathbb{N}$ there exists $\mathbf{t}_k \in I$ such that, for every $\mathbf{s} \in I \cap K_k$, there exists $x \in B$ with $\|F(\mathbf{t}_k; x) - F(\mathbf{s}; x)\|_Y > \varepsilon$. Define $\mathbf{b}_k := \mathbf{t}_k$ for all $k \in \mathbb{N}$. Due to our assumption, there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that (6.2) holds true. Since $0 \in I$, this implies the existence of a number $l_0(\varepsilon) \in \mathbb{N}$ such that

$$\|F(\mathbf{t}_{k_l}; x) - F(\mathbf{t}_{k_m}; x)\|_Y \leqslant \varepsilon, \quad l,\ m \in \mathbb{N},\ l,\ m \geqslant l_0(\varepsilon),$$

uniformly for $x \in B$. In particular, we have

$$\|F(\mathbf{t}_{k_l}; x) - F(\mathbf{t}_{k_{l_0(\varepsilon)}}; x)\|_Y \leqslant \varepsilon, \quad l \in \mathbb{N},\ l \geqslant l_0(\varepsilon),\ x \in B.$$

Therefore, $\mathbf{t}_{k_{l_0(\varepsilon)}} \notin K_l$ for all $l \in \mathbb{N}$ with $l \geqslant l_0(\varepsilon)$, which is a contradiction. Now it is quite simply to prove with the help of Cauchy criterion of convergence and the $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity of $F(\cdot;\cdot)$ that the set of translations $\{F_B(\cdot + \tau) : \tau \in I\}$ is relatively compact in $\mathrm{BUC}(B : Y)$. Applying [881, Theorem 2.2; see 1. and 3.(ii)] (see also the second part of the proof of [881, Theorem 3.3]), we see that there exist a finite cover $(T_i)_{i=1}^k$ of the set I_1 and points $\mathbf{t}_i \in T_i$ $(1 \leqslant i \leqslant k)$ such that $\|F_B(\mathbf{t}+\omega) - F_B(\mathbf{t}_i+\omega)\|_{\mathrm{BUC}(B:Y)} \leqslant \varepsilon$ for all $\omega \in I$ and $t \in T_i$ $(1 \leqslant i \leqslant k)$. Let $M_0 := l := 1 + \max\{|\mathbf{t}_i| : 1 \leqslant i \leqslant k\}$, and let $M_1 > 0$ satisfy condition (MD) with this M_0. Set $M := 2M_1 + l$. Suppose that $\mathbf{t},\ \mathbf{t} + \tau \in \mathbb{D}_M$ and $\mathbf{t}_0 \in I'_M$. Then there exists $i \in \mathbb{N}_k$ such that $\mathbf{t}_0 \in T_i$ so that $\tau = \mathbf{t}_0 - \mathbf{t}_i \in T_i - \mathbf{t}_i \in B(\mathbf{t}_0, l) \cap I'$ due to the first condition in (MD) and the obvious inequality $|\mathbf{t}_i| \leqslant l$. Furthermore, the second condition in (MD) implies $\mathbf{t} - t_i \in I$ and therefore

$$\begin{aligned}\|F_B(\mathbf{t} + \tau) - F_B(\mathbf{t})\|_{\mathrm{BUC}(B:Y)} &= \|F_B(\mathbf{t} + \mathbf{t}_0 - \mathbf{t}_i) - F_B(\mathbf{t})\|_{\mathrm{BUC}(B:Y)} \\ &= \|F_B(\mathbf{t}_0 + [\mathbf{t} - \mathbf{t}_i]) - F_B(\mathbf{t}_i + [\mathbf{t} - \mathbf{t}_i])\|_{\mathrm{BUC}(B:Y)} \leqslant \varepsilon,\end{aligned}$$

which simply completes the proof. □

Remark 6.1.41.
(i) In [881, Theorem 3.3], W. M. Ruess and W. H. Summers have considered the situation in which $I = [a, \infty)$, $X = \{0\}$ and the set of all translations $\{f(\cdot + \tau) : \tau \geqslant 0\}$ is

relatively compact in $BUC(I : Y)$. But the obtained result is a simple consequence of the corresponding result with $I = [0, \infty)$, which follows from a simple translation argument. In Theorem 6.1.40, which therefore provides a proper extension of the corresponding result from [881, Theorem 3.3] with $\mathbb{D} = I' = I = [0, \infty)$, we have decided to consider the collection R of all sequences in I, only. It might be interesting to further analyze the assumption in which the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic with R being the collection of all sequences in a certain subset I'' of $\mathbb{R}^n$ which contains 0 and satisfies $I + I'' \subseteq I$.

(ii) In the multi-dimensional framework, we cannot expect the situation in which $\mathbb{D} = I' = I$. The main problem lies in the fact that condition (MD) does not hold in this case; but, if $I = [0, \infty)^n$, for example, then the conclusion of Theorem 6.1.40 holds for any proper subsector I' of I, with the meaning clear, and $\mathbb{D} = I'$.

6.1.4 Differentiation and integration of $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic functions

Concerning the partial derivatives of ($\mathbb{D}$-asymptotically) $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic functions, we would like to state the following result which can be also formulated for the notion introduced in Definition 6.1.9.

Proposition 6.1.42.

(i) *Suppose that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, for every sequence which belongs to R_X, we find that any of its subsequences belongs to R_X, the partial derivative*

$$\frac{\partial F(\cdot;\cdot)}{\partial t_i} := \lim_{h \to 0} \frac{F(\cdot + he_i;\cdot) - F(\cdot;\cdot)}{h}$$

exists on $I \times X$ and it is uniformly continuous on $\mathcal{B}$, i. e.,

$$(\forall B \in \mathcal{B})\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ \big(\forall \mathbf{t}',\ \mathbf{t}'' \in I\big)\ (\forall x \in B)$$
$$\left(\left|\mathbf{t}' - \mathbf{t}''\right| < \delta \Rightarrow \left\| \frac{\partial F(\mathbf{t}';x)}{\partial t_i} - \frac{\partial F(\mathbf{t}'';x)}{\partial t_i} \right\| < \varepsilon\right).$$

Then the function $\partial F(\cdot;\cdot)/\partial t_i$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic.

(ii) *Suppose that, for every sequence $\mathbf{b}(\cdot)$ which belongs to R, any its subsequence belongs to R and $\mathbf{T} - \mathbf{b}(l) \in I$ whenever $\mathbf{T} \in I$ and $l \in \mathbb{N}$. Suppose, further, that there exists a sequence in R whose any subsequence is unbounded as well as that the function $F(\cdot;\cdot)$ is I-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, the partial derivative $\partial F(\mathbf{t};x)/\partial t_i$ exists for all $\mathbf{t} \in I$, $x \in X$ and it is uniformly continuous on $\mathcal{B}$. Then the function $\partial F(\cdot;\cdot)/\partial t_i$ is I-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.*

Proof. We will prove only (i) because (ii) follows similarly, by appealing to Proposition 6.1.32 in place of Proposition 6.1.8 (observe that we only need here the uniform convergence of the sequence of functions $(F_j(\cdot;\cdot))$ to the function $\partial F(\cdot;\cdot)/\partial t_i$ as $j \to +\infty$, on the individual sets $B \in \mathcal{B}$; see the proof of Proposition 6.1.8). The proof immediately follows from the fact that the sequence $(F_j(\cdot;\cdot) \equiv j[F(\cdot + j^{-1}e_i;\cdot) - F(\cdot;\cdot)])$ of $(\mathrm{R_X}, \mathcal{B})$-multi-almost periodic functions converges uniformly to the function $\partial F(\cdot;\cdot)/\partial t_i$ as $j \to +\infty$. This can be shown as in the one-dimensional case, by observing that

$$F_j(\cdot;\cdot) - \frac{\partial F(\cdot;\cdot)}{\partial t_i} = j \int_0^{1/j} \left[\frac{\partial F(\cdot + se_i;\cdot)}{\partial t_i} - \frac{\partial F(\cdot;\cdot)}{\partial t_i} \right] ds. \qquad \square$$

We continue by stating the following extension of S. M. A. Alsulami's result [47, Theorem 3.2].

Theorem 6.1.43. *Suppose that the function $F : \mathbb{R}^n \times X \to Y$ is continuous as well as that $\partial F(\cdot;\cdot)/\partial t_i : \mathbb{R}^n \times X \to Y$ is a Bohr $\mathcal{B}$-almost periodic function, where $\mathcal{B}$ is any collection of compact subsets of X. Suppose that for each $B \in \mathcal{B}$ we find that at least one of the following two conditions holds:*

(C1) *The Banach space $l_\infty(B : Y)$ does not contain c_0.*
(C2) *The range of the function $F_B(\cdot)$, given by* (6.12)*, is weakly relatively compact in $l_\infty(B : Y)$.*

Then the function $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic.

Proof. Let $B \in \mathcal{B}$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be fixed. As easily approved, it suffices to show that the mapping $F_B(\cdot)$ is almost periodic. Since we have assumed (C1) or (C2), an application of an old result of B. Basit (see, e. g., [47, Theorem 3.1]) shows that we only need to prove that the function

$$\mathbf{t} \mapsto F_B(\mathbf{t} + \mathbf{a}) - F_B(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^n,$$

is almost periodic. So, let $(\mathbf{b}_k)$ be a sequence in $\mathbb{R}^n$. Since the mapping

$$\left(\left(\frac{\partial F(\cdot;\cdot)}{\partial t_1} \right)_B, \ldots, \left(\frac{\partial F(\cdot;\cdot)}{\partial t_n} \right)_B \right) : \mathbb{R}^n \to \left(l_\infty(B : Y) \right)^n$$

is almost periodic (see Proposition 6.1.19), there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that

$$\lim_{l \to +\infty} \sup_{x \in B} \left\| \left(\frac{\partial F(\cdot;x)}{\partial t_i} \right)(\mathbf{s} + \mathbf{b}_{k_l}) - \left(\frac{\partial F(\cdot;x)}{\partial t_i} \right)(\mathbf{s}) \right\|_Y = 0, \tag{6.19}$$

uniformly in $\mathbf{s} \in \mathbb{R}^n$ and $1 \leqslant i \leqslant n$. Since, for every $x \in B$, we have

$$\begin{aligned}
&\{[F(\mathbf{t} + \mathbf{a} + \mathbf{b}_{k_l}) - F_B(\mathbf{t} + \mathbf{b}_{k_l})] - [F_B(\mathbf{t} + \mathbf{a}) - F_B(\mathbf{t})]\}(x) \\
&\quad = \sum_{i=1}^n \int_0^{a_i} F_{t_i}(s_1 + \mathbf{b}_{k_l}^1, \ldots, s_{i-1} + \mathbf{b}_{k_l}^{i-1}, s_i + \mathbf{b}_{k_l}^i + v, s_{i+1} + \mathbf{b}_{k_l}^{i+1} + a_{i+1}, \ldots, s_n + \mathbf{b}_{k_l}^n + a_n; x)\, dv \\
&\qquad - \sum_{i=1}^n \int_0^{a_i} F_{t_i}(s_1, \ldots, s_{i-1}, s_i + v, s_{i+1} + a_{i+1}, \ldots, s_n + a_n; x)\, dv,
\end{aligned}$$

applying (6.19) we simply get

$$\lim_{l\to+\infty}\big\|\big[F(\mathbf{t}+\mathbf{a}+\mathbf{b}_{k_l})-F_B(\mathbf{t}+\mathbf{b}_{k_l})\big]-\big[F_B(\mathbf{t}+\mathbf{a})-F_B(\mathbf{t})\big]\big\|_{l_\infty(B:Y)}=0,$$

uniformly in $\mathbf{t}\in\mathbb{R}^n$. The proof of the theorem is thereby complete. □

Corollary 6.1.44. *Suppose that the function $F:\mathbb{R}^n\to Y$ is continuous as well as that $\partial F(\cdot)/\partial t_i:\mathbb{R}^n\to Y$ is an almost periodic function. Suppose that at least one of the following two conditions holds:*

(C1) *The Banach space Y does not contain c_0.*

(C2) *The range of the function $F(\cdot)$ is weakly relatively compact in Y.*

Then the function $F(\cdot)$ is almost periodic.

The proof of the following extension of [47, Theorem 3.2], which has already been given in the introductory part for the scalar-valued functions, is simple and therefore omitted.

Theorem 6.1.45. *Suppose that the function $F:\mathbb{R}^n\times X\to Y$ is Bohr $\mathcal{B}$-almost periodic and for each set $B\in\mathcal{B}$ the Banach space $l_\infty(B:Y)$ does not contain c_0 or the function*

$$H(t_1,t_2,\ldots,t_n;x):=\int_0^{t_1}F(t,t_2,\ldots,t_n;x)\,dt,\quad (t_1,t_2,\ldots,t_n)\in\mathbb{R}^n,\quad x\in X,$$

satisfies the requirement that the range of the function $H_B(\cdot)$, given by (6.12) with $F=G$ therein, is weakly relatively compact in $l_\infty(B:Y)$. If there exist Bohr $\mathcal{B}$-almost periodic functions $G_i:\mathbb{R}^n\times X\to Y$ such that $F_{t_i}(t_1,t_2,\ldots,t_n;x)=(G_i)_{t_1}(t_1,t_2,\ldots,t_n;x)$ is a continuous function on $\mathbb{R}^n$ for each fixed element $x\in X$ ($2\leqslant i\leqslant n$), then the function $H:\mathbb{R}^n\times X\to Y$ is Bohr $\mathcal{B}$-almost periodic.

Corollary 6.1.46. *Suppose that the function $F:\mathbb{R}^n\to Y$ is almost periodic and the Banach space Y does not contain c_0 or the function*

$$H(t_1,t_2,\ldots,t_n):=\int_0^{t_1}F(t,t_2,\ldots,t_n)\,dt,\quad (t_1,t_2,\ldots,t_n)\in\mathbb{R}^n$$

satisfies the requirement that its range is weakly relatively compact in Y. If there exist almost periodic functions $G_i:\mathbb{R}^n\to Y$ such that $F_{t_i}(t_1,t_2,\ldots,t_n)=(G_i)_{t_1}(t_1,t_2,\ldots,t_n)$ is a continuous function on $\mathbb{R}^n$, for $2\leqslant i\leqslant n$, then the function $H:\mathbb{R}^n\to Y$ is almost periodic.

The interested reader may try to extend the results of [47, Theorem 4.1, Theorem 4.2], regarding the almost periodicity of the function

$$\mathbf{t}\mapsto\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n}F(s_1,s_2,\ldots,s_n)\,ds_1\,ds_2\cdots ds_n,\quad \mathbf{t}=(t_1,t_2,\ldots,t_n)\in\mathbb{R}^n,$$

in the above manner. The results about integration of multi-dimensional asymptotically almost periodic functions and related connections with the weak asymptotic almost periodicity, obtained in [882, Section 4], will be reconsidered elsewhere.

We can use Proposition 6.1.30 to simply deduce when the decompositions in Definition 6.2.28 are unique; Proposition 6.1.28(ii) and Proposition 6.1.32 can be reformulated in our new context, as well.

6.1.5 Composition theorems for $(\mathrm{R}, \mathcal{B})$-multi-almost periodic type functions

Suppose that $F : I \times X \to Y$ and $G : I \times Y \to Z$ are given functions. The main aim of this subsection is to analyze the almost periodic properties of the multi-dimensional Nemytskii operator $W : I \times X \to Z$ given by

$$W(\mathbf{t}; x) := G(\mathbf{t}; F(\mathbf{t}; x)), \quad \mathbf{t} \in I,\ x \in X. \tag{6.20}$$

We will first state the following generalization of [364, Theorem 4.16]; the proof is similar to the proof of the above-mentioned theorem but we will present it for the sake of completeness.

Theorem 6.1.47. *Suppose that $F : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic and $G : I \times Y \to Z$ is $(\mathrm{R}', \mathcal{B}')$-multi-almost periodic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences, as well as*

$$\mathcal{B}' := \Big\{ \bigcup_{\mathbf{t} \in I} F(t; B) : B \in \mathcal{B} \Big\}. \tag{6.21}$$

If there exists a finite constant $L > 0$ such that

$$\| G(\mathbf{t}; x) - G(\mathbf{t}; y) \|_Z \leqslant L \| x - y \|_Y, \quad \mathbf{t} \in I,\ x,\ y \in Y, \tag{6.22}$$

then the function $W(\cdot; \cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.

Proof. Let the set $B \in \mathcal{B}$ and the sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ be given. By definition, there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : I \times X \to Y$ such that (6.2) holds. Set $B' := \bigcup_{\mathbf{t} \in I} F(t; B)$ and $b' := (\mathbf{b}_{k_l})$. Then there exist a subsequence $(\mathbf{b}_{k_{l_m}} = (b_{k_{l_m}}^1, b_{k_{l_m}}^2, \ldots, b_{k_{l_m}}^n))$ of $(\mathbf{b}_{k_l})$ and a function $G^* : I \times Y \to Z$ such that

$$\lim_{m \to +\infty} \| G(\mathbf{t} + (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n); y) - G^*(\mathbf{t}; y) \|_Z = 0,$$

uniformly for $y \in B'$ and $\mathbf{t} \in I$. It suffices to show that

$$\lim_{m \to +\infty} \| G(\mathbf{t} + (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n); F(\mathbf{t} + (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n); x)) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \|_Z = 0, \tag{6.23}$$

uniformly for $x \in B$ and $\mathbf{t} \in I$. Denote $\tau_m := (b^1_{k_{l_m}}, \ldots, b^n_{k_{l_m}})$ for all $m \in \mathbb{N}$. We have ($\mathbf{t} \in I$, $x \in B$, $m \in \mathbb{N}$):

$$\begin{aligned}
&\big\|G\big(\mathbf{t}+\tau_m; F(\mathbf{t}+\tau_m;x)\big) - G^*\big(\mathbf{t}; F^*(\mathbf{t};x)\big)\big\|_Z \\
&\quad \leqslant \big\|G\big(\mathbf{t}+\tau_m; F(\mathbf{t}+\tau_m;x)\big) - G\big(\mathbf{t}+\tau_m; F^*(\mathbf{t};x)\big)\big\|_Z \\
&\qquad + \big\|G\big(\mathbf{t}+\tau_m; F^*(\mathbf{t};x)\big) - G^*\big(\mathbf{t}; F^*(\mathbf{t};x)\big)\big\|_Z \\
&\quad \leqslant L\big\|F(\mathbf{t}+\tau_m;x) - F^*(\mathbf{t};x)\big\|_Y + \big\|G\big(\mathbf{t}+\tau_m; F^*(\mathbf{t};x)\big) - G^*\big(\mathbf{t}; F^*(\mathbf{t};x)\big)\big\|_Z.
\end{aligned}$$

Since $x \in B$ and $F^*(\mathbf{t};x) \in B'$ for all $\mathbf{t} \in I$, the limit equality (6.23) holds, uniformly for $x \in B$ and $\mathbf{t} \in I$, which completes the proof of the theorem. □

Keeping in mind Proposition 6.1.17, Theorem 6.1.18, Theorem 6.1.47 and the fact that a continuous function $F : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic (Bohr $\mathcal{B}$-almost periodic) if and only if it is $(\mathrm{R}, \overline{\mathcal{B}})$-multi-almost periodic (Bohr $\overline{\mathcal{B}}$-almost periodic), where $\overline{\mathcal{B}} := \{\overline{B} : B \in \mathcal{B}\}$, we can immediately clarify the following.

Corollary 6.1.48. *Suppose that $\mathcal{B}$ is any collection of compact subsets of X, $F : \mathbb{R}^n \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic and $G : \mathbb{R}^n \times Y \to Z$ is Bohr $\mathcal{B}'$-almost periodic, where $\mathcal{B}'$ is given by* (6.21). *If there exists a finite constant $L > 0$ such that* (6.22) *holds with $I = \mathbb{R}^n$, then the function $W(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic.*

A slight modification of the proof of Theorem 6.1.47 (cf. also the proof of [364, Theorem 3.31]) shows that the following result holds true.

Theorem 6.1.49. *Suppose that $F : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic and $G : I \times Y \to Z$ is $(\mathrm{R}', \mathcal{B}')$-multi-almost periodic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences, as well as $\mathcal{B}'$ is given by* (6.21). *Set*

$$\mathcal{B}'^{*} := \bigcup_{(\mathbf{b}_k)\in \mathrm{R}; B \in \mathcal{B}} \{F^*(t;B) : \mathbf{t} \in I\},$$

with the meaning clear. If

$$\begin{aligned}
&(\forall B \in \mathcal{B})\ (\forall \varepsilon > 0)\ (\exists \delta > 0) \\
&\big(x,\ y \in \mathcal{B}' \cup \mathcal{B}'^{*} \ \textit{and}\ \|x-y\|_Y < \delta \Rightarrow \big\|G(\mathbf{t};x) - G(\mathbf{t};y)\big\|_Z < \varepsilon,\ \mathbf{t} \in I\big),
\end{aligned}$$

then the function $W(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.

Now we proceed with the analysis of composition theorems for asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions. Our first result is in a close connection with Theorem 6.1.47 and [364, Theorem 3.49].

Theorem 6.1.50. *Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, $F_0 : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, $Q_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ and $F(\mathbf{t};x) = F_0(\mathbf{t};x) + Q_0(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$. Suppose further that $G_1 : I \times Y \to Z$ is $(\mathrm{R}', \mathcal{B}')$-multi-almost periodic,* *where* R' *is a collection of all sequences* $b : \mathbb{N} \to \mathbb{R}^n$ *from* R *and all their subsequences as well as* $\mathcal{B}'$ *is defined by* (6.21) *with the function* $F(\cdot;\cdot)$ *replaced therein by the function* $F_0(\cdot;\cdot)$, $Q_1 \in C_{0,\mathbb{D},\mathcal{B}_1}(I \times Y : Z)$, *where*

$$\mathcal{B}_1 := \left\{ \bigcup_{\mathbf{t}\in I} F(t;B) : B \in \mathcal{B} \right\}, \tag{6.24}$$

and $G(\mathbf{t};x) = G_1(\mathbf{t};x) + Q_1(\mathbf{t};x)$ *for all* $\mathbf{t} \in I$ *and* $x \in Y$. *If there exists a finite constant* $L > 0$ *such that the estimate* (6.22) *holds with the function* $G(\cdot;\cdot)$ *replaced therein by the function* $G_1(\cdot;\cdot)$, *then the function* $W(\cdot;\cdot)$ *is* $\mathbb{D}$*-asymptotically* $(\mathrm{R},\mathcal{B})$*-multi-almost periodic.*

Proof. By Theorem 6.1.47, the function $(\mathbf{t};x) \mapsto G_1(\mathbf{t};F_0(\mathbf{t};x))$, $\mathbf{t} \in I$, $x \in X$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic. Furthermore, we have the following decomposition:

$$W(\mathbf{t};x) = G_1(\mathbf{t};F_0(\mathbf{t};x)) + [G_1(\mathbf{t};F(\mathbf{t};x)) - G_1(\mathbf{t};F_0(\mathbf{t};x))] + Q_1(\mathbf{t};F(\mathbf{t};x)),$$

for any $\mathbf{t} \in I$ and $x \in X$. Since

$$\|G_1(\mathbf{t};F(\mathbf{t};x)) - G_1(\mathbf{t};F_0(\mathbf{t};x))\|_Z \leqslant L\|Q_0(\mathbf{t};x)\|_Y, \quad \mathbf{t} \in I,\ x \in X,$$

we find that the function $(\mathbf{t};x) \mapsto G_1(\mathbf{t};F(\mathbf{t};x)) - G_1(\mathbf{t};F_0(\mathbf{t};x))$, $\mathbf{t} \in I$, $x \in X$ belongs to the space $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Z)$. The same holds for the function $(\mathbf{t};x) \mapsto Q_1(\mathbf{t};F(\mathbf{t};x))$, $\mathbf{t} \in I$, $x \in X$, due to our choice of the collection $\mathcal{B}_1$ in (6.24). □

It seems that we cannot remove the Lipschitz type assumptions used in Corollary 6.1.48 without imposing some additional conditions; but, this can be always done in the case that $F : \mathbb{R}^n \to Y$ is Bohr almost periodic and $G : \mathbb{R}^n \times Y \to Z$ is Bohr $\mathcal{B}$-almost periodic with $\overline{R(F)} = B \in \mathcal{B}$; see, e. g., [442, Theorem 2.11, p. 27] and its proof for the scalar-valued case. Keeping in mind Proposition 6.1.22, we can state the following extension of this result.

Theorem 6.1.51. *Suppose that the set I is admissible with respect to the almost periodic extensions. If* $F : I \to Y$ *is uniformly continuous, Bohr almost periodic and* $G : I \times Y \to Z$ *is Bohr* $\mathcal{B}$*-almost periodic with* $\overline{R(F)} = B \in \mathcal{B}$, *then the function* $W : I \to Z$ *is uniformly continuous and Bohr almost periodic, provided that the function* $G(\cdot;\cdot)$ *is uniformly continuous on* $I \times B$.

Proof. It is clear that there exists a unique almost periodic extension $\tilde{F} : \mathbb{R}^n \to Y$ of the function $F(\cdot)$ to the whole Euclidean space and there exists a unique almost periodic extension $\widetilde{G_B} : \mathbb{R}^n \to l_\infty(B : Z)$ of the function $G_B(\cdot)$ to the whole Euclidean space since the function $F(\cdot)$ is uniformly continuous and the function $G(\cdot;\cdot)$ is uniformly continuous on $I \times B$. Define

$$\tilde{W}(\mathbf{t}) := [\widetilde{G_B}(\mathbf{t})](\tilde{F}(\mathbf{t})), \quad \mathbf{t} \in \mathbb{R}^n.$$

Since $W(\mathbf{t}) = [G_B(\mathbf{t})](F(\mathbf{t}))$ for all $\mathbf{t} \in I$, it is clear that the function $\tilde{W}(\cdot)$ extends the function $W(\cdot)$ to the whole Euclidean space. Furthermore, by the proof of Theorem 6.1.37, we have $R(\tilde{F}) \subseteq B$ and there exists a sequence (τ_k) in I such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and $\lim_{k\to+\infty} G_B(\mathbf{t} + \tau_k) = \widetilde{G_B}(\mathbf{t})$, uniformly for $t \in I$. In order to see that the function $\tilde{W}(\cdot)$ is uniformly continuous on $\mathbb{R}^n$, we can use the following calculation:

$$\begin{aligned}
\|\tilde{W}(\mathbf{t}') - \tilde{W}(\mathbf{t}'')\|_Y &= \|[\widetilde{G_B}(\mathbf{t}')](\tilde{F}(\mathbf{t}')) - [\widetilde{G_B}(\mathbf{t}'')](\tilde{F}(\mathbf{t}''))\|_Y \\
&\leqslant \|[\widetilde{G_B}(\mathbf{t}')](\tilde{F}(\mathbf{t}')) - [\widetilde{G_B}(\mathbf{t}'')](\tilde{F}(\mathbf{t}'))\|_Y \\
&\quad + \|[\widetilde{G_B}(\mathbf{t}'')](\tilde{F}(\mathbf{t}')) - [\widetilde{G_B}(\mathbf{t}'')](\tilde{F}(\mathbf{t}''))\|_Y \\
&\leqslant \sup_{x\in B}\|[\widetilde{G_B}(\mathbf{t}')](x) - [\widetilde{G_B}(\mathbf{t}'')](x)\|_Y \\
&\quad + \limsup_{k\to+\infty}\|[G_B(\mathbf{t}'' + \tau_k)](\tilde{F}(\mathbf{t}')) - [G_B(\mathbf{t}'' + \tau_k)](\tilde{F}(\mathbf{t}''))\|_Y \\
&= \sup_{x\in B}\|[\widetilde{G_B}(\mathbf{t}')](x) - [\widetilde{G_B}(\mathbf{t}'')](x)\|_Y \\
&\quad + \limsup_{k\to+\infty}\|G(\mathbf{t}'' + \tau_k; \tilde{F}(\mathbf{t}')) - G(\mathbf{t}'' + \tau_k; \tilde{F}(\mathbf{t}''))\|_Y, \quad \mathbf{t}', \mathbf{t}'' \in \mathbb{R}^n,
\end{aligned}$$

the uniform continuity of $\widetilde{G_B}(\cdot)$ and the uniform continuity of $G(\cdot;\cdot)$ on $I \times B$. Due to Proposition 6.1.19, for every $\varepsilon > 0$, the functions $\tilde{F}(\cdot)$ and $\widetilde{G_B}(\cdot)$ can share the same set of ε-almost periods which is relatively dense in $\mathbb{R}^n$. Keeping in mind this fact, we can repeat almost verbatim the above calculus, with the numbers $\mathbf{t}' = \mathbf{t} \in \mathbb{R}^n$ and $\mathbf{t}'' = \mathbf{t} + \tau \in \mathbb{R}^n$ so as to conclude that the function $\tilde{W}(\cdot)$ is Bohr almost periodic on $\mathbb{R}^n$, finishing the proof. □

We can also prove the following result which corresponds to Theorem 6.1.49 and [364, Theorem 3.50].

Theorem 6.1.52. *Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, $F_0 : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, $Q_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ and $F(\mathbf{t}; x) = F_0(\mathbf{t}; x) + Q_0(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$. Suppose further that $G_1 : I \times Y \to Z$ is $(\mathrm{R}', \mathcal{B}')$-multi-almost periodic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences as well as $\mathcal{B}'$ is defined by* (6.21) *with the function $F(\cdot;\cdot)$ replaced therein by the function $F_0(\cdot;\cdot)$, $Q_1 \in C_{0,\mathbb{D},\mathcal{B}_1}(I \times Y : Z)$, where $\mathcal{B}_1$ is given through* (6.24), *and $G(\mathbf{t}; x) = G_1(\mathbf{t}; x) + Q_1(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in Y$. Set*

$$\mathcal{B}_2 := \left\{\bigcup_{\mathbf{t}\in I} F_0(t; B) : B \in \mathcal{B}\right\} \cup \bigcup_{(\mathbf{b}_k)\in\mathrm{R}; B\in\mathcal{B}} \{F_0^*(t; B) : \mathbf{t} \in I\}.$$

If

$$\begin{gathered}
(\forall B \in \mathcal{B})\ (\forall \varepsilon > 0)\ (\exists \delta > 0) \\
\big(x, y \in \mathcal{B}_1 \cup \mathcal{B}_2 \text{ and } \|x - y\|_Y < \delta \Rightarrow \|G_1(\mathbf{t}; x) - G_1(\mathbf{t}; y)\|_Z < \varepsilon,\ \mathbf{t} \in I\big),
\end{gathered}$$

then the function $W(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.

It is clear that Theorem 6.1.49 and Theorem 6.1.52 can be reformulated for Bohr $\mathcal{B}$-almost periodic functions with small terminological difficulties concerning the use of limit functions. Similar results can be established for the class of $\mathcal{B}$-uniformly recurrent functions.

6.1.6 Invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity under the actions of convolution products

This subsection investigates the invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity under the actions of convolution products. We will use the following notation: if any component of tuple $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is strictly positive, then we simply write $\mathbf{t} > \mathbf{0}$.

We start by stating the following result, which is very similar to [631, Proposition 2.6.11]; for the sake of better exposition, we will omit the proof (the main details of the proof for Stepanov generalizations will be given later).

Theorem 6.1.53. *Let $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X, Y)$ be a strongly continuous operator family such that $\int_{(0,\infty)^n} \|R(\mathbf{t})\|\, d\mathbf{t} < \infty$. If $f : \mathbb{R}^n \to X$ is almost periodic, then the function $F : \mathbb{R}^n \to Y$, given by*

$$F(\mathbf{t}) := \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{6.25}$$

is well defined and almost periodic.

For completeness, we will include the proof of following result.

Theorem 6.1.54. *Let $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X, Y)$ be a strongly continuous operator family such that $\int_{(0,\infty)^n} \|R(\mathbf{t})\|\, d\mathbf{t} < \infty$. If $f : \mathbb{R}^n \to X$ is a bounded R-almost periodic function, then the function $F : \mathbb{R}^n \to Y$, given by (6.25), is well defined and R-almost periodic.*

Proof. Let $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ be given. Then there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $f^* : \mathbb{R}^n \to X$ such that $\lim_{l\to+\infty} f(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n)) = f^*(\mathbf{t})$ uniformly for $\mathbf{t} \in \mathbb{R}^n$. Hence, the function $f^* : \mathbb{R}^n \to X$ is bounded and measurable. Clearly,

$$F(\mathbf{t}) = \int_{[0,\infty)^n} R(\mathbf{s}) f(\mathbf{t} - \mathbf{s})\, d\mathbf{s}$$

for all $\mathbf{t} \in \mathbb{R}^n$ and the integral

$$\int_{[0,\infty)^n} R(\mathbf{s}) f^*(\mathbf{t} - \mathbf{s})\, d\mathbf{s}$$

is well defined for all $\mathbf{t} \in \mathbb{R}^n$. Furthermore,

$$\lim_{l\to\infty} \int\limits_{[0,\infty)^n} R(\mathbf{s})f(\mathbf{t}+\mathbf{b}_{k_l}-\mathbf{s})\,d\mathbf{s} = \int\limits_{[0,\infty)^n} R(\mathbf{s})f^*(\mathbf{t}-\mathbf{s})\,d\mathbf{s}$$

uniformly for $\mathbf{t} \in \mathbb{R}^n$, because

$$\begin{aligned}&\Bigg\| \int\limits_{[0,\infty)^n} R(\mathbf{s})f(\mathbf{t}+\mathbf{b}_{k_l}-\mathbf{s})\,d\mathbf{s} - \int\limits_{[0,\infty)^n} R(\mathbf{s})f^*(\mathbf{t}-\mathbf{s})\,d\mathbf{s}\Bigg\|_Y \\ &\quad \leqslant \int\limits_{[0,\infty)^n} \|R(\mathbf{s})\| \cdot \|f(\mathbf{t}+\mathbf{b}_{k_l}-\mathbf{s}) - f^*(\mathbf{t}-\mathbf{s})\|\,d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,\ l \in \mathbb{N},\end{aligned}$$

which simply yields the required conclusion. □

Under certain extra conditions, we can also reformulate the above result for uniformly recurrent functions defined on $\mathbb{R}^n$. On the other hand, it seems that we must slightly strengthen the notion introduced in Definition 6.1.27 in order to investigate the invariance of $\mathbb{D}$-asymptotical multi-almost periodicity under the actions of "finite" convolution products (the various notions of asymptotical almost periodicity examined in Part I are introduced following the approach in Definition 6.1.55, so that we must confess to a little abuse of the notion here).

Definition 6.1.55. Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, and $F : I \times X \to Y$ is a continuous function. Then we say that $F(\cdot;\cdot)$ is strongly $\mathbb{D}$-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic, resp. strongly $\mathbb{D}$-asymptotically $(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodic, if and only if there exist an $(\mathrm{R},\mathcal{B})$-multi-almost periodic function $G : \mathbb{R}^n \times X \to Y$, resp. an $(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodic function $G : \mathbb{R}^n \times X \to Y$, and a function $Q \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$.

Let $I = \mathbb{R}^n$. Then it is said that $F(\cdot;\cdot)$ is strongly asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic, resp. strongly asymptotically $(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodic, if and only if $F(\cdot;\cdot)$ is strongly $\mathbb{R}^n$-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic, resp. strongly $\mathbb{R}^n$-asymptotically $(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodic. Finally, if $X = \{0\}$, then we also say that the function $F(\cdot)$ is strongly asymptotically R-multi-almost periodic, and so on and so forth.

Set, for brevity, $I_\mathbf{t} := (-\infty,t_1] \times (-\infty,t_2] \times \cdots \times (-\infty,t_n]$ and $\mathbb{D}_\mathbf{t} := I_\mathbf{t} \cap \mathbb{D}$ for any $\mathbf{t} = (t_1,t_2,\ldots,t_n) \in \mathbb{R}^n$. Now we are ready to formulate the following result.

Proposition 6.1.56. *Suppose that* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ *is a strongly continuous operator family such that* $\int_{(0,\infty)^n} \|R(\mathbf{t})\|\,d\mathbf{t} < \infty$. *If* $f : \mathbb{R}^n \to X$ *is strongly* $\mathbb{D}$*-asymptotically almost periodic (bounded strongly* $\mathbb{D}$*-asymptotically* R*-multi-almost periodic),*

$$\lim_{|\mathbf{t}|\to\infty,\mathbf{t}\in\mathbb{D}} \int\limits_{I_\mathbf{t}\cap\mathbb{D}^c} \|R(\mathbf{t}-\mathbf{s})\|\,d\mathbf{s} = 0 \qquad (6.26)$$

and for each $r > 0$ *we have*

$$\lim_{|\mathbf{t}|\to\infty,\mathbf{t}\in\mathbb{D}} \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} \|R(\mathbf{t}-\mathbf{s})\|\, d\mathbf{s} = 0, \tag{6.27}$$

then the function

$$F(\mathbf{t}) := \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})f(\mathbf{s})\, ds, \quad \mathbf{t} \in I,$$

is strongly $\mathbb{D}$*-asymptotically almost periodic (bounded strongly* $\mathbb{D}$*-asymptotically* R*-multi-almost periodic).*

Proof. By definition, we have the existence of an almost periodic function $G : \mathbb{R}^n \to X$ and a function $Q \in C_{0,\mathbb{D}}(I : X)$ such that $f(\mathbf{t}) = g(\mathbf{t}) + q(\mathbf{t})$ for all $\mathbf{t} \in I$ and $x \in X$. Clearly, we have the decomposition

$$F(\mathbf{t}) = \int_{I_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\, d\mathbf{s} + \Bigg[\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\, d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\, d\mathbf{s}\Bigg], \quad \mathbf{t} \in I.$$

Keeping in mind Theorem 6.1.53, it suffices to show that the function

$$\mathbf{t} \mapsto \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\, d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in I,$$

belongs to the class $C_{0,\mathbb{D}}(I : X)$. For the second addend, this immediately follows from the boundedness of the function $g(\cdot)$ and condition (6.26). In order to show this for the first addend, fix a number $\varepsilon > 0$. Then there exists $r > 0$ such that, for every $\mathbf{t} \in \mathbb{D}$ with $|\mathbf{t}| > r$, we have $\|q(\mathbf{t})\| < \varepsilon$. Furthermore, we have

$$\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\, d\mathbf{s} = \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\, d\mathbf{s} + \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)^c} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in I.$$

Clearly, $M := \sup_{\mathbf{t}\in\mathbb{D}} \|q(\mathbf{t})\| < \infty$ and

$$\Bigg\| \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\, d\mathbf{s}\Bigg\|_Y \leqslant M \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} \|R(\mathbf{t}-\mathbf{s})\|\, d\mathbf{s}, \quad \mathbf{t} \in I,$$

so that the first addend in the above sum belongs to the class $C_{0,\mathbb{D}}(I : X)$ due to condition (6.27). This is also clear for the second addend since

$$\Bigg\| \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\, d\mathbf{s}\Bigg\|_Y \leqslant \varepsilon \int_{(0,\infty)^n} \|R(\mathbf{s})\|\, d\mathbf{s}, \quad \mathbf{t} \in I. \qquad \square$$

If $\mathbb{D} = [\alpha_1, \infty) \times [\alpha_2, \infty) \times \cdots \times [\alpha_n, \infty)$ for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $\mathbb{D}_{\mathbf{t}} = [\alpha_1, t_1] \times [\alpha_2, t_2] \times \cdots \times [\alpha_n, t_n]$ and conditions (6.26)–(6.27) hold, as easily shown, which implies that the function $F(\mathbf{t}) = \int_{\mathbf{t}}^{\alpha} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s})\, ds$, $\mathbf{t} \in I$, is strongly $\mathbb{D}$-asymptotically almost periodic, where we accept the notation

$$\int_{\mathbf{t}}^{\alpha} \cdot = \int_{\alpha_1}^{t_1} \int_{\alpha_2}^{t_2} \cdots \int_{\alpha_n}^{t_n} . \tag{6.28}$$

6.1.7 Examples and applications to the abstract Volterra integro-differential equations

In this subsection, we apply our results in the analysis of the existence and uniqueness of the multi-dimensional almost periodic type solutions for various classes of abstract Volterra integro-differential equations.

We start with the following important examples:

1. Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1 \leqslant p < \infty$. It is well known that the Gaussian semigroup

$$(G(t)F)(x) := (4\pi t)^{-(n/2)} \int_{\mathbb{R}^n} F(x-y) e^{-\frac{|y|^2}{4t}}\, dy, \quad t > 0,\, f \in Y,\, x \in \mathbb{R}^n,$$

can be extended to a bounded analytic C_0-semigroup of angle $\pi/2$, generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y; see [82, Example 3.7.6] for more details (recall that the semigroup $(G(t))_{t>0}$ is not strongly continuous at zero on $L^\infty(\mathbb{R}^n)$). Suppose now that $\emptyset \neq I' \subseteq I = \mathbb{R}^n$ and $F(\cdot)$ is bounded Bohr $(\mathcal{B}, I')$-almost periodic, resp. bounded $(\mathcal{B}, I')$-uniformly recurrent. Then for each $t_0 > 0$ the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is likewise bounded Bohr $(\mathcal{B}, I')$-almost periodic, resp. bounded $(\mathcal{B}, I')$-uniformly recurrent. Towards see this, it suffices to recall the corresponding definitions and observe that, for every $x, \ \tau \in \mathbb{R}^n$, we have

$$\left|u(x+\tau, t_0) - u(x, t_0)\right| \leqslant (4\pi t_0)^{-(n/2)} \int_{\mathbb{R}^n} \left|F(x-y+\tau) - F(x-y)\right| e^{-\frac{|y|^2}{4t_0}}\, dy;$$

see also Proposition 6.1.5 which shows that for each $t_0 > 0$ the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is bounded, $(\mathrm{R}, \mathcal{B})$-multi-almost periodic provided that R is a certain collection of subsets in $\mathbb{R}^n$ and the function $F(\cdot)$ is bounded, $(\mathrm{R}, \mathcal{B})$-multi-almost periodic (in such a way, we have extended the conclusions obtained by S. Zaidman [1067, Example 4, p. 32] to the multi-dimensional case). Concerning this example, it should be recalled that F. Yang and C. Zhang have analyzed, in [1054, Proposition 2.4–Proposition 2.6], the existence and uniqueness of remotely almost periodic solutions of multi-dimensional heat equations following a similar approach; we will further consider the class of multi-dimensional remotely almost periodic functions somewhere else.

We can similarly clarify the corresponding results for the Poisson semigroup, which is given by

$$(T(t)F)(x) := \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{\mathbb{R}^n} F(x-y) \frac{t \cdot dy}{(t^2+|y|^2)^{(n+1)/2}}, \quad t>0,\ f \in Y,\ x \in \mathbb{R}^n.$$

Let us recall that the Fourier transform of the function

$$x \mapsto \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{t}{(t^2+|x|^2)^{(n+1)/2}}, \quad x \in \mathbb{R}^n,$$

is given by $e^{-t|\cdot|}$ for all $t > 0$ (see [82, Example 3.7.9] for more details).

2. Set

$$E_1(x,t) := (\pi t)^{-1/2} \int_0^x e^{-y^2/4t}\, dy, \quad x \in \mathbb{R},\ t > 0. \tag{6.29}$$

In connection with the homogeneous solutions of the heat equation on domain $I := \{(x,t) : x > 0,\ t > 0\}$, we would like to recall that F. Trèves [981, p. 433] has proposed the following formula:

$$u(x,t) = \frac{1}{2} \int_{-x}^{x} \frac{\partial E_1}{\partial y}(y,t) u_0(x-y)\, dy - \int_0^t \frac{\partial E_1}{\partial t}(x,t-s) g(s)\, ds, \quad x>0,\ t>0, \tag{6.30}$$

for the solution of the following mixed initial value problem:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad x>0,\ t>0; \\ &u(x,0) = u_0(x),\ x>0, \quad u(0,t) = g(t),\ t>0; \end{aligned} \tag{6.31}$$

for simplicity, we will not consider here the evolution analogues of (6.30) and the generation of various classes of operator semigroups with the help of this formula. Concerning the existence and uniqueness of multi-dimensional almost periodic type solutions of (6.31), we will present only one result which exploits the formula (6.30) with $g(t) \equiv 0$. Suppose that $0 < T < \infty$ and the function $u_0 : [0,\infty) \to \mathbb{C}$ is bounded Bohr I_0-almost periodic, resp. bounded I_0-uniformly recurrent, for a certain non-empty subset I_0 of $[0,\infty)$. Set $I' := I_0 \times (0,T)$. If $\mathbb{D}$ is any unbounded subset of I which has the property that

$$\lim_{|(x,t)|\to+\infty,(x,t)\in\mathbb{D}} \min\left(\frac{x^2}{4(t+T)}, t\right) = +\infty, \tag{6.32}$$

then the solution $u(x,t)$ of (6.31) is $\mathbb{D}$-asymptotically I'-almost periodic of type 1, resp. $\mathbb{D}$-asymptotically I'-uniformly recurrent of type 1 (see Definition 6.1.33). In order to see that, observe that the formula (6.30), in our concrete situation, reads as follows:

$$u(x,t)=\frac{1}{2}\int_{-x}^{x}(\pi t)^{-1/2}e^{-y^2/4t}u_0(x-y)\,dy,\quad x>0,\ t>0,$$

and that for any $(x,t)\in I$ and $(\tau_1,\tau_2)\in I$ we have

$$\begin{aligned}&\left|u(x+\tau_1,t+\tau_2)-u(x,t)\right|\\&\quad\leqslant\frac{\|u_0\|_\infty}{2}\int_{x}^{x+\tau_1}\left(\pi(t+\tau_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2)}\,dy\\&\qquad+\frac{\|u_0\|_\infty}{2}\int_{-(x+\tau_1)}^{-x}\left(\pi(t+\tau_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2)}\,dy\\&\qquad+\frac{1}{2}\int_{-x}^{x}\left|\left(\pi(t+\tau_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2)}u_0(x+\tau_1-y)-(\pi t)^{-1/2}e^{-y^2/4t}u_0(x-y)\right|dy.\end{aligned}\tag{6.33}$$

The considerations for both classes are similar and we will only analyze the class of $\mathbb{D}$-asymptotically I'-almost periodic functions of type 1 below. Let $\varepsilon>0$ be given. Then we know that there exists $l>0$ such that for each $x_0\in I_0$ there exists $\tau_1\in(x_0-l,x_0+l)\cap I_0$ such that

$$\left|u_0(x+\tau_1)-u_0(x)\right|\leqslant\varepsilon,\quad x\geqslant 0.\tag{6.34}$$

Furthermore, there exists a finite real number $M_0>0$ such that $\int_v^{+\infty}e^{-x^2}\,dx<\varepsilon$ for all $v\geqslant M_0$. Let $M>0$ be such that

$$\min\left(\frac{x^2}{4(t+T)},t\right)>M_0^2+\frac{1}{\varepsilon},\quad\text{provided }(x,t)\in\mathbb{D}\text{ and }\left|(x,t)\right|>M.\tag{6.35}$$

So, let $(x,t)\in\mathbb{D}$ and $|(x,t)|>M$. For the first addend in (6.33), we can use the estimates

$$\begin{aligned}\frac{\|u_0\|_\infty}{2}\int_{x}^{x+\tau_1}\left(\pi(t+\tau_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2)}\,dy&=\pi^{-1/2}\|u_0\|_\infty\int_{x/2\sqrt{t+\tau_2}}^{(x+\tau_1)/2\sqrt{t+\tau_2}}e^{-v^2}\,dv\\&\leqslant\pi^{-1/2}\|u_0\|_\infty\int_{x/2\sqrt{t+\tau_2}}^{+\infty}e^{-v^2}\,dv\\&\leqslant\pi^{-1/2}\|u_0\|_\infty\int_{x/2\sqrt{t+T}}^{+\infty}e^{-v^2}\,dv\leqslant\varepsilon\pi^{-1/2}\|u_0\|_\infty;\end{aligned}$$

the same estimate can be used for the second addend in (6.33). For the third addend in (6.33), we can use the decomposition (see (6.34))

$$\begin{aligned}
&\frac{1}{2}\int_{-x}^{x}\left|(\pi(t+\tau_2))^{-1/2}e^{-y^2/4(t+\tau_2)}u_0(x+\tau_1-y)-(\pi t)^{-1/2}e^{-y^2/4t}u_0(x-y)\right|dy\\
&\quad\leqslant\frac{1}{2}\int_{-x}^{x}(\pi(t+\tau_2))^{-1/2}e^{-y^2/4(t+\tau_2)}\left|u_0(x+\tau_1-y)-u_0(x-y)\right|dy,\\
&\qquad+\frac{1}{2}\int_{-x}^{x}\left|(\pi(t+\tau_2))^{-1/2}e^{-y^2/4(t+\tau_2)}u_0(x-y)-(\pi t)^{-1/2}e^{-y^2/4t}u_0(x-y)\right|dy,
\end{aligned}$$

which enables one to further continue the computation as follows:

$$\begin{aligned}
&\leqslant\frac{\varepsilon}{2}\int_{-x}^{x}(\pi(t+\tau_2))^{-1/2}e^{-y^2/4(t+\tau_2)}\,dy\\
&\quad+\frac{\|u_0\|_\infty}{2}\int_{-x}^{x}\left|(\pi(t+\tau_2))^{-1/2}e^{-y^2/4(t+\tau_2)}-(\pi t)^{-1/2}e^{-y^2/4t}\right|dy\\
&\leqslant\varepsilon\pi^{-1/2}\int_{-\infty}^{+\infty}e^{-v^2}\,dv\\
&\quad+\frac{\|u_0\|_\infty}{2}\int_{-x}^{x}\left|(\pi(t+\tau_2))^{-1/2}e^{-y^2/4(t+\tau_2)}-(\pi t)^{-1/2}e^{-y^2/4t}\right|dy.
\end{aligned}$$

Applying the substitution $v^2=y^2/4t$, we get

$$\begin{aligned}
&\frac{\|u_0\|_\infty}{2}\int_{-x}^{x}\left|(\pi(t+\tau_2))^{-1/2}e^{-y^2/4(t+\tau_2)}-(\pi t)^{-1/2}e^{-y^2/4t}\right|dy\\
&\quad\leqslant\pi^{-1/2}\|u_0\|_\infty\int_{-\infty}^{+\infty}\left|\sqrt{\frac{t}{t+\tau_2}}e^{-v^2\cdot\frac{t}{t+\tau_2}}-e^{-v^2}\right|dv.
\end{aligned}$$

Applying the Lagrange mean value theorem for the function $x\mapsto xe^{-v^2x^2}$, $x\in[\sqrt{t/(t+\tau_2)},1]$ ($v\in\mathbb{R}$ is fixed), we obtain

$$\begin{aligned}
&\pi^{-1/2}\|u_0\|_\infty\int_{-\infty}^{+\infty}\left|\sqrt{\frac{t}{t+\tau_2}}e^{-v^2\cdot\frac{t}{t+\tau_2}}-e^{-v^2}\right|dv\\
&\quad\leqslant\pi^{-1/2}\|u_0\|_\infty\int_{-\infty}^{+\infty}\left|\sqrt{\frac{t}{t+\tau_2}}-1\right|\max_{\zeta\in[\sqrt{\frac{t}{t+\tau_2}},1]}e^{-v^2\zeta^2}(1+2\zeta^2v^2)\,dv
\end{aligned}$$

$$\leqslant \pi^{-1/2}\|u_0\|_\infty \int\limits_{-\infty}^{+\infty} \left| \sqrt{\frac{t}{t+\tau_2}} - 1 \right| e^{-\frac{t}{t+\tau_2}v^2}\big(1+2v^2\big)\,dv$$
$$\leqslant \pi^{-1/2}\|u_0\|_\infty \left| \sqrt{\frac{t}{t+\tau_2}} - 1 \right| \int\limits_{-\infty}^{+\infty} e^{-\frac{M_0^2}{M_0^2+T}v^2}\big(1+2v^2\big)\,dv.$$

The final conclusion now follows from the estimate (6.35), by observing that

$$\left| \sqrt{\frac{t}{t+\tau_2}} - 1 \right| = \frac{\tau_2}{t+\tau_2+\sqrt{t^2+t\tau_2}} \leqslant \frac{T}{t}.$$

The following observation should be also made: If $u_0 : [0,\infty) \to \mathbb{C}$ is an essentially bounded function, then it can be easily shown that for each $x > 0$ the function $t \mapsto u(x,t)$, $t \geqslant 0$ is bounded and continuous. Furthermore, the calculus established above enables one to see that for each $x > 0$ the function $t \mapsto u(x,t)$, $t \geqslant 0$ is S-asymptotically ω-periodic for any positive real number $\omega > 0$.

The multi-dimensional almost periodic type solutions of the inhomogeneous heat equations (with respect to the space variable) will be considered somewhere else.

3. Let $\Omega = (0,\infty) \times \mathbb{R}^n$. Consider the Hamilton–Jacobi equation

$$\begin{aligned} &u_t + H(Du) = 0 \quad \text{in } \Omega,\\ &u(0,\cdot) = u_0(\cdot) \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{6.36}$$

where D is the gradient operator in space variable and H is the Hamiltonian. If we assume that $H \in C(\Omega)$ and $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$, then the Hamilton–Jacobi equation (6.36) has a unique viscosity solution. This result has been proved by M. G. Crandall and P.-L. Lions in [319, Theorem VI.2].

Theorem 6.1.57. *Suppose that $H \in C(\Omega)$ and $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$. Then for each finite real number $T > 0$ there exists a unique function $u \in C(\overline{\Omega}) \cap C_b([0,T] \times \mathbb{R}^n)$ which is a viscosity solution of* (6.36) *and satisfies*

$$\lim_{t \downarrow 0+} \|u(\cdot,t) - u_0(t)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

Moreover,

$$\big|u(t,x) - u(t,y)\big| \leqslant \sup_{\xi \in \mathbb{R}^n} \big|u_0(\xi) - u_0(\xi + y - x)\big|, \quad x,\ y \in \mathbb{R}^n,\ t \geqslant 0. \tag{6.37}$$

As a direct consequence of this result (cf. the estimate (6.37)), we find that the Bohr I'-almost periodicity (I'-uniform recurrence) of the function $u_0(\cdot)$ implies the Bohr I'-almost periodicity (I'-uniform recurrence) of viscosity solution $x \mapsto u(t,x)$, $x \in \mathbb{R}^n$ for every fixed real number $t \geqslant 0$ ($\emptyset \neq I' \subseteq \mathbb{R}^n$).

4. Consider the following Hammerstein integral equation of convolution type on $\mathbb{R}^n$ (see, e. g., [310, Section 4.3, pp. 170–180] and the references cited therein for more details on the subject):

$$y(\mathbf{t}) = g(\mathbf{t}) + \int_{\mathbb{R}^n} k(\mathbf{t} - \mathbf{s})F(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{6.38}$$

where $g : \mathbb{R}^n \to X$ is almost periodic, $F : \mathbb{R}^n \times X \to X$ is Bohr $\mathcal{B}$-almost periodic with $\mathcal{B}$ being the collection of all compact subsets of X, (6.22) holds with $F = G$ and $X = Y = Z$, $k \in L^1(\mathbb{R}^n)$ and $L\|k\|_{L^1(\mathbb{R}^n)} < 1$. Then (6.38) has a unique Bohr almost periodic solution. In actual fact, it suffices to apply the Banach contraction principle since the mapping

$$\mathrm{AP}(\mathbb{R}^n : X) \ni y \mapsto g(\cdot) + \int_{\mathbb{R}^n} k(\cdot - \mathbf{s})F(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s} \in \mathrm{AP}(\mathbb{R}^n : X)$$

is a well defined $(L\|k\|_{L^1(\mathbb{R}^n)})$-contraction, as can be easily shown by using Proposition 6.1.5, Proposition 6.1.29(v), Corollary 6.1.48 and a simple calculation.

Suppose now that R is a certain collection of sequences in $\mathbb{R}^n$ which satisfies the requirement that, for every sequence from R, any its subsequence also belongs to R. Let $\mathcal{B}'$ be the collection of all bounded subsets of X, let $F : \mathbb{R}^n \times X \to X$ be $(\mathrm{R}, \mathcal{B}')$-multi-almost periodic, (6.22) holds with $F = G$ and $X = Y = Z$, $k \in L^1(\mathbb{R}^n)$ and $L\|k\|_{L^1(\mathbb{R}^n)} < 1$. Consider the integral equation (6.38), where $g : \mathbb{R}^n \to X$ is a bounded R-multi-almost periodic function. Denote by $\mathrm{R}_b(\mathbb{R}^n : X)$ the vector space consisting of all such functions; applying Proposition 6.1.8, we see that $\mathrm{R}_b(\mathbb{R}^n : X)$ is a Banach space equipped with the sup-norm. Taking into account Proposition 6.1.5 and Theorem 6.1.47 (with $\mathrm{R}' = \mathrm{R}$), the use of Banach contraction principle enables one to see that the integral equation (6.38) has a unique bounded R-multi-almost periodic solution since the mapping

$$\mathrm{R}_b(\mathbb{R}^n : X) \ni y \mapsto g(\cdot) + \int_{\mathbb{R}^n} k(\cdot - \mathbf{s})F(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s} \in \mathrm{R}_b(\mathbb{R}^n : X)$$

is a well-defined $(L\|k\|_{L^1(\mathbb{R}^n)})$-contraction.

We can similarly analyze the existence and uniqueness of Bohr almost periodic solutions (bounded R-multi-almost periodic solutions) of the following integral equation:

$$y(\mathbf{t}) = g(\mathbf{t}) + \int_{\mathbb{R}^n} F(\mathbf{t}, \mathbf{s}, y(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

provided that $F : \mathbb{R}^{2n} \times X \to X$ is Bohr $\mathcal{B}$-almost periodic with $\mathcal{B}$ being the collection of all compact subsets of X (R is a certain collection of sequences in $\mathbb{R}^{2n}$ which satisfies the requirement that, for every sequence from R, any of its subsequences also belongs to R) and there exists a constant $L \in (0, 1)$ such that

$$\|F(\mathbf{t}, \mathbf{s}, x) - F(\mathbf{t}, \mathbf{s}, y)\| \leqslant L\|x - y\|, \quad \mathbf{t},\ \mathbf{s} \in \mathbb{R}^n;\ x,\ y \in X.$$

Details are left to the interested reader.

5. It is clear that Theorem 6.1.53 and Theorem 6.1.54 can be applied in the analysis of existence of almost periodic solutions for an essentially large class of abstract partial differential equations in Banach spaces, constructed in a little bit artificial way (even with fractional derivatives and multivalued linear operators). For example, let A_i be the infinitesimal generator of a uniformly integrable, strongly continuous semigroup $(T_i(t))_{t\geqslant 0}$ on X $(i = 1, 2)$, and let $F : \mathbb{R}^2 \to X$ be an almost periodic function. Define $T(t_1, t_2) := T_1(t_1)T_2(t_2)$, $t_1, t_2 \in \mathbb{R}$ and

$$u(t_1, t_2) := \int_{[0,\infty)^2} T_1(s_1)T_2(s_2)F(t_1 - s_1, t_2 - s_2)\, ds_1\, ds_2, \quad t_1, t_2 \in \mathbb{R}.$$

Due to Theorem 6.1.53, we find that $u : \mathbb{R}^2 \to X$ is almost periodic; furthermore, under certain conditions, we can use the Fubini theorem, interchange the order of integration and partial derivation, and use a well-known result from the one-dimensional case, in order to see that

$$\begin{aligned} u_{t_2}(t_1, t_2) &= \frac{\partial}{\partial t_2} \int_{[0,\infty)} T_1(s_1)\left(\int_0^\infty T_2(s_2)F(t_1 - s_1, t_2 - s_2)\, ds_2\right) ds_1 \\ &= \int_{[0,\infty)} T_1(s_1)\frac{\partial}{\partial t_2}\left(\int_0^\infty T_2(s_2)F(t_1 - s_1, t_2 - s_2)\, ds_2\right) ds_1 \\ &= \int_{[0,\infty)} T_1(s_1)\left(A_2 \int_0^\infty T_2(s_2)F(t_1 - s_1, t_2 - s_2)\, ds_2 + F(t_1 - s_1, t_2)\right) ds_1 \end{aligned}$$

and

$$\begin{aligned} u_{t_2 t_1}(t_1, t_2) &= \frac{\partial}{\partial t_1}\left(A_2 u(t_1, t_2) + \int_{[0,\infty)} T_1(s_1)F(t_1 - s_1, t_2)\, ds_1\right) \\ &= A_2 u_{t_1}(t_1, t_2) + A_1 \int_0^\infty T_1(s_1)F(t_1 - s_1, t_2)\, ds_1 + F(t_1, t_2), \quad t_1, t_2 \in \mathbb{R}. \end{aligned}$$

6. Consider the system of abstract partial differential equations (11) for $(s, t) \in [0, \infty)^2$, accompanied by the initial condition $u(0, 0) = x$ (since there is no risk for confusion, we will also refer to this problem as (11)). In this part, we would like to note that some partial results on the existence and uniqueness of $\mathbb{D}$-asymptotically almost periodic type solutions of this problem can be obtained by using the results from [46, Section 2.1] and some additional analyses. For simplicity, let us assume that A and B are two complex matrices of format $n \times n$, $AB = BA$, and A, resp. B, generates an exponentially decaying, strongly continuous semigroup $(T_1(s))_{s\geqslant 0}$, resp. $(T_2(t))_{t\geqslant 0}$. Let the functions $f_1(s, t)$ and $f_2(s, t)$ be continuously differentiable, let the compatibility condition $(f_2)_s - Af_2 = (f_1)_t - Bf_1$ hold $(s,\ t \geqslant 0)$, $\mathbb{D} := \{(s, t) \in [0, \infty)^2 :$

$c_1 s \leqslant t \leqslant c_2 s$ for some positive real numbers c_1 and $c_2\}$, and let the following conditions hold true:

(i) There exists a finite real constant $M > 0$ such that $|f_1(v,0)| + |f_2(0,\omega)| \leqslant M$, provided that $v,\ \omega \geqslant 0$ (here and hereafter, $|(z_1,\dots,z_n)| := (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$ if $z_i \in \mathbb{C}$ for all $i \in \mathbb{N}_n$).

(ii) The mappings $g_i : \mathbb{R}^2 \to \mathbb{C}^n$ are continuous, bounded ($i = 1,2$) and satisfy the requirement that, for every $\varepsilon > 0$, there exists $l > 0$ such that any subinterval I of $\mathbb{R}$ of length $l > 0$ contains a number $\tau \in I$ such that, for every $s,\ t \geqslant 0$, we have $|g_1(s+\tau,t) - g_1(s,t)| \leqslant \varepsilon$ and $|g_2(s,t+\tau) - g_2(s,t)| \leqslant \varepsilon$.

(iii) We find that the function $q_i : [0,\infty)^2 \to \mathbb{C}^n$ is bounded, $q_i \in C_{0,\mathbb{D}}([0,\infty)^2 : \mathbb{C}^n)$ and $f_i(s,t) = g_i(s,t) + q_i(s,t)$ for $(s,t) \in [0,\infty)^2$ and $i = 1,2$.

Then there exists a unique classical solution $u(s,t)$ of (11) (see [46, Definition 2.13]), and moreover, there exist a continuous function $u_{\mathrm{ap}}(s,t)$ on $[0,\infty)^2$ and a function $u_0 \in C_{0,\mathbb{D}}([0,\infty)^2 : \mathbb{C}^n)$ such that $u(s,t) = u_{\mathrm{ap}}(s,t) + u_0(s,t)$ for all $(s,t) \in [0,\infty)^2$, as well as for every $\varepsilon > 0$, there exists $l > 0$ such that any subinterval I of $[0,\infty)$ of length $l > 0$ contains a number $\tau \in I$ such that, for every $s,\ t \geqslant 0$, we have $|u_{\mathrm{ap}}(s+\tau,t) - u_{\mathrm{ap}}(s,t)| \leqslant \varepsilon$ and $|u_{\mathrm{ap}}(s,t+\tau) - u_{\mathrm{ap}}(s,t)| \leqslant \varepsilon$. Keeping in mind [46, Theorem 2.6, Theorem 2.16], all that we need to prove is that the above conclusion holds for the function

$$\begin{aligned} u(s,t) &= T_1(s)T_2(t)x + T_1(s)\int_0^t T_2(t-\omega)f_2(0,\omega)\,d\omega \\ &\quad + \int_0^s T_1(s-v)f_1(v,t)\,dv \\ &= T_1(s)T_2(t)x + T_2(t)\int_0^s T_1(s-v)f_1(v,0)\,dv \\ &\quad + \int_0^t T_2(t-\omega)f_2(s,\omega)\,d\omega, \quad s,\ t \geqslant 0. \end{aligned}$$

Since the quantities $s,\ t$ and $|(s,t)|$ are equivalent on $\mathbb{D}$, with the meaning clear, our assumption (i) and the exponential decaying of $(T_1(s))_{s\geqslant 0}$ $((T_2(t))_{t\geqslant 0})$ together imply that

$$\begin{aligned} &\lim_{(s,t)\in\mathbb{D},|(s,t)|\to\infty}\left[T_1(s)T_2(t)x + T_1(s)\int_0^t T_2(t-\omega)f_2(0,\omega)\,d\omega\right] \\ &= \lim_{(s,t)\in\mathbb{D},|(s,t)|\to\infty}\left[T_1(s)T_2(t)x + T_2(t)\int_0^s T_1(s-v)f_1(v,0)\,dv\right] = 0. \end{aligned}$$

Using the decomposition ($s,\ t \geqslant 0$)

$$\int_0^s T_1(s-v)f_1(v,t)\,dv$$
$$= \int_{-\infty}^s T_1(s-v)g_1(v,t)\,dv + \left[\int_0^s T_1(s-v)q(v,t)\,dv - \int_{-\infty}^0 T_1(s-v)g_1(v,t)\,dv\right],$$

the corresponding decomposition for the term $t \mapsto \int_0^t T_2(t-\omega)f_2(s,\omega)\,d\omega$, $t \geqslant 0$, our assumptions (ii)–(iii) and the argumentation contained in the proofs of [631, Proposition 2.6.11, Proposition 2.6.13; Remark 2.6.14], the required conclusion simply follows. Let us note, finally, that there exist a great number of concrete situations where the above assumptions are really satisfied. Suppose, for example, that $n = 1$, $A = B = [-1]$,

$$f_1(s,t) = \sin s + \cos s + \int_0^t \frac{e^{\xi-t}}{1+\xi^2}\,d\xi, \quad s,\ t \geqslant 0,$$

and

$$f_2(s,t) = \sin s + \frac{1}{1+t^2}, \quad s,\ t \geqslant 0;$$

see also [82, Proposition 1.3.5(d)]. Then the above requirements hold. Similarly, if we replace condition (ii) with the condition:

(ii)′ The mappings $g_i : \mathbb{R}^2 \to \mathbb{C}^n$ are continuous, bounded ($i = 1,2$) and satisfy the requirement that there exist positive real numbers $\omega_1 > 0$ and $\omega_2 > 0$ and complex numbers c_1 and c_2 such that $|c_1| = |c_2| = 1$ and, for every $s,\ t \in \mathbb{R}$, we have $g_1(s+\omega_1,t) = c_1 g_1(s,t)$ and $g_2(s,t+\omega_2) = c_2 g_2(s,t)$,

and accept all remaining assumptions, then we similarly may deduce that there exist a continuous function $u_h(s,t)$ on $[0,\infty)^2$ and a function $u_0 \in C_{0,\mathbb{D}}([0,\infty)^2 : \mathbb{C}^n)$ such that $u(s,t) = u_h(s,t) + u_0(s,t)$ for all $(s,t) \in [0,\infty)^2$, as well as that, for every $s,\ t \geqslant 0$, we have $u_h(s+\omega_1,t) = c_1 u_h(s,t)$ and $u_h(s,t+\omega_2) = c_2 u_h(s,t)$ (see also Section 7.2 for more details).

7. Concerning the big quantity of applications and techniques in the current literature which are devoted to the study of bi-almost periodic functions and bi-almost automorphic functions, we would like to note first that Z. Hu and Z. Jin [542] have analyzed almost automorphic mild solutions to the following nonautonomous evolution equation:

$$\frac{d}{dt}\big[u(t)+f(t,u(t))\big] = A(t)\big[u(t)+f(t,u(t))\big] + g(t,u(t)), \quad t \in \mathbb{R}, \tag{6.39}$$

and its generalization

$$\frac{d}{dt}[u(t)+f(t,Bu(t))]=A(t)[u(t)+f(t,Bu(t))]+g(t,Cu(t)),\quad t\in\mathbb{R}, \tag{6.40}$$

where the domains of operators $A(t)$ are not necessarily densely defined and satisfy the well-known Acquistapace–Terreni conditions, the functions $f,\ g:\mathbb{R}\times X\to X$ are almost automorphic in the first argument and Lipschitzian in the second argument as well as B and C are bounded linear operators on X. We would like to note that the statements of [542, Lemma 17, Theorem 18], concerning the existence and uniqueness of almost automorphic solutions of the problem (6.39), can be straightforwardly reformulated for almost periodicity by replacing conditions (H4) and (H5) with the corresponding almost periodic type conditions and assuming that the function $\Gamma(t,s)$ from the condition (H3) of this paper is $(\mathrm{R},\mathcal{B})$-almost periodic with R being the collection of all sequences in $\mathbb{A}:=\{(a,a):a\in\mathbb{R}\}$ and $X\in\mathcal{B}$. Similarly, the statements of [542, Lemma 20, Theorem 21], concerning the existence and uniqueness of almost automorphic solutions of the problem (6.40), can be straightforwardly reformulated for almost periodicity; see also [1034, Theorem 26, Theorem 27], where the same comment can be given, and the recent result of J. Cao, Z. Huang and G. M. N'Guérékata [240, Theorem 3.6], where a similar modification of condition (H3) for bi-almost periodicity on bounded subsets can be made.

We also stimulate the interested reader to reformulate the recent results of A. Chávez, M. Pinto and U. Zavaleta established in the third section and the fourth section of the paper [268], and the recent results of Y.-K. Chang, S. Zheng [261, Theorem 4.4] and Z. Xia, D. Wang [1037, Theorem 3.1, Theorem 3.2] for almost periodicity. It seems very plausible that all these results can be reformulated for almost periodicity by replacing the notion of bi-almost automorphicity (on bounded subsets) in their formulations and proofs with the notion of bi-almost periodicity (on bounded subsets).

6.1.8 Application to nonautonomous retarded functional evolution equations

In this subsection, we study the asymptotic behavior of bounded solutions to the following classes of time-delay function evolution equations:

$$u'(t)=A(t)u(t)+f(t,u(t-r))\quad \text{for } t\in\mathbb{R},$$

where $r>0$ is the constant time delay, $(A(t),D(A(t)))$, $t\in\mathbb{R}$ is a family of linear closed operators defined on a Banach space X. The nonlinear term $f:\mathbb{R}\times X\to X$ is assumed to be bounded and continuous with respect to t and satisfying suitable conditions with respect to the second variable. Our aim here is to prove the existence and uniqueness of almost periodic solutions to Eq. (2.17).

Let $(A(t), D(A(t))), t \in \mathbb{R}$ be the generators of a strongly continuous evolution family, i. e., $(U(t,s))_{t\geqslant s} \subseteq L(X)$ such that for $t \geqslant s$ the map $(t,s) \mapsto U(t,s)$ is strongly continuous, $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = I$ for $t \geqslant s \geqslant r$ such that the following linear Cauchy problem:

$$\begin{cases} u'(t) = A(t)u(t), & t \geqslant s,\ t,\ s \in \mathbb{R}, \\ u(s) = x \in X, \end{cases} \tag{6.41}$$

has a unique solution (at least in a mild sense) given by $u(t) := U(t,s)x$. For more details, we refer to [15, 737] and the references therein.

Let R be the collection of sequences defined in [L1]. Let us define the mapping $F : \mathbb{R}^2 \times X \to X$ by

$$F(t,s;x) := U(t,s)f(s,x), \quad t,\ s \in \mathbb{R},\ x \in X.$$

Hypotheses

Here, we list our main hypotheses:

(H1) There exists $x_0 \in X$ such that

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \|F(t,s;x_0)\|\, ds < \infty.$$

(H2) There exists a bounded function $L : \mathbb{R}^2 \to (0,\infty)$ satisfying $\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} L(t,s)\, ds < \infty$ and

$$\|F(t,s;x) - F(t,s;y)\| \leqslant L(t,s)\|x-y\|, \quad x,\ y \in X,\ t,\ s \in \mathbb{R}.$$

(H3) The mapping $(t,s;x) \in \mathbb{R}\times\mathbb{R}\times X \mapsto F(t,s;x)$ is $(\mathrm{R}, \mathcal{B})$-almost periodic (in the sense of [L1] above) and bounded on bounded subsets of X.

Hence, a mild solution of Eq. (2.17) is a continuous function $u : \mathbb{R} \to X$ such that

$$u(t) = \int_{-\infty}^{t} F\big(t,s,u(s-r)\big)\, ds, \quad t \in \mathbb{R}; \tag{6.42}$$

see [268] for more details. Notice that, in view of (H1)–(H2), the integral formula (6.42) is well defined. We have the following.

Proposition 6.1.58. *Assume that* (H1)-(H3) *are satisfied. Then the mapping* $\Gamma : \mathrm{AP}(\mathbb{R} : X) \to C_b(\mathbb{R} : X)$, *given by*

$$(\Gamma u)(t) := \int_{-\infty}^{t} F\big(t,s;u(s-r)\big)\, ds, \quad t \in \mathbb{R}, \tag{6.43}$$

is well defined and maps $\mathrm{AP}(\mathbb{R} : X)$ *into itself.*

Proof. Let $u \in AP(\mathbb{R} : X)$. Firstly, we check that the mapping $\Gamma(\cdot)$ is well defined. In fact, from (H1) and (H2), we have

$$\begin{aligned}\|(\Gamma u)(t)\| &\leqslant \int_{-\infty}^{t} \|F(t,s;u(s-r))\|\,ds \\ &\leqslant \int_{-\infty}^{t} \|F(t,s;x_0)\|\,ds + \int_{-\infty}^{t} L(t,s)\|u(s-r)-x_0\|\,ds \\ &\leqslant \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \|F(t,s;x_0)\|\,ds + (\|u\|_\infty + \|x_0\|)\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} L(t,s)\,ds.\end{aligned}$$

Let $(b_n, b_n)_n \subseteq R$ be defined as in [L1], where $(b_n)_n \subseteq \mathbb{R}$ is any scalar sequence. Since $u \in AP(\mathbb{R} : X)$, there exist a subsequence $(a_n)_n \subseteq (b_n)_n$ and a function $u^*(\cdot)$ such that

$$\lim_{n\to\infty} u(t+a_n) = u^*(t) \quad \text{uniformly in } t \in \mathbb{R}. \tag{6.44}$$

Moreover, by (H3), for (relatively compact) bounded subset $B = R(u)$ of X, there exists a function $F^*(\cdot,\cdot;\cdot)$ such that

$$\lim_{n\to\infty} F(t+a_n, s+a_n; x) = F^*(t,s;x) \quad \text{uniformly in } t,\ s \in \mathbb{R},\ x \in B. \tag{6.45}$$

It is clear that condition (H2) implies

$$\|F^*(t,s;x) - F^*(t,s;y)\| \leqslant L(t,s)\|x-y\|, \quad x,\ y \in R(u),\ t,\ s \in \mathbb{R}.$$

Define the mapping

$$(\Gamma u)^*(t) := \int_{-\infty}^{t} F^*(t,s;u^*(s-r))\,ds, \quad t \in \mathbb{R}.$$

The function $(t,s) \mapsto F(t,s;u(s-r))$, $(t,s) \in \mathbb{R}^2$ is $(\mathrm{R}, \mathcal{B})$-almost periodic since

$$\lim_{n\to\infty} F(t+a_n, s+a_n; u(s+a_n-r)) = F^*(t,s;u^*(s-r)) \quad \text{uniformly in } t,\ s \in \mathbb{R};$$

this simply follows from the estimates

$$\begin{aligned}&\|F(t+a_n, s+a_n; u(s+a_n-r)) - F^*(t,s;u^*(s-r))\| \\ &\quad\leqslant \|F(t+a_n, s+a_n; u(s+a_n-r)) - F^*(t,s;u(s+a_n-r))\| \\ &\qquad + \|F^*(t,s;u(s+a_n-r)) - F^*(t,s;u^*(s-r))\| \\ &\quad\leqslant \|F(t+a_n, s+a_n; u(s+a_n-r)) - F^*(t,s;u(s+a_n-r))\| \\ &\qquad + L(t,s)\|u(s+a_n-r) - u^*(s-r)\|\end{aligned} \tag{6.46}$$

and the boundedness of the function $L(\cdot,\cdot)$; see also Eqs. (6.44) and (6.45). A straightforward calculation yields

$$\begin{aligned}
&\|(\Gamma u)(t+a_n)-(\Gamma u)^*(t)\| \\
&\quad = \left\| \int_{-\infty}^{t+a_n} F(t+a_n,s;u(s-r))\,ds - \int_{-\infty}^{t} F^*(t,s;u^*(s-r))\,ds \right\| \\
&\quad \leqslant \int_{0}^{+\infty} \|F(t+a_n,t-s+a_n;u(t-s+a_n-r)) - F^*(t,t-s;u^*(t-s-r))\|\,ds.
\end{aligned}$$

Using the dominated convergence theorem, the estimate (6.46) with the second argument s replaced by $t-s$, and the estimate $\sup_{t,s\in\mathbb{R};x\in B}\|F(t,s;x)\|<\infty$, we obtain

$$\|(\Gamma u)(t+a_n)-(\Gamma u)^*(t)\| \to 0 \quad \text{as } n\to\infty, \text{ uniformly in } t\in\mathbb{R}. \qquad \square$$

Theorem 6.1.59. *Suppose that* (H1)-(H3) *hold. Then Eq.* (2.17) *has a unique mild almost periodic solution* $u(\cdot)$*, given by the integral formula* (6.42)*, provided that* $\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}L(t,s)\,ds<1$.

Proof. Consider the mapping $\Gamma : AP(\mathbb{R}:X)\to C_b(\mathbb{R}:X)$ defined by (6.43). By Proposition 6.1.58, we find that $\Gamma(AP(\mathbb{R}:X))\subseteq AP(\mathbb{R}:X)$. Moreover, for $p>1$, we have

$$\begin{aligned}
\|(\Gamma u)(t)-(\Gamma v)(t)\| &\leqslant \int_{-\infty}^{t} \|F(t,s;u(s-r))-F(t,s;v(s-r))\|\,ds \\
&\leqslant \int_{-\infty}^{t} L(t,s)\|u(s-r)-v(s-r)\|\,ds \\
&\leqslant \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} L(t,s)\,ds \cdot \|u-v\|_{\infty}, \quad t\in\mathbb{R}.
\end{aligned}$$

Therefore, by the Banach contraction principle, the mapping $\Gamma(\cdot)$ has a unique fixed point $u\in AP(\mathbb{R}:X)$. This proves the result. $\square$

Now we will provide an illustrative application of obtained results. Consider the following reaction–diffusion model with time-dependent diffusion and finite delay coefficients given by

$$\frac{\partial u(t,x)}{\partial t} = \delta(t)\frac{\partial^2 u(t,x)}{\partial x^2} + \alpha(t)u(t,x) + h(t,u(t-r,x)), \quad t\in\mathbb{R},\ x\in\mathbb{R}, \tag{6.47}$$

where $\delta,\ \alpha:\mathbb{R}\to\mathbb{R}$ are almost periodic functions such that $\alpha(t)\leqslant-\tilde{\omega}<0$ and there exists $\delta_0>0$ such that $\inf_{t\in\mathbb{R}}\delta(t)\geqslant\delta_0$. The nonlinear term $h:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is assumed to be almost periodic with respect to t and L-Lipschitzian with respect to the second variable with $h(t,0)\neq 0$ for all $t\in\mathbb{R}$.

Let $X := L^2(\mathbb{R})$ and let $A := \Delta$ act with its maximal distributional domain. It is well known that $(A, D(A))$ generates a contraction strongly continuous analytic semigroup $(T(t))_{t\geqslant 0}$ on X; hence, $\|T(t)\| \leqslant 1$ for all $t \geqslant 0$. Clearly, the operators

$$A(t) := \delta(t)A + \alpha(t) \quad \text{with } D(A(t)) := D(A),\ t \in \mathbb{R},$$

generate a strongly continuous evolution family given by

$$U(t,s) := e^{\int_s^t \alpha(\tau)\,d\tau} T\left(\int_s^t \delta(\tau)\,d\tau\right), \quad t \geqslant s.$$

Notice that the formula $T(\int_s^t \delta(\tau)\,d\tau)$ for $t \geqslant s$, corresponds to the mild solution for equation (6.47) with α, $f = 0$. This follows by applying the Fourier transform and the explicit representation of the diffusion semigroup; see, e. g., [82]. Set $\omega := \tilde{\omega} + \lambda\delta_0 > 0$.

It is well known that $\sigma(A) = (-\infty, 0]$. Therefore, using the spectral mapping theorem $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$, $t \geqslant 0$, we get

$$\left\| T\left(\int_s^t \delta(\tau)\,d\tau\right)\varphi \right\| \leqslant e^{-\lambda \int_s^t \delta(\tau)\,d\tau} \|\varphi\|, \quad \varphi \in X \text{ for some } \lambda \geqslant 0.$$

Hence,

$$\begin{aligned} \|U(t,s)\varphi\| &\leqslant e^{\int_s^t \alpha(\tau)\,d\tau - \lambda\int_s^t \delta(\tau)\,d\tau} \|\varphi\| \\ &\leqslant e^{-(\tilde{\omega}+\lambda\delta_0)(t-s)} \|\varphi\| = e^{-\omega(t-s)} \|\varphi\|, \quad t \geqslant s,\ \varphi \in X. \end{aligned}$$

Furthermore, we define $f : \mathbb{R} \times X \to X$ through

$$f(t,\varphi)(x) := h(t, \varphi(x)), \quad t,\ x \in \mathbb{R},\ \varphi \in X.$$

It is clear that $f(\cdot,\cdot)$ is $\mathcal{B}$-almost periodic. We also have the following.

Lemma 6.1.60. *Hypotheses* (H1) *and* (H2) *are satisfied with*

$$L(t,s) := Le^{-\omega(t-s)}, \quad t \geqslant s.$$

Proof. Define $F(t,s;\varphi) := U(t,s)f(s,\varphi)$ for all $\varphi \in X$, $t \geqslant s$. Then

$$\begin{aligned} \int_{-\infty}^t \|F(t,s;0)\|\,ds &\leqslant \int_{-\infty}^t \|U(t,s)f(s,0)\|\,ds \\ &\leqslant \int_{-\infty}^t e^{-\omega(t-s)} \|f(s,0)\|\,ds \\ &\leqslant \frac{1}{\omega}\|f(\cdot,0)\|_\infty, \quad t \in \mathbb{R}. \end{aligned}$$

Let $\varphi,\ \psi \in X$. Then the above calculation yields

$$\|F(t,s;\varphi) - F(t,s;\psi)\| \leqslant Le^{-\omega(t-s)}\|\varphi - \psi\|, \quad t \in \mathbb{R}.$$

This proves the result. □

We need to following auxiliary result.

Proposition 6.1.61. *The mapping* $(t,s) \mapsto U(t,s)$ *is* $(\mathrm{R},\mathcal{B})$*-almost periodic. Moreover,* $F(\cdot,\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B})$*-almost periodic.*

Proof. Let $B \subseteq X$ be bounded and $(b_k, b_k)_{k\geqslant 0} \in \mathrm{R}$, where $(b_k)_{k\geqslant 0}$ is any scalar sequence. Since $\delta \in AP(\mathbb{R})$ and $\alpha \in AP(\mathbb{R})$, it follows that there exist a subsequence $(a_k)_{k\geqslant 0} \subseteq (b_k)_{k\geqslant 0}$ and measurable functions $\tilde{\delta}$ and $\tilde{\alpha}$ such that

$$\lim_{k\to+\infty} \delta(t + a_k) = \tilde{\delta}(t) \quad \text{uniformly in } t \in \mathbb{R}$$

and

$$\lim_{k\to+\infty} \alpha(t + a_k) = \tilde{\alpha}(t) \quad \text{uniformly in } t \in \mathbb{R}.$$

If $\varphi \in B$, we define

$$\tilde{U}(t,s)\varphi := e^{\int_s^t \tilde{\alpha}(\tau)\,d\tau} T\Bigg(\int_s^t \tilde{\delta}(\tau)\,d\tau\Bigg)\varphi \quad \text{for all } t \geqslant s.$$

Thus, by the semigroup property of $(T(t))_{t\geqslant 0}$, we have

$$\begin{aligned}
&\big\|U(t + a_k, s + a_k)\varphi - \tilde{U}(t,s)\varphi\big\| \\
&= \Bigg\|e^{\int_{s+a_k}^{t+a_k} \alpha(\tau)\,d\tau} T\Bigg(\int_{s+a_k}^{t+a_k} \delta(\tau)\,d\tau\Bigg)\varphi - e^{\int_s^t \tilde{\alpha}(\tau)\,d\tau} T\Bigg(\int_s^t \tilde{\delta}(\tau)\,d\tau\Bigg)\varphi\Bigg\| \\
&= \Bigg\|e^{\int_s^t \alpha(\tau+a_k)\,d\tau} T\Bigg(\int_s^t \delta(\tau + a_k)\,d\tau\Bigg)\varphi - e^{\int_s^t \tilde{\alpha}(\tau)\,d\tau} T\Bigg(\int_s^t \tilde{\delta}(\tau)\,d\tau\Bigg)\varphi\Bigg\| \\
&\leqslant e^{\int_s^t \alpha(\tau+a_k)\,d\tau}\Bigg\|T\Bigg(\int_s^t \delta(\tau + a_k)\,d\tau\Bigg)\varphi - T\Bigg(\int_s^t \tilde{\delta}(\tau)\,d\tau\Bigg)\varphi\Bigg\| \\
&\quad + \big(e^{\int_s^t \alpha(\tau+a_k)\,d\tau} - e^{\int_s^t \tilde{\alpha}(\tau)\,d\tau}\big)\Bigg\|T\Bigg(\int_s^t \tilde{\delta}(\tau)\,d\tau\Bigg)\varphi\Bigg\|.
\end{aligned}$$

Therefore, by the strong continuity of the semigroup, we obtain

$$e^{\int_s^t \alpha(\tau+a_k)\,d\tau}\Bigg\|T\Bigg(\int_s^t \delta(\tau + a_k)\,d\tau\Bigg)\varphi - T\Bigg(\int_s^t \tilde{\delta}(\tau)\,d\tau\Bigg)\varphi\Bigg\| \to 0 \quad \text{as } k \to \infty,$$

uniformly in t, s. Furthermore, we find that

$$\begin{aligned}
&(e^{\int_s^t \alpha(\tau+a_k)\,d\tau} - e^{\int_s^t \tilde{\alpha}(\tau)\,d\tau})\left\|T\left(\int_s^t \tilde{\delta}(\tau)\,d\tau\right)\varphi\right\| \\
&\leqslant (e^{\int_s^t \alpha(\tau+a_k)\,d\tau} - e^{\int_s^t \tilde{\alpha}(\tau)\,d\tau})\|\varphi\| \\
&\to 0 \text{ as } k \to \infty, \quad \text{uniformly in } t,\ s.
\end{aligned}$$

So, from Proposition 6.1.61, we see that $F(\cdot,\cdot;\cdot)$ satisfies (H3). □

Hence, the following result can be deduced by applying Theorem 6.1.59.

Theorem 6.1.62. *Assume that $L < \omega$. Then the problem* (6.47) *admits a unique almost periodic solution.*

6.2 Stepanov multi-dimensional almost periodic type functions

This section investigates the generalized Stepanov multi-dimensional almost periodic type functions in Lebesgue spaces with variable exponents. With the exception of a recent paper [951] by T. Spindeler, in which the author has analyzed the Stepanov and Weyl almost periodic functions in locally compact Abelian groups, the introduced classes of functions seem to be not considered elsewhere even in the constant coefficient case (concerning Besicovitch almost periodic functions on $\mathbb{R}^n$ and general topological groups, the reader may consult the important research monograph [824] by A. A. Pankov). In our analysis of Stepanov $p(\mathbf{u})$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic functions, we assume that Ω is a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure and $p \in \mathcal{P}(\Omega)$.

The organization of this section is briefly described as follows. Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfy $\Lambda + \Omega \subseteq \Lambda$ and let $p : \Omega \in [1,\infty]$ belongs to the space $\mathcal{P}(\Omega)$, introduced in Subsection 1.1.1. At the beginning of Section 6.2, we introduce the notions of multi-dimensional Bochner transform $\hat{F}_\Omega : \Lambda \times X \to Y^\Omega$. After that, in Subsection 6.2.1, we analyze the notions of Stepanov $(\Omega, p(\mathbf{u}))$-boundedness, Stepanov distance $D^{p(\cdot)}_{S_\Omega}(F,G)$ and Stepanov norm $\|F\|_{S^{p(\mathbf{u})}_\Omega}$ for functions $F : \Lambda \times X \to Y$ and $G : \Lambda \to Y$.

At the beginning of Subsection 6.2.2, we introduce the notion of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodicity and the notion of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodicity (see Definition 6.2.4 and Definition 6.2.5, respectively). Our first structural result in connection with the introduced notion is Proposition 6.2.6, in which we analyze the Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodicity for a given tuple $(F_1,\ldots,F_k)(\cdot;\cdot)$ of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost periodic functions. After that, in Definition 6.2.7, we introduce the notions of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B},\Lambda')$-almost periodicity and Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B},\Lambda')$-uniform recurrence in a Bohr like manner. It is well known that, for every almost periodic function $F : \mathbb{R} \to \mathbb{R}$ which can be analytically extended to a strip around the real axis, its composition with the signum function is always Stepanov p-almost periodic for any finite number $p \geqslant 1$; in Example 6.2.9, we transfer and extend this statement to multi-dimensional almost periodic functions.

The most important results about the introduced classes of functions are given in Proposition 6.2.10–Proposition 6.2.12 and Theorem 6.2.13. Our first essential contributions are Theorem 6.2.14 and Theorem 6.2.15, in which we prove the uniqueness theorem for Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions and an extension type theorem for Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions. In Remark 6.2.16, we reconsider the obtained results for convex polyhedrals in $\mathbb{R}^n$. The main aim of Proposition 6.1.19 is to reconsider the issue analyzed in Proposition 6.2.6 for Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic functions. The pointwise products of Stepanov multi-dimensional almost periodic functions with Stepanov multi-dimensional scalar-valued almost periodic functions are investigated in Propositon 6.2.18 and Proposition 6.2.19. Some other results concerning Stepanov multi-dimensional almost periodic type functions are given in Theorem 6.2.21, Proposition 6.2.22, Proposition 6.2.23 and Proposition 6.2.24. Asymptotically Stepanov multi-dimensional almost periodic functions are investigated in Subsection 6.4, composition theorems for Stepanov multi-dimensional almost periodic functions in Lebesgue spaces with variable exponents are investigated in Subsection 6.1.5; we also analyze the invariance of Stepanov multi-dimensional almost periodicity under the actions of convolution products in Subsection 6.2.5. The final subsection is reserved for giving some applications of our theoretical results to the abstract Volterra integro-differential equations in Banach spaces.

In our investigations of generalized multi-dimensional almost periodicity, Λ denotes a general non-empty subset of $\mathbb{R}^n$ satisfying $\Lambda + \Omega \subseteq \Lambda$ (for the usual almost periodicity, this region has been denoted by I). We introduce the multi-dimensional Bochner transform $\hat{F}_\Omega : \Lambda \times X \to Y^\Omega$ by

$$[\hat{F}_\Omega(\mathbf{t}; x)](u) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in X.$$

6.2.1 Stepanov $(\Omega, p(\mathrm{u}))$-boundedness, Stepanov distance $D^{p(\cdot)}_{S_\Omega}(F, G)$ and Stepanov norm $\|F\|_{S^{p(\mathrm{u})}_\Omega}$

We introduce the notion of Stepanov $(\Omega, p(\mathbf{u}))$-boundedness on $\mathcal{B}$ as follows.

Definition 6.2.1. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$ and $F : \Lambda \times X \to Y$ satisfies the requirement that for each $\mathbf{t} \in \Lambda$ and $x \in X$, the function $F(\mathbf{t} + \mathbf{u}; x)$ belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$. Then we say that $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded on $\mathcal{B}$ if and only if for each $B \in \mathcal{B}$ we have

$$\sup_{\mathbf{t} \in \Lambda; x \in B} \big\| [\hat{F}_\Omega(\mathbf{t}; x)](u) \big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = \sup_{\mathbf{t} \in \Lambda; x \in B} \big\| F(\mathbf{t} + \mathbf{u}; x) \big\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty.$$

Denote by $L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ the set consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-bounded functions on $\mathcal{B}$.

If $n = 1$, $X = \{0\}$, $\Omega = [0,1]$ and $\Lambda = [0,\infty)$ or $\Lambda = \mathbb{R}$, then the notion introduced above reduces to the notion introduced recently in [372, Definition 4.1]. If $X = \{0\}$, then we abbreviate $L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ to $L_{S}^{\Omega,p(\mathbf{u})}(\Lambda : Y)$; in this case, we say that the function $F(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded and define $\|F\|_{S^{\Omega,p(\mathbf{u})}} := \sup_{\mathbf{t}\in\Lambda} \|F(\mathbf{t}+\mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega:Y)}$.

Remark 6.2.2.

(i) Condition $\Lambda+\Omega \subseteq \Lambda$ used henceforth is clearly equivalent with condition $\Lambda+\Omega = \Lambda$ if $0 \in \Omega$.

(ii) Suppose that Ω_1 is also a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure, $\Lambda + \Lambda \subseteq \Lambda$, $\Lambda + \Omega_1 \subseteq \Lambda$ and $1 \leqslant p < \infty$. It is clear that the existence of a finite subset $\{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ of Λ such that $\Omega \subseteq \bigcup_{i=1}^{k}(\mathbf{t}_i + \Omega_1)$ implies that for each $\mathbf{t} \in \Lambda$ we have $\mathbf{t} + \Omega \subseteq \bigcup_{i=1}^{k}(\mathbf{t} + \mathbf{t}_i + \Omega_1)$, so that the Stepanov $(\Omega_1, p(\mathbf{u}))$-boundedness on $\mathcal{B}$ implies the Stepanov $(\Omega, p(\mathbf{u}))$-boundedness on $\mathcal{B}$, for any function $F : \Lambda \times X \to Y$.

(iii) Let $1 \leqslant p < \infty$. In the one-dimensional case, the usual Stepanov p-boundedness of the function $F : \Lambda \to Y$, where $\Lambda = [0,\infty)$ or $\Lambda = \mathbb{R}$, is equivalent with the Stepanov (Ω, p)-boundedness of the function $F(\cdot)$, where $\Omega = [a,b]$ is any nontrivial segment in Λ.

In the general case, it is very simple to show that:

1. $\alpha F + \beta G \in L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$, provided $\alpha,\ \beta \in \mathbb{C}$ and $F,\ G \in L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$.
2. Suppose that $\tau + \Lambda \subseteq \Lambda$, $x_0 \in X$ and for each $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}$ such that $x_0 + B \subseteq B'$. Then we have $F(\cdot + \tau; \cdot + x_0) \in L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$, provided that $F(\cdot;\cdot) \in L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$.
3. If $1 \leqslant p_1(\mathbf{u}) \leqslant p(\mathbf{u})$ for a. e. $\mathbf{u} \in \Omega$ and $f \in L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$, then we have $f \in L_{S,\mathcal{B}}^{\Omega,p_1(\mathbf{u})}(\Lambda \times X : Y)$.
4. $(L_{S}^{\Omega,p(\mathbf{u})}(\Lambda : Y), \|\cdot\|_{S^{\Omega,p(\mathbf{u})}})$ is a complex Banach space.

The translation invariance stated in the point [2.] does not generally hold in the approach proposed by T. Diagana and M. Zitane in [375], as already mentioned.

Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfy $\Lambda+\Omega \subseteq \Lambda$. Suppose first that $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $F : \Lambda \to Y$ and $G : \Lambda \to Y$ are two functions for which $\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\|_Y \in L^p(\Omega)$ for all $\mathbf{t} \in \Lambda$. We define the Stepanov distance $D_{S_\Omega}^{p}(F,G)$ of functions $F(\cdot)$ and $G(\cdot)$ by

$$D_{S_\Omega}^{p}(F,G) := \sup_{\mathbf{t}\in\Lambda}\left[\left(\frac{1}{m(\Omega)}\right)^{1/p}\left\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\right\|_{L^p(\Omega:Y)}\right].$$

Suppose now that $p,\ q \in \mathcal{P}(\Omega)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$ for a. e. $\mathbf{u} \in \Omega$ and $q(\mathbf{u}) < +\infty$ for a. e. $\mathbf{u} \in \Omega$. In this case (the definition is consistent with the above given provided that $p(\mathbf{u}) \equiv p \in (1,\infty)$), we define the Stepanov distance $D_{S_\Omega}^{p(\cdot)}(F,G)$ of functions $F(\cdot)$ and

$G(\cdot)$ by

$$D_{S_\Omega}^{p(\cdot)}(F,G) := \sup_{\mathbf{t}\in\Lambda}\bigl[m(\Omega)^{-1}\|1\|_{L^{q(\mathbf{u})}(\Omega)}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^{p(\mathbf{u})}(\Omega:Y)}\bigr].$$

The use of the Hölder inequality (see Lemma 1.1.7(i)) enables one to see that the following proposition holds good.

Proposition 6.2.3. *Suppose that* $1 \leqslant p_1(\mathbf{u}) \leqslant p_2(\mathbf{u})$ *for a. e.* $\mathbf{u} \in \Omega$*, and* $\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\|_Y \in L^{p_2(\mathbf{u})}(\Omega)$ *for all* $\mathbf{t} \in \Lambda$*. Then*

$$D_{S_\Omega}^{p_1(\cdot)}(F,G) \leqslant 4D_{S_\Omega}^{p_2(\cdot)}(F,G).$$

Proof. It is clear that $\|F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u})\|_Y \in L^{p_1(\mathbf{u})}(\Omega)$ for all $\mathbf{t} \in \Lambda$. If $p_1(\mathbf{u}) = 1$ for a. e. $\mathbf{u} \in \Omega$, then we can apply the Hölder inequality once to conclude that $D_{S_\Omega}^{1}(F,G) \leqslant 2D_{S_\Omega}^{p_2(\cdot)}(F,G)$. Otherwise, if $1/p_i(\mathbf{u}) + 1/q_i(\mathbf{u}) = 1$ for a. e. $\mathbf{u} \in \Omega$ $(i = 1,2)$, then $q_2(\mathbf{u}) \leqslant q_1(\mathbf{u}) < +\infty$ for a. e. $\mathbf{u} \in \Omega$. Applying the Hölder inequality twice, we get for each $\mathbf{t} \in \Lambda$

$$\begin{aligned}
&\|1\|_{L^{q_1(\mathbf{u})}(\Omega)}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^{p_1(\mathbf{u})}(\Omega:Y)}\\
&\quad\leqslant 2\|1\|_{L^{q_1(\mathbf{u})}(\Omega)}\|1\|_{L^{(q_1(\mathbf{u})-q_2(\mathbf{u}))^{-1}}(\Omega)}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^{p_2(\mathbf{u})}(\Omega:Y)}\\
&\quad\leqslant 4\|1\|_{L^{q_2(\mathbf{u})}(\Omega)}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^{p_2(\mathbf{u})}(\Omega:Y)}.
\end{aligned}$$

This simply completes the proof. □

Clearly, if $1 \leqslant p_1(\mathbf{u}) \equiv p_1 \leqslant p_2 \equiv p_2(\mathbf{u})$ for a. e. $\mathbf{u} \in \Omega$, then we have $D_{S_\Omega}^{p_1}(F,G) \leqslant D_{S_\Omega}^{p_2}(F,G)$. If $\Omega \equiv [0,l]^n$ for some $l > 0$, then we also write $D_{S_l}^{p}(F,G) \equiv D_{S_\Omega}^{p}(F,G)$ and $D_{S_l}^{p(\cdot)}(F,G) \equiv D_{S_\Omega}^{p(\cdot)}(F,G)$.

Suppose now that $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $l_2 > l_1 > 0$. Since, for every $\mathbf{t} \in \Lambda$, we have

$$\begin{aligned}
&\left(\frac{1}{m([0,l_1]^n)}\right)^{1/p}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^p(l_1\Omega:Y)}\\
&\quad\leqslant \left(\frac{m([0,l_2]^n)}{m([0,l_1]^n)}\right)^{1/p}\left(\frac{1}{m([0,l_2]^n)}\right)^{1/p}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^p(l_2\Omega:Y)},
\end{aligned}$$

it follows that

$$D_{S_{l_1}}^{p}(F,G) \leqslant \left[\frac{l_2}{l_1}\right]^{n/p}\cdot D_{S_{l_2}}^{p}(F,G).$$

Suppose now that $l_2 = kl_1 + \theta l_1$ for some $k \in \mathbb{N}$ and $\theta \in [0,1)$. Since, for every $\mathbf{t} \in \Lambda$, we have

$$\begin{aligned}
&\left(\frac{1}{m([0,l_2]^n)}\right)^{1/p}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^p([0,l_2]^n:Y)}\\
&\quad\leqslant \left(\frac{1}{m([0,kl_1]^n)}\right)^{1/p}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^p([0,(k+1)l_1]^n:Y)}
\end{aligned}$$

$$\leqslant \left(\frac{(k+1)^n m([0,l_1]^n)}{m([0,kl_1]^n)} \right)^{1/p} \sup_{\mathbf{t}\in\Lambda} \left(\frac{1}{m([0,l_1]^n)} \right)^{1/p} \| F(\mathbf{t}+\mathbf{u}) - G(\mathbf{t}+\mathbf{u}) \|_{L^p([0,l_1]^n:Y)}$$
$$\leqslant \left(\frac{k+1}{k} \right)^{n/p} \cdot D^p_{S_{l_1}}(F,G), \tag{6.48}$$

it follows that

$$D^p_{S_{l_2}}(F,G) \leqslant \left(\frac{k+1}{k} \right)^{n/p} \cdot D^p_{S_{l_1}}(F,G).$$

Therefore, if $p(\mathbf{t}) \equiv p \in [1,\infty)$, the metrics $D^p_{S_{l_1}}(\cdot,\cdot)$ and $D^p_{S_{l_2}}(\cdot,\cdot)$ are topologically equivalent. Furthermore, the use of (6.48) enables one to see that

$$\limsup_{l\to\infty} D^p_{S_l}(F,G) \leqslant D^p_{S_{l_1}} p(F,G), \quad l_1 > 0,$$

provided that $p(\mathbf{t}) \equiv p \in [1,\infty)$. Performing the limit inferior as $l_1 \to \infty$, we get

$$\limsup_{l\to\infty} D^p_{S_l}(F,G) \leqslant \liminf_{l\to\infty} D^p_{S_l}(F,G),$$

so that the limit

$$D^p_W(F,G) := \lim_{l\to\infty} D^p_{S_l}(F,G)$$

exists. Therefore, we can define the Weyl distance $D^p_W(F,G)$ of functions $F(\cdot)$ and $G(\cdot)$; see also Subsection 6.3.1 for a slight generalization.

By $S^p_\Omega(\Lambda : Y)$ we denote the vector space of all functions $F : \Lambda \to Y$ for which $\|F(\mathbf{t}+\mathbf{u})\|_Y \in L^p(\Omega)$ for all $\mathbf{t} \in \Lambda$ and the Stepanov norm

$$\|F\|_{S^p_\Omega} := \sup_{\mathbf{t}\in\Lambda} \left[\left(\frac{1}{m(\Omega)} \right)^{1/p} \| F(\mathbf{t}+\mathbf{u}) \|_{L^p(\Omega:Y)} \right]$$

is finite; if $\Omega \equiv [0,l]^n$, then we also write $S^p_l(\Lambda : Y) \equiv S^p_\Omega(\Lambda : Y)$ and $\|\cdot\|_{S^p_l} \equiv \|\cdot\|_{S^p_\Omega}$. If $p,\ q \in \mathcal{P}(\Omega)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$ for a. e. $\mathbf{u} \in \Omega$ and $q(\mathbf{u}) < +\infty$ for a. e. $\mathbf{u} \in \Omega$, then (the definition is consistent with the above given provided that $p(\mathbf{u}) \equiv p \in (1,\infty)$), we define the Stepanov norm $\|F\|_{S^{p(\mathbf{u})}_\Omega}$ by

$$\|F\|_{S^{p(\mathbf{u})}_\Omega} := \sup_{\mathbf{t}\in\Lambda} \left[m(\Omega)^{-1} \|1\|_{L^{q(\mathbf{u})}(\Omega)} \| F(\mathbf{t}+\mathbf{u}) \|_{L^{p(\mathbf{u})}(\Omega:Y)} \right];$$

again, $S^{p(\mathbf{u})}_\Omega(\Lambda : Y)$ denotes the vector space consisting of all functions $F : \Lambda \to Y$ satisfying that $\|F(\mathbf{t}+\mathbf{u})\|_Y \in L^{p(\mathbf{u})}(\Omega)$ for all $\mathbf{t} \in \Lambda$ and $\|F\|_{S^{p(\mathbf{u})}_\Omega} < \infty$. Since Fatou's lemma holds in our framework (see, e. g., [377, p. 75]), using the arguments contained in the proof of [696, Theorem 5.2.1, p. 199] and Lemma 1.1.7(ii) we may conclude that $S^{p(\mathbf{u})}_\Omega(\Lambda : Y)$ is a Banach space equipped with the norm $\|\cdot\|_{S^{p(\mathbf{u})}_\Omega}$.

6.2.2 Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions and Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic type functions

The notion of a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic function is introduced as follows.

Definition 6.2.4. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$, (6.1) holds with the set I replaced by the set Λ therein and the function $\hat{F} : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, i. e., for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that

$$\lim_{l \to +\infty} \left\| F\left(\mathbf{t} + \mathbf{u} + \left(b_{k_l}^1, \ldots, b_{k_l}^n\right); x\right) - \left[F^*(\mathbf{t}; x)\right](\mathbf{u}) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$. By $\mathrm{APS}^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y)$ we denote the collection consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions $F : \Lambda \times X \to Y$. If $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we also say that the function $F(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-R-multi-almost periodic and abbreviate $\mathrm{APS}^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda \times X : Y)$ to $\mathrm{APS}^{\Omega,p(\mathbf{u})}_{\mathrm{R}}(\Lambda : Y)$.

In the following definition, we introduce the notion of a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic function.

Definition 6.2.5. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$ and $F : \Lambda \times X \to Y$, (6.3) holds with the set I replaced by Λ therein and the function $\hat{F} : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, i. e., for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that

$$\lim_{l \to +\infty} \left\| F\left(\mathbf{t} + \mathbf{u} + \left(b_{k_l}^1, \ldots, b_{k_l}^n\right); x + x_{k_l}\right) - \left[F^*(\mathbf{t}; x)\right](\mathbf{u}) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$. By $\mathrm{APS}^{\Omega,p(\mathbf{u})}_{(\mathrm{R}_X,\mathcal{B})}(\Lambda \times X : Y)$ we denote the space consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic functions.

The following special cases will be very important for us (see also the previous section):

L1. $\mathrm{R} = \{b : \mathbb{N} \to \mathbb{R}^n;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a, a, a, \ldots, a) \in \mathbb{R}^n : a \in \mathbb{R}\}\}$. If $n = 2$ and $\mathcal{B}$ denotes the collection of all bounded subsets of X, then we also say that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bi-almost periodic. The notion of Stepanov $(\Omega, p(\mathbf{u}))$-bi-almost periodicity seems to be new and not considered elsewhere even in the one-dimensional case $\Omega = [0,1]$ with the constant exponent $p(\mathbf{u}) \equiv p \in [1,\infty)$.

L2. R is a collection of all sequences $b(\cdot)$ in $\mathbb{R}^n$. This is the limit case in our analysis because, in this case, any Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic function, is automatically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_1, \mathcal{B})$-multi-almost periodic, resp. Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{1\mathrm{X}}, \mathcal{B})$-multi-almost periodic, for any other collection R_1 of sequences $b(\cdot)$ in $\mathbb{R}^n$, resp. any other collection $\mathrm{R}_{1\mathrm{X}}$ of sequences in $\mathbb{R}^n \times X$.

Let $k \in \mathbb{N}$ and $F_i : \Lambda \times X \to Y_i$ $(1 \leqslant i \leqslant k)$. Let us recall that we define the function $(F_1, \ldots, F_k) : \Lambda \times X \to Y_1 \times \cdots \times Y_k$ by

$$(F_1, \ldots, F_k)(\mathbf{t}; x) := \big(F_1(\mathbf{t}; x), \ldots, F_k(\mathbf{t}; x)\big), \quad \mathbf{t} \in \Lambda,\ x \in X.$$

Almost immediately from definitions, we can clarify the following analogue of Proposition 6.1.4.

Proposition 6.2.6.

(i) *Suppose that* $k \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, (6.1) *holds with* I *replaced by* Λ *therein, and for any sequence which belongs to* R *we find that any its subsequence also belongs to* R. *If the function* $F_i(\cdot; \cdot)$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-*multi-almost periodic for* $1 \leqslant i \leqslant k$, *then the function* $(F_1, \ldots, F_k)(\cdot; \cdot)$ *is also Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-*multi-almost periodic.*

(ii) *Suppose that* $k \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, (6.1) *holds with* I *replaced by* Λ *therein, and for any sequence which belongs to* R_{X} *we find that any its subsequence also belongs to* R_{X}. *If the function* $F_i(\cdot; \cdot)$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-*multi-almost periodic for* $1 \leqslant i \leqslant k$, *then the function* $(F_1, \ldots, F_k)(\cdot; \cdot)$ *is also Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-*multi-almost periodic.*

The supremum formula for Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic functions, the conditions under which the range $\{\hat{F}_{\Omega}(\mathbf{t}; x) : \mathbf{t} \in \Lambda;\ x \in B\}$, for a given set $B \in \mathcal{B}$, is relatively compact in $L^{p(\mathbf{u})}(\Omega : Y)$ and the question when for a given Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic function $F : \Lambda \times X \to Y$ and a function $\phi : Y \to Z$ we find that $\phi \circ F : \Lambda \times X \to Z$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic can be deduced by appealing to [265, Proposition 2.5, Proposition 2.6, Proposition 2.9].

Now we will introduce the following notion in a Bohr like manner.

Definition 6.2.7. Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the function $\hat{F}_{\Omega} : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous.

(i) Then we say that $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic, if $\Lambda' = \Lambda$) if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that

$$\big\|F(\mathbf{t} + \tau + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon, \quad \mathbf{t} \in \Lambda,\ x \in B.$$

By $APS^{\Omega,p(\mathbf{u})}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$ and $APS^{\Omega,p(\mathbf{u})}_{\mathcal{B}}(\Lambda \times X : Y)$ we denote the spaces consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic functions and Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic functions, respectively.

(ii) Then we say that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent (Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent, if $\Lambda' = \Lambda$) if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_n) in Λ' such that $\lim_{n\to+\infty} |\tau_n| = +\infty$ and

$$\lim_{n\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \big\|F(\mathbf{t} + \tau_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0.$$

By $URS^{\Omega,p(\mathbf{u})}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$ and $URS^{\Omega,p(\mathbf{u})}_{\mathcal{B}}(\Lambda \times X : Y)$ we denote the spaces consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent functions and Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent functions, respectively.

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-uniformly recurrent) [Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-uniformly recurrent), if $\Lambda = \Lambda'$].

Remark 6.2.8.

(i) Suppose that $p \in D_+(\Omega)$ and there exists a finite constant $L \geqslant 1$ such that

$$\big\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\big\|_Y \leqslant L\|x - y\|, \quad \mathbf{t} \in \Lambda,\ x,\ y \in X, \tag{6.49}$$

and the mapping $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined. Then it is continuous. Towards this end, let $(\mathbf{t}_n; x_n) \to (\mathbf{t}; x)$ as $n \to +\infty$. Then (6.49) implies that

$$\begin{aligned}
&\big\|F(\mathbf{t}_n + \mathbf{u}; x_n) - F(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad \leqslant \big\|F(\mathbf{t}_n + \mathbf{u}; x_n) - F(\mathbf{t}_n + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} + \big\|F(\mathbf{t}_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\quad \leqslant 2\big(1 + m(\Omega)\big) \cdot \big[L\|x_n - x\|\big]_{L^{p^+}(\Omega)} + \big\|F(\mathbf{t}_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}.
\end{aligned}$$

The first addend clearly goes to zero since $\|x_n - x\| \to 0$ as $n \to +\infty$. For the second addend, we can apply the arguments used for proving the continuity of the translation mapping from the proof of [373, Proposition 5.1].

(ii) Suppose that $F : \Lambda \times X \to Y$ is continuous and $p \in D_+(\Omega)$. Then the continuity of mapping $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ follows directly by applying the dominated convergence theorem (see Lemma 1.1.7(iv)).

Example 6.2.9 (see also Remark 2.4.19). Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Bohr Λ'-almost periodic function (Λ'-uniformly recurrent function). Define $\operatorname{sign}(0) := 0$ and $H : \mathbb{R}^n \to \mathbb{R}$ by

$H(\mathbf{t}) := \operatorname{sign}(F(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$. Then, for every $p \in D_+(\Omega)$, the function $H(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-uniformly recurrent), provided that

$$(\exists L \geqslant 1)\ (\forall \varepsilon > 0)\ \big(\forall y \in \mathbb{R}^n\big)\ m\big(\{x \in y + \Omega : |F(x)| \leqslant \varepsilon\}\big) \leqslant L\varepsilon.$$

Let $\varepsilon > 0$ be fixed. Then the required conclusion follows from the calculation

$$\begin{aligned}
&\big\|H(\mathbf{t} + \tau + \mathbf{u}; x) - H(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:\mathbb{R})} \\
&\quad \leqslant 2\big(1 + m(\Omega)\big) \cdot \big\|H(\mathbf{t} + \tau + \mathbf{u}; x) - H(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p^+}(\Omega:\mathbb{R})} \\
&\quad \leqslant 2\big(1 + m(\Omega)\big) \cdot \|1\|_{L^{p^+}((\mathbf{t}+\Omega)\cap E_\varepsilon^c:\mathbb{R})},
\end{aligned}$$

where E_ε denotes the set consisting of all tuples $y \in \mathbb{R}^n$ such that $|F(y)| \geqslant \varepsilon$ and τ is a (Λ', ε)-period od $F(\cdot)$ (the inequality stated in the last line of computation follows from the fact that for any $y \in E_\varepsilon$ and for any such a number τ we have $H(y + \tau) = H(y)$); see also [696, Theorem 5.3.1] for the first result in this direction. Suppose now that the function $F(\cdot)$ is Bohr almost periodic and there exist real numbers a and b such that $a < 0 < b$ and the function $F(\cdot)$ can be analytically extended to the region $\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \operatorname{Re} z_i \in (a, b) \text{ for all } i \in \mathbb{N}_n\}$ (in particular, this holds for any trigonometric polynomial). Then we can repeat verbatim the argumentation contained in the proof of the last mentioned theorem (see also https://math.stackexchange.com/questions/3216833/holomorphic-function-on-mathbbcn-vanishing-on-a-positive-lebesgue-measure?rq=1) in order to see that $\lim_{\varepsilon\to 0+} m(E_\varepsilon^c \cap (\mathbf{t} + \Omega)) = 0$, uniformly for $\mathbf{t} \in \mathbb{R}^n$, which combined with the above calculation shows that the function $H(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic.

In connection with the above example, it should be noted that the function $H(\cdot)$ need not be Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-uniformly recurrent) for all $p \in \mathcal{P}(\Omega)$, even in the one-dimensional case. Strictly speaking, if $\Omega := [0, 1]$, $\Lambda' := \mathbb{R}$ and $p(x) := -1 - \ln x$, $x \in (0, 1]$, then we know that the function $x \mapsto \operatorname{sign}(\sin x + \sin(\sqrt{2}x))$, $x \in \mathbb{R}$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded but not Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic. Suppose now that $\Omega = [0, 1]^n$ and $p(\mathbf{u}) := 1 - \ln(u_1 \cdot u_2 \cdots u_n)$, $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \Omega$ and $F(x_1, x_2, \ldots, x_n) := \sin(x_1 + x_2 + \cdots + x_n) + \sin(\sqrt{2}(x_1 + x_2 + \cdots + x_n))$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then $H(\cdot)$, defined as above, is essentially bounded and therefore Stepanov $(\Omega, p(\mathbf{u}))$-bounded. On the other hand, using the argumentation from the above-mentioned example, the Fubini theorem and the equality $\ln(u_1 \cdot u_2 \cdots u_n) = \ln u_1 + \ln u_2 + \cdots + \ln u_n$ for all $(u_1, u_2, \ldots, u_n) \in \Omega$, we see that, for every $\lambda \in (0, 2/e)$ and $l > 0$, we can find a ball $B(\mathbf{t_0}, l) \subseteq \mathbb{R}^n$ such that, for every $\tau \in B(\mathbf{t_0}, l)$, there exists $\mathbf{t} \in \mathbb{R}^n$ such that

$$\begin{aligned}
\int_\Omega \bigg(\frac{1}{\lambda}\bigg)^{1-\ln(u_1\cdot u_2\cdots u_n)} &\big|\operatorname{sign}\big[\sin(\mathbf{u} + \mathbf{t} + \tau) + \sin\big(\sqrt{2}(\mathbf{u} + \mathbf{t} + \tau)\big)\big] \\
&- \operatorname{sign}\big[\sin(\mathbf{u} + \mathbf{t}) + \sin\big(\sqrt{2}(\mathbf{u} + \mathbf{t})\big)\big]\big|^{1-\ln(u_1\cdot u_2\cdots u_n)}\, d\mathbf{u} = \infty.
\end{aligned}$$

This simply implies that the function $H(\cdot)$ is not Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic.

Now we will clarify the following result.

Proposition 6.2.10. *Suppose that $h \in L^1(\mathbb{R}^n)$, $p \in D_+(\Omega)$, the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic and for each bounded subset D of X there exists a constant $c_D > 0$ such that $\|F(\mathbf{t}; x)\|_Y \leqslant c_D$ for a. e. $\mathbf{t} \in \mathbb{R}^n$ and all $x \in D$. Suppose, further, that for each sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k) \in \mathrm{R}_X$ and for each set $B \in \mathcal{B}$ we find that $B + \{x_k : k \in \mathbb{N}\}$ is a bounded set in X. Then the function*

$$(h * F)(\mathbf{t}; x) := \int\limits_{\mathbb{R}^n} h(\sigma)F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X,$$

is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic and satisfies the requirement that for each bounded subset D of X there exists a constant $c_D' > 0$ such that $\|(h * F)(\mathbf{t}; x)\|_Y \leqslant c_D'$ for all $\mathbf{t} \in \mathbb{R}^n$, $x \in D$.

Proof. The prescribed assumptions imply that for each bounded subset D of X there exists a constant $c_D' > 0$ such that $\|\hat{F}_\Omega(\mathbf{t}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant c_D$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in D$, as well as that $\|(h * F)(\mathbf{t}; x)\|_Y \leqslant c_D'$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in D$. Applying Lemma 6.1.5, we see that the function $[h * \hat{F}_\Omega](\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic. The result now simply follows from the equality

$$h * \hat{F}_\Omega = h \mathbin{\hat{*}} F_\Omega \tag{6.50}$$

and a corresponding definition of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodicity. □

Using [265, Proposition 2.18] and the corresponding definition, we can immediately deduce the following result which can be also formulated for the (asymptotical) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodicity (see [265, Proposition 2.19, Proposition 2.20]).

Proposition 6.2.11. *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic. If for each $B \in \mathcal{B}$ there exists $\varepsilon_B > 0$ such that*

$$\lim_{j \to +\infty} \sup_{\mathbf{t} \in \Lambda; x \in B'} \big\|F_j(\mathbf{t} + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

where $B' \equiv B^\circ \cup \bigcup_{x \in \partial B} B(x, \varepsilon_B)$, then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic.

The subsequent result is trivial and follows almost immediately from the above definitions.

Proposition 6.2.12. *Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. Then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent) if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is Bohr $(\mathcal{B}, \Lambda')$-almost periodic ($(\mathcal{B}, \Lambda')$-uniformly recurrent).*

Since every Bohr almost periodic function $F : \mathbb{R}^n \to Y$ is immediately Bohr Δ_n-almost periodic, we may deduce from the previous proposition that a Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic function $F : \mathbb{R}^n \to Y$ is immediately Stepanov $(\Omega, p(\mathbf{u}))$-Δ_n-almost periodic. Using Lemma 6.1.18 we can simply deduce the following.

Theorem 6.2.13. *Suppose that $\hat{F}_\Omega : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous, $\mathcal{B}$ is any family of compact subsets of X and R is the collection of all sequences in $\mathbb{R}^n$. Then $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic if and only if $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic.*

Keeping in mind Proposition 6.2.12, the notion of a strong Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodicity can be introduced in the following way: a function $F : \Lambda \times X \to Y$ is said to be strongly Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is strongly almost periodic. We will skip all related details concerning this issue for brevity.

Using Lemma 6.1.38 and Proposition 6.2.12, we can deduce the following result.

Theorem 6.2.14 (The uniqueness theorem for Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions). *Suppose that $\Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda \subseteq \Lambda$ and $\mathbb{R}^n \smallsetminus [(\Lambda \cup (-\Lambda)) + (\Lambda \cup (-\Lambda))]$ is a bounded set. If $F : \mathbb{R}^n \to Y$ and $G : \mathbb{R}^n \to Y$ are two Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions and $F(\mathbf{t}) = G(\mathbf{t})$ for a. e. $\mathbf{t} \in \Lambda$, then $F(\mathbf{t}) = G(\mathbf{t})$ for a. e. $\mathbf{t} \in \mathbb{R}^n$.*

Proof. By Proposition 6.2.12, $\hat{F} : \mathbb{R}^n \to L^{p(\mathbf{u})}(\Omega : Y)$ and $\hat{G} : \mathbb{R}^n \to L^{p(\mathbf{u})}(\Omega : Y)$ are Bohr almost periodic functions. Let $\mathbf{t} \in \Lambda$ be fixed. Then our assumption implies $F(\mathbf{t} + \mathbf{u}) = G(\mathbf{t} + \mathbf{u})$ for a. e. $\mathbf{u} \in \Omega$ so that $\hat{F}(\mathbf{t}) = \hat{G}(\mathbf{t})$. Applying Lemma 6.1.38, we get $\hat{F}(\mathbf{t}) = \hat{G}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$, which simply implies that $F(\mathbf{t}) = G(\mathbf{t})$ for a. e. $\mathbf{t} \in \mathbb{R}^n$. □

Now we will render the following important result about extensions of Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic functions.

Theorem 6.2.15. *Suppose that $\Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, the set Λ' is unbounded, $m(\partial\Lambda) = 0$, $\Omega^\circ \neq \emptyset$, $F : \Lambda \to Y$ satisfies the requirement that $\hat{F}_\Omega : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is a uniformly continuous, Bohr Λ'-almost periodic function, resp. a uniformly continuous, Λ'-uniformly recurrent function, $S \subseteq \mathbb{R}^n$ is bounded and, for every $\mathbf{t}' \in \mathbb{R}^n$, there exists a finite real number $M > 0$ such that $\mathbf{t}' + \Lambda'_M \subseteq \Lambda$. Define $\Lambda_S := [(\Lambda' \cup (-\Lambda')) + (\Lambda' \cup (-\Lambda'))] \cup S$. Then there exists a Stepanov $(\Omega, p(\mathbf{u}))$-Λ_S-almost periodic, resp. a Stepanov $(\Omega, p(\mathbf{u}))$-Λ_S-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for a. e. $\mathbf{t} \in \Lambda$; furthermore, in Stepanov almost periodic case, if $\mathbb{R}^n \smallsetminus \Lambda_S$ is a bounded set and the function $\tilde{G}(\cdot)$ satisfies the same requirements as the function $\tilde{F}(\cdot)$, then there exists a set $N \subseteq \mathbb{R}^n$ such that $m(N) = 0$ and $\tilde{F}(\mathbf{t}) = \tilde{G}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n \smallsetminus N$.*

Proof. We will consider only Stepanov almost periodicity. By Proposition 6.2.12, we find that the function $\hat{F}_\Omega : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is Bohr Λ'-almost periodic. Due to the prescribed assumptions, we can apply Lemma 6.1.37 in order to see that there exists a uniformly continuous Bohr Λ_S-almost periodic function $H : \mathbb{R}^n \to L^{p(\mathbf{u})}(\Omega : Y)$ such that

$\hat{F}_\Omega(\mathbf{t}) = H(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$. Furthermore, by the corresponding proof of Lemma 6.1.37, given in [265], there exists a sequence (τ_k) in Λ' such that $H(\mathbf{t}) = \lim_{k\to+\infty} \hat{F}_\Omega(\mathbf{t} + \tau_k)$, where the limit is uniform in $\mathbf{t} \in \mathbb{R}^n$, and $\lim_{k\to+\infty} |\tau_k| = +\infty$. Now we will prove the following:

($\diamond$) Let $\mathbf{t}_1,\ \mathbf{t}_2 \in \mathbb{R}^n$ be fixed. Then there exists a set $N \subseteq \Omega$ such that $m(N) = 0$ and, for every $\mathbf{u}_1,\ \mathbf{u}_2 \in \Omega \setminus N$, the assumption $\mathbf{t}_1 + \mathbf{u}_1 = \mathbf{t}_2 + \mathbf{u}_2$ implies $[H(\mathbf{t}_1)](\mathbf{u}_1) = [H(\mathbf{t}_2)](\mathbf{u}_2)$.

In actual fact, we find that there exists a set $N_i \subseteq \Omega$ such that $m(N_i) = 0$ and $[H(\mathbf{t}_i)](\mathbf{u}_i) = \lim_{k\to+\infty} F(\mathbf{t}_i + \tau_k + \mathbf{u}_i)$ for $i = 1, 2$, so that ($\diamond$) follows immediately by plugging $N \equiv N_1 \cup N_2$. Define now $\tilde{F} : \mathbb{R}^n \to Y$ by $\tilde{F}(\mathbf{t}) := [H(\mathbf{x}_\mathbf{t})](\mathbf{t} - \mathbf{x}_\mathbf{t})$, if $\mathbf{x}_\mathbf{t} \in \mathbb{Q}^n$ and $\mathbf{t} \in \mathbf{x}_\mathbf{t} + \Omega^\circ$. Using ($\diamond$) and our assumption $\Omega^\circ \neq \emptyset$, it is very simple to prove that the function $\tilde{F}(\cdot)$ is well defined as well as that the Bochner transform of $\tilde{F}(\cdot)$ is $H(\cdot)$, i. e., that for each $\mathbf{t} \in \mathbb{R}^n$ there exists a set $N_\mathbf{t} \subseteq \Omega$ such that $\tilde{F}(\mathbf{t} + \mathbf{u}) = [H(\mathbf{t})](\mathbf{u})$ for all $\mathbf{u} \in \Omega \setminus N_\mathbf{t}$. Applying again Proposition 6.2.12, we see that the function $\tilde{F}(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-Λ_S-almost periodic. Now we will prove that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for a. e. $\mathbf{t} \in \Lambda$. By the foregoing, for every $\mathbf{t} \in \Lambda$, there exists a set $N_\mathbf{t} \subseteq \Omega$ such that $m(N_\mathbf{t}) = 0$ and

$$F(\mathbf{t} + \mathbf{u}) = [H(\mathbf{t})](\mathbf{u}) = \tilde{F}(\mathbf{t} + \mathbf{u}), \quad \mathbf{u} \in \Omega \setminus N_\mathbf{t}. \tag{6.51}$$

Let $\mathbf{x} \in \mathbb{Q}^n$ be fixed. Denote $\Lambda_k := \{\mathbf{t} \in (\mathbf{x} + \Omega) \cap \Lambda : \operatorname{dist}(\mathbf{t}, \partial\Omega) \geqslant 1/k\}$ $(k \in \mathbb{N})$. Then $[(\mathbf{x} + \Omega) \cap \Lambda] \setminus \partial\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$ so that the required statement easily follows from our assumption $m(\partial\Lambda) = 0$ and the fact that for each $k \in \mathbb{N}$ and $\mathbf{t} \in \Lambda_k$ we have $\mathbf{t} \in \Lambda^\circ$ and therefore $\Lambda_k \subseteq \bigcup_{\mathbf{t}\in(x+\Omega)\cap\Lambda}(\mathbf{t} + \Omega^\circ)$; by the Heine–Borel theorem, for every $k \in \mathbb{N}$, this implies the existence of a finite sequence of numbers $\mathbf{t}_1, \ldots, \mathbf{t}_{a_k} \in \Omega^\circ$ such that $\Lambda_k \subseteq \bigcup_{k=1}^{a_k}(\mathbf{t} + \Omega^\circ)$ and we can apply (6.51) to achieve our aims. Finally, if $\mathbb{R}^n \setminus \Lambda_S$ is a bounded set and the function $\tilde{G}(\cdot)$ satisfies the same requirements as the function $\tilde{F}(\cdot)$, then the foregoing arguments simply imply that the Bochner transform of functions $\tilde{F}(\mathbf{t})$ and $\tilde{G}(\mathbf{t})$ are equal for all $\mathbf{t} \in \Lambda$. Moreover, the Bochner transform of functions $\tilde{F}(\cdot)$ and $\tilde{G}(\cdot)$ must be Bohr almost periodic on $\mathbb{R}^n$ and therefore compactly almost automorphic so that the arguments used in [265] show that these functions are equal identically on $\mathbb{R}^n$, which completes the proof in a routine manner. □

Remark 6.2.16.

(i) It is clear that Theorem 6.2.15 is applicable provided that $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis of $\mathbb{R}^n$,

$$\Lambda' = \Lambda = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n : \alpha_i \geqslant 0 \text{ for all } i \in \mathbb{N}_n\}$$

is a convex polyhedral in $\mathbb{R}^n$ and Ω is any compact subset of Λ with non-empty interior; in this case, we find that there exists a unique Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic extension of the function $F : \Lambda \to Y$ to the whole Euclidean space. This enables one to see that Proposition 6.1.19 continues to hold with the set $\mathbb{R}^n$ replaced therein with any convex polyhedral in $\mathbb{R}^n$. It is also worth noting that Theorem 6.2.15 is applicable in the following special case: $\Lambda = [r_1, \infty) \times [r_2, \infty) \times \cdots \times [r_n, \infty)$ for some real numbers $r_i \in \mathbb{R}$ $(1 \leqslant i \leqslant n)$, $\Lambda' = [r_1', \infty) \times [r_2', \infty) \times \cdots \times [r_n', \infty)$ for some non-negative real numbers $r_i, r_i' \geqslant 0$ $(1 \leqslant i \leqslant n)$ and Ω is any compact subset of $[0, \infty)^n$ with non-empty interior, when the function $\tilde{F}(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic.

(ii) It is well known that a compact set with positive Lebesgue measure in $\mathbb{R}^n$, like the famous Smith–Volterra–Cantor set in the one-dimensional case, can have the empty interior.

Combining Proposition 6.2.6 and Theorem 6.1.18, we immediately get the following.

Proposition 6.2.17. *Suppose that $k \in \mathbb{N}$ and $\mathcal{B}$ is any family of compact subsets of X. If the function $F_i : \mathbb{R}^n \times X \to Y_i$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic for $1 \leqslant i \leqslant k$, then the function $(F_1, \ldots, F_k)(\cdot; \cdot)$ is also Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic.*

It is clear that Lemma 6.1.17(i) can be particularly used to profile when, for a given Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function $F : \Lambda \times X \to Y$ and a set $B \in \mathcal{B}$, we have $\sup_{\mathbf{t} \in \Lambda; x \in B} \|F(\mathbf{t} + \mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty$; if for every $x \in X$ we define the function $F_x : \Lambda \to Y$ by $F_x(\mathbf{t}) := F(\mathbf{t}; x)$, $\mathbf{t} \in \Lambda$, then the above means $\sup_{x \in B} \|F_x\|_{S_\Omega^{p(\cdot)}} < \infty$ for each fixed set $B \in \mathcal{B}$. Furthermore, Lemma 6.1.17(ii) can be used to describe when, for a given Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function $F : \Lambda \times X \to Y$, we find that for each $B \in \mathcal{B}$ the function $\hat{F}_\Omega(\cdot; \cdot)$ is uniformly continuous on $\Lambda \times B$.

Now we will prove the following extension of [696, Theorem 5.2.5] concerning pointwise products of multi-dimensional Stepanov $p(\mathbf{u})$-almost periodic type functions with scalar-valued Stepanov $r(\mathbf{u})$-almost periodic functions (for simplicity, we consider here case $\Lambda = \mathbb{R}^n$, only, albeit we can formulate a corresponding result in case that Λ is admissible with respect to the almost periodic extensions).

Proposition 6.2.18. *Suppose that $p, q, r \in \mathcal{P}(\Omega)$, $1/p(\mathbf{u}) + 1/r(\mathbf{u}) = 1/q(\mathbf{u})$, $f : \mathbb{R}^n \to \mathbb{C}$ is a Stepanov $(\Omega, r(\mathbf{u}))$-almost periodic function and $F : \mathbb{R}^n \times X \to Y$ is a Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function, where $\mathcal{B}$ denotes any family of compact subsets of X. Define $F_1(\mathbf{t}; x) := f(\mathbf{t})F(\mathbf{t}; x)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$. Then the function $F_1(\cdot; \cdot)$ is Stepanov $(\Omega, q(\mathbf{u}))$-$\mathcal{B}$-almost periodic.*

Proof. Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be given. We have

$$\begin{aligned} &\hat{F}_{1\Omega}(\mathbf{t}'; x') - \hat{F}_{1\Omega}(\mathbf{t}; x) \\ &\quad = \hat{f}_\Omega(\mathbf{t}') \cdot [\hat{F}_\Omega(\mathbf{t}'; x') - \hat{F}_\Omega(\mathbf{t}; x)] + [\hat{f}_\Omega(\mathbf{t}') - \hat{f}_\Omega(\mathbf{t})] \cdot \hat{F}_\Omega(\mathbf{t}; x) \end{aligned}$$

for every $\mathbf{t}, \mathbf{t}' \in \mathbb{R}^n$ and $x, x' \in X$. Since the mapping $\hat{f}_\Omega(\cdot)$ is uniformly continuous and bounded on $\mathbb{R}^n$ as well as the mapping $\hat{F}_\Omega(\cdot; \cdot)$ is continuous, we can apply the above equality and the Hölder inequality (see Lemma 1.1.7(i)) in order to see that the mapping $\hat{F}_{1\Omega}(\cdot;\cdot)$ is continuous, as well. Due to Proposition 6.2.17, there exists $l > 0$ such that for every $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l)$ such that $\|F(\mathbf{t}+\tau+\mathbf{u};x) - F(\mathbf{t}+\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon$, $\mathbf{t} \in \mathbb{R}^n$, $x \in B$ and $\|f(\mathbf{t}+\tau+\mathbf{u}) - f(\mathbf{t}+\mathbf{u})\|_{L^{r(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon$, $\mathbf{t} \in \mathbb{R}^n$. Since

$$\begin{aligned} &F_1(\mathbf{t}+\tau+\mathbf{u};x) - F_1(\mathbf{t}+\tau;x) \\ &\quad = \hat{f}_\Omega(\mathbf{t}+\tau+\mathbf{u}) \cdot \big[F(\mathbf{t}+\tau+\mathbf{u};x) - F(\mathbf{t}+\tau;x)\big] \\ &\qquad + \big[f(\mathbf{t}+\tau+\mathbf{u}) - f(\mathbf{t}+\tau)\big] \cdot F(\mathbf{t}+\mathbf{u};x) \end{aligned}$$

for every $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{u} \in \Omega$ and $x \in B$, we can apply the Hölder inequality again, along with the estimates $\sup_{\mathbf{t}\in\mathbb{R}^n} \|\hat{f}_\Omega(\mathbf{t})\|_{L^{r(\mathbf{u})}(\Omega)} < \infty$ and $\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \|\hat{F}_\Omega(\mathbf{t};x)\|_{L^{p(\mathbf{u})}(\Omega)} < \infty$, to complete the whole proof. □

We can similarly prove the following.

Proposition 6.2.19. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $f : \Lambda \to \mathbb{C}$ is Stepanov $(\Omega, r(\mathbf{u}))$-bounded and Stepanov $(\Omega, r(\mathbf{u}))$-R-multi-almost periodic and $F : \Lambda \times X \to Y$ is a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic function satisfying that $\sup_{\mathbf{t}\in\Lambda; x\in B} \|\hat{F}_\Omega(\mathbf{t}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty$. Define $F_1(\mathbf{t};x) := f(\mathbf{t})F(\mathbf{t};x)$, $\mathbf{t} \in \Lambda$, $x \in X$. Then $F_1(\cdot;\cdot)$ is Stepanov $(\Omega, q(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic, provided that for each sequence $(\mathbf{b}_k)$ in R we find that any its subsequence also belongs to R.*

Now we would like to present the following example.

Example 6.2.20. Suppose that $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and $\alpha\beta^{-1}$ is an irrational number. As we know, the functions $f_{\alpha,\beta}(\cdot)$ and $g_{\alpha,\beta}(\cdot)$, given, respectively, by (2.6) and (2.7), are Stepanov p-almost periodic but not almost periodic ($1 \leqslant p < \infty$). Suppose now that

$$F(t_1, t_2, \ldots, t_n) = f_1(t_1)f_2(t_2)\cdots f_n(t_n), \quad \mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n,$$

and for each $i \in \mathbb{N}_n$ there exist real numbers $\alpha_i, \beta_i \in \mathbb{R} \setminus \{0\}$ such that $\alpha_i\beta_i^{-1}$ is an irrational number and $f_i = f_{\alpha_i,\beta_i}$ or $f_i = g_{\alpha_i,\beta_i}$. Applying Proposition 6.2.18, we inductively may conclude that the function $\mathbf{t} \mapsto F(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ is Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic with $\Omega = [0,1]^n$ and $p \in D_+(\Omega)$.

Using Lemma 6.1.17(ii) and Theorem 6.1.18, we can repeat verbatim the argumentation used in the one-dimensional case in order to see that the following result holds.

Theorem 6.2.21. *Suppose that $\mathcal{B}$ is any family of compact subsets of X and $p \in D_+(\Omega)$. If $F : \mathbb{R}^n \times X \to Y$ is uniformly continuous and Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic, then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-multi-almost periodic.*

A sufficient condition for a function $F : \Lambda \times X \to Y$ to be Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic is given as follows.

Proposition 6.2.22. *Let $\Lambda + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $\mathcal{B}$ is any family of compact subsets of X and $F : \Lambda \times X \to Y$ satisfy the following conditions:*

(i) *For each $x \in X$, $F(\cdot;x) \in APS^{\Omega,p(\mathbf{u})}(\Lambda : Y)$.*
(ii) *$F(\cdot;\cdot)$ is $S^{p(\mathbf{u})}$-uniformly continuous with respect to the second argument on each compact subset B in $\mathcal{B}$ in the following sense: for all $\varepsilon > 0$ there exists $\delta_{B,\varepsilon} > 0$ such that for all $x_1,\ x_2 \in B$ one has*

$$\|x_1 - x_2\| \leqslant \delta_{B,\varepsilon} \Rightarrow \big\|F(\mathbf{t} + \cdot; x_1) - F(\mathbf{t} + \cdot; x_2)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon \quad \textit{for all } \mathbf{t} \in \Lambda. \tag{6.52}$$

Then $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic.

Proof. Without loss of generality, we may assume that $p(\mathbf{u}) \equiv p \in [1,\infty)$. Let $\varepsilon > 0$ and $B \subseteq X$ be a compact set. It follows that there exists a finite subset $\{x_1, \ldots, x_n\} \subseteq B$ $(n \in \mathbb{N})$ such that $B \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{B,\varepsilon})$. Therefore, for every $x \in B$, there exists $i \in \mathbb{N}_n$ satisfying $\|x - x_i\| \leqslant \delta_{B,\varepsilon}$. Let $\tau \in \Lambda$. Then we have

$$\begin{aligned}
&\Big(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau;x) - F(\mathbf{t}+\mathbf{s};x)\|_Y^p \, d\mathbf{s}\Big)^{\frac{1}{p}} \\
&\quad \leqslant \Big(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau;x) - F(\mathbf{t}+\mathbf{s}+\tau;x_i)\|_Y^p \, d\mathbf{s}\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau;x_i) - F(\mathbf{t}+\mathbf{s};x_i)\|_Y^p \, d\mathbf{s}\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s};x_i) - F(\mathbf{t}+\mathbf{s};x)\|_Y^p \, d\mathbf{s}\Big)^{\frac{1}{p}}, \quad \mathbf{t} \in \Lambda.
\end{aligned} \tag{6.53}$$

Using (i), we find that for each $i = 1, \ldots, n$ there exists $l_{B,\varepsilon} > 0$ such that for all $\mathbf{t}_0 \in \Lambda$ there exists $\tau \in B(\mathbf{t}_0, l_{B,\varepsilon})$ satisfying

$$\Big(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau;x_i) - F(\mathbf{t}+\mathbf{s};x_i)\|_Y^p \, d\mathbf{s}\Big)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda. \tag{6.54}$$

Since $\|x - x_i\| \leqslant \delta_{K,\delta}$, by (ii) we claim that

$$\Big(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s}+\tau;x) - F(\mathbf{t}+\mathbf{s}+\tau;x_i)\|_Y^p \, d\mathbf{s}\Big)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda \tag{6.55}$$

and

$$\Big(\int_{\Omega} \|F(\mathbf{t}+\mathbf{s};x) - F(\mathbf{t}+\mathbf{s};x_i)\|_Y^p \, d\mathbf{s}\Big)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda. \tag{6.56}$$

Inserting (6.54), (6.55) and (6.56) in (6.53), we obtain

$$\sup_{x\in B}\left(\int_{\Omega}\|F(\mathbf{t}+\mathbf{s}+\tau;x)-F(\mathbf{t}+\mathbf{s};x)\|_Y^p\,d\mathbf{s}\right)^{\frac{1}{p}}\leqslant\varepsilon\quad\text{for all }\mathbf{t}\in\Lambda.$$

Hence, $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic. □

Almost directly from Definition 6.2.4, we may conclude the following; the similar statements can be formulated for the notion introduced in Definition 6.2.5–Definition 6.2.7 (cf. Lemma 1.1.7).

Proposition 6.2.23. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda+\Omega\subseteq\Lambda$, $F:\Lambda\times X\to Y$ and the function $\hat{F}_\Omega:\Lambda\times X\to L^{p(\mathbf{u})}(\Omega:Y)$ is well defined and continuous.*
(i) *For every $p\in\mathcal{P}(\Omega)$, we find that $\mathrm{APS}^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X:Y)$ is a subset of $\mathrm{APS}^{\Omega,1}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X:Y)$.*
(ii) *For every $p,\ q\in\mathcal{P}(\Omega)$, we find that the assumption $q(\mathbf{u})\leqslant p(\mathbf{u})$ for a. e. $\mathbf{u}\in\Omega$ implies that $\mathrm{APS}^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X:Y)$ is a subset of $\mathrm{APS}^{\Omega,q(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X:Y)$.*
(iii) *If $p\in D_+(\Omega)$ and $1\leqslant p^-\leqslant p(\mathbf{u})\leqslant p^+<+\infty$ for a. e. $\mathbf{u}\in\Omega$, then*

$$\mathrm{APS}^{\Omega,p^+}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X:Y)\subseteq\mathrm{APS}^{\Omega,p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X:Y)\subseteq\mathrm{APS}^{\Omega,p^-}_{(\mathrm{R},\mathcal{B})}(\Lambda\times X:Y).$$

Keeping in mind Remark 6.2.8(ii) and the proof of [372, Proposition 4.5], we may deduce the following.

Proposition 6.2.24. *Suppose that $p\in D_+(\Omega)$ and the function $F:\mathbb{R}^n\times X\to Y$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic [Bohr $\mathcal{B}$-almost periodic/$\mathcal{B}$-uniformly recurrent]. Then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $p(\mathbf{u})$-$\mathcal{B}$-uniformly recurrent].*

Furthermore, we have the following simple result.

Proposition 6.2.25. *Let $F(\cdot;\cdot)$ be Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent] and $A\in L(X,Z)$. Then $AF(\cdot;\cdot)$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent].*

The main structural properties of $(\mathrm{R},\mathcal{B})$-multi-almost periodic type functions clarified above can be simply reformulated for the corresponding Stepanov classes. For example, we have the following:
(i) Suppose that $c\in\mathbb{C}$ and $F(\cdot;\cdot)$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost periodic/Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent].
Then $cF(\cdot;\cdot)$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R},\mathcal{B})$-multi-almost periodic [Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost periodic/Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega,p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent].

(ii) Suppose that $\alpha, \beta \in \mathbb{C}$ and, for every sequence which belongs to R (R_{X}), we find that any its subsequence belongs to R (R_{X}). If $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$ are Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent]. Then $(\alpha F + \beta G)(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic/Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent].

6.2.3 Asymptotically Stepanov multi-dimensional almost periodic type functions in Lebesgue spaces with variable exponents

In this subsection, we will generalize the notion introduced in Definition 6.1.26 by investigating various classes of multi-dimensional ergodic components in the Lebesgue spaces with variable exponent; the introduced notion is new even for the multi-dimensional ergodic components with constant coefficients. This would enable us to define various classes of asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions.

We start by introducing the following notion.

Definition 6.2.26. Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$ and the set $\mathbb{D}$ is unbounded. By $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ we denote the vector space consisting of all functions $Q : \Lambda \times X \to Y$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in X_{\mathcal{B}}$, we have $[\hat{Q}_{\Omega}(\mathbf{t}; x)](\mathbf{u}) \in L^{p(\mathbf{u})}(\Omega : Y)$ as well as that, for every $B \in \mathcal{B}$, we have $\lim_{\mathbf{t} \in \mathbb{D}, |\mathbf{t}| \to +\infty} [\hat{Q}_{\Omega}(\mathbf{t}; x)](\mathbf{u}) = 0$ in $L^{p(\mathbf{u})}(\Omega : Y)$, uniformly for $x \in B$. In the case that $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we abbreviate $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ to $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D}}(\Lambda : Y)$.

Using the dominated convergence theorem, it immediately follows that $C_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y) \subseteq S^{p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$. We continue by providing two illustrative examples.

Example 6.2.27.

(i) Let $1 \leqslant p < \infty$. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(s) := \begin{cases} k, & \text{if } k \leqslant s \leqslant k + k^{-p} \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

We already know that the function $f(\cdot)$ is Stepanov p-bounded in the usual sense. Fix now a number $t \geqslant 0$. Then there exists a unique integer $k \in \mathbb{N}_0$ such that $k \leqslant t < k + 1$. There exists two possibilities: $k \leqslant t < k + k^{-p}$ or $k + k^{-p} \leqslant t < k + 1$. In the first case, we have

$$\begin{aligned} \int_t^{t+1} |f(s)|^p \, ds &= \int_t^{k+k^{-p}} k^p \, ds + \int_{k+1}^{t+1} (k+1)^p \, ds \\ &= (t-k)\big[(k+1)^p - k^p\big] + 1 \geqslant 1. \end{aligned}$$

In the second case, we have

$$\int_t^{t+1} |f(s)|^p \, ds = \int_{k+1}^{t+1} (k+1)^p \, ds = (t-k)(k+1)^p \geqslant k^{-p}(k+1)^p \geqslant 1.$$

Hence, there is no unbounded set $\mathbb{D}$ such that the function $f(\cdot)$ belongs to the class $S^p_{0,\mathbb{D}}([0,\infty) : \mathbb{C})$.

(ii) Let (Ω_n) be a sequence of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}^n$, let $\Omega = [0,1]^n$ and let $f_n : \Omega_n \to Y$ $(n \in \mathbb{N})$ satisfy

$$\sup_{n\in\mathbb{N}} \|f_n(\cdot)\|_{L^\infty(\Omega_n:Y)} < \infty. \tag{6.57}$$

Define the function $f : \mathbb{R}^n \to Y$ by $f(\mathbf{t}) := 0$ if $\mathbf{t} \notin \cup_{n\in\mathbb{N}}\Omega_n$ and $f(\mathbf{t}) := f_n(\mathbf{t})$ if $\mathbf{t} \in \Omega_n$ for some $n \in \mathbb{N}$. Then it can be easily shown that the function $f(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded for any $p \in \mathcal{P}(\Omega)$, provided that there exists an integer $l \in \mathbb{N}$ such that for each $\mathbf{t} \in \mathbb{R}^n$ there exist at most l distinct positive integers s such that $(\mathbf{t} + \Omega) \cap \Omega_s \neq 0$. In actual fact, we have

$$\|F(\mathbf{t}+\mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega:X)} \leqslant 4\|F(\mathbf{t}+\mathbf{u})\|_{L^\infty(\Omega:X)} \leqslant 4l \sup_{n\in\mathbb{N}} \|f_n(\cdot)\|_{L^\infty(\Omega_n:X)}, \quad \mathbf{t} \in \mathbb{R}^n$$

and we can apply (6.57). Furthermore, if $\mathbb{D}$ is any unbounded subset of $\mathbb{R}^n$ such that $\operatorname{dist}(\mathbb{D}, \cup_{n\in\mathbb{N}}\Omega_n) \geqslant \operatorname{diam}(\Omega)$, we have $f \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D}}(\mathbb{R}^n : Y)$ for any $p \in \mathcal{P}(\Omega)$.

Now we are ready to introduce the following notion.

Definition 6.2.28.

(i) Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, $F : \Lambda \times X \to Y$ and (6.1), resp. (6.3), holds with the set I replaced by the set Λ therein. Then we say that the function $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost periodic, if and only if there exist a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic function $(H : \mathbb{R}^n \times X \to Y)$ $H : \Lambda \times X \to Y$, resp. a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost periodic function $(H : \mathbb{R}^n \times X \to Y)$ $H : \Lambda \times X \to Y$, and a function $Q \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ such that $F(\mathbf{t};x) = H(\mathbf{t};x) + Q(\mathbf{t};x)$ for a. e. $\mathbf{t} \in \Lambda$ and all $x \in X$. If $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we also say that the function $F(\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-R-multi-almost periodic.

(ii) Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded.

(ii.1) Then we say that $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic if and only if there exist a Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic function $(H : \mathbb{R}^n \times X \to Y)$ $H : \Lambda \times X \to Y$ and a function $Q \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ such that $F(\mathbf{t};x) = H(\mathbf{t};x) + Q(\mathbf{t};x)$ for a. e. $\mathbf{t} \in \Lambda$ and all $x \in X$.

(ii.2) Then we say that $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent if and only if there exist a Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent function $(H : \mathbb{R}^n \times X \to Y)$ $H : \Lambda \times X \to Y$ and a function $Q \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ such that $F(\mathbf{t}; x) = H(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for a. e. $\mathbf{t} \in \Lambda$ and all $x \in X$.

If $X \in \mathcal{B}$, then we also say that the function $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic ((strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-uniformly recurrent). If $\mathbb{D} = \Lambda$, then we omit the "prefix $\mathbb{D}$-" and say that the function $F(\cdot;\cdot)$ is (strongly) asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost periodic, for example.

We can use Proposition 6.1.30 to simply deduce when the decompositions in Definition 6.2.28 are unique; Proposition 6.1.28(ii) and Proposition 6.1.32 can be reformulated in our new context, as well.

Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$ and $\Lambda + \Omega \subseteq \Lambda$. The notion of $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodicity and the notion of $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniform recurrence can be also introduced and analyzed. We will skip all related details for brevity. For applications, we need the following definition.

Definition 6.2.29. Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a continuous function and $\Lambda + \Lambda' \subseteq \Lambda$. Then we say that:

(i) $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic of type 1 if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exist $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that

$$\|F(\mathbf{t} + \tau + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon, \quad \text{provided } \mathbf{t},\ \mathbf{t} + \tau \in \mathbb{D}_M,\ x \in B.$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-uniformly recurrent of type 1 if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_n) in Λ' and a sequence (M_n) in $(0, \infty)$ such that $\lim_{n\to+\infty} |\tau_n| = \lim_{n\to+\infty} M_n = +\infty$ and

$$\lim_{n\to+\infty} \sup_{\mathbf{t}, \mathbf{t}+\tau_n \in \mathbb{D}_{M_n}; x \in B} \|F(\mathbf{t} + \tau_n + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0.$$

If $\Lambda' = \Lambda$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent of type 1); furthermore, if $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov Λ'-uniformly recurrent of type 1). If $\Lambda' = \Lambda$ and $X \in \mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov uniformly recurrent of type 1). As before, we remove the prefix "$\mathbb{D}$-" in the case that $\mathbb{D} = \Lambda$ and remove the prefix "$(\mathcal{B},)$" in the case that $X \in \mathcal{B}$.

6.2.4 Composition theorems for Stepanov $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions in Lebesgue spaces with variable exponents

In this subsection, we will analyze the $(\mathrm{R}, \mathcal{B})$-multi-almost periodic properties of the multi-dimensional Nemytskii operator $W : \Lambda \times X \to Z$, given by (6.20). First of all, we will state and prove the following composition result for Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic functions.

Theorem 6.2.30. *Suppose that Λ is admissible with respect to the almost periodic extensions, $x : \Lambda \to X$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, and $F : \Lambda \times X \to Y$ satisfies the item* (ii) *of* Proposition 6.2.22 *as well as that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_{\Omega}(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Then the function $F(\cdot; x(\cdot))$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic.*

Proof. Without loss of generality, we may assume that $p(\mathbf{u}) \equiv p \in [1, \infty)$ and $\Lambda = \mathbb{R}^n$ (the assumptions prescribed imply that the function $x(\cdot)$ can be extended to a Bohr almost periodic function defined on $\mathbb{R}^n$ as well as that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_{\Omega}(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ can be extended to a Bohr almost periodic function defined on $\mathbb{R}^n$ so that the functions $x(\cdot)$ and the finite collection of functions of the form $\hat{F}_{\Omega}(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ can share the same ε-periods for each positive real number $\varepsilon > 0$; we only need this fact and the relative compactness of range of range of the function $x(\cdot)$ below). Let $\mathbf{t}$, $\tau \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
&\left(\int_{\Omega} \left\| F\big(\mathbf{t} + \mathbf{s} + \tau; x(\mathbf{t} + \mathbf{s} + \tau)\big) - F\big(\mathbf{t} + \mathbf{s}; x(\mathbf{t} + \mathbf{s})\big) \right\|^p d\mathbf{s} \right)^{\frac{1}{p}} \\
&\quad \leqslant \left(\int_{\Omega} \left\| F\big(\mathbf{t} + \mathbf{s} + \tau; x(\mathbf{t} + \mathbf{s} + \tau)\big) - F\big(\mathbf{t} + \mathbf{s} + \tau; x(\mathbf{t} + \mathbf{s})\big) \right\|^p d\mathbf{s} \right)^{\frac{1}{p}} \\
&\qquad + \left(\int_{\Omega} \left\| F\big(\mathbf{t} + \mathbf{s} + \tau; x(\mathbf{t} + \mathbf{s})\big) - F\big(\mathbf{t} + \mathbf{s}; x(\mathbf{t} + \mathbf{s})\big) \right\|^p d\mathbf{s} \right)^{\frac{1}{p}}.
\end{aligned}$$

Let $\varepsilon > 0$ be fixed. Due to our assumption, $K := \overline{\{x(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n\}}$ is a compact subset of X. We know that there exists $\delta_{\varepsilon,K} > 0$ such that (6.52) holds. Moreover, there exists $l_{\varepsilon} > 0$ such that every ball of center l_{ε} contains an element τ such that $\|x(\mathbf{s} + \tau) - x(\mathbf{s})\| \leqslant \delta_{\varepsilon,K}$ for all $\mathbf{s} \in \mathbb{R}^n$. Moreover, for each $\mathbf{s} \in \mathbb{R}^n$, we have $x(\mathbf{s}) \in K$. Hence,

$$\left(\int_{\Omega} \left\| F\big(\mathbf{t} + \mathbf{s} + \tau; x(\mathbf{t} + \mathbf{s} + \tau)\big) - F\big(\mathbf{t} + \mathbf{s} + \tau; x(\mathbf{t} + \mathbf{s})\big) \right\|^p d\mathbf{s} \right)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{4}. \tag{6.58}$$

Since K is compact, it follows that there exists a finite subset $\{x_1, \ldots, x_n\} \subseteq K$ ($n \in \mathbb{N}$) such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{\varepsilon,K})$. Then for all $\mathbf{t} \in \mathbb{R}^n$ there exists $i(\mathbf{t}) \in \mathbb{N}_n$ such that

$\|x(\mathbf{t}) - x_{i(\mathbf{t})}\| \leqslant \delta_{K,\varepsilon}$. Thus,

$$\left(\int_{\Omega} \big\|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s})) - F(\mathbf{t}+\mathbf{s}+\tau; x_{i(\mathbf{t})})\big\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{4} \tag{6.59}$$

and

$$\left(\int_{\Omega} \big\|F(\mathbf{t}+\mathbf{s}; x(\mathbf{t}+\mathbf{s})) - F(\mathbf{t}+\mathbf{s}; x_{i(\mathbf{t})})\big\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{4}. \tag{6.60}$$

By Proposition 6.2.22, we have

$$\left(\int_{\Omega} \big\|F(\mathbf{t}+\mathbf{s}+\tau; x_{i(\mathbf{t})}) - F(\mathbf{t}+\mathbf{s}; x_{i(\mathbf{t})})\big\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{4}. \tag{6.61}$$

Consequently, by (6.58), (6.59), (6.60) and (6.61), we obtain

$$\left(\int_{\Omega} \big\|F(\mathbf{t}+\mathbf{s}+\tau; x(\mathbf{t}+\mathbf{s}+\tau)) - F(\mathbf{t}+\mathbf{s}; x(\mathbf{t}+\mathbf{s}))\big\|^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,$$

for any $\mathbf{t} \in \mathbb{R}^n$. This proves the result. □

Now we will state the following simple consequence of Theorem 6.2.30 in which $F(\cdot;\cdot)$ is Lipschitzian with respect to the second argument; more precisely, we assume that there exists a non-negative scalar-valued function $L_F(\cdot)$ such that $\sup_{\mathbf{t}\in\Lambda} \|L_F(\mathbf{t} + \mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega)} < +\infty$ and

$$\big\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\big\| \leqslant L_F(\mathbf{t})\|x - y\|, \quad x,\ y \in X,\ \mathbf{t} \in \Lambda. \tag{6.62}$$

Corollary 6.2.31. *Suppose that Λ is admissible with respect to the almost periodic extensions, $x : \Lambda \to X$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, and $F : \Lambda \times X \to Y$ satisfies the requirement that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_{\Omega}(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Then the function $F(\cdot; x(\cdot))$ is Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-multi-almost periodic, provided that there exists a non-negative scalar-valued function $L_F(\cdot)$ such that* $\sup_{t\in\Lambda} \|L_F(\mathbf{t} + \mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega)} < +\infty$ *and* (6.62) *holds.*

The following composition principle generalizes [372, Theorem 5.4] and can be proved by using the argumentation contained in the proofs of [729, Lemma 2.1, Theorem 2.2] (the assumptions prescribed imply that we can pass to the case in which $\Lambda = \mathbb{R}^n$, as in the proof of Theorem 6.2.30).

Theorem 6.2.32. *Suppose that Λ is admissible with respect to the almost periodic extensions, $\hat{x} : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, $p \in \mathcal{P}(\Omega)$, and $F : \Lambda \times X \to Y$ satisfies the requirement that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_\Omega(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Let the following conditions hold:*

(i) *There exist a function $r \in \mathcal{P}(\Omega)$ such that $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/p(\cdot)-1)$ and a function $L_F \in L_S^{\Omega,r(\mathbf{u})}(\Lambda)$ such that*

$$\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\| \leqslant L_F(\mathbf{t})\|x - y\|_Y, \quad \mathbf{t} \in \Lambda,\ x,\ y \in Y; \tag{6.63}$$

(ii) *There exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{x(\mathbf{t}) : \mathbf{t} \in \Lambda \smallsetminus \mathrm{E}\}$ is relatively compact in X.*

Define $q \in \mathcal{P}(\Omega)$ by $q(\mathbf{u}) := p(\mathbf{u})r(\mathbf{u})/[p(\mathbf{u}) + r(\mathbf{u})]$, if $\mathbf{u} \in \Omega$ and $r(\mathbf{u}) < \infty$, $q(\mathbf{u}) := p(\mathbf{u})$, if $\mathbf{u} \in \Omega$ and $r(\mathbf{u}) = \infty$. Then $q(\mathbf{u}) \in [1, p(\mathbf{u}))$ for $\mathbf{u} \in \Omega$, $r(\mathbf{u}) < \infty$ and $F(\cdot, x(\cdot)) \in \mathrm{APS}_{\mathcal{B}}^{\Omega,q(\mathbf{u})}(\Lambda : Y)$.

The following composition principle generalizes [639, Theorem 2.1] and it is not comparable with Theorem 6.2.32 in general (see [639] for more details).

Theorem 6.2.33. *Suppose that Λ is admissible with respect to the almost periodic extensions, $\hat{x} : \Lambda \to L^{q(\mathbf{u})}(\Omega : Y)$ is a uniformly continuous, Bohr almost periodic function, $\mathcal{B}$ is any family consisting of compact subsets of X containing $\overline{R(x(\cdot))}$, $p \in \mathcal{P}(\Omega)$, and $F : \Lambda \times X \to Y$ satisfies the requirement that, for every $z \in \overline{R(x(\cdot))}$, the function $\hat{F}_\Omega(\cdot; z) : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is uniformly continuous, Bohr almost periodic. Suppose, further, that $p,\ q,\ r \in \mathcal{P}(\Omega)$, $1/p = 1/q + 1/r$ and the following conditions hold:*

(i) *There exists a function $L_F \in L_S^{\Omega,r(\mathbf{u})}(\Lambda)$ such that (6.63) holds.*
(ii) *There exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{x(\mathbf{t}) : \mathbf{t} \in \Lambda \smallsetminus \mathrm{E}\}$ is relatively compact in X.*

Then $F(\cdot, x(\cdot)) \in \mathrm{APS}_{\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda : Y)$.

Keeping in mind the above two results, we can simply extend the statements of [372, Proposition 5.5] and [639, Proposition 2.2] for $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic functions; the proofs are completely similar to the proofs of these statements given in the one-dimensional case. For simplicity, in the formulations of the following two theorems, we will assume that $\Lambda = \mathbb{R}^n$, albeit we can also assume that Λ is admissible with respect to the almost periodic extensions.

Theorem 6.2.34. *Let $\mathcal{B}$ be any family consisting of compact subsets of X, $p \in \mathcal{P}(\Omega)$ and the following conditions hold:*

(i) *$G \in \mathrm{APS}_{\mathcal{B}}^{\Omega,p(\mathbf{u})}(\mathbb{R}^n \times X : Y)$ and there exist a function $r \in \mathcal{P}(\Omega)$ such that $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/p(\cdot)-1)$ and a function $L_G \in L_S^{\Omega,r(\mathbf{u})}(\mathbb{R}^n)$ such that (6.63) holds with the function $F(\cdot; \cdot)$ replaced therein with the function $G(\cdot; \cdot)$;*

(ii) $u \in \mathrm{APS}^{\Omega,p(\mathbf{u})}(\mathbb{R}^n : X)$*, and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{u(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus \mathrm{E}\}$ *is relatively compact in* X*;*
(iii) $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$*, where* $Q \in S^{\Omega,q(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *and* $q(\cdot)$ *being defined as in the formulation of* Theorem 6.2.32*;*
(iv) $x(\mathbf{t}) = u(\mathbf{t}) + \omega(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{R}^n$*, where* $\omega \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D}}(\mathbb{R}^n : X)$*;*
(v) *there exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{x(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus E'\}$ *is relatively compact in* X*.*

Then $F(\cdot, x(\cdot)) \in \mathrm{AAPS}^{\Omega,q(\mathbf{u})}_{\mathcal{B}}(\mathbb{R}^n : Y)$*.*

Theorem 6.2.35. *Let* $\mathcal{B}$ *be any family consisting of compact subsets of* X*. Suppose that* $p,\ q,\ r \in \mathcal{P}(\Omega)$*,* $1/p = 1/q + 1/r$ *and the following conditions hold:*
(i) $G \in \mathrm{APS}^{\Omega,p(\mathbf{u})}_{\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *and there exists a function* $L_G \in L_S^{\Omega,r(\mathbf{u})}(\mathbb{R}^n)$ *such that* (6.63) *holds with the function* $F(\cdot;\cdot)$ *replaced therein with the function* $G(\cdot;\cdot)$*;*
(ii) $u \in \mathrm{APS}^{\Omega,q(\mathbf{u})}(\mathbb{R}^n : X)$*, and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{u(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus \mathrm{E}\}$ *is relatively compact in* X*;*
(iii) $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$*, where* $Q \in S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$*;*
(iv) $x(\mathbf{t}) = u(\mathbf{t}) + \omega(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{R}^n$*, where* $\omega \in S^{\Omega,q(\mathbf{u})}_{0,\mathbb{D}}(\mathbb{R}^n : X)$*;*
(v) *there exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{x(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^n \setminus E'\}$ *is relatively compact in* X*.*

Then $F(\cdot, x(\cdot)) \in \mathrm{APS}^{\Omega,p(\mathbf{u})}_{\mathcal{B}}(\mathbb{R}^n : Y)$*.*

The interested reader may try to formulate composition principles for Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-uniformly recurrent functions following the approach obeyed in [650].

6.2.5 Invariance of Stepanov $(\mathbf{R}, \mathcal{B})$-multi-almost periodicity under the actions of convolution products

Define $\Omega_{\mathbf{k}} := \Omega + \mathbf{k}$, $\mathbf{k} \in \mathbb{N}_0^n$. If any component of tuple $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is strictly positive, then we simply write $\mathbf{t} > \mathbf{0}$.

The following result is very similar to [631, Proposition 2.6.11] (see also [372, Proposition 6.1]).

Theorem 6.2.36. *Let* $\Omega = [0,1]^n$*,* $p \in D_+(\Omega)$*,* $q \in \mathcal{P}(\Omega)$*,* $1/p(x) + 1/q(x) = 1$ *for all* $x \in \Omega$*, and* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X, Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{\mathbf{k} \in \mathbb{N}_0^n} \|R(\cdot + \mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)} < \infty$*. If* $\check{f} : \mathbb{R}^n \to X$ *is Stepanov* $(\Omega, p(\mathbf{u}))$*-almost periodic, then the function* $F : \mathbb{R}^n \to Y$*, given by*

$$F(\mathbf{t}) := \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{6.64}$$

is well defined and almost periodic.

Proof. The proof of theorem can be deduced by using the argumentation given in the proof of the above-mentioned propositions and we will only present the main details. Since

$$F(\mathbf{t}) := \int_0^{+\infty}\int_0^{+\infty}\cdots\int_0^{+\infty} R(\mathbf{s})f(\mathbf{t}-\mathbf{s})\,d\mathbf{s}, \quad \mathbf{t}\in\mathbb{R}^n, \tag{6.65}$$

the Hölder inequality holds in our framework (see Lemma 1.1.7(ii)) and the function $f(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded, the above integral converges absolutely. The proof of fact that for each $\varepsilon > 0$ the set of all ε-periods of $F(\cdot)$ is relatively dense in $\mathbb{R}^n$ can be repeated verbatim. Since any element of $L^{p(\mathbf{u})}(\Omega : X)$ is absolutely continuous with respect to the norm $\|\cdot\|_{L^{p(\mathbf{u})}}$ (see [421, Definition 1.12, Theorem 1.13]) and the Bochner transform of the function $\check{f}(\cdot)$ is uniformly continuous, the proof of continuity of the function $F(\cdot)$ can be deduced along the same lines as in the one-dimensional case. □

We can similarly deduce the following result.

Theorem 6.2.37. *Let* $\Omega = [0,1]^n$, $p \in D_+(\Omega)$, $q \in \mathcal{P}(\Omega)$, $1/p(x) + 1/q(x) = 1$ *for all* $x \in \Omega$, *and* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{\mathbf{k}\in\mathbb{N}_0^n}\|R(\cdot+\mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)} < \infty$. *If* $\check{f} : \mathbb{R}^n \to X$ *is Stepanov* $(\Omega, p(\mathbf{u}))$*-bounded and Stepanov* $(\Omega, p(\mathbf{u}))$-R*-multi-almost periodic, then the function* $F : \mathbb{R}^n \to Y$, *given by* (6.64), *is well defined and* R*-multi-almost periodic.*

Now we will state and prove the following analogue of Proposition 6.1.56 for strong $\mathbb{D}$-asymptotical Stepanov $(\Omega, p(\mathbf{u}))$-almost periodicity (see also [631, Proposition 2.6.13, Remark 2.6.14]).

Proposition 6.2.38. *Suppose that* $\Omega = [0,1]^n$, $p \in D_+(\Omega)$, $q \in \mathcal{P}(\Omega)$, $1/p(x) + 1/q(x) = 1$ *for all* $x \in \Omega$, *and* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{\mathbf{k}\in\mathbb{N}_0^n}\|R(\cdot+\mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)} < \infty$. *Suppose, further, that* $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ *satisfies* $\Lambda + \Omega \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ *and the set* $\mathbb{D}$ *is unbounded. Let* $\check{g} : \mathbb{R}^n \to X$ *be Stepanov* $(\Omega, p(\mathbf{u}))$*-almost periodic (Stepanov* $(\Omega, p(\mathbf{u}))$*-bounded and Stepanov* $(\Omega, p(\mathbf{u}))$-R*-multi-almost periodic), let* $q : \Lambda \to X$, *and let* $f(\mathbf{t}) := g(\mathbf{t}) + q(\mathbf{t})$ *for all* $\mathbf{t} \in \Lambda$. *Then the function* $F : \Lambda \to Y$, *defined by*

$$F(\mathbf{t}) := \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})f(\mathbf{s})\,ds, \quad \mathbf{t}\in\Lambda,$$

is strongly $\mathbb{D}$*-asymptotically Stepanov* $(\Omega, p(\mathbf{u}))$*-almost periodic (strongly* $\mathbb{D}$*-asymptotically Stepanov* $(\Omega, p(\mathbf{u}))$-R*-multi-almost periodic), provided that*

$$\lim_{|\mathbf{t}|\to\infty, \mathbf{t}\in\mathbb{D}} \sum_{\mathbf{k}\in\mathbb{N}_0^n} \|R(\mathbf{s}+\mathbf{k})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap\mathbb{D}^c])\cap\Omega)} = 0, \tag{6.66}$$

and for each $\varepsilon > 0$ there exists $r > 0$ such that for each $\mathbf{t} \in \mathbb{D}$ with $|\mathbf{t}| \geqslant r$ there exists a finite real number $r_{\mathbf{t}} > 0$ such that

$$\sum_{\mathbf{k}\in\mathbb{N}_0^n} \{\|R(\mathbf{s}+\mathbf{k})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)} \times \|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)}\} < \varepsilon/2 \tag{6.67}$$

and

$$\sum_{\mathbf{k}\in\mathbb{N}_0^n} \{\|R(\mathbf{s}+\mathbf{k})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)} \times \|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\|_{L^{p(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)}\} < \varepsilon/2. \tag{6.68}$$

Proof. We will consider only strong $\mathbb{D}$-asymptotical Stepanov $(\Omega, p(\mathbf{u}))$-almost periodicity. Clearly, we have the decomposition

$$F(\mathbf{t}) = \int_{I_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s} + \Big[\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s}\Big], \quad \mathbf{t}\in\Lambda.$$

Keeping in mind Theorem 6.2.36, it suffices to show that the function

$$\mathbf{t} \mapsto \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s}, \quad \mathbf{t}\in\Lambda,$$

belongs to the class $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D}}(\Lambda : X)$. For the second addend, this immediately follows from the next calculus and condition (6.66):

$$\begin{aligned}\int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s} &= \int_{\mathbf{t}-[I_{\mathbf{t}}\cap\mathbb{D}^c]} R(\mathbf{s})\breve{g}(\mathbf{s}-\mathbf{t})\,d\mathbf{s}\\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^n}\int_{(\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap\mathbb{D}^c])\cap\Omega} R(\mathbf{s}+\mathbf{k})\breve{g}(\mathbf{s}+\mathbf{k}-\mathbf{t})\,d\mathbf{s}\\ &\leqslant 2\sum_{\mathbf{k}\in\mathbb{N}_0^n}\|R(\mathbf{s}+\mathbf{k})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap\mathbb{D}^c])\cap\Omega)}\cdot\sup_{\mathbf{t}\in\mathbb{R}^n}\|\hat{\breve{g}}(\mathbf{t})\|_{L^{p(\mathbf{u})}(\Omega)}.\end{aligned}$$

Let $\varepsilon > 0$ be given. Then there exists $r > 0$ such that for each $\mathbf{t} \in \mathbb{D}$ with $|\mathbf{t}| \geqslant r$ there exists a finite real number $r_{\mathbf{t}} > 0$ such that (6.67)–(6.68) hold. If $\mathbf{t} \in \mathbb{D}$ and $|\mathbf{t}| \geqslant r$, then we have

$$\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} = \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} + \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s}.$$

For the first addend in the above sum, we can use the following calculation and condition (6.67):

$$\begin{aligned}\int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} &= \int_{\mathbf{t}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})]} R(\mathbf{s})\check{q}(\mathbf{s}-\mathbf{t})\,d\mathbf{s}\\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^n}\int_{(\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega} R(\mathbf{s}+\mathbf{k})\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\,d\mathbf{s}\end{aligned}$$

$$\leqslant 2 \sum_{\mathbf{k}\in\mathbb{N}_0^n} \|R(\mathbf{s}+\mathbf{k})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)} \cdot \|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)}.$$

For the second addend in the above sum, we can use the following calculation and condition (6.68):

$$\begin{aligned}\int\limits_{\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} &= \int\limits_{\mathbf{t}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c]} R(\mathbf{s})\check{q}(\mathbf{s}-\mathbf{t})\,d\mathbf{s} \\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^n}\ \int\limits_{(\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega} R(\mathbf{s}+\mathbf{k})\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\,d\mathbf{s} \\ &\leqslant 2\sum_{\mathbf{k}\in\mathbb{N}_0^n} \|R(\mathbf{s}+\mathbf{k})\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)} \\ &\quad \cdot \|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\|_{L^{p(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)}.\end{aligned}$$

The proof of the proposition is thereby completed. □

6.2.6 Examples and applications to the abstract Volterra integro-differential equations

We start with two examples concerning Stepanov almost periodic type solutions (with respect to the space variable) of the multi-dimensional heat equations:

1. Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1 \leqslant p < \infty$. As already mentioned, the Gaussian semigroup $(G(t))_{t>0}$ can be extended to a bounded analytic C_0-semigroup of angle $\pi/2$, generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y. Suppose now that $F(\cdot)$ is bounded Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic function, where $p \in D_+(\Omega)$. Then an application of Proposition 6.2.10 shows for each $t_0 > 0$ the function $\mathbb{R}^n \ni x \mapsto u(x,t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is likewise bounded and Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic; further on, if $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, then we can use Proposition 6.2.12, Lemma 6.1.5 and Eq. (6.50) in order to conclude that for each $t_0 > 0$ the function $x \mapsto u(x,t_0)$, $x \in \mathbb{R}^n$ is bounded and Stepanov $(\Omega, p(\mathbf{u}))$-Λ'-almost periodic provided that the function $F(\cdot)$ has the same properties. Similar statements hold in the case of consideration of the Poisson semigroup.
2. Suppose that $0 < T < \infty$. Set $\Lambda := \{(x,t) : x > 0,\ t > 0\}$, the function $E_1(x,t)$ is given by (6.29) and suppose that $\mathbb{D}$ is any unbounded subset of Λ satisfying (6.32). Suppose, further, that $g(t) \equiv 0$ as well as that the function $u_0 : [0,\infty) \to \mathbb{C}$ is both Stepanov bounded and Stepanov $([0,1],1)$-Λ_0-almost periodic, resp. Stepanov bounded and Stepanov $([0,1],1)$-Λ_0-uniformly recurrent, for a certain non-empty subset Λ_0 of $[0,\infty)$. Set $\Lambda' := \Lambda_0 \times (0,T)$. We will prove that the solution $u(x,t)$ of (6.31) is $\mathbb{D}$-asymptotically Stepanov $([0,1]^2,1)$-Λ'-almost periodic of type 1,

resp. $\mathbb{D}$-asymptotically Stepanov $([0,1]^2,1)$-Λ'-uniformly recurrent of type 1 (see Definition 6.2.29). In our concrete situation, the formula (6.30) takes the following form:

$$u(x,t)=\frac{1}{2}\int_{-x}^{x}(\pi t)^{-1/2}e^{-y^2/4t}u_0(x-y)\,dy,\quad x>0,\ t>0.$$

For any $(x,t)\in\Lambda$ and $(\tau_1,\tau_2)\in\Lambda$, we have

$$\begin{aligned}
&\int_0^1\int_0^1\big|u(x+\tau_1+u_1,t+\tau_2+u_2)-u(x+u_1,t+u_2)\big|\,du_1\,du_2\\
&\quad\leqslant\frac{1}{2}\int_0^1\int_0^1\int_{x+u_1}^{x+\tau_1+u_1}\big(\pi(t+\tau_2+u_2)\big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\big|u_0(x+\tau_1+u_1-y)\big|\,dy\,du_1\,du_2\\
&\qquad+\frac{1}{2}\int_0^1\int_0^1\int_{-(x+\tau_1+u_1)}^{-(x+u_1)}\big(\pi(t+\tau_2+u_2)\big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\big|u_0(x+\tau_1+u_1-y)\big|\,dy\,du_1\,du_2\\
&\qquad+\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\big|\big(\pi(t+\tau_2+u_2)\big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0(x+\tau_1+u_1-y)\\
&\qquad-\big(\pi(t+u_2)\big)^{-1/2}e^{-y^2/4(t+u_2)}u_0(x+u_1-y)\big|\,dy\,du_1\,du_2.
\end{aligned}\tag{6.69}$$

Let $\varepsilon>0$ be given. Then we know that there exists $l>0$ such that for each $x_0\in\Lambda_0$ there exists $\tau_1\in(x_0-l,x_0+l)\cap\Lambda_0$ such that

$$\int_x^{x+1}\big|u_0(t+\tau_1)-u_0(t)\big|\,dt\leqslant\varepsilon,\quad x\geqslant 0.$$

Furthermore, there exists a finite real number $M_0>0$ such that $\int_v^{+\infty}e^{-x^2}\,dx<\varepsilon$ for all $v\geqslant M_0$. Let $M>0$ be such that

$$\min\biggl(\frac{x^2}{4(t+T)},t\biggr)>M_0^2+\frac{1}{\varepsilon},\quad\text{provided }(x,t)\in\mathbb{D}\text{ and }\big|(x,t)\big|>M.\tag{6.70}$$

So, let $(x,t)\in\mathbb{D}$ and $|(x,t)|>M$. For the first addend in (6.69), we use the Fubini theorem and the following estimates (see (6.70)):

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{x+u_1}^{x+\tau_1+u_1}\big(\pi(t+\tau_2+u_2)\big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\big|u_0(x+\tau_1+u_1-y)\big|\,dy\,du_1\,du_2\\
&\quad\leqslant\frac{1}{2}\int_0^1\int_{x+1}^{x+\tau_1}\int_0^1\big(\pi(t+\tau_2+u_2)\big)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\big|u_0(x+\tau_1+u_1-y)\big|\,du_1\,dy\,du_2
\end{aligned}$$

$$+\frac{1}{2}\int_0^1\int_x^{x+1}\int_0^1 \left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\left|u_0(x+\tau_1+u_1-y)\right|du_1\,dy\,du_2$$

$$+\frac{1}{2}\int_0^1\int_{x+\tau_1}^{x+\tau_1+1}\int_0^1 \left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\left|u_0(x+\tau_1+u_1-y)\right|du_1\,dy\,du_2$$

$$\leqslant \frac{\|u_0\|_{S^1}}{2}\int_0^1\int_x^{x+\tau_1+1}\left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\,dy\,du_2$$

$$\leqslant \frac{\|u_0\|_{S^1}}{2}\int_0^1\int_x^{\infty}\left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\,dy\,du_2$$

$$\leqslant \pi^{-1/2}\|u_0\|_{S^1}\int_{x/2\sqrt{t+T}}^{+\infty}e^{-v^2}\,dv\leqslant \pi^{-1/2}\|u_0\|_{S^1}\varepsilon. \tag{6.71}$$

The second addend in (6.69) can be estimated in the same manner as in (6.71). For the third addend in (6.69), we use the following decomposition:

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Big|\left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0(x+\tau_1+u_1-y)\\
&\quad-\left(\pi(t+u_2)\right)^{-1/2}e^{-y^2/4(t+u_2)}u_0(x+u_1-y)\Big|\,dy\,du_1\,du_2\\
&\leqslant \frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}\\
&\quad\times\left|u_0(x+\tau_1+u_1-y)-u_0(x+u_1-y)\right|dy\,du_1\,du_2\\
&\quad+\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Big|\left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0(x+u_1-y)\\
&\quad-\left(\pi(t+u_2)\right)^{-1/2}e^{-y^2/4(t+u_2)}u_0(x+u_1-y)\Big|\,dy\,du_1\,du_2.
\end{aligned}\tag{6.72}$$

The second addend in (6.72) can be estimated similarly as the first addend in (6.69) and the corresponding term from the computation given in [265]. We get

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{-(x+u_1)}^{x+u_1}\Big|\left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}u_0(x+u_1-y)\\
&\quad-\left(\pi(t+u_2)\right)^{-1/2}e^{-y^2/4(t+u_2)}u_0(x+u_1-y)\Big|\,dy\,du_1\,du_2\\
&\leqslant\frac{\|u_0\|_{S^1}}{2}\int_0^1\int_{-(x+1)}^{x+1}\Big|\left(\pi(t+\tau_2+u_2)\right)^{-1/2}e^{-y^2/4(t+\tau_2+u_2)}
\end{aligned}$$

$$
\begin{aligned}
&- \big(\pi(t+u_2)\big)^{-1/2} e^{-y^2/4(t+\tau_2+u_2)} \Big| \, dy \, du_2 \\
&\leqslant \frac{\|u_0\|_{S^1}}{2} \int_0^1 2\pi^{-1/2} \int_{-\infty}^{+\infty} \left| \sqrt{\frac{t+u_2}{t+\tau_2+u_2}} e^{-v^2 \cdot \frac{t+u_2}{t+\tau_2+u_2}} - e^{-v^2} \right| dv \, du_2 \\
&\leqslant \|u_0\|_{S^1} \pi^{-1/2} \int_0^1 \left| \sqrt{\frac{t+u_2}{t+\tau_2+u_2}} - 1 \right| du_2 \times \int_{-\infty}^{+\infty} e^{-\frac{M_0^2}{M_0^2+T} v^2} \big(1+2v^2\big) \, dv \\
&\|u_0\|_{S^1} \pi^{-1/2} \int_0^1 \frac{\tau_2}{t+u_2+\sqrt{(t+u_2)^2+(t+u_2)\tau_2}} du_2 \times \int_{-\infty}^{+\infty} e^{-\frac{M_0^2}{M_0^2+T} v^2} \big(1+2v^2\big) \, dv \\
&\leqslant \|u_0\|_{S^1} \pi^{-1/2} \frac{T}{t} \times \int_{-\infty}^{+\infty} e^{-\frac{M_0^2}{M_0^2+T} v^2} \big(1+2v^2\big) \, dv.
\end{aligned} \tag{6.73}
$$

The first addend in (6.72) can be estimated similarly to (6.73); we have

$$
\begin{aligned}
&\frac{1}{2} \int_0^1 \int_0^1 \int_{-(x+u_1)}^{x+u_1} \big(\pi(t+\tau_2+u_2)\big)^{-1/2} e^{-y^2/4(t+\tau_2+u_2)} \\
&\quad \times \big| u_0(x+\tau_1+u_1-y) - u_0(x+u_1-y) \big| \, dy \, du_1 \, du_2 \\
&\leqslant \frac{1}{2} \int_0^1 \int_{-(x+1)}^{x+1} \big(\pi(t+\tau_2+u_2)\big)^{-1/2} e^{-y^2/4(t+\tau_2+u_2)} \\
&\quad \times \left[\sup_{\xi \geqslant 0} \int_0^1 \big| u_0(\xi+\tau_1) - u_0(\xi) \big| \, du_1 \right] dy \, du_2 \\
&\leqslant \frac{\varepsilon}{2} \int_0^1 \int_{-\infty}^{+\infty} \big(\pi(t+\tau_2+u_2)\big)^{-1/2} e^{-y^2/4(t+\tau_2+u_2)} \, dy \, du_2 \leqslant \varepsilon \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-v^2} \, dv.
\end{aligned}
$$

This finally implies the required conclusion.

3. As explained in [265], Theorem 6.2.36 and Theorem 6.2.37 are applicable in the analysis of existence of almost periodic solutions for a wide class of the abstract partial differential equations, which can be constructed in a little bit artificial way. For example, let A be the infinitesimal generator of an exponentially decaying, strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on X $(i = 1, 2)$, let $\gamma \in (0, 1)$ and let $(T_\gamma(t))_{\geqslant 0}$ be the subordinated γ-times resolvent family generated by A (see [631] for more details). Suppose that $1 < p < \infty$, $F : \mathbb{R}^2 \to X$ is a Stepanov $([0, 1]^2, p)$-almost periodic function satisfying that the improper integral in (6.74) is absolutely convergent. Define

$$
u(t_1, t_2) := \int_{[0,\infty)^2} \big[-T_\gamma(s_1) + T(s_2)\big] F(t_1 - s_1, t_2 - s_2) \, ds_1 \, ds_2, \quad t_1, \ t_2 \in \mathbb{R}. \tag{6.74}
$$

Due to Theorem 6.2.36 (see also Eq. (6.65)), we find that $u : \mathbb{R}^2 \to X$ is almost periodic; furthermore, under certain conditions, we have (see also [631])

$$\begin{aligned} u_{t_2}(t_1,t_2) = &- \int_{[0,\infty)} T_\gamma(s_1)\left(\int_0^\infty \frac{\partial}{\partial t_2} F(t_1-s_1,t_2-s_2)\,ds_2\right) ds_1 \\ &+ \int_0^\infty \left(\frac{\partial}{\partial t_2}\int_0^\infty T(s_2)F(t_1-s_1,t_2-s_2)\,ds_2\right) ds_1 \\ = &\int_{[0,\infty)} T_\gamma(s_1)F_{t_2}(t_1-s_1,t_2-s_2)\,ds_2\,ds_1 \\ &+ \int_0^\infty \left(A\int_0^\infty T(s_2)F(t_1-s_1,t_2-s_2)\,ds_2 + F(t_1-s_1,t_2)\right) ds_1, \end{aligned}$$

for any $t_1,\ t_2 \in \mathbb{R}$. Since the unique solution of the abstract fractional differential equation

$$D^\gamma_{t,+}u(t) = (-A)u(t) + f(t), \quad t \in \mathbb{R},$$

is given by $t \mapsto \int_0^\infty T_\gamma(s)f(t-s)\,ds$, $t \in \mathbb{R}$, we similarly obtain

$$\begin{aligned} -D^\gamma_{t_1,+}u(t_1,t_2) = &- \int_0^\infty T(s_2)\left(\int_0^\infty D^\gamma_{t_1,+}F(t_1-s_1,t_2-s_2)\,ds_1\right) ds_2 \\ &+ \int_0^\infty \left((-A)\int_0^\infty T_\gamma(s_1)F(t_1-s_1,t_2-s_2)\,ds_1 + F(t_1,t_2-s_2)\right) ds_2, \end{aligned}$$

so that

$$\begin{aligned} u_{t_2}(t_1,t_2) - D^\gamma_{t_1,+}u(t_1,t_2) = Au(t_1,t_2) &+ \int_0^\infty F(t_1-s_1,t_2)\,ds_1 \\ &+ \int_0^\infty F(t_1,t_2-s_2)\,ds_2 + \int_{[0,\infty)} T_\gamma(s_1)F_{t_2}(t_1-s_1,t_2-s_2)\,ds_2\,ds_1 \\ &- \int_0^\infty T(s_2)\left(\int_0^\infty D^\gamma_{t_1,+}F(t_1-s_1,t_2-s_2)\,ds_1\right) ds_2, \quad t_1,\ t_2 \in \mathbb{R}. \end{aligned}$$

Unfortunately, it is very difficult to find some applications or interpretations of these types of abstract fractional PDEs in the world of real phenomena.

4. The existence and uniqueness of almost periodic solutions for a wide class of abstract semilinear integral equations of the form

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{-\infty}^{\mathbf{t}} R(\mathbf{t}-\mathbf{s})F(\mathbf{s},u(\mathbf{s}))\,d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

can be shown by using the Banach contraction principle and our results about the convolution invariance of almost periodicity under the actions of infinite convolution products and established composition principles; here, we assume that $f(\cdot)$ is almost periodic, $(R(\mathbf{t}))_{\mathbf{t}>0}$ has a similar growth rate as in Theorem 6.2.36 and $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic for a certain function $p \in D_+(\Omega)$; $\Omega \equiv [0,1]^n$. The consideration is quite similar to the corresponding considerations given in the proofs of [631, Theorem 2.7.6, Theorem 2.7.7] and therefore omitted. Observe, however, that we can similarly analyze the existence and uniqueness of asymptotically almost periodic solutions for a wide class of abstract semilinear integral equations of the form

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_0^{\mathbf{t}} R(\mathbf{t}-\mathbf{s})F(\mathbf{s}, u(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in [0,\infty)^n,$$

by using a similar argumentation containing our results about the convolution invariance of asymptotical almost periodicity under the actions of finite convolution products and established composition principles (see, e. g., [631, Theorem 2.9.10, Theorem 2.9.11], which must be slightly reformulated for our new purposes).

5. Let A generate a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X whose elements are certain complex-valued functions defined on $\mathbb{R}^n$. As we have already clarified, the function

$$u(t,x) = \big(T(t)u_0\big)(x) + \int_0^t \big[T(t-s)f(s)\big](x)\, ds, \quad t \geqslant 0,\ x \in \mathbb{R}^n,$$

is a unique classical solution of the abstract Cauchy problem

$$u_t(t,x) = Au(t,x) + F(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^n;\ u(0,x) = u_0(x),$$

where $F(t,x) := [f(t)](x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. In many concrete situations (for example, this holds for the Gaussian semigroup on $\mathbb{R}^n$), there exists a kernel $(t,y) \mapsto E(t,y)$, $t > 0$, $y \in \mathbb{R}^n$ which is integrable on any set $[0,T] \times \mathbb{R}^n$ $(T > 0)$ and satisfies $[T(t)f(s)](x) = \int_{\mathbb{R}^n} F(s, x-y)E(t,y)\, dy$, $t > 0$, $s \geqslant 0$, $x \in \mathbb{R}^n$. Suppose that this is the case and fix a positive real number $t_0 > 0$. We have already observed that the almost periodic behavior of the function $x \mapsto u_{t_0}(x) \equiv \int_0^{t_0} [T(t_0-s)f(s)](x)\, ds$, $x \in \mathbb{R}^n$ depends on the almost periodic behavior of the function $F(t,x)$ in the space variable x. Suppose, for example, that the function $F(t,x)$ is Stepanov $(\Omega,1)$-almost periodic with respect to the variable $x \in \mathbb{R}^n$, uniformly in the variable t on compact subsets of $[0,\infty)$. Then we have $(x,\ \tau \in \mathbb{R}^n;\ \mathbf{u} \in \Omega)$

$$\begin{aligned}&\big|u_{t_0}(x+\tau+\mathbf{u}) - u_{t_0}(x+\mathbf{u})\big| \\ &\quad \leqslant \int_0^{t_0}\int_{\mathbb{R}^n} \big|F(s, x+\tau-y+\mathbf{u}) - F(s, x-y+\mathbf{u})\big| \cdot \big|E(t_0,y)\big|\, dy\, ds.\end{aligned}$$

Integrating this estimate over Ω and using the Fubini theorem, we get $(x,\ \tau \in \mathbb{R}^n)$

$$\begin{aligned}
&\int_{\Omega} \big|u_{t_0}(x+\tau+\mathbf{u}) - u_{t_0}(x+\mathbf{u})\big|\, du \\
&\quad \leqslant \int_0^{t_0} \int_{\mathbb{R}^n} \Big[\int_{\Omega} \big|F(s,x+\tau-y+\mathbf{u}) - F(s,x-y+\mathbf{u})\big|\, d\mathbf{u}\Big] \cdot \big|E(t_0,y)\big|\, dy\, ds \\
&\quad \leqslant \varepsilon \int_0^{t_0} \int_{\mathbb{R}^n} \big|E(t_0,y)\big|\, dy\, ds;
\end{aligned}$$

see the corresponding definitions. It follows that the function $u_{t_0}(\cdot)$ is Stepanov $(\Omega, 1)$-almost periodic, as well.

6.3 Multi-dimensional Weyl almost periodic type functions and applications

This section aims to develop the basic theory of multi-dimensional Weyl almost periodic type functions in Lebesgue spaces with variable exponents. The introduced classes of functions seem to be not considered elsewhere even in the constant coefficient case (for the one-dimensional Weyl almost periodic type functions and their applications, we refer the reader to [4, 68, 139, 166, 171, 199, 328, 435, 503, 554, 631, 988], as well as the survey article [67] by J. Andres, A. M. Bersani, R. F. Grande, the pioneering papers by A. S. Kovanko [669–673] and the master thesis of J. Stryja [962]; concerning the multi-dimensional case, we would like to mention two recent papers [692] by D. Lenz, T. Spindeler, N. Strungaru and [951] by T. Spindeler, where the authors have analyzed the Stepanov and Weyl almost periodic functions on locally compact Abelian groups).

The organization and main ideas of this section can be briefly described as follows. In Definition 6.3.1–Definition 6.3.3 [Definition 6.3.5–Definition 6.3.7], we continue the analysis of one-dimensional case by introducing the classes $e-W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ and $e-W^{(p(\mathbf{u}),\phi,\mathbb{F})_i}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ [$e-W^{[p(\mathbf{u}),\phi,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ and $e-W^{[p(\mathbf{u}),\phi,\mathbb{F}]_i}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$] of Weyl almost periodic functions, where $i = 1, 2$. We further analyze these classes in Subsection 6.3. The main result of this subsection is Theorem 6.3.10 (see also Theorem 6.3.11), in which we investigate the convolution invariance of space $(e-)W^{(p_1(\mathbf{u}),\phi,\mathbb{F}_1)}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n\times X : Y)$; this is a crucial result for our applications to the multi-dimensional heat equation. With the exception of this result, almost all other structural results of ours are given in Subsection 6.3.1, in which we investigate the usual concept of (equi-)Weyl p-almost periodicity and the corresponding class of functions $(e-)W^p_{\mathrm{ap},\Lambda',\mathcal{B}}(\Lambda\times X : Y)$, with the constant exponent $p(\mathbf{u}) \equiv p \in [1,\infty)$. We investigate the Weyl p-distance, the Weyl p-boundedness, the Weyl p-normality and Weyl approximations by trigonometric polynomials. The main results of this subsection are Theorem 6.3.19, Proposition 6.3.20–Proposition 6.3.21, Proposition 6.3.29 and Proposition 6.3.31. We analyze the basic results about the existence of Bohr–Fourier coefficients for multi-dimensional Weyl almost periodic functions and present some applications of our theoretical results to the abstract Volterra integro-differential equations in Banach spaces. We present several useful conclusions, remarks and intriguing topics at the end of section, proposing also some open problems (special thanks go to Prof. Kamal Khalil, who proposed the use of kernel $K(t,s,\cdot,\cdot)$ in the third point of part devoted to the applications).

As before, in this section, we will always assume that $\mathcal{B}$ is a non-empty collection of certain subsets of X such that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. In the first concept, we assume that the following condition holds:

(WM1) $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Lambda)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0,\infty) \to [0,\infty)$ and $\mathbb{F} : (0,\infty) \times \Lambda \to (0,\infty)$.

We introduce the following classes of multi-dimensional Weyl almost periodic functions (the notion can be further generalized following the approach obeyed in Definition 6.3.28; all established results can be slightly generalized in this framework).

Definition 6.3.1.
(i) By $e-W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t})\phi\big(\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_Y\big)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \varepsilon. \tag{6.75}$$

(ii) By $W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\limsup_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t})\phi\big(\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_Y\big)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \varepsilon.$$

Definition 6.3.2.
(i) By $e-W^{(p(\mathbf{u}),\phi,\mathbb{F})_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t})\phi\big(\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)}\big) < \varepsilon.$$

(ii) By $W^{(p(\mathbf{u}),\phi,\mathbb{F})_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\limsup_{l\to+\infty} \sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t})\phi\big(\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)}\big) < \varepsilon.$$

Definition 6.3.3.

(i) By $e-W^{(p(\mathbf{u}),\phi,\mathbb{F})_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} \phi\big(\mathbb{F}(l,\mathbf{t})\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)}\big) < \varepsilon.$$

(ii) By $W^{(p(\mathbf{u}),\phi,\mathbb{F})_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\limsup_{l\to+\infty} \sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} \phi\big(\mathbb{F}(l,\mathbf{t})\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)}\big) < \varepsilon.$$

In some cases, it is extremely important that the function $\mathbb{F}(l,\mathbf{t})$ depends not only on $l > 0$ but also on $\mathbf{t} \in \Lambda$. We will illustrate this fact by considering the second-order partial differential equation $\Delta u = -f$, where $f \in C^2(\mathbb{R}^3)$ has a compact support. Then it is well known that the Newtonian potential of $f(\cdot)$, defined by

$$u(x) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(x-y)}{|y|}\,dy, \quad x \in \mathbb{R}^3,$$

is a unique function belonging to the class $C^2(\mathbb{R}^3)$, vanishing at infinity and satisfying $\Delta u = -f$; see e. g. [890, Theorem 3.9, pp. 126–127]. For simplicity, suppose that $p = p_1 = 1$, $\Omega = [0,1]^n$, $\Lambda' \subseteq \Lambda = \mathbb{R}^3$ and

$$\sup_{l>0;\mathbf{t}\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\mathbb{F}_1(l,\mathbf{t})}{|y|\cdot\mathbb{F}(l,\mathbf{t}-y)}\,dy < +\infty. \tag{6.76}$$

Then we have the following.

Example 6.3.4. Suppose that $f \in (e-)W^{1,x,\mathbb{F}}_{[0,1]^n,\Lambda'}(\mathbb{R}^3 : \mathbb{C})$ and (6.76) holds. Then $u \in (e-)W^{1,x,\mathbb{F}_1}_{[0,1]^n,\Lambda'}(\mathbb{R}^3 : \mathbb{C})$. Towards this end, suppose that $l > 0$ and $\mathbf{t} \in \mathbb{R}^3$ are arbitrary; consider the class $e - W^{1,x,\mathbb{F}}_{[0,1]^n,\Lambda'}(\mathbb{R}^3 : \mathbb{C})$ for brevity. Let a point $\tau \in \mathbb{R}^3$ satisfy (6.75). Using the Fubini theorem and (6.76), we have

$$\begin{aligned}\|u(\cdot+\tau) - u(\cdot)\|_{L^1(\mathbf{t}+l\Omega)} &\leqslant \frac{1}{4\pi}\int_{\mathbf{t}+l\Omega}\int_{\mathbb{R}^3} \frac{|f(x+\tau-y) - f(x-y)|}{|y|}\,dy\,dx \\ &\leqslant \frac{1}{4\pi}\int_{\mathbb{R}^3}\Big[\int_{\mathbf{t}+l\Omega} |f(x+\tau-y) - f(x-y)|\,dx\Big]\frac{dy}{|y|}\end{aligned}$$

$$= \frac{1}{4\pi}\int_{\mathbb{R}^3}\left[\int_{\mathbf{t}-y+l\Omega}\left|f(x+\tau)-f(x)\right|dx\right]\frac{dy}{|y|}$$
$$\leqslant \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\varepsilon\cdot dy}{|y|\cdot\mathbb{F}(l,\mathbf{t}-y)} \leqslant \frac{\varepsilon}{\mathbb{F}_1(l,\mathbf{t})}.$$

This simply implies the required (note that the function $y \mapsto |y|^{-1}$, $y \in \mathbb{R}^3$ does not belong to the class $L^1(\mathbb{R}^3)$ so that the results on convolution invariance of multi-dimensional Weyl almost periodicity cannot be applied here).

We can similarly analyze the two-dimensional analogue of this example by considering the logarithmic potential of $f(\cdot)$, given by

$$u(x) := \frac{(-1)}{2\pi}\int_{\mathbb{R}^2}\ln(|y|)\cdot f(x-y)\,dy, \quad x \in \mathbb{R}^2.$$

In this case, we only need to replace condition (6.76) by

$$\sup_{l>0;\mathbf{t}\in\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\ln(|y|)\cdot\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,\mathbf{t}-y)}\,dy < +\infty;$$

see also [890, Remark 3.7, p. 128].

It will be very complicated to reconsider here many other formulas from the classical theory of partial differential equations which can be employed for our purposes.

In the second concept, we aim to ensure the translation invariance of multi-dimensional Weyl almost periodic functions. We will assume now that the following condition holds:

(WM2) $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Omega)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0,\infty) \to [0,\infty)$ and $\mathbb{F} : (0,\infty)\times\Lambda \to (0,\infty)$.

We introduce the following classes of functions.

Definition 6.3.5.
(i) By $e-W^{[p(\mathbf{u}),\phi,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L)\cap\Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} l^n\mathbb{F}(l,\mathbf{t})\phi\left(\left\|F(\mathbf{t}+\tau+l\mathbf{u};x)-F(\mathbf{t}+l\mathbf{u};x)\right\|_Y\right)_{L^{p(\mathbf{u})}(\Omega)} < \varepsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L)\cap\Lambda'$ such that

$$\limsup_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} l^n\mathbb{F}(l,\mathbf{t})\phi\left(\left\|F(\mathbf{t}+\tau+l\mathbf{u};x)-F(\mathbf{t}+l\mathbf{u};x)\right\|_Y\right)_{L^{p(\mathbf{u})}(\Omega:Y)} < \varepsilon.$$

Definition 6.3.6.

(i) By $e-W^{[p(\mathbf{u}),\phi,\mathbb{F}]_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\varepsilon>0$ and $B\in\mathcal{B}$, there exist two finite real numbers $l>0$ and $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} l^n\mathbb{F}(l,\mathbf{t})\phi\big(\|F(\mathbf{t}+\tau+l\mathbf{u};x)-F(\mathbf{t}+l\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big)<\varepsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F}]_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\varepsilon>0$ and $B\in\mathcal{B}$, there exists a finite real number $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that

$$\limsup_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} l^n\mathbb{F}(l,\mathbf{t})\phi\big(\|F(\mathbf{t}+\tau+\mathbf{u};x)-F(\mathbf{t}+\mathbf{u};x)\|_{L^{p(\mathbf{u})}(l\Omega:Y)}\big)<\varepsilon.$$

Definition 6.3.7.

(i) By $e-W^{[p(\mathbf{u}),\phi,\mathbb{F}]_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\varepsilon>0$ and $B\in\mathcal{B}$, there exist two finite real numbers $l>0$ and $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda}\phi\big(l^n\mathbb{F}(l,\mathbf{t})\|F(\mathbf{t}+\tau+l\mathbf{u};x)-F(\mathbf{t}+l\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big)<\varepsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F}]_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\varepsilon>0$ and $B\in\mathcal{B}$, there exists a finite real number $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that

$$\limsup_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda}\phi\big(l^n\mathbb{F}(l,\mathbf{t})\|F(\mathbf{t}+\tau+l\mathbf{u};x)-F(\mathbf{t}+l\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big)<\varepsilon.$$

It is clear that the two concepts are equivalent in the constant coefficient case $p(\mathbf{u})\equiv p\in[1,\infty)$. Furthermore, the notion introduced here generalizes the notion introduced earlier, provided that $\Lambda'=\Lambda=I$, $\Omega=[0,1]$ and I is equal to $[0,\infty)$ or $\mathbb{R}$. Let us also note that, if a function $F:\Lambda\times X\to Y$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda')$-almost periodic, then $F\in e-W^{[p(\mathbf{u}),x,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X:Y)$ for any function $\mathbb{F}(\cdot;\cdot)$ satisfying $\mathbb{F}(1,\mathbf{t})=1$ for all $\mathbf{t}\in\Lambda$. If $X=\{0\}$ and $\mathcal{B}=\{X\}$, then we omit the term "$\mathcal{B}$" from the notation.

We continue by providing two examples.

Example 6.3.8. It can be simply shown that for each compact set $K\subseteq\mathbb{R}^n$ with a positive Lebesgue measure and for each $p\in\mathcal{P}(\mathbb{R}^n)$ the function $F(\cdot):=\chi_K(\cdot)$ belongs to the space $e-W^{(p(\mathbf{u}),x,l^{-\sigma})}_{[0,1]^n,\mathbb{R}^n}(\mathbb{R}^n:\mathbb{C})$ if and only if $\sigma>0$.

Example 6.3.9. Let $p\in[1,\infty)$. In [962], it has been proved that the Heaviside function $f(t):=\chi_{[0,\infty)}(t)$, $t\in\mathbb{R}$ is both Weyl p-normal (i. e., Weyl $(\mathrm{R},\mathcal{B},p)$-normal with $\Lambda=\Lambda'=\mathbb{R}$, $X=\{0\}$, $\mathcal{B}=\{X\}$, $Y=\mathbb{C}$ and R being the collection of all sequences in $\mathbb{R}$; see Definition 6.3.16 below) and Weyl p-almost periodic as well as that $f(\cdot)$ is not equi-Weyl p-almost periodic.

Suppose now that $F(\mathbf{t}) := \chi_{[0,\infty)^n}(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ as well as that $\Lambda := \Lambda' := \mathbb{R}^n$ and $\phi(x) \equiv x$. Then, for every $\mathbf{t}$, $\tau \in \mathbb{R}^n$ and $l > 0$, we have

$$\begin{aligned}
&\int\limits_{\mathbf{t}+l\Omega} |F(\tau+\mathbf{u}) - F(\mathbf{u})|^p \, d\mathbf{u} \\
&\quad = \int\limits_{(\mathbf{t}+l\Omega)\smallsetminus[0,\infty)^n} \big|F(\tau+\mathbf{u})\big|^p \, d\mathbf{u} + \int\limits_{(\mathbf{t}+l\Omega)\cap[0,\infty)^n} |F(\mathbf{u})|^p \, d\mathbf{u} \\
&\quad = \int\limits_{\tau+[(\mathbf{t}+l\Omega)\smallsetminus[0,\infty)^n]} |F(\mathbf{u})|^p \, d\mathbf{u} + \int\limits_{\tau+[(\mathbf{t}+l\Omega)\cap[0,\infty)^n]} |F(\mathbf{u}) - 1|^p \, d\mathbf{u} \\
&\quad \leqslant m\big((\tau + [(\mathbf{t}+l\Omega) \smallsetminus [0,\infty)^n]) \cap [0,\infty)^n\big) + m\big((\tau + [(\mathbf{t}+l\Omega) \cap [0,\infty)^n]) \smallsetminus [0,\infty)^n\big).
\end{aligned}$$

If $l > |\tau|$, then it is not difficult to prove that the latter does not exceed $2^n l^{n-1}|\tau|$, which implies that $F \in W^{(p,x,l^{-\sigma})}_{[0,1]^n,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$ if $\sigma > (n-1)/p$; this is also the best constant for σ we can obtain here. On the other hand, there is no $\sigma > 0$ such that $F \in e{-}W^{(p,x,l^{-\sigma})}_{[0,1]^n,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$.

Denote by $\mathrm{A}_{X,Y}$ any of the above introduced classes of function spaces. Then we have the following:

(i) Suppose that $c \in \mathbb{C}$ and $F(\cdot;\cdot)$ belongs to $\mathrm{A}_{X,Y}$. Then $cF(\cdot;\cdot)$ belongs to $\mathrm{A}_{X,Y}$, provided that there exists a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying that $\phi(xy) \leqslant \varphi(y)\phi(x)$, x, $y \geqslant 0$.

(ii) Suppose that $F \in \mathrm{A}_{X,Y}$, $A \in L(Y,Z)$, $\phi(\cdot)$ is monotonically increasing function and there exists a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying that $\phi(xy) \leqslant \varphi(y)\phi(x)$, x, $y \geqslant 0$. Using Lemma 1.1.7(iii), Lemma 1.1.8 and a simple argumentation, it follows that $AF \in \mathrm{A}_{X,Y}$.

(iii) (a) Suppose that $c_2 \in \mathbb{C} \smallsetminus \{0\}$ and $F(\cdot;\cdot)$ belongs to $\mathrm{A}_{X,Y}$. Then $F(\cdot;c_2\cdot)$ and $F(\cdot;\cdot)$ belong to $\mathrm{A}_{X,Y}$, where $\mathcal{B}_{c_2} \equiv \{c_2^{-1}B : B \in \mathcal{B}\}$.

(b) Suppose that $c_1 \in \mathbb{C} \smallsetminus \{0\}$, $c_2 \in \mathbb{C} \smallsetminus \{0\}$, and $F(\cdot;\cdot)$ belongs to $\mathrm{A}_{X,Y}$. Define the function $F_{c_1,c_2} : \Lambda/c_1 \times X \to Y$ by $F_{c_1,c_2}(\mathbf{t},x) := F(c_1\mathbf{t};c_2x)$, $\mathbf{t} \in \Lambda/c_1$, $x \in X$. If we assume that $\phi(\cdot)$ is a monotonically increasing function and there exists a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying that $\phi(xy) \leqslant \varphi(y)\phi(x)$, x, $y \geqslant 0$, then $F \in (e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ $[F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)]$ implies $F_{c_1,c_2} \in (e-)W^{(p_{c_1}(\mathbf{u}),\phi,\mathbb{F}_{c_1})}_{\Omega/c_1,\Lambda'/c_1,\mathcal{B}_{c_2}}((\Lambda/c_1) \times X : Y)$ $[F_{c_1,c_2} \in (e-)W^{[p_{c_1}(\mathbf{u}),\phi,\mathbb{F}_{c_1}]}_{\Omega/c_1,\Lambda'/c_1,\mathcal{B}_{c_2}}((\Lambda/c_1) \times X : Y)]$, where $p_{c_1}(\mathbf{u}) := p(c_1\mathbf{u})$, $\mathbf{u} \in \Lambda/c_1$ and $\mathbb{F}_{c_1}(x,\mathbf{t}) := \mathbb{F}(x,c_1\mathbf{t})$, $x \geqslant 0$, $\mathbf{t} \in \Lambda/c_1$. For the class $e - W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, this follows from the inequality

$$\begin{aligned}
&\big[\phi\big(\|F_{c_1,c_2}(\tau+\mathbf{u};x) - F_{c_1,c_2}(\mathbf{u};x)\|\big)\big]_{L^{p_{c_1}(\mathbf{u})}(\mathbf{t}/c_1+l\Omega/c_1:Y)} \\
&\quad \leqslant \big(1+|c_1|^{-n}\big)\big[\phi\big(\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|\big)\big]_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)}, \quad \mathbf{t} \in \Lambda,
\end{aligned}$$

which follows from a trivial computation involving the chain rule, the elementary definitions and the inequality $\varphi_{p(\mathbf{u})}(c\cdot) \leqslant |c|\varphi_{p(\mathbf{u})}(\cdot)$ for $|c| \leqslant 1$.

Similarly, if we assume that there exists a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying that $\phi(xy) \leqslant \varphi(y)\phi(x)$, $x, y \geqslant 0$ and $F \in (e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})_i}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ $[F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F}]_i}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)]$ for $i = 1$ or $i = 2$, then $F_{c_1,c_2} \in (e-)W^{(p_{c_1}(\mathbf{u}),\phi,\mathbb{F}_{c_1})_i}_{\Omega/c_1,\Lambda'/c_1,\mathcal{B}_{c_2}}((\Lambda/c_1)\times X : Y)$ $[F_{c_1,c_2} \in (e-)W^{[p_{c_1}(\mathbf{u}),\phi,\mathbb{F}_{c_1}]_i}_{\Omega/c_1,\Lambda'/c_1,\mathcal{B}_{c_2}}((\Lambda/c_1)\times X : Y)]$.

(iv) The use of Jensen integral inequality in general measure spaces (see Lemma 3.1.1) may be useful to state some inclusions about the introduced classes of functions. The consideration is similar to that established in the one-dimensional case and therefore omitted.

Regarding the convolution invariance of spaces $(e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$ and $(e-)W^{[p(\mathbf{u}),\phi,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$, we will state the following results (the corresponding proofs are very similar to the proof already given in the one-dimensional case, and we will only present the main details of the proof for Theorem 6.3.10; the results on invariance of various kinds of (equi-)Weyl almost periodicity under the actions of convolution products are not simply applied in the multi-dimensional setting and we will not reconsider these results here).

Theorem 6.3.10. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x, y \geqslant 0$, $h \in L^1(\mathbb{R}^n)$, $\Omega = [0,1]^n$, $F \in (e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$, and for each $x \in X$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$. If $\mathbb{F}_1 : (0,\infty)\times\mathbb{R}^n \to (0,\infty)$, $p_1 \in \mathcal{P}(\mathbb{R}^n)$ and if, for every $\mathbf{t} \in \mathbb{R}^n$ and $l > 0$, there exists a sequence $(a_k)_{k\in l\mathbb{Z}^n}$ of positive real numbers such that $\sum_{k\in l\mathbb{Z}^n} a_k = 1$ and*

$$\int\limits_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Bigl(2\sum_{k\in l\mathbb{Z}^n} a_k l^{-n}\bigl[\varphi\bigl(a_k^{-1}l^n h(\mathbf{u}-\mathbf{v})\bigr)\bigr]_{L^{q(\mathbf{v})}(\mathbf{u}-k+l\Omega)}\mathbb{F}_1(l,\mathbf{t})\bigl[\mathbb{F}(l,\mathbf{u}-k)\bigr]^{-1}\Bigr)\,d\mathbf{u} \leqslant 1, \tag{6.77}$$

then $h * F \in (e-)W^{(p_1(\mathbf{u}),\phi,\mathbb{F}_1)}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$.

Proof. Since $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$, $x \in X$, it is clear that the value $(h * F)(\mathbf{t};x)$ is well defined for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Furthermore, since we have assumed that the function $\phi(\cdot)$ is monotonically increasing, we have ($\mathbf{t} \in \mathbb{R}^n$, $l > 0$; $x \in X$ fixed):

$$\begin{aligned}
&\phi\bigl(\|(h*F)(\tau+\mathbf{u};x) - (h*F)(\mathbf{u};x)\|_Y\bigr)_{L^{p_1(\mathbf{u})}(\mathbf{t}+l\Omega)}\\
&= \phi\Bigl(\Bigl\|\int\limits_{\mathbb{R}^n} h(\mathbf{s})\bigl[F(\tau+\mathbf{u}-\mathbf{s};x) - F(\mathbf{u}-\mathbf{s};x)\bigr]d\mathbf{s}\Bigr\|_Y\Bigr)_{L^{p_1(\mathbf{u})}(\mathbf{t}+l\Omega)}\\
&\leqslant \phi\Bigl(\int\limits_{\mathbb{R}^n} |h(\mathbf{s})|\cdot\|F(\tau+\mathbf{u}-\mathbf{s};x) - F(\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s}\Bigr)_{L^{p_1(\mathbf{u})}(\mathbf{t}+l\Omega)}\\
&= \inf\Bigl\{\lambda > 0 : \int\limits_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Bigl(\frac{\phi(\int_{\mathbb{R}^n} |h(\mathbf{s})|\cdot\|F(\tau+\mathbf{u}-\mathbf{s};x) - F(\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s})}{\lambda}\Bigr)\,d\mathbf{u} \leqslant 1\Bigr\}.
\end{aligned}$$

But, since we have assumed that $\phi(\cdot)$ is convex and $\sum_{k\in l\mathbb{Z}^n} a_k = 1$, we have

$$\phi\Big(\sum_{k\in l\mathbb{Z}^n} a_k x_k\Big) \leqslant \sum_{k\in l\mathbb{Z}^n} a_k \phi(x_k), \tag{6.78}$$

for any sequence (x_k) of non-negative real numbers. Using (6.78), the fact that the function $\varphi_{p_1(\mathbf{u})}(\cdot)$ is monotonically increasing, the above computation, and the Jensen integral inequality and the Hölder inequality (see Lemma 1.1.7(i)), we get

$$\begin{aligned}
&\int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Big(\frac{\phi(\int_{\mathbb{R}^n} |h(\mathbf{s})| \cdot \|F(\tau+\mathbf{u}-\mathbf{s};x)-F(\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s})}{\lambda}\Big)\,d\mathbf{u} \\
&\leqslant \int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Big(\frac{\sum_{k\in l\mathbb{Z}^n} a_k \phi(\int_{k-l\Omega} a_k^{-1}|h(\mathbf{s})| \cdot \|F(\tau+\mathbf{u}-\mathbf{s};x)-F(\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s})}{\lambda}\Big)\,d\mathbf{u} \\
&\leqslant \int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Big(\frac{\sum_{k\in l\mathbb{Z}^n} a_k l^{-n} \int_{k-l\Omega} \phi(a_k^{-1} l^n|h(\mathbf{s})| \cdot \|F(\tau+\mathbf{u}-\mathbf{s};x)-F(\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s})}{\lambda}\Big)\,d\mathbf{u} \\
&= \int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Big(\frac{\sum_{k\in l\mathbb{Z}^n} a_k l^{-n} \int_{k-l\Omega} \phi(a_k^{-1} l^n|h(\mathbf{u}-\mathbf{v})| \cdot \|F(\tau+\mathbf{v};x)-F(\mathbf{v};x)\|_Y) d\mathbf{v}}{\lambda}\Big)\,d\mathbf{u} \\
&\leqslant \int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Big(\frac{\sum_{k\in l\mathbb{Z}^n} a_k l^{-n} \int_{\mathbf{u}-k+l\Omega} \varphi(a_k^{-1} l^n|h(\mathbf{u}-\mathbf{v})|)\phi(\|F(\tau+\mathbf{v};x)-F(\mathbf{v};x)\|_Y) d\mathbf{v}}{\lambda}\Big)\,d\mathbf{u} \\
&\leqslant \int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Big(\frac{\sum_{k\in l\mathbb{Z}^n} 2a_k l^{-n} [\varphi(a_k^{-1} l^n h(\mathbf{u}-\mathbf{v}))]_{L^{q(\mathbf{v})}(\mathbf{u}-k+l\Omega)}}{\lambda} \\
&\quad \times \big[\phi\big(\big\|F(\tau+\mathbf{v};x)-F(\mathbf{v};x)\big\|_Y\big)\big]_{L^{p(\mathbf{v})}(\mathbf{u}-k+l\Omega)}\Big)\,d\mathbf{u} \\
&\leqslant \int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Big(\frac{\sum_{k\in l\mathbb{Z}^n} 2a_k l^{-n} [\varphi(a_k^{-1} l^n h(\mathbf{u}-\mathbf{v}))]_{L^{q(\mathbf{v})}(\mathbf{u}-k+l\Omega)}}{\lambda \cdot \mathbb{F}(l,\mathbf{u}-k)}\Big)\,d\mathbf{u}.
\end{aligned}$$

The use of (6.77) simply completes the proof. □

Theorem 6.3.11. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x, y \geqslant 0$, $h \in L^1(\mathbb{R}^n)$, $\Omega = [0,1]^n$, $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$, and for each $x \in X$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$. If $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$, $p_1 \in \mathcal{P}(\mathbb{R}^n)$ and if, for every $\mathbf{t} \in \mathbb{R}^n$ and $l > 0$, there exists a sequence $(a_k)_{k\in l\mathbb{Z}^n}$ of positive real numbers such that $\sum_{k\in l\mathbb{Z}^n} a_k = 1$ and*

$$\int_{\Omega} \varphi_{p_1(\mathbf{u})}\Big(2\sum_{k\in l\mathbb{Z}^n} a_k l^{-n} [\varphi(a_k^{-1} l^n h(k-l\mathbf{v}))]_{L^{q(\mathbf{v})}(\Omega)} \mathbb{F}_1(l,\mathbf{t})[\mathbb{F}(l,\mathbf{t}+l\mathbf{u}-k)]^{-1}\Big)\,d\mathbf{u} \leqslant 1,$$

then $h * F \in (e-)W^{[p_1(\mathbf{u}),\phi,\mathbb{F}_1]}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$.

The interested reader may try to formulate the corresponding statements about the convolution invariance of Weyl almost periodicity for the remaining four classes of functions introduced.

Concerning the functions $\phi(\cdot)$ and $\mathbb{F}(\cdot,\cdot)$, the most important case is that one in which $\phi(x) \equiv x$, $\mathbb{F}(l,\mathbf{t}) \equiv m(l\Omega)^{-1}\|1\|_{L^{q(\mathbf{u})}(l\Omega)}$, where $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$, when we obtain the usual concept of (equi-)Weyl $p(\mathbf{u})$-almost periodicity; if this is the case, the spaces $(e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda',\mathcal{B}}$, $(e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})_1}_{\Omega,\Lambda',\mathcal{B}}$ and $(e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})_2}_{\Omega,\Lambda',\mathcal{B}}$, resp. the spaces $(e-)W^{[p(\mathbf{u}),\phi,\mathbb{F}]}_{\Omega,\Lambda',\mathcal{B}}$, $(e-)W^{[p(\mathbf{u}),\phi,\mathbb{F}]_1}_{\Omega,\Lambda',\mathcal{B}}$ and $(e-)W^{[p(\mathbf{u}),\phi,\mathbb{F}]_2}_{\Omega,\Lambda',\mathcal{B}}$, coincide. Furthermore, the use of Hölder inequality enables one to see that these spaces are contained in the corresponding spaces of functions with $p(\mathbf{u}) \equiv 1$.

6.3.1 The constant coefficient case

In this subsection, we will always assume that $\Omega = [0,1]^n$, Λ is a general non-empty subset of $\mathbb{R}^n$ satisfying $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$ and $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi(x) \equiv x$ and $p(\mathbf{t}) \equiv p \in [1,\infty)$, when the usual concept of (equi-)Weyl p-almost periodicity is obtained by plugging $\mathbb{F}(l,\mathbf{t}) \equiv l^{-n/p}$. The corresponding class of functions is denoted by $(e-)W^p_{\mathrm{ap},\Lambda',\mathcal{B}}(\Lambda \times X : Y)$.

Now we would like to present the following illustrative example.

Example 6.3.12 (see also [265, Example 2.15(i)]). Suppose that a complex-valued mapping $t \mapsto \int_0^t g_j(s)\,ds$, $t \in \mathbb{R}$ is essentially bounded and (equi-)Weyl p-almost periodic ($1 \leqslant j \leqslant n$). Define

$$F(t_1,\dots,t_{2n}) := \prod_{j=1}^{n}[g_j(t_{j+n}) - g_j(t_j)], \quad \text{where } t_j \in \mathbb{R} \text{ for } 1 \leqslant j \leqslant 2n,$$

and $\Lambda' := \{(\tau,\tau) : \tau \in \mathbb{R}^n\}$. Then the argumentation from [265, Example 2.13(ii)] shows that there exists a finite constant $M > 0$ such that

$$\begin{aligned}
&\|F(t_1+\tau_1,\dots,t_{2n}+\tau_{2n}) - F(t_1,\dots,t_{2n})\|_Y \\
&\quad \leqslant M\{|g_1(t_{n+1}+\tau_1) - g_1(t_{n+1})| + |g_1(t_1+\tau_1) - g_1(t_1)| + \cdots \\
&\qquad + |g_n(t_{2n}+\tau_n) - g_n(t_{2n})| + |g_n(t_n+\tau_n) - g_n(t_n)|\},
\end{aligned}$$

for any $(t_1,\dots,t_{2n}) \in \mathbb{R}^{2n}$ and $(\tau_1,\dots,\tau_{2n}) \in \Lambda'$. Using the corresponding definitions, the Fubini theorem and an elementary argumentation, it follows that the function $F(\cdot)$ belongs to the class $(e-)W^p_{\mathrm{ap},\Lambda'}(\mathbb{R}^{2n} : Y)$. Furthermore, in the case of consideration of equi-Weyl p-almost periodicity, when any direct product of finite number of equi-Weyl p-almost periodic functions is again equi-Weyl p-almost periodic, we can show that the function $F(\cdot)$ belongs to the class $e - W^p_{\mathrm{ap},\Lambda''}(\mathbb{R}^{2n} : Y)$, where $\Lambda'' := \{(a,a,\dots,a) \in \mathbb{R}^{2n} : a \in \mathbb{R}\}$.

Now we will say a few words about the Weyl p-distance and the Weyl p-boundedness. Suppose that $F : \Lambda \times X \to Y$ and $G : \Lambda \times X \to Y$ are two functions satisfying that $F(\mathbf{t} + \cdot; x) - G(\mathbf{t} + \cdot; x) \in L^p(l\Omega : Y)$ for all $\mathbf{t} \in \Lambda$, $x \in X$ and $l > 0$. The Stepanov distance $D^p_{S_\Omega}(F(\cdot; x), G(\cdot; x))$ of functions $F(\cdot; x)$ and $G(\cdot; x)$ is defined by

$$D^p_{S_{l\Omega}}\big(F(\cdot; x), G(\cdot; x)\big) := \sup_{\mathbf{t}\in\Lambda}\big[l^{-(n/p)}\big\|F(\mathbf{t} + \cdot; x) - G(\mathbf{t} + \cdot; x)\big\|_{L^p(l\Omega:Y)}\big],$$

for any $x \in X$ and $l > 0$. Set

$$D^p_{S_{l\Omega},B}(F, G) := \sup_{x\in B} D^p_{S_{l\Omega}}\big(F(\cdot; x), G(\cdot; x)\big) \quad (l > 0,\ B \in \mathcal{B}).$$

It is clear that the assumptions $\tau \in \mathbb{R}^n$ and $\tau + \Lambda \subseteq \Lambda$, resp. $\tau + \Lambda = \Lambda$, imply

$$D^p_{S_{l\Omega},B}\big(F(\cdot + \tau; \cdot), G(\cdot + \tau; \cdot)\big) \leqslant D^p_{S_{l\Omega},B}(F, G), \quad l > 0,\ B \in \mathcal{B}, \tag{6.79}$$

resp.

$$D^p_{S_{l\Omega},B}\big(F(\cdot + \tau; \cdot), G(\cdot + \tau; \cdot)\big) = D^p_{S_{l\Omega},B}(F, G), \quad l > 0,\ B \in \mathcal{B}. \tag{6.80}$$

Arguing as before, we may conclude to the following:

1. $$D^p_{S_{l_1\Omega},B}(F, G) \leqslant \left[\frac{l_2}{l_1}\right]^{n/p} \cdot D^p_{S_{l_2\Omega},B}(F, G),$$

   provided that $l_2 > l_1 > 0$ and $B \in \mathcal{B}$.
2. If $l_2 > l_1 > 0$, $l_2 = kl_1 + \theta l_1$ for some $k \in \mathbb{N}$ and $\theta \in [0, 1)$, then

   $$D^p_{S_{l_2\Omega},B}(F, G) \leqslant \left(\frac{k + 1}{k}\right)^{n/p} \cdot D^p_{S_{l_1\Omega},B}(F, G),$$

   provided that $B \in \mathcal{B}$.

Hence, [1.–2.] imply that for each $B \in \mathcal{B}$ we have

$$\limsup_{l\to\infty} D^p_{S_{l\Omega},B}(F, G) \leqslant D^p_{S_{l_1\Omega},B}(F, G), \quad l_1 > 0;$$

performing the limit inferior as $l_1 \to \infty$, we get

$$\limsup_{l\to\infty} D^p_{S_{l\Omega},B}(F, G) \leqslant \liminf_{l\to\infty} D^p_{S_{l\Omega},B}(F, G).$$

Hence, the limit

$$D^p_{W,B}(F, G) := \lim_{l\to\infty} D^p_{S_{l\Omega},B}(F, G)$$

exists and for each $l > 0$ we have

$$D^p_{W,B}(F,G) \leqslant D^p_{S_{l\Omega},B}(F,G), \quad B \in \mathcal{B}. \tag{6.81}$$

We call this limit the Weyl p-distance of functions $F(\cdot)$ and $G(\cdot)$ on B; the Weyl p-norm of the function $F(\cdot)$ on B, denoted by $\|F\|^p_{W,B}$, is defined by $\|F\|^p_{W,B} := D^p_{W,B}(F,0)$. Moreover, if $X \in \mathcal{B}$, then the Weyl p-norm $\|F\|^p_{W,X}$ of $F(\cdot)$ on X is also said to be the Weyl p-norm of the function $F(\cdot)$ and it is denoted by $\|F\|^p_W$.

Due to (6.79)–(6.80), we find that the assumptions $\tau \in \mathbb{R}^n$ and $\tau + \Lambda \subseteq \Lambda$, resp. $\tau + \Lambda = \Lambda$, imply

$$D^p_{W,B}\big(F(\cdot+\tau;\cdot), G(\cdot+\tau;\cdot)\big) \leqslant D^p_{W,B}(F,G), \quad B \in \mathcal{B},$$

resp.

$$D^p_{W,B}\big(F(\cdot+\tau;\cdot), G(\cdot+\tau;\cdot)\big) = D^p_{W,B}(F,G), \quad B \in \mathcal{B}.$$

We will occasionally use the following condition:

(L) The function $F : \Lambda \times X \to Y$ satisfies $\|F(\mathbf{t}+\cdot;x)\|_Y \in L^p(l\Omega)$ for all $\mathbf{t} \in \Lambda$, $x \in X$ and $l > 0$.

Definition 6.3.13. Suppose that (L) holds. Then we say that $F(\cdot;\cdot)$ is Weyl p-bounded on $\mathcal{B}$ if and only if for each $B \in \mathcal{B}$ we have $\|F\|^p_{W,B} < \infty$; moreover, if $X \in \mathcal{B}$, then we say that $F(\cdot;\cdot)$ is Weyl p-bounded.

As is well known, the space of Weyl p-bounded functions is not complete with respect to the Weyl norm $\|\cdot\|^p_W$ in the case that $X \in \mathcal{B}$. Furthermore, if (L) holds, then we set $\mathrm{B}^p_{W,B} := \{F : \Lambda \times X \to Y;\ \|F\|^p_{W,B} < +\infty\}$ ($B \in \mathcal{B}$). Let us recall that the terms "Weyl p-distance" and "Weyl p-norm" are a little bit incorrect because $D^p_{W,B}(\cdot,\cdot)$ is a pseudometric on $\mathrm{B}^p_{W,B}$, actually (for example, the function $F := \chi_{[0,1/2)}(\cdot)$ used before is a non-zero function and $\|F\|^p_W = 0$ for all $p \geqslant 1$).

The following result is well known in the one-dimensional framework.

Proposition 6.3.14. *Suppose that* (L) *holds. Then the function* $F(\cdot;\cdot)$ *is Weyl* p*-bounded on* $\mathcal{B}$ *if and only if* $F(\cdot;\cdot)$ *is Stepanov* p*-bounded on* $\mathcal{B}$.

Proof. Clearly, if $F(\cdot;\cdot)$ is Stepanov p-bounded on $\mathcal{B}$, then $F(\cdot;\cdot)$ is Weyl p-bounded on $\mathcal{B}$ due to (6.81). Suppose now that the function $F(\cdot;\cdot)$ is Weyl p-bounded on $\mathcal{B}$. Let the set $B \in \mathcal{B}$ be fixed. Then there exist two finite real constants $M > 0$ and $l \geqslant 1$ such that $D^p_{S_{l\Omega},B}(F,0) \leqslant M$, which implies that for each $\mathbf{t} \in \Lambda$ and $x \in B$ we have

$$\big\|F(\mathbf{t}+\cdot;x)\big\|_{L^p(\Omega:Y)} \leqslant \big\|F(\mathbf{t}+\cdot;x)\big\|_{L^p(l\Omega:Y)} \leqslant l^{n/p} D^p_{S_{l\Omega},B}(F,0) \leqslant l^{n/p} M.$$

This completes the proof. □

Under the previous assumptions, the quantity

$$D^p_{W,B,1}(F,G) := \sup_{x\in B} D^p_W\big(F(\cdot;x),G(\cdot;x)\big) = \sup_{x\in B}\lim_{l\to+\infty} D^p_{S_{l\Omega}}\big(F(\cdot;x),G(\cdot;x)\big)$$

also exists and we clearly have $D^p_{W,B,1}(F,G) \leqslant D^p_{W,B}(F,G)$. Finding some sufficient conditions ensuring that $D^p_{W,B,1}(F,G) \geqslant D^p_{W,B}(F,G)$ could be an interested problem; for simplicity, we will not consider the quantity $D^p_{W,B,1}(F,G)$ henceforth.

Suppose now that $F : \Lambda\times X \to Y$, $G : \Lambda\times X \to Y$ and $H : \Lambda\times X \to Y$ satisfy $F(\mathbf{t}+\cdot;x)-G(\mathbf{t}+\cdot;x) \in L^p(l\Omega : Y)$ and $G(\mathbf{t}+\cdot;x)-H(\mathbf{t}+\cdot;x) \in L^p(l\Omega : Y)$ for all $\mathbf{t}\in\Lambda$, $x\in X$ and $l>0$. Then

$$D^p_{S_{l\Omega},B}(F,G) \leqslant D^p_{S_{l\Omega},B}(F,H) + D^p_{S_{l\Omega},B}(H,G), \quad l>0,\ B\in\mathcal{B}$$

and therefore

$$D^p_{W,B}(F,G) \leqslant D^p_{W,B}(F,H) + D^p_{W,B}(H,G), \quad B\in\mathcal{B}. \tag{6.82}$$

Now we will prove the following extension of [696, Theorem 5.5.5, pp. 222–227] (cf. also [67, p. 150, l.-10; l.-5] and [696, Chapter 5, Section 9, pp. 242–247]).

Theorem 6.3.15. *Suppose that any of the functions $F_k : \Lambda\times X \to Y$ ($k\in\mathbb{N}$) and $F : \Lambda\times X \to Y$ satisfies condition* (L). *If for each set $B\in\mathcal{B}$ we have $\lim_{k\to+\infty}\|F_k-F\|^p_{W,B}=0$ and $F_k \in e-W^p_{\mathrm{ap},\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ for all $k\in\mathbb{N}$, then $F\in e-W^p_{\mathrm{ap},\Lambda',\mathcal{B}}(\Lambda\times X : Y)$.*

Proof. Let $\varepsilon>0$ and $B\in\mathcal{B}$ be fixed. Then there exists $K\in\mathbb{N}$ such that $\|F_K-F\|^p_{W,B}<\varepsilon/3$; hence, there exists $l_1>0$ such that

$$\sup_{\mathbf{t}\in\Lambda,x\in B}\big[l^{-n/p}\|F_K(\cdot;x)-F(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega:Y)}\big] < \varepsilon/3, \quad l\geqslant l_1. \tag{6.83}$$

On the other hand, since $F_K\in e-W^p_{\mathrm{ap},\Lambda,\Lambda'}(\Lambda\times X : Y)$, we have the existence of two real numbers $l_2>0$ and $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that

$$\sup_{\mathbf{t}\in\Lambda,x\in B}\big[l_2^{-n/p}\|F_K(\cdot+\tau;x)-F_K(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega:Y)}\big] < 2^{-n/p}\varepsilon/3. \tag{6.84}$$

Set $l:=\max(l_1,l_2)$, fix $\mathbf{t}\in\Lambda$ and $x\in B$. Then there exist an integer $k\in\mathbb{N}$ and a number $\theta\in[0,1)$ such that $l=kl_2+\theta l_2$. Due to (6.84), we have

$$\begin{aligned}
&\Big[l^{-n}\int_{\mathbf{t}+l\Omega}\|F_K(\mathbf{u}+\tau;x)-F_K(\mathbf{u};x)\|_Y^p\,d\mathbf{u}\Big]^{1/p}\\
&\leqslant \Big[(kl_2)^{-n}\int_{\mathbf{t}+(k+1)l_2\Omega}\|F_K(\mathbf{u}+\tau;x)-F_K(\mathbf{u};x)\|_Y^p\,d\mathbf{u}\Big]^{1/p}\\
&\leqslant \big[(kl_2)^{-n}2^{-n}(k+1)^n\varepsilon^p3^{-p}l_2^n\big]^{1/p} = 2^{-n/p}\frac{(k+1)^{n/p}}{k^{n/p}}\frac{\varepsilon}{3} \leqslant \varepsilon/3.
\end{aligned}$$

Using this estimate and (6.83), we get

$$
\begin{aligned}
& l^{-n/p} \| F(\cdot + \tau; x) - F(\cdot; x) \|_{L^p(\mathbf{t}+l\Omega:Y)} \\
& \quad \leqslant l^{-n/p} \big[\| F(\cdot + \tau; x) - F_K(\cdot + \tau; x) \|_{L^p(\mathbf{t}+l\Omega:Y)} \\
& \qquad + \| F_K(\cdot + \tau; x) - F_K(\cdot; x) \|_{L^p(\mathbf{t}+l\Omega:Y)} + \| F_K(\cdot; x) - F(\cdot; x) \|_{L^p(\mathbf{t}+l\Omega:Y)} \big] \\
& \quad \leqslant 3 \cdot \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}
$$

which completes the proof. □

Regarding the incompleteness of the space of equi-Weyl p-almost periodic functions with respect to the Weyl metric, we want only to recall that the sequence of partial sums of the series (therefore, the sequence of trigonometric polynomials)

$$
x \mapsto \sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{x}{k}, \quad x \in \mathbb{R},
$$

is a Cauchy sequence with respect to the Weyl metric W^2 but its sum, which is clearly an essentially bounded function, is not equi-Weyl 2-almost periodic; see, e. g., [696, p. 247].

Now we will investigate the Weyl p-normality and the Weyl approximations by trigonometric polynomials. We first introduce the following notion (see also [67, Definition 4.5]).

Definition 6.3.16. Suppose that (L) holds, R is a non-empty collection of sequences in $\mathbb{R}^n$ and the following holds:

$$
\text{if } \mathbf{t} \in \Lambda, \ \mathbf{b} \in \mathrm{R} \text{ and } m \in \mathbb{N}, \text{ then we have } \mathbf{t} + \mathbf{b}(m) \in \Lambda. \tag{6.85}
$$

Then we say that the function $F(\cdot;\cdot)$ is Weyl $(\mathrm{R}, \mathcal{B}, p)$-normal if and only if for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that $(F(\cdot + (b_{k_m}^1, \ldots, b_{k_m}^n); \cdot))_{m \in \mathbb{N}}$ is a Cauchy sequence with respect to the metric $D_{W,B}^p(\cdot,\cdot)$.

Remark 6.3.17. If R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$ satisfying certain conditions, then the notion of Weyl $(\mathrm{R}_X, \mathcal{B}, p)$-normality can be also introduced.

Now we are in a position to introduce the following generalization of the notion considered in [67, Definition 4.6].

Definition 6.3.18. Suppose that (L) holds. Then we say that the function $F(\cdot;\cdot)$ belongs to the space $e - \mathcal{B} - W^p(\Lambda \times X : Y)$ if and only if for every $B \in \mathcal{B}$ and for every $\varepsilon > 0$ there exist a real number $l_0 > 0$ and a trigonometric polynomial $P(\cdot;\cdot)$ such that

$$
\sup_{x \in B, \mathbf{t} \in \Lambda} \big[l^{-n/p} \| P(\mathbf{t} + \cdot; x) - F(\mathbf{t} + \cdot; x) \|_{L^p(l\Omega:Y)} \big] < \varepsilon, \quad l \geqslant l_0. \tag{6.86}
$$

If $X \in \mathcal{B}$, then we also say that $F(\cdot)$ belongs to the space $e - W^p(\Lambda \times X : Y)$.

In other words, if (L) holds, then $F \in e - \mathcal{B} - W^p(\Lambda \times X : Y)$ if and only if for every $B \in \mathcal{B}$ there exists a sequence of trigonometric polynomials $P_m(\cdot;\cdot)$ such that $\lim_{m\to+\infty} D^p_{W,B}(F, P_m) = 0$. Now we will state the following extension of [67, Theorem 4.12].

Theorem 6.3.19. *Suppose that* (L) *holds and* $F \in e - \mathcal{B} - W^p(\Lambda \times X : Y)$. *Let* R *be the collection of all sequences in* $\mathbb{R}^n$ *for which* (6.85) *holds, and let* $\mathcal{B}$ *be any collection of compact subsets of* X. *Then the function* $F(\cdot;\cdot)$ *is Weyl* $(\mathrm{R}, \mathcal{B}, p)$*-normal.*

Proof. Let $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$. Using [265, Theorem 2.17], for every $Q \in \mathbb{N}$, we can always find a sequence $((b^1_{k_{m;Q}}, \ldots, b^n_{k_{m;Q}}))_{m\in\mathbb{N}}$ and a function $F_Q : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{m\to+\infty} P_Q(\mathbf{t} + (b^1_{k_{m;Q}}, \ldots, b^n_{k_{m;Q}}); x) = F_Q(\mathbf{t}; x), \tag{6.87}$$

uniformly for $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$; furthermore, we may assume that the sequence $((b^1_{k_{m;Q}}, \ldots, b^n_{k_{m;Q}}))_{m\in\mathbb{N}}$ is a subsequence of all sequences $((b^1_{k_{m;Q'}}, \ldots, b^n_{k_{m;Q'}}))_{m\in\mathbb{N}}$ for $1 \leqslant Q' \leqslant Q$ and the initial sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n))$ as well as that $(k_{m;m})$ is a strictly increasing sequence of positive integers. Then a subsequence $(\mathbf{b}_{k_m} = (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}))$ of $(\mathbf{b}_k)$ satisfies the requirement that $(F(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); \cdot))_{m\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric $D^p_{W,B}(\cdot,\cdot)$. Indeed, there exists $s \in \mathbb{N}$ such that $D^p_{W,B}(P_s, F) < \varepsilon/3$ and we have, due to (6.82),

$$\begin{aligned}
&D^p_{W,B}(F(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); x), F(\cdot + (b^1_{k_{m';m'}}, b^2_{k_{m';m'}}, \ldots, b^n_{k_{m';m'}}); x))\\
&\quad \leqslant D^p_{W,B}(F(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); x), P_s(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); x))\\
&\qquad + D^p_{W,B}(P_s(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); x), P_s(\cdot + (b^1_{k_{m';m'}}, b^2_{k_{m';m'}}, \ldots, b^n_{k_{m';m'}}); x))\\
&\qquad + D^p_{W,B}(P_s(\cdot + (b^1_{k_{m';m'}}, b^2_{k_{m';m'}}, \ldots, b^n_{k_{m';m'}}); x), F(\cdot + (b^1_{k_{m';m'}}, b^2_{k_{m';m'}}, \ldots, b^n_{k_{m';m'}}); x))\\
&\quad \leqslant 2D^p_{W,B}(F, P_s)\\
&\qquad + D^p_{W,B}(P_s(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); x), P_s(\cdot + (b^1_{k_{m';m'}}, b^2_{k_{m';m'}}, \ldots, b^n_{k_{m';m'}}); x))\\
&\quad \leqslant 2\varepsilon/3 + D^p_{W,B}(P_s(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); x), P_s(\cdot + (b^1_{k_{m';m'}}, b^2_{k_{m';m'}}, \ldots, b^n_{k_{m';m'}}); x))\\
&\quad \leqslant 2\varepsilon/3\\
&\qquad + \sup_{y\in B, \cdot\in\Lambda} \big\| P_s(\cdot + (b^1_{k_{m;m}}, b^2_{k_{m;m}}, \ldots, b^n_{k_{m;m}}); y) - P_s(\cdot + (b^1_{k_{m';m'}}, b^2_{k_{m';m'}}, \ldots, b^n_{k_{m';m'}}); y)\big\|_Y,
\end{aligned}$$

for every $m,\ m' \in \mathbb{N}$ and $x \in B$. Since $((b^1_{k_{m;m}}, \ldots, b^n_{k_{m;m}}))_{m\in\mathbb{N}}$ is a subsequence of the sequence $((b^1_{k_{m;s}}, \ldots, b^n_{k_{m;s}}))_{m\in\mathbb{N}}$ for $s \leqslant m$, this simply implies the required statement by applying (6.87) with $Q = s$. □

Our next structural result generalizes [67, Theorem 4.7].

Proposition 6.3.20. *Suppose that* (L) *holds,* $\mathcal{B}$ *is any collection of bounded subsets of* X *and* $F \in e - \mathcal{B} - W^p(\Lambda \times X : Y)$. *Then* $F \in e - W^p_{\mathrm{ap},\Lambda,\mathcal{B}}(\Lambda \times X : Y)$.

Proof. Let a bounded set $B \in \mathcal{B}$ and a real number $\varepsilon > 0$ be given. By definition, there exist a real number $l_0 > 0$ and a trigonometric polynomial $P(\cdot;\cdot)$ such that (6.86) holds. Let

$$P(\mathbf{t};x) := \sum_{j=1}^{k} e^{i[\lambda_{1,j}t_1+\lambda_{2,j}t_2+\cdots+\lambda_{n,j}t_n]} c_j(x), \quad \mathbf{t} = (t_1,t_2,\ldots,t_n) \in \mathbb{R}^n,\ x \in X,$$

for some integer $k \in \mathbb{N}$. Since the function $c_j(\cdot)$ is continuous $(1 \leqslant j \leqslant k)$, there exists a finite real constant $M > 1$ such that

$$\sup_{x\in B}\sup_{1\leqslant j\leqslant k}\|c_j(x)\|_Y \leqslant M.$$

Since every trigonometric polynomial is almost periodic in $\mathbb{R}^n$ (cf. [265]), the existence of such a constant M and the Bochner criterion applied to the functions $e^{i[\lambda_{1,j}t_1+\lambda_{2,j}t_2+\cdots+\lambda_{n,j}t_n]}$ for $1 \leqslant j \leqslant k$ together imply the existence of a finite real number $L > 0$ such that for each point $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, L)$ which satisfies $\|P(\mathbf{t}+\tau;x) - P(\mathbf{t};x)\|_Y \leqslant (\varepsilon/3)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$. Suppose now that $\mathbf{t}_0 \in \Lambda$ and $\tau \in B(\mathbf{t}_0, L)$ is chosen as above. This yields

$$\begin{aligned}
&\|F(\tau+\cdot;x) - F(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega)} \\
&\quad\leqslant \|F(\tau+\cdot;x) - P(\tau+\cdot;x)\|_{L^p(\mathbf{t}+l\Omega)} \\
&\qquad + \|P(\tau+\cdot;x) - P(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega)} + \|P(\cdot;x) - F(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega)} \\
&\quad\leqslant \frac{2\varepsilon}{3}l^{n/p} + \|P(\tau+\cdot;x) - P(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega)} \leqslant \frac{2\varepsilon}{3}l^{n/p} + \frac{\varepsilon}{3}l^{n/p} = \varepsilon l^{n/p},
\end{aligned}$$

which completes the proof. □

Now we will extend the statement of [166, Lemma 2°, p. 83] in the following way.

Proposition 6.3.21. *Suppose that $F \in e - W^p_{\mathrm{ap},\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ and there exists a finite real number $M > 0$ such that, for every $\mathbf{t} \in \Lambda$, there exists $\mathbf{t}_0 \in \Lambda'$ such that $|\mathbf{t} + \mathbf{t}_0| \leqslant M$. Then for each $B \in \mathcal{B}$ there exist real numbers $l > 0$ and $M' > 0$ such that*

$$\sup_{\mathbf{t}\in\Lambda, x\in B}\left[l^{-(n/p)}\|F(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega:Y)}\right] \leqslant M'.$$

Proof. Let the set $B \in \mathcal{B}$ be fixed and let $\varepsilon = 1$. Then there exist real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that (6.75) holds. Fix now a point $\mathbf{t} \in \Lambda$. Due to our assumption, there exists $\mathbf{t}_0 \in \Lambda'$ such that $|\mathbf{t} + \mathbf{t}_0| \leqslant M$. Choose τ as above for this $\mathbf{t}_0$. Then $|\mathbf{t} + \tau| \leqslant |\mathbf{t} + \mathbf{t}_0| + |\mathbf{t}_0 - \tau| \leqslant M + L$, so that

$$\begin{aligned}
\|F(\cdot;x)\|_{L^p(\mathbf{t}+l\Omega:Y)} &\leqslant \|F(\cdot;x) - F(\tau+\cdot;x)\|_{L^p(\mathbf{t}+l\Omega:Y)} + \|F(\tau+\cdot;x)\|_{L^p(\mathbf{t}+l\Omega:Y)} \\
&\leqslant l^{n/p} + \|F(\cdot;x)\|_{L^p(\mathbf{t}+\tau+l\Omega:Y)} \leqslant l^{n/p} + \sup_{|\mathbf{v}|\leqslant M}\|F(\cdot;x)\|_{L^p(\mathbf{v}+l\Omega:Y)},
\end{aligned}$$

which simply completes the proof. □

Similarly we can prove the following extension of [166, Lemma 3°, p. 84].

Proposition 6.3.22. *Suppose that $F \in e - W^p_{ap,\Lambda'}(\Lambda : Y)$ and there exists a finite real number $M > 0$ such that, for every $\mathbf{t} \in \Lambda$, there exists $\mathbf{t}_0 \in \Lambda'$ such that $|\mathbf{t} + \mathbf{t}_0| \leqslant M$. Then $F(\cdot)$ is equi-W^p-uniformly continuous, i. e., for each $\varepsilon > 0$ there exist real numbers $l > 0$ and $\delta > 0$ such that, for every $\mathbf{v} \in \Lambda$ with $|\mathbf{v}| \leqslant \delta$, we have*

$$\sup_{\mathbf{t} \in \Lambda} \big[l^{-n/p} \| F(\mathbf{t} + \cdot + \mathbf{v}) - F(\mathbf{t} + \cdot) \|_{L^p(l\Omega:Y)} \big] < \varepsilon.$$

Now we are able to prove the following generalization of [166, Theorem 1°, p. 82].

Theorem 6.3.23. *Suppose that* (L) *holds with $X = \{0\}$ and $\mathcal{B} = \{X\}$. Then $F \in e - W^p_{ap,\mathbb{R}^n}(\mathbb{R}^n : Y)$ if and only if $F \in e - W^p(\mathbb{R}^n : Y)$.*

Proof. Clearly, if $F \in e - W^p(\mathbb{R}^n : Y)$, then $F \in e - W^p_{ap,\mathbb{R}^n}(\mathbb{R}^n : Y)$ due to Proposition 6.3.20. In order to prove that the assumption $F \in e - W^p_{ap,\mathbb{R}^n}(\mathbb{R}^n : Y)$ implies $F \in e - W^p(\mathbb{R}^n : Y)$, we basically follow the approach obeyed in the proof of [166, Theorem 1°, pp. 82–91] in the abstract framework developed by T. Spindeler [951] for the scalar-valued equi-Weyl p-almost periodic functions defined on the locally compact Abelian group $G = \mathbb{R}^n$, with a little abuse of notation used. First of all, we note that the sequence $(A_l \equiv l\Omega)_{l \in \mathbb{N}}$ is a van Hove sequence (see also Example 6.3.9 and the proof of Theorem 6.3.32 below) in the sense of [951, Definition 3.1] as well as that Proposition 6.3.22 implies that $F(\cdot)$ is equi-W^p-uniformly continuous, so that [951, Lemma 3.11] continues to hold in the vector-valued case. It can be simply shown that the construction of kernel $K : \mathbb{R}^n \to [0, \infty)$ holds for the vector-valued functions, so that [951, Lemma 3.12] continues to hold in the vector-valued case, as well. Furthermore, for a real number $\varepsilon > 0$ given in advance, the function

$$\Theta(\mathbf{t}) := \liminf_{l \to +\infty} l^{-n} \int_{l\Omega} F(\mathbf{t} + \mathbf{s}) K(\mathbf{s})\, d\mathbf{s} = \lim_{l \to +\infty} l^{-n} \int_{l\Omega} F(\mathbf{t} + \mathbf{s}) K(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

is almost periodic and satisfies $\lim_{m \to +\infty} D^p_{W,B}(F, \Theta) < \varepsilon$ by the same argumentation as in the proof of implication (2) $\Rightarrow$ (1) of [951, Proposition 3.13]. The remainder of the proof is trivial and therefore omitted. □

Now we would like to introduce the following notion.

Definition 6.3.24. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda + l\Omega \subseteq \Lambda$ and $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$. Then we say that Λ is admissible with respect to the (equi-)Weyl p-almost periodic extensions if and only if for any complex Banach space Y and for any function $F \in (e-)W^p_{ap,\Lambda}(\Lambda : Y)$ there exists a function $\tilde{F} \in (e-)W^p_{ap,\mathbb{R}^n}(\mathbb{R}^n : Y)$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$.

Now we are able to state the following extensions of [696, Theorem 5.5.3-Theorem 5.5.4, pp. 225–226], whose proofs are immediate consequences of Theorem 6.3.23, the fact that $e - W^p(\mathbb{R}^n : Y)$ is a vector space and the notion introduced in Definition 6.3.24.

Theorem 6.3.25. *Suppose that* $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda + l\Omega \subseteq \Lambda$ *and* $\Lambda + l\Omega \subseteq \Lambda$ *for all* $l > 0$. *If* Λ *is admissible with respect to the equi-Weyl p-almost periodic extensions, then* $e - W^p_{\mathrm{ap},\Lambda}(\Lambda : Y)$ *is a vector space.*

Theorem 6.3.26. *Suppose that* $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda + l\Omega \subseteq \Lambda$ *and* $\Lambda + l\Omega \subseteq \Lambda$ *for all* $l > 0$. *Suppose, further, that* $p, q, r \in [1, \infty)$, $1/p + 1/r = 1/q$, Λ *is admissible with respect to the equi-Weyl p-almost periodic extensions,* $f \in e - W^p_{\mathrm{ap},\Lambda}(\Lambda : \mathbb{C})$ *and* $F \in e - W^r_{\mathrm{ap},\Lambda}(\Lambda : Y)$. *Define* $F_1(\mathbf{t}) := f(\mathbf{t})F(\mathbf{t})$, $\mathbf{t} \in \Lambda$. *Then* $F_1 \in e - W^q_{\mathrm{ap},\Lambda}(\Lambda : Y)$.

Before proceeding further, let us note that Theorem 6.3.26 can be illustrated by many elaborate examples. For instance, we know that there exists a bounded scalar-valued infinitely differentiable Weyl p-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ for all $p \in [1, \infty)$ such that the regular distribution determined by this function is not almost periodic (cf. [124], [199, Main example IV, Appendix, pp. 131–133] and [631] for the notion and more details). Define now

$$F(t_1, t_2, \ldots, t_n) = f(t_1)f(t_2)\cdots f(t_n), \quad \mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n.$$

Then Theorem 6.3.26 inductively implies that $F \in e - W^p_{\mathrm{ap},\mathbb{R}^n}(\mathbb{R}^n : Y)$ for all $p \in [1, \infty)$ (even for all $p \in D_+(\Omega)$).

It is clear that the notion introduced in Definition 6.3.24 is not trivial as well as that some known results for the usual classes of multi-dimensional Bohr and Stepanov almost periodic type functions cannot be easily transferred to the corresponding Weyl classes. In connection with this problem, we would like to ask the following question, which seems to be not proposed elsewhere even in the one-dimensional setting.

Problem. Suppose that Λ is a convex polyhedral in $\mathbb{R}^n$, i. e., there exists a basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of $\mathbb{R}^n$ such that

$$\Lambda = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n : \alpha_i \geqslant 0 \text{ for all } i \in \mathbb{N}_n\}.$$

Is true that Λ is admissible with respect to the (equi-)Weyl p-almost periodic extensions?

In the remainder of this section, we assume that condition (L) holds. If $\tau \in \mathbb{R}^n$ satisfies $\tau + \Lambda \subseteq \Lambda$ and $F \in \mathrm{B}^p_{W,B}$ for all $B \in \mathcal{B}$, then $F(\cdot + \tau; \cdot) \in \mathrm{B}^p_{W,B}$ for all $B \in \mathcal{B}$. Therefore, the following notion is meaningful.

Definition 6.3.27. Suppose that $F : \Lambda \times X \to Y$ is such that (L) holds. If $\Lambda_0 \subseteq \{\tau \in \mathbb{R}^n : \tau + \Lambda \subseteq \Lambda\}$, then we say that the function $F(\cdot; \cdot)$ is $(\mathcal{B}, \Lambda_0)$-normal if and only if for each $B \in \mathcal{B}$ the set $\mathrm{F}_{\Lambda_0} \equiv \{F(\cdot + \tau; \cdot) : \tau \in \Lambda_0\}$ is totally bounded in the pseudometric space $(\mathrm{B}^p_{W,B}, D^p_{W,B})$, which means that for any $\varepsilon > 0$ and $B \in \mathcal{B}$ the set F_{Λ_0} admits a cover by finitely many open balls of radius ε in $(\mathrm{B}^p_{W,B}, D^p_{W,B})$.

Consider now the following condition:

(WM3) $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda'' \subseteq \mathbb{R}^n$, $\Omega = [0,1]^n$, $p(\mathbf{u}) \equiv p \in [1,\infty)$, $\Lambda'' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi(x) \equiv x$ and $\mathbb{F}(l,\mathbf{t}) \equiv l^{-n/p}$.

The following notion has an important role for our further investigations of the notion introduced in Definition 6.3.27.

Definition 6.3.28. Suppose that (WM3) holds.

(i) By $e-W^p_{\Omega,\Lambda',\Lambda'',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda''$ such that
$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t})\phi\big(\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_Y\big)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \varepsilon.$$

(ii) By $W^p_{\Omega,\Lambda',\Lambda'',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda''$ such that
$$\lim_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t})\phi\big(\|F(\tau+\mathbf{u};x) - F(\mathbf{u};x)\|_Y\big)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \varepsilon.$$

Now we are able to state the following result (see also [67, Corollary 4.24] and the proof of sufficiency in [67, Theorem 4.12]).

Proposition 6.3.29. *Suppose that $F : \Lambda\times X \to Y$ is such that* (L) *holds, $\Lambda_0 \subseteq \{\tau \in \mathbb{R}^n : \tau + \Lambda \subseteq \Lambda\}$, $F(\cdot;\cdot)$ is $(\mathcal{B},\Lambda_0)$-normal, $\tau + \Lambda = \Lambda$ for all $\tau \in \Lambda_0$, and condition* (WM3) *holds with $\Lambda' := -\Lambda_0$, $\Lambda'' := \Lambda_0 - \Lambda_0$. Then $F \in W^p_{\Omega,\Lambda',\Lambda'',\mathcal{B}}(\Lambda\times X : Y)$.*

Proof. Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be fixed. Due to the $(\mathcal{B},\Lambda_0)$-normality of the function $F(\cdot;\cdot)$, we find that there exist a positive integer $m \in \mathbb{N}$ and a finite subset $\{\tau_1, \tau_2, \ldots, \tau_m\}$ of Λ_0 such that for each $\mathbf{t}_0 = -\tau \in -\Lambda_0$ there exist $j \in \mathbb{N}_m$ and $l_0 > 0$ such that, for every $l \geqslant l_0$ and $x \in B$, we have
$$\sup_{\mathbf{t}\in\Lambda, x\in B}\big[l^{-n/p}\|F(\mathbf{t}+\tau+\cdot;x) - F(\mathbf{t}+\tau_j+\cdot;x)\|_{L^p(l\Omega:Y)}\big] < \varepsilon.$$
Substituting $T = \mathbf{t}+\tau$ and using the assumption that $\tau + \Lambda = \Lambda$ for all $\tau \in \Lambda_0$, the above implies
$$\sup_{\mathbf{t}\in\Lambda, x\in B}\big[l^{-n/p}\|F(\mathbf{t}+\cdot;x) - F\big(\mathbf{t}+(\tau_j-\tau)+\cdot;x\big)\|_{L^p(l\Omega:Y)}\big] < \varepsilon.$$
Set $L := \max\{|\tau_j| : j \in \mathbb{N}_m\}$. Then $\tau_j - \tau \in \Lambda_0 - \Lambda_0$ and $\tau_j - \tau \in B(\mathbf{t}_0, L)$, which simply implies the required. □

It is worth noting that Proposition 6.3.29 can be applied even in the case that the assumption $\Lambda = \Lambda_0 = \mathbb{R}^n$ is not satisfied. For example, we can take $\Lambda := \{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : x_i \geqslant 0 \text{ for all } i \in \mathbb{N}_{n-1}\}$ and $\Lambda_0 := \{(0, 0, \ldots, 0, x_n) : x_n \in \mathbb{R}\}$;

furthermore, the case in which $-\Lambda_0 \neq \Lambda_0 - \Lambda_0$ can also happen since we can take $\Lambda := \mathbb{R}^n$ and $\Lambda_0 := a + W$, where $a \neq 0$ and W is a non-trivial subspace of $\mathbb{R}^n$ (then $\Lambda_0 - \Lambda_0 = W \neq -\Lambda_0$).

Example 6.3.30 ([962]). Let $\Lambda = \Lambda' = \mathbb{R}$, $X = \{0\}$, $\mathcal{B} = \{X\}$, $Y = \mathbb{C}$ and R being the collection of all sequences in $\mathbb{R}$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := 0$ for $x \leqslant 0$, $f(x) := \sqrt{n/2}$ for $x \in (n-2, n-1]$ ($n \in 2\mathbb{N}$) and $f(x) := -\sqrt{n/2}$ for $x \in (n-1, n]$ ($n \in 2\mathbb{N}$). Then $f(\cdot)$ is Weyl 1-almost periodic, Weyl 1-unbounded, but neither equi-Weyl 1-almost periodic nor Weyl 1-normal, so that the converse of Proposition 6.3.29 does not hold, in general. Although may be interesting, we will not consider here the general case $p > 1$ as well as some more complicated relatives of Example 6.3.8-Example 6.3.9 with locally integrable functions $F : \mathbb{R}^n \to \mathbb{R}$ whose range is at most countable.

Therefore, one needs to impose some extra conditions ensuring that the inclusion $F \in W^p_{\Omega,-\Lambda_0,\Lambda_0-\Lambda_0,\mathcal{B}}(\Lambda \times X : Y)$ implies that $F(\cdot;\cdot)$ is $(\mathcal{B},\Lambda_0)$-normal. In the following result, the assumption $\Lambda = \Lambda_0 = \mathbb{R}^n$ is almost inevitable to be made (see also [669], [67, Theorem 4.22, Theorem 4.23] and the proof of necessity in [67, Theorem 4.12]; the compactness criteria for the sets in the spaces of (equi-)Weyl p-almost periodic functions have been analyzed in [671–673] with the help of Lusternik type theorems, we will not reconsider these results in the multi-dimensional framework).

Proposition 6.3.31. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is such that* (L) *holds, $\Lambda_0 = \mathbb{R}^n$ and $F \in W^p_{\Omega,\mathbb{R}^n,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$. If for each $\varepsilon > 0$ and $B \in \mathcal{B}$ there exists $\delta > 0$ such that $D^p_{W,B}(F(\cdot;\cdot), F(\cdot + \mathbf{v};\cdot)) < \varepsilon$ for every $\mathbf{v} \in \mathbb{R}^n$ with $|\mathbf{v}| \leqslant \delta$, then $F(\cdot;\cdot)$ is $(\mathcal{B},\mathbb{R}^n)$-normal.*

Proof. Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be given. Due to our assumption, we have the existence of a finite real number $l > 0$ such that, for every $\mathbf{t}_0 \in \mathbb{R}^n$, there exists $\eta \in B(\mathbf{t}_0, l)$ such that $D^p_{W,B}(F(\cdot;\cdot), F(\cdot + \eta;\cdot)) < \varepsilon/2$. Furthermore, there exists $\delta > 0$ such that $D^p_{W,B}(F(\cdot;\cdot), F(\cdot + \mathbf{v};\cdot)) < \varepsilon/2$ for every $\mathbf{v} \in \mathbb{R}^n$ with $|\mathbf{v}| \leqslant \delta$. Let $m \in \mathbb{N}$ be such that $m\delta > l$, and let S_δ denote the set consisting of all points of form $(a_1\delta, \dots, a_n\delta) \in B(0, m\delta)$, where $a_j \in \mathbb{Z}$ for all $j \in \mathbb{N}_n$. With the same notation as above, we have $-\mathbf{t}_0 + \eta \in B(0, l)$, and therefore, there exists $\zeta \in S_\delta$ such that $|\mathbf{v}| = |-\mathbf{t}_0 + \eta - \zeta| < \delta$. This implies $D^p_{W,B}(F(\cdot;\cdot), F(\cdot + [-\mathbf{t}_0 + \eta - \zeta];\cdot)) = D^p_{W,B}(F(\cdot + \zeta;\cdot), F(\cdot - \mathbf{t}_0 + \eta;\cdot)) < \varepsilon/2$. But then we have

$$\begin{aligned} &D^p_{W,B}\big(F(\cdot - \mathbf{t}_0;\cdot), F(\cdot + \zeta;\cdot)\big) \\ &\quad \leqslant D^p_{W,B}\big(F(\cdot + \zeta;\cdot), F(\cdot - \mathbf{t}_0 + \eta;\cdot)\big) + D^p_{W,B}\big(F(\cdot - \mathbf{t}_0 + \eta;\cdot), F(\cdot - \mathbf{t}_0;\cdot)\big) \leqslant 2 \cdot \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which completes the proof. □

In what follows, we analyze the existence of Bohr–Fourier coefficients for multi-dimensional Weyl almost periodic functions. First of all, we would like to emphasize that some relations presented in [67, Table 2, p. 56] seem to be stated incorrectly. The main mistake made is that the authors have interchanged at some places the class of equi-Weyl p-almost periodic functions and Weyl p-almost periodic functions, which can be simply justified by taking a closer look at the references quoted: in the research articles [171] and [199], as well as in the research monographs [166, 503] and its English translation published by Pergamon Press, Oxford in 1966, the class of Weyl p-almost periodic functions in the sense of Kovanko's approach has not been considered at all (the authors of [166, 171, 199, 503] have called an equi-Weyl p-almost periodic function simply a Weyl p-almost periodic functions therein). Therefore, there is no reasonable information which could tell us whether the class of Weyl p-almost periodic functions is contained in the class of Besicovitch p-almost periodic functions or not, as well as whether a Weyl p-almost periodic function $f : \mathbb{R} \to \mathbb{C}$ has the mean value ($1 \leqslant p < \infty$). As we will see in Theorem 8.3.8, for each finite exponent $p \geqslant 1$ there exists a real-valued Weyl p-almost periodic functions which is not Besicovitch p-almost periodic and which has no mean value.

Therefore, we must deal with the class of equi-Weyl p-almost periodic functions in order to ensure the existence of the mean value and the Bohr–Fourier coefficients for a function $F : \Lambda \times X \to Y$. The assumptions $X = \{0\}$ and $p = 1$ (due to the obvious embedding) are reasonable to be made, when we have the following.

Theorem 6.3.32. *Suppose that $\lambda \in \mathbb{R}^n$, $[0,\infty)^n = \Lambda' \subseteq \Lambda$, $\Omega = [0,1]^n$, $F : \Lambda \to Y$ is Stepanov $(\Omega, 1)$-bounded and satisfies the requirement that the function $\mathbf{t} \mapsto F_\lambda(\mathbf{t}) := e^{-i\langle \lambda, \mathbf{t}\rangle} F(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ belongs to the space $e - W^1_{ap,\Lambda}(\Lambda : Y)$. Then the Bohr–Fourier coefficient $P_\lambda(F)$ of $F(\cdot)$, defined by*

$$P_\lambda(F) := \lim_{T\to+\infty} \frac{1}{T^n} \int\limits_{\mathbf{s}+[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t}, \tag{6.88}$$

exists and does not depend on the choice of a tuple $\mathbf{s} \in [0,\infty)^n$. Moreover, for every $\varepsilon > 0$, there exists a real number $T_0(\varepsilon) > 0$ such that, for every $T \geqslant T_0(\varepsilon)$ and $\mathbf{s} \in [0,\infty)^n$, we have

$$\Bigg\| \frac{1}{T^n} \int\limits_{[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} - \frac{1}{T^n} \int\limits_{\mathbf{s}+[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} \Bigg\|_Y < \varepsilon. \tag{6.89}$$

Proof. We slightly modify the arguments contained in the proof of corresponding statement given in the one-dimensional case (see, e. g., [696, Theorem 1.3.1–Theorem 1.3.2, pp. 32–35]). Fix the numbers $\varepsilon > 0$ and $\lambda \in \mathbb{R}^n$. We know that there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in [0,\infty)^n$ there exists $\tau \in B(\mathbf{t}_0, L) \cap [0,\infty)^n$ such that

$$\sup_{\mathbf{t}\in\Lambda} \|F(\tau + \cdot) - F(\cdot)\|_{L^1(\mathbf{t}+l\Omega:Y)} < \varepsilon \cdot l^n. \tag{6.90}$$

Let $T > l$ be an arbitrary real number and let $k \in \mathbb{N}$. Denote by $A_{T,k} = \{\mathbf{s}_1, \ldots, \mathbf{s}_{k^n}\}$ the collection of all points $\mathbf{s} \in T \cdot \mathbb{N}_0^n$ such that $\mathbf{s} + [0,T]^n \subseteq [0, kT]^n$. Furthermore, let $B_{T,k} = \{\tau_1, \ldots, \tau_{k^n}\}$ be a collection of points in $[0,\infty)^n$ such that $|\tau_k - \tau_j| \leqslant L$ for all $j \in \mathbb{N}_{k^n}$ as well as that (6.90) holds with the number τ replaced therein with the number τ_j ($j \in \mathbb{N}_{k^n}$). Due to the computation following Eq. (6.84), we find that (6.90) implies $\sup_{\mathbf{t}\in\Lambda} \|F(\tau+\cdot)-F(\cdot)\|_{L^1(\mathbf{t}+T\Omega:Y)} < \varepsilon \cdot 2^n T^n$; in particular,

$$\|F(\tau+\cdot)-F(\cdot)\|_{L^1(T\Omega:Y)} < \varepsilon \cdot 2^n T^n. \tag{6.91}$$

Keeping in mind (6.91), we get

$$\begin{aligned}
&\left\| \frac{1}{T^n} \int\limits_{[0,T]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} - \frac{1}{(kT)^n} \int\limits_{[0,kT]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} \right\|_Y \\
&\leqslant \frac{\sum_{j=1}^{k^n} \| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} - \frac{1}{T^n} \int_{\mathbf{s}_j+[0,T]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} \|_Y}{k^n} \\
&= \frac{\sum_{j=1}^{k^n} \| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} - \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{s}_j+\mathbf{t})\,d\mathbf{t} \|_Y}{k^n} \\
&\leqslant \frac{\sum_{j=1}^{k^n} \| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} - \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\tau_j+\mathbf{t})\,d\mathbf{t} \|_Y}{k^n} \\
&\quad + \frac{\sum_{j=1}^{k^n} \| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\tau_j+\mathbf{t})\,d\mathbf{t} - \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{s}_j+\mathbf{t})\,d\mathbf{t} \|_Y}{k^n} \\
&\leqslant \varepsilon 2^n + \frac{\sum_{j=1}^{k^n} \| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\tau_j+\mathbf{t})\,d\mathbf{t} - \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{s}_j+\mathbf{t})\,d\mathbf{t} \|_Y}{k^n} \\
&= \varepsilon 2^n + \frac{\sum_{j=1}^{k^n} \| \frac{1}{T^n} \int_{(\tau_j+[0,T]^n)\smallsetminus(\mathbf{s}_j+[0,T]^n)} F_\lambda(\mathbf{t})\,d\mathbf{t} \|_Y}{k^n}.
\end{aligned}$$

Since $|\mathbf{s}_j - \tau_j| \leqslant L$ for all $j \in \mathbb{N}_{k^n}$, an elementary geometrical argument shows that there exists a finite real constant $c_n \in \mathbb{N}$ such that the set $(\tau_j + [0,T]^n) \smallsetminus (\mathbf{s}_j + [0,T]^n)$ can be covered by at most $\lceil LT^{n-1} \rceil$ translations of the cell $[0,1]^n$, so that the Stepanov $(\Omega,1)$-boundedness of $F(\cdot)$ implies that there exists a finite real number $T(\varepsilon) > 0$ such that

$$\begin{aligned}
&\left\| \frac{1}{T^n} \int\limits_{[0,T]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} - \frac{1}{(kT)^n} \int\limits_{[0,kT]^n} F_\lambda(\mathbf{t})\,d\mathbf{t} \right\|_Y \\
&\quad \leqslant \varepsilon 2^n + \|F\|_{S^{\Omega,1}} \frac{\lceil LT^{n-1} \rceil}{T} \leqslant \varepsilon(2^n+1), \quad T \geqslant T(\varepsilon).
\end{aligned} \tag{6.92}$$

After that, we can repeat verbatim the argumentation contained in the proof of [696, Theorem 1.3.1, p. 33] in order to see that the limit

$$\lim_{T\to+\infty} \frac{1}{T^n} \int\limits_{[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\,d\mathbf{t}$$

exists on the account of the Cauchy principle of convergence. The above geometrical argument with $\mathbf{s}_j = 0$ and $\mathbf{t}_j = 0$ implies that

$$\lim_{T\to+\infty} \frac{1}{T^n} \int_{[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} = \lim_{T\to+\infty} \frac{1}{T^n} \int_{\mathbf{s}+[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t}$$

for all $\mathbf{s} \in [0,\infty)^n$, which completes the first part of the proof. For the second part of the proof, observe that for each $\mathbf{s} \in [0,\infty)^n$ the function $\mathbf{t} \mapsto F_\lambda(\mathbf{t}+\mathbf{s})$, $\mathbf{t} \in \Lambda$ belongs to the class $e-W^1_{\mathrm{ap},\Lambda}(\Lambda : Y)$ as well as that the numbers $l > 0$ and $L > 0$ in the corresponding definition can be chosen independently of $\mathbf{s}$. Letting $k \to +\infty$ in (6.92), we get

$$\left\| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t})\, d\mathbf{t} - \lim_{T\to+\infty} \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t})\, d\mathbf{t} \right\|_Y \leqslant \varepsilon 2^n + \|F\|_{S^{\Omega,1}} \frac{\lceil LT^{n-1} \rceil}{T}. \tag{6.93}$$

By the foregoing, the same estimate holds for the function $\mathbf{t} \mapsto F_\lambda(\mathbf{t}+\mathbf{s})$, $\mathbf{t} \in \Lambda$, so that

$$\begin{aligned} &\left\| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t}+\mathbf{s})\, d\mathbf{t} - \lim_{T\to+\infty} \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t}+\mathbf{s})\, d\mathbf{t} \right\|_Y \\ &\quad \leqslant \varepsilon 2^n + \|F\|_{S^{\Omega,1}} \frac{\lceil LT^{n-1} \rceil}{T}, \quad \mathbf{s} \in [0,\infty)^n. \end{aligned} \tag{6.94}$$

After simple substitution, the first part of the proof shows that, for every $\mathbf{s} \in [0,\infty)^n$, we have

$$\lim_{T\to+\infty} \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t})\, d\mathbf{t} = \lim_{T\to+\infty} \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t}+\mathbf{s})\, d\mathbf{t}.$$

Hence, in view of (6.93) and (6.94), we get

$$\left\| \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t})\, d\mathbf{t} - \frac{1}{T^n} \int_{[0,T]^n} F_\lambda(\mathbf{t}+\mathbf{s})\, d\mathbf{t} \right\|_Y \leqslant \varepsilon 2^{n+1} + 2\|F\|_{S^{\Omega,1}} \frac{\lceil LT^{n-1} \rceil}{T},$$

which completes the proof of theorem. □

Remark 6.3.33. If we assume $\Lambda' = \Lambda = \mathbb{R}^n$ and accept all remaining requirements in Theorem 6.3.32, then we get into a classical situation in which the corresponding class is contained in the class of Besicovitch p-almost periodic functions in $\mathbb{R}^n$ (see [824, pp. 12–13]; we can use the set $\Omega = [-1,1]^n$ here producing the same results). In this case, the function $F_\lambda \in e-W^1_{\mathrm{ap},\Lambda}(\mathbb{R}^n : Y)$ if and only if $F \in e-W^1_{\mathrm{ap},\Lambda}(\mathbb{R}^n : Y)$ for each (some) $\lambda \in \mathbb{R}^n$; cf. also Theorem 6.3.26. Furthermore, the argumentation contained in the proof of Theorem 6.3.32 shows that

$$\lim_{T\to+\infty} \frac{1}{(2T)^n} \int_{\mathbf{s}+[-T,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t}$$

exists and does not depend on the choice of a tuple $\mathbf{s} \in \mathbb{R}^n$ and that, for every $\varepsilon > 0$, there exists a real number $T_0(\varepsilon) > 0$ such that, for every $T \geqslant T_0(\varepsilon)$ and $\mathbf{s} \in \mathbb{R}^n$, we have

$$\left\| \frac{1}{(2T)^n} \int\limits_{[-T,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} - \frac{1}{(2T)^n} \int\limits_{\mathbf{s}+[-T,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} \right\|_Y < \varepsilon.$$

But the restriction of the function $F(\cdot)$ to $[0,\infty)^n$ satisfies the requirements of Theorem 6.3.32 with $\Lambda' = \Lambda = [0,\infty)^n$ and we similarly see that (6.88) holds for all $\mathbf{s} \in \mathbb{R}^n$ and that (6.89) holds for all $\mathbf{s} \in \mathbb{R}^n$; plugging $\mathbf{s} = (-T/2, \ldots, -T/2)$ in this estimate, we particularly get

$$\lim_{T\to+\infty} \frac{1}{T^n} \int\limits_{\mathbf{s}+[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} = \lim_{T\to+\infty} \frac{1}{(2T)^n} \int\limits_{\mathbf{s}+[-T,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t},$$

as well as that the above limits exist and do not depend on the choice of a tuple $\mathbf{s} \in \mathbb{R}^n$. It should be also noted that there exist at most countable values of $\lambda \in \mathbb{R}^n$ for which $P_\lambda(F) \neq 0$ since $F(\cdot)$ can be uniformly approximated in the Weyl norm by trigonometric polynomials and each of them has a finite Bohr–Fourier spectrum (i. e., the set $\{\lambda \in \mathbb{R}^n : P_\lambda(F) \neq 0\}$); see also [951, Proposition 5.2]. But the function $\chi_{[0,1/2)}(\cdot)$ is equi-Weyl p-almost periodic for every $p \geqslant 1$ and its Bohr–Fourier spectrum is empty so that we cannot expect the validity of Parseval equality in our framework.

Finally, we shall apply our results in the analysis of existence and uniqueness of the multi-dimensional Weyl almost periodic type solutions for various classes of abstract Volterra integro-differential equations.

1. Let $a > 0$; then we know that the regular solution of the wave equation $u_{tt} = a^2 u_{xx}$ in domain $\{(x,t) : x \in \mathbb{R},\ t > 0\}$, equipped with the initial conditions $u(x,0) = f(x) \in C^2(\mathbb{R})$ and $u_t(x,0) = g(x) \in C^1(\mathbb{R})$, is given by the d'Alembert formula. Let us suppose that the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ belongs to the class $e - W^{(1,x,\mathbb{F})}_{[0,1],\mathbb{R}}(\mathbb{R} : \mathbb{C})$, where $g^{[1]}(\cdot) \equiv \int_0^{\cdot} g(s)\, ds$. Then the solution $u(x,t)$ can be extended to the whole real line in the time variable and this solution belongs to the class $e - W^{(1,x,\mathbb{F}_1)}_{[0,1]^2,\mathbb{R}^2}(\mathbb{R}^2 : \mathbb{C})$, provided that

$$\sup_{l>0} \sup_{(t_1,t_2)\in\mathbb{R}^2} \left[\int\limits_{t_1}^{t_1+(l/a)} \frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l, x - at_2 - l)} dx + \int\limits_{t_1}^{t_1+(l/a)} \frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l, x + at_2)} dx \right] < +\infty.$$

To verify this, fix a positive real number $\varepsilon > 0$. Then there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $t_0 \in \mathbb{R}$ there exists $\tau \in B(t_0, L)$ such that

$$\sup_{t\in\mathbb{R}} \mathbb{F}(l,t) \| f(\tau + \cdot) - f(\cdot) \|_{L^1(t+l[0,1]:\mathbb{C})} < \varepsilon \tag{6.95}$$

and that (6.95) holds with the function $f(\cdot)$ replaced therein with the function $g^{[1]}(\cdot)$. For our purposes, we choose the real numbers l/a and $L' > L$ sufficiently large. The required conclusions simply follow from the foregoing arguments, the computation

$$\begin{aligned}
&\int_{t_1}^{t_1+(l/a)} \int_{t_2}^{t_2+(l/a)} \frac{1}{2}\big|f((x-at)+(\tau_1-a\tau_2)) - f(x-at)\big|\,dx\,dt \\
&\leqslant \frac{1}{2}\int_{t_1}^{t_1+(l/a)} \int_{x-at_2-l}^{x-at_2} \big|f(z+(\tau_1-a\tau_2)) - f(z)\big|\,dz\,dx \\
&\leqslant \frac{1}{2}\int_{t_1}^{t_1+(l/a)} \frac{\varepsilon}{\mathbb{F}(l, x-at_2-l)}\,dx,
\end{aligned}$$

a similar computation for the corresponding term $f((x+at)+(\tau_1+a\tau_2)) - f(x+at)$ and the corresponding terms with the function $g^{[1]}(\cdot)$.

We continue with the following application to the Gaussian semigroup in $\mathbb{R}^n$:

2. Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1 \leqslant p < \infty$. It is well known that the Gaussian semigroup $(G(t))_{t\geqslant 0}$, already considered several times, can be extended to a bounded analytic C_0-semigroup of angle $\pi/2$, generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y. Suppose now that $1 \leqslant p < \infty$, $1/p + 1/q = 1$, $t_0 > 0$, $\emptyset \neq \Lambda' \subseteq \Lambda = \mathbb{R}^n$, $h \in L^1(\mathbb{R}^n)$, $\Omega = [0,1]^n$, $F \in (e-)W^{(p(\mathbf{u}),\phi,\mathbb{F})}_{\Omega,\Lambda'}(\mathbb{R}^n : \mathbb{C})$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$, and $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t})\| < \infty$. Suppose, further, that the functions $\mathbb{F} : (0,\infty)\times\mathbb{R}^n \to (0,\infty)$ and $\mathbb{F}_1 : (0,\infty)\times\mathbb{R}^n \to (0,\infty)$ does not depend on $\mathbf{t}$, as well as that $p_1(\mathbf{u}) \equiv 1$. If $\phi(x) = \varphi(x) = x$, $x \geqslant 0$ and for each $l > 0$ we have
$$2l^{-n/p}(4\pi t_0)^{-n/2} \sum_{k\in l\mathbb{Z}^n} e^{-\frac{(|k|-3l\sqrt{n})^2}{4t_0}} \frac{\mathbb{F}_1(l)}{\mathbb{F}(l)} \leqslant 1,$$
then Proposition 6.3.10 can be applied and shows that the function $\mathbb{R}^n \ni x \mapsto u(x,t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ belongs to the class $(e-)W^{(1,\phi,\mathbb{F}_1)}_{\Omega,\Lambda'}(\mathbb{R}^n : \mathbb{C})$. It is worth noting that this proposition can be applied even in the case that $\phi(x) = \varphi(x) = x^\alpha$, $x \geqslant 0$ for some constant $\alpha > 1$ but then we must allow that the function $\mathbb{F}_1(l)$ rapidly decays to zero as $l \to +\infty$ (notice only that the assumptions $\mathbf{u} \in \mathbf{t} + l\Omega$ and $\mathbf{v} \in \mathbf{u} - k + l\Omega$ for some $\mathbf{t} \in \mathbb{R}^n$ and $k \in l\mathbb{Z}^n$ imply $\mathbf{u} - \mathbf{v} \in k + l\Omega - l\Omega - l\Omega$ and therefore $|\mathbf{u} - \mathbf{v}| \geqslant |k| - 3l\sqrt{n}$); Proposition 6.3.11 can be also applied here.
   Here, we would like to stress that our previous analyses from [265, Example 0.1] can be also used to provide certain applications of the multi-dimensional Weyl almost periodic functions.
3. Suppose now that $Y := L^r(\mathbb{R}^n)$ for some $r \in [1,\infty)$ and $A(t) := \Delta + a(t)I$, $t \geqslant 0$, where Δ is the Dirichlet Laplacian on $L^r(\mathbb{R}^n)$, I is the identity operator on $L^r(\mathbb{R}^n)$ and $a \in L^\infty([0,\infty))$. Then it is well known that the evolution system $(U(t,s))_{t\geqslant s\geqslant 0} \subseteq L(Y)$

generated by the family $(A(t))_{t\geqslant 0}$ exists and is given by $U(t,t) := I$ for all $t \geqslant 0$ and

$$[U(t,s)F](\mathbf{u}) := \int_{\mathbb{R}^n} K(t,s,\mathbf{u},\mathbf{v})F(\mathbf{v})\,d\mathbf{v}, \quad F \in L^r(\mathbb{R}^n), \quad t > s \geqslant 0, \tag{6.96}$$

where $K(t,s,\mathbf{u},\mathbf{v})$ is given by

$$K(t,s,\mathbf{u},\mathbf{v}) := \big(4\pi(t-s)\big)^{-\frac{n}{2}} e^{\int_s^t a(\tau)\,d\tau} \exp\biggl(-\frac{|\mathbf{u}-\mathbf{v}|^2}{4(t-s)}\biggr), \quad t > s,\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n; \tag{6.97}$$

see [331] for more details. Hence, for every $\tau \in \mathbb{R}^n$, we have

$$K(t,s,\mathbf{u}+\tau,\mathbf{v}+\tau) = K(t,s,\mathbf{u},\mathbf{v}), \quad t > s \geqslant 0,\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n. \tag{6.98}$$

It is well known that, under certain assumptions, a unique mild solution of the abstract Cauchy problem $(\partial/\partial t)u(t,x) = A(t)u(t,x)$, $t > 0$; $u(0,x) = F(x)$ is given by $u(t,x) := [U(t,0)F](x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Suppose now that $F \in L^r(\mathbb{R}^n) \cap (e-)W^{(p,x,\mathbb{F})}_{[0,1]^n,\Lambda'}(\mathbb{R}^n : \mathbb{C})$, where $1 \leqslant p < \infty$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$ and the function $\mathbb{F}(l,\mathbf{t}) \equiv \mathbb{F}(l)$ does not depend on $\mathbf{t}$ (at this place, it is worth noting that, in the usual Bohr or Stepanov concept, this immediately yields $F \equiv 0$). Let $1/p + 1/q = 1$ and let $\varepsilon > 0$ be given. Then there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\sup_{\mathbf{t}\in\mathbb{R}^n} \mathbb{F}(l)|F(\tau+\mathbf{u}) - F(\mathbf{u})|_{L^p(\mathbf{t}+l[0,1]^n)} < \varepsilon.$$

Therefore, for every $t > 0$, $l > 0$ and $\mathbf{u}$, $\tau \in \mathbb{R}^n$, there exists a finite real constant $c_t > 0$ such that

$$\begin{aligned}
\big|u(t,\mathbf{u}+\tau) - u(t,\mathbf{u})\big| &= \biggl|\int_{\mathbb{R}^n} [K(t,0,\mathbf{u}+\tau,\mathbf{v}) - K(t,0,\mathbf{u},\mathbf{v})]F(\mathbf{v})\,d\mathbf{v}\biggr| \\
&= \biggl|\int_{\mathbb{R}^n} K(t,0,\mathbf{u}+\tau,\mathbf{v}+\tau)F(\mathbf{v}+\tau)\,d\mathbf{v} - \int_{\mathbb{R}^n} K(t,0,\mathbf{u},\mathbf{v})F(\mathbf{v})\,d\mathbf{v}\biggr| \\
&= \biggl|\int_{\mathbb{R}^n} K(t,0,\mathbf{u},\mathbf{v})\big[F(\mathbf{v}+\tau)\,d\mathbf{v} - F(\mathbf{v})\big]\,d\mathbf{v}\biggr| \\
&\leqslant c_t \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{u}-\mathbf{v}|^2}{4t}} |F(\mathbf{v}+\tau) - F(\mathbf{v})|\,d\mathbf{v} \\
&= c_t \sum_{k\in l\mathbb{Z}^n} \int_{k+l[0,1]^n} e^{-\frac{|\mathbf{u}-\mathbf{v}|^2}{4t}} |F(\mathbf{v}+\tau) - F(\mathbf{v})|\,d\mathbf{v} \\
&\leqslant c_t \sum_{k\in l\mathbb{Z}^n} \big\|e^{-\frac{|\mathbf{u}-\cdot|^2}{4t}}\big\|_{L^q(k+l[0,1]^n)} \big\|F(\cdot+\tau) - F(\cdot)\big\|_{L^p(k+l[0,1]^n)} \\
&\leqslant c_t \frac{\varepsilon}{\mathrm{F}(l)} \sum_{k\in l\mathbb{Z}^n} \big\|e^{-\frac{|\mathbf{u}-\cdot|^2}{4t}}\big\|_{L^q(k+l[0,1]^n)} := c_t \frac{\varepsilon}{\mathbb{F}(l)} G(l,\mathbf{u}).
\end{aligned}$$

The convergence of series defining $G(l, \mathbf{u})$ can be simply justified by the fact that for each $k \in l\mathbb{Z}^n$ with a sufficiently large absolute value we have $|\mathbf{u} - k - \mathbf{v}| \geqslant |k| - l - |\mathbf{u}|$ for all $\mathbf{v} \in l[0,1]^n$. Now we will fix a number $t > 0$ and a new exponent $p' \in [1, \infty)$. Since the function $\mathbf{u} \mapsto G(l, \mathbf{u})$, $\mathbf{u} \in \mathbb{R}^n$ is continuous and positive for every fixed $l > 0$, we can define the function $\mathbb{F}_1(\cdot; \cdot)$ by

$$\mathbb{F}_1(l, \mathbf{t}) := \frac{\mathbb{F}(l)}{(\int_{\mathbf{t}+l[0,1]^n} G(l, \mathbf{u})^{p'} \, d\mathbf{u})^{1/p'}}, \quad l > 0.$$

By the above given argumentation, we immediately get from the corresponding definition that the mapping $x \mapsto u(t, x)$, $x \in \mathbb{R}^n$ belongs to the class $(e-)W^{(p', x, \mathbb{F}_1)}_{[0,1]^n, \Lambda'}(\mathbb{R}^n : \mathbb{C})$.

Let us mention, finally, a few intriguing topics which have not been discussed here. Composition theorems for Weyl almost periodic type functions were considered by F. Bedouhene, Y. Ibaouene, O. Mellah, P. Raynaud de Fitte [139] and M. Kostić [639] in the one-dimensional setting; we have not analyzed the multi-dimensional analogues of the results established in these research studies (although considered Weyl almost periodic type functions depend on two parameters, $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$, the applications to semilinear Cauchy equations and inclusions are not examined here, as well). On the other hand, in [67, Section 6], the authors have presented several results and examples about the relationship between one-dimensional Weyl almost periodic type functions and one-dimensional Besicovitch almost periodic type functions; for the sake of brevity and better exposition, we will skip all details concerning this theme in the multi-dimensional framework. Also, many crucial properties and important counterexamples in the theory of one-dimensional Stepanov, Weyl and Besicovitch almost periodic type functions have been established by H. Bohr and E. Følner in their landmark paper [199]; let us also note that, for any real number $P > 1$, the authors of this paper have constructed a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ which is Stepanov p-almost periodic for any exponent $p \in [1, P)$ but not equi-Weyl P-almost periodic (see [199, Main example 3, pp. 83–91]). We have not been able to reconsider here such exotic examples in the multi-dimensional setting (it is also worth noting that L. I. Danilov [328] and H. D. Ursell [988] have established two interesting characterizations of equi-Weyl p-almost periodic functions as well as that the notion of Weyl almost periodicity has been investigated by A. Iwanik [554] within the field of topological dynamics (cf. also [358, 768, 790]), as emphasized earlier in [631]).

Various classes of multi-dimensional (equi-)Weyl $(p(x), \phi, F)$-uniformly recurrent type functions will not be considered here. In this monograph, we will not consider the multi-dimensional Doss-$(p(x), \phi, F)$-uniformly recurrent type functions, as well.

6.4 Weighted ergodic components in $\mathbb{R}^n$

The main aim of this section is to analyze weighted Stepanov ergodic spaces, weighted Weyl ergodic spaces and weighted pseudo-ergodic spaces in $\mathbb{R}^n$ which provide certain generalizations of the space $C_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$. To achieve our aims, we essentially employ the results from the theory of Lebesgue spaces with variable exponents $L^{p(x)}$. Before we go any further, we would like to note that several presented results seem to be new even in the case of consideration of the constant exponents $p(x) \equiv p \in [1, \infty)$ as well as that the material is taken from our joint research study [656] with B. Chaouchi and W.-S. Du.

In connection with our studies of weighted Stepanov ergodic spaces and weighted Weyl ergodic spaces, we would like to recall that R. Farwig and Y. Taniuchi have initiated, in [425], the study of backward asymptotically almost periodic-in-time solutions to Navier–Stokes equations in unbounded domains (cf. also [424, 909]). To the best of our knowledge, this was the first research article in which the asymptotic behavior of almost periodicity-in-time solutions of certain partial differential equations or ordinary differential equations has been analyzed only for sufficiently large negative values of the time variable (this is probably the unique research article in the existing literature which concerns this problematic, actually). Albeit not explicitly influenced by the results established in [425], in Theorem 6.4.2 and Theorem 6.4.11, we analyze the backward Stepanov asymptotic and the backward Weyl asymptotic of almost periodicity-in-time solutions for a general classes of abstract Volterra integro-differential equations, respectively (see also Corollary 6.4.3 and Corollary 6.4.12, which enables one to state the most important applications in the one-dimensional case). We analyze the translation invariance of introduced multi-dimensional weighted ergodic spaces and the convolution invariance of multi-dimensional weighted pseudo-ergodic spaces, which also enables us to provide certain applications to the abstract Volterra integro-differential equations. In connection with our study of multi-dimensional weighted pseudo-ergodic spaces, we would like to note that the class of pseudo-almost periodic functions in $\mathbb{R}^n$ seems to be not precisely defined and explored in the existing literature if $n \geqslant 2$ (the example concerning the d'Alembert formula, given later, indicates that an X-valued pseudo-almost periodic function on $\mathbb{R}^n$ should be defined as a sum of an X-valued almost periodic function on $\mathbb{R}^n$ and an ergodic part $Q : \mathbb{R}^n \to X$, which needs to be bounded, continuous and satisfies $\lim_{T\to+\infty} T^{-n} \int_{|\mathbf{t}|\leqslant T} \|Q(\mathbf{t})\|\, d\mathbf{t} = 0$).

Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, Ω is a compact subset of $\mathbb{R}^n$ with a positive Lebesgue measure, $\Lambda + \Omega \subseteq \Lambda$, $p \in \mathcal{P}(\Omega)$ and the set $\mathbb{D}$ is unbounded. Let $G : \Lambda \to (0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$. In Definition 6.4.1, we introduce the notion of weighted Stepanov ergodic spaces

$$S^{\Omega,p(\mathbf{u}),\phi,G}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y), \quad S^{\Omega,p(\mathbf{u}),\phi,G,1}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y) \quad \text{and} \quad S^{\Omega,p(\mathbf{u}),\phi,G,2}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y);$$

any of these spaces contains, in the set theoretical sense, the usual Stepanov ergodic space $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ from Definition 6.2.26. Our first results in Subsection 6.4.1 are already mentioned Theorem 6.4.2 and Corollary 6.4.3, in which we analyze the Stepanov asymptotically almost periodic properties at minus infinity of the function spaces introduced in Definition 6.4.1; in Corollary 6.4.3, we particularly clarify an interesting property of the infinite convolution product

$$t \mapsto \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

which has been analyzed by many authors working in the field of almost periodicity. In Subsection 6.4.2, we analyze multi-dimensional Weyl weighted ergodic components. Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, as well as the inclusions in Eq. (6.103) hold, $\phi : [0,\infty) \to [0,\infty)$ and $\mathbb{F} : (0,\infty) \times \Lambda \to (0,\infty)$. In Definition 6.4.4–Definition 6.4.6, we introduce the notion of spaces $(e-)W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$, $(e-)W^{p(\mathbf{u}),\phi,\mathbb{F},1}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ and $(e-)W^{p(\mathbf{u}),\phi,\mathbb{F},2}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$. Our first result in this subsection is Proposition 6.4.7, where we show that, under certain conditions, the ergodic function spaces introduced in Definition 6.4.1 can be embedded into the corresponding ergodic function spaces introduced in Definition 6.4.4–Definition 6.4.6. After that, we continue the analysis of the asymptotical almost periodicity at minus infinity at Theorem 6.4.11 and Corollary 6.4.12.

Subsection 6.4.3 investigates the weighted pseudo-ergodic components. Before we explain the main ideas and results of this subsection, we would like to briefly summarize first the basic facts about pseudo-almost periodic functions, weighted pseudo-almost periodic functions and double-weighted pseudo-almost periodic functions. Let us recall that the notion of a pseudo-almost periodic function was introduced by C. Zhang in his doctoral dissertation [1074]. Denote by $\mathrm{PAP}(\mathbb{R}, X, \rho)$ and $\mathrm{PAP}_0(\mathbb{R}, X, \rho)$ the space consisting of all weighted pseudo-almost periodic functions and the space of weighted ergodic components, respectively (see, e. g., [32, 184, 315, 375, 562, 710, 1074, 1094] for the notion and more details on the subject). The translation invariance of weighted pseudo-almost periodic functions and some other problems for these classes have been investigated by D. Ji and C. Zhang [562]. The space $\mathrm{PAP}(\mathbb{R}, X, \rho)$ is not convolution invariant, in general, but we know that the convolution invariance of space $\mathrm{PAP}(\mathbb{R}, X, \rho)$ is equivalent with the convolution invariance of space $\mathrm{PAP}_0(\mathbb{R}, X, \rho)$. The convolution invariance of space $\mathrm{PAP}(\mathbb{R}, X, \rho)$ was systematically analyzed by A. Coronel, M. Pinto and D. Sepúlveda in [315].

Set $\mathbb{U} := \{\rho \in L^1_{\mathrm{loc}}(\mathbb{R}) : \rho(t) > 0 \text{ a. e. } t \in \mathbb{R}\}$, $\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \inf_{x\in\mathbb{R}} \rho(x) < \infty$, $\nu(T,\rho) := \lim_{T\to+\infty} \int_{-T}^{T} \rho(t)\,dt = \infty\}$ and $\mathbb{U}_b := L^\infty(\mathbb{R}) \cap \mathbb{U}_\infty$. Then $\mathbb{U}_b \subseteq \mathbb{U}_\infty \subseteq \mathbb{U}$ and we say that weights $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are equivalent, $\rho_1 \sim \rho_2$ for short, if and only if $\rho_1/\rho_2 \in \mathbb{U}_b$. By $\mathbb{U}_T$ we denote the space consisting of all weights $\rho \in \mathbb{U}_\infty$ satisfying that ρ is equivalent with all its translations. Assume that $\rho_1, \rho_2 \in \mathbb{U}_\infty$. The space $\mathrm{PAP}_0(\mathbb{R}, X, \rho_1, \rho_2)$ of double-weighted pseudo ergodic components was introduced and analyzed for the first time by T. Diagana in [367, 368] (2011); this space is defined as follows:

$$\mathrm{PAP}_0(\mathbb{R}, X, \rho_1, \rho_2) := \left\{ f \in C_b(\mathbb{R} : X) : \lim_{T \to +\infty} \frac{1}{2\int_{-T}^{T} \rho_1(t)\,dt} \int_{-T}^{T} \|f(t)\| \rho_2(t)\,dt = 0 \right\}.$$

The space $\mathrm{PAP}_0(\mathbb{R}, X, \rho_1, \rho_2)$ was further generalized by J. Blot, P. Cieutat and K. Ezzinbi in [181–184] (2011–2012), by using certain results from the measure theory (see also the research article [370] by T. Diagana, K. Ezzinbi and M. Miraoui); if $\rho_1 = \rho_2 = \rho$, then we use the shorthand $\mathrm{PAP}_0(\mathbb{R}, X, \rho) \equiv \mathrm{PAP}_0(\mathbb{R}, X, \rho_1, \rho_2)$. It is well known that the sum of spaces $\mathrm{AP}(\mathbb{R} : X)$ and $\mathrm{PAP}_0(\mathbb{R}, X, \rho)$ need not be a closed subspace of $C_b(\mathbb{R} : X)$ although the two spaces $\mathrm{AP}(\mathbb{R} : X)$ and $\mathrm{PAP}_0(\mathbb{R}, X, \rho)$ considered separately are closed subspaces of $C_b(\mathbb{R} : X)$. In order to analyze weighted pseudo-almost periodic properties of certain classes of semilinear first-order Cauchy problems, J. Zhang, T.-J. Xiao and J. Liang [1084] have introduced the modular norm $\|\cdot\|_\rho$ on the space $\mathrm{PAP}(\mathbb{R}, X, \rho)$ by

$$\|f\|_\rho := \inf_{i \in \mathbf{I}} \Big[\sup_{t \in \mathbb{R}} \|g_i(t)\| + \sup_{t \in \mathbb{R}} \|q_i(t)\| \Big],$$

where $\mathbf{I}$ denotes the family of all possible decompositions of $f(\cdot)$ into the almost periodic component $g_i(\cdot)$ and the ergodic component $q_i(\cdot)$. The modular norm turns $\mathrm{PAP}(\mathbb{R}, X, \rho)$ into a Banach space, which enables one to further investigates the composition principles for weighted pseudo-almost periodic functions. As observed in [631], the results established in [1084] can be also formulated for semilinear Cauchy inclusions with multivalued linear operators satisfying condition (P), especially for almost sectorial operators.

Suppose that $\phi : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to [0,\infty)$, $\mathrm{F} : (0,\infty) \to (0,\infty)$ are given functions and $p \in \mathcal{P}(\Lambda)$, where $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$. In Definition 6.4.13, we introduce the ergodic function spaces $\mathrm{PAP}_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$, $\mathrm{PAP}^1_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi, \psi)$ and $\mathrm{PAP}^2_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi, \psi)$. These classes of weighted pseudo-ergodic spaces seem to be new and not considered elsewhere even in the one-dimensional case. The space $\mathrm{PAP}_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$ generalizes the spaces $\mathrm{PAP}(\mathbb{R}, X, \rho)$, $\mathrm{PAP}_0(\mathbb{R}, X, \rho_1, \rho_2)$, the spaces of the one-dimensional ergodic components considered in [631, Definition 3.3.3, Definition 3.3.4], as well as the space $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$ introduced in [370, Definition 3.10], provided that μ is the Lebesgue measure on $\Lambda = \mathbb{R}$. It is worth noting that the spaces of weighted ergodic components considered in [370] can be further generalized by examining a general measure μ on Λ; in such a way, we can generalize the notion of space $\mathcal{E}(\mathbb{R}, X, \mu, \nu)$, for example. For simplicity, we will consider here the Lebesgue measure on Λ, only. The translation invariance of spaces introduced in Definition 6.4.13 is investigated in Theorem 6.4.17, whilst the invariance of multi-dimensional weighted pseudo-ergodicity is examined in Theorem 6.4.18. The method obeyed in the proof of this result enables one to state some sufficient conditions about the convolution invariance of multi-dimensional weighted ergodic spaces in Theorem 6.4.21.

We use the standard notation throughout the section. We assume that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, Ω is a compact subset of $\mathbb{R}^n$ with a positive Lebesgue measure, and $\mathcal{B}$ is a collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. If $T > 0$ and $Q : \Lambda \times X \to Y$, then we set $\Lambda_T := \{\lambda \in \Lambda : |\lambda| \leqslant T\}$ and $\check{Q} : -\Lambda \times X \to Y$ by $\check{Q}(\mathbf{t}; x) := Q(-\mathbf{t}; x)$, $\mathbf{t} \in -\Lambda$, $x \in X$.

6.4.1 Stepanov weighted ergodic components

We can extend the notion of space $S^{\Omega,p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ in the following ways.

Definition 6.4.1. Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$, $p \in \mathcal{P}(\Omega)$ and the set $\mathbb{D}$ is unbounded. Let $G : \Lambda \to (0,\infty)$ and $\phi : [0,\infty) \to [0,\infty)$. Then we say that:

(i) a function $Q : \Lambda \times X \to Y$ belongs to the space $S^{\Omega,p(\mathbf{u}),\phi,G}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ if and only if for every $\mathbf{t} \in \Lambda$ and $x \in X$, we find that $\phi(\|Q(\mathbf{t}+\mathbf{u}; x)\|_Y) \in L^{p(\mathbf{u})}(\Omega)$ as well as that, for every $B \in \mathcal{B}$, we have

$$\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\to+\infty} G(\mathbf{t})\big[\phi\big(\big\|Q(\mathbf{t}+\mathbf{u}; x)\big\|_Y\big)\big]_{L^{p(\mathbf{u})}(\Omega)} = 0,$$

uniformly for $x \in B$;

(ii) a function $Q : \Lambda \times X \to Y$ belongs to the space $S^{\Omega,p(\mathbf{u}),\phi,G,1}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ if and only if for every $\mathbf{t} \in \Lambda$ and $x \in X$, we find that $[\hat{Q}_\Omega(\mathbf{t}; x)](\mathbf{u}) \in L^{p(\mathbf{u})}(\Omega : Y)$ and that, for every $B \in \mathcal{B}$, we have

$$\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\to+\infty} G(\mathbf{t})\phi\big(\big\|Q(\mathbf{t}+\mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big) = 0,$$

uniformly for $x \in B$;

(iii) a function $Q : \Lambda \times X \to Y$ belongs to the space $S^{\Omega,p(\mathbf{u}),\phi,G,2}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ if and only if for every $\mathbf{t} \in \Lambda$ and $x \in X$, we have $[\hat{Q}_\Omega(\mathbf{t}; x)](\mathbf{u}) \in L^{p(\mathbf{u})}(\Omega : Y)$ and, for every $B \in \mathcal{B}$, we have

$$\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\to+\infty} \phi\big(G(\mathbf{t})\big\|Q(\mathbf{t}+\mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big) = 0,$$

uniformly for $x \in B$.

Immediately from the definition, we find that the spaces introduced above are translation invariant in the following sense. Define $\mathcal{B}_{x_0} := \{-x_0 + B : B \in \mathcal{B}\}$ $(x_0 \in X)$, $G_{\mathbf{t}_0}(\mathbf{t}) := G(\mathbf{t}_0+\mathbf{t})$, $\mathbf{t} \in -\mathbf{t}_0+\Lambda$ and $Q_{\mathbf{t}_0,x_0}(\mathbf{t}, x) := Q(\mathbf{t}+\mathbf{t}_0, x+x_0)$, $\mathbf{t} \in -\mathbf{t}_0+\Lambda$, $x \in X$. Then the supposition $Q \in S^{\Omega,p(\mathbf{u}),\phi,G}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ $(Q \in S^{\Omega,p(\mathbf{u}),\phi,G,1}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$; $Q \in S^{\Omega,p(\mathbf{u}),\phi,G,2}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y))$ implies $Q_{\mathbf{t}_0,x_0} \in S^{\Omega,p(\mathbf{u}),\phi,G}_{0,-\mathbf{t}_0+\mathbb{D},\mathcal{B}_{x_0}}((-\mathbf{t}_0+\Lambda) \times X : Y)$ $(Q_{\mathbf{t}_0,x_0} \in S^{\Omega,p(\mathbf{u}),\phi,G,1}_{0,-\mathbf{t}_0+\mathbb{D},\mathcal{B}_{x_0}}((-\mathbf{t}_0+\Lambda) \times X : Y)$; $Q_{\mathbf{t}_0,x_0} \in S^{\Omega,p(\mathbf{u}),\phi,G,2}_{0,-\mathbf{t}_0+\mathbb{D},\mathcal{B}_{x_0}}((-\mathbf{t}_0+\Lambda) \times X : Y))$. The following result seems to be new even in the one-dimensional case, with the constant exponent $p(\mathbf{u}) \equiv p \in [1,\infty)$:

Theorem 6.4.2.

(i) *Suppose that* $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ *is a convex monotonically increasing function satisfying* $\phi(xy) \leqslant \varphi(x)\phi(y)$ *for all* $x,\ y \geqslant 0$, $(a_k)_{k\in\mathbb{N}_0^n}$ *is a sequence of positive real numbers such that* $\sum_{k\in\mathbb{N}_0^n} a_k = 1$, $\Omega = [0,1]^n$, $Q \in S^{[0,1]^n,p(\mathbf{u}),\phi,G}_{0,[0,\infty)^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$, *there exists a real number* $M \geqslant 1$ *such that, for every* $T \geqslant M$ *and* $\mathbf{t} \in [0,\infty)^n$ *with* $|\mathbf{t}| \geqslant M$, *we have*

$$\int_\Omega \varphi_{p(\mathbf{u})}\left(2\sum_{k\in\mathbb{N}_0^n} \frac{G_1(T)}{G(k+\mathbf{t}-\mathbf{u})} a_k \varphi(a_k^{-1})\left[\varphi(\|R(\mathbf{s}+k)\|)\right]_{L^{q(\mathbf{s})}(\Omega)}\right) d\mathbf{u} \leqslant 1, \tag{6.99}$$

where the operator function $(R(\mathbf{t}))_{\mathbf{t}>0} \subseteq L(Y,Z)$ *is strongly continuous and the function*

$$Q_1(\mathbf{t};x) \equiv \int_{-\infty}^{\mathbf{t}} R(\mathbf{t}-\mathbf{s})\check{Q}(\mathbf{s};x)\,d\mathbf{s} = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\cdots\int_{-\infty}^{t_n} R(\mathbf{t}-\mathbf{s})\,\check{Q}(\mathbf{s};x)\,d\mathbf{s} \tag{6.100}$$

is well defined for all $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$. *Then* $Q_1 \in S^{[0,1]^n,p(\mathbf{u}),\phi,G_1}_{0,-[0,\infty)^n,\mathcal{B}}(\mathbb{R}^n \times X : Z)$.

(ii) *Suppose that* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous monotonically increasing function,* $\Omega = [0,1]^n$, $Q \in S^{[0,1]^n,p(\mathbf{u}),\phi,G,1}_{0,[0,\infty)^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$, *there exists a real number* $M \geqslant 1$ *such that, for every* $T \geqslant M$ *and* $\mathbf{t} \in [0,\infty)^n$ *with* $|\mathbf{t}| \geqslant M$, *we have*

$$\int_\Omega \varphi_{p(\mathbf{u})}\left(2\sum_{k\in\mathbb{N}_0^n} \frac{\phi^{-1}(\varepsilon/G(k+\mathbf{t}-\mathbf{u}))}{\phi^{-1}(\varepsilon/G_1(T))}\|R(\mathbf{s}+k)\|_{L^{q(\mathbf{s})}(\Omega)}\right) d\mathbf{u} \leqslant 1, \quad \varepsilon > 0,$$

where the operator function $(R(\mathbf{t}))_{\mathbf{t}>0} \subseteq L(Y,Z)$ *is strongly continuous and the function* $Q_1(\cdot;\cdot)$, *given by* (6.100), *is well defined for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$. *Then we have* $Q_1 \in S^{[0,1]^n,p(\mathbf{u}),\phi,G_1,1}_{0,-[0,\infty)^n,\mathcal{B}}(\mathbb{R}^n \times X : Z)$.

(iii) *Suppose that* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous monotonically increasing function,* $\Omega = [0,1]^n$, $Q \in S^{[0,1]^n,p(\mathbf{u}),\phi,G,2}_{0,[0,\infty)^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$, *there exists a real number* $M \geqslant 1$ *such that, for every* $T \geqslant M$ *and* $\mathbf{t} \in [0,\infty)^n$ *with* $|\mathbf{t}| \geqslant M$, *we have*

$$\int_\Omega \varphi_{p(\mathbf{u})}\left(2\sum_{k\in\mathbb{N}_0^n} \frac{G_1(T)}{G(k-\mathbf{t}-\mathbf{u})}\|R(\mathbf{s}+k)\|_{L^{q(\mathbf{s})}(\Omega)}\right) d\mathbf{u} \leqslant 1,$$

where the operator function $(R(\mathbf{t}))_{\mathbf{t}>0} \subseteq L(Y,Z)$ *is strongly continuous and the function* $Q_1(\cdot;\cdot)$, *given by* (6.100), *is well defined for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$. *Then we have* $Q_1 \in S^{[0,1]^n,p(\mathbf{u}),\phi,G_1,2}_{0,-[0,\infty)^n,\mathcal{B}}(\mathbb{R}^n \times X : Z)$.

Proof. We will prove only (i). Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be given. Then we know that there exists a sufficiently large real number $M_1 \geqslant M$ such that, for every $T \geqslant M_1$ and $x \in B$, we have

$$\left[\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)\right]_{L^{p(\mathbf{u})}(\Omega:Y)} < \varepsilon/G(T). \tag{6.101}$$

It suffices to show that, for every $T \geqslant M_1$ and $x \in B$, we have

$$[\phi(\|Q_1(\mathbf{t}+\mathbf{u};x)\|_Z)]_{L^{p(\mathbf{u})}(\Omega:Y)} < \varepsilon/G_1(T).$$

It is clear that this follows if we prove that, for every $T \geqslant M_1$ and $x \in B$, we have

$$\int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q_1(\mathbf{t}+\mathbf{u};x)\|_Z)}{\varepsilon/G_1(T)}\right) d\mathbf{u} \leqslant 1. \tag{6.102}$$

But, since we have assumed that $\phi(\cdot)$ is convex and $\sum_{k\in\mathbb{N}_0^n} a_k = 1$, Eq. (6.78) holds for any sequence (x_k) of non-negative real numbers. Using (6.101), (6.78), (6.99) and the fact that the functions $\varphi_{p(\mathbf{u})}(\cdot)$ and $\phi(\cdot)$ are monotonically increasing, we obtain (6.102) from the following computation involving the Jensen integral inequality (see Lemma 3.1.1) and the Hölder inequality (see Lemma 1.1.7(i)):

$$\begin{aligned}
&\int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q_1(\mathbf{t}+\mathbf{u};x)\|_Z)}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad= \int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|\sum_{k\in\mathbb{N}_0^n} a_k \int_{k+[0,1]^n} a_k^{-1} R(\mathbf{s})\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{s};x)\,d\mathbf{s}\|_Y)}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad\leqslant \int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\phi(\sum_{k\in\mathbb{N}_0^n} a_k \int_{k+[0,1]^n} a_k^{-1} \|R(\mathbf{s})\|\cdot\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{s};x)\|_Y\,d\mathbf{s})}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad\leqslant \int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in\mathbb{N}_0^n} a_k \phi(\int_{k+[0,1]^n} a_k^{-1} \|R(\mathbf{s})\|\cdot\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{s};x)\|_Y\,d\mathbf{s})}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad\leqslant \int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in\mathbb{N}_0^n} a_k \int_{k+[0,1]^n} \phi(a_k^{-1} \|R(\mathbf{s})\|\cdot\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{s};x)\|_Y)\,d\mathbf{s}}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad\leqslant \int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in\mathbb{N}_0^n} a_k \varphi(a_k^{-1}) \int_{k+[0,1]^n} \varphi(\|R(\mathbf{s})\|)\cdot\phi(\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{s};x)\|_Y)\,d\mathbf{s}}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad= \int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in\mathbb{N}_0^n} a_k \varphi(a_k^{-1}) \int_{[0,1]^n} \varphi(\|R(\mathbf{s}+k)\|)\cdot\phi(\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{s}-k;x)\|_Y)\,d\mathbf{s}}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad= \int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in\mathbb{N}_0^n} a_k \varphi(a_k^{-1}) \int_{[0,1]^n} \varphi(\|R(\mathbf{s}+k)\|)\cdot\phi(\|Q(-\mathbf{t}-\mathbf{u}+\mathbf{s}+k;x)\|_Y)\,d\mathbf{s}}{\varepsilon/G_1(T)}\right) d\mathbf{u}\\
&\quad\leqslant \int_\Omega \varphi_{p(\mathbf{u})}\left(2\sum_{k\in\mathbb{N}_0^n} \frac{G_1(T)}{G(k-\mathbf{t}-\mathbf{u})} a_k \varphi(a_k^{-1})[\varphi(\|R(\mathbf{s}+k)\|)]_{L^{q(\mathbf{s})}(\Omega)}\right) d\mathbf{u} \leqslant 1,
\end{aligned}$$

which holds for any $\mathbf{t} \in -[0,\infty)^n$ with $|\mathbf{t}| \geqslant M_1 + \sqrt{n}$ (in actual fact, this implies $|-\mathbf{t}-\mathbf{u}+k| \geqslant |-\mathbf{t}+k| - |\mathbf{u}| \geqslant |-\mathbf{t}| - |\mathbf{u}| \geqslant |\mathbf{t}| - \sqrt{n}$ and we may apply (6.101)). □

Suppose that $1 \leqslant p < \infty$. We say that a measurable function $f : \mathbb{R} \to X$ is Stepanov asymptotically p-almost periodic at minus infinity if and only if there exist an almost periodic function $f_0 : \mathbb{R} \to X$ and a measurable function $q : \mathbb{R} \to X$ such that $f(t) = f_0(t)+q(t)$, $t \in \mathbb{R}$ and $\lim_{t\to-\infty} \|\hat{q}(t)\|_{L^p([0,1]:X)} = 0$. Now we are able to state the following corollary of Theorem 6.4.2.

Corollary 6.4.3. *Suppose that* $1 \leqslant p < \infty$, $1/p + 1/q = 1$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family, and* $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. *If* $f : \mathbb{R} \to X$ *is Stepanov p-almost periodic,* $q : \mathbb{R} \to X$ *is Stepanov p-bounded and* $\lim_{t\to+\infty} \|\hat{q}(t)\|_{L^p([0,1]:X)} = 0$, *then the function* $G : \mathbb{R} \to Y$, *given by*

$$G(t) := \int_{-\infty}^{t} R(t-s)[f(s) + \check{q}(s)]\, ds, \quad t \in \mathbb{R},$$

is well defined and Stepanov asymptotically p-almost periodic at minus infinity.

Proof. Due to [631, Proposition 2.6.11], we find that the function $t \mapsto \int_{-\infty}^{t} R(t-s)f(s)\, ds$, $t \in \mathbb{R}$ is almost periodic. It suffices to show that the Bochner transform of the function

$$t \mapsto \int_{-\infty}^{t} R(t-s)\check{q}(s)\, ds, \quad t \in \mathbb{R}$$

tends to zero in the space $L^p([0,1] : Y)$ as t goes to minus infinity. But this simply follows from Theorem 6.4.2 and a simple computation with $G_1(T) \equiv c > 0$ being a sufficiently small constant function, since we have assumed that the series $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]}$ is convergent ($\phi(x) = \varphi(x) = x$, $x \geqslant 0$ and $G(T) \equiv 1$). □

6.4.2 Weyl weighted ergodic components

We assume here that $p \in \mathcal{P}(\mathbb{R}^n)$. The following notion generalizes the corresponding notion already analyzed in the one-dimensional case.

Definition 6.4.4. Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, as well as

$$\Lambda + \Lambda \subseteq \Lambda, \quad \Lambda + l\Omega \subseteq \Lambda \quad \text{for all } l > 0, \tag{6.103}$$

$\phi : [0,\infty) \to [0,\infty)$ and $\mathbb{F} : (0,\infty) \times \Lambda \to (0,\infty)$.

(i) By $e - W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ we denote the collection of all functions $Q : \Lambda \times X \to Y$ such that, for every $\mathbf{t}$, $\mathbf{s} \in \Lambda$, $l > 0$ and $x \in X$, we find that $\phi(\|Q(\mathbf{t} + \mathbf{u}; x)\|_Y) \in L^{p(\mathbf{u})}(\mathbf{s} + l\Omega)$ as well as that, for every $B \in \mathcal{B}$, we have

$$\lim_{l\to+\infty} \limsup_{|\mathbf{t}|\to+\infty, \mathbf{t}\in\mathbb{D}} \sup_{s\in\Lambda} \big[\mathbb{F}(l,\mathbf{t})\big[\phi\big(\|Q(\mathbf{t} + \mathbf{u}; x)\|_Y\big)\big]_{L^{p(\mathbf{u})}(s+l\Omega)}\big] = 0,$$

uniformly for $x \in B$.

(ii) By $W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ we denote the collection of all functions $Q:\Lambda\times X\to Y$ such that, for every $\mathbf{t}$, $\mathbf{s}\in\Lambda$, $l>0$ and $x\in X$, we find that $\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)\in L^{p(\mathbf{u})}(\mathbf{s}+l\Omega)$ as well as that, for every $B\in\mathcal{B}$, we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\limsup_{l\to+\infty}\sup_{s\in\Lambda}\big[\mathbb{F}(l,\mathbf{t})\big[\phi\big(\big\|Q(\mathbf{t}+\mathbf{u};x)\big\|_Y\big)\big]_{L^{p(\mathbf{u})}(s+l\Omega)}\big]=0,$$

uniformly for $x\in B$.

Definition 6.4.5. Suppose that $\mathbb{D}\subseteq\Lambda\subseteq\mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, as well as (6.103) holds, $\phi:[0,\infty)\to[0,\infty)$ and $\mathbb{F}:(0,\infty)\times\Lambda\to(0,\infty)$.

(i) By $e-W^{p(\mathbf{u}),\phi,\mathbb{F},1}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ we denote the collection of all functions $Q:\Lambda\times X\to Y$ such that, for every $\mathbf{t}$, $\mathbf{s}\in\Lambda$, $l>0$ and $x\in X$, we find that $Q(\mathbf{t}+\mathbf{u};x)\in L^{p(\mathbf{u})}(\mathbf{s}+l\Omega:Y)$ as well as that, for every $B\in\mathcal{B}$, we have

$$\lim_{l\to+\infty}\limsup_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\sup_{s\in\Lambda}\big[\mathbb{F}(l,\mathbf{t})\phi\big[\big\|Q(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(s+l\Omega:Y)}\big]\big]=0,$$

uniformly for $x\in B$.

(ii) By $W^{p(\mathbf{u}),\phi,\mathbb{F},1}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ we denote the collection of all functions $Q:\Lambda\times X\to Y$ such that, for every $\mathbf{t}$, $\mathbf{s}\in\Lambda$, $l>0$ and $x\in X$, we find that $Q(\mathbf{t}+\mathbf{u};x)\in L^{p(\mathbf{u})}(\mathbf{s}+l\Omega:Y)$ and that, for every $B\in\mathcal{B}$, we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\limsup_{l\to+\infty}\sup_{s\in\Lambda}\big[\mathbb{F}(l,\mathbf{t})\phi\big[\big\|Q(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(s+l\Omega:Y)}\big]\big]=0,$$

uniformly for $x\in B$.

Definition 6.4.6. Suppose that $\mathbb{D}\subseteq\Lambda\subseteq\mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, and (6.103) holds, $\phi:[0,\infty)\to[0,\infty)$ and $\mathbb{F}:(0,\infty)\times\Lambda\to(0,\infty)$.

(i) By $e-W^{p(\mathbf{u}),\phi,\mathbb{F},2}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ we denote the collection of all functions $Q:\Lambda\times X\to Y$ such that, for every $\mathbf{t}$, $\mathbf{s}\in\Lambda$, $l>0$ and $x\in X$, we find that $Q(\mathbf{t}+\mathbf{u};x)\in L^{p(\mathbf{u})}(\mathbf{s}+l\Omega:Y)$ as well as that, for every $B\in\mathcal{B}$, we have

$$\lim_{l\to+\infty}\limsup_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\sup_{s\in\Lambda}\phi\big[\mathbb{F}(l,\mathbf{t})\cdot\big\|Q(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(s+l\Omega:Y)}\big]=0,$$

uniformly for $x\in B$.

(ii) By $W^{p(\mathbf{u}),\phi,\mathbb{F},2}_{0,\mathbb{D},\mathcal{B}}(\Lambda\times X:Y)$ we denote the collection of all functions $Q:\Lambda\times X\to Y$ such that, for every $\mathbf{t}$, $\mathbf{s}\in\Lambda$, $l>0$ and $x\in X$, we find that $Q(\mathbf{t}+\mathbf{u};x)\in L^{p(\mathbf{u})}(\mathbf{s}+l\Omega:Y)$ as well as that, for every $B\in\mathcal{B}$, we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\limsup_{l\to+\infty}\sup_{s\in\Lambda}\phi\big[\mathbb{F}(l,\mathbf{t})\cdot\big\|Q(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(s+l\Omega:Y)}\big]=0,$$

uniformly for $x\in B$.

Applying the Jensen integral inequality and imposing some additional conditions, we can simply clarify some sufficient conditions ensuring that a function $Q \in (e-)W^{1,\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ $[Q \in (e-)W^{1,\phi,\mathbb{F},1}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)]$ belongs to the space $(e-)W^{1,\phi_1,\mathbb{F}_1,1}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ $[Q \in (e-)W^{1,\phi_1,\mathbb{F}_1,2}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)]$ for appropriately chosen functions $\phi_1(\cdot)$ and $\mathbb{F}_1(\cdot;\cdot)$; this can be also done for the functions introduced in Definition 6.4.1 and Definition 6.4.13 below. Concerning the embedding results between the same classes of (equi)-Weyl multi-dimensional ergodic components with variable exponent, we can apply [377, Corollary 3.3.4], which in particular states that, if $p,\ r \in \mathcal{P}(\mathbb{R}^n)$ and $1 \leqslant r(\mathbf{u}) \leqslant p(\mathbf{u})$ for a. e. $\mathbf{u} \in \mathbb{R}^n$, then for every $F \in M(\mathbb{R}^n : \mathbb{C})$ we have

$$\|F(\mathbf{u})\|_{L^{r(\mathbf{u})}(\mathbf{s}+l\Omega)} \leqslant 2(1+l^n)\|F(\mathbf{u})\|_{L^{p(\mathbf{u})}(\mathbf{s}+l\Omega)}, \quad \mathbf{s} \in \mathbb{R}^n,\ l > 0. \tag{6.104}$$

The translation invariance of spaces of (equi)-Weyl multi-dimensional ergodic components with variable exponent holds if we impose condition (D) or (D)'.

Our next task will be to show that, under certain conditions, the ergodic function spaces introduced in Definition 6.4.1 can be embedded into the corresponding ergodic function spaces introduced in Definition 6.4.4–Definition 6.4.6 (cf. also [649, Example 3.5] for the one-dimensional case, where we have assumed that the function $G(\cdot)$ is monotonically increasing and $p \in \mathcal{P}([0,1])$).

Proposition 6.4.7. *Suppose that* $\mathbb{D} = \Lambda = [0,\infty)^n$, $\Omega = [0,1]^n$, $p \in D_+(\Omega)$, $G : \Lambda \to (0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ *and* $Q : \Lambda \times X \to Y$. *Suppose that* $0 < a < G(\mathbf{t}) \leqslant b < +\infty$ *for all* $\mathbf{t} \in \Lambda$.

(i) *Suppose that the function* $Q(\cdot;\cdot)$ *belongs to the space* $S^{\Omega,p(\mathbf{u}),\phi,G}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ *and there exists a finite real constant* $c > 0$ *such that*

$$\mathbb{F}(l,\mathbf{t})G(\mathbf{t})^{-1}l^n \leqslant c, \quad \mathbf{t} \in \Lambda,\ l > 0. \tag{6.105}$$

Then $Q(\cdot;\cdot) \in e - W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$.

(ii) *Suppose that the function* $Q(\cdot;\cdot)$ *belongs to the space* $S^{\Omega,p(\mathbf{u}),\phi,G,1}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ *and there exists a finite real constant* $c > 0$ *such that*

$$\lim_{l\to+\infty} \limsup_{|\mathbf{t}|\to+\infty} \mathbb{F}(l,\mathbf{t})l^n \sup \phi^{-1}([0,G(\mathbf{t})^{-1}]) \leqslant c, \quad \mathbf{t} \in \Lambda,\ l > 0.$$

Then $Q(\cdot;\cdot) \in e - W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$.

(iii) *Suppose that the function* $Q(\cdot;\cdot)$ *belongs to the space* $S^{\Omega,p(\mathbf{u}),\phi,G,2}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$ *and there exists a finite real constant* $c > 0$ *such that*

$$\lim_{l\to+\infty} \limsup_{|\mathbf{t}|\to+\infty} \phi(\mathbb{F}(l,\mathbf{t})G(\mathbf{t})^{-1}l^n \sup \phi^{-1}([0,1])) \leqslant c, \quad \mathbf{t} \in \Lambda,\ l > 0.$$

Then $Q(\cdot;\cdot) \in e - W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$.

Proof. We will prove only (i). Let $B \in \mathcal{B}$ be fixed. Due to our assumption $p \in D_+(\Omega)$, we have

$$\varphi_{p(\mathbf{u})}(dt) \leqslant d^{p_-+p_+}\varphi_{p(\mathbf{u})}(t), \quad d > 0,\ \mathbf{u} \in \Omega.$$

Define $p_1 : \mathbb{R}^n \to [p_-, p_+]$ by $p_1(\mathbf{u}) := p(u-k)$, if there exists $k \in \mathbb{Z}^n$ such that $\mathbf{u} \in k+\Omega^\circ$, and $p_1(\mathbf{u}) := p_-$, otherwise. Then, clearly, $p_1 \in \mathcal{P}(\mathbb{R}^n)$. Let $s \in \Lambda$ and $l > 0$ be fixed. Then there exists $k \in \mathbb{N}_0^n$ such that $s \in k + \Omega$ so that $s + l\Omega \subseteq k + (l+1)\Omega$. Denote by $k_{l,j} \in [0, \lceil l \rceil + 1]^n$ all points with integer coordinates ($1 \leqslant j \leqslant (\lceil l \rceil + 2)^n$). By our assumption, there exists $t_0 > 0$ such that for all $\mathbf{t} \in \Lambda$ with $|\mathbf{t}| \geqslant t_0$ we have

$$\big[\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)\big]_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon G(\mathbf{t})^{-1},$$

uniformly in $x \in B$, which implies

$$\int_\Omega \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u} \leqslant 1, \tag{6.106}$$

uniformly in $x \in B$. Then we have ($\mathbf{t} \in \Lambda$, $|\mathbf{t}| \geqslant t_0$; $x \in B$):

$$\begin{aligned}
&\int_{s+l\Omega} \varphi_{p_1(\mathbf{u})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u}\\
&\quad\leqslant \int_{k+(l+1)\Omega} \varphi_{p_1(\mathbf{u})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u}\\
&\quad= \int_{(l+1)\Omega} \varphi_{p_1(\mathbf{u}+k)}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u}+k;x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u}\\
&\quad\leqslant \int_{(\lceil l \rceil+1)\Omega} \varphi_{p_1(\mathbf{u}+k)}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u}+k;x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u}\\
&\quad\leqslant \sum_{j=1}^{(\lceil l \rceil+2)^n} \int_{k_{l,j}+\Omega} \varphi_{p_1(\mathbf{u}+k)}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u}+k;x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u}\\
&\quad= \sum_{j=1}^{(\lceil l \rceil+2)^n} \int_{\Omega} \varphi_{p_1(\mathbf{u}+k+k_{l,j})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u}+k+k_{l,j};x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u}\\
&\quad= \sum_{j=1}^{(\lceil l \rceil+2)^n} \int_{\Omega} \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u}+k+k_{l,j};x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}}\right) d\mathbf{u}\\
&\quad= \sum_{j=1}^{(\lceil l \rceil+2)^n} \int_{\Omega} \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u}+k+k_{l,j};x)\|_Y)}{\varepsilon G(\mathbf{t}+\mathbf{u}+k+k_{l,j})^{-1}}\,\frac{G(\mathbf{t})}{G(\mathbf{t}+\mathbf{u}+k+k_{l,j})}\right) d\mathbf{u}
\end{aligned}$$

$$\leqslant (b/a)^{p_-+p_+} \sum_{j=1}^{(\lceil l\rceil+2)^n} \int_{\Omega} \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u}+k+k_{l,j};x)\|_Y)}{\varepsilon G(\mathbf{t}+\mathbf{u}+k+k_{l,j})^{-1}}\right) d\mathbf{u}$$

$$\leqslant (b/a)^{p_-+p_+}(\lceil l\rceil+2)^n \leqslant (1+(b/a))^{p_-+p_+}(\lceil l\rceil+2)^n,$$

since $|\mathbf{t}+\mathbf{u}+k+k_{l,j}| \geqslant |\mathbf{t}|$ for $1 \leqslant j \leqslant (\lceil l\rceil+2)^n$ and (6.106) holds. This simply implies

$$\int_{s+l\Omega} \varphi_{p_1(\mathbf{u})}\left(\frac{\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)}{\varepsilon G(\mathbf{t})^{-1}[(1+(b/a))^{p_-+p_+}(\lceil l\rceil+2)^n]}\right) d\mathbf{u} \leqslant 1$$

and therefore

$$[\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)]_{L^{p(\mathbf{u})}(s+l\Omega)} \leqslant \varepsilon G(\mathbf{t})^{-1}[(1+(b/a))^{p_-+p_+}(\lceil l\rceil+2)^n], \quad \mathbf{s}\in\Lambda,\ l>0.$$

This simply completes the proof with the help of (6.105). □

Remark 6.4.8. It would be interesting to reconsider Proposition 6.4.7 for general exponents $p \in \mathcal{P}(\Omega)$.

We continue by providing two illustrative examples.

Example 6.4.9 (see also [645, Example 4.5] and [649, Example 3.7]). Let $k_1, k_2, \ldots, k_n \in \mathbb{N}_0$, $\Lambda_{k_1,k_2,\ldots,k_n} := (k_1,k_1+1)\times(k_2,k_2+1)\times\cdots\times(k_n,k_n+1)$ and $\mathbb{D} := \Lambda := [0,\infty)^n$ ($X=\{0\}$). Define the function $Q : \Lambda \to [0,\infty)$ by $Q(\mathbf{t}) := Q_{k_1,k_2,\ldots,k_n}(\mathbf{t})$ for $\mathbf{t}\in\Lambda_{k_1,k_2,\ldots,k_n}$, where $Q_{k_1,k_2,\ldots,k_n}(\mathbf{t}) := 0$ if there exists k_i ($1\leqslant i\leqslant n$) such that $k_i \notin \{n^2 : n\in\mathbb{N}_0\}$, and $Q_{k_1,k_2,\ldots,k_n}(\mathbf{t}) := 1$, otherwise. If there do not exist integers $k_1, k_2, \ldots, k_n \in \mathbb{N}_0$ such that $\mathbf{t}\in\Lambda_{k_1,k_2,\ldots,k_n}$, we set $Q(\mathbf{t}) := 0$. Since the interval $[t,t+l] \subseteq [0,\infty)$ contains at most $\sqrt{t+l}-\sqrt{t}+2$ squares of non-negative integers, it can be simply approved that the set $\mathbf{s}+\mathbf{t}+l\Omega$ contains at most

$$\left(2+\frac{l}{\sqrt{l}+\sqrt{|\mathbf{t}|}}\right)^n$$

cells $\Lambda_{k_1,k_2,\ldots,k_n}$ where the function $Q(\cdot)$ is not identically equal to zero. Using an elementary argumentation, it follows that $Q \in e-W^{p,\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda:\mathbb{C})$, provided that $\mathbb{F}(l;\mathbf{t}) := l^{-\sigma}$ for some real number $\sigma<0$ ($p(\mathbf{u})\equiv p\in[1,\infty)$). It is also clear that the function $Q(\cdot)$ is Stepanov p-bounded as well as that $Q \notin S^p_{0,\Lambda,\mathcal{B}}(\Lambda:\mathbb{C})$.

Example 6.4.10 (see also [645, Example 4.6] and [649, Example 3.6]). Let $k_1, k_2, \ldots, k_n \in \mathbb{Z}$, $\Lambda_{k_1,k_2,\ldots,k_n} := (k_1,k_1+1)\times(k_2,k_2+1)\times\cdots\times(k_n,k_n+1)$ and $\mathbb{D} := \Lambda := \mathbb{R}^n$ ($X=\{0\}$). Define the function $Q:\Lambda\to[0,\infty)$ by $Q(\mathbf{t}) := Q_{k_1,k_2,\ldots,k_n}(\mathbf{t})$ for $\mathbf{t}\in\Lambda_{k_1,k_2,\ldots,k_n}$, where $Q_{k_1,k_2,\ldots,k_n}(\mathbf{t}) := 0$ if there exists k_i ($1\leqslant i\leqslant n$) such that $k_i\notin\{n^2 : n\in\mathbb{N}_0\}$, and $Q_{k_1,k_2,\ldots,k_n}(\mathbf{t}) := \sqrt{|k_1|\cdot\cdots\cdot|k_n|}$, otherwise. If there do not exist integers $k_1, k_2, \ldots, k_n\in\mathbb{Z}$ such that $\mathbf{t}\in\Lambda_{k_1,k_2,\ldots,k_n}$, we set $Q(\mathbf{t}) := 0$. If $\mathbb{F}(l;\mathbf{t})$ does not depend on $\mathbf{t}$, it is very simple to show that

$$\limsup_{|\mathbf{t}|\to+\infty} \sup_{s\in\mathbb{R}^n} \|Q(\mathbf{u})\|_{L^p(\mathbf{s}+\mathbf{t}+l\Omega)} = +\infty, \quad l>0,$$

so that $Q(\cdot) \notin e - W^{p,\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda : \mathbb{C})$. On the other hand, a direct calculation shows that

$$\|Q(\mathbf{u})\|_{L^p(\mathbf{s}+\mathbf{t}+l\Omega)} \leqslant c_p l^{n/p}\big(1+|\mathbf{s}|+|\mathbf{t}|+l\big)^{n/2}, \quad \mathbf{s},\ \mathbf{t} \in \mathbb{R}^n,\ l > 0,$$

so that $Q(\cdot) \in W^{p,\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda : \mathbb{C})$ provided that $\mathbb{F}(l;\mathbf{t}) := l^{-\sigma}$ for some real number $\sigma > (n/p) + (n/2)$. In general case, if $p \in D_+(\mathbb{R}^n)$, the above estimate in combination with (6.104) shows that $Q(\cdot) \in W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,\mathbb{D},\mathcal{B}}(\Lambda : \mathbb{C})$ provided that $\mathbb{F}(l;\mathbf{t}) := l^{-\sigma}$ for some real number $\sigma > (n/p^+) + (3n/2)$.

In the following result, we continue our analysis from Theorem 6.4.2.

Theorem 6.4.11. *Assume that the operator function $(R(\mathbf{t}))_{\mathbf{t}>0} \subseteq L(Y,Z)$ is strongly continuous.*

(i) *Suppose that $Q : \mathbb{R}^n \times X \to Y$ satisfies the requirement that its restriction $Q_R(\cdot;\cdot)$ to the set $[0,\infty)^n \times X$ belongs to the space $e - W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,[0,\infty)^n,\mathcal{B}}([0,\infty)^n \times X : Y)$, resp. $W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,[0,\infty)^n,\mathcal{B}}([0,\infty)^n \times X : Y)$. Suppose that $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$, $(a_{k,l})_{k\in\mathbb{N}_0^n}$ is a sequence of positive real numbers such that $\sum_{k\in l\mathbb{N}_0^n} a_{k,l} = 1$ for all $l > 0$, and the value $Q_1(\mathbf{t};x)$, given by (6.100), is well defined for all $t \in -[0,\infty)^n$ and $x \in X$. If $\mathbb{F}_1 : (0,\infty) \times (-[0,\infty)^n) \to (0,\infty)$ satisfies*

$$\lim_{l\to+\infty} \limsup_{|\mathbf{t}|\to+\infty, \mathbf{t}\in-[0,\infty)^n} \sup_{s\in-[0,\infty)^n} \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(2\frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,k-\mathbf{t}-\mathbf{u})} \times \sum_{k\in l\mathbb{N}_0^n} a_{k,l} l^{-n} \varphi(l^n a_{k,l}^{-1})\big[\varphi(\|R(\mathbf{v}+k)\|)\big]_{L^{q(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u} < 1,$$

resp.

$$\limsup_{|\mathbf{t}|\to+\infty, \mathbf{t}\in-[0,\infty)^n} \lim_{l\to+\infty} \sup_{s\in-[0,\infty)^n} \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(2\frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,k-\mathbf{t}-\mathbf{u})} \times \sum_{k\in l\mathbb{N}_0^n} a_{k,l} l^{-n} \varphi(l^n a_{k,l}^{-1})\big[\varphi(\|R(\mathbf{v}+k)\|)\big]_{L^{q(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u} < 1,$$

then we have $Q_1 \in e - W^{p(\mathbf{u}),\phi,\mathbb{F}_1}_{0,-[0,\infty)^n,\mathcal{B}}((-[0,\infty)^n) \times X : Z)$, resp. $Q_1 \in W^{p(\mathbf{u}),\phi,\mathbb{F}_1}_{0,-[0,\infty)^n,\mathcal{B}}((-[0,\infty)^n) \times X : Z)$.

(ii) *Suppose that $Q : \mathbb{R}^n \times X \to Y$ satisfies the requirement that its restriction $Q_R(\cdot;\cdot)$ to the set $[0,\infty)^n \times X$ belongs to the space $e - W^{p(\mathbf{u}),\phi,\mathbb{F},1}_{0,[0,\infty)^n,\mathcal{B}}([0,\infty)^n \times X : Y)$, resp. $W^{p(\mathbf{u}),\phi,\mathbb{F},1}_{0,[0,\infty)^n,\mathcal{B}}([0,\infty)^n \times X : Y)$. Suppose that $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a continuous monotonically increasing function, and the value $Q_1(\mathbf{t};x)$, given by (6.100), is well defined for all $t \in -[0,\infty)^n$ and $x \in X$. If $\mathbb{F}_1 : (0,\infty) \times$*

$(-[0,\infty)^n) \to (0,\infty)$ *satisfies*

$$\lim_{l\to+\infty} \limsup_{|\mathbf{t}|\to+\infty, \mathbf{t}\in-[0,\infty)^n} \sup_{s\in-[0,\infty)^n} \int_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(2\sum_{k\in l\mathbb{N}_0^n} \frac{\phi^{-1}(\varepsilon/\mathbb{F}(l,k-\mathbf{t}-\mathbf{u}))}{\phi^{-1}(\varepsilon/\mathbb{F}_1(l,\mathbf{t}))} \times [\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u} < 1,$$

resp.

$$\limsup_{|\mathbf{t}|\to+\infty, \mathbf{t}\in-[0,\infty)^n} \lim_{l\to+\infty} \sup_{s\in-[0,\infty)^n} \int_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(2\sum_{k\in l\mathbb{N}_0^n} \frac{\phi^{-1}(\varepsilon/\mathbb{F}(l,k-\mathbf{t}-\mathbf{u}))}{\phi^{-1}(\varepsilon/\mathbb{F}_1(l,\mathbf{t}))} \times [\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u} < 1,$$

then we have $Q_1 \in e-W^{p(\mathbf{u}),\phi,\mathbb{F}_1,1}_{0,-[0,\infty)^n,\mathcal{B}}((-[0,\infty)^n)\times X : Z)$, *resp.* $Q_1 \in W^{p(\mathbf{u}),\phi,\mathbb{F}_1,1}_{0,-[0,\infty)^n,\mathcal{B}}((-[0,\infty)^n)\times X : Z)$.

(iii) *Suppose that* $Q : \mathbb{R}^n \times X \to Y$ *satisfies the requirement that its restriction* $Q_R(\cdot;\cdot)$ *to the set* $[0,\infty)^n \times X$ *belongs to the space* $e-W^{p(\mathbf{u}),\phi,\mathbb{F},2}_{0,[0,\infty)^n,\mathcal{B}}([0,\infty)^n \times X : Y)$, *resp.* $W^{p(\mathbf{u}),\phi,\mathbb{F},2}_{0,[0,\infty)^n,\mathcal{B}}([0,\infty)^n \times X : Y)$. *Suppose that* $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ *is a continuous monotonically increasing function, and the value* $Q_1(\mathbf{t};x)$, *given by* (6.100), *is well defined for all* $t \in -[0,\infty)^n$ *and* $x \in X$. *If* $\mathbb{F}_1 : (0,\infty)\times(-[0,\infty)^n) \to (0,\infty)$ *satisfies*

$$\lim_{l\to+\infty} \limsup_{|\mathbf{t}|\to+\infty, \mathbf{t}\in-[0,\infty)^n} \sup_{s\in-[0,\infty)^n} \int_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(2\sum_{k\in l\mathbb{N}_0^n} \frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,k-\mathbf{t}-\mathbf{u})} \times [\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u} < 1,$$

resp.

$$\limsup_{|\mathbf{t}|\to+\infty, \mathbf{t}\in-[0,\infty)^n} \lim_{l\to+\infty} \sup_{s\in-[0,\infty)^n} \int_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(2\sum_{k\in l\mathbb{N}_0^n} \frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,k-\mathbf{t}-\mathbf{u})} \times [\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u} < 1,$$

then we have $Q_1 \in e-W^{p(\mathbf{u}),\phi,\mathbb{F}_1,2}_{0,-[0,\infty)^n,\mathcal{B}}((-[0,\infty)^n)\times X : Z)$, *resp.* $Q_1 \in W^{p(\mathbf{u}),\phi,\mathbb{F}_1,2}_{0,-[0,\infty)^n,\mathcal{B}}((-[0,\infty)^n)\times X : Z)$.

Proof. We will prove only (i), for the class $e-W^{p(\mathbf{u}),\phi,\mathbb{F}}_{0,[0,\infty)^n,\mathcal{B}}([0,\infty)^n \times X : Y)$. Let $B \in \mathcal{B}$ be given, and let $\varepsilon > 0$ be sufficiently small. We know that there exists $l_0 > 0$ such that for each $l \geqslant l_0$ there exists $M_l > 0$ such that for each $\mathbf{t} \in [0,\infty)^n$ with $|\mathbf{t}| > M_l$ and for each $s \in [0,\infty)^n$ we have $\mathbb{F}(l,\mathbf{t})[\phi(\|Q(\mathbf{t}+\mathbf{u};x)\|_Y)]_{L^{p(\mathbf{u})}(s+l\Omega)} < \varepsilon$, $x \in B$. By our assumption, there exists $l_1 > 0$ such that for each $l \geqslant l_1$ there exists $M_l' > 0$ such that for each $\mathbf{t} \in -[0,\infty)^n$ with $|\mathbf{t}| > M_l'$ and for each $s \in -[0,\infty)^n$ we have

$$\int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(2\frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,k-\mathbf{t}-\mathbf{u})} \times \sum_{k\in l\mathbb{N}_0^n} a_{k,l}l^{-n}\varphi(l^n a_{k,l}^{-1})[\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u} \leqslant 1.$$

Let $l_2 = \max(l,l_1)$ and let $M_l'' = \max(M_l, M_l')$. Furthermore, let $\mathbf{t} \in -[0,\infty)^n$ with $|\mathbf{t}| > M_l''$ and let $s \in -[0,\infty)^n$. Then it suffices to show $\mathbb{F}_1(l,\mathbf{t})[\phi(\|Q_1(\mathbf{t}+\mathbf{u};x)\|_Z)]_{L^{p(\mathbf{u})}(s-l[0,1]^n)} < \varepsilon$, $x \in B$, which immediately follows if we prove that

$$\int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q_1(\mathbf{t}+\mathbf{u};x)\|_Z)}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\right)d\mathbf{u} \leqslant 1, \quad x \in B.$$

This follows from the next computation involving the Jensen integral inequality, the Hölder inequality and our assumptions on the function $\phi(\cdot)$:

$$\begin{aligned}
&\int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(\frac{\phi(\|Q_1(\mathbf{t}+\mathbf{u};x)\|_Z)}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\right)d\mathbf{u}\\
&\leqslant \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(\frac{\phi(\sum_{k\in l[0,1]^n} a_{k,l}\int_{k+l[0,1]^n} a_{k,l}^{-1}\|R(\mathbf{v})\|\cdot\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{v};x)\|_Y\,d\mathbf{v})}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\right)d\mathbf{u}\\
&\leqslant \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in l[0,1]^n} a_{k,l}\phi(\int_{k+l[0,1]^n} a_{k,l}^{-1}\|R(\mathbf{v})\|\cdot\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{v};x)\|_Y\,d\mathbf{v})}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\right)d\mathbf{u}\\
&= \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in l[0,1]^n} a_{k,l}\phi(l^{-n}\int_{k+l[0,1]^n} a_{k,l}^{-1}l^n\|R(\mathbf{v})\|\cdot\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{v};x)\|_Y\,d\mathbf{v})}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\right)d\mathbf{u}\\
&\leqslant \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(\frac{\sum_{k\in l[0,1]^n} a_{k,l}l^{-n}\varphi(a_{k,l}^{-1}l^n)\int_{k+l[0,1]^n} \phi(\|R(\mathbf{v})\|\cdot\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{v};x)\|_Y)\,d\mathbf{v}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\right)d\mathbf{u}\\
&\leqslant \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\\
&\quad\times\left(\frac{\sum_{k\in l[0,1]^n} a_{k,l}l^{-n}\varphi(a_{k,l}^{-1}l^n)\int_{k+l[0,1]^n} \varphi(\|R(\mathbf{v})\|)\cdot\phi(\|\check{Q}(\mathbf{t}+\mathbf{u}-\mathbf{v};x)\|_Y)\,d\mathbf{v}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\right)d\mathbf{u}\\
&\leqslant \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\Bigg(\frac{2\sum_{k\in l[0,1]^n} a_{k,l}l^{-n}\varphi(a_{k,l}^{-1}l^n)[\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})}\\
&\quad\times[\phi(\|Q(-\mathbf{t}-\mathbf{u}+k+\mathbf{v};x)\|_Y)]_{L^{p(\mathbf{v})}(l[0,1]^n)}\Bigg)\,d\mathbf{u}
\end{aligned}$$

$$\leqslant \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(\frac{2\sum_{k\in l[0,1]^n} a_{k,l} l^{-n} \varphi(a_{k,l}^{-1} l^n)[\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t})} \frac{\varepsilon}{\mathbb{F}(l,k-\mathbf{t}-\mathbf{u})} \right) d\mathbf{u}$$

$$= \int\limits_{s-l[0,1]^n} \varphi_{p(\mathbf{u})}\left(2\frac{\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,k-\mathbf{t}-\mathbf{u})} \sum_{k\in l\mathbb{N}_0^n} a_{k,l} l^{-n} \varphi(l^n a_{k,l}^{-1})[\varphi(\|R(\mathbf{v}+k)\|)]_{L^{q(\mathbf{v})}(l[0,1]^n)} \right) d\mathbf{u} \leqslant 1,$$

since $|k-\mathbf{t}-\mathbf{u}| \geqslant M_l''$ for all $\mathbf{u} \in s-l[0,1]^n$ and $k \in l\mathbb{N}_0^n$. □

Suppose that $1 \leqslant p < \infty$. We say that a measurable function $f : \mathbb{R} \to X$ is (equi-)Weyl asymptotically p-almost periodic at minus infinity if and only if there exist a bounded continuous (equi-)Weyl p-almost periodic function $f_0 : \mathbb{R} \to X$ and a measurable function $q : \mathbb{R} \to X$ such that $f(t) = f_0(t) + q(t)$, $t \in \mathbb{R}$ and

$$q \in (e-)W_{0,-[0,\infty)}^{p,x,l^{(-1)/p}}(-[0,\infty) : X).$$

The choice of term "bounded continuous" is a bit superfluous but in accordance with our striving to apply [631, Theorem 2.11.4]. We have the following corollary of Theorem 6.4.11.

Corollary 6.4.12. *Let* $1 \leqslant p < \infty$, $1/p + 1/q = 1$ *and let* $(R(t))_{t>0} \subseteq L(X,Y)$ *satisfy*

$$\|R(t)\|_{L(X,Y)} \leqslant \frac{Mt^{\beta-1}}{1+t^{\gamma}}, \quad t > 0 \text{ for some finite constants } \gamma > 1,\ \beta \in (0,1],\ M > 0. \tag{6.107}$$

Let a function $f : \mathbb{R} \to X$ *be equi-Weyl p-almost periodic, resp. Weyl p-almost periodic, and Weyl (equivalently, Stepanov) p-bounded, and let* $q(\beta-1) > -1$ *provided that* $p > 1$, *resp.* $\beta = 1$, *provided that* $p = 1$. *If* $q : \mathbb{R} \to X$ *is Weyl p-bounded and its restriction to* $[0,\infty)$ *is equi-Weyl p-vanishing, resp. Weyl p-vanishing, then the function* $G : \mathbb{R} \to Y$, *given by* (2.46), *is well defined and equi-Weyl asymptotically p-almost periodic at minus infinity, resp. Weyl asymptotically p-almost periodic at minus infinity.*

Proof. Due to [631, Theorem 2.11.4], we find that the function $G : \mathbb{R} \to Y$, defined through (2.46), is bounded continuous and (equi-)Weyl p-almost periodic. To complete the proof, it suffices to apply Theorem 6.4.11 since a trivial computation shows that

$$\sum_{k\geqslant 0}\left(\int\limits_{kl}^{(k+1)l} \|R(t)\|^q \, dt \right)^{1/q} < \infty, \quad \text{provided that } p > 1,$$

and

$$\sum_{k\geqslant 0} \|R(\cdot)\|_{L^{\infty}[kl,(k+1)l]} < \infty, \quad \text{provided that } p = 1.$$ □

6.4.3 Weighted pseudo-ergodic components

In this subsection, we introduce and analyze various generalizations of the space $PAP_0(\mathbb{R}, X, \rho_1, \rho_2)$, which has been analyzed in a series of research papers by now.

Definition 6.4.13. Suppose that $\phi : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to [0,\infty)$, $\mathrm{F} : (0,\infty) \to (0,\infty)$ are given functions and $p \in \mathcal{P}(\Lambda)$, where $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$. For every finite number $T > 0$, we denote by $p_T(\cdot)$ the restriction of function $p(\cdot)$ to Λ_T. We introduce the following function spaces:

$$\begin{aligned}
PAP_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi) := \Big\{ & Q : \Lambda \times X \to Y; \phi(\|Q(\mathbf{t}; x)\|_Y) \in L^{p_T(\mathbf{t})}(\Lambda_T),\ T > 0,\ x \in X \text{ and} \\
& \lim_{T \to +\infty} \mathrm{F}(T)[\phi(\|Q(\mathbf{t}; x)\|_Y)]_{L^{p_T(\mathbf{t})}(\Lambda_T)} = 0, \text{ uniformly in } x \in B \Big\}, \\
PAP^1_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi, \psi) := \Big\{ & Q : \Lambda \times X \to Y; \psi(\|Q(\mathbf{t}; x)\|_Y) \in L^{p_T(\mathbf{t})}(\Lambda_T),\ T > 0,\ x \in X \text{ and} \\
& \lim_{T \to +\infty} \mathrm{F}(T)\phi([\psi(\|Q(\mathbf{t}; x)\|_Y)]_{L^{p_T(\mathbf{t})}(\Lambda_T)}) = 0, \text{ uniformly in } x \in B \Big\},
\end{aligned}$$

and

$$\begin{aligned}
PAP^2_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi, \psi) := \Big\{ & Q : \Lambda \times X \to Y; \psi(\|Q(\mathbf{t}; x)\|_Y) \in L^{p_T(\mathbf{t})}(\Lambda_T),\ T > 0,\ x \in X \text{ and} \\
& \lim_{T \to +\infty} \phi(\mathrm{F}(T)[\psi(\|Q(\mathbf{t}; x)\|_Y)]_{L^{p_T(\mathbf{t})}(\Lambda_T)}) = 0, \text{ uniformly in } x \in B \Big\}.
\end{aligned}$$

It is clear that, if the set Λ is bounded, then there exists a finite real number $T > 0$ such that $\Lambda = \Lambda_T$ and therefore the assumption $\lim_{T \to +\infty} \mathrm{F}(T) = 0$ implies that $Q \in PAP_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$ for any function $Q \in M(\Lambda : X)$ such that $\phi(\|Q(\mathbf{t}; x)\|_Y) \in L^{p_T(\mathbf{t})}(\Lambda_T)$, $T > 0$, $x \in X$. In general, a function $F \in PAP_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$ need not be bounded. If $X = \{0\}$, then we also omit the term "$\mathcal{B}$" from the above notation.

Now we will present a few illustrative examples.

Example 6.4.14 ([710, Example 4.1]). Suppose that $X := \{0\}$, $\mathcal{B} := \{X\}$, $Y := \mathbb{C}$, $Q(t) := 2 + \sin t$, $t \in \mathbb{R}$, $\rho(t) := 2e^t$, $t \geqslant 0$, $\rho(t) := e^{-t}$, $t < 0$,

$$\mathrm{F}_\sigma(t) := t^{-\sigma} \left[\int_{-t}^{t} \rho(s)\, ds \right]^{-1}, \quad t > 0,\ \sigma \in \mathbb{R},$$

$\phi(t) := t\rho(t)$, $t \geqslant 0$ and $p(t) \equiv 1$. Then

$$\mathrm{F}_0(T)[\phi(|Q(\cdot)|)]_{L^1[-T,T]} = \frac{e^T(12 - \cos T + \sin T) - 11}{6(e^T - 1)}, \quad T > 0.$$

This implies $Q \in PAP_{0,1}(\mathbb{R}, \mathcal{B}, \mathrm{F}_\sigma, \phi)$ if and only if $\sigma > 0$.

Example 6.4.15. Suppose now that $1/q(\mathbf{t}) = 1/p(\mathbf{t})+1/r(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$, where $p, q, r \in \mathcal{P}(\mathbb{R}^n)$, $Q \in \mathrm{PAP}_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$ and a function $\mathrm{F}_1 : (0,\infty) \to (0,\infty)$ satisfies

$$\frac{\mathrm{F}_1(T)[1]_{L^{r_T(\mathbf{t})}(\Lambda_T)}}{\mathrm{F}(T)} \leqslant M, \quad T > 0,$$

for some finite constant $M > 0$. Applying the Hölder inequality and this assumption, we easily get

$$\begin{aligned}\mathrm{F}_1(T)[\phi(\|Q(\mathbf{t};x)\|_Y)]_{L^{q_T(\mathbf{t})}(\Lambda_T)} &\leqslant \frac{2\mathrm{F}_1(T)[1]_{L^{r_T(\mathbf{t})}(\Lambda_T)}}{\mathrm{F}(T)}\mathrm{F}(T)[\phi(\|Q(\mathbf{t};x)\|_Y)]_{L^{p_T(\mathbf{t})}(\Lambda_T)} \\ &\to 0, \quad T \to +\infty,\end{aligned}$$

so that $Q \in \mathrm{PAP}_{0,q}(\Lambda, \mathcal{B}, \mathrm{F}_1, \phi)$.

Example 6.4.16. Let $\Lambda = [0,\infty)$, $\phi(x) = x$ for $x \geqslant 0$, and let $p \in \mathcal{P}([0,\infty))$ be given by $p(t) := 1 - \ln t$, $t \in (0,1)$ and $p(t) := 1$, $t \geqslant 1$. Then $p \notin D_+([0,\infty))$. If $\int_0^{+\infty} |Q(t)|\,dt < 1$, $Q(t) = 0$ for $0 \leqslant t \leqslant 1/2$ and $Q(t) \geqslant 1$ for $t \in (1/2,1)$, then it can be easily shown that $Q \in \mathrm{PAP}_{0,p}(\Lambda, \mathrm{F}, \phi)$ for any function $\mathrm{F} : (0,\infty) \to (0,\infty)$ such that $\lim_{T\to+\infty} \mathrm{F}(T) = 0$.

In the next theorem, we analyze the translation invariance of spaces introduced in Definition 6.4.13 (the first part of proposition generalizes [631, Proposition 3.3.6], where we have analyzed the translation invariant properties of space $\mathrm{PAP}_0(\mathbb{R}, X, \rho_1, \rho_2)$).

Theorem 6.4.17. *Suppose that* $\phi : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to [0,\infty)$, $\mathrm{F} : (0,\infty) \to (0,\infty)$, $\mathrm{F}_1 : (0,\infty) \to (0,\infty)$ *are given functions and* $p \in \mathcal{P}(\Lambda)$, *where* $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$. *For given* $\mathbf{t}_0 \in \Lambda$ *and* $x_0 \in X$, *we set* $Q_{\mathbf{t}_0;x_0} := Q(\mathbf{t}+\mathbf{t}_0; x+x_0)$, $\mathbf{t} \in -\mathbf{t}_0+\Lambda$, $x \in X$, $\Lambda_{\mathbf{t}_0,T} := \{\mathbf{t} \in \Lambda;\ |\mathbf{t} - \mathbf{t}_0| \leqslant T\}$ *(*$T > 0$*) and* $\mathcal{B}_{x_0} := \{-x_0 + B : B \in \mathcal{B}\}$. *Define* $p_{\mathbf{t}_0} : -\mathbf{t}_0 + \Lambda \to [1,\infty]$ *by* $p_{\mathbf{t}_0}(\mathbf{t}) := p(\mathbf{t} + \mathbf{t}_0)$, $\mathbf{t} \in -\mathbf{t}_0 + \Lambda$. *Then the following holds:*
(i) *If there exists a finite real number* $c > 0$ *such that* $\mathrm{F}_1(T) \leqslant c\mathrm{F}(T)$ *for all* $T \geqslant 1$ *and*

$$\lim_{T\to+\infty} \mathrm{F}_1(T)[\phi(\|Q(\mathbf{t};x)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\setminus\Lambda_T)} = 0, \quad \textit{uniformly in } x \in B, \tag{6.108}$$

then $Q_{\mathbf{t}_0;x_0} \in \mathrm{PAP}_{0,p_{\mathbf{t}_0}}(-\mathbf{t}_0 + \Lambda, \mathcal{B}_{x_0}, \mathrm{F}_1, \phi)$.
(ii) *Suppose, in addition, that* $\phi(\cdot)$ *is monotonically increasing as well as that there exists a bounded function* $\varphi : [0,\infty)^2 \to [0,\infty)$ *such that*

$$\phi(2x + 2y) \leqslant \varphi(x,y)[\phi(x) + \phi(y)], \quad x,\ y \geqslant 0. \tag{6.109}$$

If there exists a finite real number $c > 0$ *such that* $\mathrm{F}_1(T) \leqslant c\mathrm{F}(T)$ *for all* $T \geqslant 1$ *and*

$$\lim_{T\to+\infty} \mathrm{F}_1(T)\phi([\psi(\|Q(\mathbf{t};x)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\setminus\Lambda_T)}) = 0, \quad \textit{uniformly in } x \in B, \tag{6.110}$$

then $Q_{\mathbf{t}_0;x_0} \in \mathrm{PAP}^1_{0,p_{\mathbf{t}_0}}(-\mathbf{t}_0 + \Lambda, \mathcal{B}_{x_0}, \mathrm{F}_1, \phi, \psi)$.

(iii) *Suppose, in addition, that $\phi(\cdot)$ is monotonically increasing and that there exist a bounded function $\varphi : [0,\infty)^2 \to [0,\infty)$ such that* (6.109) *holds and a function $\theta : [0,\infty) \to [0,\infty)$ such that $\phi(cx) \leqslant \theta(c)\phi(x)$, $c,\ x \geqslant 0$ and $\theta(F_1(T)/F(T)) \leqslant M$, $T \geqslant 1$ for a finite real constant $M > 0$. If*

$$\lim_{T\to+\infty} \phi\big(\mathrm{F}_1(T)[\psi(\|Q(\mathbf{t};x)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T)}\big) = 0, \quad \textit{uniformly in } x \in B, \tag{6.111}$$

then $Q_{\mathbf{t}_0;x_0} \in \mathrm{PAP}^2_{0,p_{\mathbf{t}_0}}(-\mathbf{t}_0 + \Lambda, \mathcal{B}_{x_0}, \mathrm{F}_1, \phi, \psi)$.

Proof. To prove (i), fix $x \in X$ and $B \in \mathcal{B}$ with $x \in B$. Recall that

$$\begin{aligned}
&[\phi(\|Q_{\mathbf{t}_0;x_0}\|_Y)]_{L^{(p_{\mathbf{t}_0})_T(\mathbf{t})}((-\mathbf{t}_0+\Lambda)_T)} \\
&\quad = \inf\Bigg\{\lambda > 0 : \int\limits_{(-\mathbf{t}_0+\Lambda)_T} \varphi_{(p_{\mathbf{t}_0})_T(\mathbf{t})}\bigg(\frac{\phi(\|Q(\mathbf{t}+\mathbf{t}_0; x+x_0)\|_Y)}{\lambda}\bigg)\,d\mathbf{t} \leqslant 1\Bigg\} \\
&\quad = \inf\Bigg\{\lambda > 0 : \int\limits_{\Lambda_{\mathbf{t}_0,T}} \varphi_{p(\mathbf{t})}\bigg(\frac{\phi(\|Q(\mathbf{t}; x+x_0)\|_Y)}{\lambda}\bigg)\,d\mathbf{t} \leqslant 1\Bigg\} \\
&\quad = \inf\Bigg\{\lambda > 0 : \int\limits_{\Lambda_{\mathbf{t}_0,T}\cap\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{\phi(\|Q(\mathbf{t}; x+x_0)\|_Y)}{\lambda}\bigg)\,d\mathbf{t} \\
&\qquad + \int\limits_{\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{\phi(\|Q(\mathbf{t}; x+x_0)\|_Y)}{\lambda}\bigg)\,d\mathbf{t} \leqslant 1\Bigg\}.
\end{aligned}$$

We claim that

$$\begin{aligned}
&[\phi(\|Q_{\mathbf{t}_0;x_0}\|_Y)]_{L^{(p_{\mathbf{t}_0})_T(\mathbf{t})}((-\mathbf{t}_0+\Lambda)_T)} \\
&\quad \leqslant 2[\phi(\|Q(\mathbf{t};x+x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_T)} + 2[\phi(\|Q(\mathbf{t};x+x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T)} \\
&\quad := 2A + 2B.
\end{aligned} \tag{6.112}$$

Let $\varepsilon > 0$ be arbitrary. Then there exist $\lambda_1 \in (A, A + (\varepsilon/2))$ and $\lambda_2 \in (B, B + (\varepsilon/2))$ such that

$$\int\limits_{\Lambda_{\mathbf{t}_0,T}\cap\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{\phi(\|Q(\mathbf{t}; x+x_0)\|_Y)}{\lambda_1}\bigg)\,d\mathbf{t} \leqslant 1$$

and

$$\int\limits_{\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{\phi(\|Q(\mathbf{t}; x+x_0)\|_Y)}{\lambda_2}\bigg)\,d\mathbf{t} \leqslant 1.$$

This implies $\lambda := 2\lambda_1 + 2\lambda_2 \in (2A + 2B, 2A + 2B + \varepsilon)$,

$$\begin{aligned}
&\int\limits_{\Lambda_{\mathbf{t}_0,T}\cap\Lambda_T} \varphi_{p(\mathbf{t})}\left(\frac{\phi(\|Q(\mathbf{t}; x + x_0)\|_Y)}{\lambda}\right) d\mathbf{t} \\
&\quad \leqslant \int\limits_{\Lambda_{\mathbf{t}_0,T}\cap\Lambda_T} \varphi_{p(\mathbf{t})}\left(\frac{\phi(\|Q(\mathbf{t}; x + x_0)\|_Y)}{2\lambda_1}\right) d\mathbf{t} \leqslant 1/2, \\
&\int\limits_{\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T} \varphi_{p(\mathbf{t})}\left(\frac{\phi(\|Q(\mathbf{t}; x + x_0)\|_Y)}{\lambda}\right) d\mathbf{t} \\
&\quad \leqslant \int\limits_{\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T} \varphi_{p(\mathbf{t})}\left(\frac{\phi(\|Q(\mathbf{t}; x + x_0)\|_Y)}{2\lambda_2}\right) d\mathbf{t} \leqslant 1/2,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int\limits_{\Lambda_{\mathbf{t}_0,T}\cap\Lambda_T} \varphi_{p(\mathbf{t})}\left(\frac{\phi(\|Q(\mathbf{t}; x + x_0)\|_Y)}{\lambda}\right) d\mathbf{t} \\
&\quad + \int\limits_{\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T} \varphi_{p(\mathbf{t})}\left(\frac{\phi(\|Q(\mathbf{t}; x + x_0)\|_Y)}{\lambda}\right) d\mathbf{t} \leqslant 1.
\end{aligned} \tag{6.113}$$

This implies (6.112) due to (6.113) and the fact that $\varepsilon > 0$ was arbitrary. Since we have assumed that there exists a finite real number $c > 0$ such that $\mathrm{F}_1(T) \leqslant c\mathrm{F}(T)$ for all $T \geqslant 1$ and (6.108) holds, this completes the proof of (i) in a routine manner. To prove (ii), notice first that (6.112), with the function $\phi(\cdot)$ replaced therein with the function $\psi(\cdot)$, (6.109) and our assumption that $\phi(\cdot)$ is monotonically increasing together imply that

$$\begin{aligned}
&\mathrm{F}_1(T)\phi\big([\psi(\|Q_{\mathbf{t}_0;x_0}\|_Y)]_{L^{(p_{\mathbf{t}_0})_T(\mathbf{t})}((-\mathbf{t}_0+\Lambda)_T)}\big) \\
&\quad \leqslant \mathrm{F}_1(T)\phi\big(2[\psi(\|Q(\mathbf{t}; x + x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_T)} + 2[\psi(\|Q(\mathbf{t}; x + x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T)}\big) \\
&\quad \leqslant \mathrm{F}_1(T) \cdot \varphi\big([\phi(\|Q(\mathbf{t}; x + x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_T)}, [\psi(\|Q(\mathbf{t}; x + x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T)}\big) \\
&\qquad \cdot \big[\phi\big([\psi(\|Q(\mathbf{t}; x + x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_T)}\big) + \phi\big([\psi(\|Q(\mathbf{t}; x + x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\smallsetminus\Lambda_T)}\big)\big].
\end{aligned}$$

Since we have assumed that $\varphi(\cdot,\cdot)$ is bounded as well as that there exists a finite real number $c > 0$ such that $\mathrm{F}_1(T) \leqslant c\mathrm{F}(T)$ for all $T \geqslant 1$ and (6.110) holds, this completes the proof of (ii) in a routine manner. To prove (iii), notice first that (6.112), with the function $\phi(\cdot)$ replaced therein with the function $\psi(\cdot)$, (6.109) and our assumption that $\phi(\cdot)$ is monotonically increasing together imply that

$$\begin{aligned}
&\phi\big(\mathrm{F}_1(T)[\psi(\|Q_{\mathbf{t}_0;x_0}\|_Y)]_{L^{(p_{\mathbf{t}_0})_T(\mathbf{t})}((-\mathbf{t}_0+\Lambda)_T)}\big) \\
&\quad \leqslant \phi\big(2\mathrm{F}_1(T)[\psi(\|Q(\mathbf{t}; x + x_0)\|_Y)]_{L^{p(\mathbf{t})}(\Lambda_T)}
\end{aligned}$$

$$
\begin{aligned}
&+ 2\mathrm{F}_1(T)\big[\psi\big(\|Q(\mathbf{t}; x + x_0)\|_Y\big)\big]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\setminus\Lambda_T)}\big)\\
&\leqslant \varphi\big(2\mathrm{F}_1(T)\big[\psi\big(\|Q(\mathbf{t}; x + x_0)\|_Y\big)\big]_{L^{p(\mathbf{t})}(\Lambda_T)}, 2\mathrm{F}_1(T)\big[\psi\big(\|Q(\mathbf{t}; x + x_0)\|_Y\big)\big]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\setminus\Lambda_T)}\big)\\
&\quad \cdot \big[\phi\big(\mathrm{F}_1(T)\big[\psi\big(\|Q(\mathbf{t}; x + x_0)\|_Y\big)\big]_{L^{p(\mathbf{t})}(\Lambda_T)}\big)\\
&\quad + \phi\big(\mathrm{F}_1(T)\big[\psi\big(\|Q(\mathbf{t}; x + x_0)\|_Y\big)\big]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\setminus\Lambda_T)}\big)\big]\\
&\leqslant \text{Const.} \cdot \big[\theta\big(\mathrm{F}_1(T)/\mathrm{F}(T)\big) \cdot \phi\big(\mathrm{F}(T)\big[\psi\big(\|Q(\mathbf{t}; x + x_0)\|_Y\big)\big]_{L^{p(\mathbf{t})}(\Lambda_T)}\big)\\
&\quad + \phi\big(\mathrm{F}_1(T)\big[\psi\big(\|Q(\mathbf{t}; x + x_0)\|_Y\big)\big]_{L^{p(\mathbf{t})}(\Lambda_{\mathbf{t}_0,T}\setminus\Lambda_T)}\big)\big].
\end{aligned}
$$

Since we have assumed that $\theta(\mathrm{F}_1(T)/\mathrm{F}(T)) \leqslant M$, $T \geqslant 1$ for a finite real constant $M > 0$ and (6.111) holds, this completes (iii) in a routine manner. □

In the remainder of subsection, we will consider the class $\mathrm{PAP}_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$, only, because all established results admit very simple reformulations for the classes $\mathrm{PAP}^1_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi, \psi)$ and $\mathrm{PAP}^2_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi, \psi)$. Concerning the weighted pseudo-ergodicity of the function

$$
\mathbf{t} \mapsto Q_2(\mathbf{t}; x) \equiv \int_{-\infty}^{t} R(\mathbf{t} - \mathbf{s})Q(\mathbf{s}; x)\, d\mathbf{s}, \quad t \in \mathbb{R}^n,\ x \in X,
$$

the best we can do in the present situation is to show the following.

Theorem 6.4.18. *Suppose that* $1 \leqslant p_1 < \infty$, $\Lambda = \mathbb{R}^n$, $p \in D_+(\mathbb{R}^n)$ *satisfies* $p_+ \leqslant p_1$ *and* $1/p(\mathbf{t}) + 1/q(\mathbf{t}) = 1$, $\mathbf{t} \in \mathbb{R}^n$. *If* $\varphi : [0, \infty) \to [0, \infty)$, $\phi : [0, \infty) \to [0, \infty)$ *is a convex monotonically increasing function satisfying* $\phi(xy) \leqslant \varphi(x)\phi(y)$ *for all* $x,\ y \geqslant 0$, $\psi : [0, \infty) \to [0, \infty)$, $\mathrm{F} : (0, \infty) \to (0, \infty)$, $\mathrm{F}_1 : (0, \infty) \to (0, \infty)$, $Q \in \mathrm{PAP}_{0,p_1}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$ *and the value of* $Q_2(\mathbf{t}; x)$ *is well defined for all* $t \in \mathbb{R}^n$ *and* $x \in X$. *If for each* $\varepsilon > 0$ *and* $B \in \mathcal{B}$ *there exists a finite real number* $T_1 > 0$, *as large as we want, such that for each* $\mathbf{t} \in \mathbb{R}^n$ *with* $|\mathbf{t}| \leqslant T_1$ *there exists a finite real number* $T_{\mathbf{t}} > 0$, *as large as we want, and a sequence* $(a_{k,\mathbf{t}})_{k\in T_{\mathbf{t}}\cdot\mathbb{N}_0^n}$ *such that* $\sum_{k\in T_{\mathbf{t}}\cdot\mathbb{N}_0^n} a_{k,\mathbf{t}} = 1$ *and*

$$
\begin{aligned}
&\int_{\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{a_{0,\mathbf{t}}T_{\mathbf{t}}^{-n} \int_{T_{\mathbf{t}}[0,1]^n} \varphi(a_{0,\mathbf{t}}^{-1}T_{\mathbf{t}}^n\|R(\mathbf{s})\|)\phi(\|Q(\mathbf{t} - \mathbf{s}; x)\|_Y)\, d\mathbf{s}}{\varepsilon/F_1(T)}\\
&\quad + \frac{4\sum_{k\in T_{\mathbf{t}}\cdot(\mathbb{N}_0^n\setminus\{\mathbf{0}\})} a_{k,\mathbf{t}}T_{\mathbf{t}}^{-n}\varphi(a_{k,\mathbf{t}}^{-1}T_{\mathbf{t}}^n)(1 + T_{\mathbf{t}}^n)[\varphi(\|R(\mathbf{s} + k)\|)]_{L^{q(\mathbf{s})}(T_{\mathbf{t}}[0,1]^n)}F_1(T)}{F(\sup\{|\mathbf{r}| : \mathbf{r} \in \mathbf{t} - k - T_{\mathbf{t}}[0, 1]^n\})}\bigg)\, d\mathbf{t} \leqslant 1,
\end{aligned}
\tag{6.114}
$$

for any $x \in B$ *and* $T \geqslant T_1$, *then* $Q_2 \in \mathrm{PAP}_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}_1, \phi)$. *Here, we assume that* $R : (0, \infty)^n \to L(Y, Z)$ *is strongly continuous and satisfies the requirement that all terms in* (6.114) *are well defined.*

Proof. For simplicity, we will not distinguish $p(\cdot)$ and its restrictions henceforth. Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be given. By our assumption, we know that there exists a finite real number $T_0 > 0$ such that

$$\int_{\Lambda_T} [\phi(\|Q(\mathbf{s};x)\|_Y)]^{p_1}\,d\mathbf{s} \leqslant (\varepsilon/\mathrm{F}(T))^{p_1}, \quad T \geqslant T_0,\ x \in B. \tag{6.115}$$

Let $T_1 > T_0$ be determined from our condition. We will prove that

$$\int_{\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{\phi(\|Q_2(\mathbf{t};x)\|_Z)}{\varepsilon/\mathrm{F}_1(T)}\bigg)\,d\mathbf{t} \leqslant 1, \quad T \geqslant T_1,\ x \in B, \tag{6.116}$$

which immediately implies $\mathrm{F}_1(T)[\phi(\|Q_2(\mathbf{t};x)\|_Z)]_{L^{p(\mathbf{t})}(\Lambda_T)} < \varepsilon$, $T \geqslant T_1$, $x \in B$ and completes the proof. Let $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \leqslant T_1$ be fixed, and let $T_{\mathbf{t}} > 0$ be chosen in accordance with our condition as well as condition that any component of the tuple $\mathbf{t}-T_{\mathbf{t}}(1,1,\ldots,1)$ belongs to the interval $(-\infty, -T_0/\sqrt{n}]$. Then we have

$$\sup\{|\mathbf{r}| : \mathbf{r} \in \mathbf{t} - k - T_{\mathbf{t}}[0,1]^n\} \geqslant \big|\mathbf{t} - k - T_{\mathbf{t}}(1,1,\ldots,1)\big| \geqslant \big|\mathbf{t} - T_{\mathbf{t}}(1,1,\ldots,1)\big| \geqslant T_0, \tag{6.117}$$

and therefore, due to (6.115) and Lemma 1.1.7(ii),

$$\begin{aligned}
&[\phi(\|Q(\mathbf{t}-\mathbf{s}-k;x)\|_Y)]_{L^{p(\mathbf{s})}(T_{\mathbf{t}}[0,1]^n)} \\
&\quad\leqslant 2(1+T_{\mathbf{t}}^n)\bigg(\int_{\mathbf{t}-k-T_{\mathbf{t}}[0,1]^n} [\phi(\|Q(\mathbf{s};x)\|_Y)]^{p_1}\,d\mathbf{s}\bigg)^{1/p_1} \\
&\quad\leqslant 2(1+T_{\mathbf{t}}^n)\bigg(\int_{\Lambda_{\sup\{|\mathbf{r}|:\mathbf{r}\in\mathbf{t}-k-T_{\mathbf{t}}[0,1]^n\}}} [\phi(\|Q(\mathbf{s};x)\|_Y)]^{p_1}\,d\mathbf{s}\bigg)^{1/p_1} \\
&\quad\leqslant 2(1+T_{\mathbf{t}}^n)\bigg(\frac{\varepsilon}{\mathrm{F}(\sup\{|\mathbf{r}| : \mathbf{r} \in \mathbf{t}-k-T_{\mathbf{t}}[0,1]^n\})}\bigg)^{p_1}.
\end{aligned}$$

Applying this estimate, the Jensen integral inequality, the Hölder inequality, our assumption on the function $\phi(\cdot)$, and the estimate (6.114), we obtain as before

$$\begin{aligned}
&\int_{\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{\phi(\|Q_2(\mathbf{t};x)\|_Z)}{\varepsilon/\mathrm{F}_1(T)}\bigg)\,d\mathbf{t} \\
&\quad\leqslant \int_{\Lambda_T} \varphi_{p(\mathbf{t})}\bigg(\frac{a_{0,\mathbf{t}}T_{\mathbf{t}}^{-n}\int_{T_{\mathbf{t}}[0,1]^n}\varphi(T_{\mathbf{t}}^n\|R(\mathbf{s})\|)\phi(\|Q(\mathbf{t}-\mathbf{s};x)\|_Y)\,d\mathbf{s}}{\varepsilon/\mathrm{F}_1(T)} \\
&\qquad+\frac{4\sum_{k\in T_{\mathbf{t}}\cdot(\mathbb{N}_0^n\setminus\{\mathbf{0}\})} a_{k,\mathbf{t}}T_{\mathbf{t}}^{-n}\varphi(a_{k,\mathbf{t}}^{-1}T_{\mathbf{t}}^n)(1+T_{\mathbf{t}}^n)[\varphi(\|R(\mathbf{s}+k)\|)]_{L^{q(\mathbf{s})}(T_{\mathbf{t}}[0,1]^n)}\mathrm{F}_1(T)}{\mathrm{F}(\sup\{|\mathbf{r}| : \mathbf{r} \in \mathbf{t}_j - k - T_{\mathbf{t}_j}[0,1]^n\})}\bigg)\,d\mathbf{t} \leqslant 1,
\end{aligned}$$

for any $T \geqslant T_1$ and $x \in B$, so that (6.116) holds true. □

Remark 6.4.19.

(i) The ergodicity of component in [631, Lemma 2.12.3] can be deduced from Theorem 6.4.18. In this lemma, we have assumed that $I = \mathbb{R}$ as well as that the operator family $(R(t))_{t>0} \subseteq L(X)$ satisfies the requirement that there exist real numbers $M > 0$, $c > 0$ and $\beta \in (0,1]$ such that $\|R(t)\| \leqslant Me^{-ct}t^{\beta-1}$, $t > 0$. It would be valuable to clarify some sufficient conditions for applications of Theorem 6.4.18, provided that the operator family $(R(t))_{t>0}$ has the growth order of type (6.107).

(ii) In the case that $p(\mathbf{t}) \equiv p_1$, the term "$4(1 + T_{\mathbf{t}}^n)$" in the second addend of (6.114) can be replaced with the term "2".

The method proposed in the proof of previous theorem can be used to derive some results about the convolution invariance of space $PAP_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$. Let $I_1 = [0,\infty)^n, \ldots, I_{2^n}$ be the orthants in $\mathbb{R}^n$. For each $j \in \mathbb{N}_{2^n}$, let $1 \leqslant a_j^1 < a_j^2 < \cdots < a_j^{k_j} \leqslant n$ be the corresponding axes for which all components of the points from I_j have negative values; we define

$$\Psi_j(t_1,\ldots,t_n) := \Psi\big(t_1\sigma(1),\ldots,t_n\sigma(n)\big), \quad \mathbf{t} = (t_1,\ldots,t_n) \in \mathbb{R}^n,$$

and

$$\check{Q}_j(t_1,\ldots,t_n) := Q\big(t_1\sigma(1),\ldots,t_n\sigma(n)\big) := Q(\mathbf{t}_j), \quad \mathbf{t} = (t_1,\ldots,t_n) \in \mathbb{R}^n,$$

where $\sigma(i) := -1$ if $i \notin \{a_j^1, a_j^2, \ldots, a_j^{k_j}\}$ and $\sigma(i) := 1$, if $i \in \mathbb{N}_n \setminus \{a_j^1, a_j^2, \ldots, a_j^{k_j}\}$. Using the decomposition

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Psi(\mathbf{t}-\mathbf{s})Q(\mathbf{s};x)\,d\mathbf{s} \\
&\quad = \sum_{j=1}^{2^n} \int_{\mathbf{t}+I_j} \Psi(\mathbf{t}-\mathbf{s})Q(\mathbf{s};x)\,d\mathbf{s} \\
&\quad = \sum_{j=1}^{2^n} \int_{I_1} \Psi_j(\mathbf{s})\check{Q}_j\big(t_1\sigma(1)-s_1,\ldots,t_{a_j^1}-s_{a_j^1},\ldots,t_{a_j^{k_j}}-s_{a_j^{k_j}},\ldots,t_n-s_n;x\big)\,d\mathbf{s}
\end{aligned}$$

and the argumentation contained in the proof of Theorem 6.4.18, with $\|R(\cdot)\|$ replaced therein by $\psi_j(\cdot)$, we may deduce the following (the only thing worth noting is that all variables $s_1,\ldots,s_n$ in the last sum are taken with the sign minus so that the estimate (6.117) and the computation following it can be applied again).

Theorem 6.4.20. *Suppose that* $1 \leqslant p_1 < \infty$, $\Lambda = \mathbb{R}^n$, $p \in D_+(\mathbb{R}^n)$ *satisfies* $p_+ \leqslant p_1$ *and* $1/p(\mathbf{t}) + 1/q(\mathbf{t}) = 1$, $\mathbf{t} \in \mathbb{R}^n$. *If* $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ *is a* *convex monotonically increasing function satisfying* $\phi(xy) \leqslant \varphi(x)\phi(y)$ *for all* $x,\ y \geqslant 0$, $\psi : [0,\infty) \to [0,\infty)$, $\mathrm{F} : (0,\infty) \to (0,\infty)$, $\mathrm{F}_1 : (0,\infty) \to (0,\infty)$, $\check{Q}_j \in \mathrm{PAP}_{0,p_1}(\Lambda, \mathcal{B}, \mathrm{F}, \phi)$ *for all* $j \in \mathbb{N}_{2^n}$*, the value of all integrals*

$$\int\limits_{\mathbf{t}+I_j\mathbb{R}^n} \Psi(\mathbf{t}-\mathbf{s})Q(\mathbf{s};x)\,d\mathbf{s}$$

and its sum

$$Q_3(\mathbf{t};x) \equiv \int\limits_{\mathbb{R}^n} \Psi(\mathbf{t}-\mathbf{s})Q(\mathbf{s};x)\,d\mathbf{s}$$

are well defined for all $t \in \mathbb{R}^n$ *and* $x \in X$*. If for each* $\varepsilon > 0$ *and* $B \in \mathcal{B}$ *there exists a finite real number* $T_1 > 0$*, as large as we want, such that for each* $\mathbf{t} \in \mathbb{R}^n$ *with* $|\mathbf{t}| \leqslant T_1$ *there exists a finite real number* $T_{\mathbf{t}} > 0$*, as large as we want, and a sequence* $(a_{k,\mathbf{t}})_{k\in T_{\mathbf{t}}\cdot\mathbb{N}_0^n}$ *such that* $\sum_{k\in T_{\mathbf{t}}\cdot\mathbb{N}_0^n} a_{k,\mathbf{t}} = 1$ *and*

$$\begin{aligned}\int\limits_{\Lambda_T} \varphi_{p(\mathbf{t})}\Bigg(&\sum_{j=1}^{2^n} \frac{a_{0,\mathbf{t}}2^{-n}T_{\mathbf{t}}^{-n}\int_{T_{\mathbf{t}}[0,1]^n}\varphi(a_{0,\mathbf{t}}^{-1}2^nT_{\mathbf{t}}^n|\Psi_j(\mathbf{s})|)\phi(\|\check{Q}_j(\mathbf{t}-\mathbf{s};x)\|_Y)\,d\mathbf{s}}{\varepsilon/\mathrm{F}_1(T)}\\ &+4\sum_{j=1}^{2^n}\sum_{k\in T_{\mathbf{t}}\cdot(\mathbb{N}_0^n)\smallsetminus\{0\}} a_{k,\mathbf{t}}2^{-n}T_{\mathbf{t}}^{-n}\varphi(a_{k,\mathbf{t}}^{-1}2^nT_{\mathbf{t}}^n)(1+T_{\mathbf{t}}^n)\\ &\times\big[\varphi(|\Psi_j(\mathbf{s}+k)|)\big]_{L^{q(\mathbf{s})}(T_{\mathbf{t}}[0,1]^n)}F_1(T)\big[F\big(\sup\{|\mathbf{r}| : \mathbf{r}\in\mathbf{t}_j-k-T_{\mathbf{t}}[0,1]^n\}\big)\big]^{-1}\Bigg)\,d\mathbf{t} \leqslant 1,\end{aligned} \tag{6.118}$$

holds for any $x \in B$ *and* $T \geqslant T_1$*, then* $Q_2 \in \mathrm{PAP}_{0,p}(\Lambda, \mathcal{B}, \mathrm{F}_1, \phi)$*. Here, we assume that* $\Psi(\cdot)$ *satisfies the requirement that all terms in* (6.118) *are well defined.*

We will introduce here only one general definition of an asymptotically almost periodic function with variable exponent. Let $\mathcal{X}_\Lambda$ denote any of the spaces of (Stepanov, Weyl, Besicovitch) almost periodic functions $F : \Lambda \times X \to Y$ considered in the existing literature, and let $\mathcal{Q}_\Lambda$ denote any of the spaces of weighted ergodic spaces introduced and analyzed in this section.

Definition 6.4.21. Suppose that the set $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ is unbounded and $F : \Lambda \times X \to Y$. Then we say that $F(\cdot;\cdot)$ is asymptotically $(\mathcal{X}_\Lambda, \mathcal{Q}_\Lambda)$-almost periodic if and only if there exist a function $G(\cdot;\cdot) \in \mathcal{X}_\Lambda$ and a function $Q \in \mathcal{Q}_\Lambda$ such that $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in \Lambda$ and $x \in X$; if, moreover, there exists a function $\tilde{G} \in \mathcal{X}_{\mathbb{R}^n}$ such that $\tilde{G}(\mathbf{t};x) = G(\mathbf{t};x)$ for all $\mathbf{t} \in \Lambda$ and $x \in X$, then we say that the function $F(\cdot;\cdot)$ is strongly asymptotically $(\mathcal{X}_\Lambda, \mathcal{Q}_\Lambda)$-almost periodic.

There is no need to say that the uniqueness of decomposition $F = G + Q$ in Definition 6.4.21 cannot be expected in our general framework; for example, even the decomposition of a weighted pseudo-almost periodic function into its almost periodic and ergodic component is not unique, in general (see, e. g., [710, Section 2]). In general case, it is very difficult to state some sufficient conditions ensuring that the range of $G(\cdot;\cdot)$ is contained in the closure of range of $F(\cdot;\cdot)$; see also [370, Theorem 2.16, Theorem 2.17] for some particular results in this direction. These topics will be considered somewhere else.

Basically, Corollary 6.4.3, Corollary 6.4.12 and Theorem 6.4.18 can be applied at any place where the infinite convolution product (2.46) represents the solution of a corresponding abstract Volterra integro-differential equation or inclusion.

In support of our investigation of multi-dimensional weighted ergodic components, the following examples and applications are meaningful:

1. Recall that the regular solution of the wave equation $u_{tt} = a^2 u_{xx}$ in the domain $\{(x,t) : x \in \mathbb{R},\ t > 0\}$, equipped with the initial conditions $u(x,0) = f(x) \in C^2(\mathbb{R})$ and $u_t(x,0) = g(x) \in C^1(\mathbb{R})$, is given by the d'Alembert formula (3.65). Define $g^{[1]}(x) := \int_0^x g(s)\,ds$, $x \in \mathbb{R}$ $(a > 0)$ and suppose that there exist numbers $\omega \in \mathbb{R} \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$ such that the following conditions hold:
   (i) There exists an integer $k \in \mathbb{N}$ such that $c^{k-1} = 1$ and the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is (ω, c)-periodic. Define
   $$\omega_1 := \frac{1+k}{2}\omega \quad \text{and} \quad \omega_2 := \frac{k-1}{2a}\omega.$$
   Then we know that the function $u(\cdot;\cdot)$ is (ω, c)-periodic in $\mathbb{R}^2$. Let $\mathcal{X}_{\mathbb{R}^2}$ denote the space of all (ω, c)-periodic functions from $\mathbb{R}^2$ into $\mathbb{C}$.
   (ii) Let $q_i \in \mathrm{PAP}_0(\mathbb{R} : \mathbb{C})$ for $i = 1, 2$. Then the solution $u(\cdot;\cdot)$, given by (3.65), with the functions $f(\cdot)$ and $g^{[1]}(\cdot)$ replaced therein with the functions $(f + q_1)(\cdot)$ and $(g^{[1]} + q_2)(\cdot)$, is strongly asymptotically $(\mathcal{X}_{\mathbb{R}^2}, \mathcal{Q}_{\mathbb{R}^2})$-almost periodic, where $\mathcal{Q}_{\mathbb{R}^2}$ denotes the space $\mathrm{PAP}_{0,1}(\mathbb{R}^2, \mathcal{B}, T^{-\sigma}, \phi)$ with $\phi(x) \equiv x$, if $\sigma \geqslant 2$. To see this, it suffices to show that the function
   $$(x,t) \mapsto \frac{1}{2}[q_1(x-at) + q_2(x+at)] + \frac{1}{2a}[q_2(x+at) - q_2(x-at)], \quad (x,t) \in \mathbb{R}^2$$
   belongs to the space $\mathrm{PAP}_{0,1}(\mathbb{R}^2, \mathcal{B}, T^{-2}, \phi)$; for doing so, it suffices to consider the case in which $q_2 \equiv 0$. Then the required statement follows from the next simple computation involving the corresponding definitions and the Fubini theorem:
   $$\begin{aligned} &T^{-2} \int_{|(x,t)| \leqslant T} |q_1(x-at) + q_1(x+at)|\,dx\,dt \\ &\quad \leqslant T^{-2} \int_{|x| \leqslant T, |t| \leqslant T} |q_1(x-at) + q_1(x+at)|\,dx\,dt \end{aligned}$$

$$\leqslant T^{-2} \int_{-T}^{T} \int_{-T}^{T} \big[|q_1(x-at)| + |q_1(x+at)| \big] \, dt \, dx$$
$$\leqslant 2T^{-2} \int_{-T}^{T} \int_{-(a+1)T}^{(a+1)T} |q_1(t)| \, dt \, dx, \quad T > 0.$$

It can be shown, by a large number of very simple counterexamples, that the choice $\sigma = 2$ is the best choice we can make for all choices of functions $q_i \in PAP_0(\mathbb{R} : \mathbb{C})$ for $i = 1, 2$.

3. Let $(G(t))_{t \geqslant 0}$ be the Gaussian semigroup. If $\emptyset \neq I' \subseteq I = \mathbb{R}^n$ and $F(\cdot)$ is bounded Bohr $(\mathcal{B}, I')$-almost periodic, resp. bounded $(\mathcal{B}, I')$-uniformly recurrent, then we know that for each $t_0 > 0$ the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is likewise bounded Bohr $(\mathcal{B}, I')$-almost periodic, resp. bounded $(\mathcal{B}, I')$-uniformly recurrent. Suppose now that $1 \leqslant p_1 < \infty$, $\Lambda = \mathbb{R}^n$, $p \equiv p_1$, $1/p + 1/q = 1$, $\phi(x) = \varphi(x) = x$, $x \geqslant 0$, $\mathrm{F}(T) = T^{-\sigma}$ for some real number $\sigma > 0$, $\mathrm{F}_1(T) = T^{-\sigma-1}$ $(T > 0)$, $\check{Q}_j \in PAP_{0,p}(\mathbb{R}^n, \mathcal{B}, \mathrm{F}, \phi)$ for all $j \in \mathbb{N}_{2^n}$, and for each $B \in \mathcal{B}$ we have $\sup_{x \in B} \sup_{\mathbf{t} \in \mathbb{R}^n} \|Q(\mathbf{t}; x)\|_Y < \infty$. Then a simple application of Theorem 6.4.20 shows that the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)Q)(x) \in \mathbb{C}$ belongs to the class $PAP_{0,p}(\mathbb{R}^n, \mathcal{B}, \mathrm{F}, \phi)$. Here, it is only worth noting that the value of term $[\varphi(|\Psi_j(\mathbf{s} + k)|)]_{L^q(T_{\mathbf{t}}[0,1]^n)}$ is less than or equal to $T_{\mathbf{t}}^n e^{-|k|^2/t_0}$ for $k \in T_{\mathbf{t}} \cdot (\mathbb{N}_0^n \setminus \{0\})$. Therefore, the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)[F + Q])(x) \in \mathbb{C}$ is strongly asymptotically $(\mathcal{X}_{\mathbb{R}^n}, \mathcal{Q}_{\mathbb{R}^n})$-almost periodic with $\mathcal{X}_{\mathbb{R}^n}$ being the space of Bohr $(\mathcal{B}, I')$-almost periodic functions, resp. bounded $(\mathcal{B}, I')$-uniformly recurrent functions, and $\mathcal{Q}_{\mathbb{R}^n} = PAP_{0,p}(\mathbb{R}^n, \mathcal{B}, \mathrm{F}, \phi)$.

7 Multi-dimensional (ω, c)-almost periodic type functions, multi-dimensional c-almost periodic type functions and applications

This chapter consists of three sections, Section 7.1–Section 7.3.

7.1 Multi-dimensional c-almost periodic type functions and applications

As already emphasized, the theory of almost periodic functions of several real variables has not attracted so much attention of the authors by now. The main aim of this section is to continue the research studies [265] and [586] by investigating various notions of multi-dimensional c-almost periodic type functions and related applications, where $c \in \mathbb{C} \setminus \{0\}$; for simplicity, we will not consider the corresponding Stepanov classes here. In support of our investigation, we would like to note that Example 1 and Example 2 can be very simply reformulated for the multi-dimensional c-almost periodicity.

Now we will briefly explain the organization and main ideas of this section. If $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$ and $F : I \times X \to Y$ is a continuous function, then the notions of Bohr $(\mathcal{B}, c)$-almost periodicity and $(\mathcal{B}, c)$-uniform recurrence for $F(\cdot;\cdot)$ are introduced in Definition 7.1.1. If the region I satisfies certain conditions, $F : I \times X \to Y$ is Bohr $(\mathcal{B}, c)$-almost periodic and $\mathcal{B}$ is any family of compact subsets of X, then some sufficient conditions ensuring that for each set $B \in \mathcal{B}$ we see that the set $\{F(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in B\}$ is relatively compact in Y are given in Proposition 7.1.2.

The notion introduced in Definition 6.1.14 is reexamined and extended in Definition 7.1.1, where we introduce the notions of Bohr $(\mathcal{B}, I', c)$-almost periodicity and $(\mathcal{B}, I', c)$-uniform recurrence ($\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$). Example 7.1.8, although very simple and elaborate, shows that the statement of Proposition 4.2.11 fails to be true for multi-dimensional $(\mathcal{B}, I', c)$-uniformly recurrent functions, in general. An important extension of Proposition 4.2.22 is proved in Proposition 7.1.9, where condition $I + I' = I$ is crucial for proving the fact that we always have $c = \pm 1$ provided we have the existence of a $(\mathcal{B}, I', c)$-uniformly recurrent non-zero function $F : I \to \mathbb{R}$ (if $F(\mathbf{t}) \geqslant 0$ for all $\mathbf{t} \in I$, then $c = 1$); see also Example 7.1.10. Proposition 7.1.9 is later employed in the proof of Proposition 7.1.11, where it is shown that, if the function $F : I \times X \to Y$ is Bohr $(\mathcal{B}, I', c)$-almost periodic ($(\mathcal{B}, I', c)$-uniformly recurrent), $I + I' = I$ and $F(\cdot;\cdot) \neq 0$, then $|c| = 1$.

The first example of a multi-dimensional uniformly anti-recurrent function $F : \mathbb{R}^n \to \mathbb{R}$ ($c = -1$) which is not almost periodic is presented in Example 6.1.16(iii)–(b). After that, in Proposition 7.1.13, we transfer the statement of Proposition 4.2.14 for multi-dimensional Bohr $(\mathcal{B}, c)$-almost periodic type functions (see also Corollary 7.1.14

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and Proposition 7.1.16 for similar results). We investigate the convolution invariance of Bohr $(\mathcal{B}, c)$-almost periodic type functions, invariance of Bohr c-almost periodicity and composition theorems for Bohr $(\mathcal{B}, c)$-almost periodic type functions are investigated. The main structural characterizations of $\mathbb{D}$-asymptotically c-almost periodic type functions are given in Subsection 7.1.1. In this subsection, we state and prove our main results, Theorem 7.1.25 (in which we analyze certain relations between the classes of I-asymptotically Bohr c-almost periodic functions of type 1 and I-asymptotically Bohr c-almost periodic functions) and Theorem 7.1.26 (in which we analyze the extensions of Bohr (I', c)-almost periodic functions and (I', c)-uniformly recurrent functions). The final subsection is reserved for applications of our abstract theoretical results. Unless stated otherwise, in this subsection we will always assume that $c \in \mathbb{C} \setminus \{0\}$.

We will consider the following notion.

Definition 7.1.1. Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, c)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that

$$\|F(\mathbf{t} + \tau; x) - cF(\mathbf{t}; x)\|_Y \leqslant \varepsilon, \quad \mathbf{t} \in I,\ x \in B.$$

(ii) $F(\cdot;\cdot)$ is $(\mathcal{B}, c)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \|F(\mathbf{t} + \tau_k; x) - cF(\mathbf{t}; x)\|_Y = 0.$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Bohr c-almost periodic (c-uniformly recurrent).

Unless stated otherwise, we will assume that $\emptyset \neq I \subseteq \mathbb{R}^n$ henceforth. It is clear that any Bohr ($(\mathcal{B}, c)$-)almost periodic function is ($(\mathcal{B}, c)$-)uniformly recurrent; in general, the converse statement does not hold. As already clarified, any Bohr almost periodic function $f : I \to Y$ is bounded, provided that $I = [0, \infty)$ or $I = \mathbb{R}$. In the multi-dimensional case, the things become more complicated and the best we can do is to prove the following extension of the above-mentioned result following the method proposed in the proof of Proposition 6.1.17, which is applicable in the case that $I = [0, \infty)^n$ or $I = \mathbb{R}^n$.

Proposition 7.1.2. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, I is closed, $F : I \times X \to Y$ is Bohr $(\mathcal{B}, c)$-almost periodic and $\mathcal{B}$ is any family of compact subsets of X. If*

$$(\forall l > 0)\,(\exists \mathbf{t_0} \in I)\,(\exists k > 0)\,(\forall \mathbf{t} \in I)(\exists \mathbf{t}_0' \in I)$$
$$(\forall \mathbf{t}_0'' \in B(\mathbf{t}_0', l) \cap I)\ \mathbf{t} - \mathbf{t}_0'' \in B(\mathbf{t_0}, kl) \cap I,$$

then for each $B \in \mathcal{B}$ we see that the set $\{F(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in B\}$ is relatively compact in Y; in particular, $\sup_{\mathbf{t}\in I; x\in B} \|F(\mathbf{t}; x)\|_Y < \infty$.

We continue by providing the following illustrative example.

Example 7.1.3. Suppose that $\varphi \in (-\pi, \pi] \setminus \{0\}$, $\theta \in (-\pi, \pi]$, $\mu \in \mathbb{R}^n \setminus \{0\}$ and $c = e^{i\theta}$. Then the trigonometric polynomial $\mathbf{t} \to e^{i\langle \mu, \mathbf{t}\rangle}$, $\mathbf{t} \in \mathbb{R}^n$ is c-almost periodic. Towards see this, set $S := \{j \in \mathbb{N}_n : \mu_j \neq 0\}$ and $l := \max\{2\pi|\mu_j|^{-1} : j \in S\}$. Let $\varepsilon > 0$ be fixed. Then we have ($\mathbf{t} \in \mathbb{R}^n$; $\tau \in \mathbb{R}^n$)

$$\begin{aligned}&\big|e^{i\langle \mu, \mathbf{t}+\tau\rangle} - e^{i\theta} e^{i\langle \mu, \mathbf{t}\rangle}\big| \\ &\quad = \big|e^{i[\mu_1\tau_1+\mu_2\tau_2+\cdots+\mu_n\tau_n-\theta]} - 1\big| = 2\left|\sin\left(\frac{\mu_1\tau_1 + \mu_2\tau_2 + \cdots + \mu_n\tau_n - \theta}{2}\right)\right|,\end{aligned}$$

and therefore

$$\begin{aligned}&\big|e^{i\langle \mu, \mathbf{t}+\tau\rangle} - e^{i\theta} e^{i\langle \mu, \mathbf{t}\rangle}\big| \leqslant \varepsilon, \quad \mathbf{t} \in \mathbb{R}^n \text{ if and only if there exists } k \in \mathbb{Z} \text{ such that} \\ &\mu_1\tau_1 + \mu_2\tau_2 + \cdots + \mu_n\tau_n - \theta \in \big[-\arcsin(\varepsilon/2) + k\pi, \arcsin(\varepsilon/2) + k\pi\big].\end{aligned}$$

In particular, if there exists $k \in \mathbb{Z}$ such that $\mu_1\tau_1 + \mu_2\tau_2 + \cdots + \mu_n\tau_n = k\pi + \theta$, then we have $|e^{i\langle \mu, \mathbf{t}+\tau\rangle} - e^{i\theta} e^{i\langle \mu, \mathbf{t}\rangle}| \leqslant \varepsilon$, $\mathbf{t} \in \mathbb{R}^n$. But, we can simply prove that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists a point $\tau \in B(\mathbf{t}_0, l)$ such that $\mu_1\tau_1 + \mu_2\tau_2 + \cdots + \mu_n\tau_n = k\pi + \theta$ for some $k \in \mathbb{Z}$, which simply implies the required result.

As in the case $c = 1$, we may conclude the following.

Proposition 7.1.4. *Suppose that $F : I \times X \to Y$ is Bohr $(\mathcal{B}, c)$-almost periodic/$(\mathcal{B}, c)$-uniformly recurrent, and $\phi : Y \to Z$ is uniformly continuous on $\overline{R(F)}$ and satisfies the requirement that $\phi(cy) = c\phi(y)$ for all $y \in Y$. Then $\phi \circ F : I \times X \to Z$ is Bohr $(\mathcal{B}, c)$-almost periodic/$(\mathcal{B}, c)$-uniformly recurrent.*

We continue by providing the following example.

Example 7.1.5.

(i) Suppose that $F_j : X \to Y$ is a continuous function, for each $B \in \mathcal{B}$ we have $\sup_{x \in B} \|F_j(x)\|_Y < \infty$ and the complex-valued mapping $t \mapsto (\int_0^t f_1(s)\,ds, \ldots, \int_0^t f_n(s)\,ds)$, $t \geqslant 0$ is c-almost periodic ($1 \leqslant j \leqslant n$). Set

$$F(t_1, \ldots, t_{n+1}; x) := \sum_{j=1}^{n} \int_{t_j}^{t_{j+1}} f_j(s)\,ds \cdot F_j(x) \quad \text{for all } x \in X \text{ and } t_j \geqslant 0,\ 1 \leqslant j \leqslant n.$$

Then the mapping $F : [0, \infty)^{n+1} \times X \to Y$ is Bohr $(\mathcal{B}, c)$-almost periodic.

(ii) Suppose that $F : X \to Y$ is a continuous function, for each $B \in \mathcal{B}$ we have $\sup_{x \in B} \|F(x)\|_Y < \infty$ and the complex-valued mapping $t \mapsto f_j(t)$, $t \geqslant 0$ is c-almost periodic, resp. bounded and c-uniformly recurrent ($1 \leqslant j \leqslant n$). Set

$$F(t_1, \ldots, t_n; x) := \prod_{j=1}^{n} f_j(t_j) \cdot F(x) \quad \text{for all } x \in X \text{ and } t_j \geqslant 0,\ 1 \leqslant j \leqslant n.$$

Then the mapping $F : [0,\infty)^n \times X \to Y$ is Bohr $(\mathcal{B}, c)$-almost periodic, resp. $(\mathcal{B}, c)$-uniformly recurrent.

(iii) Suppose that $G : [0,\infty)^n \to \mathbb{C}$ is c-almost periodic, resp. bounded and c-uniformly recurrent, $F : [0,\infty) \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, resp. $\mathcal{B}$-uniformly recurrent, and for each set $B \in \mathcal{B}$ we have $\sup_{t\geqslant 0; x\in B} \|F(t;x)\|_Y < \infty$. Set

$$F(t_1,\ldots,t_{n+1};x) := G(t_1,\ldots,t_n) \cdot F(t_{n+1};x)$$
$$\text{for all } x \in X \text{ and } t_j \geqslant 0,\ 1 \leqslant j \leqslant n+1.$$

Then the mapping $F : [0,\infty)^{n+1} \times X \to Y$ is Bohr $(\mathcal{B}, c)$-almost periodic, resp. $(\mathcal{B}, c)$-uniformly recurrent.

The notion introduced in Definition 6.1.9 can be extended as follows.

Definition 7.1.6. Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I', c)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that

$$\|F(\mathbf{t}+\tau;x) - cF(\mathbf{t};x)\|_Y \leqslant \varepsilon, \quad \mathbf{t} \in I,\ x \in B. \tag{7.1}$$

(ii) $F(\cdot;\cdot)$ is $(\mathcal{B}, I', c)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \|F(\mathbf{t}+\tau_k;x) - cF(\mathbf{t};x)\|_Y = 0. \tag{7.2}$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Bohr (I', c)-almost periodic ((I', c)-uniformly recurrent).

Remark 7.1.7.

(i) Let $|c| = 1$ and $F : \mathbb{R} \to Y$ be a continuous function. Then $F(\cdot)$ is c-almost periodic (c-uniformly recurrent) in the sense of our previous consideration if and only if $F(\cdot)$ is Bohr $((0,\infty), c)$-almost periodic ($((0,\infty), c)$-uniformly recurrent) in the sense of Definition 7.1.6. Albeit we will not consider here the general question concerning the existence of larger sets $I'' \supseteq I'$ for which a given a Bohr $(\mathcal{B}, I', c)$-almost periodic function $F(\cdot;\cdot)$ is also $(\mathcal{B}, I'', c)$-almost periodic (the only exception is the proof of Theorem 7.1.26), we would like to note that any Bohr $((0,\infty), c)$-almost periodic function is already Bohr $(\mathbb{R}, c)$-almost periodic. This is clear if $\arg(c)/\pi \notin \mathbb{Q}$ since we can apply then Proposition 4.2.17(i) in order to see that the function $F(\cdot)$ is also Bohr $((0,\infty), c^{-1})$-almost periodic and therefore, given $\varepsilon > 0$ in advance, we can collect all positive (ε, c)-periods of the function $F(\cdot)$ and all negative values of all positive (ε, c^{-1})-periods of the function $F(\cdot)$ (with the meaning clear), obtaining thus a relatively dense set in $\mathbb{R}$ consisting solely of (ε, c)-periods of $F(\cdot)$. The situation is similar if $\arg(c)/\pi \in \mathbb{Q}$ because then there exists $m \in \mathbb{N}$ such that $c^{m+1} = 1$

so that $c^m = c^{-1}$ and we can collect all positive (ε, c)-periods of the function $F(\cdot)$ and all negatives of all positive $(\varepsilon/m, c)$-periods of the function $F(\cdot)$ in order to obtain a relatively dense set in $\mathbb{R}$ consisting solely of (ε, c)-periods of $F(\cdot)$; observe only that the assumption $\|F(t+\tau) - cF(t)\| \leqslant \varepsilon$ for all $t \in \mathbb{R}$ and some $\tau \in \mathbb{R}$ implies

$$\begin{aligned} &\|F(t + m\tau) - c^m F(t)\| \\ &\quad \leqslant \|F(t + m\tau) - cF(t + (m-1)\tau)\| + |c| \|F(t + (m-1)\tau) - cF(t + (m-2)\tau)\| \\ &\qquad + \cdots + |c|^{m-2} \|F(t + 2\tau) - cF(t + \tau)\| + |c|^{m-1} \|F(t + \tau) - cF(t)\| \leqslant m\varepsilon, \ t \in \mathbb{R}. \end{aligned} \tag{7.3}$$

(ii) Condition $\emptyset \neq I' \subseteq I$ is a bit unnecessary and intended for considerations of regions I for which $0 \in I$; more precisely, the assumption $I + I' \subseteq I$ is mandatory and implies that for each $\mathbf{t}_0 \in I$ we have $I' \subseteq I - \mathbf{t}_0$ (take, for example $I = [1, \infty)$ and $I' = [0, \infty)$; then we do not have $I' \subseteq I$ but the notion introduced in Definition 7.1.6 is meaningful).

(iii) The main structural properties of the functions introduced in Definition 6.1.9 and Definition 7.1.6, clarified in Theorem 4.2.75, continue to hold with appropriate modifications. For example, the introduced spaces of the functions are translation invariant, in a certain sense, with respect to both variables.

Clearly, the notion from Definition 6.1.9 is recovered by plugging $I' = I$ and any $(\mathcal{B}, I', c)$-uniformly recurrent function is $(\mathcal{B}, I, c)$-uniformly recurrent provided that $I + I \subseteq I$. Concerning the statement of Proposition 4.2.11, we would like to present first the following instructive example.

Example 7.1.8. Suppose that $I := \{(x,y) \in \mathbb{R}^2 : x + y \geqslant 0\}$ $(I := \{(x,y) \in \mathbb{R}^2 : x + y \geqslant 0\})$ and $I' := \{(x,y) \in \mathbb{R}^2 : x + y = 1\}$ $(I' := \{(x,y) \in \mathbb{R}^2 : x + y = -1\})$. Set $F(x,y) := 2^{-x-y}$, $(x,y) \in I$. Then $I + I' \subseteq I + I = I$ and for every $(a,b) \in I'$ we have $F((x,y) + (a,b)) = 2^{-1}F(x,y)$, $(x,y) \in I$ $(F((x,y) + (a,b)) = 2F(x,y)$, $(x,y) \in I)$, so that $F(\cdot,\cdot)$ is both Bohr $(I', 2^{-1})$-almost periodic and 2^{-1}-uniformly recurrent (Bohr $(I', 2)$-almost periodic and 2-uniformly recurrent) but not identically equal to zero.

Furthermore, if the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I', c)$-almost periodic ($(\mathcal{B}, I', c)$-uniformly recurrent), then the function $\|F(\cdot;\cdot)\|_Y$ is Bohr $(\mathcal{B}, I', |c|)$-almost periodic ($(\mathcal{B}, I', |c|)$-uniformly recurrent). The following fact should be also clarified: If the function $F(\cdot;\cdot)$ is $(\mathcal{B}, I', c)$-uniformly recurrent, then for each $B \in \mathcal{B}$ we have

$$\sup_{\mathbf{t} \in I, x \in B} \|F(\mathbf{t}; x)\|_Y \leqslant |c|^{-1} \sup_{\mathbf{t} \in I, |\mathbf{t}| \geqslant a, \mathbf{t} \in I + I', x \in B} \|F(\mathbf{t}; x)\|_Y, \tag{7.4}$$

and for each $x \in X$ the function $F(\cdot; x)$ is identically equal to zero provided that the function $F(\cdot;\cdot)$ is $(\mathcal{B}, I', c)$-uniformly recurrent and $\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in I + I'} F(\mathbf{t}; x) = 0$.

Now we are able to state and prove the following extension of Proposition 7.1.9.

Proposition 7.1.9. *Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$ and $I + I' = I$. If the function $F : I \to \mathbb{R}$ is $(\mathcal{B}, I', c)$-uniformly recurrent and $F \neq 0$, then $c = \pm 1$. Furthermore, if $F(\mathbf{t}) \geqslant 0$ for all $\mathbf{t} \in I$, then $c = 1$.*

Proof. Since we have assumed $I + I' = I$ and $F \neq 0$, Eq. (7.4) yields the existence of a finite real number $a > 0$ and a sequence $(\mathbf{t}_k)$ in I such that $|F(\mathbf{t}_k)| > a/2$ for all $k \in \mathbb{N}$. Then the final conclusion follows by repeating verbatim the arguments contained in the proof of Proposition 7.1.9. □

Remark 7.1.10. Suppose that $c = 1/2$ in Example 7.1.8. Then the function $F(\cdot; \cdot)$ is real-valued so that the conclusion of Proposition 7.1.9 does not hold if the assumption $I + I' \neq I$ is neglected.

The most important corollary of Proposition 7.1.9, which extends the statement of Proposition 4.2.11, is stated below.

Corollary 7.1.11. *Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' = I$ and $F : I \times X \to Y$ is Bohr $(\mathcal{B}, I', c)$-almost periodic ($(\mathcal{B}, I', c)$-uniformly recurrent). If $F(\cdot; \cdot) \neq 0$, then $|c| = 1$.*

Proof. By our assumption, there exist $\mathbf{t}_0 \in I$ and $x \in X$ such that $F(\mathbf{t}_0; x) \neq 0$. Furthermore, there exists $B \in \mathcal{B}$ such that $x \in B$ and this simply implies that the function $F_x : I \to Y$ is Bohr $(\mathcal{B}, I', c)$-almost periodic ($(\mathcal{B}, I', c)$-uniformly recurrent) and not identically equal to zero. Therefore, the function $\|F_x(\cdot)\|_Y$ is Bohr $(\mathcal{B}, I', |c|)$-almost periodic ($(\mathcal{B}, I', |c|)$-uniformly recurrent) and not identically equal to zero. By Proposition 7.1.9, we get $|c| = 1$. □

If $c = \pm 1$, then we also say that the function $F(\cdot)$ is Bohr $\mathcal{B}$-almost (anti-)periodic ($\mathcal{B}$-uniformly (anti-)recurrent)/Bohr $(\mathcal{B}, I')$-almost (anti-)periodic ($(\mathcal{B}, I')$-uniformly (anti-)recurrent). Let us recall that there are a great number of very simple examples showing that the notion of $(\mathcal{B}, I')$-almost periodicity is neither stronger nor weaker than the notion of $(\mathcal{B}, I)$-almost periodicity, provided that $I + I \subseteq I$.

Similarly to before, we have the following.

Example 7.1.12.

(i) Suppose that the complex-valued mapping $t \mapsto \int_0^t f_j(s)\,ds$, $t \in \mathbb{R}$ is c-almost periodic, resp. bounded and c-uniformly recurrent $(1 \leqslant j \leqslant n)$. Set

$$F_1(t_1, \ldots, t_{2n}) := \prod_{j=1}^{n} \int_{t_j}^{t_{j+n}} f_j(s)\,ds \quad \text{and} \quad t_j \in \mathbb{R},\ 1 \leqslant j \leqslant 2n.$$

Then the mapping $F_1 : \mathbb{R}^{2n} \to \mathbb{C}$ is Bohr (I', c)-almost periodic, resp. (I', c)-uniformly recurrent, where $I' = \{(\tau, \tau) : \tau \in \mathbb{R}^n\}$; furthermore, if the function

$$t \mapsto w(t) \equiv \left(\int_0^t f_1(s)\,ds, \ldots, \int_0^t f_n(s)\,ds \right), \quad t \in \mathbb{R} \tag{7.5}$$

is c-almost periodic, resp. bounded and c-uniformly recurrent, then the function $F_1(\cdot)$ is Bohr (I'', c)-almost periodic, resp. (I'', c)-uniformly recurrent, where $I'' = \{(a, a, \ldots, a) \in \mathbb{R}^{2n} : a \in \mathbb{R}\}$.

(ii) Suppose that an X-valued mapping $t \mapsto \int_0^t f_j(s)\,ds$, $t \in \mathbb{R}$ is c-almost periodic, resp. bounded and c-uniformly recurrent, as well as that a strongly continuous operator family $(T_j(t))_{t\in\mathbb{R}} \subseteq L(X, Y)$ is uniformly bounded $(1 \leqslant j \leqslant n)$. Set

$$F_2(t_1, \ldots, t_{2n}) := \sum_{j=1}^{n} T_j(t_j - t_{j+n}) \int_{t_j}^{t_{j+n}} f_j(s)\,ds \quad \text{and}$$

$$t_j \in \mathbb{R}, \quad 1 \leqslant j \leqslant 2n.$$

Then the mapping $F_2 : \mathbb{R}^{2n} \to \mathbb{C}$ is Bohr (I', c)-almost periodic, resp. (I', c)-uniformly recurrent, where $I' = \{(\tau, \tau) : \tau \in \mathbb{R}^n\}$, but not generally Bohr c-almost periodic, in the case of consideration of almost periodicity; furthermore, if the function $t \mapsto w(t)$, $t \in \mathbb{R}$ given by (7.5), is c-almost periodic, resp. bounded and c-uniformly recurrent, then the function $F_2(\cdot)$ is Bohr I''-almost periodic, where $I'' = \{(a, a, \ldots, a) \in \mathbb{R}^{2n} : a \in \mathbb{R}\}$.

(iii) Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I_0 = [0, \infty)$ or $I_0 = \mathbb{R}$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \neq 0$ and the linear function $g(\mathbf{t}) := a_1t_1 + \cdots + a_nt_n$, $\mathbf{t} = (t_1, \ldots, t_n) \in I$ maps surjectively the region I onto I_0. Suppose, further, that $f : I_0 \to X$ is a c-uniformly recurrent function as well as that a sequence (α_k) in I_0 satisfies the requirement that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and $\lim_{k\to+\infty} \sup_{t\in I_0} \|f(t + \alpha_k) - cf(t)\| = 0$. Define $I' := g^{-1}(\{\alpha_k : k \in \mathbb{N}\})$ and $F : I \to X$ by $F(\mathbf{t}) := f(g(\mathbf{t}))$, $\mathbf{t} \in I$. Then $F(\cdot)$ is (I', c)-uniformly recurrent, and $F(\cdot)$ is not c-almost periodic provided that $f(\cdot)$ is not c-almost periodic (note that the conclusions established in Example 6.1.12 cannot be reformulated for the c-uniform recurrence).

Set $lI' := \{l\mathbf{t} : \mathbf{t} \in I'\}$ for all $l \in \mathbb{N}$. The following result extends Proposition 4.2.14 for c-almost periodic functions and c-uniformly recurrent functions.

Proposition 7.1.13. *Suppose that $l \in \mathbb{N}$, $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$ and $F : I \times X \to Y$ is Bohr $(\mathcal{B}, I', c)$-almost periodic ($(\mathcal{B}, I', c)$-uniformly recurrent). Then $lI' \subseteq I$, $I + lI' \subseteq I$ and $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, lI', c^l)$-almost periodic ($(\mathcal{B}, lI', c^l)$-uniformly recurrent).*

Proof. Since $I' \subseteq I$ and $I + I' \subseteq I$, we inductively get $jI' \subseteq I$ and $I + jI' \subseteq I$ for all $j \in \mathbb{N}$. Keeping this in mind, the proof simply follows from the corresponding definitions and the identity ($\mathbf{t} \in I$, $\tau \in I'$):

$$F(\mathbf{t} + l\tau) - c^l F(\mathbf{t}) = \sum_{j=0}^{l-1} c^j \big[F\big(\mathbf{t} + (l - j)\tau\big) - cF\big(\mathbf{t} + (l - j - 1)\tau\big)\big]. \qquad \square$$

The most important corollary of Proposition 7.1.13 follows by plugging $l = q$:

Corollary 7.1.14. *Suppose that* (4.29) *holds,* $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I+I' \subseteq I$ *and* $F : I\times X \to Y$ *is Bohr* $(\mathcal{B}, I', c)$*-almost periodic (*$(\mathcal{B}, I', c)$*-uniformly recurrent). Then the following holds:*
(i) *If p is even, then* $F(\cdot;\cdot)$ *is Bohr* $(\mathcal{B}, qI')$*-almost periodic (*$(\mathcal{B}, qI')$*-uniformly recurrent).*
(ii) *If p is odd, then* $F(\cdot;\cdot)$ *is Bohr* $(\mathcal{B}, qI')$*-almost anti-periodic (*$(\mathcal{B}, qI')$*-uniformly anti-recurrent).*

Similarly we can prove the following.

Proposition 7.1.15. *Suppose that* $|c| = 1$, $\arg(c) \in \pi\mathbb{Q}$, $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$ *and* $F : I \times X \to Y$ *is Bohr* $(\mathcal{B}, I', c)$*-almost periodic (*$(\mathcal{B}, I', c)$*-uniformly recurrent). Define* $C_c := \{l \in \mathbb{N} : c^l = 1\}$ *and* $C_{c,-1} := \{l \in \mathbb{N} : c^l = -1\}$*. If S is any finite non-empty subset of* C_c*, resp.* $C_{c,-1}$*, and* $I'_S := \bigcup_{l\in S} lI'$*, then* $F(\cdot;\cdot)$ *is Bohr* $(\mathcal{B}, I'_S)$*-almost periodic (*$(\mathcal{B}, I'_S)$*-uniformly recurrent), resp. Bohr* $(\mathcal{B}, I'_S)$*-almost anti-periodic (*$(\mathcal{B}, I'_S)$*-uniformly anti-recurrent).*

The subsequent result follows from the argumentation contained in the proof of Proposition 4.2.16(i).

Proposition 7.1.16. *Let* $|c| = 1$ *and* $\arg(c)/\pi \notin \mathbb{Q}$*. If* $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$, $lI' = I'$ *for all* $l \in \mathbb{N}$ *and* $F : I \times X \to Y$ *is a bounded, Bohr* $(\mathcal{B}, I', c)$*-almost periodic (*$(\mathcal{B}, I', c)$*-uniformly recurrent) function, then the function* $F(\cdot;\cdot)$ *is Bohr* $(\mathcal{B}, I', c)$*-almost periodic (*$(\mathcal{B}, I', c)$*-uniformly recurrent) for all* $c' \in S_1$.

Now we would like to state the following result.

Proposition 7.1.17. *Suppose that* $h \in L^1(\mathbb{R}^n)$, $\emptyset \neq I' \subseteq \mathbb{R}^n$ *and the function* $F(\cdot;\cdot)$ *is Bohr* $(\mathcal{B}, I', c)$*-almost periodic (*$(\mathcal{B}, I', c)$*-uniformly recurrent). If*
$(B)_b$ *For every* $B \in \mathcal{B}$*, there exists a finite real constant* $c_B > 0$ *such that* $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B} \|F(\mathbf{t}; x)\|_Y \leqslant c_B$,

then the function

$$(h * F)(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma)F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X,$$

is Bohr $(\mathcal{B}, I', c)$*-almost periodic (*$(\mathcal{B}, I', c)$*-uniformly recurrent) and satisfies* $(B)_b$.

Proof. Since $h \in L^1(\mathbb{R}^n)$, the prescribed assumptions imply that the function $(h*F)(\cdot;\cdot)$ is well defined and satisfies $(B)_b$. The continuity of the function $(h * F)(\cdot;\cdot)$ follows from the dominated convergence theorem, the continuity of the function $F(\cdot;\cdot)$ and condition $(B)_b$. Let $B \in \mathcal{B}$ and $\varepsilon > 0$ be fixed. Then there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that (7.1) holds with $I = \mathbb{R}^n$. Therefore,

$$\begin{aligned} &\|(h * F)(\mathbf{t} + \tau; x) - c(h * F)(\mathbf{t}; x)\|_Y \\ &\quad \leqslant \int_{\mathbb{R}^n} |h(\sigma)| \cdot \|F(\mathbf{t} + \tau - \sigma; x) - cF(\mathbf{t} - \sigma; x)\|_Y\, d\sigma, \end{aligned}$$

for any $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$. This simply implies the required result. □

Suppose that $|c| = 1$. Concerning the assertion of Theorem 4.2.28, we will first observe that any almost periodic function $F \in AP_{\mathbb{R}^n \setminus \{0\}}(\mathbb{R}^n : X)$ can be uniformly approximated by trigonometric polynomials whose frequencies belong to the set $\mathbb{R}^n \setminus \{0\}$. If we denote by $AP_{c,0}(\mathbb{R}^n : X)$ the linear span of all c-almost periodic functions $F : \mathbb{R}^n \to X$ and by $AP_{c,0}(\mathbb{R}^n : X)$ its closure in $AP(\mathbb{R}^n : X)$, then it follows from the above and our conclusion established in Example 7.1.3 that $AP_{\mathbb{R}^n \setminus \{0\}}(\mathbb{R}^n : X) \subseteq AP_{c,0}(\mathbb{R}^n : X)$. But, it is not clear how to prove or disprove the converse inclusion provided that $\arg(c) \in \pi \cdot \mathbb{Q}$.

Before moving to the next subsection, we will state and prove a composition theorem for multi-dimensional Bohr $(\mathcal{B}, c)$-almost periodic type functions. Suppose that $F : I \times X \to Y$ and $G : I \times Y \to Z$ are given functions; then the multi-dimensional Nemytskii operator $W : I \times X \to Z$ is defined by (6.20). Set $R(F) \equiv \{F(\mathbf{t}; x) : \mathbf{t} \in I,\ x \in X\}$ and suppose that there exists a finite real constant $L > 0$ such that

$$\|G(\mathbf{t}; y) - G(\mathbf{t}; y')\|_Z \leqslant L\|y - y'\|_Y, \quad \mathbf{t} \in I,\ y \in R(F),\ y' \in cR(F). \tag{7.6}$$

The following result is an extension of Theorem 4.2.36.

Theorem 7.1.18. *Suppose that the functions $F : I \times X \to Y$ and $G : I \times Y \to Z$ are continuous as well as $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$ and (7.6) holds.*

(i) *Suppose further that, for every $B \in \mathcal{B}$ and $\varepsilon > 0$, there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that (7.1) holds and*

$$\|G(\mathbf{t} + \tau; cy) - cG(\mathbf{t}; y)\|_Z \leqslant \varepsilon, \quad \mathbf{t} \in I,\ y \in R(F). \tag{7.7}$$

Then the function $W(\cdot;\cdot)$, given by (6.20), is Bohr $(\mathcal{B}, I', c)$-almost periodic.

(ii) *Suppose further that, for every $B \in \mathcal{B}$, there exists a sequence (τ_k) in I' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$, (7.2) holds and*

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \|G(\mathbf{t} + \tau_k; cF(\mathbf{t}; x)) - cG(\mathbf{t}; F(\mathbf{t}; x))\|_Y = 0. \tag{7.8}$$

Then the function $W(\cdot;\cdot)$, given by (6.20), is $(\mathcal{B}, I', c)$-uniformly recurrent.

Proof. We will prove only (i). The continuity of the function $W(\cdot;\cdot)$ is obvious. Then the final conclusion follows from the assumption made, the corresponding definition of Bohr $(\mathcal{B}, I', c)$-almost periodicity and the next simple computation:

$$\begin{aligned}
&\|G(\mathbf{t} + \tau; F(\mathbf{t} + \tau; x)) - G(\mathbf{t}; F(\mathbf{t}; x))\|_Z \\
&\quad \leqslant \|G(\mathbf{t} + \tau; F(\mathbf{t} + \tau; x)) - G(\mathbf{t} + \tau; cF(\mathbf{t}; x))\|_Z \\
&\qquad + \|G(\mathbf{t} + \tau; cF(\mathbf{t}; x)) - cG(\mathbf{t}; F(\mathbf{t}; x))\|_Z \\
&\quad \leqslant L\|F(\mathbf{t} + \tau; x) - cF(\mathbf{t}; x)\|_Y + \|G(\mathbf{t} + \tau; cF(\mathbf{t}; x)) - cG(\mathbf{t}; F(\mathbf{t}; x))\|_Z,
\end{aligned}$$

for any $\mathbf{t} \in I$, $\tau \in I'$ and $x \in X$. □

7.1.1 $\mathbb{D}$-asymptotically $(\mathcal{B}, c)$-almost periodic type functions

We open this subsection by introducing the following notion.

Definition 7.1.19. Suppose that the set $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ is unbounded, $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$ and $F : I \times X \to Y$ is a continuous function. Then we say that $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', c)$-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent, if and only if there exist a Bohr $(\mathcal{B}, I', c)$-almost periodic function $(G : \mathbb{R}^n \times X \to Y)$ $G : I \times X \to Y$, resp. a $(\mathcal{B}, I', c)$-uniformly recurrent function $(G : \mathbb{R}^n \times X \to Y)$ $G : I \times X \to Y$ and a function $Q \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$. If $I' = I$, then we also say that $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, c)$-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically $(\mathcal{B}, c)$-uniformly recurrent; if $X \in \mathcal{B}$, then we omit the term "$\mathcal{B}$" from the notation introduced, with the meaning clear.

Before we go any further, we would like to present the following extension of Theorem 4.2.37.

Theorem 7.1.20. *Suppose that the functions $F_h : I \times X \to Y$, $F_0 : I \times X \to Y$, $G_h : I \times Y \to Z$ and $G_0 : I \times Y \to Z$ are continuous, $F = F_h + F_0$, $G = G_h + G_0$ as well as $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$ and* (7.6) *holds with the functions $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$ replaced therein with the functions $F_h(\cdot;\cdot)$ and $G_h(\cdot;\cdot)$, respectively.*

(i) *Suppose further that, for every $B \in \mathcal{B}$ and $\varepsilon > 0$, there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that* (7.1) *holds with the function $F(\cdot;\cdot)$ replaced with the function $F_h(\cdot;\cdot)$ and* (7.7) *holds with the functions $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$ replaced therein with the functions $F_h(\cdot;\cdot)$ and $G_h(\cdot;\cdot)$, respectively. If $F_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ and for each $B \in \mathcal{B}$ we have $\lim_{\mathbf{t} \in \mathbb{D}, |\mathbf{t}| \to +\infty} G_0(\mathbf{t}; F(\mathbf{t}; x)) = 0$, uniformly for $x \in B$, then the function $W(\cdot;\cdot)$, given by* (6.20), *is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', c)$-almost periodic.*

(ii) *Suppose further that, for every $B \in \mathcal{B}$, there exists a sequence (τ_k) in I' such that $\lim_{k \to +\infty} |\tau_k| = +\infty$,* (7.2) *holds and* (7.8) *holds with the functions $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$ replaced therein with the functions $F_h(\cdot;\cdot)$ and $G_h(\cdot;\cdot)$, respectively. If $F_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ and for each $B \in \mathcal{B}$ we have $\lim_{\mathbf{t} \in \mathbb{D}, |\mathbf{t}| \to +\infty} G_0(\mathbf{t}; F(\mathbf{t}; x)) = 0$, uniformly for $x \in B$, then the function $W(\cdot;\cdot)$, given by* (6.20), *is $(\mathcal{B}, I', c)$-uniformly recurrent.*

Proof. We will outline all details of the proof of (i) for the sake of completeness. Clearly, the following decomposition holds true:

$$G(\cdot; F(\cdot;\cdot)) = G_h(\cdot; F_h(\cdot;\cdot)) + [G_h(\cdot; F(\cdot;\cdot)) - G_h(\cdot; F_h(\cdot;\cdot))] + G_0(\cdot; F(\cdot;\cdot)).$$

Due to Theorem 7.1.18, we see that the function $G_h(\cdot; F_h(\cdot;\cdot))$ is Bohr $(\mathcal{B}, I', c)$-almost periodic. Furthermore, the prescribed assumption implies that the function $G_0(\cdot; F(\cdot;\cdot))$ belongs to the space $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$. This also holds for the function $G_h(\cdot; F(\cdot;\cdot))$ –

$G_h(\cdot;F_h(\cdot;\cdot))$ since the function $G_h(\cdot;\cdot)$ satisfies the Lipschitz condition with respect to the first variable and $F_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$. □

Recall that $I_{\mathbf{t}} = (-\infty, t_1] \times (-\infty, t_2] \times \cdots \times (-\infty, t_n]$ and $\mathbb{D}_{\mathbf{t}} = I_{\mathbf{t}} \cap \mathbb{D}$ for any $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. Concerning the invariance of strong $\mathbb{D}$-asymptotical c-almost periodicity under the actions of finite convolution products, we will formulate the following result (the proof is similar to the proof of the corresponding result with $c = 1$ and therefore is omitted).

Proposition 7.1.21. *Suppose that $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ is a strongly continuous operator family such that $\int_{(0,\infty)^n} \|R(\mathbf{t})\|\, d\mathbf{t} < \infty$. If $f : I \to X$ is strongly $\mathbb{D}$-asymptotically c-almost periodic,*

$$\lim_{|\mathbf{t}|\to\infty, \mathbf{t}\in\mathbb{D}} \int\limits_{I_{\mathbf{t}}\cap\mathbb{D}^c} \big\|R(\mathbf{t}-\mathbf{s})\big\|\, d\mathbf{s} = 0$$

and for each $r > 0$ we have

$$\lim_{|\mathbf{t}|\to\infty, \mathbf{t}\in\mathbb{D}} \int\limits_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} \big\|R(\mathbf{t}-\mathbf{s})\big\|\, d\mathbf{s} = 0,$$

then the function

$$F(\mathbf{t}) := \int\limits_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})f(\mathbf{s})\, ds, \quad \mathbf{t} \in I,$$

is strongly $\mathbb{D}$-asymptotically c-almost periodic.

Assuming that $\mathbb{D} = [\alpha_1,\infty) \times [\alpha_2,\infty) \times \cdots \times [\alpha_n,\infty)$ for some real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $\mathbb{D}_{\mathbf{t}} = [\alpha_1,t_1] \times [\alpha_2,t_2] \times \cdots \times [\alpha_n,t_n]$. In this case, the function $F(\mathbf{t}) = \int_{\mathbf{t}}^{\alpha} R(\mathbf{t}-\mathbf{s})f(\mathbf{s})\, ds$, $\mathbf{t} \in I$ is strongly $\mathbb{D}$-asymptotically c-almost periodic, where we accept the notation (6.28).

Let $F(\cdot;\cdot)$ be I-asymptotically c-uniformly recurrent, $G : I \times X \to Y$, $Q \in C_{0,I,\mathcal{B}}(I \times X : Y)$ and $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$. Then, for every $x \in X$, we have

$$c\{G(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\} \subseteq \overline{\{F(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\}}.$$

The following proposition can be deduced as in the case that $c = 1$.

Proposition 7.1.22.

(i) *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is Bohr $(\mathcal{B},c)$-almost periodic ($(\mathcal{B},c)$-uniformly recurrent). If for each $B \in \mathcal{B}$ there exists $\varepsilon_B > 0$ such that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x\in\partial B} B(x,\varepsilon_B)$, then the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B},c)$-almost periodic ($(\mathcal{B},c)$-uniformly recurrent).*

(ii) *Suppose that for each integer* $j \in \mathbb{N}$ *the function* $F_j(\cdot;\cdot)$ *is I-asymptotically Bohr* $(\mathcal{B}, c)$*-almost periodic (I-asymptotically* $(\mathcal{B}, c)$*-uniformly recurrent). If for each* $B \in \mathcal{B}$ *there exists* $\varepsilon_B > 0$ *such that the sequence* $(F_j(\cdot;\cdot))$ *converges uniformly to a function* $F(\cdot;\cdot)$ *on the set* $B^\circ \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$*, then the function* $F(\cdot;\cdot)$ *is I-asymptotically Bohr* $(\mathcal{B}, c)$*-almost periodic (I-asymptotically* $(\mathcal{B}, c)$*-uniformly recurrent).*

Now we will introduce the following definition (recall that, for any set $\Lambda \subseteq \mathbb{R}^n$ and number $M > 0$, we set $\Lambda_M := \{\lambda \in \Lambda\,;\, |\lambda| \geqslant M\}$).

Definition 7.1.23. Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I', c)$-almost periodic of type 1 if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exist $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that

$$\|F(\mathbf{t} + \tau; x) - cF(\mathbf{t}; x)\|_Y \leqslant \varepsilon, \quad \text{provided } \mathbf{t},\ \mathbf{t} + \tau \in \mathbb{D}_M,\ x \in B. \tag{7.9}$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent of type 1 if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_k) in I' and a sequence (M_k) in $(0, \infty)$ such that $\lim_{k\to+\infty} |\tau_k| = \lim_{k\to+\infty} M_k = +\infty$ and

$$\lim_{k\to+\infty} \sup_{\mathbf{t},\mathbf{t}+\tau_k\in\mathbb{D}_{M_k};x\in B} \|F(\mathbf{t} + \tau_k; x) - cF(\mathbf{t}; x)\|_Y = 0.$$

If $I' = I$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, c)$-almost periodic of type 1 ($\mathbb{D}$-asymptotically $(\mathcal{B}, c)$-uniformly recurrent of type 1); furthermore, if $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr (I', c)-almost periodic of type 1 ($\mathbb{D}$-asymptotically (I', c)-uniformly recurrent of type 1). If $I' = I$ and $X \in \mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr c-almost periodic of type 1 ($\mathbb{D}$-asymptotically c-uniformly recurrent of type 1). As before, we remove the prefix "$\mathbb{D}$-" in the case that $\mathbb{D} = I$ and remove the prefix "$(\mathcal{B},)$" in the case that $X \in \mathcal{B}$.

Clearly, we have the following.

Proposition 7.1.24. *Suppose that* $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ *and the set* $\mathbb{D}$ *is unbounded, as well as* $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$*,* $F : I \times X \to Y$ *is a continuous function and* $I + I' \subseteq I$*. If* $F(\cdot;\cdot)$ *is* $\mathbb{D}$*-asymptotically Bohr* $(\mathcal{B}, I', c)$*-almost periodic, resp.* $\mathbb{D}$*-asymptotically* $(\mathcal{B}, I', c)$*-uniformly recurrent, then* $F(\cdot;\cdot)$ *is* $\mathbb{D}$*-asymptotically Bohr* $(\mathcal{B}, I', c)$*-almost periodic of type* 1*, resp.* $\mathbb{D}$*-asymptotically* $(\mathcal{B}, I', c)$*-uniformly recurrent of type* 1.

Concerning the converse of Proposition 7.1.24, we will state and prove the following statement which can be applied in the case that $I = [0, \infty)^n$.

Theorem 7.1.25. *Suppose that* $\emptyset \neq I \subseteq \mathbb{R}^n$*,* $I + I = I$*, I is closed and* $F : I \to Y$ *is a uniformly continuous, bounded I-asymptotically Bohr c-almost periodic function of* *type* 1, *where* $|c| = 1$. *If*

$$(\forall l > 0)\,(\forall M > 0)\,(\exists \mathbf{t_0} \in I)\,(\exists k > 0)\,(\forall \mathbf{t} \in I_{M+l})\big(\exists \mathbf{t}_\mathbf{0}' \in I\big)$$
$$\big(\forall \mathbf{t}_\mathbf{0}'' \in B(\mathbf{t}_\mathbf{0}', l) \cap I\big)\,\mathbf{t} - \mathbf{t}_\mathbf{0}'' \in B(\mathbf{t_0}, kl) \cap I_M,$$

there exists $L > 0$ *such that* $I_{kL} \setminus I_{(k+1)L} \neq \emptyset$ *for all* $k \in \mathbb{N}$ *and* $I_M + I \subseteq I_M$ *for all* $M > 0$, *then the function* $F(\cdot)$ *is* I*-asymptotically Bohr* c*-almost periodic.*

Proof. Since we have assumed that the function $F(\cdot)$ is bounded and $|c| = 1$, we can use the foregoing arguments in order to see that the function $F(\cdot)$ is I-asymptotically Bohr almost periodic function of type 1. By the foregoing, it follows that for each sequence $(\mathbf{b}_k)$ in I there exist a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ and a function $F^* : I \to Y$ such that $\lim_{l\to+\infty} F(\mathbf{t} + \mathbf{b}_{k_l}) = F^*(\mathbf{t})$, uniformly in $\mathbf{t} \in I$. We continue the proof by observing that for each integer $k \in \mathbb{N}$ there exist $l_k > 0$ and $M_k > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (7.9) holds with $c = 1$, $\varepsilon = 1/k$ and $\mathbb{D} = I$. Let τ_k be any fixed element of I such that $|\tau_k| > M_k + k^2$ and (7.9) holds with $c = 1$, $\varepsilon = 1/k$ and $\mathbb{D} = I$ ($k \in \mathbb{N}$). Then there exist of a subsequence (τ_{k_l}) of (τ_k) and a function $F^* : I \to Y$ such that

$$\lim_{l\to+\infty} F(\mathbf{t} + \tau_{k_l}) = F^*(\mathbf{t}), \quad \text{uniformly for } t \in I. \tag{7.10}$$

The mapping $F^*(\cdot)$ is clearly continuous and now we will prove that $F^*(\cdot)$ is Bohr c-almost periodic. Let $\varepsilon > 0$ be fixed, and let $l > 0$ and $M > 0$ be such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (7.9) holds with $\mathbb{D} = I$ and the number ε replaced therein by $\varepsilon/3$. Let $\mathbf{t} \in I$ be fixed, and let $l_0 \in \mathbb{N}$ be such that $|\mathbf{t} + \tau_{k_{l_0}}| \geqslant M$ and $|\mathbf{t} + \tau + \tau_{k_{l_0}}| \geqslant M$. Then we have

$$\begin{aligned}
&\big\|F^*(\mathbf{t} + \tau) - cF^*(\mathbf{t})\big\| \\
&\quad\leqslant \big\|F^*(\mathbf{t} + \tau) - F(\mathbf{t} + \tau + \tau_{k_{l_0}})\big\| + \big\|F(\mathbf{t} + \tau + \tau_{k_{l_0}}) - cF(\mathbf{t} + \tau_{k_{l_0}})\big\| \\
&\qquad + \big\|cF(\mathbf{t} + \tau_{k_{l_0}}) - cF^*(\mathbf{t})\big\| \leqslant 3 \cdot (\varepsilon/3) = \varepsilon,
\end{aligned}$$

as required. The function $\mathbf{t} \mapsto F(\mathbf{t}) - F^*(\mathbf{t})$, $\mathbf{t} \in I$ belongs to the space $C_{0,I}(I : Y)$ due to (7.10) and the fact that $F : I \to Y$ is an I-asymptotically Bohr almost periodic function of type 1, which completes the proof. □

For any set $S \subseteq \mathbb{R}^n$ and for any integer $l \in \mathbb{N}$, we define the set S_l inductively by $S_1 := S$ and $S_{l+1} := S_l + S$ ($l = 1, 2, \ldots$). Furthermore, we define $\Omega := I'$ and $\Omega_S := I' \cup S$ if $\arg(c)/\pi \notin \mathbb{Q}$. If $\arg(c)/\pi \in \mathbb{Q}$, then we take any non-empty finite set of integers $S_1 \subseteq \mathbb{Z} \setminus \{0\}$ such that $c^{m+1} = 1$ for all $m \in S_1$ and any non-empty finite set of integers $S_2 \subseteq \mathbb{N}$ such that $c^l = 1$ for all $l \in S_2$; in this case, we set $\Omega := (I' \bigcup_{m\in S_1}(-mI'))_l$ and $\Omega_S := \Omega \cup S$.

Now we are able to state and prove the following result concerning the extensions of Bohr (I', c)-almost periodic functions and (I', c)-uniformly recurrent functions.

Theorem 7.1.26. *Suppose that $I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$, the set I' is unbounded, $|c| = 1$, $F : I \to Y$ is a uniformly continuous, Bohr (I', c)-almost periodic function, resp. a uniformly continuous, (I', c)-uniformly recurrent function, $S \subseteq \mathbb{R}^n$ is bounded and condition* (AP-E) *holds. Then there exists a uniformly continuous, Bohr (Ω_S, c)-almost periodic, resp. a uniformly continuous, (Ω_S, c)-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$; furthermore, in c-almost periodic case, the uniqueness of such a function $\tilde{F}(\cdot)$ holds provided that $\mathbb{R}^n \setminus \Omega_S$ is a bounded set.*

Proof. We will consider only uniformly continuous, Bohr (I', c)-almost periodic functions. In this case, for each natural number $k \in \mathbb{N}$ there exists a point $\tau_k \in I'$ such that $\|F(\mathbf{t} + \tau_k) - cF(\mathbf{t})\|_Y \leqslant 1/k$ for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$; furthermore, since the set I' is unbounded, we may assume without loss of generality that $\lim_{k \to +\infty} |\tau_k| = +\infty$. Hence, we have

$$\lim_{k \to +\infty} F(\mathbf{t} + \tau_k) = cF(\mathbf{t}), \quad \text{uniformly for } t \in I. \tag{7.11}$$

If $\mathbf{t}' \in \mathbb{R}^n$, then there exists a finite real number $M > 0$ such that $\mathbf{t}' + I'_M \subseteq I$, and now we will prove that the sequence $(F(\mathbf{t}' + \tau_k))_{k \in \mathbb{N}}$ is Cauchy and therefore convergent. Let $\varepsilon > 0$ be fixed; then we have the existence of a number $k_0 \in \mathbb{N}$ such that $\mathbf{t}' + \tau_k \in I$ for all $k \geqslant k_0$. Suppose that $k,\ m \geqslant k_0$. Then we have

$$\begin{aligned} \|F(\mathbf{t}' + \tau_k) - F(\mathbf{t}' + \tau_m)\| \leqslant{}& \|F(\mathbf{t}' + \tau_k) - c^{-1}F(\mathbf{t}' + \tau_k + \tau)\| \\ &+ \|c^{-1}F(\mathbf{t}' + \tau_k + \tau) - c^{-1}F(\mathbf{t}' + \tau_m + \tau)\| \\ &+ \|c^{-1}F(\mathbf{t}' + \tau_m + \tau) - F(\mathbf{t}' + \tau_m)\|, \end{aligned}$$

for any $\tau \in I'$ such that $\mathbf{t}' + \tau \in I$. Since the function $F(\cdot)$ is Bohr (I', c)-almost periodic, we can always find such a number τ so that the first and the third addend in the above estimates are less than or equal to $\varepsilon/3$; for the second addend in the above estimate, we can find a sufficiently large number $k_1 \geqslant k_0$ such that

$$\|c^{-1}F(\mathbf{t}' + \tau_k + \tau) - c^{-1}F(\mathbf{t}' + \tau_m + \tau)\| < \varepsilon/3,$$

for all $k,\ m \geqslant k_1$ (see (7.11)). Therefore, $\lim_{k \to +\infty} F(\mathbf{t}' + \tau_k) := \tilde{F}(\mathbf{t}')$ exists. The function $\tilde{F}(\cdot)$ is clearly uniformly continuous because $F(\cdot)$ is uniformly continuous; furthermore, by construction, we see that $\tilde{F}(\mathbf{t})/c = F(\mathbf{t})$ for all $\mathbf{t} \in I$. Now we will prove that the function $\tilde{F}(\cdot)$ is Bohr (Ω_S, c)-almost periodic. Let a number $\varepsilon > 0$ be given. Then there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that $\|F(\mathbf{t} + \tau) - cF(\mathbf{t})\|_Y \leqslant \varepsilon/2$ for all $\mathbf{t} \in I$. Let $\mathbf{t}' \in \mathbb{R}^n$ be fixed. For any such numbers $\mathbf{t}_0 \in I'$ and $\tau \in B(\mathbf{t}_0, l) \cap I'$, we have

$$\begin{aligned} \|\tilde{F}(\mathbf{t}' + \tau) - c\tilde{F}(\mathbf{t}')\|_Y &= \Big\| \lim_{k \to +\infty} [F(\mathbf{t}' + \tau + \tau_k) - cF(\mathbf{t}' + \tau_k)] \Big\|_Y \\ &\leqslant \limsup_{k \to +\infty} \|F(\mathbf{t}' + \tau + \tau_k) - cF(\mathbf{t}' + \tau_k)\|_Y \leqslant \varepsilon/2, \quad \mathbf{t}' \in \mathbb{R}^n. \end{aligned} \tag{7.12}$$

If $\arg(c)/\pi \notin \mathbb{Q}$, this clearly implies that $F(\cdot)$ is Bohr (Ω, c)-almost periodic and therefore Bohr (Ω_S, c)-almost periodic. If $\arg(c)/\pi \in \mathbb{Q}$, then we may assume without loss of generality that the sets $S_1 = \{m\}$ and $S_2 = \{l\}$ are singletons (this follows from the corresponding definition of Bohr (I', c)-almost periodicity). Given $\varepsilon > 0$ in advance, we may assume that (7.12) holds with the number $\varepsilon/2$ replaced therein with the number $\varepsilon/l|m|$. By (7.3), we see that the number $-m\tau \in \Omega$ is an $(\varepsilon/l, c)$-period of $F(\cdot)$, with the meaning clear. Arguing as in the proof of the estimate (7.3), it readily follows that any finite sum $\tau_1 + \cdots + \tau_l$, where $\tau_i \in I' \bigcup_{m \in S_1}(-mI')$ for all $i \in \mathbb{N}_l$, is an (ε, c)-period of $F(\cdot)$. As above, this implies that $F(\cdot)$ is Bohr (Ω, c)-almost periodic and therefore Bohr (Ω_S, c)-almost periodic.

Finally, if the set $\mathbb{R}^n \smallsetminus \Omega_S$ is bounded, we can argue as before to prove the uniqueness of extension in the c-almost periodic case. □

Remark 7.1.27.
(i) It is clear that Theorem 7.1.26 strengthens Theorem 6.1.37, where we have assumed that $c = 1$ and $\Omega_S = [(I' \cup (-I')) + (I' \cup (-I'))] \cup S$.
(ii) In the case that $\arg(c)/\pi \notin \mathbb{Q}$, it is not clear whether there exists a set $\Omega_S' \supseteq \Omega_S$ such that the constructed function $\tilde{F} : \mathbb{R}^n \to Y$ is Bohr (Ω_S', c)-almost periodic. Concerning this issue, it is worth noting that the notion introduced in Definition 7.1.6 can be further extended by allowing that the set I' depends on the set B and the number $\varepsilon > 0$. This could probably fix some things here, but we will skip all related details for the sake of brevity.

Before proceeding, we would like to propose the following definition.

Definition 7.1.28. Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$ and $I + I \subseteq I$. Then we say that I is admissible with respect to the c-almost periodic extensions if and only if for any complex Banach space Y and for any uniformly continuous, Bohr c-almost periodic function $F : I \to Y$ there exists a unique Bohr c-almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in I$. If $c = \pm 1$, then we also say that the region I is admissible with respect to the almost (anti-)periodic extensions.

If $|c| = 1$, $\arg(c)/\pi \in \mathbb{Q}$, $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis of $\mathbb{R}^n$ and

$$I' = I = \{\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n : \alpha_i \geqslant 0 \text{ for all } i \in \mathbb{N}_n\}$$

is a convex polyhedral in $\mathbb{R}^n$, then $\Omega_S = \mathbb{R}^n$ and therefore the set I is admissible with respect to the c-almost periodic extensions. It is very simple to construct some sets which are not admissible with respect to the c-almost periodic extensions; for example, the set $I = [0, \infty) \times \{0\} \subseteq \mathbb{R}^2$ is not admissible with respect to the c-almost periodic extensions since there is no c-almost periodic extension of the function $F(x, y) = y$, $(x, y) \in I$ to the whole Euclidean space.

Several interesting examples and applications of our abstract theoretical results can be found in [653]. Here we will present only one application, closely related with

Theorem 4.2.40. Let (τ_k) be a sequence in $\mathbb{R}^n$, $\lim_{k\to+\infty} |\tau_k| = +\infty$ and

$$\mathrm{BUR}_{(\tau_k);c}(\mathbb{R}^n : X) := \Big\{F : \mathbb{R}^n \to X \text{ is bounded, continuous and } \lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \|F(t+\tau_k) - cf(t)\|_\infty = 0\Big\}.$$

Equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_\infty$, $\mathrm{BUR}_{(\tau_k);c}(\mathbb{R}^n : X)$ becomes a complete metric space. Define $I' := \{\tau_k : k \in \mathbb{N}\}$ and consider the following Hammerstein integral equation of convolution type on $\mathbb{R}^n$:

$$y(\mathbf{t}) = \int_{\mathbb{R}^n} k(\mathbf{t}-\mathbf{s})G(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{7.13}$$

where $G : \mathbb{R}^n \times X \to X$ is $(\mathcal{B}, I', c)$-uniformly recurrent with $\mathcal{B}$ being the collection of all bounded subsets of X. Suppose, further, that the set $\{G(\mathbf{t}, B) : \mathbf{t} \in \mathbb{R}^n\}$ is bounded for any bounded subset B of X as well as that there exists a finite real constant $L > 0$ such that (7.6) holds with $X = Z = Y$, for every $y,\ y' \in \mathbb{R}^n$, and (7.8) holds with the term $F(\mathbf{t}; x)$ replaced with the term $y(\mathbf{t})$ for any function $y \in \mathrm{BUC}_{(\tau_k);c}(\mathbb{R}^n : X)$. Applying Proposition 7.1.17 and Theorem 7.1.18(ii), we see that the mapping

$$\mathrm{BUR}_{(\tau_k);c}(\mathbb{R}^n : X) \ni y \mapsto \int_{\mathbb{R}^n} k(\cdot - \mathbf{s})G(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s} \in \mathrm{BUR}_{(\tau_k);c}(\mathbb{R}^n : X)$$

is well defined. If we additionally assume that $L \int_{\mathbb{R}^n} |k(\mathbf{t})|\, d\mathbf{t} < 1$, then an application of the Banach contraction principle shows that there exists a unique solution of (7.13) which belongs to the space $\mathrm{BUR}_{(\tau_k);c}(\mathbb{R}^n : X)$.

7.2 Multi-dimensional (ω, c)-almost periodic type functions and applications

The main aim of this section is to introduce and analyze various notions of (ω, c)-periodicity and (ω, c)-almost periodicity for vector-valued functions depending of several real variables; we provide certain applications to the abstract partial differential equations, as well [651]. In such a way, we continue our analysis of one-dimensional (ω, c)-almost periodic type functions from Section 4.1. For multi-periodic solutions of various classes of ordinary differential equations and partial differential equations, we also refer the reader to [164, 165, 450, 583, 683, 684, 775, 905, 906, 984, 985, 987]. Especially, we would like to mention the investigations of G. Nadin [806–808] concerning the space-time periodic reaction–diffusion equations, L. Rossi [879] concerning Liouville type results for almost periodic type linear operators and the investigation of B. Scarpellini [910] concerning the space almost periodic solutions of reaction–diffusion equations and the recent investigation of R. Xie, Z. Xia, J. Liu [1043] about quasi-periodic limit functions, (ω_1, ω_2)-(quasi)-periodic limit functions and their applications (two-dimensional setting).

The organization and main ideas of this section can be briefly described as follows. The main structural results concerning multi-dimensional (ω, c)-periodic functions and multi-dimensional $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic functions are obtained in Propositions 7.2.4, 7.2.5, 7.2.7, 7.2.8 and 7.2.10. The corresponding classes of asymptotically (ω, c)-almost periodic type functions are introduced in Definition 7.2.14. Subsection 7.2.1 investigates $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic type functions. In Definition 7.2.15, we introduce the notion of $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodicity, $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniform recurrence and $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-almost automorphy. The main structural features of the function spaces introduced in Definition 7.2.15 are stated in Proposition 7.2.18; we also discuss the convolution invariance of the function spaces introduced here before switching to the third subsection, which is reserved for the study of $(I', \mathbf{a}, \omega, c)$-uniform recurrence of type 1 (2) and the $(I', \mathbf{a}, \omega, c)$-almost periodicity of type 1 and type 2. Here we continue our investigation of the one-dimensional case and prove two negative results, Theorem 7.2.21 and Theorem 7.2.23, saying that the introduction of Definition 7.2.19 is basically an unsatisfactory way to extend the notion of (ω, c)-almost periodicity. In the final subsection, we provide certain applications to the abstract Volterra integro-differential equations in Banach spaces.

The following definition is crucial in our analysis.

Definition 7.2.1. Let $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$ and $\omega + I \subseteq I$. A continuous function $F : I \to X$ is said to be (ω, c)-periodic if and only if $F(\mathbf{t} + \omega) = cF(\mathbf{t})$, $\mathbf{t} \in I$.

If $F : I \to X$ is a Bloch $(\mathbf{p}, \mathbf{k})$-periodic function, then $F(\cdot)$ is $(\mathbf{p}, c)$-periodic with $c = e^{i\langle \mathbf{k}, \mathbf{p}\rangle}$; conversely, if $|c| = 1$ and $F : I \to X$ is (ω, c)-periodic, then we can always find a point $\mathbf{k} \in \mathbb{R}^n$ such that the function $F(\cdot)$ is Bloch $(\mathbf{p}, \mathbf{k})$-periodic. In the case that $|c| \neq 1$, we have the following: if $F : I \to X$ is (ω, c)-periodic, then $F(\mathbf{t} + m\omega) = c^m F(\mathbf{t})$, $\mathbf{t} \in I$, $m \in \mathbb{N}$, so that the existence of a point $\mathbf{t}_0 \in I$ such that $F(\mathbf{t}_0) \neq 0$ implies $\lim_{m\to\infty} \|F(\mathbf{t}_0 + m\omega)\| = +\infty$, provided that $|c| > 1$, and $\lim_{m\to\infty} \|F(\mathbf{t}_0 + m\omega)\| = 0$, provided that $|c| < 1$.

If $c = 1$, resp. $c = -1$, then we also say that the function $F(\cdot)$ is ω-periodic, resp. ω-anti-periodic. It is clear that, if $F(\cdot)$ is (ω, c)-periodic, $k \in \mathbb{N}$ and $c^k = 1$, resp. $c^k = -1$, then $F(\cdot)$ is $(k\omega)$-periodic, resp. $(k\omega)$-anti-periodic.

In [522, Definition 2.1], the authors have assumed that any Bloch $(\mathbf{p}, \mathbf{k})$-periodic is bounded a priori, which is a slightly redundant condition as the following example shows.

Example 7.2.2. There exists a continuous, unbounded function $F : \mathbb{R}^n \to \mathbb{R}$ which satisfies $F(\mathbf{t} + (1, 1, \ldots, 1)) = F(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$. We can simply construct such a function, with $n = 2$, as follows. Let $F_0 : \{(t_1, t_2) \in \mathbb{R}^2 : 0 \leqslant t_1 + t_2 \leqslant 2\}$ be any continuous function satisfying that:

$$F_0(t_1, t_2) = F(t_1 + 1, t_2 + 1), \quad \text{provided } (t_1, t_2) \in \mathbb{R}^2 \text{ and } t_1 + t_2 = 0, \tag{7.14}$$

$$\text{the set } \{(4k\sqrt{2}, -4k\sqrt{2}) : k \in \mathbb{N}\} \text{ is unbounded, and} \tag{7.15}$$

$$F_0((4k+2)\sqrt{2}, -(4k+2)\sqrt{2}) = 1, \quad k \in \mathbb{N}. \tag{7.16}$$

Due to condition (7.14), we can extend the function $F_0(\cdot)$ to a continuous function $F : \mathbb{R}^2 \to \mathbb{R}$ which satisfies $F(t_1+1, t_2+1) = F(t_1, t_2)$ for all $t_1,\ t_2 \in \mathbb{R}$. Clearly, this function is unbounded due to condition (7.15).

The following definition is also meaningful.

Definition 7.2.3. Let $\omega_j \in \mathbb{R} \smallsetminus \{0\}$, $c_j \in \mathbb{C} \smallsetminus \{0\}$ and $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n)$. A continuous function $F : I \to X$ is said to be $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic if and only if $F(\mathbf{t} + \omega_j e_j) = c_j F(\mathbf{t})$, $\mathbf{t} \in I$, $j \in \mathbb{N}_n$.

It is clear that, if $F : I \to X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic, then $F(\mathbf{t} + m\omega_j e_j) = c_j^m F(\mathbf{t})$, $\mathbf{t} \in I$, $m \in \mathbb{N}$, $j \in \mathbb{N}_n$, so that the existence of a point $\mathbf{t}_0 \in I$ such that $F(\mathbf{t}_0) \neq 0$ implies $\lim_{m\to\infty} ||F(\mathbf{t}_0 + m\omega_j e_j)|| = +\infty$, provided that $|c_j| > 1$, and $\lim_{m\to\infty} ||F(\mathbf{t}_0 + m\omega_j e_j)|| = 0$, provided that $|c_j| < 1$, for some $j \in \mathbb{N}_n$.

If $c_j = 1$ for all $j \in \mathbb{N}_n$, resp. $c_j = -1$ for all $j \in \mathbb{N}_n$, then we also say that the function $F(\cdot)$ is $(\omega_j)_{j \in \mathbb{N}_n}$-periodic, resp. $(\omega_j)_{j \in \mathbb{N}_n}$-anti-periodic. It is clear that, if $F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic, $k \in \mathbb{N}$ and $c_j^k = 1$ for all $j \in \mathbb{N}_n$, resp. $c_j^k = -1$ for all $j \in \mathbb{N}_n$, then $F(\cdot)$ is $(k\omega_j)_{j \in \mathbb{N}_n}$-periodic, resp. $(k\omega_j)_{j \in \mathbb{N}_n}$-anti-periodic.

The classes of (ω, c)-periodic functions and $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic functions are closed under the operation of the pointwise convergence of the functions, as easily approved. In the scalar-valued case, the following holds: If the function $F : I \to \mathbb{C} \smallsetminus \{0\}$ is (ω, c)-periodic, resp. $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic, then the function $(1/F)(\cdot)$ is $(\omega, 1/c)$-periodic, resp. $(\omega_j, 1/c_j)_{j \in \mathbb{N}_n}$-periodic. It is also clear that we have the following.

Proposition 7.2.4.

(i) *Let $\omega,\ a \in \mathbb{R}^n \smallsetminus \{0\}$, $c \in \mathbb{C} \smallsetminus \{0\}$, $\alpha \in \mathbb{C}$, $\omega + I \subseteq I$ and $a + I \subseteq I$. If the function $F : I \to X$ is (ω, c)-periodic, then $-\omega - I \subseteq -I$ and the function $\check{F} : -I \to X$, defined by $\check{F}(x) := F(-x)$, $x \in I$, is $(-\omega, c)$-periodic. Moreover, $\|F(\cdot)\|$ is $(\omega, |c|)$-periodic, the function $F_a : I \to X$ defined by $F_a(\mathbf{t}) := F(\mathbf{t} + a)$, $\mathbf{t} \in I$ is (ω, c)-periodic and the function $\alpha F(\cdot)$ is (ω, c)-periodic.*

(ii) *Let $\omega_j \in \mathbb{R} \smallsetminus \{0\}$, $c_j \in \mathbb{C} \smallsetminus \{0\}$, $\alpha \in \mathbb{C}$, $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n)$ and $a + I \subseteq I$. If a continuous function $F : I \to X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic, then $-\omega_j e_j - I \subseteq -I$ $(1 \leqslant j \leqslant n)$ and the function $\check{F} : -I \to X$ is $(-\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic. Moreover, $\|F(\cdot)\|$ is $(\omega_j, |c_j|)_{j \in \mathbb{N}_n}$-periodic, the function $F_a : I \to X$ defined above is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic and the function $\alpha F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic.*

Proposition 7.2.5. *Let $\omega_j \in \mathbb{R} \smallsetminus \{0\}$, $c_j \in \mathbb{C} \smallsetminus \{0\}$ and $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n)$. If a continuous function $F : I \to X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic, then $\omega + I \subseteq I$, where $\omega := \sum_{j=1}^n \omega_j e_j$, and the function $F(\cdot)$ is (ω, c)-periodic with $c =: \prod_{j=1}^n c_j$.*

The converse statement is not true in the general case $n > 1$, as the following simple counterexample shows.

Example 7.2.6. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ from Example 7.2.2. Then there do not exist numbers $\omega_1,\ \omega_2 \in \mathbb{R} \setminus \{0\}$ and numbers $c_1,\ c_2 \in \mathbb{C} \setminus \{0\}$ such that the function $F(\cdot)$ is $(\omega_j, c_j)_{j\in\mathbb{N}_2}$-periodic. If we assume the contrary, then we would have $F(t_1+\omega_1, 0) = c_1F(t_1, 0)$ for all $t_1 \in \mathbb{R}$. If $|c_1| \leqslant 1$, then the contradiction is obvious since (7.15) implies the unboundedness of the function $F(\cdot, 0)$, because $F(8k, 0) = F(4k\sqrt{2}, -4k\sqrt{2})$ for all $k \in \mathbb{N}$. If $|c_1| > 1$, then the contradiction is obvious due to condition (7.16), which implies that the function $F(\cdot, 0)$ cannot tend to plus infinity as the time variable tends to plus infinity (see also [586, Remark 2.4]).

Concerning the boundedness of $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic functions, we will state only one result.

Proposition 7.2.7. *Suppose that $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $M > 0$, $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n)$, the set I is closed, the function $F : I \to X$ is $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic, $|c_j| \leqslant 1$ for all $j \in \mathbb{N}_n$ and, for every $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in I$, there exist a point $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in I_M$ and integers $k_j \in \mathbb{N}$ $(1 \leqslant j \leqslant n)$ such that $t_j = k_j\omega_j + \eta_j$ $(1 \leqslant j \leqslant n)$. Then the function $F(\cdot)$ is bounded.*

Proof. Let a point $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in I$ be fixed, and let $\eta \in I$ and integers $k_j \in \mathbb{N}$ $(1 \leqslant j \leqslant n)$ satisfy the above requirements. Then we have $\mathbf{t} = \eta + \sum_{j=1}^{n} k_j\omega_j e_j$ so that $F(\mathbf{t}) = \prod_{j=1}^{n} c_j^{k_j} F(\eta)$. Since I is closed, I_M is compact and there exist a finite constant $M_1 > 0$ such that $\|F(\mathbf{x})\| \leqslant M_1$ for all $\mathbf{x} \in I_M$. Then $\|F(\mathbf{t})\| \leqslant M_1$ since $|c_j| \leqslant 1$ for all $j \in \mathbb{N}_n$. □

We profile the class of (ω, c)-periodic functions in the following way.

Proposition 7.2.8. *Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n \setminus \{0\}$, $\omega + I \subseteq I$, $c \in \mathbb{C} \setminus \{0\}$ and $S := \{i \in \mathbb{N}_n : \omega_i \neq 0\}$. Denote by A the collection of all tuples $\mathbf{a} = (a_1, a_2, \ldots, a_{|S|}) \in \mathbb{R}^{|S|}$ such that $\sum_{i\in S} a_i = 1$. Then a continuous function $F : I \to X$ is (ω, c)-periodic if and only if, for every (some) $\mathbf{a} \in \mathrm{A}$, the function $G_{\mathbf{a}} : I \to X$, defined by*

$$G_{\mathbf{a}}(t_1, t_2, \ldots, t_n) := c^{-\sum_{i\in S} \frac{a_i t_i}{\omega_i}} F(t_1, t_2, \ldots, t_n), \quad \mathbf{t} = (t_1, t_2, \ldots, t_n) \in I, \tag{7.17}$$

is $(\omega, 1)$-periodic.

Proof. Let a point $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in I$ be fixed. Then it is clear that $G_{\mathbf{a}}(\mathbf{t} + \omega) = G_{\mathbf{a}}(\mathbf{t})$ if and only if

$$c^{-\sum_{i\in S} \frac{a_i(t_i+\omega_i)}{\omega_i}} F(t_1 + \omega_1, t_2 + \omega_2, \ldots, t_n + \omega_n) = c^{-\sum_{i\in S} \frac{a_i t_i}{\omega_i}} F(t_1, t_2, \ldots, t_n)$$

if and only if $F(\mathbf{t} + \omega) = cF(\mathbf{t})$. □

We illustrate Proposition 7.2.8 with the following example.

Example 7.2.9 (see also [522, pp. 22–23]). Suppose that $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n \setminus \{0\}$, $\mathbf{k} \in \mathbb{R}^n \setminus \{0\}$, $\mathbf{a} \in \mathrm{A}$, (b_l) is any sequence of complex numbers such that $|b_l| = O(l^{-2})$, $\langle \mathbf{k}, \omega \rangle = 2\pi/3$ and

$$F(t_1, t_2, \ldots, t_n) = c^{\sum_{i \in S} \frac{a_i t_i}{\omega_i}} \sum_{l \in 1+3\mathbb{N}} b_l e^{il\langle \mathbf{t}, \mathbf{k} \rangle}, \quad \mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n.$$

Then $F(\cdot)$ is $(3\omega, c)$-almost periodic.

Similarly, we can prove the following.

Proposition 7.2.10. *Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n)$ and the function $F : I \to X$ is continuous. For each $j \in \mathbb{N}_n$, we define the function $G_j : I \to X$ by*

$$G_j(t_1, t_2, \ldots, t_n) := c_j^{-\frac{t_j}{\omega_j}} F(t_1, t_2, \ldots, t_n), \quad \mathbf{t} = (t_1, t_2, \ldots, t_n) \in I. \tag{7.18}$$

Then $F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic if and only if, for every $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in I$ and $j \in \mathbb{N}_n$, we have

$$G_j(t_1, t_2, \ldots, t_j + \omega_j, \ldots, t_n) = G_j(t_1, t_2, \ldots, t_j, \ldots, t_n).$$

Therefore, we have the following.

Example 7.2.11. Let $c_j \in \mathbb{C} \setminus \{0\}$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \ldots, t_n) := \prod_{j=1}^{n} c_j^{\frac{t_j}{2\pi}} \sin t_j, \quad \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

is $(2\pi, c_j)_{j \in \mathbb{N}_n}$-periodic.

If $\omega \in \mathbb{R}^n \setminus \{0\}$, $c_i \in \mathbb{C} \setminus \{0\}$ for $i = 1, 2$, $\omega + I \subseteq I$, the function $G : I \to \mathbb{C}$ is (ω, c_1)-periodic and the function $H : I \to X$ is (ω, c_2)-periodic, then the function $F(\cdot) := G(\cdot)H(\cdot)$ is $(\omega, c_1 c_2)$-periodic. For the class of $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic functions, we can clarify the following result.

Proposition 7.2.12. *Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_{j,i} \in \mathbb{C} \setminus \{0\}$ and $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n, 1 \leqslant i \leqslant 2)$. Suppose that the function $G : I \to \mathbb{C}$ is $(\omega_j, c_{j,1})_{j \in \mathbb{N}_n}$-periodic and the function $H : I \to X$ is $(\omega_j, c_{j,2})_{j \in \mathbb{N}_n}$-periodic. Set $c_j := c_{j,1} c_{j,2}$, $1 \leqslant j \leqslant n$. Then the function $F(\cdot) := G(\cdot)H(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic.*

Now we would like to state and prove the following result.

Proposition 7.2.13. *Suppose that $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $S = \{i \in \mathbb{N}_n : \omega_i \neq 0\}$, $\mathbf{a} = (a_1, a_2, \ldots, a_{|S|}) \in \mathbb{R}^{|S|}$ and $\sum_{i \in S} a_i = 1$, resp. $\omega_j \in \mathbb{R} \setminus \{0\}$ and $c_j \in \mathbb{C} \setminus \{0\}$ $(1 \leqslant j \leqslant n)$. Suppose, further, that $F : \mathbb{R}^n \to X$ is (ω, c)-periodic and the function $G_{\mathbf{a}}(\cdot)$, defined through (7.17) is bounded, resp. $F : \mathbb{R}^n \to X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic and for each $j \in \mathbb{N}_n$ the* *function* $G_j(\cdot)$*, defined through* (7.18) *is bounded. If the function* $c^{-\sum_{i\in S} a_i t_i/\omega_i} h(t_1, \ldots, t_n)$ *belongs to the space* $L^1(\mathbb{R}^n)$*, resp. for each* $j \in \mathbb{N}_n$ *the function* $c^{-t_j/\omega_j} h(t_1, \ldots, t_n)$ *belongs to the space* $L^1(\mathbb{R}^n)$*, then the function*

$$(h * F)(\mathbf{t}) := \int_{\mathbb{R}^n} h(y)F(\mathbf{t} - y)\, dy, \quad \mathbf{t} \in \mathbb{R}^n,$$

is (ω, c)*-periodic, resp.* $(\omega_j, c_j)_{j\in\mathbb{N}_n}$*-periodic.*

Proof. We will consider only (ω, c)-periodicity. By Proposition 7.2.8, it suffices to show that

$$c^{-\sum_{i\in S} \frac{a_i(t_i+\omega_i)}{\omega_i}} (h * F)(\mathbf{t} + \omega) = c^{-\sum_{i\in S} \frac{a_i t_i}{\omega_i}} (h * F)(\mathbf{t}) \tag{7.19}$$

for every fixed point $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$. Note first that the value $(h * F)(\mathbf{t})$ is well defined, since we have assumed that the function $c^{-\sum_{i\in S} a_i t_i/\omega_i} h(t_1, \ldots, t_n)$ belongs to the space $L^1(\mathbb{R}^n)$, and that the function $G_{\mathbf{a}}(\cdot)$, defined through (7.17), is bounded and

$$\begin{aligned} &c^{-\sum_{i\in S} a_i t_i/\omega_i} (h * F)(\mathbf{t}) \\ &\quad = \int_{\mathbb{R}^n} \big[c^{-\sum_{i\in S} a_i y_i/\omega_i} h(y_1, \ldots, y_n)\big] \cdot \big[c^{-\sum_{i\in S} a_i(t_i - y_i)/\omega_i} F(\mathbf{t} - y)\big]\, dy. \end{aligned} \tag{7.20}$$

Keeping in mind (7.20) and the dominated convergence theorem, we see that the function $(h * F)(\cdot)$ is continuous. Similarly, by plugging $\mathbf{t} + \omega$ in place of $\mathbf{t}$ in (7.20), we see that (7.19) holds because $\sum_{i\in S} a_i = 1$. □

Concerning asymptotically (ω, c)-periodic type functions, we will use the following definition, only.

Definition 7.2.14. Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega + I \subseteq I$, $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n, 1 \leqslant i \leqslant 2)$ and $F : I \to X$. Then we say that the function $F(\cdot)$ is $\mathbb{D}$-asymptotically (ω, c)-periodic, resp. $\mathbb{D}$-asymptotically $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic, if and only if there exists a (ω, c)-periodic, resp. $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic, function $F_0 : I \to X$ and a function $C_{0,\mathbb{D},\mathcal{B}}(I : X)$ such that $F(\mathbf{t}) = F_0(\mathbf{t}) + Q(\mathbf{t})$, $\mathbf{t} \in I$.

7.2.1 $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-Almost periodic type functions

We can introduce and analyze several various generalizations of the class of multi-dimensional $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic functions with the help of Proposition 7.2.10. For example, suppose that $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$ and $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n)$; if a function $F : I \to X$ is $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic, then for each $j \in \mathbb{N}_n$ and for every $k \in \mathbb{N}$ we have $F(\mathbf{t} + k\omega_j e_j) = c_j^k F(\mathbf{t})$, $\mathbf{t} \in I$, $j \in \mathbb{N}_n$ and $G_j(\mathbf{t} + k\omega_j e_j) = G_j(\mathbf{t})$, $\mathbf{t} \in I$, $j \in \mathbb{N}_n$. Set

$W_+ := \{j \in \mathbb{N}_n : \omega_j > 0\}$ and $W_- := \{j \in \mathbb{N}_n : \omega_j < 0\}$, as well as $I_{j,\mathbf{t}} := \{x \geqslant 0 : \mathbf{t} + xe_j \in I\}$ if $j \in W_+$, resp. $I_{j,\mathbf{t}} := \{x \geqslant 0 : \mathbf{t} - xe_j \in I\}$ if $j \in W_-$ ($\mathbf{t} \in I$), and $G_{j,\mathbf{t}}(x) := G_j(\mathbf{t} + xe_j)$, $x \in I_{j,\mathbf{t}}$ if $j \in W_+$, resp. $G_{j,\mathbf{t}}(x) := G_j(\mathbf{t} - xe_j)$, $x \in I_{j,\mathbf{t}}$ if $j \in W_-$ ($\mathbf{t} \in I$). Then we can generalize the class of $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic functions as follows.

Definition 7.2.15. Suppose that $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$ ($1 \leqslant j \leqslant n$) and $F : I \to X$ is a continuous function. Let $r_j \in \mathbb{C} \setminus \{0\}$ for $1 \leqslant j \leqslant n$, and let $\emptyset \neq I'_{j,\mathbf{t}} \subseteq I_{j,\mathbf{t}} \subseteq \mathbb{R}$, $I_{j,\mathbf{t}} + I'_{j,\mathbf{t}} \subseteq I_{j,\mathbf{t}}$ for $1 \leqslant j \leqslant n$, $\mathbf{t} \in I$. Set $\mathbb{I}'_j := \{I'_{j,\mathbf{t}} : \mathbf{t} \in I\}$. Then we say that the function $F(\cdot)$ is:

(i) $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic if and only if, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, the function $G_{j,\mathbf{t}}(\cdot)$ defined above is $(I'_{j,\mathbf{t}}, r_j)$-almost periodic;

(ii) $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent if and only if, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, the function $G_{j,\mathbf{t}}(\cdot)$ is $(I'_{j,\mathbf{t}}, r_j)$-uniformly recurrent.

Suppose that $I = \mathbb{R}^n$. Then we say that the function $F(\cdot)$ is:

(iii) $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-almost automorphic if and only if, for every $j \in \mathbb{N}_n$, for every $\mathbf{t} \in \mathbb{R}^n$ and for every real sequence (b_k), there exist a subsequence (a_k) of (b_k) and a function $F^*_{j,\mathbf{t}} : \mathbb{R} \to X$ such that

$$\lim_{k\to+\infty} G_j(\mathbf{t} + (x + a_k)e_j) = F^*_{j,\mathbf{t}}(x) \quad \text{and} \quad \lim_{k\to+\infty} F^*_{j,\mathbf{t}}(x - a_k) = G_j(\mathbf{t} + xe_j), \tag{7.21}$$

pointwise for $x \in \mathbb{R}$; if, moreover, the convergence in (7.21) is uniform in the variable x on compact subsets of $\mathbb{R}$, then we say that the function $F(\cdot)$ is compactly $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-almost automorphic.

Remark 7.2.16.

(i) It is clear that $I = \mathbb{R}^n$ is equivalent to saying that $I + \eta e_j \subseteq I$ for all $\eta \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{N}_n$.

(ii) It is clear that (i) implies (ii) and that the almost periodicity of the function $\widetilde{G_{j,\mathbf{t}}}(x) := G_j(\mathbf{t} + xe_j)$, $x \in \mathbb{R}$ for all $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$ implies (iii), which is equivalent to saying that the function $\widetilde{G_{j,\mathbf{t}}}(\cdot)$ defined above is almost automorphic for all $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$.

Now we will provide an illustrative example in which we have $I = \mathbb{R}^n$, $\omega_j = c_j = 1$ and $I_{j,\mathbf{t}} = I'_{j,\mathbf{t}} = [0, \infty)$ for all $j \in \mathbb{N}_n$ and $\mathbf{t} \in \mathbb{R}^n$:

Example 7.2.17.

(i) Suppose that $r_j = 1$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \ldots, t_n) := \prod_{j=1}^{n} [\sin t_j + \sin(\sqrt{2} t_j)], \quad \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic but not $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic.

(ii) Suppose that $r_j = -1$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \dots, t_n) := \prod_{j=1}^{n} \left[(\sin t_j) \cdot \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\left(\frac{t_j}{3^n} \right) \right], \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n,$$

is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent but not $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic.

(iii) Suppose that $r_j = 1$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \dots, t_n) := \prod_{j=1}^{n} \sin\left(\frac{1}{2 + \sin t_j + \sin(\sqrt{2} t_j)} \right), \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n,$$

is $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-almost automorphic but not $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic.

The function spaces introduced in Definition 7.2.15 are translation invariant and closed under the pointwise multiplications with complex scalars. Furthermore, if the function $F(\cdot)$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic, then it can be easily proved that the function $\|F(\cdot)\|$ is $(\omega_j, |c_j|; |r_j|, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic. Suppose now that the function $F : I \to \mathbb{C} \setminus \{0\}$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic, $|F(\mathbf{t})| \geqslant m > 0$ for all $\mathbf{t} \in I$, and $|c_j| = 1$ for all $j \in \mathbb{N}_n$. Then the function $(1/F)(\cdot)$ is $(\omega_j, 1/c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic, which can be simply proved as follows. Let $j \in \mathbb{N}_n$, $\mathbf{t} \in I$ and $\varepsilon > 0$ be fixed; without loss of generality, we may assume that $j \in W_+$. Let $\tau \in I'_{j,\mathbf{t}}$ and $|G_{j,\mathbf{t}}(x+\tau) - r_j G_{j,\mathbf{t}}(x)| < \varepsilon$, $x \geqslant 0$. After multiplication with $c_j^{t_j/\omega_j}$, we get

$$\left| c_j^{-\frac{x+\tau}{\omega_j}} F(t_1, t_2, \dots, t_j + x, \dots, t_n) - r_j F\big(t_1, t_2, \dots, t_j + (x+\tau), \dots, t_n\big) \right| \leqslant \varepsilon, \quad x \geqslant 0. \tag{7.22}$$

Hence, for every $x \geqslant 0$, we have

$$\begin{aligned} &\left| \frac{c_j^{\frac{t_j+x+\tau}{\omega_j}}}{F(t_1, t_2, \dots, t_j + x + \tau, \dots, t_n)} - r_j^{-1} \frac{c_j^{\frac{t_j+x}{\omega_j}}}{F(t_1, t_2, \dots, t_j + x, \dots, t_n)} \right| \\ &\quad = \left| c_j^{\frac{\tau}{\omega_j}} F(t_1, t_2, \dots, t_j + x, \dots, t_n) - r_j^{-1} F(t_1, t_2, \dots, t_j + x + \tau, \dots, t_n) \right| \\ &\qquad \cdot \frac{1}{|F(t_1, t_2, \dots, t_j + x + \tau, \dots, t_n)| \cdot |F(t_1, t_2, \dots, t_j + x, \dots, t_n)|} \leqslant m^{-2} |r_j|^{-1} \varepsilon, \end{aligned}$$

where we have employed (7.22) in the last estimate. This simply implies the required result.

By a careful examination of the notion introduced in Definition 7.2.15 and the paragraph preceding it, we may deduce that the $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodicity of the function $F : I \to X$ implies the $(-\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodicity of the function $\check{F}(\cdot)$. We leave all details concerning the proof of this fact to the interested reader.

Using already proved results about multi-dimensional c-almost periodic functions and corresponding definitions, we may deduce the following proposition.

Proposition 7.2.18.

(i) *Suppose that, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, we have $I_{j,\mathbf{t}} + I'_{j,\mathbf{t}} = I_{j,\mathbf{t}}$ and the function $F : I \to X$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent. Then, for every $j \in \mathbb{N}_n$, we have $r_j = \pm 1$; if, additionally, $F(\mathbf{t}) \geqslant 0$ for all $\mathbf{t} \in I$, then, for every $j \in \mathbb{N}_n$, we have $r_j = 1$.*

(ii) *Suppose that $l \in \mathbb{N}$, and $F : I \to X$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic ($(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent). Then, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, we have $lI'_{j,\mathbf{t}} \subseteq I_{j,\mathbf{t}}$, $I_{j,\mathbf{t}} + lI'_{j,\mathbf{t}} \subseteq I_{j,\mathbf{t}}$ and $F(\cdot)$ is $(\omega_j, c_j; r_j^l, l\mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic ($(\omega_j, c_j; r_j^l, l\mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent), where $l\mathbb{I}'_j := \{lI'_{j,\mathbf{t}} : \mathbf{t} \in I\}$ for all $j \in \mathbb{N}_n$.*

(iii) *Suppose that (4.29) holds and $F : I \to X$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic ($(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent). Then the following holds:*
   (a) *If p is even, then $F(\cdot)$ is $(\omega_j, c_j; 1, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic ($(\omega_j, c_j; 1, q\mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent), where $q\mathbb{I}'_j := \{qI'_{j,\mathbf{t}} : \mathbf{t} \in I\}$ for all $j \in \mathbb{N}_n$.*
   (b) *If p is odd, then $F(\cdot)$ is $(\omega_j, c_j; -1, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic ($(\omega_j, c_j; -1, q\mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent).*

(iv) *Let $|c| = 1$ and $\arg(c)/\pi \notin \mathbb{Q}$. If, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, $lI'_{j,\mathbf{t}} = I'_{j,\mathbf{t}}$ for all $l \in \mathbb{N}$ and $F : I \to X$ is a bounded, $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic ($(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent) function, then the function $F(\cdot)$ is $(\omega_j, c_j; r'_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-almost periodic ($(\omega_j, c_j; r'_j, \mathbb{I}'_j)_{j\in\mathbb{N}_n}$-uniformly recurrent) for all $(r'_1, \ldots, r'_n) \in (S_1)^n$.*

Concerning the convolution invariance of spaces introduced in Definition 7.2.15, the following important fact should be emphasized: we have introduced the notion of $(I'_{j,\mathbf{t}}, r_j)$-almost periodicity, for example, by requiring that, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, the function $G_{j,\mathbf{t}}(\cdot)$ is $(I'_{j,\mathbf{t}}, r_j)$-almost periodic. Unfortunately, sometimes we need to assume that, for every $j \in \mathbb{N}_n$, the function $G_{j,\mathbf{t}}(\cdot)$ is $(I'_{j,\mathbf{t}}, r_j)$-almost periodic uniformly in the variable $\mathbf{t} \in I$, in a certain sense. For simplicity, let us assume that $I = \mathbb{R}^n$, which immediately implies that, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in \mathbb{R}^n$, we have $I_{j,\mathbf{t}} = [0, \infty)$. Assume, further, that for each $j \in \mathbb{N}_n$ there exists a set $A_j \subseteq [0, \infty)$ such that $A_j = I'_{j,\mathbf{t}}$ for every $\mathbf{t} \in \mathbb{R}^n$, as well as that for each $\varepsilon > 0$ there exists $l > 0$ such that for each $x_0 \in A_j$ we have the existence of a number $x \in B(x_0, l) \cap A_j$ such that

$$\|G_{j,\mathbf{t}}(x + \tau) - r_j G_{j,\mathbf{t}}(x)\| \leqslant \varepsilon, \quad x \geqslant 0,\ \mathbf{t} \in \mathbb{R}^n,$$

i. e.,

$$\left\|c_j^{-\frac{t_j+x+\tau}{\omega_j}} F(\mathbf{t} + (x + \tau)e_j) - r_j c_j^{-\frac{t_j+x}{\omega_j}} F(\mathbf{t} + xe_j)\right\| \leqslant \varepsilon, \quad x \geqslant 0,\ \mathbf{t} \in \mathbb{R}^n. \tag{7.23}$$

If $h \in L^1(\mathbb{R}^n)$ and $F(\cdot)$ is a bounded, continuous function, then the function $(h * F)(\cdot)$ is well defined on $\mathbb{R}^n$, bounded and continuous. If we assume, in addition to all above, that $|c_j| = 1$ for all $j \in \mathbb{N}_n$, then the estimate (7.23) will be invariant under the action of convolution $h * \cdot$, since

$$\begin{aligned}
&\Big\| c_j^{-\frac{t_j+x+\tau}{\omega_j}} (h * F)\big(\mathbf{t} + (x+\tau)e_j\big) - r_j c_j^{-\frac{t_j+x}{\omega_j}} (h * F)(\mathbf{t} + xe_j)\Big\| \\
&\leqslant \int_{\mathbb{R}^n} |h(y)| \Big\| c_j^{-\frac{t_j+x+\tau}{\omega_j}} F\big(\mathbf{t} + (x+\tau)e_j - y\big) - r_j c_j^{-\frac{t_j+x}{\omega_j}} F(\mathbf{t} + xe_j - y)\Big\| \, dy \\
&= \int_{\mathbb{R}^n} |h(y)| \Big\| c_j^{-\frac{t_j-y_j+x+\tau}{\omega_j}} F\big(\mathbf{t} + (x+\tau)e_j - y\big) - r_j c_j^{-\frac{t_j-y_j+x}{\omega_j}} F(\mathbf{t} + xe_j - y)\Big\| \, dy \\
&\leqslant \varepsilon, \quad x \geqslant 0,\ \mathbf{t} \in \mathbb{R}^n,
\end{aligned}$$

where we have employed (7.23) in the last estimate. We close this subsection with the observation that a similar result can be established for the $(\omega_j, c_j; r_j, \mathbb{I}_j')_{j\in\mathbb{N}_n}$-uniform recurrence and the $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-almost automorphy.

7.2.2 Further generalizations of the concepts (ω, c)-periodicity and $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodicity

Unless stated otherwise, in this subsection we will assume that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$, $\omega \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$. Define $S := \{i \in \mathbb{N}_n : \omega_i \neq 0\}$ and A to be the collection of all tuples $\mathbf{a} = (a_1, a_2, \ldots, a_{|S|}) \in \mathbb{R}^{|S|}$ such that $\sum_{i\in S} a_i = 1$. Let $\mathbf{a} \in$ A.

We introduce the following notion.

Definition 7.2.19. We say that a continuous function $F : I \to X$ is:

(i) $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1, resp. $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 2, if and only if there exists a sequence $(\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n}))$ in I' such that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I} \Big\| F(\mathbf{t} + \alpha_k) - c^{\sum_{i\in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t}) \Big\| = 0, \tag{7.24}$$

resp.

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I} \Big\| c^{-\sum_{i\in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t} + \alpha_k) - F(\mathbf{t}) \Big\| = 0; \tag{7.25}$$

(ii) $(I', \mathbf{a}, \omega, c)$-almost periodic of type 1, resp. $(I', \mathbf{a}, \omega, c)$-almost periodic of type 2, if and only if for each $\varepsilon > 0$ and $\mathbf{t}_0 \in I'$ there exist a finite number $l > 0$ and a point $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that

$$\sup_{\mathbf{t}\in I} \Big\| F(\mathbf{t} + \tau) - c^{\sum_{i\in S} \frac{a_i \tau_i}{\omega_i}} F(\mathbf{t}) \Big\| < \varepsilon,$$

resp.

$$\sup_{\mathbf{t}\in I}\left\| c^{-\sum_{i\in S}\frac{a_i\tau_i}{\omega_i}} F(\mathbf{t}+\tau) - F(\mathbf{t})\right\| < \varepsilon. \tag{7.26}$$

If $|c| = 1$, then the concept $(I', \mathbf{a}, \omega, c)$-uniform recurrence of type 1 and the concept $(I', \mathbf{a}, \omega, c)$-uniform recurrence of type 2 coincide, as easily approved by multiplying (7.24) with $c^{-\sum_{i\in S}\frac{a_i\alpha_{k,i}}{\omega_i}}$; this also holds for the concepts $(I', \mathbf{a}, \omega, c)$-almost periodicity of type 1 and $(I', \mathbf{a}, \omega, c)$-almost periodicity of type 2, but then we can say a little bit more. In actual fact, we can multiply the two sides of (7.26) with $c^{-\sum_{i\in S}\frac{a_i t_i}{\omega_i}}$ in order to see that $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-almost periodic of type 2 (1) if and only if the function $G_{\mathbf{a}}(\cdot)$, defined through (7.17), is I'-almost periodic; in the usually considered case $I = I' = \mathbb{R}^n$, this is equivalent to saying that the function $F(\cdot)$ is almost periodic.

The function spaces introduced in Definition 7.2.15 are translation invariant and closed under the pointwise multiplications with complex scalars; if $I = \mathbb{R}^n$ and $F(\cdot)$ is a bounded, continuous function which belongs to any of the above introduced function spaces, then for each $h \in L^1(\mathbb{R}^n)$ the function $(h * F)(\cdot)$ is also bounded and belongs to the same space. Furthermore, if the function $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1, resp. $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 2 [$(I', \mathbf{a}, \omega, c)$-almost periodic of type 1, resp. $(I', \mathbf{a}, \omega, c)$-almost periodic of type 2], then $F(\cdot)$ is $(I', \mathbf{a}, \omega, |c|)$-uniformly recurrent of type 1, resp. $(I', \mathbf{a}, \omega, |c|)$-uniformly recurrent of type 2 [$(I', \mathbf{a}, \omega, |c|)$-almost periodic of type 1, resp. $(I', \mathbf{a}, \omega, |c|)$-almost periodic of type 2]. Concerning the invariance of the function spaces under the operation of uniform convergence, we will only state that the assumptions $a_j\omega_j > 0$ for all $j \in S$, $|c| \leqslant 1$, $I' \subseteq [0, \infty)^n$ and the sequence (F_k) of $(I', \mathbf{a}, \omega, c)$-uniformly recurrent functions of type 1 [$(I', \mathbf{a}, \omega, c)$-almost periodic functions of type 1] uniformly converging to a function $F : I \to X$ imply that the function $F(\cdot)$ is likewise $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1 [$(I', \mathbf{a}, \omega, c)$-almost periodic of type 1].

For the sequel, we need the following lemma.

Lemma 7.2.20. *Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I = -I$, $I + I' = I$ and the function $F : I \to X$ is continuous. Then $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1 [$(I', \mathbf{a}, \omega, c)$-almost periodic of type 1] if and only if $\check{F}(\cdot)$ is $(I', \mathbf{a}, \omega, 1/c)$-uniformly recurrent of type 2 [$(I', \mathbf{a}, \omega, 1/c)$-almost periodic of type 2].*

Proof. We will present the main details of the proof provided that $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1. Then there exists a sequence $(\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n}))$ in I' such that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and (7.24) holds. Since we have assumed $I = -I$ and $I + I' = I$, the proof simply follows from the next computation ($k \in \mathbb{N}$):

$$\begin{aligned}
&\sup_{\mathbf{t}\in I}\left\| (1/c)^{-\sum_{i\in S}\frac{a_i\alpha_{k,i}}{\omega_i}} \check{F}(\mathbf{t}+\alpha_k) - \check{F}(\mathbf{t})\right\| \\
&\quad = \sup_{\mathbf{t}\in I}\left\| (1/c)^{-\sum_{i\in S}\frac{a_i\alpha_{k,i}}{\omega_i}} F(-\mathbf{t}-\alpha_k) - F(-\mathbf{t})\right\|
\end{aligned}$$

$$= \sup_{\mathbf{t}\in-(I+I')} \left\| c^{\sum_{i\in S} \frac{a_i\alpha_{k,i}}{\omega_i}} F(\mathbf{t}) - F(\mathbf{t}+\alpha_k)\right\|$$
$$= \sup_{\mathbf{t}\in-I} \left\| c^{\sum_{i\in S} \frac{a_i\alpha_{k,i}}{\omega_i}} F(\mathbf{t}) - F(\mathbf{t}+\alpha_k)\right\|$$
$$= \sup_{\mathbf{t}\in I} \left\| c^{\sum_{i\in S} \frac{a_i\alpha_{k,i}}{\omega_i}} F(\mathbf{t}) - F(\mathbf{t}+\alpha_k)\right\|. \qquad \square$$

Now we will state and prove the following result.

Theorem 7.2.21. *Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, I' is unbounded, $F : I \to X$ is continuous, $I + I' = I$, $\omega \in \mathbb{R}^n \setminus \{0\}$, $|c| > 1$, $S = \mathbb{N}_n$ and any component of a tuple $\mathbf{a} \in \mathrm{A}$ is positive. Suppose further that, for every $\mathbf{t} \in I$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geqslant 0$. Then the following assertions are equivalent:*

(i) *The function $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1.*
(ii) *The function $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 2.*
(iii) *There exists a sequence $(\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n}))$ in I' such that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and the function $G_{\mathbf{a}} : I \to X$, defined through (7.17), satisfies $G_{\mathbf{a}}(\mathbf{t} + \alpha_k) = G_{\mathbf{a}}(\mathbf{t})$ for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$.*
(iv) *There exists a sequence $(\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n}))$ in I' such that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and*

$$F(\mathbf{t} + \alpha_k) = c^{\sum_{i\in S} \frac{a_i\alpha_{k,i}}{\omega_i}} F(\mathbf{t}), \quad \mathbf{t} \in I,\ k \in \mathbb{N}. \tag{7.27}$$

(v) *There exists a point $\omega \in I' \setminus \{0\}$ such that*

$$F(\mathbf{t} + \omega) = c^{\sum_{i\in S} \frac{a_i\omega_i}{\omega_i}} F(\mathbf{t}), \quad \mathbf{t} \in I. \tag{7.28}$$

Proof. If $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1, then our assumptions $a_j > 0$ and $\alpha_{k,j}/\omega_i > 0$ ($k \in \mathbb{N}$, $j \in \mathbb{N}_n$) imply that

$$\left|c^{-\sum_{j=1}^n \frac{a_j\alpha_{k,j}}{\omega_j}}\right| \leqslant 1,$$

so that (7.24) implies (7.25) after multiplication with $c^{-\sum_{j=1}^n \frac{a_j\alpha_{k,j}}{\omega_j}}$; hence, (i) implies (ii). Suppose now that $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$-uniformly recurrent of type 2 and the sequence (α_k) in I' satisfies (7.25). Let $k \in \mathbb{N}$ be fixed. Then (7.25) implies the existence of a finite real number $M \geqslant 1$ such that

$$\sup_{\mathbf{t}\in I} \left\|G_{\mathbf{a}}(\mathbf{t} + \alpha_k) - G_{\mathbf{a}}(\mathbf{t})\right\| \leqslant M|c|^{-\sum_{j=1}^n a_j t_j/\omega_j}.$$

Since we have assumed that $a_j > 0$ and $t_j/\omega_j > 0$ for all $j \in \mathbb{N}_n$, the above estimate yields

$$\left\|G_{\mathbf{a}}(\mathbf{t} + \alpha_k) - G_{\mathbf{a}}(\mathbf{t})\right\| \leqslant M|c|^{-\min\{a_j:j\in\mathbb{N}_n\}[\max\{\omega_j:j\in\mathbb{N}_n\}]^{-1}[|t_1|+\cdots+|t_n|]},$$

for all $\mathbf{t} \in I$, which implies that $\lim_{|t|\to\infty} \|G_{\mathbf{a}}(\mathbf{t}+\alpha_k)-G_{\mathbf{a}}(\mathbf{t})\| = 0$. On the other hand, (7.25) implies

$$\lim_{m\to+\infty}\|G_{\mathbf{a}}(\mathbf{t}+\alpha_m+\alpha_k)-G_{\mathbf{a}}(\mathbf{t}+\alpha_m)\| = 0.$$

Therefore, the function $\mathbf{t} \mapsto G_{\mathbf{a}}(\mathbf{t}+\alpha_k) - G_{\mathbf{a}}(\mathbf{t})$, $\mathbf{t} \in I$ is I'-uniformly recurrent and tends to zero as $|\mathbf{t}| \to +\infty$. Since we have assumed that $I + I' = I$, by the foregoing it follows that $G_{\mathbf{a}}(\mathbf{t}+\alpha_k) = G_{\mathbf{a}}(\mathbf{t})$ for all $\mathbf{t} \in I$, which implies (iii). The implications (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (v) are trivial. To complete the proof, it suffices to show that (v) implies (iv). This follows by plugging $\alpha_k := k\omega$ for all $k \in \mathbb{N}$ since (7.28) implies inductively

$$F(\mathbf{t}+k\omega) = c^{\sum_{i\in S}\frac{a_i k\omega_i}{\omega_i}} F(\mathbf{t}), \quad \mathbf{t} \in I,\ k \in \mathbb{N}. \qquad \square$$

Remark 7.2.22.

(i) Since I' is unbounded, it is clear that the $(I', \mathbf{a}, \omega, c)$-almost periodicity of type 1 implies the $(I', \mathbf{a}, \omega, c)$-uniform recurrence of type 1 for $F(\cdot)$ as well as that the $(I', \mathbf{a}, \omega, c)$-almost periodicity of type 1 implies the $(I', \mathbf{a}, \omega, c)$-almost periodicity of type 2 for $F(\cdot)$, which further implies the $(I', \mathbf{a}, \omega, c)$-uniform recurrence of type 2 for $F(\cdot)$.

(ii) Let (α_k) be a sequence from (iv). Then it is clear that (iv) implies that for each number $k \in \mathbb{N}$ the function $F(\cdot)$ is $(I'_k, \mathbf{a}, \omega, c)$-almost periodic of type 1, where $I'_k := \{m\alpha_k : m \in \mathbb{N}\}$. Keeping Theorem 7.2.21 and this observation in mind, we have extended so the first part of Theorem 4.1.13(i), where we have assumed that $I = [0, \infty)$.

Concerning the statement of Theorem 4.1.13(i) with the interval $I = \mathbb{R}$, we would like to note that it can be extended to the higher dimensions as follows. Suppose that $I = I_0 \cup I_1$, where $\emptyset \neq I'_0 \subseteq I_0 \subseteq \mathbb{R}^n$, $I_0 + I'_0 = I_0$ and the function $F : I \to X$ is $(I_0, \mathbf{a}, \omega, c)$-uniformly recurrent of type 2, where $|c| > 1$, $S = \mathbb{N}_n$, any component of a tuple $\mathbf{a} \in \mathrm{A}$ is positive and, for every $\mathbf{t} \in I_0$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geqslant 0$. Then the restriction of the function $F(\cdot)$ to the interval I_0 is $(I'_0, \mathbf{a}, \omega, c)$-uniformly recurrent of type 2, as well, so that we can apply Theorem 7.2.21 in order to conclude that (7.27) holds for every $\mathbf{t} \in I_0$ and $k \in \mathbb{N}$. To show the validity of this condition for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$, we may assume additionally that:

(a) For every $\mathbf{t} \in I_1$, there exists $m_0 \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$ with $m \geqslant m_0$, we have $\mathbf{t} + \alpha_m \in I_0$.

Applying (7.27) twice, with $\mathbf{t} + \alpha_m$ and $\mathbf{t}$ the first time, and with $\mathbf{t} + \alpha_k + \alpha_m$ and $\mathbf{t} + \alpha_k$ the second time, we easily see that (7.27) holds for every $\mathbf{t} \in I$. Therefore, we have proved the following.

Theorem 7.2.23. *Suppose that $\emptyset \neq I'_0 \subseteq I_0 \subseteq \mathbb{R}^n$, I'_0 is unbounded, $I_0 + I'_0 = I_0$, $I = I_0 \cup I_1$, condition* (a) *holds and $F : I \to X$ is continuous. Suppose that $\omega \in \mathbb{R}^n \setminus \{0\}$, $|c| > 1$,*

$S = \mathbb{N}_n$ and any component of a tuple $\mathbf{a} \in \mathrm{A}$ is positive. Suppose further that, for every $\mathbf{t} \in I_0$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geqslant 0$. Then the following assertions are equivalent:

(i) *The function $F(\cdot)$ is $(I_0', \mathbf{a}, \omega, c)$-uniformly recurrent of type 1.*

(ii) *The function $F(\cdot)$ is $(I_0', \mathbf{a}, \omega, c)$-uniformly recurrent of type 2.*

(iii) *There exists a sequence $(\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n}))$ in I_0' such that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and the function $G_{\mathbf{a}} : I \to X$, defined through (7.17), satisfies $G_{\mathbf{a}}(\mathbf{t} + \alpha_k) = G_{\mathbf{a}}(\mathbf{t})$ for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$.*

(iv) *There exists a sequence $(\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n}))$ in I_0' such that $\lim_{k\to+\infty} |\alpha_k| = +\infty$ and (7.27) holds.*

(v) *There exists a point $\omega \in I' \smallsetminus \{0\}$ such that (7.28) holds.*

Suppose now that $|c| < 1$, $S := \mathbb{N}_n$, any component of a tuple $\mathbf{a} \in \mathrm{A}$ is positive and, for every $\mathbf{t} \in I_0$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geqslant 0$. Applying Lemma 7.2.20, we can simply extend the statement of Theorem 4.1.13(ii) to the higher dimensions, provided that condition (a) holds with $I_1 = -I_0$. Details can be left to the interested reader.

In the case that $a_j\omega_j > 0$ for all $j \in S = \mathbb{N}_n$, $|c| < 1$, $I = I' = [0, \infty)^n$, then it can be simply proved as in the one-dimensional case that the function $F : I \to X$ is $(I_0', \mathbf{a}, \omega, c)$-almost periodic of type 1 if and only if there exists a finite constant $M \geqslant 1$ such that

$$\|F(\mathbf{t})\| \leqslant M|c|^{\sum_{i\in S} a_i t_i/\omega_i}, \quad \mathbf{t} \in I;$$

the statement of Proposition 4.1.22 can be also extended to the higher dimensions provided that the function $F(\cdot)$ is bounded, $a_j\omega_j > 0$ for all $j \in S = \mathbb{N}_n$ and $|c| < 1$. Without any essential changes of the proof of Proposition 4.1.23, we may deduce the following (the study of vector-valued Levitan N-almost periodic functions on topological (semi-)groups and multi-dimensional vector-valued Levitan N-almost periodic functions will be carried out somewhere else; see also the interesting article [1064] by R. Yuan).

Proposition 7.2.24. *Suppose that $a_j\omega_j > 0$ for all $j \in S = \mathbb{N}_n$, $|c| < 1$ and $I = I' = [0, \infty)^n$. Then a continuous function $F : I \to X$ is $(I_0', \mathbf{a}, \omega, c)$-almost periodic of type 2 if and only if the function $\mathbf{t} \mapsto G(\mathbf{t}) \equiv c^{\sum_{i\in S} -a_i t_i/\omega_i} F(\mathbf{t})$, $\mathbf{t} \in I$ is bounded, continuous and satisfies the requirement that for each $\varepsilon > 0$, $\mathbf{t}_0 \in I$ and $N > 0$ there exist a finite number $l > 0$ and a point $\tau \in B(\mathbf{t}_0, l) \cap I$ such that*

$$\|G(\mathbf{t} + \tau) - G(\mathbf{t})\| \leqslant \varepsilon, \quad \mathbf{t} \in I_N.$$

In our previous research studies of the multi-dimensional almost periodicity, we have also analyzed the invariance of almost periodicity under the actions of the finite convolution products and the infinite convolution products. In the one-dimensional case, this theme is crucially important for giving the most intriguing applications in the qualitative analysis of almost periodic type solutions for various classes of the abstract Volterra integro-differential equations. In the multi-dimensional case, the results obtained so far are not so easily applicable and, because of that, we will skip all related details with regards to this question.

Finally, we will present some illustrative examples and applications of our results to the abstract Volterra integro-differential equations in Banach spaces. We start with the observation that all established results on the convolution invariance of introduced function spaces can be applied to the Gaussian semigroup and the Poisson semigroup; see [82, Example 3.7.9] and [265] for more details. Concerning the strongly continuous semigroups, we would like to note that our recent consideration of Example 1 can be used to justify the introduction of analyzed function spaces, as well.

1. Suppose that $a > 0$ and the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is c-almost periodic, where $g^{[1]}(\cdot) \equiv \int_0^{\cdot} g(s)\,ds$. Then the unique regular solution of the wave expression $u(x,t)$, given by the d'Alembert formula, can be extended to the whole real line in the time variable and this solution is c-almost periodic in $(x,t) \in \mathbb{R}^2$.

   1.1. We assume here that there exist numbers $\omega \in \mathbb{R} \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$ such that the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is (ω, c)-periodic. Then it is clear that the solution $u(x,t)$ can be extended to the whole real line in the time variable and now we will prove that, for every $\omega_2 \in \mathbb{R}$, we have

   $$u(x+\omega, t) = cu(x,t), \quad x,\ t \in \mathbb{R},$$

   i. e., the function $u(\cdot;\cdot)$ is $((\omega, 0), c)$-periodic. But, the last equality simply follows from the next calculation:

   $$\begin{aligned} u(x+\omega,t) &= \frac{1}{2}\big[f(x-at+\omega)+f(x+at+\omega)\big] \\ &\quad + \frac{1}{2a}\big[g^{[1]}(x+at+\omega)-g^{[1]}(x-at+\omega)\big] \\ &= \frac{1}{2}\big[cf(x-at)+cf(x+at)\big] \\ &\quad + \frac{c}{2a}\big[g^{[1]}(x+at)-g^{[1]}(x-at)\big] = cu(x,t), \quad x,\ t \in \mathbb{R}. \end{aligned}$$

   1.2. We assume here that there exist numbers $\omega \in \mathbb{R} \setminus \{0\}$, $k \in \mathbb{N}$ and $c \in \mathbb{C} \setminus \{0\}$ such that $c^{k-1} = 1$ and the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is (ω, c)-periodic. Set

   $$\omega_1 := \frac{1+k}{2}\omega \quad \text{and} \quad \omega_2 := \frac{k-1}{2a}\omega.$$

   Then $(\omega_1, \omega_2) \neq (0,0)$, $\omega_1 - a\omega_2 = \omega$, $\omega_1 + a\omega_2 = k\omega$, $c^k = c$, $f(x+\omega) = cf(x) = c^k f(x) = f(x+k\omega)$, $g^{[1]}(x+\omega) = cg^{[1]}(x) = c^k g^{[1]}(x) = g^{[1]}(x+k\omega)$ for all $x \in \mathbb{R}$, and we can simply show as above that

   $$u(x+\omega_1, t+\omega_2) = cu(x,t), \quad x,\ t \in \mathbb{R},$$

   i. e., the function $u(\cdot;\cdot)$ is $((\omega_1, \omega_2), c)$-periodic.

1.3. Let the assumptions of the previous point hold. Assume, further, that the function $x \mapsto (f_0(x), g_0^{[1]}(x))$, $x \in \mathbb{R}$ satisfies $\lim_{x\to\pm\infty} f_0(x) = \lim_{x\to\pm\infty} g_0^{[1]}(x) = 0$. Set $B := \{(x,t) \in \mathbb{R}^2 : x = \pm at\}$. If $\mathbb{D}$ is any subset of $\mathbb{R}^2$ satisfying

$$\lim_{|(x,t)|\to+\infty,(x,t)\in\mathbb{D}} \operatorname{dist}\big((x,t);B\big) = +\infty,$$

then the solution given by the d'Alembert formula, with the functions $f(\cdot)$ and $g(\cdot)$ replaced therein with the functions $(f+f_0)(\cdot)$ and $(g+g_0)(\cdot)$, is $\mathbb{D}$-asymptotically $((\omega_1,\omega_2),c)$-periodic.

2. Let $\omega \in \mathbb{R}^n \setminus \{0\}$ and $|c| = 1$. Equipped with the sup-norm, the space $B_{\omega,c}(\mathbb{R}^n : X)$ consisting of all X-valued, bounded and (ω,c)-periodic functions becomes a Banach space. Consider the following Hammerstein integral equation of convolution type on $\mathbb{R}^n$:

$$y(\mathbf{t}) = g(\mathbf{t}) + \int_{\mathbb{R}^n} k(\mathbf{t}-\mathbf{s})G\big(\mathbf{s},y(\mathbf{s})\big)\,d\mathbf{s}, \quad \mathbf{t}\in\mathbb{R}^n. \tag{7.29}$$

Suppose now that $g : \mathbb{R}^n \to X$ is bounded and (ω,c)-periodic, $k \in L^1(\mathbb{R}^n)$, $G : \mathbb{R}^n \times X \to X$ is continuous and satisfies the requirement that for each bounded subset of X we see that the set $\{G(\mathbf{t},x) : \mathbf{t}\in\mathbb{R}^n, x \in B\}$ is bounded as well as that $G(\mathbf{t}+\omega,x) = cG(\mathbf{t},x)$ for all $\mathbf{t}\in\mathbb{R}^n$ and $x \in X$. If there exists a finite real constant $L \geqslant 1$ such that

$$\big\|G(\mathbf{t},x) - G(\mathbf{t},y)\big\| \leqslant L\|x-y\|, \quad \mathbf{t}\in\mathbb{R}^n;\ x,\ y\in X,$$

and $L\int_{\mathbb{R}^n}|k(\mathbf{y})|\,d\mathbf{y} < 1$, then we can apply the Banach contraction principle and Proposition 7.2.13 in order to see that there exists a unique solution of the integral equation (7.29) which belongs to the space $B_{\omega,c}(\mathbb{R}^n : X)$.

7.3 Generalized c-almost periodic type functions in $\mathbb{R}^n$

In this section, we analyze multi-dimensional quasi-asymptotically c-almost periodic functions and their Stepanov generalizations as well as multi-dimensional Weyl c-almost periodic type functions [652]. We also analyze several important subclasses of the class of multi-dimensional quasi-asymptotically c-almost periodic functions and reconsider the notion of semi-c-periodicity in the multi-dimensional setting, working in the general framework of Lebesgue spaces with variable exponent. We provide certain applications of our results to the abstract Volterra integro-differential equations in Banach spaces.

The organization of this section can be briefly described as follows. In Subsection 7.3.1 we introduce and analyze $(S,\mathbb{D})$-asymptotically (ω,c)-periodic type functions and S-asymptotically $(\omega_j,c_j,\mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodic type functions; Subsection 7.3.2 investigates semi-$(c_j, \mathcal{B})_{j \in \mathbb{N}_n}$-periodic functions. Here, it is worth noting that the notion of $(S, \mathbb{D})$-asymptotical (ω, c)-periodicity seems to be new even in the one-dimensional setting. Various classes of multi-dimensional quasi-asymptotically c-almost periodic functions are examined in Subsection 7.3.3 following the approach in [588] and [658], while the Stepanov generalizations of multi-dimensional quasi-asymptotically c-almost periodic type functions are examined in Subsection 7.3.4 (the introduced classes seem to be new and were not considered elsewhere even in the case that the exponent $p(\cdot)$ has a constant value). The main aim of Subsection 7.3.5 is to continue our analysis of Weyl c-almost periodic type functions from [588] in the multi-dimensional setting; some applications of our results to the abstract Volterra integro-differential equations are presented at the end of this subsection. We also provide numerous illustrative examples.

7.3.1 $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic type functions and $(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic type functions

This section investigates the classes of $(S, \mathbb{D})$-asymptotically (ω, c)-periodic type functions and S-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic type functions. In the following two definitions, we extend the recently introduced notion of S_c-asymptotical periodicity (cf. M. T. Khalladi, M. Kostić, M. Pinto, A. Rahmani and D. Velinov [588, Definition 3.1], where the authors have considered the case in which $X = \{0\}$ and $I = \mathbb{D} = \mathbb{D}_1$ is $\mathbb{R}$ or $[0, \infty)$) and its subnotions: the S-asymptotical Bloch (ω, c)-periodicity, resp. S-asymptotical ω-anti-periodicity (see [259, Definition 3.1, Definition 3.2], where Y.-K. Chang and Y. Wei have considered the particular cases $|c| = 1$, resp. $c = -1$, $X = \{0\}$ and $I = \mathbb{R} = \mathbb{D} = \mathbb{D}_1$).

Definition 7.3.1. Let $\omega \in \mathbb{R}^n \smallsetminus \{0\}$, $c \in \mathbb{C} \smallsetminus \{0\}$, $\omega + I \subseteq I$, $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. A continuous function $F : I \times X \to Y$ is said to be $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic if and only if for each $B \in \mathcal{B}$ we have

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}} \|F(\mathbf{t} + \omega; x) - cF(\mathbf{t}; x)\|_Y = 0, \quad \text{uniformly in } x \in B.$$

Definition 7.3.2. Let $\omega_j \in \mathbb{R} \smallsetminus \{0\}$, $c_j \in \mathbb{C} \smallsetminus \{0\}$, $\omega_j e_j + I \subseteq I$, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded $(1 \leqslant j \leqslant n)$. A continuous function $F : I \times X \to Y$ is said to be $(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic if and only if for each $j \in \mathbb{N}_n$ we have

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}_j} \|F(\mathbf{t} + \omega_j e_j; x) - c_j F(\mathbf{t}; x)\|_Y = 0, \quad \text{uniformly in } x \in B.$$

Before going any further, we will present an illustrative example.

Example 7.3.3. Let $X := c_0$ be the Banach space of all numerical sequences tending to zero, equipped with the sup-norm. Suppose that $\omega_j = 2\pi$, $c_j \in \mathbb{C}$ and $|c_j| = 1$ for all

$j \in \mathbb{N}_n$. By the foregoing, we know that the function

$$F_1(t_1,\dots,t_n) := \prod_{j=1}^{n} c_j^{\frac{t_j}{2\pi}} \sin t_j, \quad \mathbf{t} = (t_1,\dots,t_n) \in [0,\infty)^n,$$

is $(2\pi, c_j)_{j\in\mathbb{N}_n}$-periodic. On the other hand, from [531, Example 3.1] and [647, Example 2.6], we know that the function

$$f(t) := \left(\frac{4k^2t^2}{(t^2+k^2)^2}\right)_{k\in\mathbb{N}}, \quad t \geqslant 0,$$

is S-asymptotically ω-periodic for any positive real number $\omega > 0$, as well as that its range is not relatively compact in X and $f(\cdot)$ is uniformly continuous; let us only note here that R. Xie and C. Zhang have constructed, in [1042, Example 17], an example of an S-asymptotically ω-periodic function which is not uniformly continuous. Set

$$F(t_1,\dots,t_n,t_{n+1}) := F_1(t_1,\dots,t_n)\cdot f(t_{n+1}), \quad (t_1,\dots,t_n,t_{n+1}) \in [0,\infty)^{n+1}.$$

Then the function $F(\cdot)$ is S-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j\in\mathbb{N}_{n+1}}$-periodic, where $c_{n+1} = 1$, $\omega_{n+1} > 0$ being arbitrary, $\mathbb{D}_j = [0,\infty)^{n+1}$ for $1 \leqslant j \leqslant n$ and $\mathbb{D}_{n+1} = K \times [0,\infty)$ ($\emptyset \neq K \subseteq [0,\infty)^n$ is a compact set), as easily approved. See also [647, Example 2.16, Example 2.17, Example 2.18].

Immediately from the corresponding definitions, we have the following result.

Proposition 7.3.4.

(i) *Let $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega + I \subseteq I$, $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. If $\omega + \mathbb{D} \subseteq \mathbb{D}$ and the function $F : I \times X \to Y$ is $(\mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic, then the function $F(\cdot;\cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic.*

(ii) *Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded ($1 \leqslant j \leqslant n$). If $\omega e_j + \mathbb{D} \subseteq \mathbb{D}$ and the function $F : I \times X \to Y$ is $(\mathbb{D}, \mathcal{B})$-asymptotically $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic, then the function $F(\cdot;\cdot)$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodic with $\mathbb{D}_j \equiv \mathbb{D}$ for all $j \in \mathbb{N}_n$.*

We will provide the proof of the first part of the following simple result for the sake of completeness.

Proposition 7.3.5.

(i) *Let $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega + I \subseteq I$, $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. If for each $B \in \mathcal{B}$ there exists $\varepsilon_B > 0$ such that the sequence $(F_k(\cdot;\cdot))$ of $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic functions converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$, then $F(\cdot;\cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic.*

(ii) *Let* $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ *and the set* $\mathbb{D}_j$ *be unbounded* ($1 \leqslant j \leqslant n$). *If for each* $B \in \mathcal{B}$ *there exists* $\varepsilon_B > 0$ *such that the sequence* $(F_k(\cdot;\cdot))$ *of* $(S, \mathcal{B})$*-asymptotically* $(\omega_j, c_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$*-periodic functions converges uniformly to a function* $F(\cdot;\cdot)$ *on the set* $B^\circ \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$, *then the function* $F(\cdot;\cdot)$ *is* $(S, \mathcal{B})$*-asymptotically* $(\omega_j, c_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$*-periodic.*

Proof. The validity of (i) can be deduced as follows. By the foregoing arguments, it follows that the function $F(\cdot;\cdot)$ is continuous. Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be fixed. Then there exists $k_0 \in \mathbb{N}$ such that $\|F_{k_0}(\mathbf{t};x) - F(\mathbf{t};x)\|_Y \leqslant \varepsilon/3(1 + |c|)$ for all $(\mathbf{t}, x) \in I \times B$. Furthermore, there exists $M > 0$ such that the assumptions $|\mathbf{t}| > M$, $\mathbf{t} \in \mathbb{D}$ and $x \in B$ imply $\|F_{k_0}(\mathbf{t} + \omega; x) - cF_{k_0}(\mathbf{t};x)\|_Y < \varepsilon/3$. Then the final conclusion follows from the well-known decomposition and estimates

$$\begin{aligned}
&\|F(\mathbf{t} + \omega; x) - cF(\mathbf{t};x)\|_Y \\
&\quad \leqslant \|F(\mathbf{t} + \omega; x) - F_{k_0}(\mathbf{t};x)\|_Y + \|F_{k_0}(\mathbf{t} + \omega; x) - cF_{k_0}(\mathbf{t};x)\|_Y \\
&\qquad + |c| \cdot \|F_{k_0}(\mathbf{t} + \omega; x) - F(\mathbf{t};x)\|_Y \leqslant 3 \cdot (\varepsilon/3) = \varepsilon.
\end{aligned}$$

□

The convolution invariance of function spaces introduced in Definition 7.3.1 and Definition 7.3.2 can be shown under very mild assumptions.

Theorem 7.3.6. *Suppose that* $h \in L^1(\mathbb{R}^n)$ *and* $F : \mathbb{R}^n \times X \to Y$ *is a continuous function satisfying that for each* $B \in \mathcal{B}$ *there exists a finite real number* $\varepsilon_B > 0$ *such that* $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^\cdot} \|F(\mathbf{t}, x)\|_Y < +\infty$, *where* $B^\cdot \equiv B^\circ \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$.

(i) *Suppose that* $\mathbb{D} = \mathbb{R}^n$. *Then the function*

$$(h * F)(\mathbf{t};x) := \int_{\mathbb{R}^n} h(\sigma)F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X, \tag{7.30}$$

is well defined and for each $B \in \mathcal{B}$ *we have* $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^\cdot} \|(h*F)(\mathbf{t};x)\|_Y < +\infty$; *furthermore, if* $F(\cdot;\cdot)$ *is* $(S, \mathbb{R}^n, \mathcal{B})$*-asymptotically* (ω, c)*-periodic, then the function* $(h*F)(\cdot;\cdot)$ *is* $(S, \mathbb{R}^n, \mathcal{B})$*-asymptotically* (ω, c)*-periodic.*

(ii) *Suppose that* $\mathbb{D}_j = \mathbb{R}^n$ *for all* $j \in \mathbb{N}_n$. *Then the function* $(h * F)(\cdot;\cdot)$, *given by* (7.30), *is well defined and for each* $B \in \mathcal{B}$ *we have* $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^\cdot} \|(h * F)(\mathbf{t};x)\|_Y < +\infty$; *moreover, if the function* $F(\cdot;\cdot)$ *is* $(S, \mathcal{B})$*-asymptotically* $(\omega_j, c_j, \mathbb{R}^n)_{j\in\mathbb{N}_n}$*-periodic, then the function* $(h * F)(\cdot;\cdot)$ *is likewise* $(S, \mathcal{B})$*-asymptotically* $(\omega_j, c_j, \mathbb{R}^n)_{j\in\mathbb{N}_n}$*-periodic.*

Proof. We will prove only (i). It is clear that the function $(h * F)(\cdot;\cdot)$ is well defined as well as that $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^\cdot} \|(h * F)(\mathbf{t};x)\|_Y < +\infty$ for all $B \in \mathcal{B}$. Its continuity at the fixed point $(\mathbf{t}_0; x_0) \in \mathbb{R}^n \times X$ follows from the existence of a set $B \in \mathcal{B}$ such that $x_0 \in B$, the assumption $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^\cdot} \|F(\mathbf{t};x)\|_Y < +\infty$ and the dominated convergence theorem. Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be fixed. Then there exists a sufficiently large real number $M > 0$ such that $\|F(\mathbf{t} + \omega; x) - cF(\mathbf{t};x)\|_Y < \varepsilon/2$, provided $|\mathbf{t}| > M_1$ and $x \in B$. Therefore, there exists a finite constant $c_B \geqslant 1$ such that

$$\begin{aligned}
&\left\|(h * F)(\mathbf{t} + \omega; x) - c(h * F)(\mathbf{t}; x)\right\|_Y \\
&\quad \leqslant \int_{\mathbb{R}^n} \left|h(\sigma)\right| \cdot \left\|F(\mathbf{t} + \omega - \sigma; x) - cF(\mathbf{t} - \sigma; x)\right\|_Y \, d\sigma \\
&\quad = \int_{|\sigma| \leqslant M_1} \left|h(\mathbf{t} - \sigma)\right| \cdot \left\|F(\sigma + \omega; x) - cF(\sigma; x)\right\|_Y \, d\sigma \\
&\qquad + \int_{|\sigma| \geqslant M_1} \left|h(\mathbf{t} - \sigma)\right| \cdot \left\|F(\sigma + \omega; x) - cF(\sigma; x)\right\|_Y \, d\sigma \\
&\quad \leqslant \varepsilon/2 + \int_{|\sigma| \geqslant M_1} \left|h(\mathbf{t} - \sigma)\right| \cdot \left\|F(\sigma + \omega; x) - cF(\sigma; x)\right\|_Y \, d\sigma \\
&\quad \leqslant \varepsilon/2 + c_B \int_{|\sigma| \geqslant M_1} \left|h(\mathbf{t} - \sigma)\right| \, d\sigma.
\end{aligned}$$

On the other hand, there exists a finite real number $M_2 > 0$ such that $\int_{|\sigma| \geqslant M_2} |h(\sigma)| \, d\sigma < \varepsilon/2c_B$. If $|\mathbf{t}| > M_1 + M_2$, then for each $\sigma \in \mathbb{R}^n$ with $|\sigma| \leqslant M_1$ we have $|\mathbf{t} - \sigma| \geqslant M_2$. This simply implies the required conclusion. □

The following result will allow us to stretch the connections between the notion introduced in Definition 7.3.1 and Definition 7.3.2.

Proposition 7.3.7. *Let $\omega_j \in \mathbb{R} \smallsetminus \{0\}$, $c_j \in \mathbb{C} \smallsetminus \{0\}$, $\omega_j e_j + I \subseteq I$, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded ($1 \leqslant j \leqslant n$). If $F : I \times X \to Y$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic and the set $\mathbb{D}$ consisting of all tuples $\mathbf{t} \in \mathbb{D}_n$ such that $\mathbf{t} + \sum_{i=j+1}^n \omega_i e_i$ for all $j \in \mathbb{N}_{n-1}$ is unbounded in $\mathbb{R}^n$, then the function $F(\cdot; \cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic, with $\omega := \sum_{j=1}^n \omega_j e_j$ and $c := \prod_{j=1}^n c_j$.*

Proof. The proof simply follows from the corresponding definitions and the next estimates:

$$\begin{aligned}
&\left\|F(\mathbf{t} + \omega; x) - cF(\mathbf{t}; x)\right\| \\
&\quad = \left\|F(t_1 + \omega_1, \ldots, t_n + \omega_n; x) - c_1 \cdots c_n F(t_1, \ldots, t_n; x)\right\| \\
&\quad \leqslant \left\|F(t_1 + \omega_1, t_2 + \omega_2, \ldots, t_n + \omega_n; x) - c_1 F(t_1, t_2 + \omega_2, \ldots, t_n + \omega_n; x)\right\| \\
&\qquad + |c_1| \cdot \left\|F(t_1, t_2 + \omega_2, \ldots, t_n + \omega_n; x) - c_2 \cdots c_n F(t_1, \ldots, t_n; x)\right\| \\
&\quad \leqslant \left\|F(t_1 + \omega_1, t_2 + \omega_2, \ldots, t_n + \omega_n; x) - c_1 F(t_1, t_2 + \omega_2, \ldots, t_n + \omega_n; x)\right\| \\
&\qquad + |c_1| \cdot \left[\left\|F(t_1, t_2 + \omega_2, \ldots, t_n + \omega_n; x) - c_2 F(t_1, t_2, \ldots, t_n + \omega_n; x)\right\|\right. \\
&\qquad + \left.|c_2| \cdot \left\|F(t_1, t_2, \ldots, t_n + \omega_n; x) - c_3 \cdots c_n F(t_1, t_2, \ldots, t_n; x)\right\|\right] \\
&\quad \leqslant \cdots .
\end{aligned}$$

□

The proof of following proposition is simple and therefore is omitted.

Proposition 7.3.8. *Let ω, $a \in \mathbb{R}^n \smallsetminus \{0\}$, $c \in \mathbb{C} \smallsetminus \{0\}$, $\alpha \in \mathbb{C}$, $\omega + I \subseteq I$ and $a + I \subseteq I$. Suppose that the functions $F : I \times X \to Y$ and $G : I \times X \to Y$ are $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic ($(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic). Then we have the following:*

(i) *The function $\check{F}(\cdot;\cdot)$ is $(S, -\mathbb{D}, \mathcal{B})$-asymptotically $(-\omega, c)$-periodic ($(S, \mathcal{B})$-asymptotically $(-\omega_j, c_j, -\mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic), where $\check{F}(\mathbf{t}; x) := F(-\mathbf{t}; x)$, $\mathbf{t} \in -I$, $x \in X$.*

(ii) *The functions $\|F(\cdot;\cdot)\|$, $[F + G](\cdot;\cdot)$ and $\alpha F(\cdot;\cdot)$ are $(S, \mathbb{D}, \mathcal{B})$-asymptotically $(\omega, |c|)$-periodic ($(S, \mathcal{B})$-asymptotically $(\omega_j, |c_j|, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic).*

(iii) *If $a + \mathbb{D} \subseteq \mathbb{D}$ ($a + \mathbb{D}_j \subseteq \mathbb{D}_j$ for all $j \in \mathbb{N}_n$) and $y \in X$, then the function $F_{a,y} : I \times X \to Y$ defined by $F_{a,y}(\mathbf{t}; x) := F(\mathbf{t} + a; x + y)$, $\mathbf{t} \in I$, $x \in X$ is $(S, \mathbb{D}, \mathcal{B}_y)$-asymptotically (ω, c)-periodic ($(S, \mathcal{B}_y)$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic), where $\mathcal{B}_y := \{-y + B : B \in \mathcal{B}\}$.*

(iv) *If $\omega \in \mathbb{R}^n \smallsetminus \{0\}$, $c_i \in \mathbb{C} \smallsetminus \{0\}$ for $i = 1, 2$, $\omega + I \subseteq I$, the function $G : I \times X \to \mathbb{C}$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c_1)-periodic and the function $H : I \times X \to Y$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c_2)-periodic, then the function $F(\cdot) := G(\cdot)H(\cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically $(\omega, c_1 c_2)$-periodic, provided that for each set $B \in \mathcal{B}$ we have* $\sup_{\mathbf{t} \in I; x \in B}[|G(\mathbf{t}; x)| + \|F(\mathbf{t}; x)\|_Y] < \infty$.

(v) *Let $\omega_j \in \mathbb{R} \smallsetminus \{0\}$, $c_{j,i} \in \mathbb{C} \smallsetminus \{0\}$ and $\omega_j e_j + I \subseteq I$ ($1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant 2$). Suppose that the function $G : I \times X \to \mathbb{C}$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, c_{j,1}, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic and the function $H : I \to X$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, c_{j,2}, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic. Set $c_j := c_{j,1} c_{j,2}$, $1 \leqslant j \leqslant n$. Then the function $F(\cdot) := G(\cdot)H(\cdot)$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic, provided that for each set $B \in \mathcal{B}$ we have* $\sup_{\mathbf{t} \in I; x \in B}[|G(\mathbf{t}; x)| + \|F(\mathbf{t}; x)\|_Y] < \infty$.

Using the already proved characterizations of the classes of (ω, c)-periodic functions and $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic functions, we can introduce various spaces of pseudo-like $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic type functions and pseudo-like $(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic type functions following the method proposed in [48, Definition 2.4, Definition 2.5]; we will skip all related details for simplicity. The interested reader may also try to formulate some extensions of [588, Proposition 3.1, Corollary 3.1-Corollary 3.2] in the multi-dimensional setting.

7.3.2 Semi-$(c_j, \mathcal{B})_{j \in \mathbb{N}_n}$-periodic functions

In this subsection, we will exhibit the main results about the class of multi-dimensional semi-$(c_j, \mathcal{B})_{j \in \mathbb{N}_n}$-periodic functions. For the sake of brevity, we will always assume here that the region I has the form $I = I_1 \times I_2 \times \cdot \ldots \times I_n$, where each set I_j is equal to $\mathbb{R}$, $(-\infty, a_j]$ or $[a_j, \infty)$ for some real number $a_j \in \mathbb{N}$ ($1 \leqslant j \leqslant n$).

We will use the following definition.

Definition 7.3.9. Suppose that $F : I \times X \to Y$ is a continuous function and $c_j \in \mathbb{C} \smallsetminus \{0\}$ ($1 \leqslant j \leqslant n$). Then we say that $F(\cdot;\cdot)$ is semi-$(c_j, \mathcal{B})_{j \in \mathbb{N}_n}$-periodic if and only if, for every

$\varepsilon > 0$ and $B \in \mathcal{B}$, there exist real numbers $\omega_j \in \mathbb{R} \smallsetminus \{0\}$ such that $\omega_j e_j + I \subseteq I$ $(1 \leqslant j \leqslant n)$ and

$$\left\|F(\mathbf{t} + m\omega_j e_j; x) - c_j^m F(\mathbf{t}; x)\right\| \leqslant \varepsilon, \quad m \in \mathbb{N},\ j \in \mathbb{N}_n,\ \mathbf{t} \in \mathbb{R}^n,\ x \in B. \tag{7.31}$$

The function $F(\cdot;\cdot)$ is said to be semi-$\mathcal{B}$-periodic if and only if $F(\cdot;\cdot)$ is semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic with $c_j = 1$ for all $j \in \mathbb{N}_n$.

Suppose that $j \in \mathbb{N}_n$, $x \in X$ and $|c_j| \neq 1$. Fix the variables $t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n$. Then there exist three possibilities:

1. $I_j = \mathbb{R}$. Then, due to (7.31), the function $f : \mathbb{R} \to Y$ given by $f(t) := F(t_1, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_n)$, $t \in \mathbb{R}$ is semi-c_j-periodic of type 1_+ in the sense of [587, Definition 3(i)] and therefore $f(\cdot)$ is c_j-periodic due to [587, Theorem 1]. Hence, the function $F(\cdot; x)$ is c_j-periodic in the variable t_j.
2. $I_j = [a_j, +\infty)$ for some real number $a_j \in \mathbb{R}$. Then the function $F(\cdot; x)$ is c_j-periodic in the variable t_j, which follows from the same argumentation applied to the function $f(t) := F(t_1, \ldots, t_{j-1}, t - a_j, t_{j+1}, \ldots, t_n)$, $t \geqslant 0$.
3. $I_j = (-\infty, a_j]$ for some real number $a_j \in \mathbb{R}$. Then the function $F(\cdot; x)$ is c_j-periodic in the variable t_j, which follows from the same argumentation applied to the function $f(t) := F(t_1, \ldots, t_{j-1}, -t - |a_j|, t_{j+1}, \ldots, t_n)$, $t \geqslant 0$.

In the remainder of this subsection, we will assume that $|c_j| = 1$ for all $j \in \mathbb{N}_n$. Then any semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic function $F : I \times X \to Y$ is bounded on any subset B of the collection $\mathcal{B}$, as easily approved; even in the one-dimensional setting, this function need not be periodic in the usual sense (see [587, p. 2]). Furthermore, if for each integer $k \in \mathbb{N}$ the function $F_k : I \times X \to Y$ is semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic and for each $B \in \mathcal{B}$ there exists a finite real number $\varepsilon_B > 0$ such that $\lim_{k\to+\infty} F_k(\mathbf{t}; x) = F(\mathbf{t}; x)$ for all $\mathbf{t} \in I$, uniformly in $x \in B^{\cdot} \equiv B^{\circ} \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$, then the function $F(\cdot;\cdot)$ is likewise semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic.

Let $B \in \mathcal{B}$ be fixed. Let us recall that the Banach space $l_\infty(B : Y)$ consists of all bounded functions $f : B \to Y$ and is equipped with the sup-norm. Suppose that the function $F : I \times X \to Y$ is semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic. We define the function $F_B : I \to l_\infty(B : Y)$ as before; then the mapping $F_B(\cdot)$ is well defined and semi-$(c_j)_{j\in\mathbb{N}_n}$-periodic. Using now an insignificant modification of the proofs of [69, Lemma 1, Theorem 1], we may conclude that for each set $B \in \mathcal{B}$ there exists a sequence of $(c_j)_{j\in\mathbb{N}_n}$-periodic functions $(F_k : I \times X \to Y)_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} F_k(\mathbf{t}; x) = F(\mathbf{t}; x)$ for all $\mathbf{t} \in I$, uniformly in $x \in B$. The converse statement is also true; hence, we have the following important result.

Theorem 7.3.10. *Suppose that $F : I \times X \to Y$ is continuous. Then the function $F(\cdot;\cdot)$ is semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic if and only if for each set $B \in \mathcal{B}$ there exists a sequence of $(c_j)_{j\in\mathbb{N}_n}$-periodic functions $(F_k : I \to l_\infty(B : Y))_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} F_k(\mathbf{t}) = F_B(\mathbf{t})$ uniformly in $\mathbf{t} \in I$.*

Now we would like to present the following illustrative applications of Theorem 7.3.10.

Example 7.3.11. Suppose that $q_1, \ldots, q_n$ are odd natural numbers. Define $F : \mathbb{R}^n \to \mathbb{C}$ by

$$F(t_1, t_2, \ldots, t_n) := \sum_{l=(l_1,l_2,\ldots,l_n)\in\mathbb{N}^n} \frac{e^{it_1/(2l_1q_1+1)}e^{it_2/(2l_2q_2+1)}\cdots e^{it_n/(2l_nq_n+1)}}{l_1!l_2!\cdots l_n!},$$

for any $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. Then $F(\cdot)$ is a semi-$(-1, -1, \ldots, -1)$-periodic function since it is a uniform limit of $(-1, -1, \ldots, -1)$-periodic functions

$$F_k(\mathbf{t}) := \sum_{|l|\leqslant k} \frac{e^{it_1/(2l_1q_1+1)}e^{it_2/(2l_2q_2+1)}\cdots e^{it_n/(2l_nq_n+1)}}{l_1!l_2!\cdots l_n!}, \quad \mathbf{t} \in \mathbb{R}^n,\ k \in \mathbb{N}.$$

Example 7.3.12. It is worth noticing that, in many concrete situations, the solutions of PDEs on rectangular domains, constructed by the well known method of separation of variables, are restrictions of semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic functions defined on the whole Euclidean space. For example, a unique solution of the wave equation $u_{tt} = u_{xx}$ in the rectangle $0 < x < 1$ and $t > 0$, equipped with the initial conditions $u(0, t) = u(1, t) = 0$, $u(x, 0) = x(1 - x)$ and $u_t(x, 0) = 0$ is given by

$$u(x, t) = \frac{8}{\pi^3} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi t) \cdot \sin((2k-1)\pi x)}{(2k-1)^3}.$$

It is clear that this solution can be extended to the whole plane by the same formula, which defines a semi-$(-1, -1)$-periodic function there.

We continue with the observation that the statements of Proposition 2.5, Proposition 2.7, Proposition 2.8, Proposition 2.9, Proposition 2.12, Theorem 2.13 and Proposition 2.17 of [586] admit very simple reformulations in the multi-dimensional setting. For example, if $F : I \to \mathbb{R}$ is semi-$(c_j)_{j\in\mathbb{N}_n}$-periodic, then $c_j \in \{-1, 1\}$ for all $j \in \mathbb{N}_n$; furthermore, if $F(\mathbf{t}) \geqslant 0$ for all $\mathbf{t} \in I$, then $c_j = 1$ for all $j \in \mathbb{N}_n$.

Any semi-$(c_j)_{j\in\mathbb{N}_n}$-periodic function $F : I \to Y$ can be extended uniquely to a semi-$(c_j)_{j\in\mathbb{N}_n}$-periodic function $\tilde{F} : \mathbb{R}^n \to Y$ and therefore it has a mean value as an almost periodic function; see e. g., the proof of [265, Theorem 2.36]. Furthermore, any semi-$(c_j)_{j\in\mathbb{N}_n}$-periodic function $F : I \to Y$ is semi-periodic. In the one-dimensional case, [69, Lemma 2] tells us that there exists a positive real number $\theta > 0$ such that $\sigma(F) \subseteq \theta \cdot \mathbb{Q}$, which enables one to construct a great deal of almost periodic functions which are not semi-periodic. If we put ourselves in a similar situation in the multi-dimensional setting, then we have the following.

Proposition 7.3.13. *Suppose that the function $F : I \to Y$ is semi-$(c_j)_{j\in\mathbb{N}_n}$-periodic, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(F)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \sigma(F)$. Then there exist non-zero real numbers $\omega_j \in \mathbb{R} \smallsetminus \{0\}$ $(1 \leqslant j \leqslant n)$ such that $\lambda_j\omega_j \in 2\pi\mathbb{Z}$ and $\mu_j\omega_j \in 2\pi\mathbb{Z}$ for all $j \in \mathbb{N}_n$.*

Proof. By the foregoing, we may assume that $I = \mathbb{R}^n$, $\lambda = \mu$ and $c_j = 1$ for all $j \in \mathbb{N}_n$. We will follow the proof of [82, Corollary 4.5.4(d)] with appropriate modifications. First of all, note that

$$\lim_{k\to+\infty} k^{-1} \sum_{j=0}^{k-1} z^j = 0,$$

if $|z| = 1$ and $z \neq 1$, while

$$\lim_{k\to+\infty} k^{-1} \sum_{j=0}^{k-1} z^j = 1,$$

if $z = 1$. Our assumption is that

$$\lim_{T\to+\infty} \frac{1}{T^n} \int\limits_{[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} \neq 0 \quad \text{and} \quad \lim_{T\to+\infty} \frac{1}{T^n} \int\limits_{[0,T]^n} e^{-i\langle\mu,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} \neq 0.$$

By Theorem 7.3.10, the proof of [69, Lemma 2] and continuity, we may assume without loss of generality that $F(\cdot)$ is $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic for some non-zero real numbers $\omega_j \in \mathbb{R} \setminus \{0\}$ $(1 \leqslant j \leqslant n)$. We have

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{T^n} \int\limits_{[0,T]^n} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} \\
&= \lim_{T\to+\infty} \frac{1}{T^n} \sum_{j_1=0}^{\lfloor T/|\omega_1|\rfloor} \cdots \sum_{j_n=0}^{\lfloor T/|\omega_n|\rfloor} \int\limits_{\prod_{k=1}^n [j_k|\omega_k|,(j_k+1)|\omega_k|]} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} \\
&= \lim_{T\to+\infty} \frac{1}{T^n} \sum_{j_1=0}^{\lfloor T/|\omega_1|\rfloor} \cdots \sum_{j_n=0}^{\lfloor T/|\omega_n|\rfloor} \int\limits_{[0,|\omega_1|]\times\cdots\times[0,|\omega_n|]} e^{i[\lambda_1 j_1|\omega_1|+\cdots+\lambda_n j_n|\omega_n|]} e^{-i\langle\lambda,\mathbf{t}\rangle} F(\mathbf{t})\, d\mathbf{t} \\
&= \lim_{T\to+\infty} \left\{ \left[\frac{1}{T} \sum_{j_1=0}^{\lfloor T/|\omega_1|\rfloor} \left(e^{i\lambda_1|\omega_1|}\right)^{j_1} \right] \cdots \left[\frac{1}{T} \sum_{j_n=0}^{\lfloor T/|\omega_n|\rfloor} \left(e^{i\lambda_n|\omega_n|}\right)^{j_n} \right] \right\} \\
&= \lim_{T\to+\infty} \left[\frac{1}{T} \sum_{j_1=0}^{\lfloor T/|\omega_1|\rfloor} \left(e^{i\lambda_1|\omega_1|}\right)^{j_1} \right] \cdots \lim_{T\to+\infty} \left[\frac{1}{T} \sum_{j_n=0}^{\lfloor T/|\omega_n|\rfloor} \left(e^{i\lambda_n|\omega_n|}\right)^{j_n} \right].
\end{aligned}$$

The final conclusion follows by observing that the product of the above limits, which exist in $\mathbb{C}$, is not equal to zero if and only if $\exp(i\lambda_j|\omega_j|) = 1$ for all $j \in \mathbb{N}_n$, and that the same calculation can be given for the tuple μ. □

The Stepanov classes of semi-$(c_j, \mathcal{B})_{j\in\mathbb{N}_n}$-periodic functions can be also analyzed.

7.3.3 Multi-dimensional quasi-asymptotically c-almost periodic type functions

We will first introduce the notion of $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', c)$-almost periodicity and recall the notion of $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', c)$-uniform recurrence here (it can be easily shown that the notion of quasi-asymptotical uniform recurrence introduced before is equivalent with the corresponding notion introduced in the second part of the following definition; concerning the first part of this definition, it extends the notion of one-dimensional quasi-asymptotical c-almost periodicity).

Definition 7.3.14. Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, the sets $\mathbb{D}$ and I' are unbounded, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that there exists a finite real number $M(\varepsilon, \tau) > 0$ such that

$$\|F(\mathbf{t} + \tau; x) - cF(\mathbf{t}; x)\|_Y \leqslant \varepsilon, \quad \text{provided } \mathbf{t},\ \mathbf{t} + \tau \in \mathbb{D}_{M(\varepsilon,\tau)},\ x \in B. \tag{7.32}$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_k) in I' and a sequence (M_k) in $(0, \infty)$ such that $\lim_{k\to+\infty} |\tau_k| = \lim_{k\to+\infty} M_k = +\infty$ and

$$\lim_{k\to+\infty} \sup_{\mathbf{t},\mathbf{t}+\tau_k \in \mathbb{D}_{M_k}; x\in B} \|F(\mathbf{t} + \tau_k; x) - cF(\mathbf{t}; x)\|_Y = 0. \tag{7.33}$$

If $I' = I$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, c)$-almost periodic ($\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, c)$-uniformly recurrent); furthermore, if $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically (I', c)-almost periodic ($\mathbb{D}$-quasi-asymptotically (I', c)-uniformly recurrent). If $I' = I$ and $X \in \mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically c-almost periodic ($\mathbb{D}$-quasi-asymptotically c-uniformly recurrent). We remove the prefix "$\mathbb{D}$-" in the case that $\mathbb{D} = I$, remove the prefix "$(\mathcal{B},)$" in the case that $X \in \mathcal{B}$ and remove the prefix "c-" if $c = 1$.

We have already analyzed the notion of $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I', c)$-almost periodicity of type 1, which is a special case of $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', c)$-almost periodicity. The notion of $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', c)$-uniform recurrence, which generalizes the notion of $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', c)$-almost periodicity has been also introduced under the slightly different name of $\mathbb{D}$-asymptotical $(\mathcal{B}, I', c)$-uniform recurrence of type 1. It is evident that the notion of $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I', c)$-almost periodicity of type 1 (see Definition 6.1.33) is a special case of the notion of $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', c)$-almost periodicity introduced in Definition 7.3.14(i). The following generalization of [658, Proposition 2] can be deduced straightforwardly (we can simply formulate an extension of [658, Proposition 3] in the multi-dimensional setting, as well).

Proposition 7.3.15. *Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, $c \in \mathbb{C} \setminus \{0\}$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. If the function $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-almost periodic, resp. $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent, and $Q \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$,* *then* $[F+Q](\cdot;\cdot)$ *is* $\mathbb{D}$-*quasi-asymptotically* $(\mathcal{B}, I', c)$-*almost periodic, resp.* $\mathbb{D}$-*quasi-asymptotically* $(\mathcal{B}, I', c)$-*uniformly recurrent.*

We continue by providing an example.

Example 7.3.16. The function $F : \mathbb{R}^n \to \mathbb{R}$, given by $F(\mathbf{t}) := \sin(\ln(1+|\mathbf{t}|))$, $\mathbf{t} \in \mathbb{R}^n$, is quasi-asymptotically almost periodic but not asymptotically uniformly recurrent; this can be shown as in the one-dimensional case. Furthermore, it can be easily shown that $F(\cdot)$ is quasi-asymptotically c-almost periodic for some $c \in \mathbb{C} \setminus \{0\}$ if and only if $c = 1$.

In the following result, we show that the notion introduced in the previous subsection can be viewed as a particular case of the notion introduced in Definition 7.3.14(i), under some very reasonable assumptions (in the second part, we can also consider the situation in which $I' := \omega_j e_j \cdot \mathbb{N}$ for some $j \in \mathbb{N}_n$).

Proposition 7.3.17.
(i) *Let* $\omega \in I \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $|c| \leqslant 1$, $\omega + I \subseteq I$ *and* $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$. *Set* $I' := \omega \cdot \mathbb{N}$. *If a continuous function* $F : I \times X \to Y$ *is* $(S, \mathbb{D}, \mathcal{B})$-*asymptotically* (ω, c)-*periodic, then the function* $F(\cdot;\cdot)$ *is* $\mathbb{D}$-*quasi-asymptotically* $(\mathcal{B}, I', c)$-*almost periodic.*
(ii) *Let* $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$, $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$, *the set* $\mathbb{D}_j$ *be unbounded* $(1 \leqslant j \leqslant n)$ *and the set* $\mathbb{D}$ *consisting of all tuples* $\mathbf{t} \in \mathbb{D}_n$ *such that* $\mathbf{t} + \sum_{i=j+1}^{n} \omega_i e_i$ *for all* $j \in \mathbb{N}_{n-1}$ *be unbounded in* $\mathbb{R}^n$. *Set* $\omega := \sum_{j=1}^{n} \omega_j e_j$, $I' := \omega \cdot \mathbb{N}$ *and* $c := \prod_{j=1}^{n} c_j$. *If* $F : I \times X \to Y$ *is* $(S, \mathcal{B})$-*asymptotically* $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-*periodic,* $|c| \leqslant 1$ *and* $\omega \in I$, *then the function* $F(\cdot;\cdot)$ *is* $\mathbb{D}$-*quasi-asymptotically* $(\mathcal{B}, I', c)$-*almost periodic.*

Proof. The proof of (i) is very similar to the proof of [588, Proposition 3.2]. First of all, note that our assumptions $\omega \in I \setminus \{0\}$ and $\mathbb{D} + \omega \cdot \mathbb{N}_0 \subseteq \mathbb{D}$ imply that the set $\mathbb{D}$ is unbounded, while the assumptions $\omega \in I \setminus \{0\}$ and $\omega + I \subseteq I$ imply that I' is an unbounded subset of I and $I + I' \subseteq I$. Let $B \in \mathcal{B}$ and $\varepsilon > 0$ be fixed. Then we can take $l = 2|\omega|$ in Definition 7.3.14(i) since for each $\mathbf{t}_0 = n'\omega \in I'$, where $n' \in \mathbb{N}$, there exists $\tau = n\omega \in B(\mathbf{t}_0, l) \cap I'$, with $n' = n+1$. Since the function $F(\cdot;\cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic, we have the existence of a finite real number $M > 0$ such that the assumptions $|\mathbf{t}| > M$ and $\mathbf{t} \in \mathbb{D}$ imply $\|F(\mathbf{t}+\omega;x) - cF(\mathbf{t};x)\| < \varepsilon/n$ for all $x \in B$. Let $\mathbf{t} \in \mathbb{D}$ and $|\mathbf{t}| > M(\varepsilon,\tau) \equiv M + n|\omega|$. Then (7.32) holds since the assumptions $\mathbf{t}$, $\mathbf{t}+\tau \in \mathbb{D}_{M(\varepsilon,\tau)}$ and $x \in B$ imply

$$\begin{aligned}\|F(\mathbf{t}+\tau;x) - cF(\mathbf{t};x)\|_Y &\leqslant \sum_{k=0}^{n-1} |c|^{n-k-1} \|F(\mathbf{t}+(k+1)\omega;x) - cF(\mathbf{t}+k\omega;x)\|_Y \\ &\leqslant \sum_{k=0}^{n-1} \|F(\mathbf{t}+(k+1)\omega;x) - cF(\mathbf{t}+k\omega;x)\|_Y \leqslant n(\varepsilon/n) = \varepsilon,\end{aligned}$$

as claimed. To deduce (ii), it suffices to observe that our assumptions imply by Proposition 7.3.7 that the function $F(\cdot;\cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic, with $\omega = \sum_{j=1}^{n} \omega_j e_j$. After that, we can apply the first part of proposition. □

The spaces introduced in Definition 7.3.14 do not form vector spaces under the pointwise addition of functions and these spaces are not closed under the pointwise multiplication with scalar-valued functions of the same type, as is well known in the one-dimensional case [647]. The introduced spaces are homogeneous and, under certain reasonable assumptions, these spaces are translation invariant, invariant under the homotheties with ratio $b > 0$ and the reflections at zero with respect to the first variable. Furthermore, we have the following statements stated here without simple proofs (see also [265, Proposition 2.7, Proposition 2.8]).

Proposition 7.3.18.
(i) *Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, the sets $\mathbb{D}$ and I' are unbounded, $F : I \times X \to \mathbb{C}$ is a continuous function and $I + I' \subseteq I$.*
  (a) *If $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-almost periodic and, for every $B \in \mathcal{B}$, there exists a real number $c_B > 0$ such that $|F(\mathbf{t}; x)| \geqslant c_B$ for all $x \in B$ and $\mathbf{t} \in I$, then the function $1/F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', 1/c)$-almost periodic.*
  (b) *$F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_k) in I' and a sequence (M_k) in $(0, \infty)$ such that $\lim_{k\to+\infty} |\tau_k| = \lim_{k\to+\infty} M_k = +\infty$ and*

$$\lim_{k\to+\infty} \sup_{\mathbf{t}, \mathbf{t}+\tau_k \in \mathbb{D}_{M_k}; x \in B} \left\| F(\mathbf{t} + \tau_k; x) - cF(\mathbf{t}; x) \right\|_Y = 0.$$

(ii) *If $(F_k(\cdot;\cdot))$ is a sequence of $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-almost periodic functions, resp. $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent functions, such that for each $B \in \mathcal{B}$ there exists a finite real number $\varepsilon_B > 0$ such that $\lim_{k\to+\infty} F_k(\mathbf{t}; x) = F(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}$, uniformly in $x \in B^{\cdot} \equiv B^{\circ} \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$, then the function $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-almost periodic, resp. $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent.*

The proof of the following result is very similar to that of Theorem 7.3.6 and therefore omitted (the assumption on the compact support of the function $h(\cdot)$ made in [658] for the class of quasi-asymptotically uniformly recurrent functions is superfluous).

Theorem 7.3.19. *Suppose that $h \in L^1(\mathbb{R}^n)$, $\emptyset \neq I' \subseteq \mathbb{R}^n$ is unbounded and $F : \mathbb{R}^n \times X \to Y$ is a continuous function satisfying that for each $B \in \mathcal{B}$ there exists a finite real number $\varepsilon_B > 0$ such that $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^{\cdot}} \|F(\mathbf{t}, x)\|_Y < +\infty$, where $B^{\cdot} \equiv B^{\circ} \cup \bigcup_{x\in\partial B} B(x, \varepsilon_B)$. Suppose that $\mathbb{D} = \mathbb{R}^n$. Then the function $(h * F)(\cdot;\cdot)$, given by (7.30), is well defined and for each $B \in \mathcal{B}$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^{\cdot}} \|(h * F)(\mathbf{t}; x)\|_Y < +\infty$; furthermore, if $F(\cdot;\cdot)$ is $\mathbb{R}^n$-quasi-asymptotically $(\mathcal{B}, I', c)$-almost periodic, resp. $\mathbb{R}^n$-quasi-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent, then the function $(h * F)(\cdot;\cdot)$ is likewise $\mathbb{R}^n$-quasi-asymptotically $(\mathcal{B}, I', c)$-almost periodic, resp. $\mathbb{R}^n$-quasi-asymptotically $(\mathcal{B}, I', c)$-uniformly recurrent.*

Accepting the notation employed in [647] and [658], we have the following ($I = \mathbb{R}$ or $I = [0, \infty)$; $\omega \in I$):

(i) Suppose that $f \in SAP_{\omega}(\mathbb{R} : X) \cap AAA(\mathbb{R} : X)$, resp. $f \in SAP_{\omega}(I : X) \cap AAP(I : X)$. Then $f \in AP_{\omega}(\mathbb{R} : X)$, resp. $f \in AP_{\omega}(I : X)$.
(ii) Suppose that $f \in SAP_{\omega}(\mathbb{R} : X) \cap AA(\mathbb{R} : X)$, resp. $f \in SAP_{\omega}(I : X) \cap AP(I : X)$. Then $f \in C_{\omega}(\mathbb{R} : X)$, resp. $f \in C_{\omega}(I : X)$.
(iii) $AAA(\mathbb{R} : X) \cap Q - AAP(\mathbb{R} : X) = AAP(\mathbb{R} : X)$ and $[AAA(\mathbb{R} : X) \setminus AAP(\mathbb{R} : X)] \cap Q - AAP(\mathbb{R} : X) = \emptyset$.
(iv) $AA(\mathbb{R} : X) \cap Q - AAP(\mathbb{R} : X) = AP(\mathbb{R} : X)$.
(v) Let $F(I : X)$ be any space of functions $h : I \to X$ satisfying that for each $\tau \in I$ the supremum formula holds for the function $h(\cdot + \tau) - h(\cdot)$, i. e.,

$$\sup_{t \in I} \|h(\cdot + \tau) - h(\cdot)\| = \sup_{t \in I,\ |t| \geqslant a} \|h(\cdot + \tau) - h(\cdot)\|, \quad a \in I.$$

Then we have $[F(I : X) + C_0(I : X)] \cap Q - AUR(I : X) \subseteq AUR(I : X)$ and $F(I : X) \cap Q - AUR(I : X) \subseteq UR(I : X)$.

Furthermore, the above statements can be reformulated for the corresponding Stepanov classes.

We will only note here that these statements admit very simple generalizations in the multi-dimensional setting. For example, if $I = \mathbb{R}^n$ or $I = [0, \infty)^n$ and the function $F : I \to Y$ is both S-asymptotically $(\omega_j, c_j, I)_{j \in \mathbb{N}_n}$-periodic and I-asymptotically Bohr $(I, 1)$-almost periodic, then the function $F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$-periodic (this can be used to provide certain examples of compactly almost automorphic functions in $\mathbb{R}^n$ which are not quasi-asymptotically uniformly recurrent). Essential is that the proof of [658, Theorem 1] works in the multi-dimensional setting (see the item (v) above).

7.3.4 Stepanov classes of quasi-asymptotically *c*-almost periodic type functions

In this subsection, we investigate the Stepanov quasi-asymptotically c-almost periodic type functions (the Weyl and Besicovitch generalizations of quasi-asymptotically c-almost periodic type functions can be also introduced and analyzed but we will skip all related details concerning this issue here). We will always assume that $c \in \mathbb{C} \setminus \{0\}$, Ω is a fixed compact subset of $\mathbb{R}^n$ with a positive Lebesgue measure, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, the sets $\mathbb{D}$ and Λ' are unbounded, as well as $\Lambda + \Lambda' \subseteq \Lambda$.

We employ the following conditions:

$(MD - B)_S$ $\phi : [0, \infty) \to [0, \infty)$, $p \in \mathcal{P}(\Omega)$, $\mathsf{F} : \Lambda \times (0, \infty) \times \Lambda' \to (0, \infty)$, $\mathbf{F} : \Lambda \times \mathbb{N} \to (0, \infty)$ and $\mathsf{F} : \Lambda \to (0, \infty)$.

We will follow the approach obeyed for introduction of the notion in [658, Definition 13–Definition 15], only, in which we do not lose the valuable information about the translation invariance of introduced spaces.

Definition 7.3.20. Let $(MD-B)_S$ hold.

(i) A function $F : \Lambda \times X \to Y$ is called Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathsf{F}, c]$-quasi-asymptotically almost periodic, resp. Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathbf{F}, c]$-quasi-asymptotically uniformly recurrent, if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that there exists a finite real number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{\mathbf{t} \in \mathbb{D}_{M(\varepsilon,\tau)} : \mathbf{t}+\tau \in \mathbb{D}_{M(\varepsilon,\tau)}; x \in B} \mathsf{F}(\mathbf{t}, \varepsilon, \tau)\phi\big(\|F(\cdot + \mathbf{t} + \tau; x) - cF(\cdot + \mathbf{t}; x)\|_Y\big)_{L^{p(\cdot)}(\Omega)} \leqslant \varepsilon, \tag{7.34}$$

resp. there exist a strictly increasing sequence (τ_k) in Λ' whose norms tending to plus infinity and a sequence (M_k) of positive real numbers tending to plus infinity such that

$$\lim_{k \to +\infty} \sup_{\mathbf{t} \in \mathbb{D}_{M_k} : \mathbf{t}+\tau_k \in \mathbb{D}_{M_k}; x \in B} \mathbf{F}(\mathbf{t}, k)\phi\big(\|F(\cdot + \mathbf{t} + \tau_k; x) - cF(\cdot + \mathbf{t}; x)\|_Y\big)_{L^{p(\cdot)}(\Omega)} = 0.$$

(ii) Let $\omega \in \mathbb{R}^n \smallsetminus \{0\}$, $c \in \mathbb{C} \smallsetminus \{0\}$, $\omega + \Lambda \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. A function $F : \Lambda \times X \to Y$ is said to be Stepanov $[S, \Omega, \mathcal{B}, \mathbb{D}, p, \phi, \mathsf{F}]$-asymptotically (ω, c)-periodic if and only if for each $B \in \mathcal{B}$ we have

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}} \mathsf{F}(\mathbf{t})\phi\big(\|F(\mathbf{t} + \omega + \cdot; x) - cF(\mathbf{t} + \cdot; x)\|_Y\big)_{L^{p(\cdot)}(\Omega)} = 0, \quad \text{uniformly in } x \in B.$$

(iii) Let $\omega_j \in \mathbb{R} \smallsetminus \{0\}$, $c_j \in \mathbb{C} \smallsetminus \{0\}$, $\omega_j e_j + \Lambda \subseteq \Lambda$, $\mathbb{D}_j \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded $(1 \leqslant j \leqslant n)$. A function $F : \Lambda \times X \to Y$ is said to be $[S, \Omega, \mathcal{B}, \mathbb{D}, p, \phi, \mathsf{F}]$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic if and only if for each $j \in \mathbb{N}_n$ we have

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}_j} \mathsf{F}(\mathbf{t})\phi\big(\|F(\mathbf{t} + \omega_j e_j + \cdot; x) - c_j F(\mathbf{t} + \cdot; x)\|_Y\big)_{L^{p(\cdot)}(\Omega)} = 0, \quad \text{uniformly in } x \in B.$$

Definition 7.3.21. Let $(MD-B)_S$ hold.

(i) A function $F : \Lambda \times X \to Y$ is called Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathsf{F}, c]$-quasi-asymptotically almost periodic of type 1, resp. Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathbf{F}, c]$-quasi-asymptotically uniformly recurrent of type 1, if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that there exists a finite real number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{\mathbf{t} \in \mathbb{D}_{M(\varepsilon,\tau)} : \mathbf{t}+\tau \in \mathbb{D}_{M(\varepsilon,\tau)}; x \in B} \mathsf{F}(\mathbf{t}, \varepsilon, \tau)\phi\big(\|F(\cdot + \mathbf{t} + \tau; x) - cF(\cdot + \mathbf{t}; x)\|_{L^{p(\cdot)}(\Omega : Y)}\big) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (τ_k) in Λ' whose norms tending to plus infinity and a sequence (M_k) of positive real numbers tending to plus infinity such that

$$\lim_{k \to +\infty} \sup_{\mathbf{t} \in \mathbb{D}_{M_k} : \mathbf{t}+\tau_k \in \mathbb{D}_{M_k}; x \in B} \mathbf{F}(\mathbf{t}, k)\phi\big(\|F(\cdot + \mathbf{t} + \tau_k; x) - cF(\cdot + \mathbf{t}; x)\|_{L^{p(\cdot)}(\Omega : Y)}\big) = 0.$$

(ii) Let $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega + \Lambda \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. A function $F : \Lambda \times X \to Y$ is said to be Stepanov $[S, \Omega, \mathcal{B}, \mathbb{D}, p, \phi, \mathsf{F}]$-asymptotically (ω, c)-periodic of type 1 if and only if for each $B \in \mathcal{B}$ we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}} \mathsf{F}(\mathbf{t})\phi\big(\|F(\mathbf{t} + \omega + \cdot; x) - cF(\mathbf{t} + \cdot; x)\|_{L^{p(\cdot)}(\Omega:Y)}\big) = 0, \quad \text{uniformly in } x \in B.$$

(iii) Let $\omega_j \in \mathbb{R}\setminus\{0\}$, $c_j \in \mathbb{C}\setminus\{0\}$, $\omega_j e_j + \Lambda \subseteq \Lambda$, $\mathbb{D}_j \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded $(1 \leqslant j \leqslant n)$. A function $F : \Lambda \times X \to Y$ is said to be $[S, \Omega, \mathcal{B}, \mathbb{D}, p, \phi, \mathsf{F}]$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodic of type 1 if and only if for each $j \in \mathbb{N}_n$ we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}_j} \mathsf{F}(\mathbf{t})\phi\big(\|F(\mathbf{t} + \omega_j e_j + \cdot; x) - c_j F(\mathbf{t} + \cdot; x)\|\big)_{L^{p(\cdot)}(\Omega:Y)} = 0, \quad \text{uniformly in } x \in B.$$

Definition 7.3.22. Let $(MD - B)_S$ hold.

(i) A function $F : \Lambda \times X \to Y$ is called Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathsf{F}, c]$-quasi-asymptotically almost periodic of type 2, resp. Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathbf{F}, c]$-quasi-asymptotically uniformly recurrent of type 2, if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that there exists a finite real number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{\mathbf{t}\in\mathbb{D}_{M(\varepsilon,\tau)}:\mathbf{t}+\tau\in\mathbb{D}_{M(\varepsilon,\tau)};x\in B} \phi\big(\mathsf{F}(\mathbf{t}, \varepsilon, \tau)\|F(\cdot + \mathbf{t} + \tau; x) - cF(\cdot + \mathbf{t}; x)\|_{L^{p(\cdot)}(\Omega:Y)}\big) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (τ_k) in Λ' whose norms tending to plus infinity and a sequence (M_k) of positive real numbers tending to plus infinity such that

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in\mathbb{D}_{M_k}:\mathbf{t}+\tau_k\in\mathbb{D}_{M_k};x\in B} \phi\big(\mathbf{F}(\mathbf{t}, k)\|F(\cdot + \mathbf{t} + \tau_k; x) - cF(\cdot + \mathbf{t}; x)\|_{L^{p(\cdot)}(\Omega:Y)}\big) = 0.$$

(ii) Let $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega + \Lambda \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. A function $F : \Lambda \times X \to Y$ is said to be Stepanov $[S, \Omega, \mathcal{B}, \mathbb{D}, p, \phi, \mathsf{F}]$-asymptotically (ω, c)-periodic of type 2 if and only if for each $B \in \mathcal{B}$ we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}} \phi\big(\mathsf{F}(\mathbf{t})\|F(\mathbf{t} + \omega + \cdot; x) - cF(\mathbf{t} + \cdot; x)\|_{L^{p(\cdot)}(\Omega:Y)}\big) = 0, \quad \text{uniformly in } x \in B.$$

(iii) Let $\omega_j \in \mathbb{R}\setminus\{0\}$, $c_j \in \mathbb{C}\setminus\{0\}$, $\omega_j e_j + \Lambda \subseteq \Lambda$, $\mathbb{D}_j \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded $(1 \leqslant j \leqslant n)$. A function $F : \Lambda \times X \to Y$ is said to be $[S, \Omega, \mathcal{B}, \mathbb{D}, p, \phi, \mathsf{F}]$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j\in\mathbb{N}_n}$-periodic of type 2 if and only if for each $j \in \mathbb{N}_n$ we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}_j} \phi\big(\mathsf{F}(\mathbf{t})\|F(\mathbf{t} + \omega_j e_j + \cdot; x) - c_j F(\mathbf{t} + \cdot; x)\|_{L^{p(\cdot)}(\Omega:Y)}\big) = 0, \quad \text{uniformly in } x \in B.$$

Remark 7.3.23. If $\mathbb{D}+\Lambda' \subseteq \mathbb{D}$ (this is always true provided that $\mathbb{D} = \Lambda$ due to our standing assumption), then it is irrelevant whether we will write $\sup_{\mathbf{t}\in\mathbb{D}_{M_k}:\mathbf{t}+\tau_k\in\mathbb{D}_{M_k}} \cdot$ or only

$\sup_{\mathbf{t}\in \mathbb{D}_{M_k}} \cdot$ in Definition 7.3.20(ii); a similar comment holds for the notion introduced in Definition 7.3.20(i), Definition 7.3.21 and Definition 7.3.22.

Without any doubt, the most intriguing case is that in which we have $p(x) \equiv p \in [1, \infty)$, $\phi(x) \equiv x$, $\Omega = [0, 1]^n$, and the functions F, F, F are identically equal to one. In this case, we can simply reformulate a great number of statements clarified by now for the Stepanov classes of the functions introduced in this section by using the notion of multi-dimensional Bochner transform. If $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Lambda : Y)$ is well defined and continuous, then the function $F : \Lambda \times X \to Y$ will be, e. g., Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathsf{F}, c]$-quasi-asymptotically almost periodic if and only if the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Lambda : Y)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, \Lambda', c)$-almost periodic. In the case that the functions F, F, F are only bounded and not necessarily identically equal to one, then we can simply transfer the statements of [658, Proposition 4, Corollary 1] to the multi-dimensional setting.

Using the trivial inequalities and Lemma 1.1.7, we can clarify a great number of inclusions for the introduced classes of functions. The main result of this subsection, Theorem 7.3.24, can be reworded for all other classes of the functions introduced in Definition 7.3.20(ii)–(iii), Definition 7.3.21 and Definition 7.3.22.

Theorem 7.3.24. *Let a function* $F : \mathbb{R}^n \times X \to Y$ *be Stepanov* $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, F, c]$*-quasi-asymptotically almost periodic, resp. Stepanov* $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathbf{F}, c]$*-quasi-asymptotically uniformly recurrent, where* $\Omega = [0, 1]^n$, $\mathbb{D} = \mathbb{R}^n$, $\phi : [0, \infty) \to [0, \infty)$ *is a convex, monotonically increasing function which additionally satisfies the requirement that there exists a function* $\varphi : [0, \infty) \to [0, \infty)$ *such that* $\phi(xy) \leqslant \varphi(x)\phi(y)$ *for all* x, $y \geqslant 0$. *Let* $h \in L^1(\mathbb{R}^n)$ *and let for each set* $B \in \mathcal{B}$ *we have* $\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \|F(\mathbf{t}; x)\|_Y < \infty$. *Suppose that there exists a continuous function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0+) = 0+$ *and a sequence* $(a_k)_{k\in\mathbb{Z}^n}$ *of strictly increasing positive reals such that* $\sum_{k\in\mathbb{Z}^n} a_k = 1$ *and for each* $\varepsilon > 0$ *and* $\tau \in \Lambda'$*, resp. for each* $n \in \mathbb{N}$ *and* $\tau \in \Lambda'$*, there exists* $M'(\varepsilon, \tau) > 0$*, resp.* $M'(n, \tau) > 0$*, such that for each* $\mathbf{t} \in \mathbb{R}^n$ *with* $|\mathbf{t}| \geqslant M'(\varepsilon, \tau)$*, resp.* $|\mathbf{t}| \geqslant M'(n, \tau)$*, we have*

$$\int\limits_{[0,1]^n} \varphi_{p(\mathbf{u})}\bigg(F_1(\mathbf{t}, \varepsilon, \tau)\bigg(\varphi(2) \sum_{k\in\mathbb{Z}^n} \frac{a_k \varphi(a_k^{-1})[\varphi(|h(\mathbf{t} - \sigma - k)|)]_{L^{q(\sigma)}(\Omega)}}{F(\mathbf{u} + k, \varepsilon, \tau)} + g(\varepsilon)\bigg)\bigg)\, d\mathbf{u} \leqslant 1, \tag{7.35}$$

resp.

$$\int\limits_{[0,1]^n} \varphi_{p(\mathbf{u})}\bigg(\mathbf{F}_1(\mathbf{t}, n)\bigg(\varphi(2) \sum_{k\in\mathbb{Z}^n} \frac{a_k \varphi(a_k^{-1})[\varphi(|h(\mathbf{t} - \sigma - k)|)]_{L^{q(\sigma)}(\Omega)}}{F(\mathbf{u} + k, n)} + g(1/n)\bigg)\bigg)\, d\mathbf{u} \leqslant 1.$$

Then the function $(h * F)(\cdot; \cdot)$ *is Stepanov* $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, F_1, c]$*-quasi-asymptotically almost periodic, resp. Stepanov* $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathbf{F}_1, c]$*-quasi-asymptotically uniformly recurrent.*

Proof. We will prove the result only for the class of Stepanov $[\Omega, \mathcal{B}, \Lambda', \mathbb{D}, p, \phi, \mathrm{F}, c]$-quasi-asymptotically almost periodic functions. It is clear that the function $(h * F)(\cdot;\cdot)$ is well defined. Let $\varepsilon > 0$ and $B \in \mathcal{B}$ be fixed. Due to our assumption, there exists $l > 0$ s. t. for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ s. t. there exists a finite real number $M(\varepsilon, \tau) > 0$ s. t. (7.34) holds. Let such a point τ be fixed. Then we know that there exists $M'(\varepsilon, \tau) > 0$ such that for each $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \geqslant M'(\varepsilon, \tau)$ we have (7.35). Let $M_1(\varepsilon, \tau) \geqslant M(\varepsilon, \tau) + M'(\varepsilon, \tau) + |\tau|$. Arguing as in the proof of Theorem 7.3.6, the continuity of the function $\phi(\cdot)$ at the point $t = 0$ implies that there exists a finite real number $M_3(\varepsilon, \tau) \geqslant M_1(\varepsilon, \tau)$ such that

$$\varphi(2c_B)\frac{1}{2}\phi\Bigg(\int\limits_{|\sigma|\leqslant M_2(\varepsilon,\tau)} \big|h(\mathbf{t}-\sigma)\big|\, d\sigma\Bigg) \leqslant \varepsilon g(\varepsilon). \tag{7.36}$$

Keeping in mind (7.35) and the definition of the norm in $L^{p(\cdot)}(\Omega)$, with $\lambda = \varepsilon/\mathrm{F}_1(\mathbf{t}, \varepsilon, \tau)$ and the meaning clear, it suffices to show that, for every fixed element $x \in B$ and for every fixed point $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \geqslant M_4(\varepsilon, \tau) \equiv M_3(\varepsilon, \tau) + |\tau|$, we have

$$\begin{aligned}
&\phi\big(\|(h * F)(\mathbf{t} + \mathbf{u} + \tau; x) - c(h * F)(\mathbf{t} + \mathbf{u}; x)\|_Y\big) \\
&\quad = \phi\Bigg(\Bigg\|\int\limits_{\mathbb{R}^n} h(\mathbf{t} - \sigma) \cdot \big[F(\sigma + \mathbf{u} + \tau; x) - cF(\sigma + \mathbf{u}; x)\big]\, d\sigma\Bigg\|_Y\Bigg) &(7.37)\\
&\quad \leqslant \varepsilon\varphi(2) \sum_{k\in\mathbb{Z}^n} \frac{a_k \varphi(a_k^{-1})[\varphi(|h(\mathbf{t} - \sigma - k)|)]_{L^{q(\sigma)}(\Omega)}}{\mathrm{F}(\mathbf{u} + k, \varepsilon, \tau)} + \varepsilon g(\varepsilon). &(7.38)
\end{aligned}$$

Towards this end, observe first that there exists a finite constant $c_B > 0$ such that (see (7.37))

$$\begin{aligned}
&\phi\Bigg(\Bigg\|\int\limits_{\mathbb{R}^n} h(\mathbf{t} - \sigma) \cdot \big[F(\sigma + \mathbf{u} + \tau; x) - cF(\sigma + \mathbf{u}; x)\big]\, d\sigma\Bigg\|_Y\Bigg) \\
&\quad \leqslant \phi\Bigg(2\frac{1}{2}\int\limits_{|\sigma|\geqslant M_4(\varepsilon,\tau)} \big|h(\mathbf{t} - \sigma)\big|\big\|F(\mathbf{t} + \sigma + \tau; x) - cF(\mathbf{u} + \sigma)\big\|_Y\, d\sigma \\
&\qquad + \frac{c_B}{2}\int\limits_{|\sigma|\leqslant M_4(\varepsilon,\tau)} \big|h(\mathbf{t} - \sigma)\big|\, d\sigma\Bigg) \\
&\quad \leqslant \varphi(2)\frac{1}{2}\phi\Bigg(\int\limits_{|\sigma|\geqslant M_4(\varepsilon,\tau)} \big|h(\mathbf{t} - \sigma)\big|\big\|F(\mathbf{t} + \sigma + \tau; x) - cF(\mathbf{u} + \sigma)\big\|_Y\, d\sigma\Bigg) \\
&\qquad + \varphi(2c_B)\frac{1}{2}\phi\Bigg(\int\limits_{|\sigma|\leqslant M_4(\varepsilon,\tau)} \big|h(\mathbf{t} - \sigma)\big|\, d\sigma\Bigg).
\end{aligned}$$

Then (7.38) follows from the last estimate, (7.36) and the next computation involving the Jensen inequality:

$$
\begin{aligned}
&\phi\Big(\int_{|\sigma|\geqslant M_4(\varepsilon,\tau)} \big|h(\mathbf{t}-\sigma)\big| \big\|F(\mathbf{t}+\sigma+\tau;x)-cF(\mathbf{u}+\sigma)\big\|_Y \, d\sigma\Big) \\
&= \phi\Big(\sum_{k\in\mathbb{Z}^n} a_k \int_{\sigma\in k+\Omega;|\sigma|\geqslant M_4(\varepsilon,\tau)} a_k^{-1}\big|h(\mathbf{t}-\sigma)\big| \big\|F(\mathbf{t}+\sigma+\tau;x)-cF(\mathbf{u}+\sigma)\big\|_Y \, d\sigma\Big) \\
&\leqslant \sum_{k\in\mathbb{Z}^n} a_k \varphi(a_k^{-1}) \int_{\sigma\in\Omega;|\sigma+k|\geqslant M_4(\varepsilon,\tau)} \varphi\big(|h(\mathbf{t}-\sigma)|\big)\phi\big(\big\|F(\mathbf{t}+\sigma+\tau;x)-cF(\mathbf{u}+\sigma)\big\|_Y\big)\, d\sigma,
\end{aligned}
$$

and a simple application of the Hölder inequality after that. □

7.3.5 Multi-dimensional Weyl *c*-almost periodic type functions

In this subsection, we will introduce and analyze the multi-dimensional Weyl c-almost periodic type functions; we will always assume that the following condition holds:

(WM2) $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Omega)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0,\infty) \to [0,\infty)$ and $\mathbb{F} : (0,\infty) \times \Lambda \to (0,\infty)$.

Definition 7.3.25.

(i) By $e-W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\big(\big\|F(\mathbf{t}+\tau+l\mathbf{u};x)-cF(\mathbf{t}+l\mathbf{u};x)\big\|_Y\big)_{L^{p(\mathbf{u})}(\Omega)} < \varepsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\limsup_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\big(\big\|F(\mathbf{t}+\tau+l\mathbf{u};x)-cF(\mathbf{t}+l\mathbf{u};x)\big\|_Y\big)_{L^{p(\mathbf{u})}(\Omega:Y)} < \varepsilon.$$

Definition 7.3.26.

(i) By $e-W^{[p(\mathbf{u}),\phi,\mathbb{F},c]_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\big(\big\|F(\mathbf{t}+\tau+l\mathbf{u};x)-cF(\mathbf{t}+l\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big) < \varepsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F},c]_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\limsup_{l\to+\infty} \sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\big(\|F(\mathbf{t}+\tau+\mathbf{u};x) - cF(\mathbf{t}+\mathbf{u};x)\|_{L^{p(\mathbf{u})}(l\Omega:Y)}\big) < \varepsilon.$$

Definition 7.3.27.

(i) By $e - W^{[p(\mathbf{u}),\phi,\mathbb{F},c]_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda\times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} \phi\big(l^n \mathbb{F}(l,\mathbf{t})\|F(\mathbf{t}+\tau+l\mathbf{u};x) - cF(\mathbf{t}+l\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big) < \varepsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F},c]_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that

$$\limsup_{l\to+\infty} \sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} \phi\big(l^n \mathbb{F}(l,\mathbf{t})\|F(\mathbf{t}+\tau+l\mathbf{u};x) - cF(\mathbf{t}+l\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big) < \varepsilon.$$

It is clear that the notion from the second parts of the above definitions extends the corresponding notion from the first parts of these definitions. In many concrete situations, the situation in which $\Lambda' \neq \Lambda$ may occur, as we have already clarified in the case that $c = 1$.

It is clear that all introduced spaces are invariant under the pointwise multiplications with complex scalars provided that $\phi : [0,\infty) \to [0,\infty)$ is a convex, monotonically increasing function which additionally satisfies the requirement that there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$. The translation invariance of spaces introduced in Definition 7.3.25 and Definition 7.3.26 holds provided that, for every $\tau \in \Lambda$, we have

$$\sup_{l>0,\mathbf{t}\in\Lambda} \frac{\mathbb{F}(l,\mathbf{t})}{\mathbb{F}(l,\mathbf{t}+\tau)} < +\infty,$$

while the translation invariance of spaces introduced in Definition 7.3.27 holds provided this condition and the assumption on the existence of the function $\varphi(\cdot)$ above. Furthermore, it can be simply shown that for any scalar-valued function $F(\cdot;\cdot)$ which is bounded away from zero on elements of the collection $\mathcal{B}$, the function $1/F(\cdot;\cdot)$ is well defined and belongs to the same space of functions as $F(\cdot;\cdot)$, with the constant c replaced by $1/c$ in the corresponding space and the meaning clear.

The conclusions from the following result can be also formulated for the classes of functions introduced in Definition 7.3.26 and Definition 7.3.27.

Proposition 7.3.28.

(i) *Suppose that the function $\phi(\cdot)$ is monotonically increasing and*

$$F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y).$$

Then we have $\|F(\cdot;\cdot)\|_Y \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},|c|]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$.

(ii) *Suppose that $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$. Then we have*

$$\check{F} \in (e-)W^{[p_1(\mathbf{u}),\phi,\mathbb{F}_1,c]}_{-\Omega,-\Lambda',\mathcal{B}}((-\Lambda) \times X : Y),$$

where $p_1(\cdot) := p(-\cdot)$ and $\mathbb{F}_1(\cdot;\cdot) := \mathbb{F}(\cdot;-\cdot)$.

Proof. The proof of (i) simply follows from Lemma 1.1.7(iii), our assumption that the function $\phi(\cdot)$ is monotonically increasing and the inequality

$$\big|\|F(\mathbf{t}+\tau+l\mathbf{u};x)\|_Y - |c|\|F(\mathbf{t}+l\mathbf{u};x)\|_Y\big| \leqslant \|F(\mathbf{t}+\tau+l\mathbf{u};x) - cF(\mathbf{t}+l\mathbf{u};x)\|_Y,$$

with the notation and meaning clear. The proof of (ii) follows from the chain rule, the definition of norm in $L^{p_1(\cdot)}(-\Omega)$ and the next equalities:

$$\begin{aligned}
&l^n\mathbb{F}(l,-\mathbf{t})\big[\phi\big(\|F(-\mathbf{t}-\tau-l\mathbf{u};x) - cF(-\mathbf{t}-l\mathbf{u};x)\|_Y\big)\big]_{L^{p_1(\cdot)}(-\Omega)} \\
&\quad = l^n\mathbb{F}(l,-\mathbf{t}) \inf\Big\{\lambda > 0 : \int\limits_{-\Omega} \varphi_{p(-\mathbf{u})}\Big(\frac{\phi(\|F(-\mathbf{t}-\tau-l\mathbf{u};x) - cF(-\mathbf{t}-l\mathbf{u};x)\|_Y)}{\lambda}\Big)\, d\mathbf{u} \leqslant 1\Big\} \\
&\quad = l^n\mathbb{F}(l,-\mathbf{t}) \inf\Big\{\lambda > 0 : \int\limits_{\Omega} \varphi_{p(\mathbf{u})}\Big(\frac{\phi(\|F(-\mathbf{t}-\tau+l\mathbf{u};x) - cF(-\mathbf{t}+l\mathbf{u};x)\|_Y)}{\lambda}\Big)\, d\mathbf{u} \leqslant 1\Big\},
\end{aligned}$$

with the notation and meaning clear. □

In what follows, we will extend the statements of [588, Proposition 2.3, Corollary 2.1, Proposition 2.4] to the multi-dimensional setting.

Theorem 7.3.29. *Suppose that the function $\mathbb{F}(\cdot;\cdot)$ does not depend on the second argument.*

(i) *Suppose that $m \in \mathbb{N}$, $j\Lambda' + \Lambda + l\Omega \subseteq \Lambda$ for all $l \geqslant 0$ and $j \in \mathbb{N}$, as well as that condition* (F) *holds and there exists a finite real constant $c_m > 0$ such that*

$$\phi(x_1 + \cdots + x_m) \leqslant c_m[\phi(x_1) + \cdots + \phi(x_m)], \quad x_i \geqslant 0 \quad (i \in \mathbb{N}_m). \tag{7.39}$$

Suppose, further, that $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, resp. $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]_i}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ for $i = 1,2$. Then $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c^m]}_{\Omega,m\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, resp. $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c^m]_i}_{\Omega,m\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ provided $i = 1,2$ and the function $\phi(\cdot)$ is monotonically increasing.

(ii) *Suppose that* $m \in 2\mathbb{Z} \setminus \{0\}$, $p \in \mathbb{N}$, $(m,n) = 1$, $|c| = 1$ *and* $\arg(c) = \pi m/p$ *[*$m \in 2\mathbb{Z}+1$, $p \in \mathbb{N}$, $(m,n) = 1$, $|c| = 1$ *and* $\arg(c) = \pi m/p$*],* $m \in \mathbb{N}$, $j\Lambda' + \Lambda + l\Omega \subseteq \Lambda$ *for all* $l \geqslant 0$ *and* $j \in \mathbb{N}$, *as well as that condition* (F) *holds and there exists a finite real constant* $c_m > 0$ *such that* (7.39) *holds. If* $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, *resp.* $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]_i}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *for* $i = 1,2$, *then* $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},1]}_{\Omega,m\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *[*$F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},-1]}_{\Omega,m\Lambda',\mathcal{B}}(\Lambda \times X : Y)$*], resp.* $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},1]_i}_{\Omega,m\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *[*$F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},-1]_i}_{\Omega,m\Lambda',\mathcal{B}}(\Lambda \times X : Y)$*], provided* $i = 1,2$ *and the function* $\phi(\cdot)$ *is monotonically increasing.*

(iii) *Suppose that* $|c| = 1$, $\arg(c) \notin \pi\mathbb{Q}$, $j\Lambda' + \Lambda + l\Omega \subseteq \Lambda$ *for all* $l \geqslant 0$ *and* $j \in \mathbb{N}$, *as well as that condition* (F) *holds and for each* $m \in \mathbb{N}$ *there exists a finite real constant* $c_m > 0$ *such that* (7.39) *holds. Let the function* $\varphi(\cdot)$ *be continuous at zero. Suppose, further, that* $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, *resp.* $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]_i}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *for* $i = 1,2$. *Then* $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c']}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, *provided that for each set* $B \in \mathcal{B}$ *the following condition holds*

$$\sup_{l>1,\mathbf{t}\in\Lambda;x\in B} l^n\mathbb{F}(l)\big[\phi\big(\|F(\mathbf{t}+l\mathbf{u};x)\|_Y\big)\big]_{L^{p(\mathbf{u})}(\Omega)} < +\infty, \tag{7.40}$$

resp. $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c']_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *[*$F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c']_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$*] provided that the function* $\phi(\cdot)$ *is monotonically increasing and for each set* $B \in \mathcal{B}$ *the following condition holds:*

$$\sup_{l>1,\mathbf{t}\in\Lambda;x\in B} l^n\mathbb{F}(l)\phi\big(\|F(\mathbf{t}+l\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big) < +\infty$$
$$\Big[\sup_{l>1,\mathbf{t}\in\Lambda;x\in B} \phi\big(l^n\mathbb{F}(l)\|F(\mathbf{t}+l\mathbf{u};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}\big) < +\infty\Big].$$

Proof. We will prove the statements (i) and (iii) for the class $(e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, only. Clearly, we have the following decomposition ($\mathbf{t} \in \Lambda$; $\mathbf{u} \in \Omega$; $l > 0$):

$$\begin{aligned} &F(\mathbf{t}+m\tau+l\mathbf{u};x) - c^mF(\mathbf{t}+l\mathbf{u};x)\\ &\quad= \sum_{j=0}^{m-1} c^j\big[F\big(\mathbf{t}+(m-j)\tau+l\mathbf{u};x\big) - cF\big(\mathbf{t}+(m-j-1)+l\mathbf{u};x\big)\big].\end{aligned}$$

Therefore, our assumptions imply

$$\begin{aligned} &\phi\big(\|F(\mathbf{t}+m\tau+l\mathbf{u};x) - c^mF(\mathbf{t}+l\mathbf{u};x)\|_Y\big)_{L^{p(\mathbf{u})}(\Omega)}\\ &\quad\leqslant c_m\sum_{j=0}^{m-1}\varphi(c^j)\phi\big(\|F\big(\mathbf{t}+(m-j)\tau+l\mathbf{u};x\big) - cF\big(\mathbf{t}+(m-j-1)\tau+l\mathbf{u};x\big)\|_Y\big)_{L^{p(\mathbf{u})}(\Omega)}\end{aligned}$$

and $\mathbf{t} + (m-j-1)\tau \in \Lambda$ for all $\mathbf{t} \in \Lambda$ and $0 \leqslant j \leqslant m-1$. The final conclusion of (i) simply follows from the above. To prove (iii), it should be only recalled that the set

$\{c^m : m \in \mathbb{N}\}$ is dense in the unit circle $S_1 \equiv \{z \in \mathbb{C} : |z| = 1\}$ so that there exists a strictly increasing sequence (l_k) of positive integers such that $\lim_{l\to+\infty} c^{l_k} = c'$. Then the conclusion follows similarly to the proof of [588, Proposition 2.3], by applying the first part of this theorem, our assumption with $m = 2$ and the estimate (7.40). □

We will revisit once more the characteristic function of the region $[0,\infty)^n$:

Example 7.3.30. Let $\Omega = [0,1]^n$.

(i) Suppose that $\emptyset \neq K \subseteq \mathbb{R}^n$ and $F(\mathbf{t}) := \chi_K(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$. We will prove that for each $p \in D_+(\Omega)$ and $c \in \mathbb{C} \setminus \{0\}$ we have $F \in e - W^{[p(\mathbf{u}),x,l^{-\sigma},c]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$. Keeping in mind Lemma 1.1.7(ii), we get $(\tau \in \mathbb{R}^n; l > 0)$

$$\begin{aligned}
&\sup_{\mathbf{t}\in\mathbb{R}^n} l^{n-\sigma}\|\chi_K(\mathbf{t}+\tau+l\mathbf{u}) - c\chi_K(\mathbf{t}+l\mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega)} \\
&\quad \leqslant 4\sup_{\mathbf{t}\in\mathbb{R}^n} l^{n-\sigma}\|\chi_K(\mathbf{t}+\tau+l\mathbf{u}) - c\chi_K(\mathbf{t}+l\mathbf{u})\|_{L^{p^+}(\Omega)} \\
&\quad = 4\sup_{\mathbf{t}\in\mathbb{R}^n} l^{-\sigma}\|\chi_K(\mathbf{t}+\tau+\mathbf{u}) - c\chi_K(\mathbf{t}+\mathbf{u})\|_{L^{p^+}(l\Omega)} \\
&\quad \leqslant 4\sup_{\mathbf{t}\in\mathbb{R}^n} l^{-\sigma}\big[\|\chi_K(\cdot)\|_{L^{p^+}(l\Omega\cap[K-\mathbf{t}-\tau])} + |c|\|\chi_K(\cdot)\|_{L^{p^+}(l\Omega\cap[K-\mathbf{t}])}\big] \\
&\quad \leqslant 4l^{-\sigma}(1+|c|)m(K).
\end{aligned}$$

This simply implies the required result.

(ii) Set $F(\mathbf{t}) := \chi_{[0,\infty)^n}(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$. We already know that $F \in W^{[p,x,l^{-\sigma},1]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$ if and only if $\sigma > (n-1)/p$, as well as that there is no $\sigma > 0$ such that $F \in e - W^{[p,x,l^{-\sigma},1]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$; similarly, we see that there is no $\sigma > 0$ and $c \in \mathbb{C} \setminus \{0\}$ such that $F \in e - W^{[p,x,l^{-\sigma},c]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$. Since

$$\sup_{\mathbf{t}\in\mathbb{R}^n}\|\chi_{[0,\infty)^n}(\mathbf{t}+\tau+l\mathbf{u}) - c\chi_{[0,\infty)^n}(\mathbf{t}+l\mathbf{u})\|_{L^p(\Omega)} \geqslant |1-c|,$$

as easily proved, we see that there is no $c \in \mathbb{C} \setminus \{0,1\}$ such that $F \in W^{[p,x,l^{-\sigma},c]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$ for $n \geqslant \sigma > (n-1)/p$. This is also the optimal result we can obtain because for any $\sigma > 0$ and any essentially bounded function $F(\cdot)$ we have $F \in e - W^{[p,x,l^{-\sigma},c]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n : \mathbb{C})$.

Regarding the convolution invariance of spaces introduced in this section, we will clarify just one result for the class $(e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$; the proof is omitted.

Theorem 7.3.31. *Suppose that $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying that there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x, y \geqslant 0$. Suppose, further, that $h \in L^1(\mathbb{R}^n)$, $\Omega = [0,1]^n$, $F \in (e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},c]}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$, and for each $x \in X$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n}\|F(\mathbf{t}; x)\|_Y < \infty$. If $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$, $p_1 \in \mathcal{P}(\mathbb{R}^n)$ and if, for every $\mathbf{t} \in \mathbb{R}^n$ and $l > 0$, there exists a sequence $(a_k)_{k\in l\mathbb{Z}^n}$ of positive real numbers such that*

$\sum_{k\in l\mathbb{Z}^n} a_k = 1$ *and*

$$\int_\Omega \varphi_{p_1(\mathbf{u})}\Big(2\sum_{k\in l\mathbb{Z}^n} a_k l^{-n}[\varphi(a_k^{-1}l^n h(k-l\mathbf{v}))]_{L^{q(\mathbf{v})}(\Omega)}\mathbb{F}_1(l,\mathbf{t})\big[\mathbb{F}(l,\mathbf{t}+l\mathbf{u}-k)\big]^{-1}\Big)\,d\mathbf{u} \leqslant 1,$$

then $h * F \in (e-)W^{[p_1(\mathbf{u}),\phi,\mathbb{F}_1,c]}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$.

If $p \in [1,\infty)$, then any Stepanov (p,c)-quasi-asymptotically almost periodic function is Weyl (p,c)-almost periodic, which also holds for the corresponding classes of uniformly recurrent functions. The generalized Weyl uniform recurrence in Lebesgue spaces with variable exponents has been already analyzed in the one-dimensional setting and here we will only mention the following multi-dimensional notion here: Let (WM2) hold. Then we say that a function $F : \Lambda \times X \to Y$ is Weyl $[\Omega,\mathcal{B},\Lambda',p,\phi,\mathbf{F},c]$-uniformly recurrent if and only if for each set $B \in \mathcal{B}$ we can find a sequence (τ_k) in Λ' such that $\lim_{k\to+\infty}|\tau_k| = +\infty$ as well as that

$$\lim_{k\to+\infty}\limsup_{l\to+\infty}\sup_{\mathbf{t}\in\Lambda;x\in B}\big[\mathbf{F}(l,\mathbf{t})\phi\big(\|F(\cdot l+\mathbf{t}+\tau_k;x)-cF(\cdot l+\mathbf{t};x)\|_Y\big)_{L^{p(\cdot)}(\Omega)}\big] = 0.$$

The above-mentioned result on the set-theoretical embedding of space of Stepanov (p,c)-quasi-asymptotically almost periodic functions into the space of Weyl (p,c)-almost periodic functions can be generalized in many different directions; in [658, Proposition 6], e. g., we have shown that any Stepanov (p,ϕ,F)-quasi-asymptotically uniformly recurrent function is Weyl $(p(x),\phi,F_1)$-uniformly recurrent under certain assumptions. This result can be formulated in the multi-dimensional setting but we will consider here only the constant coefficient case $p(\cdot) \equiv p \in [1,\infty)$ for brevity.

Proposition 7.3.32. *Suppose that* $(MD-B)_S$ *holds and a function* $F : \Lambda \times X \to Y$ *is Stepanov* $[\Omega,\mathcal{B},\Lambda',\Lambda,p,\phi,\mathbf{F},c]$*-quasi-asymptotically uniformly recurrent. If* $\mathbf{F}_1 : (0,\infty)\times\Lambda \to (0,\infty)$ *satisfies*

$$\lim_{k\to+\infty}\limsup_{l\to+\infty}\sup_{\mathbf{t}\in\Lambda}\mathbf{F}_1(l,\mathbf{t})\sum_{a\in\mathbb{Z}^n\cap[0,l]^n}\frac{1}{F(\mathbf{t}+a,k)} < \infty$$

and

$$\lim_{l\to+\infty}\sup_{\mathbf{t}\in\Lambda}\mathbf{F}_1(l,\mathbf{t}) = 0,$$

then the function $F(\cdot;\cdot)$ *is Weyl* $[\Omega,\mathcal{B},\Lambda',p,\phi,\mathbf{F},c]$*-uniformly recurrent.*

Now we will present some applications of our theoretical results to the abstract Volterra integro-differential equations.

1. We start by noting that all established applications made in the application part of Section 6.3 including applications to the d'Alembert formula, the Gaussian semigroups in $\mathbb{R}^n$ and the nonautonomous differential equations of the first or-

der, can be straightforwardly formulated for the corresponding classes of multi-dimensional (equi-)Weyl c-almost periodic type functions considered in this section. In this part, we will present the following illustrative application of Theorem 7.3.24, only: Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1 \leqslant p < \infty$. Suppose that $t_0 > 0$ is a fixed real number, $\Omega = [0,1]^n$, $\mathbb{D} = \Lambda = \mathbb{R}^n$ and the function $F : \mathbb{R}^n \to \mathbb{C}$ is Stepanov $[\Omega, \Lambda', \mathbb{R}^n, p, x, \mathsf{F}, c]$-quasi-asymptotically almost periodic, resp. Stepanov $[\Omega, \Lambda', \mathbb{R}^n, p, x, \mathbf{F}, c]$-quasi-asymptotically uniformly recurrent. Then the function $x \mapsto (G(t_0)F)(x)$, $x \in \mathbb{R}^n$, where $(G(t))_{t \geqslant 0}$ is the Gaussian semigroup, is Stepanov $[\Omega, \Lambda', \mathbb{R}^n, p, x, \mathsf{F}_1, c]$-quasi-asymptotically almost periodic, resp. Stepanov $[\Omega, \Lambda', \mathbb{R}^n, p, x, \mathbf{F}_1, c]$-quasi-asymptotically uniformly recurrent provided that there exists a continuous function $g : [0, \infty) \to [0, \infty)$ with $g(0+) = 0+$ such that for each $\varepsilon > 0$ and $\tau \in \Lambda'$, resp. for each $n \in \mathbb{N}$ and $\tau \in \lambda'$, there exists $M(\varepsilon, \tau) > 0$, resp. $M(n, \tau) > 0$, such that for each $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \geqslant M(\varepsilon, \tau)$, resp. $|\mathbf{t}| \geqslant M(n, \tau)$, we have

$$\int\limits_{[0,1]^n} \Bigg[\mathsf{F}_1(\mathbf{t}, \varepsilon, \tau) \Bigg(\sum_{k \in \mathbb{Z}^n} \frac{e^{-|\mathbf{t}-k|^2}}{\mathsf{F}(\mathbf{u} + k, \varepsilon, \tau)} + g(\varepsilon) \Bigg) \Bigg]^p \, d\mathbf{u} \leqslant 1,$$

resp.

$$\int\limits_{[0,1]^n} \Bigg[\mathbf{F}_1(\mathbf{t}, n) \Bigg(\sum_{k \in \mathbb{Z}^n} \frac{e^{-|\mathbf{t}-k|^2}}{\mathbf{F}(\mathbf{u} + k, n)} + g(1/n) \Bigg) \Bigg]^p \, d\mathbf{u} \leqslant 1.$$

However, this is a pure theoretical condition which cannot be so simply verified in some practical situations; see also Theorem 7.3.6 and Theorem 7.3.19 which can be also applied here.

2. Concerning the regular solutions of the inhomogeneous wave equations given by the d'Alembert formula, we would like to note that the analysis carried out in the corresponding issues of section concerning multi-dimensional (ω, c)-periodic type functions can be also used to justify the introduction of the notion in Definition 7.3.1 and Definition 7.3.2. More precisely, suppose that $\omega \in \mathbb{R} \smallsetminus \{0\}$, $k \in \mathbb{N}$ and $c \in \mathbb{C} \smallsetminus \{0\}$ satisfies $c^{k-1} = 1$. Recall that the regular solution of the wave equation $u_{tt} = a^2 u_{xx}$ in domain $\{(x,t) : x \in \mathbb{R},\ t > 0\}$, equipped with the initial conditions $u(x,0) = f(x) \in C^2(\mathbb{R})$ and $u_t(x,0) = g(x) \in C^1(\mathbb{R})$, is given by (3.65). Suppose that $\mathbb{D}$ is any unbounded set in the plane $\mathbb{R}^2$ such that ($g^{[1]}(\cdot) \equiv \int_0^{\cdot} g(s)\, ds$):

$$\begin{aligned} \lim_{|(x,t)| \to +\infty, (x,t) \in \mathbb{D}} \Bigg[& \big| f(x - at + \omega) - cf(x - at) \big| + \big| g^{[1]}(x - at + \omega) - cg^{[1]}(x - at) \big| \\ & + \sum_{j=1}^{k} \big(\big| f(x + at + j\omega) - cf(x + at + (j-1)\omega) \big| \\ & + \big| g^{[1]}(x + at + j\omega) - cg^{[1]}(x + at + (j-1)\omega) \big| \big) \Bigg] = 0, \end{aligned}$$

and that

$$\omega_1 := \frac{1+k}{2}\omega \quad \text{and} \quad \omega_2 := \frac{k-1}{2a}\omega.$$

Then $(\omega_1, \omega_2) \neq (0,0)$, $\omega_1 - a\omega_2 = \omega$, $\omega_1 + a\omega_2 = k\omega$, $c^k = c$ and a simple use of the estimate

$$\left|f(x+k\omega) - c^k f(x)\right| \leqslant \sum_{j=1}^{k} \left|f(x+j\omega) - cf(x+(j-1)\omega)\right|, \quad x \in \mathbb{R}$$

shows that the function $(x,t) \mapsto u(x,t)$, $(x,t) \in \mathbb{R}^2$ is $(S, \mathbb{D})$-asymptotically (ω, c)-periodic. In the particular case $a = 1$ and $\mathbb{D} := \{(x,t) \in \mathbb{R}^2 : x \geqslant 0,\ t \geqslant 0,\ x \geqslant t^2+1\}$, e. g., it suffices to assume that the restrictions of the functions $f(\cdot)$ and $g^{[1]}(\cdot)$ to the interval $[0,\infty)$ are S-asymptotically (ω, c)-periodic.

3. We will reconsider here the semilinear Hammerstein integral equation of convolution type on $\mathbb{R}^n$. By the foregoing, we know that the space $SP(\mathbb{R}^n : X)$ of all semi-periodic functions $F : \mathbb{R}^n \to X$ is convolution invariant (it is not a Banach space but only a complete metric space). Under certain assumptions, we are able to show that the following semilinear Hammerstein integral equation

$$y(\mathbf{t}) = \int\limits_{\mathbb{R}^n} k(\mathbf{t}-\mathbf{s}) G(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

where $G : \mathbb{R}^n \times X \to X$ is semi-$(c_j, \mathcal{B})_{j \in \mathbb{N}_n}$-periodic with $\mathcal{B}$ being the collection of all bounded subsets of X and $c_j = 1$ for all $j \in \mathbb{N}_n$, has a unique semi-periodic solution. Let us assume that there exists a finite real constant $L > 0$ such that

$$\|G(\mathbf{t}; y) - G(\mathbf{t}; y')\|_X \leqslant L\|y - y'\|_X, \quad \mathbf{t} \in \mathbb{R}^n,\ y \in X,\ y' \in X.$$

It can be simply shown that for any semi-periodic function $y : \mathbb{R}^n \to X$ we see that the mapping $\mathbf{t} \mapsto G(\mathbf{t}; y(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$ is semi-periodic, as well. Since the space of semi-periodic functions in $\mathbb{R}^n$ is convolution invariant, it follows that the mapping

$$SP(\mathbb{R}^n : X) \ni y \mapsto \int\limits_{\mathbb{R}^n} k(\cdot - \mathbf{s}) G(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s} \in SP(\mathbb{R}^n : X)$$

is well defined. If we assume that $L \int_{\mathbb{R}^n} |k(\mathbf{t})|\, d\mathbf{t} < 1$, then the use of Banach contraction principle shows that there exists a unique solution of (7.13) which belongs to the space $SP(\mathbb{R}^n : X)$.

8 Multi-dimensional almost automorphic type functions and applications

This chapter consists of three sections, Section 8.1–Section 8.3.

8.1 Multi-dimensional almost automorphic functions and applications

In 1955, S. Bochner discovered the concept of almost automorphy while he was studying problems related to differential geometry [188]; after that, it was proved that the almost automorphy is a generalization of the almost periodicity (see [189–191] and the references therein). Starting presumably with the papers of W. A. Veech [993, 994], many authors have deeply investigated this concept on various classes of (semi-)topological groups.

Suppose that $F : \mathbb{R}^n \to X$ is continuous. Then it is said that $F(\cdot)$ is almost automorphic if and only if for every sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$ there exist a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$ and a map $G : \mathbb{R}^n \to X$ such that

$$\lim_{k\to\infty} F(\mathbf{t} + \mathbf{a}_k) = G(\mathbf{t}) \quad \text{and} \quad \lim_{k\to\infty} G(\mathbf{t} - a_k) = F(\mathbf{t}).$$

The strong motivational factor for genesis of article [267], from which the matrial of this section is taken, presents the fact that almost nothing has been said by now about the space almost automorphic solutions to the (abstract) Volterra integro-differential equations. In support of our investigations of multi-dimensional almost automorphic type functions, we also want to note that we have not been able to find any relevant reference in the existing literature which throws light on some striking peculiarities of almost automorphic functions in $\mathbb{R}^n$ different from those already known for the almost automorphic functions on general topological groups.

The almost automorphic solutions with respect to the time variable for various classes of the (abstract) Volterra integro-differential equations have been intensively sought in numerous research studies (see, e. g., [231, 240, 261] and the references therein). Let us recall here that some almost periodic systems do not necessarily carry almost periodic dynamics (see, e. g., [819, 927]), while such systems may have bounded oscillating solutions which belong to a broader class of almost automorphic functions (see also the research article of R. A. Johnson [567], who proved the existence of a linear almost periodic system of ordinary differential equations which admits an almost automorphic solution but no almost periodic solution).

Let us recall that the solutions to nonautonomous evolution differential equations satisfy certain integral equations in which the integral kernels are expressed by means of two-parameter evolution families $(U(t,s))_{t\geqslant s\geqslant 0}$. In the case of nonautonomous evolution differential equations with almost automorphic dynamics, the

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notion of bi-almost automorphy of the evolution operator $(U(t,s))_{t\geqslant s\geqslant 0}$ is essential in the research studies of the existence and uniqueness of almost automorphic mild solutions. The notion of a (positively) bi-almost automorphic function was introduced by T. J. Xiao et al. in [1041, (2009)]; in this paper, the authors have obtained some sufficient conditions for the existence of pseudo-almost automorphic mild solutions of the following equations in $\mathbb{R}$:

$$x'(t) = A(t)x(t) + f(t, x(t)),$$
$$x'(t) = A(t)x(t) + f(t, x(t-h)),$$
$$x'(t) = A(t)x(t) + f(t, x(\alpha(t, x(t)))).$$

Three years later, Z. Chen and W. Lin employed this notion in their investigation of nonautonomous stochastic evolution equations [281]; see also [263, 264] and [364, Appendix A.3], where the authors have analyzed the notion of bi-almost automorphic sequences. More precisely, in [281], the authors have introduced the notion of a square-mean bi-almost automorphic function for a stochastic processes and analyzed the existence of square-mean almost automorphic solutions of the following non-autonomous linear stochastic evolution equation:

$$dx(t) = A(t)x(t)dt + f(t)dt + \gamma(t)\, dW(t),$$

with f, γ being stochastic processes and W being a two-sided standard one-dimensional Brownian motion. In [263, 264], the authors have analyzed the notion of discrete bi-almost automorphy and prove several results concerning the non-autonomous difference equations appearing in the dynamics of the following hybrid system of differential equations:

$$x'(t) = A(t)x(t) + B(t)x(\lfloor t\rfloor) + f(t, x(t), x(\lfloor t\rfloor)).$$

We also mention that, in [268], the authors have used the notion of bi-almost automorphy and the notion of λ-boundedness in their studies of the following nonlinear abstract integral equations of advanced and delayed type:

$$y(t) = f(t, y(t), y(a_0(t))) + \int_{-\infty}^{t} C_1(t, s, y(s), y(a_1(s)))\, ds + \int_{t}^{+\infty} C_2(t, s, y(s), y(a_2(s)))\, ds.$$

Besides the above-mentioned papers, we would like to quote the research studies [261] by Y.-K. Chang, S. Zheng, [542] by Z. Hu, Z. Jin, [862] by L. Qi and R. Yuan, [1034] by Z. Xia and [1037] by Z. Xia, D. Wang. Observing the previous works (and the references cited therein), we emphasize that the notion of bi-almost automorphy is crucial in the study of almost automorphic dynamics for various classes of differential, integro-differential and difference equations.

The notion of $\mathbb{Z}$-almost automorphy and the notion of bi-almost automorphy, which have been analyzed in the above-mentioned papers, are special cases of the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy, which is a crucial object of our investigations (for example, the notion of bi-almost automorphy is obtained with the collection R of all sequences in $\Delta_2 \equiv \{(w, w) : w \in \mathbb{R}\}$, the diagonal of $\mathbb{R}^2$). Furthermore, the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy is a special case of the notion of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy, which has been introduced and analyzed in this section following the previous investigations of almost automorphic functions on (semi-)topological groups. We aim to develop here the basic theory of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic type functions as well as to provide some concrete applications to the abstract Volterra integro-differential equations and partial differential equations such as the classical heat equation and the wave equation. It is our strong belief that the research study [267] is only the beginning of serious investigations of space almost automorphic solutions of integro-differential equations.

The organization of the present section is as follows. We introduce the classes of (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions (Definition 8.1.1), $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions and $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions (Definition 8.1.2); here, we assume that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k) \in \mathrm{R}$ we have $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. A real novelty of the introduced class of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions is marked in Example 8.1.5 because we present here an example of an $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function $F : \mathbb{R}^2 \to X$ (R is the collection of all sequences in Δ_2 and $\mathcal{B}$ denotes the collection of all bounded subsets of X) in which the convergence of limits in Eqs. (8.1)–(8.2) below is uniform not on the whole space (the almost periodic case) and not only on compact subsets of $\mathbb{R}^n$ (the compact almost automorphic case); this example is important for a better understanding of the notion $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy we are working with.

After illustrating this notion with some other examples, we introduce the notions of $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost automorphy, $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy and $(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy in Definition 8.1.8. In Proposition 8.1.9, we investigate the relative compactness of range of a two-parameter $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost automorphic function $F : \mathbb{R}^n \times X \to Y$. After that, we divide the remainder of the second section into several separate subsections. The main aim of Subsection 8.1.1 is to thoroughly study the compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions; in Subsection 8.1.2, we continue our study by clarifying several new structural characterizations of $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost automorphic type functions. Subsection 8.1.3 investigates $\mathbb{D}$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions; composition theorems for $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions are analyzed in Subsection 8.1.4, while the invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic properties under the actions of convolution products are analyzed in Subsection 8.1.5. Subsection 8.1.6 is reserved for applications of our theoretical results to the various classes of abstract

Volterra integro-differential equations (it should be noted that we revisit here the theory of integrated solution operator families, C-regularized solution operator families and their applications to the abstract ill-posed Cauchy problems). We analyze almost automorphic solutions to the abstract semilinear Volterra integral equations and some applications to the classical heat equation and the wave equation. We do not cover many important subjects; for example, we will not consider here the notion of a positively $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy and its generalizations [1041].

Before we go any further, we would like to note the conclusions established in Example 2 can reformulated in our new framework, which presents a strong motivational factor for the investigation of multi-dimensional almost automorphic type functions, as well; see [267] for more details.

In this section, we investigate almost automorphic analogues of $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions and $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic functions. We start with the following definition, which seems to be new even in the one-dimensional setting.

Definition 8.1.1. Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{l \to +\infty} F(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x) = F^*(\mathbf{t}; x) \tag{8.1}$$

and

$$\lim_{l \to +\infty} F^*(\mathbf{t} - (b_{k_l}^1, \ldots, b_{k_l}^n); x) = F(\mathbf{t}; x), \tag{8.2}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. If for each $x \in B$ the above limits converge uniformly on compact subsets of $\mathbb{R}^n$, then we say that $F(\cdot;\cdot)$ is compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. By $AA_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n \times X : Y)$ and $AA_{(\mathrm{R},\mathcal{B},\mathbf{c})}(\mathbb{R}^n \times X : Y)$ we denote the spaces consisting of all $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions and compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, respectively.

In the case that $X = \{0\}$ and $\mathcal{B} = \{X\}$, i. e., if we consider the function $F : \mathbb{R}^n \to Y$, then we also say that $F(\cdot)$ is (compactly) R-multi-almost automorphic and denote the corresponding spaces by $AA_{\mathrm{R}}(\mathbb{R}^n : Y)$ and $AA_{\mathrm{R},\mathbf{c}}(\mathbb{R}^n : Y)$ [in the remainder of section, we will tacitly omit the term "$\mathcal{B}$" from the notation in such situations].

The following definition also seems to be new in the one-dimensional setting.

Definition 8.1.2. Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function as well as that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ we have $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then we say that $F(\cdot;\cdot)$ is:

(i) $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (8.1)–(8.2) hold pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ and such that for each $x \in B$ the convergence in $\mathbf{t}$ is uniform for any element of the collection $W_{B,(\mathbf{b}_k)}(x)$;

(ii) $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (8.1)–(8.2) hold pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ and such that the convergence in (8.1)–(8.2) is uniform in $(\mathbf{t}; x)$ for any set of the collection $\mathrm{P}_{B,(\mathbf{b}_k)}$.

Before we go any further, we would like to present the following illustrative example of the notion introduced above.

Example 8.1.3 (R. Terras [976], P. Milnes [772]). Let us write the set $\mathbb{R}$ as the disjoint union of intervals $\bigcup_{k=1}^{\infty} V_k$, where $V_k := \bigcup_{m\in\mathbb{Z}}([0,1) + s_k + 2^k m)$ and $s_k := ((-2)^{k-1} - 1)/3$ for all $k \in \mathbb{N}$. After that, we define a continuous function $f : \mathbb{R} \to \mathbb{R}$ through $f(t) := \sin(2^k \pi t)$ if $t \in V_k$ for some $k \in \mathbb{N}$. We know that the function $f(\cdot)$ is almost automorphic as well as that the sequence of translations $(f(\cdot + s_k))_{k\in\mathbb{N}}$ does not converge uniformly on the set $[0,1]$, so that $f(\cdot)$ is not uniformly continuous and not compactly almost automorphic. If $f_1(\cdot), \ldots, f_{n-1}(\cdot)$ are almost automorphic complex-valued functions, then we set $F(t_1, \ldots, t_{n-1}, t_n) := f_1(t_1)\cdots f_{n-1}(t_{n-1}) f(t_n)$, $\mathbf{t} = (t_1, \ldots, t_{n-1}, t_n) \in \mathbb{R}^n$. It can be easily shown that $F(\cdot)$ is an almost automorphic function which is not compactly almost automorphic, as well as that $F(\cdot)$ cannot be $(\mathrm{R}, W_{\mathrm{R}})$-multi-almost automorphic for any collection of sequences in $\mathbb{R}^n$ which contains the sequence $(\mathbf{b}_k = (0, \ldots, 0; s_k))_{k\in\mathbb{N}}$ and for any collection $W_{\mathbf{b}_k}$ of subsets of $\mathbb{R}^n$ which contains the set $S \times [0,1]$, where $S = (t_1^0, \ldots, t_n^0) \in \mathbb{R}^{n-1}$ and $f_1(t_1^0)\cdots f_{n-1}(t_{n-1}^0) \neq 0$.

The notion in which R is not the collection of all sequences in $\mathbb{R}^n$ is far from being comparable with the usual almost automorphy (see, e. g., Proposition 9.0.26 below). In several important research studies of spatially almost periodic solutions of (abstract) Volterra integro-differential equations, the Bochner criterion is essentially employed with the collection R of all sequences in $\mathbb{R}^n$; here we would like to emphasize, without going into full details, that some established results concerning this problematic can be further extended by allowing that R is an arbitrary collection of sequences (in $\mathbb{R}^n$) in their formulations.

Example 8.1.4. It is well known that the Euler equations in $\mathbb{R}^n$, where $n \geqslant 2$, describe the motion of perfect incompressible fluids. It is problem to find the unknown functions $u = u(x,t) = (u^1(x,t), \ldots, u^n(x,t))$ and $p = p(x,t)$ denoting the velocity field and the pressure of the fluid, respectively, such that

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\
&\operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\
&u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n,
\end{aligned} \tag{8.3}$$

where $u_0 = u_0(x) = (u_0^1(x), \dots, u_0^n(x))$ denotes the given initial velocity field. There are many results concerning the well-posedness of (8.3) in the case that the initial velocity field $u_0(x)$ belongs to some direct product of (fractional) Sobolev spaces. For our observation, it is crucial to remind the reader of the research article [823] by H. C. Pak and Y. J. Park, who investigated the well-posedness of (8.3) in the case that the initial velocity field $u_0(x)$ belongs to the space $B^1_{\infty,1}(\mathbb{R}^n)^n$, where $B^1_{\infty,1}(\mathbb{R}^n)$ denotes the usual Besov space (see, e. g., [908, Definition 2.1]). The authors have proved that for any function $u_0 \in B^1_{\infty,1}(\mathbb{R}^n)^n$ such that div $u_0 = 0$ there exists a finite real number $T > 0$ such that there exists a solution $u \in C([0,T] : B^1_{\infty,1}(\mathbb{R}^n)^n)$ of (8.3). Using some known results proved by H. C. Pak, Y. J. Park and the fact that a function $f \in B^0_{\infty,1}(\mathbb{R}^n)$ is almost periodic in $\mathbb{R}^n$ if and only if the set of all its translations is relatively compact in $B^0_{\infty,1}(\mathbb{R}^n)$ (see [908, Lemma 4.2]), O. Sawada and R. Takada have proved, in [908, Theorem 1.5], that the assumption that $u_0(x)$ is almost periodic in $\mathbb{R}^n$ implies that the solution $u(\cdot,t)$ of (8.3) is almost periodic in $\mathbb{R}^n$ for all $t \in [0,T]$. Let R denote an arbitrary collection of sequences in $\mathbb{R}^n$, and let $u_0(\cdot)$ have the property that for each sequence $(\mathbf{b}_k)$ in R there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that the sequence of translations $(u_0(\cdot + \mathbf{b}_{k_l}))$ is convergent in the space $B^0_{\infty,1}(\mathbb{R}^n)^n$. Then for each sequence $(\mathbf{b}_k)$ in R there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that, for every $t \in [0,T]$, the sequence of translations $(u(\cdot + \mathbf{b}_{k_l}, t))$ is convergent in the space $B^0_{\infty,1}(\mathbb{R}^n)^n$; let us only note that the assumptions on function $u_0(\cdot)$ used here can serve one to introduce a new notion of multi-dimensional R-almost automorphy which is not so simply connected, in the general case, with the notion introduced in Definition 8.1.1 and Definition 8.1.2 (more details will appear elsewhere). See also the research studies [471] by Y. Giga, A. Mahalov, B. Nicolaenko, [698] by C. Li and [969] by Y. Taniuchi, T. Tashiro, T. Yoneda for further information concerning spatially almost periodic solutions of (abstract) Volterra integro-differential equations.

In what follows, we will provide several elaborate examples illustrating the concept of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy and the concepts introduced in Definition 8.1.2.

Example 8.1.5. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a (compactly) almost automorphic function, and let $(T(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ be a strongly continuous operator family. Suppose first that R is the collection of all sequences in Δ_2 as well as that $X \in \mathcal{B}$. Define a function $G : \mathbb{R}^2 \times X \to Y$ by

$$F(t,s;x) := e^{\int_s^t \varphi(\tau)\,d\tau} T(t-s)x, \quad (t,s) \in \mathbb{R}^2,\ x \in X. \tag{8.4}$$

The function $F(\cdot,\cdot;\cdot)$ is (compactly) bi-almost automorphic, which can be simply shown (see also [281, Example 7.1] and [1041, Example 4.1]).

Suppose now that $\varphi : \mathbb{R} \to \mathbb{C}$ is almost periodic and that R is the collection of all sequences in Δ_2 and $\mathcal{B}$ denotes the collection of all bounded subsets of X. Let for each bounded subset B of X and for each sequence $(\mathbf{b}_k = (b_k, b_k))$ in R the collection

$P_{B,(\mathbf{b}_k)}$ be constituted of all sets of form $\{(t,s) \in \mathbb{R}^2 : |t-s| \leqslant L\} \times B$, where $L > 0$. Then the function $F(\cdot,\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, which can be deduced as follows. Let a real number $L > 0$ and a bounded subset B of X be fixed, and let $(t,s) \in \mathbb{R}^2$ satisfy $|t-s| \leqslant L$. By Bochner's criterion, there exist a subsequence (b_{k_l}, b_{k_l}) of (b_k, b_k) and a function $\varphi^* : \mathbb{R} \to \mathbb{C}$ such that $\lim_{l\to+\infty} \varphi(r + b_{k_l}) = \varphi^*(r)$, uniformly in $r \in \mathbb{R}$. Set

$$F^*(t,s;x) := e^{\int_s^t \varphi^*(\tau)\,d\tau} T(t-s)x, \quad (t,s) \in \mathbb{R}^2,\ x \in X.$$

Then the function $\varphi^*(\cdot)$ is bounded and there exists a finite real constant $c_{L,B} > 0$ such that

$$\begin{aligned}
&\big\|e^{\int_{s+b_{k_l}}^{t+b_{k_l}} \varphi(\tau)\,d\tau} T(t-s)x - e^{\int_s^t \varphi^*(\tau)\,d\tau} T(t-s)x\big\|_Y \\
&\leqslant c_{L,B}\big|e^{\int_{s+b_{k_l}}^{t+b_{k_l}} \varphi(\tau)\,d\tau} - e^{\int_s^t \varphi^*(\tau)\,d\tau}\big| \leqslant c_{L,B}\big|e^{\int_s^t \varphi(\tau+b_{k_l})\,d\tau} - e^{\int_s^t \varphi^*(\tau)\,d\tau}\big| \\
&\leqslant c_{L,B} e^{L\|\varphi^*\|_\infty}\big|e^{\int_s^t [\varphi(\tau+b_{k_l})-\varphi^*(\tau)]\,d\tau} - 1\big| \\
&\leqslant c_{L,B} e^{L\|\varphi^*\|_\infty}\left|\int_s^t \big[\varphi(\tau+b_{k_l}) - \varphi^*(\tau)\big]\,d\tau\right| e^{|\int_s^t [\varphi(\tau+b_{k_l})-\varphi^*(\tau)]\,d\tau|} \\
&\leqslant c_{L,B} e^{L\|\varphi^*\|_\infty} L\varepsilon e^{L\varepsilon}, \quad l \geqslant l_0(\varepsilon),
\end{aligned}$$

which simply implies the required (we have used the well-known inequality $|e^z - 1| \leqslant |z| \cdot e^{|z|}$, $z \in \mathbb{C}$ here). A large class of relatively simple examples shows that the function $F(\cdot,\cdot;\cdot)$ is not $(\mathrm{R}, \mathcal{B})$-multi-almost periodic in general (let us only note here that the obtained conclusions can be simply applied to some partial differential equations in the distributional spaces as well as that it would be very difficult to aggregate all such applications; put e. g. $\varphi \equiv 0$ in (8.4)).

We can simply construct the corresponding analogue of this example in the higher dimensions $n > 2$; for example, if $\varphi_j : \mathbb{R} \to \mathbb{R}$ is (compactly) almost automorphic or almost periodic and $(T(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ is a strongly continuous operator family $(1 \leqslant j \leqslant n-1)$, resp., if $\varphi_j : \mathbb{R} \to \mathbb{R}$ is (compactly) almost automorphic or almost periodic and $(T(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ is a strongly continuous operator family $(1 \leqslant j \leqslant n)$, then the similar conclusions hold for the function $F : \mathbb{R}^n \times X \to X$ defined through

$$F(t_1,t_2,\ldots,t_n;x) := \sum_{j=1}^{n-1} T_j(t_{j+1}-t_j) e^{\int_{t_j}^{t_{j+1}} \varphi_j(\xi)\,d\xi} x, \quad (t_1,t_2,\ldots,t_n) \in \mathbb{R}^n,\ x \in X,$$

with $\mathrm{R} := \{b : \mathbb{N} \to \mathbb{R}^n;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a,a,a,\ldots,a) \in \mathbb{R}^n : a \in \mathbb{R}\}\}$, resp., for the function

$$F(t_1,t_2,\ldots,t_{2n};x) := \sum_{j=1}^{n} T_j(t_{2j}-t_{2j-1}) e^{\int_{t_{2j-1}}^{t_{2j}} \varphi_j(\xi)\,d\xi} x, \quad (t_1,t_2,\ldots,t_{2n}) \in \mathbb{R}^{2n},\ x \in X,$$

with $\mathrm{R} := \{b : \mathbb{N} \to \mathbb{R}^n;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a_1, a_1, a_2, a_2, \ldots, a_n, a_n) \in \mathbb{R}^{2n} : a_i \in \mathbb{R}\}\}$.

Example 8.1.6. Let $f_j : \mathbb{R} \to \mathbb{R}$ be a (compactly) almost automorphic function ($1 \leqslant j \leqslant n$). The function $F : \mathbb{R}^{2n} \to \mathbb{R}$, defined by

$$(s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n) \mapsto F(s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n) := \prod_{j=1}^{n} \int_{s_j}^{t_j} f_j(\xi)\, d\xi,$$

is (compactly) R-multi-almost automorphic, where $\mathrm{R} := \{b : \mathbb{N} \to \mathbb{R}^n \times \mathbb{R}^n;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \times \mathbb{R}^n : a_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n\}\}$. We have already analyzed case in which the functions $t \mapsto \int_0^t f_j(s)\, ds$, $t \in \mathbb{R}$ are almost periodic ($1 \leqslant j \leqslant n$); if we assume that the functions $t \mapsto f_j(t)$, $t \in \mathbb{R}$ are almost periodic ($1 \leqslant j \leqslant n$), then we can simply prove that the function $F(\cdot)$ will be $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-multi-almost automorphic, where for each sequence $b \in \mathrm{R}$ the collection P_{R} consists of all sets of the form $\{(s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n) \in \mathbb{R}^{2n} : |s_i - t_i| \leqslant L_i$ for all $i \in \mathbb{N}_n\}$ with $L_i > 0$ for all $i \in \mathbb{N}_n$.

Example 8.1.7. This example substantially generalizes the previous one. Let R be any collection of sequences in $\mathbb{R}^n$ such that each subsequence of a sequence $(\mathbf{b}_k) \in \mathrm{R}$ also belongs to R, and let R' be any collection of sequences in $\mathbb{R}^m$ such that each subsequence of a sequence $(\mathbf{b}'_k) \in \mathrm{R}'$ also belongs to R'. Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a bounded, (compactly) R-almost automorphic function ($1 \leqslant i \leqslant p$), and let $g_j : \mathbb{R}^m \to \mathbb{R}$ be a bounded, (compactly) R'-almost automorphic function ($1 \leqslant j \leqslant q$). Define the functions $F : \mathbb{R}^n \to \mathbb{R}^q$ by $F(\mathbf{t}) := \sum_{i=1}^{p} f_i(\mathbf{t}) e_i$ and $G : \mathbb{R}^m \to \mathbb{R}^q$ by $G(\mathbf{s}) := \sum_{j=1}^{q} g_j(\mathbf{s}) e_j$. Now, we define the function $F \otimes G : \mathbb{R}^n \times \mathbb{R}^m \to M_{p\times q}(\mathbb{R})$ by ($\mathbf{t} \in \mathbb{R}^n$, $\mathbf{s} \in \mathbb{R}^m$)

$$F \otimes G\,(\mathbf{t}, \mathbf{s}) := \begin{pmatrix} f_1(\mathbf{t})g_1(\mathbf{s}) & f_1(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_1(\mathbf{t})g_q(\mathbf{s}) \\ f_2(\mathbf{t})g_1(\mathbf{s}) & f_2(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_2(\mathbf{t})g_q(\mathbf{s}) \\ \vdots & \vdots & \ddots & \vdots \\ f_p(\mathbf{t})g_1(\mathbf{s}) & f_p(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_p(\mathbf{t})g_q(\mathbf{s}) \end{pmatrix}, \tag{8.5}$$

where $M_{p\times q}(\mathbb{R})$ denotes the set of all real matrices of format $p \times q$. It is not difficult to prove that $F \otimes G$ is (compactly) $(\mathrm{R} \times \mathrm{R}')$-almost automorphic, where $\mathrm{R} \times \mathrm{R}' := \{(\mathbf{b}, \mathbf{b}') : \mathbf{b} \in \mathrm{R}, \mathbf{b}' \in \mathrm{R}'\}$. Furthermore, if for each $i \in \mathbb{N}_p$ $f_i : \mathbb{R}^n \to \mathbb{R}$ is a bounded $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-almost automorphic function as well as that for each $j \in \mathbb{N}_q$ we see that $g_j : \mathbb{R}^n \to \mathbb{R}$ is a bounded $(\mathrm{R}', \mathrm{P}'_{\mathrm{R}'})$-almost automorphic function, then the function $F \otimes G$ is $(\mathrm{R} \times \mathrm{R}', \mathrm{P}''_{\mathrm{R}\times\mathrm{R}'})$-almost automorphic function, provided that for each sequence $\mathbf{b}$ from R ($\mathbf{c}$ from R') each set of the collection $\mathrm{P}_{\mathbf{b}}$ ($\mathrm{P}_{\mathbf{c}}$) belongs to the collection $\mathrm{P}_{\mathbf{b}'}$ ($\mathrm{P}_{\mathbf{c}'}$) for any subsequence $\mathbf{b}'$ of $\mathbf{b}$ ($\mathbf{c}'$ of $\mathbf{c}$) and for each sequence $(\mathbf{b}; \mathbf{c})$ belonging to $\mathrm{R} \times \mathrm{R}'$ the collection $\mathrm{P}''_{(\mathbf{b};\mathbf{c})}$ consists of all direct products of sets from the collections $\mathrm{P}_{\mathbf{b}}$ and $\mathrm{P}'_{\mathbf{c}}$.

From the viewpoint of the theory of differential equations with piecewise constant argument (see, e. g., the references quoted in [263, 264]), the continuity of the function $F(\cdot;\cdot)$ in Definition 8.1.1 is a slightly redundant condition; we will not go into further details with regard to this question here. Furthermore, the notion introduced in Definition 8.1.1 is a special case of the notion introduced in the following definition (with $\mathrm{R_X} := \{b : \mathbb{N} \to \mathbb{R}^n \times X; (\exists a \in \mathrm{R})\, b(l) = (a(l); 0) \text{ for all } l \in \mathbb{N}\}$); this is an extremely important notion because, in the case that $X \in \mathcal{B}$ and R_X denotes the collection of all sequences in $\mathbb{R}^n \times X$, the notion of $(\mathrm{R_X}, \mathcal{B})$-multi-almost automorphy is equivalent with the usual notion of almost automorphy on the topological group $\mathbb{R}^n \times X$.

Definition 8.1.8. Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R_X}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R_X}$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{m\to+\infty} F(\mathbf{t} + (b_{k_m}^1, \ldots, b_{k_m}^n); x + x_{k_m}) = F^*(\mathbf{t}; x) \tag{8.6}$$

and

$$\lim_{l\to+\infty} F^*(\mathbf{t} - (b_{k_l}^1, \ldots, b_{k_l}^n); x - x_{k_l}) = F(\mathbf{t}; x), \tag{8.7}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. We say that the function $F(\cdot;\cdot)$ is compactly $(\mathrm{R_X}, \mathcal{B})$-multi-almost automorphic if and only if the convergence of limits in (8.6)–(8.7) is uniform on any compact subset K of $\mathbb{R}^n \times X$ which belongs to $\mathbb{R}^n \times B$. By $AA_{(\mathrm{R_X},\mathcal{B})}(\mathbb{R}^n \times X : Y)$ and $AA_{(\mathrm{R_X},\mathcal{B},\mathbf{c})}(\mathbb{R}^n \times X : Y)$ we denote the spaces consisting of all $(\mathrm{R_X}, \mathcal{B})$-multi-almost automorphic functions and compactly $(\mathrm{R_X}, \mathcal{B})$-multi-almost automorphic functions, respectively.

Furthermore, let for each $B \in \mathcal{B}$ and $(\mathbf{b};\mathbf{x}) = ((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)_k) \in \mathrm{R_X}$ us have $W_{B,(\mathbf{b};\mathbf{x})} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$. Then the following notion generalizes the corresponding notion from Definition 8.1.2; we say that $F(\cdot;\cdot)$ is:

(i) $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)_k) \in \mathrm{R_X}$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (8.6)–(8.7) hold pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that for each $x \in B$ the convergence in (8.6)–(8.7) is uniform in $\mathbf{t}$ for any set of the collection $W_{B,(\mathbf{b};\mathbf{x})}(x)$;

(ii) $(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R_X}$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (8.6)–(8.7) hold pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that the convergence in (8.6)–(8.7) is uniform in $(\mathbf{t}; x)$ for any set of the collection $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})}$.

It is clear that the assumption $X \in \mathcal{B}$ implies that a continuous function $F : \mathbb{R}^n \times X \to Y$ is (compactly) $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic if and only if the above requirements hold for any sequence $((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_{\mathrm{X}}$ and the set $B = X$.

The following result holds true.

Proposition 8.1.9.

(i) *Suppose that $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function, where R denotes the collection of all sequences in $\mathbb{R}^n$ and $\mathcal{B}$ denotes any collection of compact subsets of X. If for every $B \in \mathcal{B}$ there exists a finite real constant $L_B > 0$ such that, for every $x, y \in B$ and $\mathbf{t} \in \mathbb{R}^n$, we have*

$$\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\|_Y \leqslant L_B \|x - y\|, \tag{8.8}$$

then, for every set $B \in \mathcal{B}$, we see that the set $\{F(\mathbf{t}, x) : \mathbf{t} \in \mathbb{R}^n, x \in B\}$ is relatively compact in Y.

(ii) *Suppose that $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic function, where R_X denotes the collection of all sequences in $\mathbb{R}^n \times X$ and $\mathcal{B}$ denotes any collection of compact subsets of X. Then, for every set $B \in \mathcal{B}$, we see that the set $\{F(\mathbf{t}, x) : \mathbf{t} \in \mathbb{R}^n, x \in B\}$ is relatively compact in Y.*

Proof. To prove (i), it suffices to show that, for every sequence $((\mathbf{t}_k; x_k))_{k \in \mathbb{N}}$ in $\mathbb{R}^n \times B$, there exists a subsequence $((\mathbf{t}_{k_l}; x_{k_l}))_{l \in \mathbb{N}}$ which converges for topology of Y. Since B is compact, we may assume without loss of generality that $x_k \to x$, $k \to +\infty$ for some element $x \in B$. Applying the definition of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy, we can find a subsequence $((\mathbf{t}_{k_l}; x_{k_l}))_{l \in \mathbb{N}}$ of $((\mathbf{t}_k; x_k))_{k \in \mathbb{N}}$ such that $F(0 + \mathbf{t}_{k_l}; x) = F(\mathbf{t}_{k_l}; x)$ converges to some element $y \in Y$ as $l \to +\infty$. Then the final conclusion follows from (8.8) and the decomposition

$$\begin{aligned} \|F(\mathbf{t}_{k_l}; x_{k_l}) - y\|_Y &\leqslant \|F(\mathbf{t}_{k_l}; x_{k_l}) - F(\mathbf{t}_{k_l}, x)\|_Y + \|F(\mathbf{t}_{k_l}; x) - y\|_Y \\ &\leqslant L_B \|x_{k_l} - x\| + \|F(\mathbf{t}_{k_l}; x) - y\|_Y. \end{aligned}$$

The proof of (ii) is similar but, in this part, we do not need any Lipschitz type condition because there exists a subsequence of the sequence $((\mathbf{t}_{k_l}; x_{k_l}))_{l \in \mathbb{N}} \in \mathrm{R}_X$ of $((\mathbf{t}_k; x_k))_{n \in \mathbb{N}}$ obeying the properties in the definition of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy. □

Before switching to Subsection 8.1.1, we would like to note that it is very simple to show that the assumption $X \in \mathcal{B}$ implies that a continuous function $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic if and only if for every sequence $((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_{\mathrm{X}}$ there exists a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$ such that

$$\lim_{l \to +\infty} \lim_{m \to +\infty} F(\mathbf{t} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x - x_{k_l} + x_{k_m}) = F(\mathbf{t}; x),$$

pointwise for all $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$; in the general case ($X \in \mathcal{B}$ or $X \notin \mathcal{B}$), the $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy of a continuous function $F : \mathbb{R}^n \times X \to Y$ is equivalent to saying that for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that

$$\lim_{l\to+\infty}\lim_{m\to+\infty} F(\mathbf{t}-\mathbf{b}_{k_l}+\mathbf{b}_{k_m};x)=F(\mathbf{t};x), \tag{8.9}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

8.1.1 Compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions

In this subsection, we analyze compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. The following result is crucial.

Theorem 8.1.10. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic function as well as that, for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k) \in \mathrm{R}_X$, there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (8.6)–(8.7) hold pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. Let for each $B \in \mathcal{B}$ and $(\mathbf{b};\mathbf{x}) \in \mathrm{R}_X$ we have $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$. Suppose also that the following conditions hold:*
(a) *if $(\mathbf{b};\mathbf{x}) \in \mathrm{R}_X$, then every subsequence of $(\mathbf{b};\mathbf{x})$ also belongs to R_X;*
(b) *if $B \in \mathcal{B}$, $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_X$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$, then $D \in P_{B,((\mathbf{b};\mathbf{x})_{k_l})}$ for every subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$.*

Then the following holds:
(i) *If $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, then the following statements are equivalent:*
  (c) *for every $B \in \mathcal{B}$ and $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_X$, the limit function $F^*(\cdot;\cdot)$ is uniformly continuous on any set D of the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$;*
  (d) *for every $\varepsilon > 0$, $B \in \mathcal{B}$, $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_X$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$, there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$, an integer $l_0 \in \mathbb{N}$ and a finite real number $\delta > 0$ such that, for every $(\mathbf{t};x)$, $(\mathbf{t}';x') \in D$ with $|\mathbf{t}-\mathbf{t}'|+\|x-x'\| \leqslant \delta$ and for every integer $l \geqslant l_0$, we have*

$$\left\|F(\mathbf{t}+b_{k_l};x+x_{k_l})-F(\mathbf{t}'+b_{k_l};x'+x_{k_l})\right\|_Y \leqslant \varepsilon. \tag{8.10}$$

  Moreover, (c) and (d) hold provided that condition (Q) holds, where:
  (Q) *For every $B \in \mathcal{B}$ and $(\mathbf{b};\mathbf{x}) \in \mathrm{R}_X$, we see that every set D of the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$ is compact in $\mathbb{R}^n \times X$.*

(ii) *If (Q) holds, then the validity of condition (d) and*
  $(d)_s$ *for every $\varepsilon > 0$, $B \in \mathcal{B}$, $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_X$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$, there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$, integers l_0, $m_0 \in \mathbb{N}$, and a finite real number $\delta > 0$ such that, for every $(\mathbf{t};x)$, $(\mathbf{t}';x') \in D$ with $|\mathbf{t}-\mathbf{t}'|+\|x-x'\| \leqslant \delta$ and for every integers $l \geqslant l_0$ and $m \geqslant m_0$, we have $x - x_{k_l} \in B$ and*

$$\left\|F(\mathbf{t}-b_{k_l}+b_{k_m};x-x_{k_l}+x_{k_m})-F(\mathbf{t}'-b_{k_l}+b_{k_m};x'-x_{k_l}+x_{k_m})\right\|_Y \leqslant \varepsilon$$

  implies that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic.

Proof. We will firstly prove that (d) implies (c). Let $\varepsilon > 0$, $B \in \mathcal{B}$, $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_\mathrm{X}$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$. Furthermore, let a subsequence $((\mathbf{b};\mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ be such that (8.6)–(8.7) hold pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. Then $((\mathbf{b};\mathbf{x})_{k_l})$ is a sequence which belongs to the collection R_X and $D \in P_{B,((\mathbf{b};\mathbf{x})_{k_l})}$ due to conditions (a) and (b). Since (d) holds, we may assume without loss of generality that there exist an integer $l_0 \in \mathbb{N}$ and a finite real number $\delta > 0$ such that, for every $(\mathbf{t};x),\ (\mathbf{t}';x') \in D$ with $|\mathbf{t} - \mathbf{t}'| + \|x - x'\| \leqslant \delta$ and for every integer $l \geqslant l_0$, we have (8.10) with the number ε replaced therein with the number $\varepsilon/3$. Since $F(\cdot;\cdot)$ is $(\mathrm{R}_\mathrm{X}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, (c) simply follows from the decomposition

$$\begin{aligned}
&\|F^*(\mathbf{t};x) - F^*(\mathbf{t}';x')\|_Y \\
&\quad\leqslant \|F^*(\mathbf{t};x) - F(\mathbf{t} + b_{k_l}; x + x_{k_l})\|_Y + \|F(\mathbf{t} + b_{k_l}; x + x_{k_l}) - F(\mathbf{t}' + b_{k_l}; x' + x_{k_l})\|_Y \\
&\qquad + \|F(\mathbf{t}' + b_{k_l}; x' + x_{k_l}) - F^*(\mathbf{t}';x')\|_Y \\
&\quad\leqslant 2\varepsilon/3 + \|F(\mathbf{t} + b_{k_l}; x + x_{k_l}) - F(\mathbf{t}' + b_{k_l}; x' + x_{k_l})\|_Y \leqslant \varepsilon, \quad l \geqslant l_0.
\end{aligned}$$

The proof of implication (c) $\Rightarrow$ (d) is similar and follows from the decomposition:

$$\begin{aligned}
&\|F(\mathbf{t} + b_{k_l}; x + x_{k_l}) - F(\mathbf{t}' + b_{k_l}; x' + x_{k_l})\|_Y \\
&\quad\leqslant \|F^*(\mathbf{t};x) - F(\mathbf{t} + b_{k_l}; x + x_{k_l})\|_Y + \|F^*(\mathbf{t};x) - F^*(\mathbf{t}';x')\|_Y \\
&\qquad + \|F^*(\mathbf{t}';x') - F(\mathbf{t}' + b_{k_l}; x' + x_{k_l})\|_Y.
\end{aligned}$$

Assume now that (Q) holds and $\varepsilon > 0$. Then, for every fixed set $B \in \mathcal{B}$ and for every sequence $(\mathbf{b};\mathbf{x}) \in \mathrm{R}_X$, we see that every set D of the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$ is compact. Furthermore, the above argumentation shows that there exists an integer $l_0 \in \mathbb{N}$ such that, for every $(\mathbf{t};x),\ (\mathbf{t}';x') \in D$, we have

$$\|F^*(\mathbf{t};x) - F^*(\mathbf{t}';x')\|_Y \leqslant 2\varepsilon/3 + \|F(\mathbf{t} + b_{k_{l_0}}; x + x_{k_{l_0}}) - F(\mathbf{t}' + b_{k_{l_0}}; x' + x_{k_{l_0}})\|_Y.$$

Since the function $F(\cdot;\cdot)$ is uniformly continuous on the compact set $D + (\mathbf{b}_{k_{l_0}}; x_{k_{l_0}})$, the above estimate simply implies (c). In order to show (ii), suppose again that condition (Q) holds. Let (d) hold, and let $\varepsilon > 0$ be fixed. We need to prove that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_\mathrm{X}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic. If the set D from the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$ is fixed, then (d) implies the existence of a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$, an integer $l_0 \in \mathbb{N}$ and a finite real number $\delta_1 > 0$ such that, for every $(\mathbf{t};x),\ (\mathbf{t}';x') \in D$ with $|\mathbf{t} - \mathbf{t}'| + \|x - x'\| \leqslant \delta_1$ and for every integer $l \geqslant l_0$, we have (8.10) with the number ε replaced therein with the number $\varepsilon/3$. Since (c) holds, there exists a number $\delta \in (0, \delta_1]$ such that, for every $(\mathbf{t};x),\ (\mathbf{t}';x') \in D$ with $|\mathbf{t} - \mathbf{t}'| + \|x - x'\| \leqslant \delta$, we have

$$\|F^*(\mathbf{t};x) - F^*(\mathbf{t}';x')\|_Y \leqslant \varepsilon/3.$$

Moreover, since D is compact and $F(\cdot;\cdot)$ is uniformly continuous on D, there exists a finite net $\{(\mathbf{t}_i;x_i)\}_{1\leqslant i\leqslant n}$ in D such that, for every $(\mathbf{t};x)\in D$, we have the existence of a number $i\in\mathbb{N}_n$ such that $|\mathbf{t}-\mathbf{t}_i|+\|x-x_i\|\leqslant\delta$ and

$$\|F(\mathbf{t}_i;x_i)-F(\mathbf{t};x)\|_Y\leqslant\varepsilon/3.$$

Then there exists an integer $l_0\in\mathbb{N}$ such that, for every integer $l\geqslant l_0$ and for every tuple $(\mathbf{t};x)\in D$, we have

$$\begin{aligned}&\|F(\mathbf{t}+b_{k_l};x+x_{k_l})-F^*(\mathbf{t};x)\|_Y\\&\quad\leqslant\|F(\mathbf{t}+b_{k_l};x+x_{k_l})-F(\mathbf{t}_i+b_{k_l};x_i+x_{k_l})\|_Y\\&\qquad+\|F(\mathbf{t}_i+b_{k_l};x_i+x_{k_l})-F^*(\mathbf{t}_i;x_i)\|_Y+\|F^*(\mathbf{t}_i;x_i)-F^*(\mathbf{t};x)\|_Y\\&\quad\leqslant 2\varepsilon/3+\|F(\mathbf{t}_i+b_{k_l};x_i+x_{k_l})-F^*(\mathbf{t}_i;x_i)\|_Y\leqslant 2\varepsilon/3+\varepsilon/3=\varepsilon,\end{aligned}$$

due to condition (d). Moreover, we have

$$\begin{aligned}&\|F^*(\mathbf{t}-b_{k_l};x-x_{k_l})-F(\mathbf{t};x)\|_Y\\&\quad\leqslant\|F^*(\mathbf{t}-b_{k_l};x-x_{k_l})-F^*(\mathbf{t}_i-b_{k_l};x_i-x_{k_l})\|_Y\\&\qquad+\|F^*(\mathbf{t}_i-b_{k_l};x_i-x_{k_l})-F(\mathbf{t}_i;x_i)\|_Y+\|F(\mathbf{t}_i;x_i)-F(\mathbf{t};x)\|_Y\\&\quad\leqslant\|F^*(\mathbf{t}-b_{k_l};x-x_{k_l})-F^*(\mathbf{t}_i-b_{k_l};x_i-x_{k_l})\|_Y\\&\qquad+\|F^*(\mathbf{t}_i-b_{k_l};x_i-x_{k_l})-F(\mathbf{t}_i;x_i)\|_Y+\varepsilon/3\\&\quad\leqslant\|F^*(\mathbf{t}-b_{k_l};x-x_{k_l})-F^*(\mathbf{t}_i-b_{k_l};x_i-x_{k_l})\|_Y+2\varepsilon/3\quad(l\geqslant l_0)\\&\quad=\Big\|\lim_{m\to+\infty}[F(\mathbf{t}-b_{k_l}+b_{k_m};x-x_{k_l}+x_{k_m})-F(\mathbf{t}_i-b_{k_l}+b_{k_m};x_i-x_{k_l}+x_{k_m})]\Big\|_Y\\&\qquad+2\varepsilon/3\leqslant\varepsilon,\quad l\geqslant l_0,\ m\geqslant m_0,\end{aligned}$$

where we have applied $(d)_s$ in the last estimate. □

Now we would like to state the following important corollary of Theorem 8.1.10.

Corollary 8.1.11. *Suppose that $F:\mathbb{R}^n\times X\to Y$ is an $(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost automorphic function, $X\in\mathcal{B}$ and R_{X} denotes the collection of all sequences in $\mathbb{R}^n\times X$. Then $F(\cdot;\cdot)$ is compactly $(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost automorphic if and only if $F(\cdot;\cdot)$ is uniformly continuous.*

Proof. Without loss of generality, we may assume that $\mathcal{B}=\{X\}$ and that, for every sequence $(\mathbf{b};\mathbf{x})$ in $\mathbb{R}^n\times X$, $P_{B,(\mathbf{b};\mathbf{x})}$ is the collection of all compact sets in $\mathbb{R}^n\times X$. Let $F(\cdot;\cdot)$ be uniformly continuous. Then conditions (d) and $(d)_s$ hold, so that the conclusion simply follows from Theorem 8.1.10. Assume that $F(\cdot;\cdot)$ is compactly $(\mathrm{R}_{\mathrm{X}},\mathcal{B})$-multi-almost automorphic and not uniformly continuous. Then there exist $\varepsilon>0$ and two sequences $(\mathbf{b}_k;x_k)$ and $(\mathbf{b}_k';x_k')$ in $\mathbb{R}^n\times X$ such that, for every $k\in\mathbb{N}$, we have $|\mathbf{b}_k-\mathbf{b}_k'|+\|x_k-x_k'\|\leqslant 1/k$ and $\|F(\mathbf{b}_k;x_k)-F(\mathbf{b}_k';x_k')\|\geqslant\varepsilon$. The set $D:=\{(0;0)\}\cup\{(\mathbf{b}_k'-\mathbf{b}_k;x_k'-x_k):k\in\mathbb{N}\}$ is compact in $\mathbb{R}^n\times X$ and this violates condition (d) from Theorem 8.1.10 with the number $\varepsilon>0$, $B=X$, and the sequence $(\mathbf{b}_k;x_k)$. □

Similarly we can prove the following result (see also [149, Lemma 5.1, Theorem 5.1], [443, Lemma 1] and [495, Theorem 2.6] for some particular cases of Theorem 8.1.10 and Corollary 8.1.11–Corollary 8.1.12, as well as [149, Definition 5.2, Definition 5.3] where the notion of compact almost automorphy has been defined for the first time).

Corollary 8.1.12. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function, where R denotes the collection of all sequences in $\mathbb{R}^n$ and $X \in \mathcal{B}$. Then $F(\cdot;\cdot)$ is compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if for every fixed element $x \in X$ we see that the function $F(\cdot; x)$ is uniformly continuous on $\mathbb{R}^n$.*

Before proceeding further, we would like to note that the notion of a compact(ly) almost automorphic function $F : \mathbb{R} \times X \to X$ has been introduced by E. H. Ait Dads, F. Boudchich and B. Es-sebbar [29, Definition 5] in a slightly artificial way, following the results obtained in the previous two corollaries. The approach of these authors can be also used for the introduction of several new types of compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions which will not be considered here. For compactly almost automorphic solutions of evolution equations, we may refer also to [414] and [416].

We close the subsection with the following example, which has been already considered in the almost periodic case.

Example 8.1.13. Suppose that $f : \mathbb{R}^n \to X$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are (compactly) almost automorphic functions. Define the function

$$F(\mathbf{t}) := f\big(\mathbf{t} - g(\mathbf{t})\big), \quad \mathbf{t} \in \mathbb{R}^n.$$

Then the function $F(\cdot)$ is (compactly) almost automorphic, as well. This can be shown as in [29, Lemma 7], where the corresponding statement has been analyzed in the one-dimensional setting.

8.1.2 Further properties of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions

In this subsection, we further explore the class of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. First of all, it is clear that we have the following: Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. If $\mathcal{B}'$ is a certain collection of subsets of X which contains $\mathcal{B}$, R_X' is a certain collection of sequences in $\mathbb{R}^n \times X$ which contains R_X and $F(\cdot;\cdot)$ is (compactly) $(\mathrm{R}_X', \mathcal{B}')$-multi-almost automorphic, then $F(\cdot;\cdot)$ is (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic. This also holds for any other class of functions introduced so far.

It is very simple to deduce the following result, which can be also reformulated for $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy by assuming additionally that $X \in \mathcal{B}$; see also (8.9) and [697, Property 4, p. 3].

Proposition 8.1.14. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic, resp. $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}})$-multi-almost automor-*

phic] and $\phi : Y \to Z$ is continuous, resp. $\phi : Y \to Z$ is continuous and satisfies the requirement that, for every $B \in \mathcal{B}$ as well as for every element $x \in B$, for every sequence $(\mathbf{b}_k) \in \mathrm{R}$ and every its subsequence $(\mathbf{b}_{k_l})$, there exists an integer $s \in \mathbb{N}$ such that the function $\phi(\cdot)$ is uniformly continuous on the closure of the set $\{F(\mathbf{t}+b_{k_m};x) : m \geqslant s,\ \mathbf{t} \in W_{B,(\mathbf{b}_k)}(x)\} \cup \{F(\mathbf{t}-b_{k_l}+b_{k_m};x) : m,\ l \geqslant s,\ \mathbf{t} \in W_{B,(\mathbf{b}_k)}(x)\}$ [$\phi : Y \to Z$ is continuous and satisfies the requirement that, for every $B \in \mathcal{B}$ as well as for every sequence $(\mathbf{b}_k) \in \mathrm{R}$ and every its subsequence $(\mathbf{b}_{k_l})$, there exists an integer $s \in \mathbb{N}$ such that the function $\phi(\cdot)$ is uniformly continuous on the closure of the set $\{F(\mathbf{t}+b_{k_m};x) : m \geqslant s,\ (\mathbf{t};x) \in P_{B,(\mathbf{b}_k)}\} \cup \{F(\mathbf{t}-b_{k_l}+b_{k_m};x) : m,\ l \geqslant s,\ (\mathbf{t};x) \in P_{B,(\mathbf{b}_k)}\}$]. Then $\phi \circ F : \mathbb{R}^n \times X \to Z$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic, resp. $(\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R},\mathcal{B},P_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic].

In [631, Lemma 3.9.9], we have clarified the supremum formula for the one-dimensional almost automorphic functions. This formula can be extended in our framework as follows.

Proposition 8.1.15 (The supremum formula). *Let $F : \mathbb{R}^n \times X \to Y$ be $(\mathrm{R},\mathcal{B})$-multi-almost automorphic. Suppose that there exists a sequence $b(\cdot)$ in R whose any subsequence is unbounded. Then for any $a \geqslant 0$ we have*

$$\sup_{\mathbf{t}\in\mathbb{R}^n, x\in X} \|F(\mathbf{t};x)\|_Y = \sup_{\mathbf{t}\in\mathbb{R}^n, |t|\geqslant a, x\in X} \|F(\mathbf{t};x)\|_Y. \tag{8.11}$$

Proof. We will include all relevant details of the proof for the sake of completeness. Let $\varepsilon > 0$, $a \geqslant 0$ and $x \in X$ be given. Then (8.11) holds if we prove that

$$\|F(\mathbf{t};x)\|_Y \leqslant \varepsilon + \sup_{\mathbf{t}\in\mathbb{R}^n, |t|\geqslant a} \|F(\mathbf{t};x)\|_Y. \tag{8.12}$$

By assumption, there exists $B \in \mathcal{B}$ with $x \in B$. Let $b(\cdot)$ be a sequence in R whose any subsequence is unbounded. Then we have (8.9), and consequently, there exist two integers $l_0 \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that

$$\|F(\mathbf{t};x)\|_Y \leqslant \varepsilon + \|F(\mathbf{t}-(b^1_{k_l},\ldots,b^n_{k_l})+(b^1_{k_m},\ldots,b^n_{k_m});x)\|_Y, \quad l \geqslant l_0,\ m \geqslant m_0.$$

In particular,

$$\|F(\mathbf{t};x)\|_Y \leqslant \varepsilon + \|F(\mathbf{t}-(b^1_{k_{l_0}},\ldots,b^n_{k_{l_0}})+(b^1_{k_m},\ldots,b^n_{k_m});x)\|_Y, \quad m \geqslant m_0.$$

Since the sequence $(b^1_{k_m},\ldots,b^n_{k_m})_{m\geqslant m_0}$ is unbounded, (8.12) follows immediately. □

Arguing similarly to the almost periodic case, we may deduce the following.

Proposition 8.1.16.

(i) *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is $(\mathrm{R}_\mathrm{X},\mathcal{B})$-multi-almost automorphic and, for every sequence which belongs to R_X, any its subsequence also* *belongs to* $\mathrm{R_X}$. *If the sequence* $(F_j(\cdot;\cdot))$ *converges uniformly to a function* $F(\cdot;\cdot)$ *on* X, *then the function* $F(\cdot;\cdot)$ *is* $(\mathrm{R_X},\mathcal{B})$*-multi-almost automorphic. If, additionally, for each* $B\in\mathcal{B}$ *and* $(\mathbf{b};\mathbf{x})\in \mathrm{R_X}$ *we have* $\mathrm{W}_{B,(\mathbf{b};\mathbf{x})}: B\to P(P(\mathbb{R}^n))$, $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})}\in P(P(\mathbb{R}^n\times B))$, $W_{B,(\mathbf{b};\mathbf{x})}(x)\subseteq W_{B,(\mathbf{b};\mathbf{x})'}(x)$ *and* $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})}\subseteq \mathrm{P}_{B,(\mathbf{b};\mathbf{x})'}$ *for any* $x\in B$ *and any subsequence* $(\mathbf{b};\mathbf{x})'$ *of* $(\mathbf{b};\mathbf{x})$, *and* $F_j(\cdot;\cdot)$ *is* $(\mathrm{R}_X,\mathcal{B},W_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic, resp.* $(\mathrm{R}_X,\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic, then the function* $F(\cdot;\cdot)$ *is likewise* $(\mathrm{R}_X,\mathcal{B},W_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic, resp.* $(\mathrm{R}_X,\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic.*

(ii) *Suppose that for each integer* $j\in\mathbb{N}$ *the function* $F_j(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B})$*-multi-almost automorphic and, for every sequence which belongs to* R, *any its subsequence also belongs to* R. *If for each* $B\in\mathcal{B}$ *there exists* $\varepsilon_B>0$ *such that the sequence* $(F_j(\cdot;\cdot))$ *converges uniformly to a function* $F(\cdot;\cdot)$ *on the set* $B^\circ\cup\bigcup_{x\in\partial B}B(x,\varepsilon_B)$, *then the function* $F(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B})$*-multi-almost automorphic. If, additionally, for each* $B\in\mathcal{B}$ *and* $(\mathbf{b}_k)\in\mathrm{R}$ *we have* $W_{B,(\mathbf{b}_k)}: B\to P(P(\mathbb{R}^n))$, $\mathrm{P}_{B,(\mathbf{b}_k)}\in P(P(\mathbb{R}^n\times B))$, $W_{B,(\mathbf{b})}(x)\subseteq W_{B,(\mathbf{b})'}(x)$ *and* $\mathrm{P}_{B,(\mathbf{b})}\subseteq \mathrm{P}_{B,(\mathbf{b})'}$ *for any* $x\in B$ *and any subsequence* $(\mathbf{b})'$ *of* $(\mathbf{b})$, *and* $F_j(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic, resp.* $(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic, then the function* $F(\cdot;\cdot)$ *is likewise* $(\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic, resp.* $(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic.*

Concerning the convolution invariance of space consisting of all $(\mathrm{R}_X,\mathcal{B})$-multi-almost automorphic functions, we would like to state the following result.

Proposition 8.1.17. *Suppose that* $h\in L^1(\mathbb{R}^n)$ *and* $F:\mathbb{R}^n\times X\to Y$ *is an* $(\mathrm{R}_X,\mathcal{B})$*-multi-almost automorphic function satisfying that for each* $B\in\mathcal{B}$ *there exists a finite real number* $\varepsilon_B>0$ *such that* $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|F(\mathbf{t},x)\|_Y<+\infty$, *where* $B^\cdot\equiv B^\circ\cup\bigcup_{x\in\partial B}B(x,\varepsilon_B)$. *Let condition* (CI) *holds, where:*
(CI) $\mathrm{R_X}=\mathrm{R}$, *or* $X\in\mathcal{B}$ *and* R_X *is general.*

Then the function

$$(h*F)(\mathbf{t};x):=\int\limits_{\mathbb{R}^n}h(\sigma)F(\mathbf{t}-\sigma;x)\,d\sigma,\quad \mathbf{t}\in\mathbb{R}^n,\ x\in X,$$

is well defined, $(\mathrm{R}_X,\mathcal{B})$*-multi-almost automorphic, and for each* $B\in\mathcal{B}$ *we have* $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|(h*F)(\mathbf{t};x)\|_Y<+\infty$.

Proof. It is clear that the function $(h*F)(\cdot;\cdot)$ is well defined and that $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|(h*F)(\mathbf{t};x)\|_Y<+\infty$ for all $B\in\mathcal{B}$. The continuity of the function $(h*F)(\cdot;\cdot)$ at the fixed point $(\mathbf{t}_0;x_0)\in\mathbb{R}^n\times X$ follows from the continuity of the function $F(\cdot;\cdot)$ at this point, the existence of a set $B\in\mathcal{B}$ such that $x_0\in B$, the assumption $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|F(\mathbf{t};x)\|_Y<+\infty$ and the dominated convergence theorem. We will prove the remainder provided that the second part of condition (CI) holds. Let $((\mathbf{b};\mathbf{x})_k)\in\mathrm{R_X}$ be fixed. Then we know that there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l}=((b^1_{k_l},b^2_{k_l},\ldots,b^n_{k_l});x_{k_l}))$ of $((\mathbf{b};\mathbf{x})_k)$ and a function

$F^* : \mathbb{R}^n \times X \to Y$ such that (8.6)–(8.7) hold pointwise for all $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$. It is not difficult to prove that the function $F^*(\cdot; x)$ is measurable for every fixed element $x \in X$. Clearly, the function

$$(h * F)^*(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma) F^*(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in B,$$

is well defined. Using the dominated convergence theorem, it can be simply shown that we have

$$\lim_{m\to+\infty} (h * F)\big(\mathbf{t} + (b^1_{k_m}, \ldots, b^n_{k_m}); x + x_{k_m}\big) = (h * F)^*(\mathbf{t}; x)$$

and

$$\lim_{l\to+\infty} (h * F)^*\big(\mathbf{t} - (b^1_{k_l}, \ldots, b^n_{k_l}); x - x_{k_l}\big) = (h * F)(\mathbf{t}; x),$$

pointwise for all $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$. This completes the proof. □

8.1.3 $\mathbb{D}$-Asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions

This subsection investigates $\mathbb{D}$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. We start by introducing the following notion.

Definition 8.1.18. Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, $i = 1, 2$ and $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic if and only if there exist a function $G(\cdot;\cdot)$ which is (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic and a function $Q \in C_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$.

It is said that $F(\cdot;\cdot)$ is asymptotically (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic if and only if $F(\cdot;\cdot)$ is $\mathbb{R}^n$-asymptotically (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic.

We similarly introduce the classes of $\mathbb{D}$-asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions and asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, as well as the corresponding classes of functions in which the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy ($(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy) is replaced with some of the notions introduced in Definition 8.1.2 or Definition 8.1.8. We will not consider here the notion in which the space $C_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ is replaced with some space of weighted ergodic components in $\mathbb{R}^n$.

The proof of the following proposition can be given as for the usually considered almost automorphic functions [492]; all clarifications also hold if the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy ($(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy) is replaced with some of the notions introduced in Definition 8.1.2 or Definition 8.1.8.

Proposition 8.1.19.

(i) *Suppose that* $\tau \in \mathbb{R}^n$, $x_0 \in X$ *and* $F(\cdot;\cdot)$ *is (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic. Then* $F(\cdot + \tau; \cdot + x_0)$ *is (compactly)* $(\mathrm{R}_X, \mathcal{B}_{x_0})$*-multi-almost automorphic, where* $\mathcal{B}_{x_0} \equiv \{-x_0 + B : B \in \mathcal{B}\}$*. Furthermore, if* $F(\cdot;\cdot)$ *is* $\mathbb{D}$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $F(\cdot+\tau;\cdot+x_0)$ *is* $(\mathbb{D}-\tau)$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B}_{x_0})$*-multi-almost automorphic.*

(ii) *Suppose that* $c_1 \in \mathbb{C} \setminus \{0\}$, $c_2 \in \mathbb{C} \setminus \{0\}$*, and* $F(\cdot;\cdot)$ *is (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic. Then* $F(c_1\cdot; c_2\cdot)$ *is (compactly)* $((\mathrm{R}_{c_1})_X, \mathcal{B}_{c_2})$*-multi-almost automorphic, where* $(\mathrm{R}_{c_1})_X \equiv \{c_1^{-1}\mathbf{b}(\cdot) : \mathbf{b} \in \mathrm{R}_X\}$ *and* $\mathcal{B}_{c_2} \equiv \{c_2^{-1}B : B \in \mathcal{B}\}$*. Furthermore, if* $F(\cdot;\cdot)$ *is* $\mathbb{D}$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $F(c_1\cdot; c_2\cdot)$ *is* $\mathbb{D}/c_1$*-asymptotically (compactly)* $((\mathrm{R}_{c_1})_X, \mathcal{B}_{c_2})$*-multi-almost automorphic.*

(iii) *Suppose that* α, $\beta \in \mathbb{C}$ *and, for every sequence which belongs to* R_X*, we see that any its subsequence belongs to* R_X*. If* $F(\cdot;\cdot)$ *and* $G(\cdot;\cdot)$ *are (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $\alpha F(\cdot;\cdot) + \beta G(\cdot;\cdot)$ *is also (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic. The same holds for* $\mathbb{D}$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic functions.*

(iv) *If* $X \in \mathcal{B}$ *and* $F(\cdot;\cdot)$ *is asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic, then* $F(\cdot;\cdot)$ *is bounded in case* [L3]*; furthermore, if* $F(\cdot;\cdot)$ *is asymptotically* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $F(\cdot;\cdot)$ *is bounded in the case that* R_X *denotes the collection of all sequences in* $\mathbb{R}^n \times X$*.*

Using Proposition 8.1.19(iv) and the supremum formula clarified in Proposition 8.1.15 (see also the estimate (8.12)), we can simply deduce that the decomposition in Definition 8.1.18 is unique (the same holds for the class of $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, where $\mathbb{D}$ contains the complement of a ball centered at the origin).

Proposition 8.1.20. *Suppose that there exist a function* $G_i(\cdot;\cdot)$ *which is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic and a function* $Q_i \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *such that* $F(\mathbf{t}; x) = G_i(\mathbf{t}; x) + Q_i(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$ *(*$i = 1, 2$*). Then we have* $G_1 \equiv G_2$ *and* $Q_1 \equiv Q_2$*, provided that the collection* R *satisfies the following two conditions:*

D1. *There exists a sequence in* R *whose any subsequence is unbounded.*

D2. *For every sequence which belongs to* R*, we see that any its subsequence belongs to* R*.*

Furthermore, arguing as in the proof of [364, Lemma 4.28], we may deduce the following.

Lemma 8.1.21. *Suppose that there exist an* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic function* $G(\cdot;\cdot)$ *and a function* $Q \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *such that* $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ *for all*

$\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Then we have

$$\overline{\{G(\mathbf{t}; x) : \mathbf{t} \in \mathbb{R}^n,\ x \in X\}} \subseteq \overline{\{F(\mathbf{t}; x) : \mathbf{t} \in \mathbb{R}^n,\ x \in X\}},$$

provided that condition [D1] *holds.*

Proposition 8.1.22. *Suppose that conditions* [D1]-[D2] *hold and for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. If the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$, then the function $F(\cdot;\cdot)$ is asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.*

Proof. Due to Proposition 8.1.20, we know that there exist a uniquely determined function $G(\cdot;\cdot)$ which is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic and a uniquely determined function $Q \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Furthermore, we have

$$F_j(\mathbf{t}; x) - F_m(\mathbf{t}; x) = \big[G_j(\mathbf{t}; x) - G_m(\mathbf{t}; x)\big] + \big[Q_j(\mathbf{t}; x) - Q_m(\mathbf{t}; x)\big],$$

for all $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ and $j,\ m \in \mathbb{N}$. Due to Proposition 8.1.19(iv), we see that the function $F_j(\cdot;\cdot) - F_m(\cdot;\cdot)$ is asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic as well as that the function $G_j(\cdot;\cdot) - G_m(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic ($j,\ m \in \mathbb{N}$). Keeping in mind this fact, Lemma 8.1.21 and the argumentation used in the proof of [364, Theorem 4.29], we get

$$\begin{aligned} 3 &\sup_{\mathbf{t}\in\mathbb{R}^n, x\in X} \big\|F_j(\mathbf{t}; x) - F_m(\mathbf{t}; x)\big\|_Y \\ &\geqslant \sup_{\mathbf{t}\in\mathbb{R}^n, x\in X} \big\|G_j(\mathbf{t}; x) - G_m(\mathbf{t}; x)\big\|_Y + \sup_{\mathbf{t}\in\mathbb{R}^n, x\in X} \big\|Q_j(\mathbf{t}; x) - Q_m(\mathbf{t}; x)\big\|_Y, \end{aligned}$$

for any $j,\ m \in \mathbb{N}$. This implies that the sequences $(G_j(\cdot;\cdot))$ and $(Q_j(\cdot;\cdot))$ converge uniformly to the functions $G(\cdot;\cdot)$ and $Q(\cdot;\cdot)$, respectively. Due to Proposition 8.1.16, we see that the function $G(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. The final conclusion follows from the obvious equality $F = G + Q$ and the fact that $C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ is a Banach space. □

Remark 8.1.23. The previous proposition is also true in the one-dimensional case, with $\mathbb{D} = [0,\infty)$ and R being any collection of sequences in $[0,\infty)$ satisfying conditions [D1]–[D2].

Concerning the partial derivatives of (asymptotically) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions, we will state and prove only one partial result.

Proposition 8.1.24.

(i) *Suppose that the function $F(\cdot;\cdot)$ is (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic,* [D2] *holds, the partial derivative*

$$\frac{\partial F(\cdot;\cdot)}{\partial t_i} := \lim_{h\to 0} \frac{F(\cdot + he_i;\cdot) - F(\cdot;\cdot)}{h}, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X,$$

exists and it is uniformly continuous on $\mathcal{B}$, i. e.,

$$(\forall B \in \mathcal{B})\,(\forall \varepsilon > 0)\,(\exists \delta > 0)\,\big(\forall \mathbf{t}',\ \mathbf{t}'' \in \mathbb{R}^n\big)\,(\forall x \in B)$$
$$\left(\left|\mathbf{t}' - \mathbf{t}''\right| < \delta \Rightarrow \left\|\frac{\partial F(\mathbf{t}';x)}{\partial t_i} - \frac{\partial F(\mathbf{t}'';x)}{\partial t_i}\right\|_Y < \varepsilon\right).$$

Then the function $\partial F(\cdot;\cdot)/\partial t_i$ is (compactly) $(\mathrm{R},\mathcal{B})$-multi-almost automorphic.

(ii) *Suppose that the function $F(\cdot;\cdot)$ is asymptotically (compactly) $(\mathrm{R},\mathcal{B})$-multi-almost automorphic,* [D1]–[D2] *hold, the partial derivative $\partial F(\mathbf{t};x)/\partial t_i$ exists for all $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ and it is uniformly continuous on $\mathcal{B}$. Then the function $\partial F(\cdot;\cdot)/\partial t_i$ is asymptotically (compactly) $(\mathrm{R},\mathcal{B})$-multi-almost automorphic.*

Proof. We will prove only (i) because (ii) follows similarly, by appealing to Proposition 8.1.22 instead of Proposition 8.1.16. The proof immediately follows from the fact that the sequence $(F_j(\cdot;\cdot) \equiv j[F(\cdot + j^{-1}e_i;\cdot) - F(\cdot;\cdot)])$ of (compactly) $(\mathrm{R},\mathcal{B})$-multi-almost automorphic functions converges uniformly to the function $\partial F(\cdot;\cdot)/\partial t_i$ as $j \to +\infty$. This can be shown as in the one-dimensional case, by observing that

$$F_j(\cdot;\cdot) - \frac{\partial F(\cdot;\cdot)}{\partial t_i} = j\int_0^{1/j}\left[\frac{\partial F(\cdot + se_i;\cdot)}{\partial t_i} - \frac{\partial F(\cdot;\cdot)}{\partial t_i}\right] ds. \qquad \square$$

8.1.4 Composition theorems for $(\mathrm{R},\mathcal{B})$-multi-almost automorphic functions

Suppose that $F : \mathbb{R}^n \times X \to Y$ and $G : \mathbb{R}^m \times Y \to Z$ are given functions, where $m \in \mathbb{N}$. The main aim of this subsection is to analyze the $(\mathrm{R},\mathcal{B})$-multi-almost automorphic properties of the following multi-dimensional Nemytskii operator $W : \mathbb{R}^n \times X \to Z$, given by $W(\mathbf{t};x) := G(\mathbf{t};F(\mathbf{t};x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$.

We will first state the following generalization of [364, Theorem 4.16]; the proof is similar to the proof of the above-mentioned theorem but we will present all details for the sake of completeness.

Theorem 8.1.25. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic and $G : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}',\mathcal{B}')$-multi-almost automorphic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences, as well as*

$$\mathcal{B}' := \left\{B' \equiv \overline{\bigcup_{\mathbf{t}\in\mathbb{R}^n} F(\mathbf{t};B)} : B \in \mathcal{B}\right\}. \tag{8.13}$$

If there exists a finite constant $L > 0$ such that

$$\left\|G(\mathbf{t};x) - G(\mathbf{t};y)\right\|_Z \leqslant L\|x - y\|_Y, \quad \mathbf{t} \in \mathbb{R}^n,\ x,\ y \in Y, \tag{8.14}$$

then the function $W(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic. Furthermore, let $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then we have the following:

(i) *Suppose that* $F(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic, for every* $B\in\mathcal{B}$*,* $x\in B$ *and* $(\mathbf{b}_k)\in\mathrm{R}$*, we see that any set of collection* $W_{B,(\mathbf{b}_k)}(x)$ *is an element of the collection* $W_{B,(\mathbf{b}_{k_l})}(x)$ *for any subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*. If the condition*
(DB) *for every* $B\in\mathcal{B}$*,* $(\mathbf{b}_k)\in\mathrm{R}$*,* $x\in B$*,* $D\in W_{B,(\mathbf{b}_k)}(x)$ *as well as for every subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*, we can find a subsequence* $(\mathbf{b}_{k_{l_m}})$ *of* $(\mathbf{b}_{k_l})$ *and a function* $G^*:\mathbb{R}^n\times Y\to Z$ *such that*

$$\lim_{m\to+\infty}\left\|G(\mathbf{t}+(b^1_{k_{l_m}},\dots,b^n_{k_{l_m}});y)-G^*(\mathbf{t};y)\right\|_Z=0 \tag{8.15}$$

holds uniformly for $(\mathbf{t},y)\in D\times\overline{F([D\times\{x\}]+\{(\mathbf{b}_{k_{l_m}};0):m\in\mathbb{N}\})}$ *and*

$$\lim_{m\to+\infty}\left\|G^*(\mathbf{t}-(b^1_{k_{l_m}},\dots,b^n_{k_{l_m}});y)-G(\mathbf{t};y)\right\|_Z=0 \tag{8.16}$$

holds uniformly for $(\mathbf{t};y)\in D\times F(D\times\{x\})$

holds, then the function $W(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic.*

(ii) *Suppose that* $F(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic and, for every* $B\in\mathcal{B}$ *and* $(\mathbf{b}_k)\in\mathrm{R}$*, we see that any set of collection* $\mathrm{P}_{B,(\mathbf{b}_k)}$ *is an element of the collection* $\mathrm{P}_{B,(\mathbf{b}_{k_l})}$ *for any subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*. If the condition*
(DB1) *for every* $B\in\mathcal{B}$*,* $(\mathbf{b}_k)\in\mathrm{R}$*,* $D\in\mathrm{P}_{B,(\mathbf{b}_k)}$ *as well as for every subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*, we can find a subsequence* $(\mathbf{b}_{k_{l_m}})$ *of* $(\mathbf{b}_{k_l})$ *and a function* $G^*:\mathbb{R}^n\times Y\to Z$ *such that* (8.15) *holds uniformly for* $(\mathbf{t},y)\in D\times\overline{F(D+\{(\mathbf{b}_{k_{l_m}};0):m\in\mathbb{N}\})}$ *and* (8.16) *holds uniformly for* $(\mathbf{t};y)\in D\times F(D\times\{x\})$

holds, then the function $W(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic.*

Proof. Let the set $B\in\mathcal{B}$ and the sequence $(\mathbf{b}_k=(b^1_k,b^2_k,\dots,b^n_k))\in\mathrm{R}$ be given. By definition, there exist a subsequence $(\mathbf{b}_{k_l}=(b^1_{k_l},b^2_{k_l},\dots,b^n_{k_l}))$ of $(\mathbf{b}_k)$ and a function $F^*:\mathbb{R}^n\times X\to Y$ such that (8.1)–(8.2) hold true. Then there exist a subsequence $(\mathbf{b}_{k_{l_m}}=(b^1_{k_{l_m}},b^2_{k_{l_m}},\dots,b^n_{k_{l_m}}))$ of $(\mathbf{b}_{k_l})$ and a function $G^*:\mathbb{R}^n\times X\to Y$ such that (8.15)–(8.16) hold pointwise for all $y\in B'$ and $\mathbf{t}\in\mathbb{R}^n$. Using (8.14) and (8.15), we get

$$\left\|G^*(\mathbf{t};x)-G^*(\mathbf{t};y)\right\|_Z\leqslant L\|x-y\|_Y,\quad \mathbf{t}\in\mathbb{R}^n,\ x,\ y\in B'. \tag{8.17}$$

In order to see that the function $W(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic, it suffices to show that

$$\lim_{m\to+\infty}\left\|G(\mathbf{t}+(b^1_{k_{l_m}},\dots,b^n_{k_{l_m}});F(\mathbf{t}+(b^1_{k_{l_m}},\dots,b^n_{k_{l_m}});x))-G^*(\mathbf{t};F^*(\mathbf{t};x))\right\|_Z=0 \tag{8.18}$$

and

$$\lim_{m\to+\infty}\left\|G^*(\mathbf{t}-(b^1_{k_{l_m}},\dots,b^n_{k_{l_m}});F^*(\mathbf{t}-(b^1_{k_{l_m}},\dots,b^n_{k_{l_m}});x))-G(\mathbf{t};F(\mathbf{t};x))\right\|_Z=0, \tag{8.19}$$

pointwise for $t\in\mathbb{R}^n$ and $x\in B$. The proof of (8.18) goes as follows. For simplicity, denote $\tau_{\mathbf{m}}:=(b^1_{k_{l_m}},\dots,b^n_{k_{l_m}})$ for all $m\in\mathbb{N}$. We have ($\mathbf{t}\in\mathbb{R}^n$, $x\in B$, $m\in\mathbb{N}$):

$$\begin{aligned}&\left\|G(\mathbf{t}+\tau_{\mathbf{m}};F(\mathbf{t}+\tau_{\mathbf{m}};x))-G^*(\mathbf{t};F^*(\mathbf{t};x))\right\|_Z\\&\quad\leqslant\left\|G(\mathbf{t}+\tau_{\mathbf{m}};F(\mathbf{t}+\tau_{\mathbf{m}};x))-G(\mathbf{t}+\tau_{\mathbf{m}};F^*(\mathbf{t};x))\right\|_Z\end{aligned}$$

$$
\begin{aligned}
&+ \| G(\mathbf{t} + \tau_{\mathbf{m}}; F^*(\mathbf{t}; x)) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \|_Z \\
\leqslant\ & L \| F(\mathbf{t} + \tau_{\mathbf{m}}; x) - F^*(\mathbf{t}; x) \|_Y + \| G(\mathbf{t} + \tau_{\mathbf{m}}; F^*(\mathbf{t}; x)) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \|_Z .
\end{aligned}
$$

Since $x \in B$ and $F^*(\mathbf{t}; x) \in B'$ for all $\mathbf{t} \in \mathbb{R}^n$, (8.18) follows by applying (8.1) and (8.15). Keeping in mind the estimate (8.17) and the estimate

$$
\begin{aligned}
&\| G^*(\mathbf{t} - \tau_l; F^*(\mathbf{t} - \tau_l; x)) - G(\mathbf{t}; F(\mathbf{t}; x)) \|_Z \\
&\quad \leqslant \| G^*(\mathbf{t} - \tau_l; F^*(\mathbf{t} - \tau_l; x)) - G^*(\mathbf{t} - \tau_l; F(\mathbf{t}; x)) \|_Z \\
&\quad\quad + \| G^*(\mathbf{t} - \tau_l; F(\mathbf{t}; x)) - G(\mathbf{t}; F(\mathbf{t}; x)) \|_Z ,
\end{aligned}
$$

the proof of (8.19) is quite analogous, which completes the proof of the first part of theorem. The proofs of (i)–(ii) follow from the already shown part and an elementary argumentation involving the corresponding definitions and the prescribed conditions. □

In the one-dimensional case, some composition principles for compactly almost automorphic functions are stated in [364, Lemma 4.36, Lemma 4.37, Lemma 4.38] and [29]. We will clarify only one, almost immediate, corollary of Theorem 8.1.25 for compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic type functions.

Corollary 8.1.26. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic and $G : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathrm{R},\mathcal{B}})$-multi-almost automorphic, where R is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$, $\mathcal{B}$ is the collection of all compact subsets of X, and for every $B \in \mathcal{B}$ we see that $\mathrm{P}_{\mathrm{R},\mathcal{B}}(B)$ is the collection of all compact subsets of $\mathbb{R}^n \times X$, and there exists a finite constant $L > 0$ such that (8.14) holds. Then the function $W(\cdot;\cdot)$ is compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.*

A slight modification of the proof of Theorem 8.1.25 (cf. also the proof of [364, Theorem 4.17]) shows that the following result holds true.

Theorem 8.1.27. *Suppose that $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic and $G : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}', \mathcal{B}')$-multi-almost automorphic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences, as well as $\mathcal{B}'$ be given by (8.13). If*

$$
\begin{aligned}
&(\forall B \in \mathcal{B})\ (\forall \varepsilon > 0)\ (\exists \delta > 0) \\
&\big(x,\ y \in B' \text{ and } \|x - y\|_Y < \delta \Rightarrow \| G(\mathbf{t}; x) - G(\mathbf{t}; y) \|_Z < \varepsilon,\ \mathbf{t} \in \mathbb{R}^n\big),
\end{aligned}
$$

then the function $W(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. Furthermore, let $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then we have the following:
(i) *The requirements in* (i) *of* Theorem 8.1.25 *imply that the function $W(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic.*
(ii) *The requirements in* (ii) *of* Theorem 8.1.25 *imply that the function $W(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic.*

Now we proceed with the analysis of composition theorems for asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost automorphic functions. Our first result corresponds to Theorem 8.1.25 and [364, Theorem 4.34].

Theorem 8.1.28. *Suppose that $F_0 : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic, $Q_0 \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ and $F(\mathbf{t};x) = F_0(\mathbf{t};x) + Q_0(\mathbf{t};x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Suppose further that $G_1 : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}',\mathcal{B}')$-multi-almost automorphic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences as well as $\mathcal{B}'$ is defined by (8.13) with the function $F(\cdot;\cdot)$ replaced therein by the function $F_0(\cdot;\cdot)$, $Q_1 \in C_{0,\mathbb{R}^n,\mathcal{B}_1}(\mathbb{R}^n \times Y : Z)$, where*

$$\mathcal{B}_1 := \Big\{ \bigcup_{\mathbf{t}\in\mathbb{R}^n} F(\mathbf{t};B) : B \in \mathcal{B} \Big\}, \tag{8.20}$$

and $G(\mathbf{t};x) = G_1(\mathbf{t};x) + Q_1(\mathbf{t};x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. If there exists a finite constant $L > 0$ such that the estimate (8.14) holds with the function $G(\cdot;\cdot)$ replaced therein by the function $G_1(\cdot;\cdot)$, then the function $W(\cdot;\cdot)$ is asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost automorphic.

Proof. Using the above assumptions and Theorem 8.1.25, we see that the function $(\mathbf{t};x) \mapsto G_1(\mathbf{t};F_0(\mathbf{t};x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic. Furthermore, we have the following decomposition:

$$W(\mathbf{t};x) = G_1(\mathbf{t};F_0(\mathbf{t};x)) + [G_1(\mathbf{t};F(\mathbf{t};x)) - G_1(\mathbf{t};F_0(\mathbf{t};x))] + Q_1(\mathbf{t};F(\mathbf{t};x)),$$

for any $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Since

$$\|G_1(\mathbf{t};F(\mathbf{t};x)) - G_1(\mathbf{t};F_0(\mathbf{t};x))\|_Z \leqslant L\|Q_0(\mathbf{t};x)\|_Y, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X,$$

we see that the function $(\mathbf{t};x) \mapsto G_1(\mathbf{t};F(\mathbf{t};x)) - G_1(\mathbf{t};F_0(\mathbf{t};x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ belongs to the space $C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Z)$. The same holds for the function $(\mathbf{t};x) \mapsto Q_1(\mathbf{t};F(\mathbf{t};x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ because of our choice of the collection $\mathcal{B}_1$ in (8.20). The proof of the theorem is thereby complete. □

Similarly we can prove the following result, which corresponds to Theorem 8.1.27 and [364, Theorem 4.35].

Theorem 8.1.29. *Suppose that $F_0 : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic, $Q_0 \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ and $F(\mathbf{t};x) = F_0(\mathbf{t};x) + Q_0(\mathbf{t};x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Suppose further that $G_1 : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}',\mathcal{B}')$-multi-almost automorphic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences and $\mathcal{B}'$ is defined by (8.13) with the function $F(\cdot;\cdot)$ replaced therein by the function $F_0(\cdot;\cdot)$, $Q_1 \in C_{0,\mathbb{R}^n,\mathcal{B}_1}(\mathbb{R}^n \times Y : Z)$, where $\mathcal{B}_1$ is given through (8.20), and $G(\mathbf{t};x) = G_1(\mathbf{t};x) + Q_1(\mathbf{t};x)$*

for all $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$. *For every* $B \in \mathcal{B}$, *we set* $B' := \overline{\bigcup_{\mathbf{t}\in\mathbb{R}^n} F_0(\mathbf{t}; B)}$. *If*

$$(\forall B \in \mathcal{B})\ (\forall \varepsilon > 0)\ (\exists \delta > 0)$$
$$\big(x,\ y \in B' \text{ and } \|x-y\|_Y < \delta \Rightarrow \big\|G_1(\mathbf{t};x) - G_1(\mathbf{t};y)\big\|_Z < \varepsilon,\ \mathbf{t} \in \mathbb{R}^n\big),$$

then the function $W(\cdot;\cdot)$ *is asymptotically* $(\mathrm{R}, \mathcal{B})$-*multi-almost automorphic.*

The statements of Theorem 8.1.28 and Theorem 8.1.29 can be reformulated for the asymptotical $(\mathrm{R}, \mathcal{B}, W_{B,(\mathbf{b}_k)})$-multi-almost automorphy and the asymptotical $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{B,(\mathbf{b}_k)})$-multi-almost automorphy by taking into consideration conditions (i) and (ii) from the formulation of Theorem 8.1.25.

8.1.5 Invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost automorpic properties under actions of convolution products

Recall, if $\mathbf{t} = (t_1, t_2, \ldots, t_n)$, then we use the notation $\mathcal{I}_{\mathbf{t}} = (-\infty, t_1] \times (-\infty, t_2] \times \cdots \times (-\infty, t_n]$. We impose the following condition:

(E1) $(R(\mathbf{t}))_{\mathbf{t}\in(0,\infty)^n} \subseteq L(X, Y)$ is a strongly continuous operator family and $\int_{(0,\infty)^n} \|R(\mathbf{t})\|_{L(X,Y)}\, d\mathbf{t} < +\infty$.

The main results of this subsection, Theorem 8.1.30 and Theorem 8.1.32, are new even in the one-dimensional setting. This enables one to provide numerous applications in the analysis of time almost automorphic solutions of the abstract (degenerate) Volterra integro-differential equations [631].

Theorem 8.1.30. *Let* $f : \mathbb{R}^n \to X$ *be a bounded* R-*multi-almost automorphic function and* (E1) *hold. Define*

$$F(\mathbf{t}) := \int_{\mathcal{I}_{\mathbf{t}}} R(\mathbf{t}-\eta) f(\eta)\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n.$$

Then $F(\cdot)$ *is a bounded* R-*multi-almost automorphic function. Furthermore, if* $f : \mathbb{R}^n \to X$ *is a bounded* $(\mathrm{R}, W_{\mathrm{R}})$-*multi-almost automorphic function, then* $F(\cdot)$ *is likewise a bounded* $(\mathrm{R}, W_{\mathrm{R}})$-*multi-almost automorphic function provided that, for every set* $D \in W_{\mathrm{R}}$ *and for every compact set* $K \subseteq [0,\infty)^n$, *we see that* $D - K \subseteq D'$ *for some set* $D' \in W_{\mathrm{R}}$.

Proof. First of all, observe that the Lebesgue dominated convergence theorem implies in view of condition (E1) that $F(\cdot)$ is a continuous function on $\mathbb{R}^n$; it is also clear that (E1) implies that the function $F(\cdot)$ is bounded on $\mathbb{R}^n$. On the other hand, since $f(\cdot)$ is R-multi-multi-almost automorphic, given a sequence $(b_n) \in \mathrm{R}$, there exist a subsequence (c_n) of (b_n) and a function $\tilde{f}(\cdot)$ such that $\lim_{n\to\infty} f(\mathbf{t}+c_n) = \tilde{f}(\mathbf{t})$ and $\lim_{n\to\infty} \tilde{f}(\mathbf{t}-c_n) = f(\mathbf{t})$ pointwise for all $\mathbf{t} \in \mathbb{R}^n$. It is clear that the function $\tilde{f}(\cdot)$ is measurable and bounded. Now, let us define

$$F^*(\mathbf{t}) := \int_{\mathcal{I}_{\mathbf{t}}} R(\mathbf{t}-\eta)\tilde{f}(\eta)\,d\eta, \quad \mathbf{t} \in \mathbb{R}^n.$$

Then we have

$$\begin{aligned}\|F(\mathbf{t}+c_n) - F^*(t)\|_Y &= \left\| \int_{\mathcal{I}_{\mathbf{t}+c_n}} R(\mathbf{t}+c_n-\eta)f(\eta)\,d\eta - \int_{\mathcal{I}_{\mathbf{t}}} R(\mathbf{t}-\eta)\tilde{f}(\eta)\,d\eta \right\| \\ &\leqslant \int_{\mathcal{I}_{\mathbf{t}}} \|R(\mathbf{t}-\eta)\|_{L(X,Y)} \cdot \|f(\eta+c_n) - \tilde{f}(\eta)\|\,d\eta, \quad \mathbf{t} \in \mathbb{R}^n.\end{aligned}$$

Using condition (E1), the above estimate and the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} F(\mathbf{t}+c_n) = F^*(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^n.$$

Similarly we get

$$\lim_{n\to\infty} F^*(\mathbf{t}-c_n) = F(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^n,$$

which completes the proof of the first part of theorem. Suppose now that $f : \mathbb{R}^n \to X$ is a bounded $(\mathrm{R}, W_{\mathrm{R}})$-multi-almost automorphic function, $\varepsilon > 0$ and $D \in W_{\mathrm{R}}$. Then there exists $L > 0$ such that

$$\|f\|_\infty \int_{\eta\in[0,\infty)^n;|\eta|\geqslant L} \|R(\eta)\|_{L(X,Y)}\,d\eta < \varepsilon/4.$$

Due to our assumption, we have the existence of a set $D' \in W_{\mathrm{R}}$ such that $D - \{\mathbf{t} \in [0,\infty)^n : |\mathbf{t}| \leqslant L\} \subseteq D'$. Choose after that a natural number $n_0 \in \mathbb{N}$ such that

$$\|f(\mathbf{t}+c_n-\eta) - \tilde{f}(\mathbf{t}-\eta)\| < \frac{\varepsilon}{2(1+\int_{\eta\in[0,\infty)^n;|\eta|\leqslant L} \|R(\eta)\|_{L(X,Y)}\,d\eta)}.$$

Arguing as above, we get

$$\begin{aligned}\|F(\mathbf{t}+c_n) - F^*(t)\|_Y &\leqslant \int_{(0,\infty)^n} \|R(\eta)\|_{L(X,Y)} \|f(\mathbf{t}+c_n-\eta) - \tilde{f}(\mathbf{t}-\eta)\|\,d\eta \\ &\leqslant 2\|f\|_\infty \int_{\eta\in[0,\infty)^n;|\eta|\geqslant L} \|R(\eta)\|_{L(X,Y)}\,d\eta \\ &\quad + \int_{\eta\in(0,\infty)^n;|\eta|\leqslant L} \|R(\eta)\|_{L(X,Y)} \|f(\mathbf{t}+c_n-\eta) - \tilde{f}(\mathbf{t}-\eta)\|\,d\eta \\ &\leqslant (\varepsilon/2) + (\varepsilon/2) = \varepsilon, \quad \mathbf{t} \in D.\end{aligned}$$

We can similarly prove that $\lim_{n\to\infty} F^*(\mathbf{t}-c_n) = F(\mathbf{t})$, uniformly in $\mathbf{t} \in D$. □

Remark 8.1.31. It is clear that the above requirements hold if W_{R} denotes the collection of all compact subsets of $\mathbb{R}^n$, so that Theorem 8.1.30 transfers the well-known result of H. R. Henríquez and C. Lizama [529, Lemma 3.1] to the multi-dimensional setting. On the other hand, W_{R} need not consist of compact sets; for example, in our previous analyses, we have analyzed case in which W_{R} is a collection of sets of the form $\{(x,y) \in \mathbb{R}^2 : |x-y| \leqslant L\}$, when $L > 0$ ($n = 2$). Then the requirements of Theorem 8.1.30 and Theorem 8.1.32 are also satisfied.

Let $\mathbb{D}$ be an unbounded subset of $\mathbb{R}^n$. For the invariance of $\mathbb{D}$-asymptotical R-multi-almost automorphy under the actions of multi-dimensional finite convolution products, we impose the following conditions:

(E2) $\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}} \int_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}^c} \|R(\mathbf{t}-\eta)\|_{L(X,Y)}\,d\eta = 0$;

(E3) there exists $r_0 > 0$ such that, for every $r > 0$, we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}} \int\limits_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}\cap B(0,r)} \|R(\mathbf{t}-\eta)\|_{L(X,Y)}\,d\eta = 0.$$

The following theorem can be shown as in the almost periodic case.

Theorem 8.1.32. *Let conditions* (E1)–(E3) *be fulfilled, let for every set $D \in W_{\mathrm{R}}$ and for every compact set $K \subseteq [0,\infty)^n$, we see that $D-K \subseteq D'$ for some set $D' \in W_{\mathrm{R}}$, and let $f = f_a + f_0$, where $f_a(\cdot)$ is a bounded* R*-multi-almost automorphic function (bounded* $(\mathrm{R}, W_{\mathrm{R}})$*-multi-almost automorphic function) and $f_0(\cdot) \in C_{0,\mathbb{D}}(\mathbb{R}^n : X) \cap L^\infty(\mathbb{R}^n : X)$. Define*

$$\Gamma f(\mathbf{t}) := \int\limits_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\eta)f(\eta)\,d\eta, \quad \mathbf{t} \in \mathbb{R}^n.$$

Then $\Gamma f(\cdot)$ can be written as a sum of a bounded R*-multi-almost automorphic function (bounded* $(\mathrm{R}, W_{\mathrm{R}})$*-multi-almost automorphic function) and a bounded function belonging to the space $C_{0,\mathbb{D}}(\mathbb{R}^n : Y)$.*

Remark 8.1.33. In $\mathbb{R}^2$, let us consider the set $\mathbb{D}$ formed by the union of lines containing a fixed point $p \in \mathbb{R}^2$. Then we have

$$\int\limits_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\eta)f(\eta)\,d\eta = 0,$$

for any $\mathbf{t} \in \mathbb{R}^2$. More generally, if $\mathbb{D}$ consists of sets contained in the Euclidean spaces of dimension less than n, after the canonical embedding of this space into $\mathbb{R}^n$ we get

$$\int\limits_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\eta)f(\eta)\,d\eta = 0,$$

for any $\mathbf{t} \in \mathbb{R}^n$. Therefore, in the formulation of previous theorem, it seems very reasonable to assume that there exists a point $\mathbf{t_0} \in \mathbb{R}^n$ such that $\operatorname{int}(\mathbb{D}_{\mathbf{t_0}}) \neq \emptyset$.

Example 8.1.34. Let α, β be positive real numbers and consider the kernel function $K_e : \mathbb{R}^2 \to \mathbb{R}$ given by $K_e(x,y) := \exp(-\alpha x)\exp(-\beta y)$. Suppose that $\mathbb{D}$ is the first quadrant $[0,+\infty) \times [0,+\infty)$ and denote $\mathbf{t} = (x,y)$. Consider the integral operator

$$F(\mathbf{t}) = \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x-s,y-r)f(s,r)\,ds\,dr$$

with $f(\mathbf{t}) = 1 + e^{-(\alpha x+\beta y)}$ and R being any collection of sequences. Then

$$\begin{aligned}
F(\mathbf{t}) &= \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x-s,y-r)\big(1+e^{-(\alpha s+\beta r)}\big)\,ds\,dr \\
&= \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x-s,y-r)\,ds\,dr + \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x-s,y-r)e^{-(\alpha s+\beta r)}\,ds\,dr \\
&= \iint\limits_{\mathcal{I}_{\mathbf{t}}} K_e(x-s,y-r)\,ds\,dr - \iint\limits_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}^c} K_e(x-s,y-r)\,ds\,dr + \\
&\quad + \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x-s,y-r)e^{-(\alpha s+\beta r)}\,ds\,dr \\
&= F_1(\mathbf{t}) + F_2(\mathbf{t}),
\end{aligned}$$

where

$$F_1(\mathbf{t}) := \iint\limits_{\mathcal{I}_{\mathbf{t}}} K_e(x-s,y-r)\,ds\,dr$$

and

$$F_2(\mathbf{t}) := \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x-s,y-r)e^{-(\alpha s+\beta r)}\,ds\,dr - \iint\limits_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}^c} K_e(x-s,y-r)\,ds\,dr.$$

We see that $F_1(\cdot)$ is R-multi-almost periodic (note that $K_e(\cdot;\cdot)$ satisfies condition (E1)). On the other hand, $F_2(\cdot)$ is not R-multi-almost automorphic because for $(x_0,y) \in \mathbb{D}$, with fixed $x_0 \in (0,+\infty)$, we have

$$\lim_{|(x_0,y)|\to+\infty} F_2(x_0,y) \neq 0.$$

8.1.6 Applications to the abstract Volterra integro-differential equations

In this subsection, we present some applications of our abstract results in the qualitative analysis of solutions for various classes of the abstract Volterra integro-differential equations.

Applications to semilinear Volterra integral equations

First of all, we will present some applications of established composition results and the results about the invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy under the actions of multi-dimensional convolution products. We start by stating the following result.

Theorem 8.1.35. *Let $F, G : \mathbb{R}^n \times X \to X$ be two $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, where $\mathcal{B}$ is the collection of all bounded subsets of X, R is any collection of sequences in $\mathbb{R}^n$ satisfying that for each sequence $(\mathbf{b}_k)$ in R any its subsequence also belongs to R. Suppose that, for every bounded subset B of X, we have*

$$\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \big[\|F(\mathbf{t}; x)\| + \|G(\mathbf{t}; x)\|\big] < \infty. \tag{8.21}$$

If [E1] *holds with $Y = X$, then there exists a unique bounded R-multi-almost automorphic solution of the integral equation*

$$u(\mathbf{t}) = F(\mathbf{t}; u(\mathbf{t})) + \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t} - \eta) G(\eta, u(\eta))\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n, \tag{8.22}$$

provided that the function $G(\cdot;\cdot)$ satisfies the estimate (8.14) *with some finite real constant $L > 0$, the function $F(\cdot;\cdot)$ satisfies the estimate* (8.14) *with some finite real constant $L_F > 0$ and the meaning clear, and*

$$L_F + L \int_{(0,\infty)^n} \|K(\eta)\|_{L(X)}\, d\eta < 1. \tag{8.23}$$

Proof. Due to Proposition 8.1.16(ii), the vector space $\mathcal{X}$ of all bounded R-multi-almost automorphic functions $u : \mathbb{R}^n \to X$ endowed with the sup-norm is a Banach space. Furthermore, Theorem 8.1.25 in combination with the estimate (8.21) implies that, for every function $u : \mathbb{R}^n \to X$ which belongs to $\mathcal{X}$, the functions $\mathbf{t} \mapsto F(\mathbf{t}; u(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{t} \mapsto G(\mathbf{t}; u(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$ are bounded R-multi-almost automorphic. Applying after that Theorem 8.1.30, we see that the integral operator

$$\mathbf{t} \mapsto (\Gamma u)(\mathbf{t}) := F(\mathbf{t}; u(\mathbf{t})) + \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t} - \eta) G(\eta, u(\eta))\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n,$$

is well defined and maps the space $\mathcal{X}$ into itself. The final conclusion simply follows from the Banach contraction principle and a simple calculation involving the estimate (8.23). □

Without any substantial difficulties, we can similarly consider the existence and uniqueness of bounded compactly R-multi-almost automorphic solutions of the integral equation (8.22), provided that the functions $F(\cdot;\cdot)$ and $G(\cdot;\cdot)$ satisfy conditions sufficient for applying Corollary 8.1.26. Furthermore, we can similarly consider the existence and uniqueness of bounded $(\mathrm{R}, W_{\mathrm{R}})$-multi-almost automorphic solutions of the equation (and its semilinear analogues)

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t}-\eta)u(\eta)\,d\eta, \quad \mathbf{t} \in \mathbb{R}^n,$$

where $f(\cdot)$ is bounded $(\mathrm{R}, W_{\mathrm{R}})$-multi-almost automorphic, (E1) holds and, for every set $D \in W_{\mathrm{R}}$ and for every compact set $K \subseteq [0,\infty)^n$, we see that $D - K \subseteq D'$ for some set $D' \in W_{\mathrm{R}}$; cf. also the formulation of Theorem 8.1.30.

It is worth noting that Eq. (8.22) can be used for modeling of some two-dimensional nonlinear Volterra integral equations of convolution type of the second kind with infinite delay; see [103] for some examples in the absence of delay and [318, Chapter 10] for some other results in this direction. In actual fact, we can consider the well-posedness of the equation

$$u(x,y) = g(x,y) + \int_{-\infty}^{x}\int_{-\infty}^{y} K\big(x,y,s,t,u(s,t)\big)\,ds\,dt, \quad (x,y) \in \mathbb{R}^2,$$

provided that $K(x,y,s,t,u(s,t))$ has the form

$$K\big(x,y,s,t,u(s,t)\big) = k(s-x,t-y)h\big(s,t,u(s,t)\big);$$

our results about the invariance of $\mathbb{D}$-asymptotical $(\mathrm{R},\mathcal{B})$-multi-almost automorphy can be applied in the qualitative analysis of solutions to the following two-dimensional nonlinear Volterra integral equation ($\mathbf{t} = (x,y)$):

$$f(\mathbf{t}) = g\big(\mathbf{t}; f(\mathbf{t})\big) + \int_{0}^{x}\int_{0}^{y} K(\mathbf{t}-\eta)h\big(\eta, f(\eta)\big)\,d\eta,$$

as well.

We close this part by observing that we can easily transfer the results established for the Hammerstein integral equation of the convolution type (6.38) to the multi-dimensional almost automorphic functions. For example, assume that $g : \mathbb{R}^n \to X$ is (compactly) almost automorphic, R is the collection of all sequences in $\mathbb{R}^n$, $\mathcal{B}$ is the collection of all compact subsets of X, $F(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathrm{R},\mathcal{B}})$-multi-almost automorphic, where for each $B \in \mathcal{B}$ we see that $\mathrm{P}_{\mathrm{R},\mathcal{B}}(B)$ is the collection of all compact subsets of $\mathbb{R}^n \times X$, and there exists a finite constant $L > 0$ such that (8.14) holds with the function $G(\cdot;\cdot)$ replaced therein with the function $F(\cdot;\cdot)$. If $k \in L^1(\mathbb{R}^n)$ and $L\|k\|_{L^1(\mathbb{R}^n)} < 1$, then (6.38) has a unique (compactly) almost automorphic solution (see Proposition 8.1.17 and Corollary 8.1.26).

Applications to the heat equation and the wave equation

In this part, we will first study the initial value problem for the homogeneous heat equation with nonlocal diffusion

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } [0,+\infty) \times \mathbb{R}^n, \\ u(0,x) &= F(x) \quad \text{in } \mathbb{R}^n \times \{0\}. \end{aligned}$$

Suppose for simplicity that $X = \mathrm{BUC}(\mathbb{R}^n : \mathbb{C})$. Then it is well known that (the Gaussian semigroup is denoted by $(G(t))_{t\geqslant 0}$) the unique solution of (8.40) is given by $(t,x) \mapsto (G(t)F)(x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Suppose now that a number $t_0 > 0$ is fixed. Then Proposition 8.1.17 shows that the function $\mathbb{R}^n \ni x \mapsto u(x,t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is bounded, R-multi-almost automorphic provided that R is any non-empty collection of sequences in $\mathbb{R}^n$ and the function $F(\cdot)$ is bounded, R-multi-almost automorphic. We can similarly apply Proposition 8.1.17 to the Poisson semigroup in $\mathbb{R}^n$.

Now we will revisit the classical theory of partial differential equations of second order and provide some new applications in the qualitative analysis of solutions of the wave equations in $\mathbb{R}^3$:

$$u_{tt}(t,x) = d^2\Delta_x u(t,x), \quad x \in \mathbb{R}^3,\ t > 0; \quad u(0,x) = g(x), \quad u_t(0,x) = h(x), \tag{8.24}$$

where $d > 0$, $g \in C^3(\mathbb{R}^3 : \mathbb{R})$ and $h \in C^2(\mathbb{R}^3 : \mathbb{R})$. By the famous Kirchhoff formula (see, e. g., [890, Theorem 5.4, pp. 277-278]; we will use the same notion and notation), the function

$$\begin{aligned} u(t,x) &:= \frac{\partial}{\partial t}\left[\frac{1}{4\pi d^2 t}\int\limits_{\partial B_{dt}(x)} g(\sigma)\,d\sigma\right] + \frac{1}{4\pi d^2 t}\int\limits_{\partial B_{dt}(x)} g(\sigma)\,d\sigma \\ &= \frac{1}{4\pi}\int\limits_{\partial B_1(0)} g(x+dt\omega)\,d\omega + \frac{dt}{4\pi}\int\limits_{\partial B_1(0)} \nabla g(x+dt\omega)\cdot\omega\,d\omega \\ &\quad + \frac{t}{4\pi}\int\limits_{\partial B_1(0)} h(x+dt\omega)\,d\omega, \quad t \geqslant 0,\ x \in \mathbb{R}^3, \end{aligned} \tag{8.25}$$

is a unique solution of problem (8.24) which belongs to the class $C^2([0,\infty) \times \mathbb{R}^3)$. Fix now a number $t_0 > 0$. Then the function $x \mapsto u(t_0,x)$, $x \in \mathbb{R}^3$ is Bohr c-almost periodic (c-uniformly recurrent) provided that the functions $g(\cdot)$, $\nabla g(\cdot)$ and $h(\cdot)$ are c-almost periodic (c-uniformly recurrent), where $c \in \mathbb{C} \setminus \{0\}$. Similarly, let us assume that the functions $g(\cdot)$, $\nabla g(\cdot)$ and $h(\cdot)$ are bounded R-multi-almost automorphic, where R is any collection of sequences in $\mathbb{R}^3$ such that, for every sequence $(\mathbf{b}_k) \in \mathrm{R}$, any subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ also belongs to R (the last condition is superfluous in the case that $g \equiv 0$). If we replace the functions $g(\cdot)$ and $\nabla g(\cdot)$ in (8.25) with the corresponding limit functions $g^*(\cdot)$ and $\nabla g^*(\cdot)$ for the sequence $(\mathbf{b}_k)$ from the definition of R-multi-almost automorphy, then the use of the dominated convergence theorem shows that the function $x \mapsto u(t_0,x)$, $x \in \mathbb{R}^3$ is likewise bounded R-multi-almost automorphic; furthermore, the same statement holds for the notion of bounded $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-multi-almost automorphy provided that the following hold:

(i) For every sequence $(\mathbf{b}_k) \in \mathrm{R}$ and for every subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$, we have $P_{(\mathbf{b}_k)} \subseteq P_{(\mathbf{b}_{k_l})}$.

(ii) For every sequence $(\mathbf{b}_k) \in \mathrm{R}$, for every set $D \in P_{(\mathbf{b}_k)}$ and for every compact set $K \subseteq \mathbb{R}^3$, we have the existence of a set $D' \in P_{(\mathbf{b}_k)}$ such that $D + K \subseteq D'$.

We can similarly provide some applications in the qualitative analysis of solutions of the wave equations in $\mathbb{R}^2$:

$$u_{tt}(t,x) = d^2\Delta_x u(t,x), \quad x \in \mathbb{R}^2,\ t > 0; \quad u(0,x) = g(x), \quad u_t(0,x) = h(x), \tag{8.26}$$

where $d > 0$, $g \in C^3(\mathbb{R}^2 : \mathbb{R})$ and $h \in C^2(\mathbb{R}^2 : \mathbb{R})$. By the Poisson formula (see, e. g., [890, Theorem 5.5, pp. 280–281]), we see that the function

$$\begin{aligned} u(t,x) &:= \frac{\partial}{\partial t}\left[\frac{1}{2\pi d}\int\limits_{\partial B_{dt}(x)} \frac{g(\sigma)}{\sqrt{d^2t^2 - |x-y|^2}}\,d\sigma\right] + \frac{1}{2\pi d}\int\limits_{\partial B_{dt}(x)} \frac{h(\sigma)}{\sqrt{d^2t^2 - |x-y|^2}}\,d\sigma \\ &= d\int\limits_{B_1(0)} \frac{g(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\,d\sigma + d^2t\int\limits_{B_1(0)} \frac{\nabla g(x+dt\sigma)\cdot\sigma}{\sqrt{1-|\sigma|^2}}\,d\sigma \\ &\quad + dt\int\limits_{B_1(0)} \frac{h(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\,d\sigma, \quad t \geqslant 0,\ x \in \mathbb{R}^2, \end{aligned}$$

is a unique solution of problem (8.26) which belongs to the class $C^2([0,\infty)\times\mathbb{R}^3)$. Then we can argue as in the three-dimensional case.

Concerning the one-dimensional case, it should be recalled that the unique regular solution of wave equation

$$u_{tt}(t,x) = d^2\Delta_x u(t,x), \quad x \in \mathbb{R}^2,\ t > 0; \quad u(0,x) = g(x), \quad u_t(0,x) = h(x),$$

where $d > 0$, $g \in C^2(\mathbb{R}^2 : \mathbb{R})$ and $h \in C^1(\mathbb{R}^2 : \mathbb{R})$, is given by the d'Alembert formula. If we assume, in the corresponding formula used multiple times before, that the functions $g(\cdot)$ and $h^{[1]}(\cdot) \equiv \int_0^{\cdot} h(s)\,ds$ are almost automorphic, then the solution $u(x,t)$ is almost automorphic in $(x,t) \in \mathbb{R}^2$. Details are left to the interested reader.

Consider now the inhomogeneous wave equation

$$\begin{aligned} &u_{tt}(t,x) - d^2\Delta_x u(t,x) = f(t,x), \quad x \in \mathbb{R}^2,\ t > 0; \\ &u(0,x) = g(x), \quad u_t(0,x) = h(x), \end{aligned} \tag{8.27}$$

where $d > 0$, $f(t,x)$ is continuously differentiable in the variable $t \in \mathbb{R}$ and continuous in the variable $x \in \mathbb{R}$, $g \in C^2(\mathbb{R}^2 : \mathbb{R})$ and $h \in C^1(\mathbb{R}^2 : \mathbb{R})$. Using the d'Alembert formula and the Duhamel principle (we will not consider the higher dimensions here for simplicity), the unique solution of (8.27) is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big[g(x-at) + g(x+at)\big] + \frac{1}{2a}\int\limits_{x-at}^{x+at} h(s)\,ds + \frac{1}{2d}\int\limits_0^t \left[\int\limits_{x-d(t-s)}^{x+d(t-s)} f(r,s)\,dr\right] ds \\ &:= u_h(x,t) + \frac{1}{2d}\int\limits_0^t \left[\int\limits_{x-d(t-s)}^{x+d(t-s)} f(r,s)\,dr\right] ds, \quad x \in \mathbb{R},\ t > 0. \end{aligned}$$

If we assume that the functions $g(\cdot)$ and $h^{[1]}(\cdot) \equiv \int_0^{\cdot} h(s)\,ds$ are almost automorphic, then we see from the above that the solution $u_h(x,t)$ is almost automorphic in $(x,t) \in \mathbb{R}^2$. It is clear that the function

$$(x,t) \mapsto u_p(x,t) \equiv \frac{1}{2d} \int_0^t \left[\int_{x-d(t-s)}^{x+d(t-s)} f(r,s)\,dr \right] ds$$

can be defined for all $(x,t) \in \mathbb{R}^2$. Suppose now that $L > 0$ and the function $f(\cdot,\cdot)$ has the property that $\lim_{|x|\to+\infty} f(x,t) = 0$, uniformly in $t \in [0,L]$. Set $\mathbb{D} := \{(x,t) \in \mathbb{R}^2 : t \in [0,L]\}$. Then $u_p \in C_{0,\mathbb{D}}(\mathbb{R}^2 : \mathbb{R})$ since

$$u_p(x,t) = \frac{1}{2d} \int_0^t \left[\int_{x-ds}^{x+ds} f(r,t-s)\,dr \right] ds, \quad x \in \mathbb{R},\ t \in \mathbb{R}$$

and there exists a sufficiently large real number $x_0 > 0$ such that, for every $x \in \mathbb{R}$ with $|x| \geqslant x_0$, for every $t \in [0,L]$ and for every $s \in [0,t]$, we have $|f(r,t-s)| \leqslant \varepsilon$ for all $r \in [x-dL, x+dL]$ and therefore

$$\left|u_p(x,t)\right| \leqslant \varepsilon \cdot L^2, \quad (x,t) \in \mathbb{D},\ |x| \geqslant x_0.$$

Hence, the solution $u(x,t)$ obtained by a combination of the d'Alembert formula and the Duhamel principle will be $\mathbb{D}$-asymptotically R-multi-almost automorphic with R being the collection of all sequences in $\mathbb{R}^2$.

Let us note that H.-S. Ding, T.-J. Xiao and J. Liang have investigated, in [385], the asymptotically almost automorphic solutions of the following integro-differential equation (with nonlocal initial data), which models the heat conduction in materials with memory:

$$u'(t) = Au(t) + \int_0^t B(t-s)u(s)\,ds + f\big(t,u(t)\big),\ t \geqslant 0, \tag{8.28}$$

$$u(0) = u_0 + g(u); \tag{8.29}$$

here, $u_0 \in X$, A and $(B(t))_{t\geqslant 0}$ are linear, closed and densely defined operators on X. Some results about the existence and uniqueness of the asymptotically almost automorphic solutions to the integro-differential equation (8.28)–(8.29) have been established in [268], as well. It could be of some importance to reconsider the statement of [385, Theorem 2.7], given in the one-dimensional setting, for asymptotically R-almost automorphic type functions, where R denotes a certain collection of sequences in R which has the property that, for every sequence $(b_k) \in$ R, any subsequence (b_{k_l}) of (b_k) also belongs to R. It seems that this can be done with some obvious modifications, not only in the case of consideration [385, Theorem 2.7], but also in the case of consideration of many other structural results obtained so far regarding the time almost automorphic solutions of the abstract PDEs.

Applications to the abstract ill-posed Cauchy problems
In the final part of this section, we will revisit once more the theory of integrated solution operator families, C-regularized solution operator families and their applications to the abstract ill-posed Cauchy problems. For more details about the notion used, we refer the reader to the monographs [82] by W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander and [629, 630] by M. Kostić.

Without going into full details, which is almost impossible, we will only present two illustrative examples which strongly justify the introduction of function spaces analyzed in this chapter (similar conclusions hold for the corresponding classes of multi-dimensional almost periodic type functions). In order to achieve our aims, we mainly apply Proposition 8.1.17 concerning the convolution invariance of introduced function spaces (the use of symbol D is clear from the context).

1. Suppose that $k \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, $0 \leqslant |\alpha| \leqslant k$, $a_\alpha \neq 0$ for some α with $|\alpha| = k$, $P(x) = \sum_{|\alpha|\leqslant k} a_\alpha i^{|\alpha|} x^\alpha$, $x \in \mathbb{R}^n$, $P(\cdot)$ is an elliptic polynomial, i. e., there exist $C > 0$ and $L > 0$ such that $|P(x)| \geqslant C|x|^k$, $|x| \geqslant L$, $\omega := \sup_{x\in\mathbb{R}^n} \operatorname{Re}(P(x)) < \infty$, and X is one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$. Define
$$P(D) := \sum_{|\alpha|\leqslant k} a_\alpha f^{(\alpha)} \quad \text{and} \quad D(P(D)) := \{f \in E : P(D)f \in E \text{ distributionally}\},$$
$n_X := n|(1/2) - (1/p)|$, if $X = L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$ and $n_X > n/2$, otherwise. Then we know that the operator $P(D)$ generates an exponentially bounded r-times integrated semigroup $(S_r(t))_{t\geqslant 0}$ in X for any $r > n_X$ as well as that the operator $P(D)$ generates an exponentially bounded n_X-times integrated semigroup $(S_{n_X}(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^n)$ provided $p \in (1, \infty)$; see, e. g., [629, Example 2.8.6] and the references therein. We will consider the general case $r > n/2$ and the spaces $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ below; in the setting of L^p-spaces, certain applications can be given for the multi-dimensional Weyl almost periodic functions and the multi-dimensional Weyl almost automorphic solutions. It is well known that for each $t \geqslant 0$ there exists a function $f_t \in L^1(\mathbb{R}^n)$ such that
$$[S_r(t)f](x) := (f_t * f)(x), \quad x \in \mathbb{R}^n, f \in X.$$
Let us fix a number $t_0 \geqslant 0$, and let us assume that the function $X \ni f$ is R-multi-almost automorphic, where R is any non-empty collection of sequences in $\mathbb{R}^n$. Applying Proposition 8.1.17, we see that the function $x \mapsto [S_r(t_0)f](x)$, $x \in \mathbb{R}^n$ is R-multi-almost automorphic and belongs to X. In terms of the corresponding abstract first-order Cauchy problem, this means that there exists a unique X-valued continuous function $t \mapsto u(t)$, $t \geqslant 0$ such that $\int_0^t u(s)\,ds \in D(P(D))$ for every $t \geqslant 0$ and
$$u(t) = P(D)\int_0^t u(s)\,ds - \frac{t^r}{\Gamma(r+1)} f, \quad t \geqslant 0;$$

furthermore, the solution $t \mapsto u(t)$, $t \geqslant 0$ of this abstract Cauchy problem has the property that its orbit consists solely of R-multi-almost automorphic functions. Suppose now that the collection R additionally satisfies the requirement that for each sequence $\mathbf{b} \in$ R any its subsequence also belongs to R and consider, for simplicity, case in which $r \in \mathbb{N}$. If we assume that $f \in D(P(D)^r)$ and all functions

$$f,\ P(D)f,\ \ldots,\ P(D)^r f$$

are R-multi-almost automorphic, then it is well known that the function

$$u(t) := S_r(t)P(D)^r f + \frac{t^{r-1}}{(r-1)!}P(D)^{r-1}f + \cdots + tP(D)f + f, \quad t \geqslant 0, \tag{8.30}$$

is a unique continuous X-valued function which satisfies the requirement that $\int_0^t u(s)\,ds \in D(P(D))$ for every $t \geqslant 0$ and

$$u(t) = P(D)\int_0^t u(s)\,ds - f, \quad t \geqslant 0;$$

due to the representation formula (8.30) and our assumptions, the solution $t \mapsto u(t)$, $t \geqslant 0$ of this abstract Cauchy problem has the property that its orbit consists solely of R-multi-almost automorphic functions; see [630, Subsection 2.9.7] for more details regarding the existence and growth of mild solutions of operators generating fractionally integrated C-semigroups and fractionally integrated C-cosine functions in locally convex spaces.

2. Suppose now that X is $C_b(\mathbb{R}^n)$ or $\mathrm{BUC}(\mathbb{R}^n)$, $m \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$ for $0 \leqslant |\alpha| \leqslant k$ and $a_\alpha \neq 0$ for some α with $|\alpha| = k$. Consider the operator $P(D)$ with its maximal distributional domain and its associated polynomial $P(x)$ defined as above. Set

$$h_{t,\beta}(x) := \left(1+|x|^2\right)^{-\beta/2}\sum_{j=0}^{\infty}\frac{t^{2j}P(x)^j}{(2j)!}, \quad x \in \mathbb{R}^n,\ t \geqslant 0,\ \beta \geqslant 0,$$

$\Omega(\omega) := \{\lambda^2 : \operatorname{Re}\lambda > \omega\}$, if $\omega > 0$ and $\Omega(\omega) := \mathbb{C} \setminus (-\infty, \omega^2]$, if $\omega \leqslant 0$. Assume $r \in [0,k]$, $\omega \in \mathbb{R}$ and condition [630, (W); Example 2.2.14] holds. Then, for every $\beta > (k - \frac{r}{2})\frac{n}{4}$, $P(D)$ generates an exponentially bounded $C_\beta(0)$-regularized cosine function $(C_\beta(t))_{t\geqslant 0}$ in X satisfying

$$C_\beta(t)f = \mathcal{F}^{-1}h_{t,\beta} * f, \quad t \geqslant 0,\ f \in X,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in $\mathbb{R}^n$. Since $\mathcal{F}^{-1}h_{t,\beta} \in L^1(\mathbb{R}^n)$ for every $t \geqslant 0$, we can repeat verbatim the arguments from the first application. For example, suppose that the function $X \ni f$ is R-multi-almost automorphic. Then the function $t \mapsto C_\beta(t)f$, $t \geqslant 0$ is a unique continuous X-valued function which satisfies $\int_0^t (t-s)u(s)\,ds \in D(P(D))$ for every $t \geqslant 0$ and

$$u(t) = P(D)\int_0^t (t-s)u(s)\,ds - C_\beta(0)f, \quad t \geqslant 0;$$

as above, for each fixed number $t \geqslant 0$ we see that $u(t)$ is a spatially R-multi-almost automorphic function which belongs to X. See also [630, Section 2.5], where we have analyzed the generation of fractional resolvent families by (non-)coercive differential operators; the obtained results can be applied with the obvious choice of operators $A_j \equiv -i\partial/\partial x_j$ $(1 \leqslant j \leqslant n)$.

8.2 Stepanov multi-dimensional almost automorphic type functions

In [373, Definition 6, Definition 7], we have recently made the first steps in the analysis of (asymptotical) Stepanov $p(x)$-almost automorphy in the one-dimensional setting. This study set out to provide the first systematic account of Stepanov multi-dimensional almost automorphic type functions in Lebesgue spaces with variable exponents (see [662] for more details). Among many other topics, we investigate here the pointwise products of Stepanov multi-dimensional almost automorphic functions, the convolution invariance of Stepanov multi-dimensional almost automorphy and provide several illustrative examples. We also provide certain applications of our results to the abstract Volterra integro-differential equations in Banach spaces, considering primarily the multi-dimensional heat equation and the multi-dimensional wave equation. Although we work with Lebesgue spaces with variable exponents, it is worth noting that the introduced classes of Stepanov multi-dimensional almost automorphic functions seem to be not analyzed elsewhere even in the case that the exponent $p(\cdot)$ has a constant value.

We will occasionally use the following condition:

(ST) The function $F : \mathbb{R}^n \times X \to Y$ satisfies the requirement that the Bochner transform $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is well defined and continuous. For each $B \in \mathcal{B}$ and $\mathbf{b} = (\mathbf{b}_k) \in \mathrm{R}$ we see that $W_{B,\mathbf{b}} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,\mathbf{b}} \in P(P(\mathbb{R}^n \times B))$; for each $B \in \mathcal{B}$ and $(\mathbf{b};\mathbf{x}) = ((\mathbf{b};\mathbf{x})_k) \in \mathrm{R}_{\mathrm{X}}$ we have $W_{B,(\mathbf{b};\mathbf{x})} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$.

If this condition holds, then we can simply introduce the following classes of Stepanov multi-dimensional almost automorphic type functions.

Definition 8.2.1. Suppose that (ST) holds. Then we say that the function $F(\cdot;\cdot)$ is:

(i) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic;

(ii) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic] if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R}, \mathcal{B}, \mathrm{R}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic];

(iii) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost automorphic if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_\mathrm{X}, \mathcal{B})$-multi-almost automorphic;

(iv) Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic] if and only if the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic].

For the functions of the form $F : \mathbb{R}^n \to Y$, we will omit the term "$\mathcal{B}$" from the notation henceforth.

Without any doubt, the most important case in our analysis is that one in which we see that R (R_X) is a collection of all sequences $\mathbf{b}(\cdot)$ in $\mathbb{R}^n$ ($(\mathbf{b}; \mathbf{x})$ in $\mathbb{R}^n \times X$). If this is the case and $\Omega = [0, 1]^n$, then we will simply say that the function $F : \mathbb{R}^n \to Y$ is Stepanov $p(\cdot)$-almost automorphic.

Let $k \in \mathbb{N}$ and $F_i : \mathbb{R}^n \times X \to Y_i$ $(1 \leqslant i \leqslant k)$. Then we define the function $(F_1, \dots, F_k) : \mathbb{R}^n \times X \to Y_1 \times \cdots \times Y_k$ as before, by $(F_1, \dots, F_k)(\mathbf{t}; x) := (F_1(\mathbf{t}; x), \dots, F_k(\mathbf{t}; x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$.

Keeping in mind the introduced notion, we immediately get the following.

Proposition 8.2.2. *Suppose that* $k \in \mathbb{N}$, (ST) *and the following condition hold:*

(C1) *for each set* $B \in \mathcal{B}$, *for each sequence* $(\mathbf{b}; \mathbf{x}) = ((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_\mathrm{X}$ *and for every subsequence* $(\mathbf{b}; \mathbf{x})'$ *of* $(\mathbf{b}; \mathbf{x})$ *we have* $W_{B,(\mathbf{b};\mathbf{x})}(x) \subseteq W_{B,(\mathbf{b};\mathbf{x})'}(x)$ *for all* $x \in B$ *and* $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \subseteq \mathrm{P}_{B,(\mathbf{b};\mathbf{x})'}$.

If the function $F_i : \mathbb{R}^n \times X \to Y_i$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_\mathrm{X}, \mathcal{B})$*-multi-almost automorphic [Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic; Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic] for* $1 \leqslant i \leqslant k$, *then the function* $(F_1, \dots, F_k)(\cdot; \cdot)$ *has the same property.*

Clearly, we also have the following.

Proposition 8.2.3. *Suppose that* (ST) *holds.*

(i) *If the function* $F : \mathbb{R}^n \times X \to Y$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, then* $F(\cdot; \cdot)$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic, where for each* $B \in \mathcal{B}$ *and* $\mathbf{b} \in \mathrm{R}$ *we have* $\mathrm{P}_{B,\mathbf{b}} = \{\{\mathbb{R}^n \times B\}\}$.

(ii) *Suppose that for each set* $B \in \mathcal{B}$ *and sequence* $(\mathbf{b}_k; x_k) \in \mathrm{R}_X$ *we see that there exists an integer* $k_0 \in \mathbb{N}$ *such that, for every integer* $k \geqslant k_0$, *we have* $B - x_k \subseteq B$. *If the function* $F : \mathbb{R}^n \times X \to Y$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$*-multi-almost periodic, then* $F(\cdot; \cdot)$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic, where for each* $B \in \mathcal{B}$ *and* $(\mathbf{b}; \mathbf{x}) \in \mathrm{R}_X$ *we have* $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} = \{\{\mathbb{R}^n \times B\}\}$.

We continue by providing two indicative examples; the first one is a slight modification of Example 8.1.5 and the second one is a slight modification of Example 8.1.7.

Example 8.2.4. Suppose that $\varphi : \mathbb{R} \to \mathbb{C}$ is an almost periodic function, $\Omega := [0,1]^2$ and $(T(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ is an operator family which is strongly locally integrable and not strongly continuous at zero. Suppose, further, that there exist a finite real number $M \geqslant 1$ and a real number $\gamma \in (0,1)$ such that

$$\|T(t)\|_{L(X,Y)} \leqslant \frac{M}{|t|^{\gamma}}, \quad t \in \mathbb{R} \setminus \{0\}, \tag{8.31}$$

as well as that R is the collection of all sequences in $\Delta_2 \equiv \{(t,t) : t \in \mathbb{R}\}$ and $\mathcal{B}$ is the collection of all bounded subsets of X. Define a function $F : \mathbb{R}^2 \times X \to Y$ by

$$F(t,s;x) := e^{\int_s^t \varphi(\tau)\,d\tau} T(t-s)x, \quad (t,s) \in \mathbb{R}^2,\ x \in X.$$

Let for each bounded subset B of X and for each sequence $(\mathbf{b}_k = (b_k, b_k))$ in R the collection $P_{B,(\mathbf{b}_k)}$ be constituted of all sets of form $\{(t,s) \in \mathbb{R}^2 : |t-s| \leqslant L\} \times B$, where $L > 0$. Then the function $F(\cdot,\cdot;\cdot)$ is Stepanov $(\Omega,1)$-$(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, which can be deduced as follows. First of all, it can be simply shown with the help of Fubini theorem and our assumption (8.31) that for each real numbers $s,\ t \in \mathbb{R}$ and for each element $x \in X$ we have ($\mathbf{u} = (u_1,u_2)$):

$$(u_1,u_2) \mapsto F(t+u_1, s+u_2; x) \equiv e^{\int_{s+u_2}^{t+u_1} \varphi(r)\,dr} T\big(t-s+(u_1-u_2)\big)x \in L^1\big([0,1]^2 : Y\big).$$

Furthermore, it can be simply shown with the help of the Fubini theorem, the dominated convergence theorem and an elementary argumentation that the function $\hat{F}_{\Omega} : \mathbb{R}^2 \times X \to Y$ is continuous. Let a real number $L > 0$ and a bounded subset B of X be fixed, and let $(t,s) \in \mathbb{R}^2$ satisfy $|t-s| \leqslant L$. By Bochner's criterion, there exist a subsequence (b_{k_l}, b_{k_l}) of (b_k, b_k) and a function $\varphi^* : \mathbb{R} \to \mathbb{C}$ such that $\lim_{l\to+\infty} \varphi(r + b_{k_l}) = \varphi^*(r)$, uniformly in $r \in \mathbb{R}$. Set

$$[F^*(t,s;x)](u_1,u_2) := e^{\int_{s+u_2}^{t+u_1} \varphi^*(r)\,dr} T\big(t-s+(u_1-u_2)\big)x, \quad (t,s) \in \mathbb{R}^2,\ x \in X.$$

Arguing as in Example 8.1.5, we can simply show that this is the right choice of the required limit function and that the function $F(\cdot,\cdot;\cdot)$ is Stepanov $(\Omega,1)$-$(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, as claimed. Observe, finally, that the function $F(\cdot,\cdot;\cdot)$ is not $(\mathrm{R},\mathcal{B})$-multi-almost automorphic in general since it is not necessarily continuous in general as well as that the higher-dimensional analogue of this example can be constructed in the same way as in final part of the above-mentioned example. It would be valuable to reconsider the conclusions established in Example 8.1.6, provided that the functions $f_j(\cdot)$ from this example are Stepanov p-almost automorphic (Stepanov p-almost periodic) for some finite real exponent $p \geqslant 1$.

Example 8.2.5. Suppose that R is any collection of sequences in $\mathbb{R}^n$ such that each subsequence of a sequence $(\mathbf{b}_k) \in$ R also belongs to R, as well as that R′ is any collection of sequences in $\mathbb{R}^m$ such that each subsequence of a sequence $(\mathbf{b}'_k) \in$ R′ also belongs to R′. Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a Stepanov 1-bounded, Stepanov $(\Omega_1, 1)$-R-almost automorphic function $(1 \leqslant i \leqslant p)$, and $g_j : \mathbb{R}^m \to \mathbb{R}$ be a Stepanov 1-bounded, Stepanov $(\Omega_1, 1)$-R′-almost automorphic function $(1 \leqslant j \leqslant q)$. Define the functions $F : \mathbb{R}^n \to \mathbb{R}^q$ by $F(\mathbf{t}) := \sum_{i=1}^{p} f_i(\mathbf{t})e_i$ and $G : \mathbb{R}^m \to \mathbb{R}^q$ by $G(\mathbf{s}) := \sum_{j=1}^{q} g_j(\mathbf{s})e_j$. Define also the function $F \otimes G : \mathbb{R}^n \times \mathbb{R}^m \to M_{p\times q}(\mathbb{R})$ by (8.5), for any $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{s} \in \mathbb{R}^m$. Set, for every $\mathbf{u} \in \Omega_1$, $\mathbf{v} \in \Omega_2$, $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{s} \in \mathbb{R}^m$,

$$[F \otimes G(\mathbf{t}, \mathbf{s})](\mathbf{u}, \mathbf{v})$$
$$:= \begin{pmatrix} [f_1^*(\mathbf{t})](\mathbf{u})[g_1^*(\mathbf{s})](\mathbf{v}) & [f_1^*(\mathbf{t})](\mathbf{u})[g_2^*(\mathbf{s})](\mathbf{v}) & \cdots & [f_1^*(\mathbf{t})](\mathbf{u})[g_q^*(\mathbf{s})](\mathbf{v}) \\ [f_2^*(\mathbf{t})](\mathbf{u})[g_1^*(\mathbf{s})](\mathbf{v}) & [f_2^*(\mathbf{t})](\mathbf{u})[g_2^*(\mathbf{s})](\mathbf{v}) & \cdots & [f_2^*(\mathbf{t})](\mathbf{u})[g_q^*(\mathbf{s})](\mathbf{v}) \\ \vdots & \vdots & \ddots & \vdots \\ [f_p^*(\mathbf{t})](\mathbf{u})[g_1^*(\mathbf{s})](\mathbf{v}) & [f_p^*(\mathbf{t})](\mathbf{u})[g_2^*(\mathbf{s})](\mathbf{v}) & \cdots & [f_p^*(\mathbf{t})](\mathbf{u})[g_q^*(\mathbf{s})](\mathbf{v}) \end{pmatrix}.$$

Then it is not difficult to prove that $F \otimes G$ is Stepanov $(\Omega_1 \times \Omega_2, 1)$-(R × R′)-almost automorphic, where $\mathrm{R} \times \mathrm{R}' := \{(\mathbf{b}, \mathbf{b}') : \mathbf{b} \in \mathrm{R}, \mathbf{b}' \in \mathrm{R}'\}$. If, in addition to the above, for each $i \in \mathbb{N}_p$ we see that $f_i : \mathbb{R}^n \to \mathbb{R}$ is a Stepanov $(\Omega_1, 1)$-$(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-almost automorphic function as well as that for each $j \in \mathbb{N}_q$ we see that $g_j : \mathbb{R}^n \to \mathbb{R}$ is a Stepanov $(\Omega_2, 1)$-$(\mathrm{R}', \mathrm{P}'_{\mathrm{R}'})$-almost automorphic function, then the function $F \otimes G$ is Stepanov $(\Omega_1 \times \Omega_2, 1)$-$(\mathrm{R} \times \mathrm{R}', \mathrm{P}''_{\mathrm{R}\times\mathrm{R}'})$-almost automorphic function, provided that for each sequence $\mathbf{b}$ from R ($\mathbf{c}$ from R′) each set of the collection $\mathrm{P}_{\mathbf{b}}$ ($\mathrm{P}_{\mathbf{c}}$) belongs to the collection $\mathrm{P}_{\mathbf{b}'}$ ($\mathrm{P}_{\mathbf{c}'}$) for any subsequence $\mathbf{b}'$ of $\mathbf{b}$ ($\mathbf{c}'$ of $\mathbf{c}$) and for each sequence $(\mathbf{b}; \mathbf{c})$ belonging to R × R′ the collection $\mathrm{P}''_{(\mathbf{b};\mathbf{c})}$ consists of all direct products of sets from the collections $\mathrm{P}_{\mathbf{b}}$ and $\mathrm{P}'_{\mathbf{c}}$.

Furthermore, let us consider the notion introduced in Definition 8.2.1(iii). If the function $\hat{F} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic, then for each set $B \in \mathcal{B}$ and sequence $(\mathbf{b}_k; x_k) \in \mathrm{R}_X$ there exist a subsequence $(\mathbf{b}_{k_l}; x_{k_l})$ of $(\mathbf{b}_k; x_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{l\to+\infty} \left\| F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_l}; x + x_{k_l}) - [F^*(\mathbf{t}; x)](\mathbf{u}) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0 \tag{8.32}$$

and

$$\lim_{l\to+\infty} \left\| [F^*(\mathbf{t} + \mathbf{u} - \mathbf{b}_{k_l}; x - x_{k_l}) - F(\mathbf{t} + \mathbf{u}; x) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. In the general case, it is very difficult to deduce the existence of a function $G : \mathbb{R}^n \times X \to Y$ such that $G(\mathbf{t} + \mathbf{u}; x) = [F^*(\mathbf{t}; x)](\mathbf{u})$ for all $x \in B$ and a. e. $\mathbf{t} \in \mathbb{R}^n$, $u \in \Omega$. But this can be always done provided that $\Omega = [0, 1]^n$, which can be simply deduced by using the first limit equality (8.32) and the proof of

[373, Proposition 3.1] with appropriate modifications (we can write down the set $\mathbb{R}^n$ as the union of sets $\mathbf{k} + \Omega$ when $\mathbf{k} \in \mathbb{Z}^n$ and define after that $G(\mathbf{t}; x) := [F^*(\mathbf{k}; x)](\mathbf{t} - \mathbf{k})$ if $\mathbf{t} \in \mathbf{k} + \Omega$ for some $\mathbf{k} \in \mathbb{Z}^n$).

Up to now, we have clarified several embedding type results for the spaces of Stepanov multi-dimensional almost periodic functions. These results can be reformulated for the corresponding spaces of Stepanov multi-dimensional almost automorphic functions since their proofs simply follow by applying Lemma 1.1.7. For example, if $p \in D_+(\Omega)$ and $1 \leqslant p_- \leqslant p(\mathbf{u}) \leqslant p^+ < +\infty$ for a. e. $\mathbf{u} \in \Omega$, then any Stepanov (Ω, p^+)-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic function is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic and any Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic function is Stepanov (Ω, p^-)-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic; in particular, any Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic function is Stepanov $(\Omega, 1)$-$(\mathrm{R}_X, \mathcal{B})$-almost automorphic (this statement actually holds for any $p \in \mathcal{P}(\Omega)$).

Now we will clarify the following simple result.

Proposition 8.2.6. *Suppose that the function $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic, $p \in D_+(\Omega)$ and for each set $B \in \mathcal{B}$ we have $\sup_{\mathbf{t} \in \mathbb{R}^n; x \in B} \|F(\mathbf{t}; x)\|_Y < +\infty$. Then the function $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-almost automorphic.*

Proof. Due to our conclusion from Remark 6.2.8(ii), the function $\hat{F}_\Omega : \mathbb{R}^n \times X \to Y$ is continuous. Since $p \in D_+(\Omega)$, we have $1 \in E^{p(\mathbf{u})}(\Omega)$ so that the final conclusion simply follows by applying the dominated convergence theorem and our assumption that for each set $B \in \mathcal{B}$ we have $\sup_{\mathbf{t} \in \mathbb{R}^n; x \in B} \|F(\mathbf{t}; x)\|_Y < +\infty$. □

We continue by stating the following examples.

Example 8.2.7. Let $\mathcal{F} := \{F \in L^1_{\mathrm{loc}}(\mathbb{R}^n : Y) ; \operatorname{supp}(F) \text{ is compact}\}$, let $p \in \mathcal{P}(\Omega)$ and let R denote any collection of sequences in $\mathbb{R}^n$ such that there exists a sequence $(\mathbf{b}_k) \in \mathrm{R}$ of which any subsequence is unbounded. Then a non-trivial function $F \in \mathcal{F}$ cannot be Stepanov $(\Omega, p(\mathbf{u}))$-R-multi-almost automorphic, which can be shown arguing as in [641, Example 1, Example 2]. On the other hand, as we have already seen, a non-trivial function $F \in \mathcal{F}$ can belong to certain classes of equi-Weyl multi-dimensional almost periodic functions.

Example 8.2.8. Suppose that $\Omega := [0, 1]^n$ and $p(\mathbf{u}) := 1 - \ln(u_1 \cdot u_2 \cdots u_n)$, where $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \Omega$, and $F(x_1, x_2, \ldots, x_n) := \sin(x_1 + x_2 + \cdots + x_n) + \sin(\sqrt{2}(x_1 + x_2 + \cdots + x_n))$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Set $H(\mathbf{t}) := \operatorname{sign}(F(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$. Then the function $H(\cdot)$ is essentially bounded and therefore Stepanov $(\Omega, p(\mathbf{u}))$-bounded. On the other hand, we know that the function $H(\cdot)$ cannot be Stepanov $(\Omega, p(\mathbf{u}))$-almost periodic; the argumentation used for proving this fact in combination with the argumentation used in [641, Example 2] shows that the function $H(\cdot)$ cannot be Stepanov $(\Omega, p(\mathbf{u}))$-R-almost automorphic, where R denotes the collection of all sequences in $\mathbb{R}^n$.

Concerning the pointwise products of Stepanov multi-dimensional almost periodic type functions, we have the following result.

Proposition 8.2.9. *Suppose that* (ST) *and* (C1) *hold with* $\mathrm{R}_X = \mathrm{R}$ *as well as that* $p,\ q,\ r \in \mathcal{P}(\Omega)$ *and* $1/p(\mathbf{u}) + 1/r(\mathbf{u}) = 1/q(\mathbf{u})$. *Suppose, further, that:*

(i) $f : \mathbb{R}^n \to \mathbb{C}$ *is Stepanov-*$(\Omega, r(\mathbf{u}))$*-bounded and Stepanov* $(\Omega, r(\mathbf{u}))$*-*R*-multi-almost automorphic [Stepanov* $(\Omega, r(\mathbf{u}))$*-*$(\mathrm{R}, W^f_{\mathrm{R}})$*-multi-almost automorphic];*

(ii) $F : \mathbb{R}^n \times X \to Y$ *is a Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic [Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, W^F_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic; Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, \mathrm{P}^F_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic] function satisfying*

$$\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \big\|\hat{F}_\Omega(\mathbf{t}; x)\big\|_{L^{p(\mathbf{u})}(\Omega)} < \infty. \tag{8.33}$$

Define

$$F_1(\mathbf{t}; x) := f(\mathbf{t})F(\mathbf{t}; x), \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X,$$

and let

(iii) $W^{F_1}_{\mathcal{B},(\mathbf{b}_k)}(x)$ *be the collection of all sets of the form* $D \cap D'$*, where* $D \in W^F_{\mathcal{B},(\mathbf{b}_k)}(x)$ *and* $D' \in W^f_{(\mathbf{b}_k)}$ *for all* $B \in \mathcal{B}$*,* $(\mathbf{b}_k) \in \mathrm{R}$ *and* $x \in B$ *[*$\mathrm{P}^{F_1}_{\mathcal{B},(\mathbf{b}_k)}$ *be the collection of all sets of the form* $D \cap D'$*, where* $D \in \mathrm{R}^F_{\mathcal{B},(\mathbf{b}_k)}$ *and* $D' \in W^f_{(\mathbf{b}_k)}$ *for all* $B \in \mathcal{B}$ *and* $(\mathbf{b}_k) \in \mathrm{R}$*].*

Then $F_1(\cdot;\cdot)$ *is Stepanov-*$(\Omega, q(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic [Stepanov* $(\Omega, q(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, W^{F_1}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic; Stepanov* $(\Omega, q(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B}, \mathrm{P}^{F_1}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic].*

Proof. Let $(\mathbf{b}_k) \in \mathrm{R}$ and $B \in \mathcal{B}$ be given. Then we have

$$\begin{aligned}&\hat{F}_{1\Omega}(\mathbf{t}'; x') - \hat{F}_{1\Omega}(\mathbf{t}; x)\\ &\quad = \hat{f}_\Omega(\mathbf{t}') \cdot \big[\hat{F}_\Omega(\mathbf{t}'; x') - \hat{F}_\Omega(\mathbf{t}; x)\big] + \big[\hat{f}_\Omega(\mathbf{t}') - \hat{f}_\Omega(\mathbf{t})\big] \cdot \hat{F}_\Omega(\mathbf{t}; x)\end{aligned}$$

for every $\mathbf{t},\ \mathbf{t}' \in \mathbb{R}^n$ and $x,\ x' \in X$. Since the mapping $\hat{f}_\Omega(\cdot) \in L^{r(\mathbf{u})}(\Omega : \mathbb{C})$ is continuous and the mapping $\hat{F}_\Omega(\cdot;\cdot)$ is continuous, the above equality in combination with the Hölder inequality (see Lemma 1.1.7(i)) shows that the mapping $\hat{F}_{1\Omega}(\cdot;\cdot) \in L^{p(\mathbf{u})}(\Omega : \mathbb{C})$ is continuous, as well. In the remainder of proof, we will consider only the general class of Stepanov-$(\Omega, q(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions. Since (C1) holds, we know that there exist a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ and two functions $f^* : \mathbb{R}^n \to \mathbb{C}$ and $F^* : \mathbb{R}^n \times X \to Y$ such that, for every $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, we have

$$\lim_{l\to+\infty} f(\mathbf{t} + \mathbf{b}_{k_l} + \mathbf{u}) = f^*(\mathbf{t} + \mathbf{u}), \quad \text{and} \quad \lim_{l\to+\infty} f^*(\mathbf{t} + \mathbf{u} - \mathbf{b}_{k_l}) = f^*(\mathbf{t} + \mathbf{u}),$$

for the topology of $L^{r(\mathbf{u})}(\mathbb{R}^n : \mathbb{C})$,

$$\lim_{l\to+\infty} F(\mathbf{t} + \mathbf{b}_{k_l} + \mathbf{u}; x) = \big[F^*(\mathbf{t}; x)\big](\mathbf{u}),$$

and

$$\lim_{l\to+\infty}\left[F^*(\mathbf{t}-\mathbf{b}_{k_l};x)\right](\mathbf{u})=F(\mathbf{t}+\mathbf{u};x),$$

for the topology of $L^{p(\mathbf{u})}(\mathbb{R}^n : Y)$. Define

$$\left[F_1^*(\mathbf{t};x)\right](\mathbf{u}) := \left[f^*(\mathbf{t})\right](\mathbf{u})\cdot\left[F(\mathbf{t};x)\right](\mathbf{u}), \quad \mathbf{t}\in\mathbb{R}^n,\ x\in X,\ u\in\Omega.$$

Since

$$\begin{aligned}F_1(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u};x)&-\left[F_1(\mathbf{t};x)\right](\mathbf{u})\\ &=\left[f(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u})-f^*(\mathbf{t}+\mathbf{u})\right]\cdot F(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u};x)\\ &\quad+f^*(\mathbf{t}+\mathbf{u})\cdot\left[F(\mathbf{t}+\mathbf{b}_{k_l}+\mathbf{u};x)-\left[F^*(\mathbf{t};x)\right](\mathbf{u})\right]\end{aligned}$$

for every $\mathbf{t}\in\mathbb{R}^n$, $\mathbf{u}\in\Omega$ and $x\in B$, the first limit equality follows from the Hölder inequality, the obvious estimate $\sup_{\mathbf{t}\in\mathbb{R}^n}\|f^*(\mathbf{t})\|_{L^{r(\mathbf{u})}(\Omega)}<\infty$ and (8.33). The second limit equality can be proved similarly. □

Now we would like to present the following illustrative example.

Example 8.2.10. Suppose that $\Omega=[0,1]^n$, R denotes the collection of all sequences in $\mathbb{R}^n$ and, for every $i\in\mathbb{N}_n$, the function $f_i(\cdot)$ is Stepanov (Ω,p)-R-almost automorphic for every finite exponent $p\in[1,\infty)$. Set

$$F(t_1,t_2,\ldots,t_n):=f_1(t_1)f_2(t_2)\cdots f_n(t_n),\quad \mathbf{t}=(t_1,t_2,\ldots,t_n)\in\mathbb{R}^n.$$

Applying Proposition 8.2.9, we see that the function $\mathbf{t}\mapsto F(\mathbf{t})$, $\mathbf{t}\in\mathbb{R}^n$ is Stepanov-$(\Omega, p(\mathbf{u}))$-R-almost automorphic for any $p\in D_+(\Omega)$.

We also have the following result.

Proposition 8.2.11. *Suppose that $h\in L^1(\mathbb{R}^n)$, $p\in D_+(\Omega)$ and $F:\mathbb{R}^n\times X\to Y$ is a Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost automorphic function satisfying that for each $B\in\mathcal{B}$ there exists a finite real number $\varepsilon_B>0$ such that*

$$\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^{\cdot}}\left\|F(\mathbf{t},x)\right\|_Y<+\infty,$$

where $B^{\cdot}\equiv B^{\circ}\cup\bigcup_{x\in\partial B}B(x,\varepsilon_B)$. Let condition (CI) *hold, where:*
(CI) $\mathrm{R}_X=\mathrm{R}$*, or $X\in\mathcal{B}$ and R_X is general.*

Then the function

$$(h*F)(\mathbf{t};x):=\int_{\mathbb{R}^n}h(\sigma)F(\mathbf{t}-\sigma;x)\,d\sigma,\quad \mathbf{t}\in\mathbb{R}^n,\ x\in X,$$

is well defined, Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathrm{R}_X,\mathcal{B})$-multi-almost automorphic, and for each $B\in\mathcal{B}$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^{\cdot}}\|(h*F)(\mathbf{t};x)\|_Y<+\infty$.

Proof. Arguing similarly to before, we may conclude that the function $(h * F)(\cdot;\cdot)$ is well defined and that $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^{\cdot}} \|(h * F)(\mathbf{t};x)\|_Y < +\infty$ for all $B \in \mathcal{B}$. By definition, the function $\hat{F}_\Omega(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic; furthermore, since we have assumed that $p \in D_+(\Omega)$, an application of Lemma 1.1.7(ii) shows that for each set $B \in \mathcal{B}$ we have

$$\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^{\cdot}} \|\hat{F}_\Omega(\mathbf{t};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)} < +\infty.$$

This enables one to conclude that the function $h * \hat{F}_\Omega(\cdot;\cdot)$ is well defined and $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic. Now the final conclusion follows from the equality

$$h * \hat{F}_\Omega = h \hat{*} F_\Omega$$

and a corresponding definition of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy. □

For the sequel, we need the following auxiliary lemma.

Lemma 8.2.12. *Suppose that* (ST) *and* (C1) *hold. If for every* $\varepsilon > 0$, $B \in \mathcal{B}$ *and* $(\mathbf{b}_k) \in \mathrm{R}$, *there exist a subsequence* $(\mathbf{b}_{k_l}^{\varepsilon})$ *of* $(\mathbf{b}_k)$ *and a function* $F^{*,\varepsilon} : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ *such that, for every* $x \in B$ *and* $\mathbf{t} \in \mathbb{R}^n$ *[for every* $x \in B$, *for every* $D \in W_{B,(\mathbf{b}_k)}(x)$ *and for every* $\mathbf{t} \in D$; *for every* $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ *and for every* $(\mathbf{t};x) \in D$*], there exists* $l_0 \in \mathbb{N}$ *such that, for every* $l \geqslant l_0$, *we have*

$$\left\|F(\mathbf{t} + \mathbf{u} + (b_{k_l}^1, \dots, b_{k_l}^n)^{\varepsilon}; x) - [F^{*,\varepsilon}(\mathbf{t};x)](\mathbf{u})\right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon/2$$

and

$$\left\|[F^{*,\varepsilon}(\mathbf{t} - (b_{k_l}^1, \dots, b_{k_l}^n)^{\varepsilon}; x)](\mathbf{u}) - F(\mathbf{t} + \mathbf{u}; x)\right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon/2,$$

then $F(\cdot;\cdot)$ *is Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic [Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic; Stepanov* $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic].*

Proof. Let $B \in \mathcal{B}$ and $(\mathbf{b}_k) \in \mathrm{R}$ be fixed. Suppose that $s \in \mathbb{N}$. Using our assumption, we see that there exist a subsequence $(\mathbf{b}_k^s)$ of $(\mathbf{b}_k)$ and a function $F_s^* : \mathbb{R}^n \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ such that, for every $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [for every $x \in B$, for every $D \in W_{B,(\mathbf{b}_k)}(x)$ and for every $\mathbf{t} \in D$; for every $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ and for every $(\mathbf{t};x) \in D$], there exists $k_0 \in \mathbb{N}$ such that, for every $k \geqslant k_0$, we have

$$\left\|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_k^s; x) - [F_s^*(\mathbf{t};x)](\mathbf{u})\right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant 1/s$$

and

$$\left\|[F_s^*(\mathbf{t} - \mathbf{b}_k^s; x)](\mathbf{u}) - F(\mathbf{t} + \mathbf{u}; x)\right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant 1/s;$$

furthermore, since (C1) holds, we may assume that $(\mathbf{b}_k^{s+1})$ is a subsequence of $(\mathbf{b}_k^s)$ for all $s \in \mathbb{N}$. It is not difficult to prove that, for every fixed point $\mathbf{t} \in \mathbb{R}^n$ and element $x \in X$, the sequence $(F_s^*(\mathbf{t}; x))$ is a Cauchy sequence in $L^{p(\mathbf{u})}(\Omega : Y)$ and therefore convergent; indeed, let $\varepsilon > 0$ be given and let $s_0 \in \mathbb{N}$ satisfy $1/2s < \varepsilon$. Suppose that $s_1 \geqslant s$ and $s_2 \geqslant s$. Then there exist two sufficiently large integers $k,\ k' \in \mathbb{N}$ such that

$$\begin{aligned}
&\big\|[F_{s_1}^*(\mathbf{t}; x)](\mathbf{u}) - [F_{s_2}^*(\mathbf{t}; x)](\mathbf{u})\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\quad\leqslant \big\|[F_{s_1}^*(\mathbf{t}; x)](\mathbf{u}) - F(\mathbf{t}+\mathbf{u}+\mathbf{b}_k^{s_1}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\qquad+ \big\|F(\mathbf{t}+\mathbf{u}+\mathbf{b}_k^{s_1}; x) - F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k'}^{s_2}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\qquad+ \big\|F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k'}^{s_2}; x) - [F_{s_2}^*(\mathbf{t}; x)](\mathbf{u})\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\quad= \big\|[F_{s_1}^*(\mathbf{t}; x)](\mathbf{u}) - F(\mathbf{t}+\mathbf{u}+\mathbf{b}_k^{s_1}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\qquad+ \big\|F(\mathbf{t}+\mathbf{u}+\mathbf{b}_{k'}^{s_2}; x) - [F_{s_2}^*(\mathbf{t}; x)](\mathbf{u})\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\quad\leqslant (1/s_1) + (1/s_2) \leqslant 1/2s < \varepsilon.
\end{aligned}$$

Set $F^*(\mathbf{t}; x) := \lim_{s\to+\infty} F_s^*(\mathbf{t}; x)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in B$ and $\mathbf{c}_k := \mathbf{b}_k^k$, $k \in \mathbb{N}$ (with a little loss of generality; we can always use here the well known diagonal procedure). Observe that, in the case of consideration of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy, for every $x \in B$ and for every $D \in W_{B,(\mathbf{b}_k)}(x)$, the above limit is uniform in $\mathbf{t} \in D$, as well as that, in the case of consideration of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy, for every $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$, the above limit is uniform in $(\mathbf{t}; x) \in D$. Furthermore, for every $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [for every $x \in B$, for every $D \in W_{B,(\mathbf{b}_k)}(x)$ and for every $\mathbf{t} \in D$; for every $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ and for every $(\mathbf{t}; x) \in D$], we have the existence of a sufficiently large integer $k \in \mathbb{N}$ such that

$$\begin{aligned}
&\big\|F(\mathbf{t}+\mathbf{c}_k+\mathbf{u}) - [F^*(\mathbf{t}; x)](\mathbf{u})\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\quad\leqslant \big\|F(\mathbf{t}+\mathbf{c}_k+\mathbf{u}) - [F_k^*(\mathbf{t}; x)](\mathbf{u})\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\\
&\qquad+ \big\|[F_k^*(\mathbf{t}; x)](\mathbf{u}) - [F^*(\mathbf{t}; x)](\mathbf{u})\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant (\varepsilon/2) + (\varepsilon/2) = \varepsilon.
\end{aligned}$$

This completes the proof of lemma. ☐

Now we are able to state the following result.

Theorem 8.2.13. *Suppose that $k \in \mathbb{N}$, (ST) and (C1) hold. Let $\mathcal{B}$ be any family of compact subsets of X, let $\mathcal{B}_f$ be the collection of all finite subsets of X, and let $F : \mathbb{R}^n \times X \to Y$ satisfy the following conditions:*

(i) *The function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f)$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f, W^f_{\mathcal{B}_f,\mathrm{R}})$-multi-almost automorphic; Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f, \mathrm{P}^f_{\mathcal{B}_f,\mathrm{R}})$-multi-almost automorphic].*
(ii) *For every $B \in \mathcal{B}$, $(\mathbf{b}) = (\mathbf{b}_k) \in \mathrm{R}$ and $\varepsilon > 0$, there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that, for every point $\mathbf{t} \in \mathbb{R}^n$ and for every two*

elements $x', x'' \in B$, there exist two integers $m_0, l_0 \in \mathbb{N}$ such that, for every integer $m \geqslant m_0$, we have

$$\|x' - x''\| \leqslant \delta \\ \implies \|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x') - F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x'')\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon/2 \tag{8.34}$$

and, for every integer $m \geqslant m_0$ and $l \geqslant l_0$, we have

$$\|x' - x''\| \leqslant \delta \\ \implies \|F(\mathbf{t} + \mathbf{u} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x') - F(\mathbf{t} + \mathbf{u} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x'')\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon/2 \tag{8.35}$$

[for each $B \in \mathcal{B}$, $(\mathbf{b}_k) \in \mathrm{R}$ and $\varepsilon > 0$, there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that, for every two elements $x', x'' \in B$, set $D \in W_{B;(\mathbf{b}_k)}(x')$ and point $\mathbf{t} \in D$, there exist two integers $m_0, l_0 \in \mathbb{N}$ such that, for every integer $m \geqslant m_0$, the implication (8.34) *holds as well as that, for every integer $l \geqslant l_0$ we have $D - \mathbf{b}_{k_l} \subseteq W_{B,(\mathbf{b}_k)}(x') \cap W_{B,(\mathbf{b}_k)}(x'')$ and* (8.35)*; for each $B \in \mathcal{B}$, $(\mathbf{b}_k) \in \mathrm{R}$ and $\varepsilon > 0$, there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that, for every $D \in \mathrm{P}_{(\mathbf{b};\mathbf{x})}$, $(\mathbf{t}; x') \in D$ and $x'' \in B$, there exist two integers $m_0, l_0 \in \mathbb{N}$ such that, for every integer $m \geqslant m_0$, the implication* (8.34) *holds as well as that, for every integer $l \geqslant l_0$ we have $D - (\mathbf{b}_{k_l}, 0) \subseteq \mathrm{P}_{B,(\mathbf{b}_k)}$ and* (8.35)*].*

(iii) *For each set $B \in \mathcal{B}$ and for each finite subset B' of B, we have $W^f_{B';(\mathbf{b}_k)}(x) \supseteq W_{B';(\mathbf{b}_k)}(x)$ for all $x \in B$, $x' \in B'$ and $\mathrm{P}^f_{B';(\mathbf{b}_k)} \supseteq \mathrm{P}_{B;(\mathbf{b})}$.*

Then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic; Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic].

Proof. We may assume that $p(\mathbf{u}) \equiv p \in [1, \infty)$; the proof in the general case can be deduced similarly. Let $\varepsilon > 0$, $B \in \mathcal{B}$ and $(\mathbf{b}; \mathbf{x}) = ((\mathbf{b}_k; x_k)) \in \mathrm{R}_X$ be given. Then there exist a subsequence $(\mathbf{b}_{k_l}) \in \mathrm{R}$ of $(\mathbf{b}_k)$ and a real number $\delta > 0$ such that (ii) holds, which implies that there exists a finite subset $\{x_1', \ldots, x_s'\} \subseteq B$ ($s \in \mathbb{N}$) such that $B \subseteq \bigcup_{i=1}^{l} B(x_i', \delta)$. Due to (i) and (C1), we have the existence of a subsequence of $(\mathbf{b}_{k_l})$ [w. l. o. g. we may assume that this subsequence is equal to the initial sequence $(\mathbf{b}_{k_l})$] and a function $F^* : \mathbb{R}^n \times X \to L^p(\Omega : Y)$ such that, for every $\mathbf{t} \in \mathbb{R}^n$, there exists an integer $m_0 \in \mathbb{N}$ such that, for every $m \geqslant m_0$, we have

$$\left(\int_\Omega \|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x_i') - [F^*(\mathbf{t}; x_i')](\mathbf{u})\|_Y^p \, d\mathbf{u} \right) \leqslant \varepsilon/2, \quad m \geqslant m_0, \ i \in \mathbb{N}_s, \tag{8.36}$$

and

$$\left(\int_\Omega \|[F^*(\mathbf{t} - \mathbf{b}_{k_m}; x_i')](\mathbf{u}) - F(\mathbf{t} + \mathbf{u}; x_i')\|_Y^p \, d\mathbf{u} \right) \leqslant \varepsilon/2, \quad m \geqslant m_0, \ i \in \mathbb{N}_s. \tag{8.37}$$

Let $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ be fixed [let $x \in B$, $D \in W_{B,(\mathbf{b}_k)}(x)$ and $\mathbf{t} \in D$ be fixed; let $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ and $(\mathbf{t}; x) \in D$ be fixed]. By the foregoing, there exists $i \in \mathbb{N}_s$ such that $\|x - x_i'\| \leqslant \delta$. By (ii), we have the existence of an integer $m_1 \in \mathbb{N}$ such that, for every integer $m \geqslant m_1$, one has

$$\|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x) - F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x_i')\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leqslant \varepsilon/2. \tag{8.38}$$

Assume first that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}_f)$-multi-almost automorphic. Then we have

$$\begin{aligned}
&\Big(\int_\Omega \|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x) - [F^*(\mathbf{t}; x_i')](\mathbf{u})\|_Y^p \, d\mathbf{u}\Big)^{\frac{1}{p}} \\
&\quad \leqslant \Big(\int_\Omega \|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x) - F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x_i')\|_Y^p \, d\mathbf{u}\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_\Omega \|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_m}; x_i') - [F^*(\mathbf{t}; x_i')](\mathbf{u})\|_Y^p \, d\mathbf{u}\Big)^{\frac{1}{p}} \\
&\quad \leqslant (\varepsilon/2) + (\varepsilon/2) = \varepsilon, \quad m \geqslant m_0 + m_1,
\end{aligned} \tag{8.39}$$

where (8.39) follows by applying (8.36) and (8.38); in the case of consideration of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy [Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy], we also need to apply condition (iii) and the corresponding assumptions from the issue (ii). For the second limit equation, we use the estimates

$$\begin{aligned}
&\Big(\int_\Omega \|[F^*(\mathbf{t} - \mathbf{b}_{k_l}; x)](\mathbf{u}) - F(\mathbf{t} + \mathbf{u}; x)\|_Y^p \, d\mathbf{u}\Big)^{\frac{1}{p}} \\
&\quad \leqslant \Big(\int_\Omega \|[F^*(\mathbf{t} - \mathbf{b}_{k_l}; x)](\mathbf{u}) - [F^*(\mathbf{t} - \mathbf{b}_{k_l}; x_i')](\mathbf{u})\|_Y^p \, d\mathbf{u}\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_\Omega \|[F^*(\mathbf{t} - \mathbf{b}_{k_l}; x_i')](\mathbf{u}) - F(\mathbf{t} + \mathbf{u}; x)\|_Y^p \, d\mathbf{u}\Big)^{\frac{1}{p}}, \quad m \geqslant m_0 + m_1,\ l \geqslant l_0,
\end{aligned}$$

where we have applied (8.35), (8.37), the limit equality

$$\lim_{l \to +\infty} F(\mathbf{t} + \mathbf{u} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x') = [F^*(\mathbf{t} - \mathbf{b}_{k_l}; x_i')](\mathbf{u}),$$

and the corresponding limit equality with the element x_i' replaced therein with the element x. Then the final conclusion follows from Lemma 8.2.12. □

It is clear that some known statements for multi-dimensional almost automorphic functions can be straightforwardly extended to the corresponding Stepanov classes by using the properties of the Bochner transform. For example, suppose that $F : \mathbb{R}^n \times X \to Y$ is a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function, where R denotes the collection of all sequences in $\mathbb{R}^n$ and $\mathcal{B}$ denotes any collection of compact subsets of X. If there exists a finite real constant $L > 0$ such that (6.49) holds, then, for every set $B \in \mathcal{B}$, we see that the set $\{\hat{F}_\Omega(\mathbf{t}, x) : \mathbf{t} \in \mathbb{R}^n,\ x \in B\}$ is relatively compact in $L^{p(\mathbf{u})}(\Omega : Y)$. Similarly, we can clarify the supremum formula for Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions and some sufficient conditions ensuring the invariance of various types of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphy under the composition with continuous functions.

Suppose now that $\mathbb{D} \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. Any of the function spaces from Definition 8.2.1 can be extended by introducing the corresponding space of $\mathbb{D}$-asymptotically Stepanov multi-almost automorphic functions; for example, we can introduce the following notion.

Definition 8.2.14. We say that the function $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if there exist a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function $H : \mathbb{R}^n \times X \to Y$ and a function $Q \in S^{\Omega, p(\mathbf{u})}_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ such that $F(\mathbf{t}; x) = H(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for a. e. $\mathbf{t} \in \mathbb{R}^n$ and all $x \in X$. If $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we also say that the function $F(\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-R-multi-almost automorphic.

Using the Bochner transform, we can formulate a great number of corresponding statements for ($\mathbb{D}$-asymptotically) Stepanov multi-dimensional almost automorphic functions. For example, we can prove the following.

Proposition 8.2.15. *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. If for each $B \in \mathcal{B}$ there exists $\varepsilon_B > 0$ such that*

$$\lim_{j \to +\infty} \sup_{\mathbf{t} \in \Lambda; x \in B^{\cdot}} \left\| F_j(\mathbf{t} + \mathbf{u}; x) - F(\mathbf{t} + \mathbf{u}; x) \right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

where $B^{\cdot} \equiv B^{\circ} \cup \bigcup_{x \in \partial B} B(x, \varepsilon_B)$, then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.

It is worth noting that any such space of $\mathbb{D}$-asymptotically Stepanov multi-dimensional almost automorphic functions has the linear vector structure provided that the collection R (R_X) has the property that, for every sequence which belongs to R (R_X), any its subsequence belongs to R (R_X); under certain conditions, the decomposition of an $\mathbb{D}$-asymptotically Stepanov multi-almost automorphic function into its Stepanov multi-almost automorphic part and the corrective part is unique, which simply follows from an application of the deduced supremum formula. We can simply transfer the corresponding parts of the above-mentioned proposition to $\mathbb{D}$-asymptotically Stepanov multi-almost automorphic functions.

Now we would like to clarify the following result.

Theorem 8.2.16.

(i) *Suppose that* $\Omega = [0,1]^n$*, the collection* R_X *(*R*) has the property that, for every sequence which belongs to* R_X *(*R*), any its subsequence belongs to* R_X *(*R*) and the function* $F : \mathbb{R}^n \times X \to Y$ *is Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic (Stepanov* $(\Omega, p(\mathbf{u}))$*-*$(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic). If the function* $F(\cdot;\cdot)$ *is uniformly convergent on* $\mathbb{R}^n \times X$ *(if for each* $B \in \mathcal{B}$ *there exists* $\varepsilon_B > 0$ *such that the function* $F(\cdot;\cdot)$ *is uniformly convergent on* $\mathbb{R}^n \times B^{\cdot}$*, where* $B^{\cdot} := B^{\circ} \cup \bigcup_{x \in \partial B} B(x, \varepsilon_B)$*), then the function* $F(\cdot;\cdot)$ *is* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic ((*$\mathrm{R}, \mathcal{B}$*)-multi-almost automorphic).*

(ii) *Suppose that* $\Omega = [0,1]^n$ *and the function* $F : \mathbb{R}^n \to Y$ *is a Stepanov* $\mathbb{R}^n$*-asymptotically* $(\Omega, p(\mathbf{u}))$*-*R*-multi-almost automorphic, where* R *denotes the collection of all sequences in* $\mathbb{R}^n$*. If the function* $F(\cdot)$ *is uniformly continuous, then* $F(\cdot)$ *is asymptotically almost automorphic.*

Proof. We will prove the part (i) only for $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. Define, for every $s \in \mathbb{N}$, $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$, $F_s(\mathbf{t}; x) := \int_\Omega F(\mathbf{t} + (\mathbf{u}/s); x)\, d\mathbf{u}$. Suppose that an integer $s \in \mathbb{N}$ is fixed. In order to prove that the function $F_s(\cdot;\cdot)$ is continuous at the fixed point $(\mathbf{t}; x) \in \mathbb{R}^n \times X$, let us take arbitrary real number $\varepsilon > 0$ and choose after that a set $B \in \mathcal{B}$ such that $x \in B$. Then there exists a real number $\varepsilon_B > 0$ such that the function $F(\cdot;\cdot)$ is uniformly convergent on $\mathbb{R}^n \times B^{\cdot}$. Using this fact, the required continuity of the function $F_s(\cdot;\cdot)$ at $(\mathbf{t}; x)$ follows from the equality

$$F_s(\mathbf{t}; x) - F_s(\mathbf{t}'; x') = \int_\Omega [F(\mathbf{t} + (\mathbf{u}/s); x) - F(\mathbf{t}' + (\mathbf{u}/s); x')]\, d\mathbf{u}$$

and the fact that for each sufficiently small real number $\delta > 0$ we see that $B(x', \delta) \subseteq B^{\cdot}$. Similarly we can prove that for each set $B \in \mathcal{B}$ the sequence $(F_s(\cdot;\cdot))$ converges uniformly to the function $F(\cdot;\cdot)$ on the set $\mathbb{R}^n \times B^{\circ}$. In the remainder of the proof of (i) we may assume without loss of generality (see Lemma 1.1.7(ii)) that $p(\mathbf{u}) \equiv 1$. It suffices to show that the function $F_s(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic for all $s \in \mathbb{N}$. Let a sequence $(\mathbf{b}_k) \in \mathrm{R}$ and a set $B \in \mathcal{B}$ be given. By our assumption, we have the existence a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to L^1(\Omega : Y)$ such that $\lim_{l \to +\infty} F(\mathbf{t} + \cdot + \mathbf{b}_{k_l}; x) = [F^*(\mathbf{t}; x)](\cdot)$ and $\lim_{l \to +\infty} [F^*(\mathbf{t} - \mathbf{b}_{k_l}; x)](\cdot) = F(\mathbf{t} + \cdot; x)$ for the topology of $L^1(\Omega : Y)$. Define, for fixed integer $s \in \mathbb{N}$,

$$F_s^*(\mathbf{t}; x) := s^n \int_{\Omega/s} [F^*(\mathbf{t}; x)](\mathbf{u})\, d\mathbf{u}, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X.$$

Then, for every $s \in \mathbb{N}$, $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, we have

$$\begin{aligned} \|F_s(\mathbf{t} + \mathbf{b}_{k_l}) - F_s^*(\mathbf{t}; x)\|_Y &= \Big\|s^n \int_{\Omega/s} [F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_l}; x)\, d\mathbf{u} - s^n \int_{\Omega/s} [F^*(\mathbf{t}; x)](\mathbf{u})\, d\mathbf{u}\Big\|_Y \\ &\leqslant s^n \int_\Omega \|F(\mathbf{t} + \mathbf{u} + \mathbf{b}_{k_l}; x)\, d\mathbf{u} - [F^*(\mathbf{t}; x)](\mathbf{u})\|_Y\, d\mathbf{u} \to 0, \quad l \to +\infty. \end{aligned}$$

We can similarly prove the second limit equation. The proof of (ii) follows similarly to the one-dimensional case. □

The composition principles for one-dimensional Stepanov p-almost automorphic type functions ($1 \leqslant p < \infty$), established by Z. Fan, J. Liang, T.-J. Xiao [422] and H.-S. Ding, J. Liang, T.-J. Xiao [387], have recently been reconsidered and slightly generalized by T. Diagana and M. Kostić [373, Section 4] for one-dimensional Stepanov $p(x)$-almost automorphic type functions (see also the research article [639]). The above-mentioned results admit straightforward reformulations in the multi-dimensional setting and, because of that, we will not reconsider these results here (cf. also [266, Section 4] for more details given in the almost periodic case). For simplicity, we will not consider here various questions about the invariance of Stepanov multi-dimensional almost automorphic properties under the actions of convolution products.

Considering applications, we would like to make a few noteworthy observations concerning the homogeneous heat equation with nonlocal diffusion,

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } [0, +\infty) \times \mathbb{R}^n, \\ u(0, x) = F(x) \quad & \text{in } \mathbb{R}^n \times \{0\}. \end{aligned} \tag{8.40}$$

Let $X = C_b(\mathbb{R}^n : \mathbb{C})$, the Banach space of bounded continuous functions on $\mathbb{R}^n$ equipped with the sup-norm. Then we know that the Gaussian semigroup $(G(t))_{t\geqslant 0}$ is a bounded holomorphic semigroup which is not strongly continuous at zero, generated by the Laplacian Δ_x with maximal distributional domain (see [82, Example 3.7.6, Example 3.7.8] for more details). Under certain conditions, the unique solution of (8.40) is given by $(t, x) \mapsto (G(t)F)(x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Let a number $t_0 > 0$ be fixed, and let $p \in D_+(\Omega)$. Then Proposition 8.2.11 shows that the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is bounded, Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic provided that R is any non-empty collection of sequences in $\mathbb{R}^n$ and the function $F(\cdot)$ is bounded, Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic. We can similarly reconsider the conclusions obtained in Example 1 for Stepanov multi-dimensional almost automorphic type inhomogeneities.

Let us consider now the wave equation (8.24) in $\mathbb{R}^3$. Assume, in the already considered Kirchhoff formula, that a number $t_0 > 0$ is fixed as well as that the functions $g(\cdot)$, $\nabla g(\cdot)$ and $h(\cdot)$ are bounded and Stepanov $([0,1]^3, 1)$-R-multi-almost automorphic, where R is any collection of sequences in $\mathbb{R}^3$ such that, for every sequence $(\mathbf{b}_k) \in \mathrm{R}$, any subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ also belongs to R. Using the dominated convergence theorem and the Fubini theorem, we can simply conclude that the function $x \mapsto u(t_0, x)$, $x \in \mathbb{R}^3$ is likewise bounded and Stepanov $([0,1]^3, 1)$-R-multi-almost automorphic. We can similarly consider the Poisson formula and the wave equation in $\mathbb{R}^2$.

Let us finally consider the one-dimensional case. Then the unique regular solution of wave equation is given by d'Alembert formula. If we suppose that the functions $g(\cdot)$

and $h^{[1]}(\cdot) \equiv \int_0^{\cdot} h(s)\,ds$ are Stepanov 1-almost automorphic, then we can simply prove with the help of the dominated convergence theorem and the Fubini theorem that the solution $u(x,t)$ is Stepanov 1-almost automorphic in the variable $(x,t) \in \mathbb{R}^2$.

8.3 Weyl almost automorphic functions and applications

In this section, we reconsider the notion of Weyl p-almost automorphy introduced by S. Abbas [4] in 2012 and propose the following notions of Weyl p-almost automorphy: the Weyl p-almost automorphy of type 1, the Weyl p-almost automorphy of type 2 and the joint Weyl p-almost automorphy ($1 \leqslant p < \infty$). Furthermore, we introduce and analyze the multi-dimensional analogues of these concepts by using the definitions and results from the theory of Lebesgue spaces with variable exponents. Several illustrative examples, open problems and applications to the abstract Volterra integro-differential equations are presented. It should be also noted that we present some new results about the (equi-)Weyl almost periodic functions here; for example, we prove by a simple example that for each finite number $p \geqslant 1$ there exists a Weyl p-almost periodic function $f : \mathbb{R} \to [0,\infty)$ satisfying that $f(\cdot)$ is Weyl p-almost automorphic, neither Weyl p-almost automorphic of type 1 nor jointly Weyl p-almost automorphic, as well as that $f(\cdot)$ is not Besicovitch-p-almost periodic (Besicovitch p-bounded) and has no finite mean value. See Theorem 8.3.8 and Example 8.3.20 for more details (in Theorem 8.3.10, we analyze the Weyl almost automorphic properties of the Heaviside function; both results, Theorem 8.3.8 and Theorem 8.3.10, can be formulated as examples but we have decided to formulate them as theorems because of their indisputable theoretical novelties).

The organization and main ideas of this section can be briefly summarized as follows. The core is Subsection 8.3.1, in which we introduce three new concepts of Weyl p-almost automorphy for vector-valued functions depending on one real variable. Here we reconsider and give some constructive criticism about the notion introduced by S. Abbas, providing also numerous important examples and relations between the notions of Weyl p-almost automorphy, Weyl p-almost automorphy of type 1, Weyl p-almost automorphy of type 2 and joint Weyl p-almost automorphy (it is worth noting that, in Subsection 8.3.2, we define the notion of Weyl p-almost automorphy of type 2 without using limit functions, which seems to be completely new, not being considered elsewhere in the existing literature; this is motivated by the fact that the spaces of equi-Weyl p-almost periodic functions are not complete with respect to the Weyl metric). We continue our analysis in Subsection 8.3, where we investigate multi-dimensional Weyl almost automorphic functions in Lebesgue spaces with variable exponent; in Subsection 8.3.4, we specifically analyze Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions, which are most important for applications. In Subsection 8.3.5, we apply our theoretical results in the qualitative analysis of solutions for various classes of the abstract Volterra integro-differential equations in Banach spaces; we also present some conclusions, remarks and further perspectives for the investigations of Weyl and Besicovitch classes of almost automorphic functions.

In this section, we deal with the double limits. We write $\lim_{k\to\infty}\lim_{l\to+\infty} a_{k,l} = a$ if for each $k \in \mathbb{N}$ the limit $\lim_{l\to+\infty} a_{k,l}$ exists as well as that for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}_0$ with $k \geqslant k_0$ we see that the limit $\lim_{l\to+\infty} a_{k,l}$ does not exceed ε; the notion of double limit $\lim_{l\to\infty}\lim_{k\to+\infty} a_{k,l} = a$ is understood similarly. See the research article [505] by E. Habil for more details about the subject.

8.3.1 Weyl almost automorphic functions of one real variable

In our previous considerations, we have dealt with the notion of Weyl p-almost automorphic function $f : \mathbb{R} \to X$, where $p \geqslant 1$ (see Definition 2.3.5). The investigation of Weyl p-almost automorphy, introduced by our friend and colleague S. Abbas, has strongly motivated us to carry out many research studies of this intriguing notion by now. Regrettably, we are obliged to emphasize some unclear places in Definition 2.3.5 and the main result of [4].

Remark 8.3.1.
1. In [4, Definition 0.4], it is assumed but not explicitly stated that the integration is taken with respect to the variable x because the limits as $l \to +\infty$ in this definition must tend to zero as $k \to +\infty$, pointwise for every fixed $t \in \mathbb{R}$; hence, t cannot be the variable under which the integration is taken twice. Also, there are two extra left brackets in the integrals mentioned and the considered function has range in X, which are only small typographical errors.
2. The class of Weyl pseudo-almost automorphic functions has been introduced and analyzed in [4], as well. But the above observation becomes crucial in this point because the proof of deduced composition theorem for Weyl pseudo-almost automorphic functions is based on the wrong arguments. In this proof, the integration is taken over the variable t and the author has operated with $\sup_{-\infty<x<+\infty}\cdot$ in the expressions appearing on the right hand side of page 3 (see [4, l.-8, -9]), which is completely meaningless because the integration (if we want to take the pointwise limits for $t \in \mathbb{R}$ as $k \to +\infty$) must be taken with the variable x.
3. Suppose that the function $f(\cdot)$ is Stepanov p-bounded as well as that, for a sequence (s_n) given in advance, the limit function $f^*(\cdot)$ from Definition 2.3.5 is also Stepanov p-bounded. Then for each $k \in \mathbb{N}$ the limits

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\|f(t+s_{n_k}+x)-f^*(t+x)\|^p\,dx$$

and

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f^*(t-s_{n_k}+x)-f(t+x)\right\|^p\,dx$$

from Definition 2.3.5 are equal, which cannot be satisfied in any reasonable definition of Weyl p-almost automorphy. In actual fact, we have

$$\begin{aligned}
&\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f(t+s_{n_k}+x)-f^*(t+x)\right\|^p\,dx\\
&=\lim_{l\to+\infty}\frac{1}{2l}\int_{t-l+s_{n_k}}^{t+l+s_{n_k}}\left\|f(x)-f^*(x-s_{n_k})\right\|^p\,dx\\
&=\lim_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\left\|f(x)-f^*(x-s_{n_k})\right\|^p\,dx \qquad (8.41)\\
&=\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f^*(t-s_{n_k}+x)-f(t+x)\right\|^p\,dx,
\end{aligned}$$

where (8.41) follows from a simple computation involving the Stepanov p-boundedness of functions $f(\cdot)$ and $f^*(\cdot)$.

The genesis of paper [654], from which we take the material of this section, is strongly motivated by the above observations and the following question.

Question 8.3.2. Let $1 \leqslant p < \infty$. If we accept the notion introduced in Definition 2.3.5, is it true that a (compactly, Stepanov p-) almost automorphic function is Weyl p-almost automorphic?

In any expected notion of Weyl p-almost automorphy, this must be satisfied. But, unfortunately, there is no reasonable argumentation which could tell us straightforwardly that the answer to Question 8.3.2 is affirmative. Therefore, we are getting into some unexpected troubles; how to proceed? Our first idea is to replace the limits in Eqs. (2.25)–(2.26).

Definition 8.3.3. Let $p \geqslant 1$. Then we say that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ is Weyl p-almost automorphic of type 1 if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ such that

$$\lim_{l\to+\infty}\lim_{k\to\infty}\frac{1}{2l}\int_{-l}^{l}\left\|f(t+s_{n_k}+x)-f^*(t+x)\right\|^p\,dx=0$$

and

$$\lim_{l\to+\infty}\lim_{k\to\infty}\frac{1}{2l}\int_{-l}^{l}\|f^*(t-s_{n_k}+x)-f(t+x)\|^p\,dx=0$$

for each $t\in\mathbb{R}$. The set of all such functions is denoted by $W^p\mathrm{AA}_1(\mathbb{R}:X)$.

Accepting this definition, it is very simple to show that a Stepanov p-almost automorphic function is Weyl p-almost automorphic of type 1 because for every fixed real numbers t and l we have

$$\lim_{k\to\infty}\frac{1}{2l}\int_{-l}^{l}\|f(t+s_{n_k}+x)-f^*(t+x)\|^p\,dx=0$$

and

$$\lim_{k\to\infty}\frac{1}{2l}\int_{-l}^{l}\|f^*(t-s_{n_k}+x)-f(t+x)\|^p\,dx=0. \tag{8.42}$$

In actual fact, we have

$$\frac{1}{2l}\int_{-l}^{l}\|f(t+s_{n_k}+x)-f^*(t+x)\|^p\,dx\leqslant\frac{1}{2l}\sum_{j=0}^{\lfloor 2l\rfloor}\int_{t-l+j}^{t-l+j+1}\|f(s_{n_k}+x)-f^*(x)\|^p\,dx,$$

which simply implies by definition of Stepanov p-almost automorphy that for any real number $\varepsilon>0$ we can always find a positive integer $k_0\in\mathbb{N}$ such that

$$\frac{1}{2l}\int_{-l}^{l}\|f(t+s_{n_k}+x)-f^*(t+x)\|^p\,dx\leqslant\frac{1+\lfloor 2l\rfloor}{2l}\frac{\varepsilon}{2}\leqslant\varepsilon,\quad k\geqslant k_0;$$

we can similarly prove the limit equation (8.42). On the other hand, it can be easily shown by using the Bochner criterion [631] that any Stepanov p-almost periodic function $f:\mathbb{R}\to X$ is Weyl p-almost automorphic, Weyl p-almost automorphic of type 1, as well as jointly Weyl p-almost automorphic in the sense of the following definition (with the limit function $f^*\equiv f$).

Definition 8.3.4. Let $p\geqslant 1$. Then we say that a function $f\in L^p_{\mathrm{loc}}(\mathbb{R}:X)$ is jointly Weyl p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^*\in L^p_{\mathrm{loc}}(\mathbb{R}:X)$ such that

$$\lim_{(l,k)\to\infty}\frac{1}{2l}\int_{-l}^{l}\|f(t+s_{n_k}+x)-f^*(t+x)\|^p\,dx=0 \tag{8.43}$$

and

$$\lim_{(l,k)\to\infty} \frac{1}{2l} \int_{-l}^{l} \|f^*(t - s_{n_k} + x) - f(t + x)\|^p \, dx = 0 \tag{8.44}$$

for each $t \in \mathbb{R}$. The set of all such functions is denoted by $W^p\mathrm{AA}_j(\mathbb{R} : X)$.

All above conclusions regarding Stepanov p-almost automorphic (periodic) functions and Weyl p-almost automorphic type functions can be formulated in the multidimensional setting (we leave it to the reader to make this precise). Furthermore, it can be simply verified that the joint Weyl p-almost automorphy of the function $f \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ implies its Weyl p-almost automorphy provided that for each $k \in \mathbb{N}$ the two limits in Eqs. (2.25)–(2.26) exist as $l \to +\infty$; a similar comment holds for the notion of Weyl p-almost automorphy of type 1 (see also [505, Theorem 2.13]). Before proceeding, we would like to note that the joint Weyl p-almost automorphy of a function $f \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ does not imply its Stepanov p-almost automorphy.

Example 8.3.5. Let $p \geqslant 1$. Then it is well known that the function $f(t) := \chi_{[0,1/2]}(t)$, $t \in \mathbb{R}$ is not Stepanov p-almost automorphic as well as that this function is equi-Weyl p-almost periodic. It is also jointly Weyl p-almost automorphic with the limit function $f^* \equiv 0$, as easily shown (furthermore, this function is jointly Weyl p-almost automorphic in the sense of Definition 8.3.17(iii) below for any function $\mathbb{F}(l)$ satisfying $\lim_{l\to+\infty} \mathbb{F}(l) = 0$, with the meaning clear). The use of the zero limit function shows that $L^p(\mathbb{R} : X) \subseteq W^p\mathrm{AA}_j(\mathbb{R} : X)$ and $L^p(\mathbb{R} : X) \subseteq \mathrm{AA}W^{\mathbb{F},p,j}_{\mathrm{R}}(\mathbb{R} : X)$, provided that $\lim_{l\to+\infty} \mathbb{F}(l) = 0$ and R denotes the collection of all real sequences; see Definition 8.3.17(iii) for the notion. The above conclusions remain valid for the Weyl p-almost automorphy and the Weyl p-almost automorphy of type 1, with the same choice of the limit function.

From the application point of view, the main drawback of the notion of Weyl p-almost automorphy (Weyl p-almost automorphy of type 1) is presented by the fact that we cannot so simply state satisfactory results about the invariance of Weyl p-almost automorphy (Weyl p-almost automorphy of type 1) under the actions of convolution products. Concerning the joint Weyl p-almost automorphy, the best we can do in the present situation is to state the following result with $p = 1$; the proof is very similar to that of [641, Proposition 7] and therefore is omitted.

Proposition 8.3.6. *Suppose that $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying*

$$\int_0^\infty (1 + t)\|R(t)\|_{L(X,Y)} \, dt < \infty.$$

Let $f \in W^1\mathrm{AA}_j(\mathbb{R} : X)$, and let $f(\cdot)$ be essentially bounded. Then the function $F(\cdot)$, given by (2.46)*, is bounded and belongs to the class $W^1\mathrm{AA}_j(\mathbb{R} : Y)$.*

The situation in which the exponent p is strictly greater that one is a bit complicated. Regarding this problematic, we would like to ask the following.

Question 8.3.7. Can we deduce an analogue of [435, Theorem 1] for joint Weyl p-almost automorphy?

Furthermore, it should be noted that the Weyl p-almost automorphy does not imply Weyl p-almost automorphy of type 1 or joint Weyl p-almost automorphy.

Theorem 8.3.8. *Suppose that $\sigma \in (0,1)$, $p \in [1,\infty)$, $(1-\sigma)p < 1$ and $a > 1-(1-\sigma)p$. Define $f(x) := |x|^{\sigma}$, $x \in \mathbb{R}$. Then the function $f(\cdot)$ is Weyl p-almost automorphic, not Weyl p-almost automorphic of type* 1 *or joint Weyl p-almost automorphic; furthermore, the function $f(\cdot)$ is Weyl p-almost periodic, Besicovitch p-unbounded and has no mean value.*

Proof. It is clear that, for every real numbers ω and t, we have

$$\lim_{l\to+\infty} l^{-a}\int_{-l}^{l}\big||t+x+\omega|^{\sigma}-|t+x|^{\sigma}\big|^{p}\,dx=0,$$

which implies that the function $f(\cdot)$ is Weyl p-almost automorphic with the limit function $f^{*}\equiv f$ (moreover, $f\in AW_{\mathrm{R}}^{\mathbb{F},p}(\mathbb{R}:\mathbb{C})$ in the sense of Definition 8.3.17(i) with $\mathbb{F}(l)\equiv l^{-a}$ and R being the collection of all real sequences). In order to see that, we can apply the Lagrange mean value theorem and the following computation ($l>\max(|t|,|t\pm\omega|)$):

$$\begin{aligned}
&l^{-a}\int_{t-l}^{t+l}\big||x+\omega|^{\sigma}-|x|^{\sigma}\big|^{p}\,dx\\
&\leqslant l^{-a}\sigma^{p}|\omega|^{p}\int_{t-l}^{t+l}\max_{v\in[|x|,|x+\omega|]\cup[|x+\omega|,|x|]}v^{(\sigma-1)p}\,dx\\
&\leqslant l^{-a}\sigma^{p}|\omega|^{p}\int_{t-l}^{t+l}\big[|x|^{(\sigma-1)p}+|x+\omega|^{(\sigma-1)p}\big]\,dx\\
&=l^{-a}\sigma^{p}|\omega|^{p}\big[(t+l)^{1-(1-\sigma)p}+(l-t)^{1-(1-\sigma)p}+(l-t)^{1-(1-\sigma)p}\\
&\quad+(t+l+\omega)^{1-(1-\sigma)p}+(l-t-\omega)^{1-(1-\sigma)p}\big]\\
&\leqslant l^{-a}\sigma^{p}|\omega|^{p}\big[4t^{1-(1-\sigma)p}+4l^{1-(1-\sigma)p}+2\omega^{1-(1-\sigma)p}\big]\to 0,\quad l\to+\infty.
\end{aligned}$$

To see that $f(\cdot)$ is not Weyl p-almost automorphic of type 1, it suffices to show that, for every $l>0$, for every $f^{*}\in L^{p}_{\mathrm{loc}}(\mathbb{R}:\mathbb{C})$ and for every strictly increasing real sequence (s_k) tending to plus infinity, we have

$$\lim_{k\to+\infty}\int_{-l}^{l}\big||x+s_k|^{\sigma}-f^{*}(x)\big|^{p}\,dx=+\infty.$$

This follows from the next computation:

$$\int_{-l}^{l} \big||x+s_k|^{\sigma} - f^*(x)\big|^p \, dx \geqslant \int_{0}^{l} \big|(x+s_k)^{\sigma} - f^*(x)\big|^p \, dx$$
$$\geqslant \int_{0}^{l} 2^{1-p}(x+s_k)^{\sigma p} \, dx - \int_{0}^{l} \big|f^*(x)\big|^p \, dx$$
$$\geqslant 2^{1-p} l s_k^{\sigma p} - \int_{0}^{l} \big|f^*(x)\big|^p \, dx \to +\infty, \quad k \to +\infty.$$

We can similarly prove that $f(\cdot)$ is not jointly Weyl p-almost automorphic.

Concerning the Weyl almost periodic properties of the function $f(\cdot)$, we first observe that this function is not equi-Weyl (p, x, F)-almost periodic for any function $F(l, t)$ which does not depend on t because, for every real numbers $l > 0$ and $t \in \mathbb{R}$, we have

$$\lim_{\tau \to +\infty} \int_{t}^{t+l} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big| \, dx = +\infty.$$

On the other hand, if $a > (1-(1-\sigma)p)/p$, then the function $f(\cdot)$ is Weyl (p, x, F)-almost periodic with $F(l, t) \equiv l^{-a}$. Towards this end, we will prove the following estimate:

$$\int_{t}^{t+l} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx$$
$$\leqslant \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} \big[1 + 2^{1-(\sigma-1)p}\big] + |\tau|^{\sigma p+1} \cdot (2^{\sigma}+1)^p, \tag{8.45}$$

provided $t,\ \tau \in \mathbb{R}$ and $l > |\tau|$. This estimate is clearly satisfied for $\tau = 0$ and, since the right hand side of estimate does not depend on $t \in \mathbb{R}$, it suffices to verify its validity for $\tau > 0$ (we can apply the substitution $x \mapsto x + \tau$). Let it be the case; then we recognize the following subcases:

1. $t \leqslant -\tau$ and $t + l \leqslant 0$. Then we have two possibilities:
   1.1. $-\tau \leqslant t + l$. Then we have $0 \leqslant -(t+\tau) \leqslant l$ and

$$\int_{t}^{t+l} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx \leqslant \int_{t}^{0} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx$$
$$= \int_{t}^{-\tau} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx + \int_{-\tau}^{0} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx$$
$$\leqslant \int_{t}^{-\tau} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx + |\tau|^{\sigma p+1} \cdot (2^{\sigma}+1)^p.$$

Applying the Lagrange mean value theorem, we can continue the computation as follows:

$$
\begin{aligned}
&\leqslant \sigma^p|\tau|^p \int_t^{-\tau} \max_{v\in(-x-\tau,-x)} v^{p(\sigma-1)}\,dx + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p \\
&= \sigma^p|\tau|^p \int_t^{-\tau} (-x-\tau)^{p(\sigma-1)}\,dx + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p \\
&= \sigma^p|\tau|^p \frac{[-(t+\tau)]^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p \\
&\leqslant \sigma^p|\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p.
\end{aligned}
$$

1.2. $t+l \leqslant -\tau$. Then we have $0 \leqslant -(t+\tau)$, $0 \leqslant -(t+\tau+l)$ and we can apply the Lagrange mean value theorem in order to see that

$$
\begin{aligned}
&\int_t^{t+l} \big||x+\tau|^\sigma - |x|^\sigma\big|^p\,dx \\
&\quad\leqslant \sigma^p|\tau|^p \int_t^{t+l} \max_{v\in(-x-\tau,-x)} v^{p(\sigma-1)}\,dx = \sigma^p|\tau|^p \int_t^{t+l} (-x-\tau)^{(\sigma-1)p}\,dx \\
&\quad= \sigma^p|\tau|^p \frac{(-t-\tau)^{1-(1-\sigma)p} - (-t-\tau-l)^{1-(1-\sigma)p}}{1-(1-\sigma)p} \leqslant \sigma^p|\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p}.
\end{aligned}
$$

2. $t \leqslant -\tau$ and $t+l > 0$. Then we have $l \geqslant |t| \geqslant |\tau|$ and arguing as in case 1.1, we get

$$
\begin{aligned}
&\int_t^{t+l} \big||x+\tau|^\sigma - |x|^\sigma\big|^p\,dx \\
&\quad\leqslant \int_t^{-\tau} \big||x+\tau|^\sigma - |x|^\sigma\big|^p\,dx + \int_{-\tau}^{0} \big||x+\tau|^\sigma - |x|^\sigma\big|^p\,dx + \int_0^{t+l} \big||x+\tau|^\sigma - |x|^\sigma\big|^p\,dx \\
&\quad\leqslant \sigma^p|\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p + \int_0^{t+l} \big[(x+\tau)^\sigma - x^\sigma\big]^p\,dx \\
&\quad\leqslant \sigma^p|\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p + \sigma^p|\tau|^p \int_0^{t+l} x^{(\sigma-1)p}\,dx \\
&\quad\leqslant \sigma^p|\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p + \sigma^p|\tau|^p \frac{(t+l)^{1-(1-\sigma)p}}{1-(1-\sigma)p} \\
&\quad\leqslant \sigma^p|\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p} + |\tau|^{\sigma p+1}\cdot(2^\sigma+1)^p + \sigma^p|\tau|^p \frac{(2l)^{1-(1-\sigma)p}}{1-(1-\sigma)p}.
\end{aligned}
$$

3. $t > -\tau$ and $t + l > 0$ (case $t > -\tau$ and $t + l \leqslant 0$ cannot happen because then we would have $-\tau < t < t + l \leqslant 0$, which contradicts our assumption $l > |\tau|$). We consider the following two subcases of this case:
 3.1. $t \geqslant 0$. Then the situation is clear since

$$\begin{aligned}
&\int_t^{t+l} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx \\
&\quad = \int_t^{t+l} \big[(x+\tau)^{\sigma} - x^{\sigma}\big]^p \, dx \leqslant \sigma^p |\tau|^p \int_t^{t+l} x^{(\sigma-1)p} \, dx \\
&\quad = \sigma^p |\tau|^p \frac{(t+l)^{1-(1-\sigma)p} - t^{1-(1-\sigma)p}}{1-(1-\sigma)p} \leqslant \sigma^p |\tau|^p \frac{l^{1-(1-\sigma)p}}{1-(1-\sigma)p}.
\end{aligned}$$

 3.2. $-\tau < t < 0$. Then $l > |t|$ and we have

$$\begin{aligned}
&\int_t^{t+l} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx \\
&\quad \leqslant \int_{-\tau}^{0} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx + \int_0^{2l} \big[(x+\tau)^{\sigma} - x^{\sigma}\big]^p \, dx \\
&\quad \leqslant |\tau|^{\sigma p+1} \cdot \big(2^{\sigma}+1\big)^p + \sigma^p |\tau|^p \frac{(2l)^{1-(1-\sigma)p}}{1-(1-\sigma)p}.
\end{aligned}$$

Therefore, the estimate (8.45) is proved. Fix now $\tau \in \mathbb{R}$ and $l > |\tau|$. The estimate (8.45) implies

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} l^{-a} \Bigg[\int_t^{t+l} \big||x+\tau|^{\sigma} - |x|^{\sigma}\big|^p \, dx \Bigg]^{1/p} \\
&\quad \leqslant l^{-a} \Bigg[\sigma|\tau| \frac{l^{(1-(1-\sigma)p)/p}}{(1-(1-\sigma)p)^{1/p}} \big[1 + 2^{1-(\sigma-1)p}\big]^{1/p} + |\tau|^{(\sigma p+1)/p} \cdot \big(2^{\sigma}+1\big) \Bigg]. \qquad (8.46)
\end{aligned}$$

It is clear that (8.46) implies the required conclusion, because for any $\varepsilon > 0$ in the corresponding definition of Weyl (p, x, F)-almost periodicity we can take $L = 1$ and after that, for any $\tau \in I'$ we can take

$$l \geqslant \max\Big(\big(\varepsilon|\tau|^{-\frac{\sigma p+1}{p}}\big)^{(-1)/a}, \big(\varepsilon|\tau|^{-1}\big)^{\frac{p}{1-(1-\sigma)p-ap}}\Big).$$

In particular, $f(\cdot)$ is Weyl p-almost periodic; furthermore, this function is not Besicovitch p-bounded since

$$\lim_{l\to+\infty} \frac{1}{l} \int_0^l x^{\sigma p} \, dx = \lim_{l\to+\infty} \frac{1}{l} \frac{l^{\sigma p+1}}{\sigma p+1} = +\infty.$$

Therefore, the function $f(\cdot)$ is not Besicovitch-p-almost periodic (Besicovitch–Doss p-almost periodic, equivalently, see Definition 2.2.7) and the function $f(\cdot)$ has no finite mean value since

$$\lim_{l\to+\infty}\frac{1}{l}\int_0^l x^\sigma\,dx=\lim_{l\to+\infty}\frac{1}{l}\frac{l^{\sigma+1}}{\sigma+1}=+\infty. \qquad \square$$

Before going any further, we would like to note that we can prove that a function $f(\cdot)$ is not Weyl p-almost automorphic of type 1 by applying the following general result.

Proposition 8.3.9. *Suppose $p\geqslant 1$ and $f:\mathbb{R}\to Y$ is not Stepanov p-bounded. Then $f(\cdot)$ is not Weyl p-almost automorphic of type 1 and not jointly Weyl p-almost automorphic.*

Proof. We will only prove that $f(\cdot)$ is not Weyl p-almost automorphic of type 1. Suppose the contrary. Then there exist a real number $l\geqslant 1$ and a p-locally integrable function $f^*:\mathbb{R}\to Y$ such that

$$\lim_{k\to+\infty}\int_{-l}^{l}\|f(x+s_{n_k})-f^*(x)\|^p\,dx\leqslant 1. \tag{8.47}$$

Since $f(\cdot)$ is not Stepanov p-bounded, we may assume without loss of generality that there exists a strictly increasing sequence (l_n) tending to plus infinity such that $\lim_{n\to+\infty}\|f(x)\|^p\,dx=+\infty$. Define $s_n:=l+l_n$, $n\in\mathbb{N}$. Then for each subsequence (s_{n_k}) of (s_n) we have

$$\int_{-l+s_{n_k}}^{l+s_{n_k}}\|f(x)\|^p\,dx=\int_{l_{n_k}}^{l_{n_k}+2l}\|f(x)\|^p\,dx\to+\infty,$$

as $k\to+\infty$. This contradicts (8.47) since

$$\begin{aligned}\frac{1}{2l}\int_{-l}^{l}\|f(x+s_{n_k})-f^*(x)\|^p\,dx&\geqslant\frac{1}{2l}\left[2^{1-p}\int_{-l}^{l}\|f(x+s_{n_k})\|^p\,dx-\int_{-l}^{l}\|f^*(x)\|^p\,dx\right]\\&\to+\infty,\quad k\to+\infty.\end{aligned} \qquad \square$$

Let $p\geqslant 1$. Then we know that the Heaviside function $f(t):=\chi_{[0,\infty)}(t)$, $t\in\mathbb{R}$ is not equi-Weyl p-almost periodic and that this function is Weyl p-almost periodic. Regarding the Weyl p-almost automorphic properties of the Heaviside function, we have the following.

Theorem 8.3.10. *The Heaviside function $f(\cdot)$ is not jointly Weyl p-almost automorphic but $f(\cdot)$ is both Weyl p-almost automorphic and Weyl p-almost automorphic of type 1.*

Proof. Suppose that $f(\cdot)$ is jointly Weyl p-almost automorphic; let (s_n) be any strictly increasing sequence of real numbers tending to plus infinity and let $\varepsilon \in (0, 2^{-p}/3)$ be given. By definition, we know that there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ such that (8.43) and (8.44) hold true. Hence, there exists a finite real number $m > 0$ such that, for every $l \geqslant m$ and for every $k \in \mathbb{N}$ with $k \geqslant m$, we have

$$\frac{1}{2l}\left[\int_{-l}^{l}\left|f(t+s_{n_k}+x)-f^*(t+x)\right|^p\,dx+\int_{-l}^{l}\left|f^*(t-s_{n_k}+x)-f(t+x)\right|^p\,dx\right]<\varepsilon.$$

This implies

$$\begin{aligned}
&\frac{1}{2l}\Bigg[\int\limits_{[-l,l]\cap(-\infty,-t-s_{n_k}]}\left|f^*(t+x)\right|^p\,dx+\int\limits_{[-l,l]\cap[-t-s_{n_k},+\infty)}\left|1-f^*(t+x)\right|^p\,dx\\
&\quad+\int\limits_{[-l,l]\cap(-\infty,-t]}\left|f^*(t+x-s_{n_k})\right|^p\,dx+\int\limits_{[-l,l]\cap[-t,+\infty)}\left|1-f^*(t+x-s_{n_k})\right|^p\,dx\Bigg]\\
&=\frac{1}{2l}\Bigg[\int\limits_{[-l+t,l+t]\cap(-\infty,-s_{n_k}]}\left|f^*(x)\right|^p\,dx+\int\limits_{[-l+t,l+t]\cap[-s_{n_k},+\infty)}\left|1-f^*(x)\right|^p\,dx\\
&\quad+\int\limits_{[-l+t-s_{n_k},l+t-s_{n_k}]\cap(-\infty,-s_{n_k}]}\left|f^*(x)\right|^p\,dx\\
&\quad+\int\limits_{[-l+t-s_{n_k},l+t-s_{n_k}]\cap[-s_{n_k},\infty)}\left|1-f^*(x)\right|^p\,dx\Bigg]<\varepsilon.
\end{aligned}$$

This particularly holds with $t = 0$, so that letting $k \to +\infty$ in the last estimate (cf. the second addend) yields

$$\frac{1}{2l}\int_{-l}^{l}\left|1-f^*(x)\right|^p\,dx<\varepsilon,\quad l\geqslant m.$$

With fixed $k = \lceil m\rceil$, the last estimate in the previous calculation also yields (cf. the first addend) that

$$\frac{1}{2l}\int_{-l}^{-s_{n_{\lceil m\rceil}}}\left|f^*(x)\right|^p\,dx<\varepsilon,\quad l\geqslant m,$$

so that there exists a finite real number $m_1 > m$ such that

$$\frac{1}{2l}\int_{-l}^{l}\left|1-f^*(x)\right|^p\,dx+\frac{1}{2l}\int_{-l}^{0}\left|f^*(x)\right|^p\,dx<3\varepsilon,\quad l\geqslant m_1.$$

As a consequence, we have

$$\frac{1}{2l}\int_{-l}^{0}\left|1-f^{*}(x)\right|^{p}dx+\frac{1}{2l}\int_{-l}^{0}\left|f^{*}(x)\right|^{p}dx<3\varepsilon,\quad l\geqslant m_1.$$

This contradicts the following estimate:

$$\frac{1}{2l}\int_{-l}^{0}\left|1-f^{*}(x)\right|^{p}dx+\frac{1}{2l}\int_{-l}^{0}\left|f^{*}(x)\right|^{p}dx\geqslant\frac{1}{2l}\int_{-l}^{0}2\cdot 2^{-p}\,dx=2^{-p},\quad l\geqslant m_1,$$

so that $f(\cdot)$ is not jointly Weyl p-almost automorphic. In order to see that $f(\cdot)$ is Weyl p-almost automorphic, we can take $f^{*}\equiv f$ in the corresponding definition and here it is only worth noting that, for every fixed real numbers t and a, we see that the mapping

$$l\mapsto\int_{-l}^{l}\left|f(t+x+a)-f(t+x)\right|^{p}dx,\quad l\in\mathbb{R},$$

is bounded, which follows from a simple analysis concerning the support of the function $x\mapsto f(t+x+a)-f(t+x)$, $x\in\mathbb{R}$; let us also stress that the above also shows that $f\in AAW_{\mathrm{R}}^{\mathbb{F},p}(\mathbb{R}:\mathbb{C})$, provided that $\lim_{l\to+\infty}\mathbb{F}(l)=0$ and R denotes the collection of all real sequences (see Definition 8.3.17(i) for the notion). It remains to be proved that $f(\cdot)$ is Weyl p-almost automorphic of type 1. More generally, let $\mathbb{F}:(0,\infty)\to(0,\infty)$ be such that $\lim_{l\to+\infty}\mathbb{F}(l)=0$ and let R denote the collection of all real sequences; we will prove that $f\in AAW_{\mathrm{R}}^{\mathbb{F},p,1}(\mathbb{R}:\mathbb{C})$. If the sequence (s_n) is bounded, then the situation is very simple and we can take $f^{*}(\cdot)$ to be a certain translation of $f(\cdot)$ after applying the Bolzano–Weierstrass theorem. If the sequence (s_n) is unbounded, then it has a strictly monotone subsequence (s_n') tending to plus infinity or minus infinity. The considerations in both cases are similar and we may assume without loss of generality that $\lim_{n\to+\infty}s_n'=+\infty$ and $s_n'>0$ for all $n\in\mathbb{N}$. Choose any strictly increasing sequence (a_n) of positive real numbers such that $\lim_{n\to+\infty}a_n=+\infty$. After that, we construct inductively a subsequence (s_{n_k}) of (s_n') so that $s_{n_1}=s_1'$ and the following conditions are satisfied:

$$s_{n_{k+1}}>s_{n_k}+2a_{n_k},\quad k\in\mathbb{N};\tag{8.48}$$

$$\left|\mathbb{F}(v)\right|<\frac{1}{(a_{n_1}+a_{n_2}+\cdots+a_{n_k})^{2}},\quad\text{provided } v\geqslant s_{n_k}\text{ and } k\in\mathbb{N}\setminus\{1\}.\tag{8.49}$$

Define now $f^{*}(t):=1$ for $t\geqslant -s_{n_1}$ or $t\in[-s_{n_{k+1}},-a_{n_k}-s_{n_k}]$ for some $k\geqslant 2$, and $f^{*}(t):=0$ if there exists $k\in\mathbb{N}$ such that $t\in(-a_{n_k}-s_{n_k},-s_{n_k})$. By the corresponding definition, the dominated convergence theorem and a simple argumentation, it suffices to show that

$$\lim_{l\to+\infty}\mathbb{F}(l)\int_{-l}^{l}\left|1-f^{*}(t+x)\right|^{p}dx=0\tag{8.50}$$

and

$$\forall l > 0 : \lim_{k\to+\infty} \int_{-l}^{l} \left|f^*(t + x - s_{n_k}) - f(t + x)\right|^p dx = 0. \tag{8.51}$$

In order to prove (8.51), it suffices to repeat verbatim the argumentation used in the proof of joint Weyl p-almost automorphy of $f(\cdot)$. We actually need to prove that there exists $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$ we have

$$\begin{aligned}&\int_{[-l+t-s_{n_k}, l+t-s_{n_k}]\cap(-\infty,-s_{n_k}]} \left|f^*(x)\right|^p dx \\ &\quad + \int_{[-l+t-s_{n_k}, l+t-s_{n_k}]\cap[-s_{n_k},\infty)} \left|1 - f^*(x)\right|^p dx < \varepsilon.\end{aligned}$$

But for sufficiently large values of parameter k we have $a_{n_k} > l + |t| + |l - t|$ so that the above sum is equal to zero due to our construction of the function f^* and condition (8.48). To prove (8.50), it suffices to consider the case $t = 0$ due to the boundedness of the function $f^*(\cdot)$. We need to prove that

$$\lim_{l\to+\infty} \mathbb{F}(l) \int_{-l}^{-s_{n_1}} \left|1 - f^*(x)\right|^p dx = 0.$$

Let $l > s_{n_3}$ and let $l \in [s_{n_{k+1}}, s_{n_{k+2}})$ for some $k \in \mathbb{N} \smallsetminus \{1\}$. Taking into account (8.49), we have

$$\begin{aligned}\mathbb{F}(l) \int_{-l}^{-s_{n_1}} \left|1 - f^*(x)\right|^p dx &\leqslant \Big[\max_{v\geqslant s_{n_k}} \mathbb{F}(v)\Big] \cdot \sum_{j=1}^{n_k} a_{n_j} \\ &\leqslant \frac{1}{(a_{n_1} + a_{n_2} + \cdots + a_{n_k})^2} \cdot \sum_{j=1}^{n_k} a_{n_j} \leqslant \frac{1}{(a_{n_1} + a_{n_2} + \cdots + a_{n_k})}.\end{aligned}$$

This simply implies the required result. □

8.3.2 Concept without limit functions

The set $W^p AA(\mathbb{R} : X)$, equipped with the usual operations of pointwise addition of functions and multiplication of functions with scalars, has a linear vector structure. As we have observed in [631], it is not clear how one can prove (see also [4, p. 5, l.2–3]) that an equi-Weyl p-almost periodic function $f : \mathbb{R} \to X$ is Weyl p-almost automorphic. The main problem lies in the fact that it is not clear how one can prove that, for a given real sequence (s_n), there exist a subsequence (s_{n_k}) of (s_n) and a p-locally integrable function $f^* : \mathbb{R} \to X$ such that the sequence of translations $(f(\cdot + s_{n_k}))$ converges to $f^*(\cdot)$ in the Weyl metric, i. e., that

$$\lim_{k\to+\infty} \lim_{l\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{2l} \int_{-l}^{l} \|f(t + x + s_{n_k}) - f^*(t + x)\|^p \, dx = 0.$$

We can only prove that the family of translations $\{f(\cdot + h) : h \in \mathbb{R}\}$ is totally bounded with respect to the Weyl metric (see, e. g., [68, Theorem 2] for case $p = 1$) as well as that, for a given real sequence (s_n), there exists a subsequence (s_{n_k}) of (s_n) such that the sequence of translations $(f(\cdot + s_{n_k}))$ is a Cauchy sequence in the Weyl metric), i. e., that for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that, for every $k, m \in \mathbb{N}$, the assumptions $k \geqslant k_0$ and $m \geqslant k_0$ imply that

$$\lim_{l\to+\infty} \sup_{t\in\mathbb{R}} \left[\frac{1}{2l} \int_{-l}^{l} \|f(t + s_{n_k} + x) - f(t + s_{n_m} + x)\|^p \, dx \right] < \varepsilon. \tag{8.52}$$

Before proceeding any further, we would like to note that the existence of a subsequence (s_{n_k}) of (s_n) such that the sequence of translations $(f(\cdot + s_{n_k}))$ is a Cauchy sequence in the Weyl metric does not imply that the function $f(\cdot)$ is equi-Weyl p-almost periodic; for example, this is not true for the Heaviside function $f(\cdot)$.

Example 8.3.11. Let (s_n) be a real sequence, let (s_{n_k}) be the same as (s_n), and let $\varepsilon > 0$. We choose $k_0 = 1$ in the above definition. Then, for every $k, m \in \mathbb{N}$, we have

$$\lim_{l\to+\infty} \sup_{t\in\mathbb{R}} \left[\frac{1}{2l} \int_{-l}^{l} \|f(t + s_{n_k} + x) - f(t + s_{n_m} + x)\|^p \, dx \right] = 0, \tag{8.53}$$

and therefore, (8.52) is satisfied. In order to see that (8.53) holds, observe that for each $t \in \mathbb{R}$ the integral

$$\int_{-l}^{l} \|f(t + s_{n_k} + x) - f(t + s_{n_m} + x)\|^p \, dx \tag{8.54}$$

is taken with respect to the variable x which belongs to the interval $[-l, l]$ but the integrand is not equal to zero only for those values of x for which the numbers $t + s_{n_k} + x$ and $t + s_{n_m} + x$ have different signs; hence, the measure of set of integration in (8.54) is less than or equal to $2|s_{n_k} - s_{n_m}|$. Since the essential bound of the integrand is less than or equal to 1 for each $t \in \mathbb{R}$, we get

$$\sup_{t\in\mathbb{R}} \left[\int_{-l}^{l} \|f(t + s_{n_k} + x) - f(t + s_{n_m} + x)\|^p \, dx \right] \leqslant 2|s_{n_k} - s_{n_m}|,$$

which simply implies the required result.

In our new concept, which generalizes the concept of equi-Weyl p-almost periodicity, we do not use the limit functions with the respect to the Weyl metric but only Cauchy sequences with respect to the Weyl metric (our idea is, in fact, to remove the operation $\sup_{t\in\mathbb{R}}\cdot$ from Eq. (8.52)).

Definition 8.3.12. Let $p \geqslant 1$ and $f \in L^p_{\text{loc}}(\mathbb{R} : X)$. Then we say that $f(\cdot)$ is Weyl p-almost automorphic of type 2 if and only if for each real sequence (s_n) there exists a subsequence (s_{n_k}) of (s_n) such that for each $\varepsilon > 0$ and $t \in \mathbb{R}$ there exists $k_0 \in \mathbb{N}$ such that, for every $k,\ m \in \mathbb{N}$ with $k \geqslant k_0$ and $m \geqslant k_0$, there exists $l_0 > 0$ such that

$$\frac{1}{2l}\int_{-l}^{l}\left\|f(t+s_{n_k}+x)-f(t+s_{n_m}+x)\right\|^p\,dx < \varepsilon, \quad l \geqslant l_0. \tag{8.55}$$

It is clear that the Weyl p-almost automorphic functions of type 2 form a vector space under the usual operations as well as that [631, Lemma 2.2.13] implies that any Weyl p'-almost automorphic function of type 2 is Weyl p-almost automorphic of type 2, provided that $1 \leqslant p \leqslant p' < +\infty$ (the same holds for all other classes of Weyl almost automorphic functions considered so far).

We have the following result.

Proposition 8.3.13. *Suppose that $p \geqslant 1$ and $f \in L^p_{\text{loc}}(\mathbb{R} : X)$ is Weyl p-almost automorphic or jointly Weyl p-almost automorphic. Then $f(\cdot)$ is Weyl p-almost automorphic of type* 2.

Proof. We will consider the class of Weyl p-almost automorphic functions, only. Let a real sequence (s_n) be given. Then there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{\text{loc}}(\mathbb{R} : X)$ such that (2.25) holds. Let the numbers $\varepsilon > 0$ and $t \in \mathbb{R}$ be given. Then we have the existence of a positive integer $k_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ with $k \geqslant k_0$ there exists $l_k > 0$ such that for every $l \geqslant l_k$ we have $|a_{k,l}| < \varepsilon/(2(2^p-1))$, where

$$a_{k,l} := \frac{1}{2l}\int_{-l}^{l}\left\|f(t+s_{n_k}+x)-f^*(t+x)\right\|^p\,dx.$$

Let $k',\ k'' \geqslant k_0$. Then there exists a finite real number $l_0 := \max(l_{k'}, l_{k''}) > 0$ such that for every $l \geqslant l_0$ we have $|a_{k',l}| < \varepsilon/(2(2^p-1))$ and $|a_{k'',l}| < \varepsilon/(2(2^p-1))$. Using the inequality $(a+b)^p \leqslant 2^{p-1}(a^p+b^p)$, $a,\ b \geqslant 0$, the above simply implies (8.55) with the numbers k and m replaced therein with the numbers k' and k'', which completes the proof. □

The proof of Proposition 8.3.13 does not work for Weyl p-almost automorphic functions of type 1 and we would like to ask the following.

Question 8.3.14. Suppose that $p \geqslant 1$ and $f : \mathbb{R} \to Y$ is (Stepanov p-almost automorphic) Weyl p-almost automorphic functions of type 1. Is it true that $f(\cdot)$ is Weyl p-almost automorphic of type 2?

The following question is also meaningful.

Question 8.3.15. Suppose $p \geqslant 1$. Construct, if possible, a jointly Weyl p-almost automorphic function which is not (equi-)Weyl p-almost periodic?

Before switching to the next subsection, we shall revisit some already considered examples.

Example 8.3.16. Suppose that $p \geqslant 1$. Let us recall that the function $f(\cdot)$, given by (2.28), is bounded, uniformly continuous, uniformly recurrent, Besicovitch 1-unbounded and Weyl p-almost automorphic. By Proposition 8.3.9, we immediately see that $f(\cdot)$ is not Weyl p-almost automorphic of type 1 nor jointly Weyl p-almost automorphic. On the other hand, an application of Proposition 8.3.13 shows that $f(\cdot)$ is Weyl p-almost automorphic of type 2. In the present situation, we do not know to tell whether, for a given real sequence (s_n), there exists a subsequence (s_{n_k}) of (s_n) such that the sequence of translations $(f(\cdot + s_{n_k}))$ is a Cauchy sequence in the Weyl metric (cf. also the remarkable example by H. Bohr [196, pp. 113–115, part I], which will not be reexamined here). Concerning the already examined example of J. de Vries, [358, point 6., p. 208; Figure 3.7.3, p. 208], we will only prove here the following new property of the function $f(\cdot)$ (we use the same notation):

$$\limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \left|1 - f(x)\right| dx = \limsup_{l\to+\infty} \frac{1}{l} \int_{0}^{l} \left|1 - f(x)\right| dx \geqslant \frac{1}{4}. \tag{8.56}$$

In order to do that, fix a positive integer $i \in \mathbb{N}$ and consider the straight line $y = x/p_{i+1}$ and the straight line $y = [(-1)/p_i](x - 2p_i)$ connecting the points $(p_i, 1)$ and $(2p_i, 0)$. The intersection of these lines is the point $(2p_i p_{i+1}/(p_i + p_{i+1}), 2p_i/(p_i + p_{i+1}))$. Set $l_i := 2p_i p_{i+1}/(p_i + p_{i+1})$; then $\lim_{i\to+\infty} l_i = +\infty$ and $f(x) \geqslant f_i(x)$, $x \in [p_i, l_i]$. This implies (8.56), because

$$\begin{aligned} \frac{1}{l_i} \int_{0}^{l_i} \left|1 - f(x)\right| dx &\geqslant \frac{1}{l_i} \int_{p_i}^{l_i} \left|1 - f(x)\right| dx \\ &= \frac{1}{2} \frac{p_i + p_{i+1}}{2p_i p_{i+1}} \left[\frac{2p_i p_{i+1}}{p_i + p_{i+1}} - p_i\right] \cdot \left[1 - \frac{2p_i}{p_i + p_{i+1}}\right] \\ &= \frac{1}{2} \frac{p_i p_{i+1} - p_i^2}{2p_i p_{i+1}} \cdot \left[1 - \frac{2p_i}{p_i + p_{i+1}}\right] \to \frac{1}{4}, \quad i \to +\infty. \end{aligned}$$

Finally, we would like to ask whether the function $f(\cdot)$ is (equi-)Weyl p-almost periodic [(jointly) Weyl p-almost automorphic (of type 1, 2)] for some (each) finite exponent $p \geqslant 1$.

8.3.3 Multi-dimensional Weyl almost automorphy in Lebesgue spaces with variable exponent

The main aim of this subsection is to introduce and analyze multi-dimensional Weyl almost automorphy in Lebesgue spaces with variable exponent. Here, we basically follow the approach used in our investigations of (equi-)Weyl $(p, \phi, \mathbb{F})$-almost periodic functions and our considerations from Subsection 8.3.1. Unless stated otherwise, we will always assume henceforth that $\Omega := [-1,1]^n \subseteq \mathbb{R}^n$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $\mathbb{F} : (0,\infty)\times\mathbb{R}^n \to (0,\infty)$; in contrast with the above-mentioned research, we will always assume here $\phi(x) \equiv x$ for simplicity.

We start by introducing the following notion.

Definition 8.3.17. Suppose that $F : \mathbb{R}^n \times X \to Y$ satisfies the requirement that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$. Let for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, and

(i)

$$\lim_{m\to+\infty}\lim_{l\to+\infty}\mathbb{F}(l,\mathbf{t})\big\|F\big(\mathbf{t}+\mathbf{u}+(b_{k_m}^1,\ldots,b_{k_m}^n);x\big)-F^*(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)}=0 \tag{8.57}$$

and

$$\lim_{m\to+\infty}\lim_{l\to+\infty}\mathbb{F}(l,\mathbf{t})\big\|F^*\big(\mathbf{t}+\mathbf{u}-(b_{k_m}^1,\ldots,b_{k_m}^n);x\big)-F(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)}=0, \tag{8.58}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, or

(ii)

$$\lim_{l\to+\infty}\lim_{m\to+\infty}\mathbb{F}(l,\mathbf{t})\big\|F\big(\mathbf{t}+\mathbf{u}+(b_{k_m}^1,\ldots,b_{k_m}^n);x\big)-F^*(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)}=0$$

and

$$\lim_{l\to+\infty}\lim_{m\to+\infty}\mathbb{F}(l,\mathbf{t})\big\|F^*\big(\mathbf{t}+\mathbf{u}-(b_{k_m}^1,\ldots,b_{k_m}^n);x\big)-F(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)}=0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, or

(iii)

$$\lim_{(l,m)\to+\infty}\mathbb{F}(l,\mathbf{t})\big\|F\big(\mathbf{t}+\mathbf{u}+(b_{k_m}^1,\ldots,b_{k_m}^n);x\big)-F^*(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)}=0$$

and

$$\lim_{(l,m)\to+\infty}\mathbb{F}(l,\mathbf{t})\big\|F^*\big(\mathbf{t}+\mathbf{u}-(b_{k_m}^1,\ldots,b_{k_m}^n);x\big)-F(\mathbf{t}+\mathbf{u};x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)}=0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

In the case that (i), resp. [(ii); (iii)], holds, then we say that the function $F(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, resp. [Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic of type 1; jointly Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic]. By $AAW^{\mathbb{F},p(\mathbf{u})}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n \times X : Y)$, resp. [$AAW^{\mathbb{F},p(\mathbf{u}),1}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n \times X : Y)$; $AAW^{\mathbb{F},p(\mathbf{u}),j}_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n \times X : Y)$] we denote the collection of all Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic [Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic of type 1; jointly Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic] functions $F : \mathbb{R}^n \times X \to Y$.

From our analysis of multi-dimensional Weyl almost periodicity, it follows that the case $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $\mathbb{F}(l,\mathbf{t}) \equiv l^{-n/p}$ is the most important, when we say that the function $F : \mathbb{R}^n \times X \to Y$ is (jointly) Weyl p-$(\mathrm{R}, \mathcal{B})$-multi-almost automorphic (of type 1).

In the next definition, we continue our analysis from Subsection 8.3.2 by introducing the following non-trivial class of functions.

Definition 8.3.18. Suppose that $\emptyset \neq W \subseteq \mathbb{R}^n$, $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$ and $F : \mathbb{R}^n \times X \to Y$ satisfies the requirement that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t} + \mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$. If for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\varepsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every m, $m' \in \mathbb{N}$ with $m \geqslant m_0$ and $m' \geqslant m_0$, there exists $l_0 > 0$ such that, for every $l \geqslant l_0$ and $w \in lW$, we have

$$\begin{aligned} &\big\|F(\mathbf{t} + \mathbf{u} + (b_{k_m}^1, \ldots, b_{k_m}^n) - w; x) - F(\mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w; x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} \\ &\quad < \varepsilon/\mathbb{F}(l, \mathbf{t} - w), \end{aligned} \tag{8.59}$$

then we say that $F(\cdot;\cdot)$ is Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic of type 2.

We can also introduce the concepts in which the parameters $x \in B$ and $\mathbf{t} \in \mathbb{R}$ are separated with respect to the use of quantifiers, as well, but we will not go into further details concerning this notion. If $F : \mathbb{R}^n \to Y$, then we omit the term "$\mathcal{B}$" from the notation, as accepted before.

Remark 8.3.19.

(i) Since the introduced classes of almost automorphic functions are translation invariant, we do not need to follow the second approach, obeyed for the introduction of various classes of (equi-)Weyl $[p, \phi, \mathbb{F}]$-almost periodic functions. In this concept, we assume that $\Omega := [-1,1]^n \subseteq \mathbb{R}^n$, $p \in \mathcal{P}(\Omega)$ and $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$. We can consider the following notion: Suppose that $F : \mathbb{R}^n \times X \to Y$ satisfies the requirement that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t} + l\mathbf{u}; x) \in L^{p(\mathbf{u})}(\Omega : Y)$. We say that the function $F(\cdot;\cdot)$ is Weyl $[\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B}]$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each

$x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t} + l\mathbf{u}; x) \in L^{p(\mathbf{u})}(\Omega : Y)$, as well as

$$\lim_{m\to+\infty} \lim_{l\to+\infty} \mathbb{F}(l, \mathbf{t}) \big\| F(\mathbf{t} + l\mathbf{u} + (b^1_{k_m}, \ldots, b^n_{k_m}); x) - F^*(\mathbf{t} + l\mathbf{u}; x) \big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0$$

and

$$\lim_{m\to+\infty} \lim_{l\to+\infty} \mathbb{F}(l, \mathbf{t}) \big\| F^*(\mathbf{t} + l\mathbf{u} - (b^1_{k_m}, \ldots, b^n_{k_m}); x) - F(\mathbf{t} + l\mathbf{u}; x) \big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0,$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. For the sake of brevity, we will skip all related details concerning this class of functions and related classes of functions defined in a similar way, with the replaced limits or just one joint limit.

(ii) Suppose that $F : \mathbb{R}^n \times X \to Y$ satisfies the requirement that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t} + \mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as that, for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b^1_k, b^2_k, \ldots, b^n_k)) \in \mathrm{R}$, there exist a subsequence $(\mathbf{b}_{k_m} = (b^1_{k_m}, b^2_{k_m}, \ldots, b^n_{k_m}))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t} + \mathbf{u}; x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, and

$$\lim_{m\to+\infty} \lim_{l\to+\infty} \sup_{\mathbf{t}\in\mathbb{R}^n, x\in B} \mathbb{F}(l) \big\| F(\mathbf{t} + \mathbf{u} + (b^1_{k_m}, \ldots, b^n_{k_m}); x) - F^*(\mathbf{t} + \mathbf{u}; x) \big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0.$$

Using the substitution $\mathbf{t} \mapsto \mathbf{t} + (b^1_{k_m}, \ldots, b^n_{k_m})$, $\mathbf{t} \in \mathbb{R}^n$ we get

$$\lim_{m\to+\infty} \lim_{l\to+\infty} \sup_{\mathbf{t}\in\mathbb{R}^n, x\in B} \mathbb{F}(l) \big\| F^*(\mathbf{t} + \mathbf{u} - (b^1_{k_m}, \ldots, b^n_{k_m}); x) - F(\mathbf{t} + \mathbf{u}; x) \big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} = 0.$$

Hence, $F(\cdot,\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. It is also worth noting that, in the case that $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $\mathbb{F}(l) \equiv l^{-n/p}$, the assumptions used in this remark imply the Weyl $(\mathrm{R}, \mathcal{B}, p)$-normality of the function $F(\cdot,\cdot)$.

We continue by providing the following examples.

Example 8.3.20 (J. Stryja [962, pp. 42–47]; see also [67, Example 4.28]). Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := 0$ for $x \leqslant 0$, $f(x) := \sqrt{n/2}$ if $x \in (n-2, n-1]$ for some $n \in 2\mathbb{N}$ and $f(x) := -\sqrt{n/2}$ if $x \in (n-1, n]$ for some $n \in 2\mathbb{N}$. It is clear that the function $f(\cdot)$ is not Stepanov 1-bounded, which immediately implies that the function $f(\cdot)$ is not asymptotically Stepanov 1-almost automorphic, not Weyl 1-almost automorphic of type 1 and not jointly Weyl 1-almost automorphic; furthermore, arguing in the same way as in the proof of Proposition 8.3.9, we see that the function $f(\cdot)$ is not Weyl $(\mathbb{F}, 1, \mathrm{R})$-multi-almost automorphic if R is any collection of real sequences containing a strictly increasing sequence (s_n) tending to plus infinity. We already know that the function $f(\cdot)$ is not equi-Weyl 1-almost periodic as well as that the function $f(\cdot)$ is Weyl 1-almost periodic and not Weyl 1-normal, as well as that for each $n \in 2\mathbb{N}$ we have

$$\lim_{l\to+\infty} \frac{1}{2l} \sup_{t\in\mathbb{R}} \int_{-l}^{l} \big| f(t + n + x) - f(t + x) \big| \, dx = 0, \tag{8.60}$$

so that the function $f(\cdot)$ is Weyl $(\mathbb{F}, 1, \mathrm{R})$-multi-almost automorphic with $\mathbb{F}(l,t) \equiv 1/l$ and R being the collection of all real sequences (a_m) satisfying that $a_m \in 2\mathbb{N}$ for all $m \in \mathbb{N}$. Now we will prove that the function $f(\cdot)$ is not Weyl $(\mathbb{F}, 1, \mathrm{R})$-multi-almost automorphic provided that there exists a real sequence (s_n) from R which contains only a finite number of even numbers; in particular, the function $f(\cdot)$ is not Weyl 1-almost automorphic. Towards this end, it suffices to show that, for every fixed real number $\omega \notin 2\mathbb{Z}$, we have

$$\lim_{l\to+\infty} \frac{1}{2l} \int_0^l \left|f(x+\omega) - f(x)\right| dx = +\infty, \tag{8.61}$$

where $\omega' \in 2\mathbb{Z}$ denotes the nearest even number to ω. Without loss of generality, we may assume that $\omega' < \omega$ so that $h := \omega - \omega' \in (0,1]$. Keeping in mind the triangle inequality and the estimate (8.60) with $t = 0$ and $n = \omega'$, we only need to show that

$$\lim_{l\to+\infty} \frac{1}{2l} \int_0^l \left|f(x+\omega) - f(x+\omega')\right| dx = +\infty. \tag{8.62}$$

This follows from the following calculation ($l > 2 + |\omega'|$):

$$\begin{aligned}
&\frac{1}{2l} \int_0^l \left|f(x+\omega) - f(x+\omega')\right| dx \\
&= \frac{1}{2l} \int_{\omega'}^{\omega'+l} \left|f(x+h) - f(x)\right| dx \\
&= 2\frac{\omega'+l}{l} \frac{1}{2(\omega'+l)} \int_0^{\omega'+l} \left|f(x+h) - f(x)\right| dx - \frac{1}{2l} \int_0^{\omega'} \left|f(x+h) - f(x)\right| dx \\
&\geqslant 2\frac{\omega'+l}{l} \left[\frac{4}{3} h \sqrt{\left\lfloor \frac{l+\omega'}{2} \right\rfloor - 1}\right] - \frac{1}{2l} \int_0^{\omega'} \left|f(x+h) - f(x)\right| dx \to +\infty, \quad l \to +\infty,
\end{aligned}$$

where the estimate used above follows by applying the estimate proved on [67, p. 149, l.7–9], which is valid for $h \in (0,1]$ and $L \geqslant 2$. Using a similar argumentation involving the estimates (8.61)–(8.62), it follows that the function $f(\cdot)$ is not 1-$(1/l, \mathrm{R}, \{0\})$-multi-almost automorphic provided that there exists a sequence (s_n) from R satisfying the requirement that, for every its subsequence (s_{n_k}), there exist two arbitrarily large positive integers k' and k'' such that the difference $s_{n_{k'}} - s_{n_{k''}}$ is not an even number; in particular, the function $f(\cdot)$ is not Weyl 1-almost automorphic of type 2. It is also worth noting that the function $f(\cdot)$ is not Besicovitch 1-bounded, not Besicovitch 1-almost periodic and has no finite mean value (cf. also Theorem 8.3.8 above).

Example 8.3.21.

(i) Similarly to the one-dimensional case, it can be proved that for any compact set $K \subseteq \mathbb{R}^n$ and for any $p \in \mathcal{P}(\mathbb{R}^n)$ we see that the function $\chi_K(\cdot)$ belongs to any of the function spaces introduced in Definition 8.3.17 and Definition 8.3.18 with $\mathbb{F}(t,\mathbf{t}) \equiv l^{-\sigma}$, with $\sigma > 0$.

(ii) Suppose now that $F(\mathbf{t}) := \chi_{[0,\infty)^n}(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$. Let R denote the collection of all sequences in $\mathbb{R}^n$, and let $1 \leqslant p < \infty$. Arguing as before, we can prove that the function $F(\cdot)$ is Weyl $(\mathbb{F}, p, \mathrm{R})$-multi-almost automorphic with $\mathbb{F}(l,\mathbf{t}) \equiv l^{-\sigma}$ if and only if $\sigma > (n-1)/p$, as well as that there is no $\sigma > 0$ such that $F(\cdot)$ is Weyl $(\mathbb{F}, p, \mathrm{R})$-multi-almost automorphic with $\mathbb{F}(l,\mathbf{t}) \equiv l^{-\sigma}$. On the other hand, our analysis from Example 8.3.11 can be repeated with minor modifications in order to see that for each real number $\sigma > 0$ the function $F(\cdot)$ is Weyl p-$(\mathbb{F}, \mathrm{R}, \mathbb{R}^n)$-multi-almost automorphic with $\mathbb{F}(l,\mathbf{t}) \equiv l^{-\sigma}$. An insignificant modification of the construction established in the corresponding part of the proof of Theorem 8.3.10 shows that the function $F(\cdot)$ is also Weyl $(\mathbb{F}, p, \mathrm{R})$-multi-almost automorphic of type 1 for any function $\mathbb{F}(l)$ satisfying that $\lim_{l\to+\infty} \mathbb{F}(l) = 0$.

Example 8.3.22. The following example is a simple modification of the corresponding example examined in the Weyl almost periodic case. Suppose that $1 \leqslant p < \infty$, the complex-valued mapping $t \mapsto g_j(t) \in Y$, $t \in \mathbb{R}$ is essentially bounded and jointly Weyl $(\mathbb{F}_j, p, \mathrm{R}_j)$-almost automorphic, where R_j denotes the collection of all real sequences $(1 \leqslant j \leqslant n)$. Define

$$F(t_1,\ldots,t_{2n}) := \prod_{j=1}^{n} [g_j(t_{j+n}) - g_j(t_j)], \quad \text{where } t_j \in \mathbb{R} \text{ for } 1 \leqslant j \leqslant 2n,$$

and $\Lambda' := \{(\tau,\tau) : \tau \in \mathbb{R}^n\}$. Then we know there exists a finite constant $M > 0$ such that

$$\begin{aligned}
&\|F(t_1+\tau_1,\ldots,t_{2n}+\tau_{2n}) - F(t_1,\ldots,t_{2n})\|_Y \\
&\quad \leqslant M\{\|g_1(t_{n+1}+\tau_1) - g_1(t_{n+1})\| + \|g_1(t_1+\tau_1) - g_1(t_1)\| + \cdots \\
&\qquad + \|g_n(t_{2n}+\tau_n) - g_n(t_{2n})\| + \|g_n(t_n+\tau_n) - g_n(t_n)\|\},
\end{aligned}$$

for any $(t_1,\ldots,t_{2n}) \in \mathbb{R}^{2n}$ and $(\tau_1,\ldots,\tau_{2n}) \in \Lambda'$. Suppose that $c \in (0,\infty)$, $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to Y$ and

$$\sum_{j=1}^{n} \left[\frac{1}{F_j(l,t_j)} + \frac{1}{F_j(l,t_{j+n})}\right] \leqslant \frac{c}{F(l,t_1,\ldots,t_{2n})}, \quad l > 0,\ (t_1,\ldots,t_{2n}) \in \mathbb{R}^{2n}.$$

Using the corresponding definition and the above estimates, it follows that the function $F(\cdot)$ is jointly Weyl $(\mathbb{F}, p, \mathrm{R})$-almost automorphic, where R denotes the collection of all sequences in Λ'.

Immediately from the above definitions, we have the following simple proposition which can be clarified for all other classes of functions introduced above.

Proposition 8.3.23.
(i) *Suppose that $c \in \mathbb{C}$ and the function $F(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. Then $cF(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic.*
(ii) *Suppose that $\tau \in \mathbb{R}^n$, $x_0 \in X$, the function $\mathbb{F}(\cdot,\cdot)$ does not depend on the second argument, and $F(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. Then $F(\cdot+\tau;\cdot+x_0)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, where $\mathcal{B}_{x_0} \equiv \{-x_0 + B : B \in \mathcal{B}\}$.*
(iii) (a) *Suppose that $c_2 \in \mathbb{C} \smallsetminus \{0\}$, and $F(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic. Then $F(\cdot;c_2\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, where $\mathcal{B}_{c_2} \equiv \{c_2^{-1}B : B \in \mathcal{B}\}$.*
(b) *Suppose that $c_1 \in \mathbb{C} \smallsetminus \{0\}$, $c_2 \in \mathbb{C} \smallsetminus \{0\}$, and $F(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic, with some constant exponent $p \geqslant 1$. Then the function $F(c_1\cdot;c_2\cdot)$ is Weyl $(\mathbb{F}_{c_1}, p, \mathrm{R}, \mathcal{B})$-multi-almost automorphic, where $\mathrm{R}_{c_1} \equiv \{c_1^{-1}\mathbf{b} : \mathbf{b} \in \mathrm{R}\}$ and $\mathbb{F}_{c_1}(l,\mathbf{t}) \equiv \mathbb{F}(l, c_1\mathbf{t})$.*

We have the following simple result, which can be also clarified for all other classes of functions introduced in this section so far.

Proposition 8.3.24. *Suppose that $F(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic and $A \in L(Y, Z)$. Then $(A \circ F)(\cdot;\cdot)$ is Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, \mathcal{B})$-multi-almost automorphic.*

Proof. Let $x \in X$, $\mathbf{t} \in \mathbb{R}^n$, $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ be fixed. Then we know that $F(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, so that there exists a finite real number $\lambda > 0$ such that

$$\int_{l\Omega} \varphi_{p(\mathbf{u})}\left(\frac{\|F(\mathbf{t}+\mathbf{u};x)\|_Y}{\lambda}\right) d\mathbf{u} \leqslant 1.$$

This easily implies that

$$\int_{l\Omega} \varphi_{p(\mathbf{u})}\left(\frac{\|A(F(\mathbf{t}+\mathbf{u};x))\|_Z}{\lambda'}\right) d\mathbf{u} \leqslant 1,$$

with $\lambda' = \|A\| \cdot \lambda$. Hence, $A(F(\mathbf{t}+\mathbf{u};x)) \in L^{p(\mathbf{u})}(l\Omega : Z)$. Furthermore, we know that there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as that (8.57)–(8.58) hold. By the foregoing, we see that $A(F^*(\mathbf{t}+\mathbf{u};x)) \in L^{p(\mathbf{u})}(l\Omega : Z)$. Using Lemma 1.1.8, it can be simply shown that (8.57)–(8.58) hold with the functions F and F^* replaced therein with the functions $A \circ F$ and $A \circ F^*$, respectively, finishing the proof of the proposition. □

8.3.4 Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy of type 2 and joint Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy

In this subsection, we investigate the Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions, primarily from their invaluable importance in applications.

In Definition 8.3.18, we can also assume that the set W depends on l but the situation is more complicated then. In Subsection 8.3.2 we have $W = \{0\}$; case $W \neq \{0\}$ is also important to be analyzed, as the following result about the convolution invariance of Weyl $p(\mathbf{u})$-$(\mathrm{R}, \mathcal{B}, W)$-multi-almost automorphy of type 2 shows (the choice of sets $W_1 = (2\mathbb{Z}+1)^n$ and $W_2 \subseteq (2\mathbb{Z})^n$ strongly depends on the choice of region $\Omega = [-1,1]^n$ here; the things certainly can be arranged in a slightly different, generalized way, the reader may try to make this more precise).

Theorem 8.3.25. *Suppose that $h \in L^1(\mathbb{R}^n)$ and $F : \mathbb{R}^n \times X \to Y$ is Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, (2\mathbb{Z}+1)^n)$-multi-almost automorphic of type 2. Let $p_1,\ q \in \mathcal{P}(\mathbb{R}^n)$, let $1/p(\mathbf{u})+1/q(\mathbf{u}) \equiv 1$, and let $\mathbb{F}_1 : (0,\infty)\times\mathbb{R}^n \to (0,\infty)$. Suppose that, for every $x \in X$, we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$, as well as that $\emptyset \neq W_2 \subseteq (2\mathbb{Z})^n$ and for every $\mathbf{t} \in \mathbb{R}^n$ there exists $l_1 > 0$ such that, for every $l \geqslant l_1$ and $w \in lW_2$, we have*

$$\int_{l\Omega} \varphi_{p_1(\mathbf{u})}\Bigg(2\mathbb{F}_1(l, \mathbf{t}+w) \sum_{k\in l(2\mathbb{Z}+1)^n} \frac{\|h(\mathbf{u}+k-\mathbf{v})\|_{L^{q(\mathbf{v})}(l\Omega)}}{\mathbb{F}(l, \mathbf{t}-k+w)}\Bigg)\,du \leqslant 1. \tag{8.63}$$

Then the function $h * F : \mathbb{R}^n \times X \to Y$, defined by

$$(h * F)(\mathbf{t};x) := \int_{\mathbb{R}^n} h(\sigma)F(\mathbf{t}-\sigma;x)\,d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X, \tag{8.64}$$

is Weyl $p_1(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W_2)$-multi-almost automorphic of type 2.

Proof. It can be simply verified that the function $(h * F)(\cdot;\cdot)$ is well defined because we have assumed that, for every $x \in X$, we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$. By our assumption, for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$, there exists a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\varepsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every $m,\ m' \in \mathbb{N}$ with $m \geqslant m_0$ and $m' \geqslant m_0$, there exists $l \geqslant l_0$ such that, for every $l \geqslant l_0$ and $w \in l(2\mathbb{Z}+1)^n$ (8.59) holds with $W \equiv W_1 \equiv (2\mathbb{Z}+1)^n$. Let $l \geqslant \max(l_0, l_1)$ and $w \in lW_2$. Since the mapping $\varphi_{p_1(\mathbf{u})}(\cdot)$ is monotonically increasing, we have

$$\begin{aligned}
&\big\|(h * F)(\mathbf{t}+w+\mathbf{b}_{k_m}+\mathbf{u};x) - (h * F)(\mathbf{t}+w+\mathbf{b}_{k_{m'}}+\mathbf{u};x)\big\|_{L^{p_1(\mathbf{u})}(l\Omega:Y)} \\
&\quad = \Bigg\| \int_{\mathbb{R}^n} h(\mathbf{s})\big[F(\mathbf{t}+w+\mathbf{b}_{k_m}+\mathbf{u}-\mathbf{s};x) - F(\mathbf{t}+w+\mathbf{b}_{k_{m'}}+\mathbf{u}-\mathbf{s};x)\big]d\mathbf{s}\Bigg\|_{L^{p_1(\mathbf{u})}(l\Omega:Y)} \\
&\quad \leqslant \Bigg(\int_{\mathbb{R}^n} |h(\mathbf{s})| \cdot \|F(\mathbf{t}+w+\mathbf{b}_{k_m}+\mathbf{u}-\mathbf{s};x) - F(\mathbf{t}+w+\mathbf{b}_{k_{m'}}+\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s}\Bigg)_{L^{p_1(\mathbf{u})}(l\Omega)},
\end{aligned}$$

which is equal to the infimum of all positive real numbers $\lambda > 0$ such that

$$\int_{l\Omega} \varphi_{p_1(\mathbf{u})}\Bigg(\frac{\int_{\mathbb{R}^n} |h(\mathbf{s})| \cdot \|F(\mathbf{t}+w+\mathbf{b}_{k_m}+\mathbf{u}-\mathbf{s};x) - F(\mathbf{t}+w+\mathbf{b}_{k_{m'}}+\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s}}{\lambda}\Bigg)\,d\mathbf{u} \leqslant 1.$$

By definition of norm in $L^{p_1(\mathbf{u})}(l\Omega)$, it suffices to show that the last estimate holds with $\lambda = \varepsilon/\mathbb{F}_1(l, \mathbf{t}+w)$. This follows from the next computation involving the Hölder inequality (see Lemma 1.1.7(i)) as well as our assumptions $W_2 \subseteq (2\mathbb{Z})^n$ and (8.63):

$$\int_{l\Omega} \varphi_{p_1(\mathbf{u})}\left(\frac{\int_{\mathbb{R}^n} |h(\mathbf{s})| \cdot \|F(\mathbf{t}+w+\mathbf{b}_{k_m}+\mathbf{u}-\mathbf{s};x) - F(\mathbf{t}+w+\mathbf{b}_{k_{m'}}+\mathbf{u}-\mathbf{s};x)\|_Y d\mathbf{s}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t}+w)} \right) d\mathbf{u}$$

$$= \int_{l\Omega} \varphi_{p_1(\mathbf{u})}\left(\frac{\int_{\mathbb{R}^n} |h(\mathbf{s}+\mathbf{u})| \cdot \|F(\mathbf{t}+w+\mathbf{b}_{k_m}-\mathbf{s};x) - F(\mathbf{t}+w+\mathbf{b}_{k_{m'}}-\mathbf{s};x)\|_Y d\mathbf{s}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t}+w)} \right) d\mathbf{u}$$

$$= \int_{l\Omega} \varphi_{p_1(\mathbf{u})} \times \left(\sum_{k\in l(2\mathbb{Z}+1)^n} \frac{\int_{k-l\Omega} |h(\mathbf{s}+\mathbf{u})| \cdot \|F(\mathbf{t}+w+\mathbf{b}_{k_m}-\mathbf{s};x) - F(\mathbf{t}+w+\mathbf{b}_{k_{m'}}-\mathbf{s};x)\|_Y d\mathbf{s}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t}+w)} \right) d\mathbf{u}$$

$$\leqslant \int_{l\Omega} \varphi_{p_1(\mathbf{u})} \times \left(\sum_{k\in lW_1} \int_{l\Omega} \frac{|h(\mathbf{u}-\mathbf{v}+k)| \cdot \|F(\mathbf{t}+w+\mathbf{b}_{k_m}+\mathbf{v}-k;x) - F(\mathbf{t}+w+\mathbf{b}_{k_{m'}}+\mathbf{v}-k;x)\|_Y d\mathbf{v}}{\varepsilon/\mathbb{F}_1(l,\mathbf{t}+w)} \right) d\mathbf{u}$$

$$\leqslant \int_{l\Omega} \varphi_{p_1(\mathbf{u})}\left(2 \sum_{k\in lW_1} \frac{\|h(\mathbf{u}+k-\mathbf{v})\|_{L^{q(\mathbf{v})}(l\Omega)} \cdot (\varepsilon/\mathbb{F}(l,\mathbf{t}-k+w))}{\varepsilon/\mathbb{F}_1(l,\mathbf{t}+w)} \right) d\mathbf{u}$$

$$\leqslant \int_{l\Omega} \varphi_{p_1(\mathbf{u})}\left(2F_1(l,\mathbf{t}+w) \sum_{k\in l(2\mathbb{Z}+1)^n} \frac{\|h(\mathbf{u}+k-\mathbf{v})\|_{L^{q(\mathbf{v})}(l\Omega)}}{\mathbb{F}(l,\mathbf{t}-k+w)} \right) d\mathbf{u} \leqslant 1. \qquad \square$$

Remark 8.3.26. In contrast with the one-dimensional case, the set $\{l \geqslant l_0 : \mathbf{t} - l(2\mathbb{Z}+1)^n\}$ cannot be bounded in $\mathbb{R}^n$ for any $\mathbf{t} \in \mathbb{R}^n$, as easily approved ($n \geqslant 2$). But even in the one-dimensional setting, the requirements of Theorem 8.3.25 do not imply the equi-Weyl p-almost periodicity of function under our consideration (see Example 8.3.11). However, it is not clear whether the requirements of Theorem 8.3.25 imply the Weyl p-almost periodicity of considered function or not.

Concerning the invariance of Weyl $p_1(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, W)$-multi-almost automorphy of type 2 under the actions of infinite convolution products, we will only investigate the one-dimensional case for simplicity (the statements of Theorem 8.3.27 can be also formulated in the multi-dimensional setting, with minor complications).

Theorem 8.3.27. *Suppose that* $p,\ q \in \mathcal{P}(\mathbb{R})$, $F : (0,\infty) \times X \to (0,\infty)$, $F_1 : (0,\infty) \times X \to (0,\infty)$, $\emptyset W_2 \subseteq 2\mathbb{Z}$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $f : \mathbb{R} \to X$ *is a measurable function such that*

$$\int_{-\infty}^{t} \|R(t-s)\| \cdot \|f(s)\|\, ds < \infty, \quad t \in \mathbb{R}.$$

If $f(\cdot)$ is Weyl $p(u)$-$(\mathbb{F}, \mathrm{R}, 2\mathbb{Z}+1)$-multi-almost automorphic of type 2 and

$$\int_{-l}^{l} \varphi_{p(u)}\Bigg(2\mathbb{F}_1(l,t-w)\Bigg[\sum_{m\in\mathbb{N}} \frac{\|R(-v+2ml-u)\|_{L^{q(v)}[-l,l]}}{\mathbb{F}(l,t-w-2ml)} + \frac{\|R(-v+u)\|_{L^{q(v)}[-l,l]}}{\mathbb{F}(l,t-w)}\Bigg]\Bigg)\,du \leqslant 1, \tag{8.65}$$

then the function $F(\cdot)$, given by (2.46), is Weyl $p(u)$-$(\mathbb{F}_1, \mathrm{R}, W_2)$-multi-almost automorphic of type 2.

Proof. It is clear that the function $F(\cdot)$, given by (2.46), is well defined since the integral defining this function is absolutely convergent. Let $(b_k) \in \mathrm{R}$. Then we know that there exists a subsequence (b_{k_m}) of (b_k) such that for each $\varepsilon > 0$ and $t \in \mathbb{R}$ there exists $m_0 \in \mathbb{N}$ such that, for every $m, m' \in \mathbb{N}$ with $m \geqslant m_0$ and $m' \geqslant m_0$, there exists $l \geqslant l_0$ such that, for every $l \geqslant l_0$ and $w \in l(2\mathbb{Z}+1)$, (8.59) holds with $W \equiv W_1 \equiv (2\mathbb{Z}+1)$ and $n = 1$. Let $l \geqslant \max(l_0, l_1)$ and $w \in lW_2$. Then the final conclusion follows similarly to Theorem 8.3.25, by using (8.65) and the next estimate:

$$\int_{-l}^{l} \varphi_{p(u)}\Bigg(\mathbb{F}_1(l,t-w)\int_0^{\infty}\|R(s)\|\big\|f(t+u+b_{k_m}-w-s)-f(t+u+b_{k_{m'}}-w-s)\big\|_Y\,ds\Bigg)\,du \leqslant 1.$$

In order to see the last estimate is valid, we first conclude that

$$\begin{aligned}
&\int_{-l}^{l} \varphi_{p(u)}\Bigg(\mathbb{F}_1(l,t-w)\int_0^{\infty}\|R(s)\|\big\|f(t+u+b_{k_m}-w-s)-f(t+u+b_{k_{m'}}-w-s)\big\|_Y\,ds\Bigg)\,du \\
&= \int_{-l}^{l} \varphi_{p(u)}\Bigg(\mathbb{F}_1(l,t-w)\int_{-l}^{u}\|R(-s+u)\|\big\|f(t+b_{k_m}-w+s)-f(t+b_{k_{m'}}-w+s)\big\|_Y\,ds \\
&\quad + \sum_{m=1}^{\infty}\int_{-l}^{l}\|R(-v+2ml-u)\| \\
&\quad \cdot \big\|f(t+b_{k_m}-w-2ml+v)-f(t+b_{k_{m'}}-w-2ml+v)\big\|_Y\Bigg)\,du.
\end{aligned}$$

After that, we can apply the Hölder inequality, our assumption on the function $f(\cdot)$ and the estimate (8.65). □

It is worth noting that an analogue of Theorem 8.3.25 can be formulated for the following slight generalization of the class introduced in Definition 8.3.17(iii).

Definition 8.3.28. Suppose that $\emptyset \neq W \subseteq \mathbb{R}^n$ and $F : \mathbb{R}^n \times X \to Y$ satisfies the requirement that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$. Let for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \dots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \dots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, as well as for each $\varepsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, there exists $p > 0$ such that, for every $l \in [p, +\infty)$, $m \in \mathbb{N}$ with $m \geqslant p$ and $w \in lW$, we have

$$\mathbb{F}(l, \mathbf{t}-w)\big\|F\big(\mathbf{t}+\mathbf{u}+\big(b_{k_m}^1, \dots, b_{k_m}^n\big)-w;x\big)-F^*(\mathbf{t}+\mathbf{u}-w;x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} < \varepsilon \tag{8.66}$$

and

$$\mathbb{F}(l, \mathbf{t}-w)\big\|F^*\big(\mathbf{t}+\mathbf{u}-\big(b_{k_m}^1, \dots, b_{k_m}^n\big)-w;x\big)-F(\mathbf{t}+\mathbf{u}-w;x)\big\|_{L^{p(\mathbf{u})}(l\Omega:Y)} < \varepsilon, \tag{8.67}$$

then we say that the function $F(\cdot;\cdot)$ is jointly Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic.

It is clear that Lemma 1.1.7(ii) implies that any jointly Weyl $(\mathbb{F}_q, q(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic function $F(\cdot;\cdot)$ is jointly Weyl $(\mathbb{F}_p, p(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic, provided that $p,\ p' \in \mathcal{P}(\mathbb{R}^n)$, $1 \leqslant p \leqslant p'$ a.e. on $\mathbb{R}^n$ and $\mathbb{F}_p(l,\mathbf{t}) := (1+l^n)^{-1}\mathbb{F}_q(l,\mathbf{t})$ for $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$. Furthermore, if we assume that for each sequence belonging to R any its subsequence belongs to R, then the jointly Weyl $(\mathbb{F}, p(\mathbf{u}), \mathrm{R}, W)$-multi-almost automorphic functions form a vector space with the usual operations (the same holds for all other classes of functions introduced in this section).

Furthermore, we have the following result.

Theorem 8.3.29. *Suppose that $h \in L^1(\mathbb{R}^n)$ and $F : \mathbb{R}^n \times X \to Y$ is jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, \mathbb{Z}^n)$-multi-almost automorphic. Let $p_1,\ q \in \mathcal{P}(\mathbb{R}^n)$, let $1/p(\mathbf{u}) + 1/q(\mathbf{u}) \equiv 1$, and let $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$. Suppose that, for every $x \in X$, we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$, as well as that for every $\mathbf{t} \in \mathbb{R}^n$ there exists $l_1 > 0$ such that, for every $l \geqslant l_1$ and $w \in l\mathbb{Z}^n$, the estimate (8.63) holds. If, for every compact set $K \subseteq \mathbb{R}^n$, $x \in X$ and $l > 0$, there exists a finite real constant $c > 0$ such that*

$$\big\|h(\mathbf{u}-\mathbf{v})\big\|_{L^{q(\mathbf{v})}(l\Omega)} \leqslant \Big(\mathbb{F}(l,0)^{-1} + \sup_{\mathbf{t}\in\mathbb{R}^n}\big\|F(\mathbf{t};x)\big\|_Y \cdot \|1\|_{L^{p(\mathbf{u})}(l\Omega)}\Big)^{-1}, \quad \mathbf{u} \in K, \tag{8.68}$$

then $h * F : \mathbb{R}^n \times X \to Y$ (cf. (8.64)), is a well-defined, jointly Weyl $p_1(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, \mathbb{Z}^n)$-multi-almost automorphic function.

Proof. The proof of theorem is very similar to the proof of Theorem 8.3.25 and we will only emphasize the most important differences. First of all, it is clear that the function $h * F : \mathbb{R}^n \times X \to Y$ is well defined. Fix $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \dots, b_k^n)) \in \mathrm{R}$. Then there exists a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \dots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ we have $F^*(\mathbf{t}+\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$, and for each $\varepsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$, there exists $p > 0$ such that, for every $l \in [p, +\infty)$, $m \in \mathbb{N}$ with $m \geqslant p$ and $w \in lW$, we have (8.66) and (8.67). Let $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$ be fixed; we will prove that the value $(h * F^*)(\mathbf{t}; x)$ is well defined. It suffices to prove that

$$\int_{\mathbb{R}^n} |h(\mathbf{t}-\mathbf{s})| \|F^*(\mathbf{s};x)\|_Y \, d\mathbf{s} := \lim_{l\to+\infty} \int_{l\Omega} |h(\mathbf{t}-\mathbf{s})| \|F^*(\mathbf{s};x)\|_Y \, d\mathbf{s} < +\infty.$$

Since the mapping $l \mapsto \int_{l\Omega} |h(\mathbf{t}-\mathbf{s})| \|F^*(\mathbf{s};x)\|_Y \, d\mathbf{s}$, $l > 0$ is monotonically increasing, it suffices to show its boundedness for $l > 0$. This follows from the fact that $F^*(\mathbf{u};x) \in L^{p(\mathbf{u})}(l\Omega : Y)$ for all $l > 0$ (this is a consequence of (8.66) with $\mathbf{t} = \omega = 0$ and our assumption that, for every $x \in X$, we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$), the Hölder inequality and the assumption that, for every $\mathbf{u} \in \mathbb{R}^n$ and $l > 0$, there exists a finite real constant $c > 0$ such that (8.68) holds. The remainder of proof can be given by copying the corresponding part of proof of Theorem 8.3.25. □

In order to relax our exposition, we will only note that an analogue of Theorem 8.3.27 can be formulated for jointly Weyl $p(u)$-$(\mathbb{F}, \mathrm{R}, W)$-multi-almost automorphic functions following the method proposed in the proofs of Theorem 8.3.27 and Theorem 8.3.29, by assuming condition of type (8.68) for the resolvent family $(R(t))_{t>0} \subseteq L(X, Y)$ under consideration. Details can be left to the reader.

Concerning the pointwise products of Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions with the scalar-valued functions of the same type, we will clarify only the following result.

Proposition 8.3.30. *Assume that for each sequence belonging to* R *any its subsequence belongs to* R.

(i) *Suppose that* $\emptyset \neq W \subseteq \mathbb{R}^n$, $p \in \mathcal{P}(\mathbb{R}^n)$, $g : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$, $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$, $f : \mathbb{R}^n \to \mathbb{C}$ *is essentially bounded and Weyl* $p(\mathbf{u})$-(g, R, W)-*multi-almost automorphic of type* 2, $F : \mathbb{R}^n \times X \to Y$ *is Weyl* $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-*multi-almost automorphic of type* 2 *and for each* $x \in X$ *we have* $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$. *Suppose that* $\mathbb{F}_1 : (0,\infty)\times\mathbb{R}^n \to (0,\infty)$ *satisfies the requirement that there exist real numbers* $c > 0$ *and* $l_0 > 0$ *such that*

$$\frac{1}{\mathbb{F}(l,\mathbf{t})} + \frac{1}{g(l,\mathbf{t})} \leqslant \frac{c}{\mathbb{F}_1(l,\mathbf{t})}, \quad l \geqslant l_0,\ \mathbf{t} \in \mathbb{R}^n. \tag{8.69}$$

Then the function $F_1(\mathbf{t};x) := f(\mathbf{t}) \cdot F(\mathbf{t};x)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ *is Weyl* $p(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W)$-*multi-almost automorphic of type* 2.

(ii) *Suppose that* $\emptyset \neq W \subseteq \mathbb{R}^n$, p, $q \in \mathcal{P}(\mathbb{R}^n)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) \equiv 1$, $g : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$, $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$, $f : \mathbb{R}^n \to \mathbb{C}$ *is essentially bounded and jointly Weyl* $p(\mathbf{u})$-(g, R, W)-*multi-almost automorphic,* $F : \mathbb{R}^n \times X \to Y$ *is jointly Weyl* $q(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-*multi-almost automorphic and for each* $x \in X$ *we have* $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t};x)\|_Y < \infty$. *Suppose that* $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$ *satisfies the* *requirement that for each $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$ there exist real numbers $c > 0$ and $l_0 > 0$ such that, for every $l \geqslant l_0$, we have*

$$\begin{aligned}\mathbb{F}_1(l, \mathbf{t}-w)\Bigg[&\frac{1}{g(l, \mathbf{t}-w)} \sup_{\mathbf{t}\in\mathbb{R}^n}\|F(\mathbf{t}; x)\|_Y \|1\|_{L^{q(\mathbf{u})}(l\Omega)} \\ &+ \frac{1}{\mathbb{F}(l, \mathbf{t}-w)}\left(\frac{1}{g(l, \mathbf{t}-w)} + \|f\|_\infty \cdot \|1\|_{L^{p(\mathbf{u})}(l\Omega)}\right)\Bigg] \leqslant c. \end{aligned} \tag{8.70}$$

Then the function $F_1(\mathbf{t}; x) := f(\mathbf{t}) \cdot F(\mathbf{t}; x)$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ is jointly Weyl 1-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic.

Proof. Let $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ be given. Since we have assumed that, for every sequence which belongs to R, any its subsequence also belongs to R, by the corresponding definition we get the existence of a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\varepsilon > 0$, $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every m, $m' \in \mathbb{N}$ with $m \geqslant m_0$ and $m' \geqslant m_0$, there exists $l \geqslant l_0$ such that, for every $l \geqslant l_0$ and $w \in lW$, we have

$$\begin{aligned}&\|f(\mathbf{t}+\mathbf{u}+(b_{k_m}^1, \ldots, b_{k_m}^n) - w) - f(\mathbf{t}+\mathbf{u}+(b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w)\|_{L^{p(\mathbf{u})}(l\Omega:Y)} \\ &\quad < \varepsilon / g(l, \mathbf{t}-w),\end{aligned} \tag{8.71}$$

and (8.59). Since we have assumed that the function $f(\cdot)$ is essentially bounded as well as that for each $x \in X$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$, the estimates (8.59), (8.69)–(8.71) and the decomposition

$$\begin{aligned}&F_1(\mathbf{t}+\mathbf{u}+(b_{k_m}^1, \ldots, b_{k_m}^n) - w; x) - F_1(\mathbf{t}+\mathbf{u}+(b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w) \\ &\quad = f(\mathbf{t}+\mathbf{u}+(b_{k_m}^1, \ldots, b_{k_m}^n) - w) \\ &\qquad \times [F(\mathbf{t}+\mathbf{u}+(b_{k_m}^1, \ldots, b_{k_m}^n) - w; x) - F(\mathbf{t}+\mathbf{u}+(b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w; x)] \\ &\qquad + F(\mathbf{t}+\mathbf{u}+(b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w; x) \\ &\qquad \times [f(\mathbf{t}+\mathbf{u}+(b_{k_m}^1, \ldots, b_{k_m}^n) - w) - f(\mathbf{t}+\mathbf{u}+(b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w)]\end{aligned}$$

simply imply that $F_1(\cdot; \cdot)$ is Weyl $p(\mathbf{u})$-$(\mathbb{F}_1, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic of type 2. The second part of proposition follows from a similar decomposition with the limit functions, by applying the Hölder inequality, the estimate (8.70) and a simple estimate for the function $f^*(\cdot)$ obtained from (8.66). □

The interested reader may try to reformulate the statement of [660, Theorem 3.4] for Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions of type 2 and jointly Weyl $p(\mathbf{u})$-$(\mathbb{F}, \mathrm{R}, \mathcal{B}, W)$-multi-almost automorphic functions.

8.3.5 Applications to the abstract Volterra integro-differential equations

In this subsection, we will present some applications of the obtained theoretical results in the qualitative analysis of solutions for various classes of the abstract Volterra integro-differential equations in Banach spaces.

1. Besides many other applications, we would like to note that Proposition 8.3.6 takes effect in the qualitative analysis of jointly Weyl 1-almost automorphic solutions of the fractional Poisson heat equation $D_{t,+}^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x)$, $t \in \mathbb{R}$, $x \in \Omega$; $v(t,x) = 0$, $v(t,x) \in [0,\infty) \times \partial\Omega$ in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a. e. $x \in \Omega$, $m \in L^{\infty}(\Omega)$, $\gamma \in (0,1)$ and $1 < p < \infty$; in the case of consideration of general exponent $p \in \mathcal{P}(\mathbb{R})$, we can also apply Theorem 8.3.27. See [631] for more details.
2. Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1 \leqslant p < \infty$. Consider again the Gaussian semigroup $(G(t))_{t \geqslant 0}$. Suppose now that the number $t_0 > 0$ is fixed and that $F : \mathbb{R}^n \to \mathbb{C}$ is both essentially bounded and Weyl p-$(\mathbb{F}, \mathrm{R}, (2\mathbb{Z}+1)^n)$-multi-almost automorphic of type 2, where $p(\mathbf{u}) \equiv p \in [1,\infty)$ and $\mathbb{F} : (0,\infty) \to (0,\infty)$. Let $p_1 \in [1,\infty)$, let $1/p + 1/q = 1$, and let $\mathbb{F}_1 : (0,\infty) \to (0,\infty)$. Then the function $x \mapsto (G(t_0)F)(x)$, $x \in \mathbb{R}^n$ is essentially bounded. Suppose, further, that $\emptyset \neq W_2 \subseteq (2\mathbb{Z})^n$ and for every $\mathbf{t} \in \mathbb{R}^n$ there exists $l_1 > 0$ such that, for every $l \geqslant l_1$ and $w \in lW_2$, we have
$$(4\pi t_0)^{-(n/2)}\left(\frac{\mathbb{F}_1(l)}{\mathbb{F}(l)}\right)^{p_1} \int\limits_{l\Omega}\Bigg[\sum_{k \in l(2\mathbb{Z}+1)^n}\Bigg(\int\limits_{l\Omega} e^{-\frac{q|(\mathbf{u}+k-\mathbf{v})|^2}{4t_0}}\, d\mathbf{v}\Bigg)^{1/q}\Bigg]^{p_1} d\mathbf{u} \leqslant 1. \tag{8.72}$$
Then Theorem 8.3.25 implies that the function $x \mapsto (G(t_0)F)(x)$, $x \in \mathbb{R}^n$ is Weyl p_1-$(\mathbb{F}_1, \mathrm{R}, W_2)$-multi-almost automorphic of type 2. Here we only want to note that the series in (8.72) converges because, for every $\mathbf{u} \in l\Omega$, $\mathbf{v} \in l\Omega$ and $k \in l(2\mathbb{Z}+1)^n$, we have $|\mathbf{u} + k - \mathbf{v}| \geqslant |k - 2l\sqrt{n}|$. Note that Theorem 8.3.29 is also applicable here.
3. (cf. also the corresponding application already considered in the part about multidimensional Weyl almost periodic functions) Suppose that $Y := L^r(\mathbb{R}^n)$ for some $r \in [1,\infty)$ and $A(t) := \Delta + a(t)I$, $t \geqslant 0$, where Δ is the Dirichlet Laplacian on $L^r(\mathbb{R}^n)$, I is the identity operator on $L^r(\mathbb{R}^n)$ and $a \in L^{\infty}([0,\infty))$. Then the evolution system $(U(t,s))_{t \geqslant s \geqslant 0} \subseteq L(Y)$ generated by the family $(A(t))_{t \geqslant 0}$ exists and is given by $U(t,t) := I$ for all $t \geqslant 0$ and
$$[U(t,s)F](\mathbf{u}) := \int\limits_{\mathbb{R}^n} K(t,s,\mathbf{u},\mathbf{v})F(\mathbf{v})\, d\mathbf{v}, \quad F \in L^r(\mathbb{R}^n),\ t > s \geqslant 0,$$
where
$$K(t,s,\mathbf{u},\mathbf{v}) := \big(4\pi(t-s)\big)^{-\frac{n}{2}} e^{\int_s^t a(\tau)\, d\tau} \exp\left(-\frac{|\mathbf{u}-\mathbf{v}|^2}{4(t-s)}\right), \quad t > s,\ \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n.$$

Under certain assumptions, a unique mild solution of the abstract Cauchy problem $(\partial/\partial t)u(t,x) = A(t)u(t,x)$, $t > 0$; $u(0,x) = F(x)$ is given by $u(t,x) := [U(t,0)F](x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Suppose now that $F \in L^r(\mathbb{R}^n)$ and $F(\cdot)$ is Weyl p-$(\mathbb{F}, \mathrm{R}, (2\mathbb{Z}+1)^n)$-multi-almost automorphic of type 2, where $1 \leqslant p < \infty$ and the function $\mathbb{F}(l,\mathbf{t}) \equiv \mathbb{F}(l)$ does not depend on $\mathbf{t}$. Let $1/p + 1/q = 1$, let $\varepsilon > 0$ be given, and let $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$. Then we know that there exists a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ such that for each $\varepsilon > 0$ and $\mathbf{t} \in \mathbb{R}^n$ there exists $m_0 \in \mathbb{N}$ such that, for every m, $m' \in \mathbb{N}$ with $m \geqslant m_0$ and $m' \geqslant m_0$, there exists $l_0 > 0$ such that, for every $l \geqslant l_0$ and $w \in l(2\mathbb{Z})^n$, we have (8.59). Let a number $t_0 > 0$ be fixed. Arguing as before, we see that there exists a finite constant $c_{t_0} > 0$ such that

$$\begin{aligned}
&\bigl|u(t_0, \mathbf{t} + \mathbf{u} + (b_{k_m}^1, \ldots, b_{k_m}^n) - w) - u(t_0, \mathbf{t} + \mathbf{u} + (b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w)\bigr| \\
&\quad \leqslant c_{t_0} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{u}-\mathbf{v}|^2}{4t_0}} \bigl|F(\mathbf{v} + \mathbf{t} + (b_{k_m}^1, \ldots, b_{k_m}^n) - w) - F(\mathbf{v} + \mathbf{t} + (b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w)\bigr|\, d\mathbf{v} \\
&\quad = c_{t_0} \sum_{k \in l(2\mathbb{Z}+1)^n} \int_{k+l[-1,1]^n} e^{-\frac{|\mathbf{u}-\mathbf{v}|^2}{4t_0}} \\
&\qquad \times \bigl|F(\mathbf{v} + \mathbf{t} + (b_{k_m}^1, \ldots, b_{k_m}^n) - w) - F(\mathbf{v} + \mathbf{t} + (b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w)\bigr|\, d\mathbf{v} \\
&\quad \leqslant c_{t_0} \sum_{k \in l\mathbb{Z}^n} \bigl\|e^{-\frac{|\mathbf{u}-\cdot|^2}{4t_0}}\bigr\|_{L^q(k+l[-1,1]^n)} \\
&\qquad \times \bigl\|F(\cdot + \mathbf{t} + (b_{k_m}^1, \ldots, b_{k_m}^n) - w - k) - F(\cdot + \mathbf{t} + (b_{k_{m'}}^1, \ldots, b_{k_{m'}}^n) - w - k)\bigr\|_{L^p(l[-1,1]^n)} \\
&\quad \leqslant c_{t_0} \frac{\varepsilon}{\mathbb{F}(l)} \sum_{k \in l\mathbb{Z}^n} \bigl\|e^{-\frac{|\mathbf{u}-\cdot|^2}{4t_0}}\bigr\|_{L^q(k+l[-1,1]^n)} := c_{t_0} \frac{\varepsilon}{\mathbb{F}(l)} G(l, \mathbf{u}).
\end{aligned}$$

Let $1 \leqslant p' < \infty$. Define the function $\mathbb{F}_1(\cdot)$ by

$$\mathbb{F}_1(l, \mathbf{t}) := \frac{\mathbb{F}(l)}{(\int_{l[-1,1]^n} G(l, \mathbf{u})^{p'}\, d\mathbf{u})^{1/p'}}, \quad l > 0.$$

By the foregoing, we see that the mapping $x \mapsto u(t_0, x)$, $x \in \mathbb{R}^n$ is Weyl p'-$(\mathbb{F}_1, \mathrm{R}, (2\mathbb{Z})^n)$-multi-almost automorphic of type 2.

Question 8.3.2, Question 8.3.7, Question 8.3.14, Question 8.3.15 and the following remain open after this study.

Question 8.3.31. Let $I = \mathbb{R}$ and $p \geqslant 1$. Does there exist a Weyl p-almost automorphic function of type 1 which is not Weyl p-almost automorphic?

The results of Subsection 8.3.3, which are formulated for the functions of the form $F : \mathbb{R}^n \times X \to Y$, will be the basis of our further investigations of composition principles for Weyl almost automorphic type functions and related abstract semilinear Cauchy problems.

Regarding the invariance of generalized almost periodicity and automorphy under the action of infinite convolution products, we would like to note that the notion of equi-Weyl p-normality (see Subsection 8.3.2) can be also modified following the approach obeyed in this paper; for example, in the one-dimensional setting, we can analyze the following notions:

1. A p-locally integrable function $f : \mathbb{R} \to X$ is said to be equi-Weyl p-normal of type 1 if and only if for any real sequence (s_n) there exist a subsequence (s_{n_k}) of (s_n) and a p-locally integrable function $f^* : \mathbb{R} \to X$ such that
$$\lim_{l\to+\infty} \lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{2l} \int_{-l}^{l} \|f(t+x+s_{n_k}) - f^*(t+x)\|^p \, dx = 0.$$
2. A p-locally integrable function $f : \mathbb{R} \to X$ is said to be jointly equi-Weyl p-normal if and only if for any real sequence (s_n) there exist a subsequence (s_{n_k}) of (s_n) and a p-locally integrable function $f^* : \mathbb{R} \to X$ such that
$$\lim_{(k,l)\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{2l} \int_{-l}^{l} \|f(t+x+s_{n_k}) - f^*(t+x)\|^p \, dx = 0.$$

Then it is possible to state some results about the invariance of Weyl p-almost normality and jointly Weyl p-almost normality under the actions of convolution products, like [641, Proposition 7]. It is also worth noting that the characteristic function of any fixed compact subset of $\mathbb{R}$ is jointly equi-Weyl p-normal, with the limit function $f^* \equiv 0$.

The class of Besicovitch p-almost automorphic functions can be further generalized by replacing the $\limsup \cdot$ in the corresponding definition with $\liminf \cdot$.

Definition 8.3.32. Let $p \geqslant 1$. Then we say that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ is weakly Besicovitch p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ such that
$$\lim_{k\to\infty} \liminf_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \|f(t+s_{n_k}+x) - f^*(t+x)\|^p \, dx = 0$$
and
$$\lim_{k\to\infty} \liminf_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \|f^*(t-s_{n_k}+x) - f(t+x)\|^p \, dx = 0$$
for each $t \in \mathbb{R}$.

Multi-dimensional analogues of (weak) Besicovitch almost automorphic type functions can be also introduced but we will analyze this topic elsewhere.

9 Notes and appendices to Part II

In this chapter, we will briefly consider several important topics about multi-dimensional almost periodic type functions and multi-dimensional almost automorphic type functions which have not been discussed so far.

Almost periodicity and homogenization theory

For a brief introduction to the mathematical theory of homogenization, the reader may consult the monographs [152] by A. Bensoussan, J. L. Lions, G. Papanicolau, [215] by A. Braides, A. Defraceschi, [297] by D. Cioranescu, P. Donato, [970] by L. Tartar and [1099] by V. Zhikov, S. Kozlov, O. Oleinik. Roughly, homogenization extracts homogeneous effective parameters from models of disordered media, when it is often called statistical homogenization, or heterogeneous media. The study of asymptotic behavior of oscillating structures has been carried on successfully under certain hypothesis of periodicity, in a great deal of papers in the field of calculus of variations (see also [18, 74–76, 90, 91, 552, 861] for some relevant articles of mathematicians from the former USSR). It seems that the corresponding problems in the almost periodic setting were analyzed for the first time by S. M. Kozlov [675] in 1979.

It would be really difficult to summarize here all relevant results obtained so far with regards to the homogenization problems for various types of partial differential operators, equations and systems of equations in the almost periodic setting. In this part, we will briefly describe the main results of research articles [338] by R. De Arcangelis, [551] by H. Ishii and [738] by B. Luo (see also [212–214, 216, 275, 339, 812, 817, 868, 931, 942, 1044]).

In the homogenization theory, numerous research articles investigate the asymptotic behavior of the solutions of the problem

$$\inf\left\{\int_{\Omega} f(hx, Du) + \int_{\Omega} \psi x : u(\cdot) \text{ Lipschitz continuous and } u = 0 \text{ on } \partial\Omega\right\}, \tag{9.1}$$

where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open bounded set, $\psi(\cdot)$ is essentially bounded on Ω, and $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ satisfies the usual Carathéodory conditions:

$$f(x,z) \quad \text{is measurable in } x \text{ and convex in } z, \tag{9.2}$$

$$f(\cdot,z) \quad \text{is } [0,1]^n\text{-periodic for every } z \in \mathbb{R}^n, \tag{9.3}$$

and

$$\begin{aligned} &0 \leqslant w(x)|z|^p \leqslant f(x,z) \leqslant W(x)(1+|z|^p) \quad \text{for a. e. } x \in \mathbb{R}^n \text{ and every } z \in \mathbb{R}^n; \\ &p > 1; \quad w^{(-1)/(p-1)},\ W \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \end{aligned}$$

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Under some extra assumptions, including the Lipschitz type boundary of Ω, G. de Giorgi has proved in [345] that the values in (9.1) converge to

$$\inf\left\{\int_{\Omega} f_{\infty}(Du)+\int_{\Omega}\psi x : u(\cdot)\text{ Lipschitz continuous and } u=0 \text{ on } \partial\Omega\right\}, \tag{9.4}$$

where $f_{\infty} : \mathbb{R}^n \to [0,\infty)$ is a convex function defined by

$$f_{\infty}(x) := \lim_{s\to\infty} s^{-n}\inf\left\{\int_{(0,s)^n} f(x,z+Du) : u(\cdot)\text{ Lipschitz continuous and } u=0 \text{ on } \partial\big((0,s)^n\big)\right\}.$$

Condition (9.3) has been replaced with certain almost periodic assumptions in many research articles. In [338], the author has assumed that (9.2) holds, $f(\cdot,z)\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for every $z\in\mathbb{R}^n$, $|z|\leqslant f(x,z)$ for a. e. $x\in\mathbb{R}^n$ and every $z\in\mathbb{R}^n$, and the following almost periodic type condition: For every $\varepsilon>0$, there exists a finite real number $L_{\varepsilon}>0$ such that, for every $x_0\in\mathbb{R}^n$, there exists $\tau\in x_0+B(0,L_{\varepsilon})$ such that

$$\big|f(x+\tau,z)-f(x,z)\big|\leqslant\varepsilon\big(1+f(x,z)\big),\quad \text{for a. e. } x\in\mathbb{R}^n \text{ and every } z\in\mathbb{R}^n.$$

Then, for every open convex set Ω and for every essentially bounded function $\psi(\cdot)$ on Ω, the values in (9.1) converges to the value in (9.4).

In [551], H. Ishii has analyzed the asymptotic behavior, as the parameter ε tends to 0+, of the solution u^{ε} of the Hamilton–Jacobi equation

$$u(x)+H\big(x,x/\varepsilon,Du(x)\big)=0,\quad x\in\mathbb{R}^n, \tag{9.5}$$

where $\varepsilon>0$ is a positive real number. This equation describes a sort of distance functions in the space where the Riemannian metric is oscillatory (for more details about the generalized solutions of the Hamilton–Jacobi equations, we refer the reader to the monograph [715] by P. L. Lions). The basic assumption made in [551] is that the Hamiltonian $H(x,y,p)$ is almost periodic with respect to the variable y; since (9.5) does not have classical solutions, the author has considered certain types of viscosity solutions of this equation. If the assumptions [551, (A0)–(A4)] hold, then there exists a unique bounded Lipschitz continuous solution $u(\cdot)$ of (9.5). The existence of a constant $\lambda>0$, whose existence is proved in [551, Theorem 2], enables the author to introduce the notion of an effective Hamiltonian $\overline{H} : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, which satisfies the following estimates:

$$\inf_{y\in\mathbb{R}^n} H(x,y,p)\leqslant\overline{H}(x,y)\leqslant\sup_{y\in\mathbb{R}^n} H(x,y,p),\quad x,\ p\in\mathbb{R}^n.$$

The effective Hamiltonian is continuous on its domain and the main result of paper is Theorem 5 which asserts that $u^{\varepsilon} \to u$ locally uniformly as ε tends to 0+, where $u(\cdot)$ is a unique bounded, uniformly continuous solution of the equation

$$u(x) + \overline{H}\big(x, Du(x)\big) = 0, \quad x \in \mathbb{R}^n.$$

Reference [738] investigates the homogenization limit of the following parabolic equation:

$$u_t = a(u_x)u_{xx} + f(u_x), \quad -1 < x < 1,\ t > 0, \tag{9.6}$$

accompanied by the nonlinear boundary conditions:

$$u_x(-1,t) = g\big(u(-1,t)/\varepsilon\big), \quad u_x(1,t) = -g\big(u(1,t)/\varepsilon\big), \tag{9.7}$$

where $\varepsilon > 0$ is a real parameter and $g(\cdot)$ is a function which takes values near its supremum "frequently". It is shown that a time-global solution u^{ε} of (9.6)–(9.7) converges as $\varepsilon \to 0+$ to the solution μ of (9.6) accompanied by the linear boundary conditions:

$$\mu_x(-1,t) = \sup g, \quad \mu_x(1,t) = -\sup g,$$

provided $\mu(\cdot)$ increases monotonically. In the case that the function $g(\cdot)$ is almost periodic, we have the existence of a unique almost periodic traveling wave U_{ε} of (9.6)–(9.7) and the homogenization limit of U_{ε} is a classical traveling wave of (9.6)–(9.7).

n-Parameter strongly continuous semigroups

The notion of a semigroup over topological monoid naturally generalizes the notion of usually considered one-parameter strongly continuous semigroup of bounded linear operators. This broad class of semigroups includes the semigroups defined on the set $[0,\infty)^n$, which are oftenly called multiparameter semigroups (this class of semigroups was introduced by E. Hille in 1944; see [237] and [536]).

So, let $(M,+)$ be a topological monoid with the neutral element 0. By a semigroup over a Banach space X defined over a monoid M we mean any operator-valued function $T : M \to L(X)$ such that $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t,\ s \in M$. A semigroup $T : M \to L(X)$, which we also denote by $(T(t))_{t\in M}$, is called strongly continuous if and only if the mapping $t \mapsto T(t)x$, $t \in M$ is strongly continuous at $t = 0$. In [324], R. Dahya has extended a well-known result saying that every weakly continuous semigroup $(T(t))_{t\geqslant 0}$ is strongly continuous to the semigroups over topological monoids.

Furthermore, if $M = [0,\infty)^n$ and $(T(\mathbf{t}))_{\mathbf{t}\in M}$ is strongly continuous, then we denote by $T_i(t) := T(te_i)$, $t \geqslant 0$, the corresponding one-parameter strongly continuous semigroup ($i \in \mathbb{N}_n$). Let $T_i(\cdot)$ be generated by A_i ($i \in \mathbb{N}_n$). Then the tuple $(A_1, A_2, \ldots, A_n)$ is said to be the infinitesimal generator of $(T(\mathbf{t}))_{\mathbf{t}\in[0,\infty)^n}$. We can simply prove that

$(T(\mathbf{t}))_{\mathbf{t}\in[0,\infty)^n}$ is strongly continuous (uniformly continuous) if and only if for each $i \in \mathbb{N}_n$ the one-parameter semigroup $(T_i(t))_{t\geqslant 0}$ is strongly continuous (uniformly continuous). A strongly continuous semigroup $(T(\mathbf{t}))_{\mathbf{t}\in[0,\infty)^n}$ is always exponentially bounded in the sense that there exist two finite real constants $M \geqslant 1$ and $\omega > 0$ such that $\|T(\mathbf{t})\| \leqslant Me^{\omega|\mathbf{t}|}$ for all $\mathbf{t} \in [0,\infty)^n$; see e. g., Theorem 1 in the paper [108] by V. A. Babalola, where the author has considered generalizations of the Hille–Yosida–Phillips theorem for abstract-parameter semigroups. Furthermore, the following hold [237, 536]:

(i) If $i \in \mathbb{N}_n$ and $x \in D(A_i)$, then $T(\mathbf{t})x \in D(A_i)$ for all $\mathbf{t} \in [0,\infty)^n$ and $T(\mathbf{t})A_ix = A_iT(\mathbf{t})x$, $\mathbf{t} \in [0,\infty)^n$;
(ii) $\bigcap_{i\in\mathbb{N}_n} D(A_i)$ is dense in X;
(iii) If $i,\ j \in \mathbb{N}_n$, then $D(A_i) \cap D(A_iA_j) \subseteq D(A_jA_i)$ and for each $x \in D(A_i) \cap D(A_iA_j)$ we have $A_iA_jx = A_jA_ix$.

Set $I := [0,T_1] \times [0,T_2] \times \cdots \times [0,T_n]$ for some $(T_1,T_2,\ldots,T_n) \in (0,\infty)^n$. The well-posedness of the homogeneous n-parameter abstract Cauchy problem

$$\text{(ACP)}: \begin{cases} u \in C(I:X) \cap C^1(I^\circ : X), \\ u_{t_i}(\mathbf{t}) = A_iu(\mathbf{t}) + F_i(\mathbf{t}), \quad \mathbf{t} \in I^\circ,\ 1 \leqslant i \leqslant n, \\ u(0) = x, \quad x \in \bigcap_{i\in\mathbb{N}_n} D(A_i), \end{cases}$$

has been analyzed by M. Janfada and A. Niknam in [557, Theorem 2.1], who proved that, if $(A_1,A_2,\ldots,A_n)$ is the infinitesimal generator of a strongly continuous semigroup $(T(\mathbf{t}))_{\mathbf{t}\in[0,\infty)^n}$, then (ACP) has a unique solution $u(\mathbf{t}) = T(\mathbf{t})x$, $t \in I$ for all initial values $x \in \bigcap_{i\in\mathbb{N}_n} D(A_i)$; a converse of this statement has been analyzed in [557, Theorem 2.2] (see also Theorem 2.5 in this paper, where the authors have shown a negative result about the uniqueness of solutions of the abstract Cauchy problem (ACP) as well as the paper [592] where the authors have considered the special case $n = 2$ by using the notion of a two-parameter integrated semigroup; the special case $n = 2$ has been also analyzed in [556] and [595], where the authors have introduced the notion of a two-parameter C-regularized semigroup and the notion of a two-parameter N-times integrated semigroup, respectively, where the operator $C \in L(X)$ is injective and $N \in \mathbb{N}$). The extensions of the Lumer–Philips theorem for two-parameter C_0-semigroups have been analyzed by R. Abazari, A. Niknam and M. Hassani in [3], while a Hille–Yosida type theorem for multiparameter semigroups has been analyzed by Yu. S. Mishura and A. S. Lavréntev in [776]. The different notions of generators of two parameter semigroups have been analyzed by Sh. Al-Sharif, R. Khalil [45] and S. Arora, S. Sharda [88]; see also an interesting generalization of R. Datko's result [548, Theorem 1.3] to nonlinear two parameter semigroups established by A. Ichikawa in [548, Theorem 2.1]. For some applications of multiparameter strongly continuous semigroups in the analysis of partial differential equations in the spaces of almost periodic functions, we refer the reader to [180] and [1005]; for some applications to the stochastic differential equations, see [205, 899, 1032] and the monograph [1099].

To the best of our knowledge, the notion of an n-parameter α-times integrated semigroup, the notion of an n-parameter C-regularized semigroup and the notion of an n-parameter α-times integrated C-regularized semigroup have not been introduced so far. The degenerate case also remains still very unexplored, even for degenerate two-parameter strongly continuous semigroups (in the current literature, we have not been able to locate any research paper regarding this issues).

Concerning some applications of multiparameter semigroups in the analysis of multi-dimensional almost periodic type solutions of the abstract partial differential equations and their systems, the situation is similar: there are only a few research papers devoted to the study of variation of parameters formulas for multiparameter semigroups (fractional multiparameter resolvent families have not been analyzed elsewhere either) but the existence and uniqueness of almost periodic type solutions of the abstract partial differential equations and their systems have not still been analyzed with the help of the theory of multiparameter semigroups. With regards to this intriguing topic, we want to mention only the investigation of M. Khanehgir, M. Janfada and A. Niknam [593], where the authors have examined the well-posedness of the following inhomogeneous abstract Cauchy problem:

$$(\mathrm{ACP})_2 : \begin{cases} u \in C(I : X) \cap C^1(I^{\cdot} : X), \\ u_{t_i}(\mathbf{t}) = A_i u(\mathbf{t}) + F(\mathbf{t}), \quad \mathbf{t} \in I,\ i = 1, 2, \\ u(0) = x, \quad x \in \bigcap_{i \in \mathbb{N}_2} D(A_i), \end{cases}$$

assuming that the pair (A_1, A_2) generates a strongly continuous semigroup $(T(t_1, t_2))_{t_1 \geqslant 0, t_2 \geqslant 0}$ on X. In their analysis, the same inhomogeneity has appeared for $i = 1$ and $i = 2$, which forces upon us a very unpleasant condition:

$$F_{t_1}(t_1, t_2) - F_{t_2}(t_1, t_2) = (A_1 - A_2)F(t_1, t_2), \quad t_1,\ t_2 > 0.$$

Despite this, the following formula for a solution of $(\mathrm{ACP})_2$ has been proposed:

$$u(t_1, t_2) = T(t_1, t_2)x + \int_0^{t_1} T(t_1 - t, t_2)F(t, 0)\, dt + \int_0^{t_2} T(0, t - t_2)F(t_1, 0)\, dt,$$

for any $t_1,\ t_2 > 0$. It could be interesting to formulate some results about the asymptotically almost periodic solutions of this solution provided that the semigroup $(T(t_1, t_2))_{t_1 \geqslant 0, t_2 \geqslant 0}$ is exponentially decaying and the function $F(\cdot; \cdot)$ is asymptotically almost periodic in a certain sense.

The multiparameter semigroups play an important role in the study of the approximations of periodic functions of several real variables (A. P. Terehin [975]) and the study of diffusion equations in space-time dynamics (S. V. Zelik [1072]). For some other questions about multiparameter semigroups, we refer the reader to the research articles [10, 343, 520, 553, 557, 558] and [593, 594, 732, 733, 975, 983].

Multivariate trigonometric polynomials and approximations of periodic functions of several real variables

Without any doubt, trigonometric polynomials of several real variables, sometimes also called multivariate trigonometric polynomials, presents the best explored class of almost periodic functions of several real variables. Multivariate trigonometric polynomials have an invaluable importance in the theory of approximations of periodic functions of several real variables, especially in the two-dimensional case. For the basic source of information about this subject, the reader may consult the research monographs [403] by B. Dumitrescu, [405] by D. Dung, V. Temlyakov, T. Ullrich, [971] and [972] by V. Temlyakov (see also the research studies [44, 160, 357, 402, 491, 578, 621, 923, 973, 1045, 1058]; for some other questions about multivariate trigonometric polynomials, we refer the reader to the research articles [131, 406, 469, 470, 486, 802, 838, 856, 876] and references quoted therein).

In this part, we will briefly explain the main results and ideas of papers [109] by A. M.-B. Babayev, [838] by L. Pfister, Y. Bresler and [578] by L. Kämmerer, D. Potts, T. Volkmer. If $f : \mathbb{R} \to \mathbb{R}$ belongs to the space $C_{2\pi}$ of all real continuous functions of period 2π, then it is well known that the Vallee–Poussin singular integral $V_k(\cdot)$, defined by

$$V_k(x) := \frac{1}{2\pi}\frac{(2k)!!}{(2k-1)!!}\int_{-\pi}^{\pi} f(t)\cos^{2k}\frac{t-x}{2}\,dt, \quad x \in \mathbb{R}\ (k \in \mathbb{N}),$$

has the property that $\lim_{k\to+\infty} V_k(x) = f(x)$, uniformly for $x \in \mathbb{R}$. This result of Vallee–Poussin improves the classical Weierstrass second theorem on the density of trigonometric polynomials in the spaces of continuous functions. The two-dimensional Vallee–Poussin singular integral $V_{k,m}(\cdot)$, defined for each $x \in \mathbb{R}$ by $(k,\ m \in \mathbb{N})$,

$$V_{k,m}(x,y) := \frac{1}{(2\pi)^2}\frac{(2k)!!}{(2k-1)!!}\frac{(2m)!!}{(2m-1)!!}\int_{-\pi}^{\pi} f(t,\tau)\cos^{2k}\frac{t-x}{2}\cos^{2k}\frac{\tau-y}{2}\,d\tau,$$

has been introduced in [109, Definition 2]. In the same paper, the author has shown that $\lim_{k\to+\infty}\lim_{m\to+\infty} V_{k,m}(x,y) = f(x,y)$, uniformly for $(x,y) \in \mathbb{R}^2$ as well as that $V_{k,m}(x,y)$ is a trigonometric polynomial in variables x and y, for all $k,\ m \in \mathbb{N}$ (see [109, Theorem 2]). For proving the last fact, the author has used a lemma clarifying that the product of two trigonometric polynomials of two variables is also the trigonometric polynomial of two variables whose order equals the sum of order of cofactors as well as that any even trigonometric polynomial $T(x,y)$, i. e., a trigonometric polynomial $T(x,y)$ which satisfies $T(-x,-y) = T(x,y)$, $T(-x,y) = T(x,y)$ and $T(x,-y) = T(x;y)$ identically for $(x,y) \in \mathbb{R}^2$, may be represented in the form

$$T(x,y) = A + \sum_{k=1}^{m}\sum_{l=1}^{n}(a_{kl}\cos kx\cos ly + b_{kl}\cos kx + c_{kl}\cos ly), \quad (x,y) \in \mathbb{R}^2,$$

which does not contain the sines of multiple arcs (see [109, Lemma 3, Lemma 4]). We would like to note that the obtained results continue to hold in the vector-valued case.

In [838], L. Pfister and Y. Bresler have investigated bounding multivariate trigonometric polynomials and given some applications to the problems of filter bank design. Denote

$$T_l^n := \operatorname{span}\Big\{e^{i\langle \mathbf{k},\lambda\rangle} : \lambda \in [0,2\pi]^n,\ \mathbf{k} \in \mathbb{Z}^n, \|\mathbf{k}\| := \sup_{1\leqslant i\leqslant n} |k_i| \leqslant l\Big\} \quad (l \in \mathbb{N})$$

and

$$\Theta_N := \{2\pi k/N : k = 0,1,\dots,N-1\} \quad (N \in \mathbb{N}).$$

For any $N \in \mathbb{N}$ and for any real-valued trigonometric polynomial

$$P(\lambda) := \sum_{k_1=-l}^{l}\sum_{k_2=-l}^{l}\cdots\sum_{k_n=-l}^{l} c_{k_1k_2\cdots k_n} e^{i\langle k,\lambda\rangle} \in T_l^n,$$

i. e., the multivariate trigonometric polynomial $P(\cdot)$ for which $c_{k_1,k_2,\dots,k_n} = c^*_{-k_1,-k_2,\dots,-k_n}$ ($\|\mathbf{k}\| \leqslant l$; the star denotes complex conjugation), we define

$$\|P\|_\infty := \max_{\lambda\in[0,2\pi]^n}\big|P(\lambda)\big| \quad \text{and} \quad \|P\|_{N^n,\infty} := \max_{\lambda\in\Theta_N^n}\big|P(\lambda)\big|.$$

Then two well-known results of the approximation theory state that

$$\|P\|_\infty \leqslant \|P\|_{(2l+1)^n,\infty}\big(1 + 4\pi^{-1} + 2\pi^{-1}\ln(2l+1)\big)^n$$

and, in the one-dimensional case,

$$\|P\|_\infty \leqslant \frac{\|P\|_{N,\infty}}{\sqrt{1-(2l/N)}}.$$

In [838, Theorem 1], the authors have shown that the assumptions $N \geqslant 2l+1$ and $\alpha = 2l/N$ yield the existence of a positive real constant $C_{N,l}^n \in [0,(1-\alpha)^{-(n/2)}]$ such that

$$\|P\|_\infty \leqslant C_{N,l}^n\|P\|_{N^n,\infty}, \quad P \in T_l^n,$$

and $C_{N,l}^n\|P\|_{N^n,\infty} - \|P\|_\infty = O(\ln/N)$, $P \in T_l^n$. In order to achieve their aims, the authors have used the de la Vallée–Poussin kernels and the tensor products of one-dimensional Dirichlet kernels.

In [578], the authors have investigated certain algorithms for the approximation of multivariate periodic functions by trigonometric polynomials, which are based on the use of a single one-dimensional fast Fourier transform and the so-called method of sampling of multivariate functions on rank-1 lattices. In their analysis, the authors have used periodic Sobolev spaces of generalized mixed smoothness and presented some advantages of their method compared to the method based on the trigonometric interpolations on generalized sparse grids. Some numerical results and tests are presented up to dimension $n = 10$, as well.

Almost periodic pseudo-differential operators and Gevrey classes

Almost periodic pseudo-differential operators have been analyzed by numerous mathematicians including L. A. Coburn, R. D. Moyer, I. M. Singer [305], P. E. Dedik [344], R. Iannacci, A. M. Bersani, G. Dell'Acqua, P. Santucci [546], A. A. Pankov [825], M. A. Shubin [937, 939, 940] and P. Wahlberg [1008]. In this section, we will present the main ideas and results of research study [818] by A. Oliaro, L. Rodino and P. Wahlberg, only.

It is well known that M. A. Shubin has proved that almost periodic pseudo-differential operators act continuously on the space of smooth almost periodic functions as well as that the operator norm on L^2 equals that on the Hilbert space $B^2(\mathbb{R}^n)$ of Besicovitch almost periodic functions whose Fourier coefficients are square summable. It is also well known that M. A. Shubin has introduced, for every exponent $p \in [1, \infty]$ and for every real number $t \in \mathbb{R}$, the space $W_t^p(\mathbb{R}^n)$ of almost periodic functions and proved the continuity of any almost periodic pseudo-differential operator $A : W_t^2(\mathbb{R}^n) \to W_{t-m}^2(\mathbb{R}^n)$, with arbitrary $t \in \mathbb{R}$, provided that the symbol of A belongs to the class $S_{\rho,\delta}^m$ $(0 \leqslant \delta < \rho \leqslant 1)$. In the papers of M. A. Shubin, some regularity results for formally hypoelliptic almost periodic pseudo-differential operators have been examined on the space $W_{-\infty}^2(\mathbb{R}^n) := \bigcup_{t \in \mathbb{R}} W_t^2(\mathbb{R}^n)$.

In [818], the authors have sought for ultradistributional analogues of the above-mentioned results, working with almost periodic functions that are Gevrey regular of order $s \geqslant 1$ (the difference between the real analytic case $s = 1$ and the pure ultradistributional case $s > 1$ should be emphasized here). If $\emptyset \neq \Omega \subseteq \mathbb{R}^n$, then the space of all Gevrey functions of order $s \geqslant 1$, denoted by $G^s(\Omega)$, is defined as a collection of all infinitely differentiable functions $F : \mathbb{R}^n \to \mathbb{C}$ such that for each compact set $K \subseteq \mathbb{R}^n$ there exists a finite real constant $C_K > 0$ such that

$$|D^\alpha F(\mathbf{t})| \leqslant C_K^{1+|\alpha|} \alpha!^s$$

for all $\mathbf{t} \in K$ and $\alpha \in \mathbb{N}_0^n$. It is natural to ask whether an almost periodic function $F : \mathbb{R}^n \to \mathbb{C}$ which belongs to the space $G^s(\Omega)$ obeys the property of the existence of a global real constant $C > 0$ such that

$$|D^\alpha F(\mathbf{t})| \leqslant C^{1+|\alpha|} \alpha!^s$$

for all $\mathbf{t} \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$? An instructive counterexample in the one-dimensional setting, with $s > 1$, is given in [818, Example 2.1], showing that this is not true in general: Set $g_s(x) := \exp(-x^{1/(1-s)})$, $x > 0$, $g_s(x) := 0$, $x \leqslant 0$, $\psi_s(x) := g_s(x) g_s(1-x)$, $x \in \mathbb{R}$, $\psi_{s,n}(x) := \psi_s(nx)$, $x \in \mathbb{R}$ and $\varphi_{s,n}(x) := \sum_{k \in \mathbb{Z}} \psi_s(x - 2^n(2k+1))$, $x \in \mathbb{R}$ $(n \in \mathbb{N})$. It has been shown that the function

$$F_s(x) := \sum_{n=1}^{\infty} n^{-1/4} \varphi_{s,n}(x), \quad x \in \mathbb{R},$$

is well defined, as well as that the above series is uniformly convergent in the variable $x \in \mathbb{R}$, so that the function $F_s(\cdot)$ is actually, semi-periodic, since the function $\varphi_{s,n}(\cdot)$ is of period 2^{n+1} ($n \in \mathbb{N}$). We also have $F_s \in G^s(\mathbb{R})$ and $F_s \notin G^s_{ap}(\mathbb{R})$; see the notion explained below. Albeit not explicitly constructed in [818], it is our strong belief that also this example can be transferred to the multi-dimensional setting without any serious difficulties (more to the point, the case $s = 1$ has not been considered in [818, Example 2.1] and deserves further analysis).

After providing this counterexample, the authors have paid a special attention to the analysis of almost periodic functions $F : \mathbb{R}^n \to \mathbb{C}$ belonging to the space $G^s(\mathbb{R}^n)$ and obeying the property that there exists a real constant $C > 0$ such that $|D^\alpha F(\mathbf{t})| \leqslant C^{1+|\alpha|}\alpha!^s$ for all $\mathbf{t} \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$. The union of these functions, denoted by $G^s_{ap}(\mathbb{R}^n)$, is equipped with the usual inductive limit topology as a union of Banach spaces. The authors have introduced after that the corresponding classes of symbols, pseudo-differential operators and continued their non-trivial analysis; see [818] for more details.

Periodic generalized functions

It would be rather unpleasant to recapitulate here all relevant methods and already established results about periodic generalized functions. We will only say a few words about scalar-valued periodic distributions, scalar-valued periodic ultradistributions and quote some references about periodic Colombeau hyperfunctions. A detailed study of multi-dimensional periodic generalized functions with values in Banach spaces and general topological spaces will be carried out in our forthcoming research studies.

1. **Periodic distributions.** It is well known that a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ is called periodic of period $\mathbf{T} = (T_1, \ldots, T_n) > 0$ if and only if for each $i \in \mathbb{N}_n$ we have $F(x_1, \ldots, x_i + T_i, \ldots, x_n) = F(x_1, \ldots, x_i, \ldots, x_n)$. The vector space consisting of all distributions $F \in \mathcal{D}'(\mathbb{R}^n)$ which are periodic of period $\mathbf{T} > 0$ are usually denoted by $\mathcal{D}'_T(\mathbb{R}^n)$. The most simple example of a periodic non-regular distribution of period $\mathbf{T} > 0$ is given by $\delta_T(x) := \sum_{k\in\mathbb{Z}^n} \delta(x + kT)$; many similar examples can be found in the paper [869] by N. Reckoski, V. Reckovski and V. Manova-Erakovikj.
   We know that any distribution $F \in \mathcal{D}'_T(\mathbb{R}^n)$ is tempered as well as that F can be expanded into a corresponding Fourier series which converges to F in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. More precisely, if $F \in \mathcal{D}'_T(\mathbb{R}^n)$ and $\varphi : \mathbb{R}^n \to \mathbb{C}$ is an infinitely differentiable function of period $\mathbf{T} > 0$, then we define $\langle F, \varphi\rangle := \langle F, \psi_0\varphi\rangle$, where $\psi_0 : \mathbb{R}^n \to \mathbb{C}$ is an arbitrary infinitely differentiable function satisfying that $\sum_{k\in\mathbb{Z}^n} \psi_0(x + k\mathbf{T}) = 1$ for all $x \in \mathbb{R}^n$. It is well known that the value of $\langle F, \varphi\rangle$ does not depend on the choice of the function $\psi_0(\cdot)$, and that the special choice of the function $\psi_0(\cdot)$ as in the book of V. Vladimirov [1004], shows that for any regular distribution $F \in \mathcal{D}'_T(\mathbb{R}^n)$ we have $\langle F, \varphi\rangle = \int_0^T F(x)\varphi(x)\,dx$. For simplicity, we will assume now that $T_i = 1$ for all $i \in \mathbb{N}_n$. In this case, with each $F \in \mathcal{D}'_1(\mathbb{R}^n)$

we associate its formal Fourier series

$$\sum_{k\in\mathbb{Z}^n} a_k(F)e^{i\sum_{j=1}^n 2\pi k_j x_j},$$

where

$$a_k(F) := \int_0^1 \cdots \int_0^1 e^{-i\sum_{j=1}^n 2\pi k_j x_j} F(x_1,\ldots,x_n)\,dx_1\cdots\,dx_n, \quad k\in\mathbb{Z}^n.$$

This series converges to F in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$; see [847, Section 10.6] for more details.
It is worth noting that K. N. Khan, W. Lamb and A. C. McBride have developed, in [591], two equivalent approaches for defining fractional derivatives of periodic distributions in the one-dimensional setting, following the approach of A. H. Zemanian from [1073, Chapter 9], with $I = (0, 2\pi)$. The first approach is a distributional version of the Weyl approach for ordinary functions, whilst the second approach is based on the Grünwald–Letnikov formula for defining a fractional derivative of a locally integrable function; the authors have provided an interesting application in the study of distributional solutions of the fractional diffusion equation (see also the research monograph [220] by A. C. McBride for further information on fractional calculus of generalized functions). Let us also mention that E. L. Korotyaev has considered, in [625], the KdV equation on the Sobolev space of periodic distributions (cf. also [622–624]).

2. **Periodic ultradistributions**. Within the Komatsu theory of ultradistributions, V. I. Gorbachuk [477] and V. I. Gorbachuk, M. L. Gorbachuk [478] were the first who introduced the classes of periodic ultradistributions of Beurling and Roumieu type (1981–1982). As noticed by S. Pilipović in [845], who structurally characterized the spaces analyzed in [477, 478], a more general space of periodic generalized functions was considered by A. Sźaz [966] in 1978. In [845], it has been assumed that a sequence $(M_p)_{p\in\mathbb{N}_0}$ of positive real numbers satisfies conditions (M.1),
(M.2)* $M_{p+1} \leqslant AH^p M_p$, $p\in\mathbb{N}_0$, for some constants A, $H > 1$,
and
(M.3)* $\lim_{p\to+\infty}\sqrt[p]{M_p} = +\infty$,
which is a consequence of the already analyzed condition (M.3$'$); in particular, these conditions are satisfied for the sequence $M_p := p!^{\alpha}$, where $\alpha > 0$. The space $\mathcal{D}(M_p, L)$ is defined for each positive real number $L > 0$ as a collection of all infinitely differentiable functions $\varphi : [0, 2\pi] \to \mathbb{C}$ for which

$$\|\varphi\|_{L,\infty} := \sup\left\{\frac{\|\varphi^{(p)}\|_\infty}{L^p M_p} : p\in\mathbb{N}_0\right\} < +\infty.$$

After that, the spaces $\mathcal{D}(M_p)$ and $\mathcal{D}\{M_p\}$ are defined by

$$\mathcal{D}(M_p) := \operatorname{proj} \lim_{L \to 0+} \mathcal{D}(M_p, L) \quad \text{and} \quad \mathcal{D}\{M_p\} := \operatorname{ind} \lim_{L \to +\infty} \mathcal{D}(M_p, L).$$

The spaces of periodic ultradistributions of Beurling type and the Roumieu type are defined, respectively, as the strong duals of these spaces; therefore, we have

$$\mathcal{D}'(M_p) = \operatorname{ind} \lim_{L \to 0+} \mathcal{D}'(M_p, L) \quad \text{and} \quad \mathcal{D}'\{M_p\} := \operatorname{proj} \lim_{L \to +\infty} \mathcal{D}'(M_p, L).$$

The first serious observation made in [845] was that the spaces analyzed in [477, 478] present very exceptional cases of the spaces of periodic generalized function spaces considered by A. H. Zemanian in [1073, Chapter 9]; see [845, Theorem 1] for more details. The representation theorem established in [845, Theorem 2] says that for any element $f \in \mathcal{D}'(M_p)$ we can always find a positive integer $n \in \mathbb{N}$ and a bounded sequence $(f_p)_{p \in \mathbb{N}_0}$ in $L^2[0, 2\pi]$ such that $f = \sum_{p=0}^{\infty} (n^p f_p^{(p)} / M_p)$ as well as that for any positive integer $n \in \mathbb{N}$ and any bounded sequence $(f_p)_{p \in \mathbb{N}_0}$ in $L^2[0, 2\pi]$ the above expression determines an element $f \in \mathcal{D}'(M_p)$. For the Roumieu class $\mathcal{D}'\{M_p\}$, the main representation theorem [845, Theorem 3] says that for any element $f \in \mathcal{D}'\{M_p\}$ we can always find a sequence $(f_p)_{p \in \mathbb{N}_0}$ in $L^2[0, 2\pi]$ such that $\sum_{n=0}^{\infty} \|n^p M_p f_p\|_{L^2[0,2\pi]} < \infty$ for any positive integer $n \in \mathbb{N}$ and $f = \sum_{p=0}^{\infty} f_p^{(p)}$ as well as that for any sequence $(f_p)_{p \in \mathbb{N}_0}$ in $L^2[0, 2\pi]$ such that $\sum_{n=0}^{\infty} \|n^p M_p f_p\|_{L^2[0,2\pi]} < \infty$ for any positive integer $n \in \mathbb{N}$, the expression $f = \sum_{p=0}^{\infty} f_p^{(p)}$ determines an element $f \in \mathcal{D}'\{M_p\}$.

Generalized differential algebras containing the spaces of periodic ultradistributions have recently been investigated by A. Debrouwere in [341], while the Fourier coefficients of periodic functions of Gevrey classes and ultradistributions defined on the torus have been analyzed by Y. Taguchi in [968]. For the sequential approach to the theory of (periodic) ultradistributions, we refer the reader to the doctoral dissertation of P. Sokolski [945] (for the sequential approach to the theory of distributions and the basic theory of periodic distributions, the classic monograph [73] by P. Antosik, J. Mikusiński and R. Sikorski is also of incredible importance). Within the Braun-Meise-Taylor theory of ultradistributions [217], it is worth mentioning the research article [944] by B. K. Sohn, who studied the classes of periodic tempered distributions of Beurling type and periodic ultradifferentiable functions with arbitrary support.

Concerning periodic Colombeau generalized functions, the reader may consult [291, 293, 350, 608, 992]; for more details about periodic generalized function spaces, we also refer the reader to [135, 393, 591, 753, 816, 878] and [846, 884, 907, 919, 967, 991, 1071].

Stepanov-like almost periodicity in mixed Lebesgue spaces

Let us recall that the notion of a mixed Lebesgue space can be traced back to the paper of L. Hörmander [540], where he investigated the estimates for translation invariant operators (1960). The mixed Lebesgue spaces (or the Lebesgue spaces with vector exponents $L^{\vec{p}}$) are considered as a natural generalization of the classical Lebesgue space L^p via replacing the constant exponent p by a vector exponent $\vec{p} := (p_1, \ldots, p_n) \in (0, +\infty]^n$. A first detailed study of the mixed Lebesgue spaces is carried out by A. Benedek and R. Panzone in [150] (1961); see also [39, 150, 303, 439, 464, 467, 544, 570, 960, 1090] and the references cited therein for more details about the subject.

The definition goes as follows. For any $\vec{p} := (p_1, \ldots, p_n) \in [1, +\infty]^n$, we denote by $\vec{q} := (q_1, \ldots, q_n)$ its conjugate exponent, i. e., $\frac{1}{\vec{p}} + \frac{1}{\vec{q}} = 1$; namely, for any $j \in \{1, \ldots, n\}$, $\frac{1}{p_j} + \frac{1}{q_j} = 1$. If $\vec{p} = (p_1, \ldots, p_n)$ and $\vec{q} = (q_1, \ldots, q_n)$ are two vectors in $[1, +\infty]^n$, then we write $\vec{p} \leqslant \vec{q}$ ($\vec{p} < \vec{q}$) if and only if $p_j \leqslant q_j$ ($p_j < q_j$) for any $j \in \{1, \ldots, n\}$.

Definition 9.0.1. Let $\emptyset \neq \Lambda_j \subseteq \mathbb{R}$ be a Lebesgue measurable set ($1 \leqslant j \leqslant n$), let $\Lambda = \prod_{j=1}^n \Lambda_j \subseteq \mathbb{R}^n$, and let $\vec{p} = (p_1, \ldots, p_n) \in [1, +\infty]^n$. The mixed Lebesgue space $L^{\vec{p}}(\Lambda : X)$ is defined to be the set of all Lebesgue measurable functions $F : \Lambda \to X$ such that

$$\|f\|_{L^{\vec{p}}(\Lambda:X)} := \left(\int\limits_{\Lambda_n} \cdots \left(\int\limits_{\Lambda_2} \left(\int\limits_{\Lambda_1} \|F(s_1, \ldots, s_n)\|_X^{p_1} \, ds_1 \right)^{p_2/p_1} ds_2 \right)^{p_3/p_2} \cdots \, ds_n \right)^{1/p_n} < \infty,$$

with the usual modifications when $p_j = +\infty$ for some $j \in \{1, \ldots, n\}$.

In the case that $p_1 = \cdots = p_n = p$, with some $p \in [1, +\infty]$, the space $L^{\vec{p}}(\Lambda : X)$ coincides with the usual Lebesgue space $L^p(\Lambda : X)$.

Applying successively Minkowski's inequality, for $F, G \in L^{\vec{p}}(\Lambda : X)$, $\vec{p} \in [1, +\infty]^n$, we obtain the following Minkowski inequality in $L^{\vec{p}}(\Lambda : X)$:

$$\|F + G\|_{L^{\vec{p}}(\Lambda:X)} \leqslant \|F\|_{L^{\vec{p}}(\Lambda:X)} + \|G\|_{L^{\vec{p}}(\Lambda:X)}. \tag{9.8}$$

Similarly, for $F \in L^{\vec{p}}(\Lambda : X)$ and $G \in L^{\vec{q}}(\Lambda : X)$, we have $FG \in L^1(\Lambda : X)$ and the successive applications of the usual Hölder inequality gives the following Hölder inequality in $L^{\vec{p}}(\Lambda : X)$:

$$\|FG\|_{L^1(\Lambda:X)} \leqslant \|F\|_{L^{\vec{p}}(\Lambda:X)} \|G\|_{L^{\vec{q}}(\Lambda:X)}, \tag{9.9}$$

for any $\vec{p} \in [1, +\infty]^n$ and $\vec{q} \in [1, +\infty]^n$ satisfying $\frac{1}{\vec{p}} + \frac{1}{\vec{q}} = 1$.

As a consequence, $(L^{\vec{p}}(\Lambda : X), \| \cdot \|_{L^{\vec{p}}(\Lambda:X)})$ is a Banach space for any $\vec{p} \in [1, +\infty]^n$.

Before proceeding to the next section, we would like to recall that the mixed Lebesgue spaces play an important role in the abstract harmonic analysis, especially in the theory of Wiener amalgam spaces and the theory of modulation spaces. For example, the mixed Lebesgue norm appears in definitions of the generalized modulation space $M^m_{p,q}(\mathbb{R}^d)$ introduced by H. Feichtinger and K. Gröchening in [432, Definition 2.3], the amalgam space $W(L^p; L^q_\omega)$ introduced by C. Heil in [524, Definition 11.3.1], the mixed Lebesgue space $L^{p,q}(v)$ introduced by H. Rauhut in [867, Section 6], and the general ultramodulation space $M^{\omega_\gamma}_{p,q}$ introduced by N. Teofanov in [974, Definition 4, p. 36]; see also the research monographs [153] by A. Bényi, K. Okoudjou, [489] by K. Gröchenig, [763] by Y. Meyer, [1027] by N. Wiener and the doctoral dissertation of C. Heil [523] for more details about the subject.

Now we will extend the concept of S^p-almost periodicity to the Lebesgue spaces $L^{\vec{p}}(\Lambda : X)$ with vector exponent $\vec{p} \in [1, +\infty)^n$. Unless stated otherwise, in the sequel of this part we will always assume that $\Omega = [0,1]^n$.

First of all, we will investigate the notion of Stepanov $\vec{p}$-boundedness.

Definition 9.0.2. Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ and $\vec{p} = (p_1, \ldots, p_n) \in [1, +\infty)^n$. A function $F : \Lambda \times X \to Y$ is said to be $\vec{p}$-locally integrable on $\mathcal{B}$ if and only if, for every $B \in \mathcal{B}$ and for every sequence $(K_j)_{1 \leqslant j \leqslant n}$ of compact subsets of $\mathbb{R}$ such that $K_1 \times K_2 \times \cdots \times K_n \subseteq \Lambda$, we have

$$\begin{aligned} &\|F\|_{L^{\vec{p}}(\Lambda:X),(K_j)_{1\leqslant j\leqslant n},B} \\ &:= \sup_{x\in B}\Bigg(\int\limits_{K_n}\cdots\Bigg(\int\limits_{K_2}\Bigg(\int\limits_{K_1}\|F(s_1,\ldots,s_n;x)\|_X^{p_1}\,ds_1\Bigg)^{p_2/p_1} ds_2\Bigg)^{p_3/p_2}\cdots ds_n\Bigg)^{1/p_n} < \infty. \end{aligned}$$

The set of all $\vec{p}$-locally integrable functions on Λ is denoted by $L^{\vec{p},\mathcal{B}}_{\mathrm{loc}}(\Lambda \times X : Y)$.

In this part, the multi-dimensional Bochner transform will be also denoted by $F^b : \Lambda \times X \to Y^\Omega$; hence, if function $F : \Lambda \times X \to Y$ is given, then

$$[F^b(\mathbf{t}; x)](\mathbf{u}) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in X.$$

Now we are ready to introduce the following notion.

Definition 9.0.3. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + [0,1]^n \subseteq \Lambda$ and let $\vec{p} = (p_1, \ldots, p_n) \in [1, +\infty)^n$. Let a function $F : \Lambda \times X \to Y$ be $\vec{p}$-locally integrable on $\mathcal{B}$. Then we say that $F(\cdot;\cdot)$ is Stepanov $\vec{p}$-bounded on $\mathcal{B}$ if and only if for every $B \in \mathcal{B}$ there exists a finite real constant $M > 0$ such that

$$\|F\|_{S^{\vec{p}},B} := \sup_{\mathbf{t}\in\Lambda; x\in B}\Bigg(\int\limits_0^1\Bigg(\int\limits_0^1\Bigg(\int\limits_0^1\|F(t_1+s_1,\ldots,t_n+s_n;x)\|^{p_1}\,ds_1\Bigg)^{p_2/p_1} ds_2\Bigg)^{p_3/p_2}\cdots ds_n\Bigg)^{1/p_n} < M$$

for any $\mathbf{t} = (t_1, \ldots, t_n) \in \Lambda$ and $x \in B$. The collection of these functions will be denoted by $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$.

By applying Minkowski's inequality (9.8), it is easy to see that $L^{\vec{p},\mathcal{B}}_{\mathrm{loc}}(\Lambda \times X : Y)$ and $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$ are vector spaces. Let $F(\cdot;\cdot)$ be Stepanov $\vec{p}$-bounded on $\mathcal{B}$, and let $B \in \mathcal{B}$ be fixed. Then it is easy to see that $\|\cdot\|_{S^{\vec{p}},B}$ is a norm on $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$; furthermore, we have the following expected result.

Proposition 9.0.4. *Let $\vec{p} \in [1,+\infty)^n$. Then $(BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y), \|\cdot\|_{S^{\vec{p},B}})$ is a Banach space.*

Proof. Let $(F_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$. Then we have

$$\forall \varepsilon > 0 \ \exists j_0 \in \mathbb{N} \ \forall i, j \geqslant j_0 \Rightarrow \|F_i - F_j\|_{S^{\vec{p},B}} \leqslant \varepsilon.$$

This shows that $(F_j(\mathbf{t}; x))_{j\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^{\vec{p}}(\prod_{k=1}^n [t_k, t_k + 1] : Y)$ uniformly with respect to $\mathbf{t} = (t_1, \ldots, t_n) \in \Lambda$ and $x \in B$, so there exists a function $F : \Lambda \times X \to Y$ such that

$$\sup_{(t_1,\ldots,t_n)\in\Lambda; x\in B} \left(\int_{t_n}^{t_n+1} \cdots \left(\int_{t_2}^{t_2+1} \left(\int_{t_1}^{t_1+1} \|F_j(s_1, \ldots, s_n; x) - F(s_1, \ldots, s_n; x)\|_X^{p_1} \, ds_1 \right)^{p_2/p_1} ds_2 \right)^{p_3/p_2} \cdots ds_n \right)^{1/p_n} \underset{j\to+\infty}{\longrightarrow} 0.$$

Minkowski's inequality allows us to conclude that

$$\|F\|_{S^{\vec{p},B}} \leqslant \|F_j - F\|_{S^{\vec{p},B}} + \|F_j\|_{S^{\vec{p},B}} < \infty, \quad j \in \mathbb{N},$$

which shows that $F \in BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$ since $(F_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $BS^{\vec{p}}(\Lambda \times X : Y)$ and therefore bounded. □

Under certain very reasonable assumptions, we see that $BS^{\vec{p},\mathcal{B}}(\Lambda \times X : Y)$ is translation invariant in both arguments. Furthermore, if we assume that $\vec{1} \leqslant \vec{q} \leqslant \vec{p}$, then we can use the Hölder inequality (9.9) in order to see that there exist two finite real constants $c_1 > 0$ and $c_2 > 0$ such that the following estimates hold true:

$$\|F\|_{S^{\vec{q},B}} \leqslant c_1 \|F\|_{S^{\vec{p},B}} \leqslant c_2 \|F\|_{S^{\vec{1},B}},$$

whenever the above expressions make a sense; here, of course, $\vec{1} := (1, 1, \ldots, 1)$.

We introduce the following spaces of mixed Lebesgue–Stepanov-like almost periodic functions.

Definition 9.0.5. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the following condition holds:

$$\text{If } \mathbf{t} \in \Lambda, \ \mathbf{b} \in \mathrm{R} \text{ and } l \in \mathbb{N}, \text{ then we have } \mathbf{t} + \mathbf{b}(l) \in \Lambda.$$

Let the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ be well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\vec{p}, \mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, i. e., for every $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ such that

$$\lim_{l\to+\infty} \left\| F(\mathbf{t} + \mathbf{u} + (b_{k_l}^1, \ldots, b_{k_l}^n); x) - [F^*(\mathbf{t}; x)](\mathbf{u}) \right\|_{L^{\vec{p}}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$.

Definition 9.0.6. Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the following condition holds:

$$\text{If } \mathbf{t} \in \Lambda,\ (\mathbf{b}; \mathbf{x}) \in \mathrm{R}_{\mathrm{X}} \text{ and } l \in \mathbb{N},\ \text{then we have } \mathbf{t} + \mathbf{b}(l) \in \Lambda.$$

Let the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ be well defined and continuous. Then we say that the function $F(\cdot;\cdot)$ is Stepanov $(\vec{p}, \mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic if and only if the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ is $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic, i. e., for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_{\mathrm{X}}$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ such that

$$\lim_{l \to +\infty} \big\| F(\mathbf{t} + \mathbf{u} + (b_{k_l}^1, \ldots, b_{k_l}^n); x + x_{k_l}) - [F^*(\mathbf{t}; x)](\mathbf{u}) \big\|_{L^{\vec{p}}(\Omega:Y)} = 0,$$

uniformly for all $x \in B$ and $\mathbf{t} \in \Lambda$.

Definition 9.0.7. Suppose that $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a continuous function and $\Lambda + \Lambda' \subseteq \Lambda$. Then we say that:

(i) $F(\cdot;\cdot)$ is Stepanov $(\vec{p}, \mathcal{B}, \Lambda')$-almost periodic (Stepanov $(\vec{p}, \mathcal{B})$-almost periodic, if $\Lambda = \Lambda'$) if and only if for every $B \in \mathcal{B}$ and $\varepsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that

$$\big\| F(\mathbf{t} + \tau + \cdot; x) - F(\mathbf{t} + \cdot; x) \big\|_{L^{\vec{p}}(\Omega:Y)} \leqslant \varepsilon, \quad \mathbf{t} \in \Lambda,\ x \in B.$$

(ii) $F(\cdot;\cdot)$ is Stepanov $(\vec{p}, \mathcal{B}, \Lambda')$-uniformly recurrent ($(\vec{p}, \mathcal{B})$-uniformly recurrent, if $\Lambda = \Lambda'$) if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_n) in Λ' such that $\lim_{n \to +\infty} |\tau_n| = +\infty$ and

$$\lim_{n \to +\infty} \sup_{\mathbf{t} \in \Lambda; x \in B} \big\| F(\mathbf{t} + \tau_n + \cdot; x) - F(\mathbf{t} + \cdot; x) \big\|_{L^{\vec{p}}(\Omega:Y)} = 0.$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Stepanov $(\vec{p}, \Lambda')$-almost periodic ($(\vec{p}, \Lambda')$-uniformly recurrent) [Stepanov almost periodic (uniformly recurrent), if $\Lambda = \Lambda'$].

The use of space $L^{\vec{p}}(\Omega : X)$ in our approach is new but very similar to the use of space $L^{p(\mathbf{u})}(\Omega : Y)$ in Section 6.2. Keeping in mind the proofs of our structural results from [266], it becomes clear from the introduced notion and the fact that the Minkowski inequality and the Hölder inequality hold in mixed Lebesgue spaces, that many structural results from Section 6.2 remain true in our framework without any essential changes of the notion and notation (see also [266]). For example, the statements of [266, Proposition 2.6, Propositions 2.10–2.12,Theorems 2.13–2.15, Theorem 2.17, Propositions 2.18–2.19, Proposition 2.25] can be straightforwardly formulated for the Stepanov classes of functions introduced in Definition 9.0.5–Definition 9.0.7; similarly, the supremum formula for Stepanov-like almost periodic functions in mixed

Lebesgue spaces, the relative compactness of range $\{F^b(\mathbf{t}; x) : \mathbf{t} \in \Lambda;\ x \in B\}$, for a given set $B \in \mathcal{B}$, and some results about the composition of Stepanov-like almost periodic functions in mixed Lebesgue spaces, can be achieved similarly as in [266] (see also Section 6.2). In particular, it is worth noting that any Stepanov $(\vec{p}, \mathcal{B})$-almost periodic function defined on $\mathbb{R}^n$ must be Stepanov $\vec{p}$-bounded, provided that $\mathcal{B}$ is a collection of compact subsets of X, and that, in this case, the space $\mathrm{APS}^{\vec{p}}_{\mathcal{B}}(\mathbb{R}^n : Y)$, consisting of all Stepanov $(\vec{p}, \mathcal{B})$-almost periodic functions $F : \mathbb{R}^n \times X \to Y$, is densely and continuously embedded in the space $BS^{\vec{p}}_{\mathcal{B}}(\mathbb{R}^n : Y)$.

The result about the convolution invariance of Stepanov-like almost periodic functions in mixed Lebesgue spaces, which can be simply obtained by reformulating [266, Proposition 2.10], can be applied to the Gaussian semigroup in $\mathbb{R}^n$.

For example, we have the following analogue of [266, Proposition 2.22].

Proposition 9.0.8. *Let $\Lambda + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $\mathcal{B}$ is any family of compact subsets of X and $F : \Lambda \times X \to Y$ satisfy the following conditions:*
(i) *For each $x \in X$, the function $F(\cdot; x)$ is Stepanov $(\vec{p}, \Lambda)$-almost periodic.*
(ii) *For each $\varepsilon > 0$ there exists $\delta_{B,\varepsilon} > 0$ such that for all $x_1, x_2 \in B$ one has*

$$\|x_1 - x_2\| \leqslant \delta_{B,\varepsilon} \Rightarrow \|F(\mathbf{t} + \cdot; x_1) - F(\mathbf{t} + \cdot; x_2)\|_{L^{\vec{p}}(\Omega:Y)} \leqslant \varepsilon \quad \textit{for all } \mathbf{t} \in \Lambda.$$

Then $F(\cdot;\cdot)$ is Stepanov $(\vec{p}, \mathcal{B})$-almost periodic.

The following statement can be formulated for all other classes of functions introduced in Definition 9.0.5–Definition 9.0.7.

Proposition 9.0.9. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, $F : \Lambda \times X \to Y$ and the function $F^b : \Lambda \times X \to L^{\vec{p}}(\Omega : Y)$ is well defined and continuous. If $\vec{1} \leqslant \vec{q} \leqslant \vec{p}$, then*

$$\mathrm{APS}^{\vec{q}}_{\mathcal{B}}(\mathbb{R}^n : Y) \subseteq \mathrm{APS}^{\vec{p}}_{\mathcal{B}}(\mathbb{R}^n : Y) \subseteq \mathrm{APS}^{\overrightarrow{1}}_{\mathcal{B}}(\mathbb{R}^n : Y).$$

Using Proposition 9.0.9 and [266, Theorem 2.21] with $p(\mathbf{u}) \equiv 1$, we immediately get the following.

Proposition 9.0.10. *Suppose that $\mathcal{B}$ is any family of compact subsets of X. If $F : \mathbb{R}^n \times X \to Y$ is uniformly continuous and Stepanov $(\vec{p}, \mathcal{B})$-almost periodic, then $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic.*

Furthermore, we can use Proposition 9.0.9 and our analysis from [266, Example 2.9] in order to conclude that for each $\vec{p} \in [1, \infty)^n$ we have the following (just take $p(\mathbf{u}) \equiv \max\{p_i : 1 \leqslant i \leqslant n\}$ in the above-mentioned example):

Example 9.0.11. Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is a Bohr Λ'-almost periodic function (Λ'-uniformly recurrent function). Define $H : \mathbb{R}^n \to \mathbb{R}$ by $H(\mathbf{t}) := \mathrm{sign}(F(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$. Then the function $H(\cdot)$ is Stepanov $(\vec{p}, \Lambda')$-almost periodic (Stepanov $(\vec{p}, \Lambda')$-uniformly recurrent), provided that

$$(\exists L \geqslant 1)\,(\forall \varepsilon > 0)\,(\forall y \in \mathbb{R}^n)\; m(\{x \in y + \Omega : |F(x)| \leqslant \varepsilon\}) \leqslant L\varepsilon.$$

In particular, the last estimate holds for any multivariate trigonometric polynomial.

We will not consider the composition principles for Stepanov-like almost periodic functions in mixed Lebesgue spaces, as well as the invariance of Stepanov-like almost periodicity in mixed Lebesgue spaces under the actions of convolution products and many other topics here. Finally, we will present the following illustrative application to close this section (see also the third application in Subsection 8.1.6; the final conclusion of Example 9.0.12 can be deduced by assuming that $\vec{p} = \vec{1}$ from the very beginning):

Example 9.0.12. Suppose that $Y := L^r(\mathbb{R}^n)$ for some $r \in [1, \infty)$ and $A(t) := \Delta + a(t)I$, $t \geqslant 0$, where Δ is the Dirichlet Laplacian on $L^r(\mathbb{R}^n)$, I is the identity operator on $L^r(\mathbb{R}^n)$ and $a \in L^\infty([0, \infty))$. Then we know that the evolution system $(U(t, s))_{t\geqslant s\geqslant 0} \subseteq L(Y)$ generated by the family $(A(t))_{t\geqslant 0}$ exists and is given by $U(t, t) := I$ for all $t \geqslant 0$ and (6.96) for $t > s \geqslant 0$, where the corresponding kernel $K(t, s, \mathbf{u}, \mathbf{v})$ is given through (6.97). It is clear that, for every $\tau \in \mathbb{R}^n$, we have (6.98). Furthermore, under certain assumptions, a unique mild solution of the abstract Cauchy problem $(\partial/\partial t)u(t, x) = A(t)u(t, x)$, $t > 0$; $u(0, x) = F(x)$ is given by $u(t, x) := [U(t, 0)F](x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Suppose now that $F : \mathbb{R}^n \to \mathbb{C}$ is Stepanov $(\vec{p}, \mathbb{R}^n)$-almost periodic. Let $1/\vec{p} + 1/\vec{q} = 1$, and let $t > 0$ be fixed. Then there exists a finite real constant $c_t > 0$ such that, for every $\mathbf{t}$, $\tau \in \mathbb{R}^n$ and $\mathbf{u} \in \Omega$, we have

$$\begin{aligned}
&\big|u(t, \mathbf{t} + \mathbf{u} + \tau) - u(t, \mathbf{t} + \mathbf{u})\big| \\
&\quad = \Big|\int_{\mathbb{R}^n} [K(t, 0, \mathbf{t} + \mathbf{u} + \tau, \mathbf{v}) - K(t, 0, \mathbf{t} + \mathbf{u}, \mathbf{v})]F(\mathbf{v})\, d\mathbf{v}\Big| \\
&\quad = \Big|\int_{\mathbb{R}^n} K(t, 0, \mathbf{t} + \mathbf{u} + \tau, \mathbf{v} + \tau)F(\mathbf{v} + \tau)\, d\mathbf{v} - \int_{\mathbb{R}^n} K(t, 0, \mathbf{t} + \mathbf{u}, \mathbf{v})F(\mathbf{v})\, d\mathbf{v}\Big| \\
&\quad = \Big|\int_{\mathbb{R}^n} K(t, 0, \mathbf{t} + \mathbf{u}, \mathbf{v})[F(\mathbf{v} + \tau)\, d\mathbf{v} - F(\mathbf{v})]\, d\mathbf{v}\Big| \\
&\quad \leqslant c_t \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{t}+\mathbf{u}-\mathbf{v}|^2}{4t}} |F(\mathbf{v} + \tau) - F(\mathbf{v})|\, d\mathbf{v} \\
&\quad = c_t \sum_{k\in\mathbb{Z}^n} \int_{[0,1]^n} e^{-\frac{|\mathbf{t}+\mathbf{u}-\mathbf{v}-k|^2}{4t}} |F(k + \mathbf{v} + \tau) - F(k + \mathbf{v})|\, d\mathbf{v} \\
&\quad \leqslant c_t \sum_{k\in\mathbb{Z}^n} \big\|e^{-\frac{|\mathbf{t}+\mathbf{u}-\cdot-k|^2}{4t}}\big\|_{L^{\vec{q}}([0,1]^n)} \big\|F(k + \cdot + \tau) - F(k + \cdot)\big\|_{L^{\vec{p}}([0,1]^n)} \\
&\quad \leqslant c_t \sum_{k\in\mathbb{Z}^n} \big\|e^{-\frac{|\mathbf{t}+\mathbf{u}-\cdot-k|^2}{4t}}\big\|_{L^\infty([0,1]^n)} \big\|F(k + \cdot + \tau) - F(k + \cdot)\big\|_{L^{\vec{p}}([0,1]^n)}.
\end{aligned}$$

If $\tau \in \mathbb{R}^n$ satisfies $\|F(\mathbf{t}+\cdot+\tau)-F(\mathbf{t}+\cdot)\|_{L^{\hat{p}}([0,1]^n)} < \varepsilon$ for all $\mathbf{t} \in \mathbb{R}^n$, then the above implies

$$\big|u(t,\mathbf{t}+\mathbf{u}+\tau)-u(t,\mathbf{t}+\mathbf{u})\big| \leqslant c_t\varepsilon \sum_{k\in\mathbb{Z}^n} \big\|e^{-\frac{|\mathbf{t}+\mathbf{u}-\cdot-k|^2}{4t}}\big\|_{L^\infty([0,1]^n)}, \quad \mathbf{t}\in\mathbb{R}^n,\ \mathbf{u}\in\Omega.$$

A very simple computation involving the Cauchy–Schwartz inequality shows that

$$\big\|e^{-\frac{|\mathbf{t}+\mathbf{u}-\cdot-k|^2}{4t}}\big\|_{L^\infty([0,1]^n)} \leqslant e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}}, \quad \mathbf{t}\in\mathbb{R}^n,\ k\in\mathbb{Z}^n,\ \mathbf{u}\in\Omega,$$

so that

$$\big|u(t,\mathbf{t}+\mathbf{u}+\tau)-u(t,\mathbf{t}+\mathbf{u})\big| \leqslant c_t\varepsilon \sum_{k\in\mathbb{Z}^n} e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}}, \quad \mathbf{t}\in\mathbb{R}^n,\ \mathbf{u}\in\Omega. \tag{9.10}$$

Since the function defined by the above series is continuous in the variable $\mathbf{t} \in \mathbb{R}^n$, there exists a finite real number $M_t \geqslant 1$ such that

$$\sum_{k\in\mathbb{Z}^n} e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}} \leqslant M_t, \quad |\mathbf{t}| \leqslant 1;$$

furthermore, if $|\mathbf{t}| > 1$, then we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n} e^{-\frac{|\mathbf{t}-k|^2-4\sqrt{n}|\mathbf{t}-k|}{4t}} &\leqslant \sum_{k\in\mathbb{Z}^n} e^{-\frac{|k|^2-2|k||\mathbf{t}|+|\mathbf{t}|^2-4\sqrt{n}|k|-4\sqrt{n}|\mathbf{t}|}{4t}} \\ &\leqslant e^{-\frac{|\mathbf{t}|^2-4\sqrt{n}|\mathbf{t}|}{4t}} \sum_{k\in\mathbb{Z}^n} e^{-\frac{|k|^2-2|k|-4\sqrt{n}||k|}{4t}},\end{aligned}$$

which simply implies along with the estimate (9.10) that the function $x \mapsto u(t,x)$, $x \in \mathbb{R}^n$ is Bohr almost periodic in the usual sense since it is continuous.

Slowly oscillating and remotely c-almost periodic type functions in $\mathbb{R}^n$

The material presented here is a part of our recent joint research study [661] with V. Kumar. We start this part by introducing the following notion (see also [82, Definition 4.2.1, p. 247] for a slightly different notion of a one-dimensional slowly oscillating function, and [902] for the notion of a slowly oscillating function $f : [0,\infty) \to \mathbb{C}$ at 0 and $+\infty$):

Definition 9.0.13. Let $c \in \mathbb{C} \setminus \{0\}$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. Define

$$A_I := \{\omega \in \mathbb{R}^n \setminus \{0\} : \omega + I \subseteq I\}.$$

Then we say that a continuous function $F : I \times X \to Y$ is $(\mathbb{D},\mathcal{B})$-slowly oscillating if and only if for each $B \in \mathcal{B}$ and $\omega \in A_I$, we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}} \big\|F(\mathbf{t}+\omega;x)-F(\mathbf{t};x)\big\|_Y = 0, \quad \text{uniformly in } x \in B. \tag{9.11}$$

In other words, a continuous function $F : I \times X \to Y$ is $(\mathbb{D}, \mathcal{B})$-slowly oscillating if and only if $F(\cdot;\cdot)$ is $(S, \mathbb{D}, \mathcal{B})$-asymptotically $(\omega, 1)$-periodic for all $\omega \in A_I$. Clearly, we have $kA_I \subseteq A_I$ for all $k \in \mathbb{N}$.

If $X \in \mathcal{B}$, then we omit the term "$\mathcal{B}$" from the notation and, if $\mathbb{D} = I$, then we omit the term "$\mathbb{D}$" from the notation; for example, if $\mathbb{D} = I$ and $F : I \to Y$ is $(\mathbb{D}, \mathcal{B})$-slowly oscillating with $X = \{0\}$, then we simply say that the function $F(\cdot)$ is slowly oscillating.

We would like to note that it is not so logical to study the class of $(\mathbb{D}, \mathcal{B}, c)$-slowly oscillating functions by replacing the term $\|F(\mathbf{t}+\omega; x) - F(\mathbf{t}; x)\|_Y$ in (2.4.29) by the term $\|F(\mathbf{t} + \omega; x) - cF(\mathbf{t}; x)\|_Y$, where $c \in \mathbb{C} \setminus \{0\}$. In actual fact, we have the following result which is clearly applicable if $\mathbb{D} = I = [0, \infty)^n$ or $\mathbb{D} = I = \mathbb{R}^n$.

Proposition 9.0.14. *Let $c \in \mathbb{C} \setminus \{0\}$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. Suppose that $A_I \subseteq 2A_I$ and $\omega' + \mathbb{D} \subseteq \mathbb{D}$ for all $\omega' \in A_I/2$. Then the following hold:*
(i) *If a continuous function $F : I \times X \to Y$ is $(\mathbb{D}, \mathcal{B}, c)$-slowly oscillating, then $F \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$.*
(ii) *If, in addition to the above, we have $\omega + \mathbb{D} \subseteq \mathbb{D}$ for all $\omega \in A_I$, then a continuous function $F : I \times X \to Y$ is $(\mathbb{D}, \mathcal{B}, c)$-slowly oscillating if and only if $F \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$.*

Proof. To prove (i), suppose that $\omega' \in A_I$ and $B \in \mathcal{B}$; then there exists $\omega \in A_I$ such that $\omega' = 2\omega$. We have ($\mathbf{t} \in I$; $x \in B$):

$$\begin{aligned} F(\mathbf{t} + \omega'; x) - c^2 F(\mathbf{t}; x) &= F(\mathbf{t} + 2\omega; x) - c^2 F(\mathbf{t}; x) \\ &= [F(\mathbf{t} + 2\omega; x) - cF(\mathbf{t} + \omega; x)] + c[F(\mathbf{t} + \omega; x) - cF(\mathbf{t}; x)]. \end{aligned}$$

The prescribed assumption $(A_I/2) + \mathbb{D} \subseteq \mathbb{D}$ implies $\mathbf{t} + \omega \in \mathbb{D}$, $\mathbf{t} \in \mathbb{D}$ and

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}} \|F(\mathbf{t} + \omega'; x) - c^2 F(\mathbf{t}; x)\|_Y = 0, \quad \text{uniformly in } x \in B.$$

Subtracting the terms in the above limit equality and the limit equality (9.11), with the number ω replaced therein with the number ω', we get

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}} \|(c^2 - c) \cdot F(\mathbf{t}; x)\|_Y = 0, \quad \text{uniformly in } x \in B.$$

This immediately implies (i) since $c \neq 1$. To prove (ii), it suffices to apply (i) and observe that the assumption $\omega + \mathbb{D} \subseteq \mathbb{D}$ for all $\omega \in A_I$ implies

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}} \|F(\mathbf{t} + \omega; x)\|_Y = 0, \quad \text{uniformly in } x \in B.$$ □

Concerning the notion of a $(\mathbb{D}, \mathcal{B})$-slowly oscillating function, we would like to note that we do not require any kind of boundedness of the function $F(\cdot;\cdot)$ here. In the classical approach, developed by D. Sarason [904] for the functions of form $f : [0, \infty) \to \mathbb{C}$, the boundedness of the function $f(\cdot)$ is required a priori, which is not a direct consequence of definition since the function $f(t) := t^{\alpha}$, $t \geqslant 0$ satisfies (9.11) if $\alpha \in (0,1)$; the boundedness is obtained by applying the function $e^{i\cdot}$ after that (in other words, the function $t \mapsto e^{it^{\alpha}}$, $t \geqslant 0$ is slowly oscillating in the sense of [904], for any $\alpha \in (0,1)$). It is also worth noting that the global boundedness of the function $f(\cdot)$ has not been used in the proof of [904, Proposition 1], and that the argumentation contained in the proof of this theorem can serve to deduce the following result in the multi-dimensional setting; we will include all details of the proof for the sake of completeness.

Proposition 9.0.15. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$ is an unbounded, closed set and the function $F : I \to Y$ is slowly oscillating. Suppose, further, that the following condition holds:*

(DS1) *For every $r > 0$ and $\delta > 0$, for every points $\mathbf{t}$, $\mathbf{t}' \in I \setminus I_r$ with $|\mathbf{t} - \mathbf{t}'| < \delta$, and for every point $z \in (A_I + \mathbf{t} - \mathbf{t}') \cup (A_I + \mathbf{t}' - \mathbf{t})$, there exists $\eta_z > 0$ such that $B(z, \eta_z) \subseteq A_I$. Here, $I_r \equiv \{\mathbf{t} \in I : |\mathbf{t}| \leqslant r\}$ $(r > 0)$.*

Then the function $F(\cdot)$ is uniformly continuous.

Proof. Suppose that the function $F(\cdot)$ is not uniformly continuous. Since the set I is closed, the set $I \cap B(0,r)$ is compact for all positive real numbers $r > 0$; hence, the following holds:

(DS2) There exists a positive real number $\varepsilon > 0$ such that, for every positive real numbers $\delta > 0$ and $r > 0$, there exist $\mathbf{t}$, $\mathbf{t}' \in I \setminus I_r$ such that $|\mathbf{t} - \mathbf{t}'| < \delta$ and $\|F(\mathbf{t}) - F(\mathbf{t}')\|_Y > \varepsilon$.

Using conditions (DS1) and (DS2), as well as the fact that the function $F(\cdot)$ is slowly oscillating, we can inductively construct the sequences (ω_k) in A_I, $(\mathbf{t}_k)$ in I and (η_k) in $(0,\infty)$ such that $\lim_{k\to+\infty} \eta_k = 0$, $\lim_{k\to\infty} |\mathbf{t}_k| = +\infty$, $B(\omega_k, \eta_k) \subseteq B(\omega_{k+1}, \eta_{k+1}) \subseteq A_I$, $k \in \mathbb{N}$ and $\|F(\mathbf{t}_k) - F(\mathbf{t}_k + \mathbf{t})\|_Y \geqslant \varepsilon/2$, provided $k \in \mathbb{N}$ and $\mathbf{t} \in B(\omega_k, \eta_k)$; it is only worth noting here that, in each step of this construction, we can choose the point ω_{k+1} to be $\omega_k + (\mathbf{t_k} - \mathbf{t}_k')$ or $\omega_k + (\mathbf{t_k}' - \mathbf{t}_k)$, where the points $\mathbf{t}_k$ and $\mathbf{t}_k'$ are already chosen points from I with sufficiently large absolute values, satisfying additionally that $\|F(\mathbf{t}_k) - F(\mathbf{t}_k')\|_Y > \varepsilon$ and $|\mathbf{t_k}' - \mathbf{t}_k| \leqslant 1/k$. Due to the Cantor theorem, there exists a unique number $\mathbf{t}' \in \bigcap_{k\in\mathbb{N}} B(\omega_k, \eta_k)$. This implies $\|F(\mathbf{t}_k) - F(\mathbf{t}_k + \mathbf{t}')\|_Y \geqslant \varepsilon/2$ for all $k \in \mathbb{N}$, which is a contradiction since the function $F(\cdot)$ is slowly oscillating and $\mathbf{t}' \in A_I$. □

Using this result, the interested reader may simply transfer the statement of [904, Proposition 2] to the higher dimensions, as well; details can be left to the interested reader. For more details about the life and professional work of D. Sarason, we refer the reader to the communication paper [465] by S. R. Garcia.

We continue by observing that, in the infinite-dimensional setting, there exists a bounded, uniformly continuous, slowly oscillating function $F : [0,\infty) \to Y$ whose range is not relatively compact in Y; see e. g., Example 2.4.25.

The following notion is also meaningful.

Definition 9.0.16. Let $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}_j$ be unbounded ($1 \leqslant j \leqslant n$). Define

$$B_I := \{(\omega_1, \ldots, \omega_n) \in (\mathbb{R} \setminus \{0\})^n : \omega_j e_j + I \subseteq I \text{ for all } j \in \mathbb{N}_n\}.$$

Then we say that a continuous function $F : I \times X \to Y$ is $(\mathcal{B}, (\mathbb{D}_j)_{j \in \mathbb{N}_n})$-slowly oscillating if and only if for each $(\omega_1, \ldots, \omega_n) \in B_I$ and $j \in \mathbb{N}_n$ we have

$$\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}_j} \|F(\mathbf{t} + \omega_j e_j; x) - F(\mathbf{t}; x)\|_Y = 0, \quad \text{uniformly in } x \in B.$$

In other words, a continuous function $F : I \times X \to Y$ is $(\mathcal{B}, (\mathbb{D}_j)_{j \in \mathbb{N}_n})$-slowly oscillating if and only if $F(\cdot; \cdot)$ is $(S, \mathcal{B})$-asymptotically $(\omega_j, 1, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic for all tuples $(\omega_1, \ldots, \omega_n) \in B_I$. Clearly, we have $kB_I \subseteq B_I$ for all $k \in \mathbb{N}$.

In our previous work, we have investigated the following topics in connection with $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, c)-periodic functions and $(S, \mathcal{B})$-asymptotically $(\omega_j, c_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$-periodic functions:
(i) the invariance under the operation of uniform convergence,
(ii) the convolution invariance,
(iii) the invariance under reflections at zero,
(iv) the translation invariance,
(v) the pointwise products with the scalar-valued functions of the same type, etc.

All these statements can be simply reformulated for the notion introduced in Definition 9.0.13 and Definition 9.0.16 (with $c = 1$; $c_j = 1$ for all $j \in \mathbb{N}_n$). We will skip all applications based on the use of results concerning the above-mentioned topics, like those established to d'Alembert formula and the heat equation in $\mathbb{R}^n$.

Concerning the usual class of one-dimensional slowly oscillating functions, we would like to note that the statement of [221, Lemma 2.1], which has recently been established by D. Brindle in his doctoral dissertation and which plausibly holds for uniformly integrable resolvent operator families under consideration, and the statement of [259, Theorem 3.9] with $k = 0$, which has recently been proved by Y.-K. Chang and Y. Wei, can be used to profile important results concerning the invariance of slowly oscillating property under the actions of finite convolution products and the actions of infinite convolution products, respectively (the multi-dimensional analogues can be deduced without any substantial difficulties). Such results enable one to analyze the existence and uniqueness of slowly oscillating solutions for various classes of the abstract Volterra integro-differential equations considered in [631].

Now we would like to state some relations between quasi-asymptotical c-almost periodicity and remote c-almost periodicity. Let us take a closer look at the equations (7.32) and (7.33). We first observe that it is completely irrelevant whether we will write that there exists a finite real number $M(\varepsilon, \tau) > 0$ such that (7.32) holds, or more concisely,

$$\limsup_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \mathbb{D}} \sup_{x \in B} \|F(\mathbf{t} + \tau; x) - cF(\mathbf{t}; x)\|_Y \leqslant \varepsilon, \tag{9.12}$$

i. e.,

$$\lim_{s\to+\infty}\sup_{|\mathbf{t}|\geqslant s,\mathbf{t}\in\mathbb{D};x\in B}\|F(\mathbf{t}+\tau;x)-cF(\mathbf{t};x)\|_Y\leqslant\varepsilon.$$

It is also very simple to show that it is completely irrelevant whether we will write that there exists a finite real number $M(\varepsilon,\tau)>0$ such that (7.33) holds, or more concisely,

$$\lim_{k\to+\infty}\limsup_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}}\sup_{x\in B}\|F(\mathbf{t}+\tau_k;x)-cF(\mathbf{t};x)\|_Y=0,$$

i. e.,

$$\lim_{k\to+\infty}\lim_{s\to+\infty}\sup_{|\mathbf{t}|\geqslant s,\mathbf{t}\in\mathbb{D};x\in B}\|F(\mathbf{t}+\tau_k;x)-cF(\mathbf{t};x)\|_Y=0.$$

The special case $c=1$, $X\in\mathcal{B}$ and $\mathbb{D}=I=I'=\mathbb{R}^n$ has been considered in [588, 647, 652], where a $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',c)$-almost periodic function is simply called quasi-asymptotically almost periodic. In this case, the above consideration shows that the notion of quasi-asymptotical almost periodicity is equivalent with the notion of remote almost periodicity considered by F. Yang and C. Zhang in [1054, Definition 1.1; (1) and (3)]; see also the pioneering paper [903], where D. Sarason has analyzed the complex-valued remotely almost periodic functions defined on the real line, and the paper [1081], where C. Zhang and L. Jiang have analyzed the class of remotely almost periodic sequences (see also [864]).

As our former analyses show (see also the research article [1042] by R. Xie and C. Zhang), a quasi-asymptotically almost periodic function $F:\mathbb{R}^n\to\mathbb{C}$ need not be uniformly continuous, so that the notion introduced in [1054, Definition 1.1; (2)] is not satisfactory to a certain extent (see also S. Zhang, D. Piao [1085, Definition 2.1] and the first sentence after [1054, Definition 1.1], where the authors have assumed a priori that a remotely almost periodic function $F:\mathbb{R}^n\to X$ is uniformly continuous). It is our strong belief that it is much better to analyze both: the general classes of $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',c)$-almost periodic type functions which are not uniformly continuous on $\mathcal{B}$ and the corresponding classes of $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',c)$-almost periodic type functions which are uniformly continuous on $\mathcal{B}$ (in a certain sense).

Definition 9.0.17. Suppose that $F:I\times X\to Y$ is a continuous function.

(i) It is said that $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B},I',c)$-almost periodic if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',c)$-almost periodic and for each $B\in\mathcal{B}$ the function $F(\cdot;\cdot)$ is uniformly continuous on $I\times B$; that is,

$$(\forall B\in\mathcal{B})\,(\forall\varepsilon>0)\,(\exists\delta>0)\,(\forall\mathbf{t}',\,\mathbf{t}''\in I)\,(\forall x',\,x''\in B)$$
$$(|\mathbf{t}'-\mathbf{t}''|+\|x-x'\|<\delta\Rightarrow\|F(\mathbf{t}';x')-F(\mathbf{t}'';x'')\|_Y<\varepsilon).$$

(ii) It is said that $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-uniformly recurrent if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B}, I', c)$-uniformly recurrent and for each $B \in \mathcal{B}$ the function $F(\cdot;\cdot)$ is uniformly continuous on $I \times B$.

(iii) It is said that $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-almost periodic of type 1 if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B}, I', c)$-almost periodic and

$$(\forall B \in \mathcal{B})\,(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall \mathbf{t}',\ \mathbf{t}'' \in I)\,(\forall x \in B)$$
$$(|\mathbf{t}' - \mathbf{t}''| < \delta \Rightarrow \|F(\mathbf{t}'; x) - F(\mathbf{t}''; x)\|_Y < \varepsilon). \tag{9.13}$$

(iv) It is said that $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-uniformly recurrent of type 1 if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B}, I', c)$-uniformly recurrent and (9.13) holds.

It is clear that any $\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-almost periodic ($\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-uniformly recurrent) function is $\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-almost periodic of type 1 ($\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-uniformly recurrent) of type 1. The converse statement holds provided that the function $F(\cdot;\cdot)$ is Lipshitzian with respect to the second argument.

Proposition 9.0.18. *Suppose that $F : I \times X \to Y$ is a continuous function and for each set $B \in \mathcal{B}$ there exists a finite real constant $L_B > 0$ such that*

$$\|F(\mathbf{t}; x') - F(\mathbf{t}; x'')\|_Y \leqslant L_B\|x' - x''\|, \quad \mathbf{t} \in I,\ x',\ x'' \in B. \tag{9.14}$$

If $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-almost periodic of type 1 ($\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-uniformly recurrent of type 1), then $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-almost periodic ($\mathbb{D}$-remotely $(\mathcal{B}, I', c)$-uniformly recurrent).

Proof. Let the set $B \in \mathcal{B}$ be given and let $L_B > 0$ satisfy (9.14). The proof is a simple consequence of the corresponding definitions and the following decomposition ($\mathbf{t}$, $\mathbf{t} \in I$; x', $x'' \in B$):

$$\begin{aligned}\|F(\mathbf{t}'; x') - F(\mathbf{t}''; x'')\|_Y &\leqslant \|F(\mathbf{t}'; x') - F(\mathbf{t}'; x'')\|_Y + \|F(\mathbf{t}'; x'') - F(\mathbf{t}''; x'')\|_Y \\ &\leqslant L_B\|x' - x''\| + \|F(\mathbf{t}'; x'') - F(\mathbf{t}''; x'')\|_Y.\end{aligned}$$ □

Furthermore, we want to notice that we do not require any type of boundedness of the function $F(\cdot)$ in Definition 7.3.14 and Definition 9.0.17; for example, an application of the Lagrange mean value theorem shows that for each fixed real number $\sigma \in (0, 1)$ we have $|(t + \tau)^\sigma - t^\sigma| \leqslant \tau\sigma t^{\sigma-1}$, $t > 0$, $\sigma \geqslant 0$, so that the function $t \mapsto t^\sigma$, $t \geqslant 0$ is remotely almost periodic in the sense of Definition 9.0.17, as we have already discussed for slowly oscillating functions. In connection with the unboundedness of the function $F(\cdot)$ in these definitions, we would like to note that an application of the supremum formula for almost periodic functions implies that the unbounded function $f(\cdot)$, given by (2.28), is not quasi-asymptotically almost periodic in the sense of Definition 7.3.14 and [588, Definition 3.3].

If we denote by $Q - \mathrm{AAP}_{\mathrm{buc}}(\mathbb{R}^n : Y)$ the space consisting of all bounded, uniformly continuous quasi-asymptotically almost periodic functions $F : \mathbb{R}^n \to Y$, then we know from the foregoing that $Q - \mathrm{AAP}_{\mathrm{buc}}(\mathbb{R}^n : Y)$ coincides with the space of all uniformly continuous (usually, we assume this as a blank hypothesis) remotely almost periodic functions $\mathcal{RAP}(\mathbb{R}^n : Y)$. We know therefore that $Q - \mathrm{AAP}_{\mathrm{buc}}(\mathbb{R}^n : \mathbb{C})$ is exactly the closed subalgebra of $C_b(\mathbb{R}^n : \mathbb{C})$ generated by the space of all almost periodic functions $F : \mathbb{R}^n \to \mathbb{C}$ and the space of all slowly oscillating functions $F : \mathbb{R}^n \to \mathbb{C}$; this means that, for every $\varepsilon > 0$ and for every $F \in Q - \mathrm{AAP}_{\mathrm{buc}}(\mathbb{R}^n : \mathbb{C})$, we can always find two almost periodic functions $G_i : \mathbb{R}^n \to \mathbb{C}$ $(i = 1, 2)$ and two slowly oscillating functions $Q_i : \mathbb{R}^n \to \mathbb{C}$ $(i = 1, 2)$ such that $\|F - [G_1 + Q_1 + G_2 Q_2]\|_\infty < \varepsilon$ [903, 1054]. The proof of this important result is based on the use of certain results from the theory of C^*-algebras concerning the Gelfand spaces of multiplicative linear functionals of $\mathcal{RAP}(\mathbb{R}^n : \mathbb{C})$; it could be very enticing to extend this result for the functions defined on the general regions $I \subseteq \mathbb{R}^n$.

The results obtained in [1082, Proposition 2.1, Proposition 2.2] provide new characterizations of bounded, uniformly continuous quasi-asymptotically almost periodic functions $F : \mathbb{R}^n \to Y$, while [1082, Proposition 2.3] and [1085, Proposition 2.3] provide new characterizations of bounded, uniformly continuous quasi-asymptotically almost periodic functions $F : \mathbb{R} \to Y$. On the other hand, the results obtained in [588, Theorem 3.1, Theorem 3.2, Proposition 3.4], the composition principles obtained in [588, Theorem 3.3, Theorem 3.4] and the result obtained in [647, Proposition 2.15] provide new characterizations of remotely (c-)almost periodic functions $F : I \to Y$, $I \subseteq \mathbb{R}$ (it is worth noting that [588, Proposition 3.4(ii)] can be used to substantially shorten the proof of [1085, Lemma 3.6]), while the results obtained in [652, Proposition 3.2, Proposition 3.5, Theorem 3.6] provide new characterizations of remotely (c-)almost periodic functions $F : I \to Y$, $I \subseteq \mathbb{R}^n$ (and certain two-parameter analogues). For example, using [588, Theorem 3.1(ii)] with $c = 1$ and our analysis contained in the final paragraph of [652, Section 3], we immediately get

$$\mathrm{AA}(\mathbb{R}^n : Y) \cap \mathcal{RAP}(\mathbb{R}^n : Y) = \mathrm{BUC}(\mathbb{R}^n : Y),$$

where $\mathrm{BUC}(\mathbb{R}^n : Y)$ and $\mathrm{AA}(\mathbb{R}^n : Y)$ denote the space of all almost periodic functions from $\mathbb{R}^n$ into Y and the space of all almost automorphic functions from $\mathbb{R}^n$ into Y, respectively.

From application point of view, it is incredibly important to emphasize that [588, Proposition 3.4] can be used to profile some statements concerning the invariance of remote c-almost periodicity under the actions of convolution products, since the uniform continuity is preserved under the actions of convolution products in the equations [588, (3.1); (3.2)]; these results seem to new and not considered elsewhere even for the usual remote almost periodicity ($c = 1$). This enables one to provide numerous important applications in the study of time-remotely almost periodic solutions for various classes of the abstract (degenerate) Volterra integro-differential equations (see also [647, Section 4], where we have analyzed quasi-asymptotically almost periodic solutions of the abstract nonautonomous differential equations of first order; the question whether the obtained solutions are uniformly continuous is not so simple to be answered and requires further analysis).

Let us observe that, if a continuous function $F : I \times X \to Y$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I_i', c)$-almost periodic for $i = 1, 2$, then the function $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I_1' \cup I_2', c)$-almost periodic (a similar statement holds for $\mathbb{D}$-quasi-asymptotical $(\mathcal{B}, I', c)$-uniform recurrence). Keeping this in mind, the subsequent result follows immediately from Proposition 7.3.17.

Proposition 9.0.19.

(i) *Let $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ be unbounded. If a continuous function $F : I \times X \to Y$ is $(\mathbb{D}, \mathcal{B})$-slowly oscillating, then the function $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I')$-almost periodic with*

$$I' := \{\omega \cdot \mathbb{N}; \omega \in A_I\}.$$

(ii) *Let $\mathbb{D}_j \subseteq I \subseteq \mathbb{R}^n$, the set $\mathbb{D}_j$ be unbounded ($1 \leqslant j \leqslant n$) and for each tuple $\omega = (\omega_1, \ldots, \omega_n) \in B_I$ the set $\mathbb{D}_\omega$ consisting of all tuples $\mathbf{t} \in \mathbb{D}_n$ such that $\mathbf{t} + \sum_{i=j+1}^n \omega_i e_i \in \mathbb{D}_j$ for all $j \in \mathbb{N}_{n-1}$ be unbounded in $\mathbb{R}^n$. Suppose that the set $\mathcal{D} \equiv \bigcap_{\omega \in B_I} \mathbb{D}_\omega$ is unbounded,*

$$I' := \{\omega \cdot \mathbb{N}; \omega \in B_I \cap I\},$$

and $c := \prod_{j=1}^n c_j$. If $F : I \times X \to Y$ is $(\mathcal{B}, (\mathbb{D}_j)_{j \in \mathbb{N}_n})$-slowly oscillating, then the function $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically $(\mathcal{B}, I')$-almost periodic.

It is clear that every slowly oscillating function $F : I \to Y$, where I is $[0, \infty)^n$ or $\mathbb{R}^n$, is quasi-asymptotically almost periodic, which immediately follows from Proposition 9.0.19.

We will not consider here the differentiation and integration of multi-dimensional remotely c-almost periodic functions (see [265, Subsection 2.4] for the related results concerning multi-dimensional $(\mathrm{R}, \mathcal{B})$-almost periodic type functions, and [1085, Proposition 2.3] for a result concerning the first anti-derivatives of one-dimensional remotely almost periodic functions). Concerning the existence of mean value, the boundedness of a remotely c-almost periodic function $F(\cdot)$ is almost inevitable to be assumed in order to ensure the existence of finite mean value of $F(\cdot)$. We feel it is our duty to emphasize that the proof of [1085, Proposition 2.4], a statement which considers the existence and properties of mean value of one-dimensional remotely almost periodic functions defined on the whole real line, is not correct and contains several important mistakes:

1. The estimate directly after the equation [1085, (2.12)] is not correct since the term "$2G(l+s_0)$" has to be written here as "$2G(2l+s_0+a)$", which causes several serious and unpleasant consequences for the remainder of the proof.

2. It is not clear the meaning of the number T_0 in the equations [1085, (2.13)–(2.14)].
3. The existence of mean value, stated in the equation [1085, (2.15)], is given without any reasonable explanation; see also the proof of [696, Theorem 1.3.1, pp. 32-34], where the correct proof of the existence of mean value is given for the usually considered class of almost periodic functions (besides these observations, we would like to note that the uniform continuity of the function $f(\cdot)$ has not been used in the proof of the above-mentioned proposition).

Keeping in mind these observations, it follows that the problem of existence or nonexistence of mean value of remotely almost periodic functions is still unsolved. In the following example, we will prove the existence of a bounded, uniformly continuous slowly oscillating function $f : [0,\infty) \to c_0$ which does not have mean value, which clearly marks that the calculations given in [1085, Proposition 2.4] are not true:

Example 9.0.20. Define $f : [0,\infty) \to c_0$ by $f(t) := (e^{-t/n})_{n\in\mathbb{N}}$, $t \geqslant 0$. In [221, Example 2.2], D. Brindle has proved that the function $f(\cdot)$ is bounded, uniformly continuous and slowly oscillating (albeit we have found some minor typographical errors in this example, the obtained conclusions are correct; we can use the inequality $1 - e^{-x} \leqslant x$, $x \geqslant 0$ here). If we assume that the limit

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t f(s)\, ds$$

exists in c_0, then it can be simply approved that this limit has to be equal to the zero sequence, so that we would have

$$\lim_{t\to+\infty} \sup_{n\in\mathbb{N}} \left[\frac{n}{t}\left(1 - e^{-(t/n)}\right) \right] = 0. \tag{9.15}$$

If we assume that $t \geqslant 1$ and $t \in [n, n+1)$ for some integer $n \in \mathbb{N}$, then we have

$$\frac{n}{t}\left(1 - e^{-(t/n)}\right) \geqslant \frac{n}{n+1}\left(1 - e^{-(n/n)}\right) \geqslant \frac{1}{2}\left(1 - e^{-1}\right)$$

and therefore

$$\sup_{n\in\mathbb{N}} \left[\frac{n}{t}\left(1 - e^{-(t/n)}\right) \right] \geqslant \frac{1}{2}\left(1 - e^{-1}\right), \quad t \geqslant 1,$$

which clearly contradicts (9.15).

The interested reader may try to construct an example of a bounded, uniformly continuous slowly oscillating function $f : [0,\infty) \to \mathbb{C}$ without mean value (it is our strong belief that such a function really exists; see also [221, Section 2.2], [222, 223, 379, 380]).

At the end of this part, we would like to point out that we will not consider here the extensions of multi-dimensional (c-)almost periodic type functions. Without going into full details, let us only note that we can construct many different extensions of a slowly oscillating function $F : I \to Y$ to the whole Euclidean space $\mathbb{R}^n$; for example, if a slowly oscillating function $f : [0,\infty) \to Y$ is given in advance, we can extend it linearly to the interval $[-r,0]$, where $r > 0$ is an arbitrary real number, and after that we can extend the obtained function by zero outside the interval $[-r,\infty)$.

Now we will analyze some application of our results from this part.

1. In [1082, Theorem 3.4], C. Zhang and L. Jiang have analyzed remotely almost periodic solutions of the perturbed heat equation

$$\begin{aligned} &u_t = \sum_{i=1}^{m+n} [u_{x_i x_i} + b_i(x,t)u_{x_i}] - c(x,t)u = f(x,t), \quad (x,t) \in \mathbb{R}_T^{n+m}; \\ &u(x,0) = \varphi(x), \quad x \in \mathbb{R}^{n+m}, \end{aligned} \tag{9.16}$$

following the method proposed by A. Friedman [454]; see also the boundary value problem considered in [1054, Lemma 3.3], which can be also reconsidered in our context. Since [1082, Lemma 3.1] (see also the proof of [1082, Proposition 2.41]) and [1082, Corollary 3.2, Lemma 3.3] can be reformulated for multi-dimensional remotely c-almost periodic functions, the argumentation contained in the proof of [1082, Theorem 3.4] shows that the following holds (we define the spaces $\mathcal{R}\mathrm{BUC}_c(\mathbb{R}^n \times \overline{\mathbb{R}_T^m})$ and $\mathcal{R}\mathrm{BUC}_c(\mathbb{R}^{n+m})$ similarly to [1082], with the use of difference $\cdot - c\cdot$ in place of difference $\cdot - \cdot$).

Theorem 9.0.21. *If the functions $f(x,t)$, $b_i(x,t)$, $\partial b_i/\partial x_j(x,t)$ ($j = 1,\ldots,n+m$) and $c(x,t)$ belong to the space $\mathcal{R}\mathrm{BUC}_c(\mathbb{R}^n \times \overline{\mathbb{R}_T^m})$ and the functions φ, $\partial\varphi/\partial x_j$ belong to the space $\mathcal{R}\mathrm{BUC}_c(\mathbb{R}^{n+m})$, then there exists a unique solution $u(x,t)$ of (9.16) which can be written as a finite sum of functions belonging to the space $\mathcal{R}\mathrm{BUC}_c(\mathbb{R}^n \times \overline{\mathbb{R}_T^m})$.*

Let us also point out that the statement of [1082, Proposition 2.2] does not hold for remotely c-almost periodic functions unless $c = 1$. We will not analyze the inverse parabolic problems here.

2. The convolution invariance of multi-dimensional quasi-asymptotically c-almost periodic functions has been analyzed in Theorem 7.3.19. We want also to note that the uniform continuity of the function $F(\cdot)$ in the formulation of this theorem implies the uniform continuity of the function $(h * F)(\cdot)$, as can be simply shown; using this observation, we can simply reconsider the application to the ill-posed abstract Cauchy problems (see the related part of Subsection 8.1.6) for remotely c-almost periodic functions.
3. Consider again the Richard–Chapman equation (5.5) with an external perturbation $f(\cdot)$. We need the following auxiliary lemma, which generalizes [1054, Lemma 3.5] (see also [757, Lemma 4]).

Lemma 9.0.22. *Suppose that $\alpha > 0$, the functions $a : \mathbb{R} \to [\alpha, \infty)$ and $f : \mathbb{R} \to \mathbb{R}$ are slowly oscillating. Then also the function*

$$t \mapsto F(t) \equiv \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} f(s)\,ds, \quad t \in \mathbb{R}$$

is slowly oscillating.

Proof. Let $\omega \in \mathbb{R} \setminus \{0\}$. The proof that the function $F(t)$ is slowly oscillating as $t \to -\infty$ follows from the existence of a sufficiently large number $t_0 > 0$ such that $|a(t+\omega) - a(t)| + |f(t+\omega) - f(t)| < \varepsilon$ provided that $|t| > t_0$, and from the following calculation:

$$\begin{aligned}
&\big|F(t+\omega) - F(t)\big| \\
&= \left| \int_{-\infty}^{0} e^{-\int_{s+t}^{t} a(r+\omega)\,dr} f(t+s+\omega)\,ds - \int_{-\infty}^{0} e^{-\int_{s+t}^{t} a(r)\,dr} f(t+s)\,ds \right| \\
&\leqslant \int_{-\infty}^{0} e^{-\int_{s+t}^{t} a(r+\omega)\,dr} \big|f(t+s+\omega) - f(t+s)\big|\,ds \\
&\quad + \|f\|_\infty \int_{-\infty}^{0} \big|e^{-\int_{s+t}^{t} a(r+\omega)\,dr} - e^{-\int_{s+t}^{t} a(r)\,dr}\big|\,ds \\
&\leqslant (\varepsilon/\alpha) + \|f\|_\infty \int_{-\infty}^{0} e^{\alpha s} \big|1 - e^{\int_{s+t}^{t}[a(r+\omega)-a(r)]\,dr}\big|\,ds \\
&\leqslant (\varepsilon/\alpha) + \|f\|_\infty \int_{-\infty}^{0} e^{\alpha s} \left| \int_{s+t}^{t} [a(r+\omega) - a(r)]\,dr \right| e^{|\int_{s+t}^{t}[a(r+\omega)-a(r)]\,dr|}\,ds \\
&\leqslant (\varepsilon/\alpha) + \|f\|_\infty \varepsilon \int_{-\infty}^{0} |s| e^{(\alpha+\varepsilon)s}\,ds = (\varepsilon/\alpha) + \|f\|_\infty \varepsilon (\alpha+\varepsilon)^2, \quad t < -t_0.
\end{aligned}$$

The proof that the function $F(t)$ is slowly oscillating as $t \to +\infty$ is a little incorrect in [1054, Lemma 3.5] but we can apply a trick from [1080, Remark 2.2] here. Strictly speaking, we can use the same decomposition and calculation as above but we need to divide first the interval of integration $(-\infty, 0]$ into two subintervals $(-\infty, -M]$ and $[-M, 0]$, where $M > 0$ is a sufficiently large real number such that $\int_{-\infty}^{0} e^{\alpha s}\,ds < \varepsilon/2$. □

Keeping in mind Lemma 9.0.22, the fact that the space of real-valued slowly oscillating functions is closed under pointwise products and sums, as well as the fact that for each positive slowly oscillating function $f : \mathbb{R} \to (0, \infty)$ and for each real number $r > 0$ the function $f^r : \mathbb{R} \to (0, \infty)$ is also slowly oscillating, we can repeat verbatim the argumentation contained in the proof of Theorem 5.0.18 in order to see that the following result holds true.

Theorem 9.0.23. *Suppose that the hypotheses* (H1)–(H3) *hold. Then the equation* (5.5) *has a unique slowly oscillating solution* $\phi^*(t)$ *satisfying* $\gamma^{-1/\theta} \leqslant \phi^*(t) \leqslant \omega^{-1/\theta}$ *for all* $t \in \mathbb{R}$.

Homogenization in algebras with mean value, generalized Besicovitch spaces and applications

There are numerous recent research studies regarding homogenization in algebras with mean value, generalized Besicovitch spaces and their applications. In this section, we will briefly explain the main ideas of papers [1030] and [1031] by J. L. Woukeng, only.

Homogenization in algebras with mean value, generalized Besicovitch spaces and their applications are investigated in [1031]. We say that a closed, translation invariant subalgebra A of the C^*-algebra of bounded uniformly continuous functions $\mathrm{BUC}(\mathbb{R}^n)$ is an algebra with mean value if and only if A contains all constants and satisfies the requirement that each element of A has a mean value in the following sense: For each $u \in \mathrm{A}$, the sequence $(u_\varepsilon)_{\varepsilon>0}$ (where $u_\varepsilon(x) = u(x/\varepsilon_1(\varepsilon))$, $x \in \mathbb{R}^n$) weakly $*$-converges in $L^\infty(\mathbb{R}^n)$ to some constant function $M(u) \in \mathbb{C}$ as $\varepsilon \to 0+$, and $\varepsilon_1(\cdot)$ is a positive function satisfying $\lim_{\varepsilon\to 0+} \varepsilon_1(\varepsilon) = 0$.

Endowed with the sup-norm topology, A is a commutative C^*-algebra with identity. By A^m we denote the space of all elements $\psi \in \mathrm{A}$ such that $D^\alpha\psi \in \mathrm{A}$ for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant m$ $(m \in \mathbb{N}_0)$. We endow A with the norm $\|\psi\|_m := \sup_{x\in\mathbb{R}^n,|\alpha|\leqslant m} |D^\alpha\psi(x)|$ $(\psi \in \mathrm{A}^m)$. By A^∞ we denote the projective limit of spaces A^m as $m \to +\infty$. Furthermore, by B^p_{A} we denote the Besicovitch space associated to A, i. e., the closure of A with respect to the Besicovitch seminorm

$$\|u\|_p := \left(\limsup_{l\to+\infty} \frac{1}{(2l)^n} \int\limits_{l[-1,1]^n} |u(x)|^p \, dx\right)^{1/p}.$$

Then B^p_{A} is a complete seminormed vector space and $B^q_{\mathrm{A}} \subseteq B^p_{\mathrm{A}}$ if $1 \leqslant p \leqslant q < +\infty$. By B^∞_{A} we denote the space of all functions $\psi(\cdot)$ which belongs to the intersection of all spaces B^p_{A} when $1 \leqslant p < +\infty$ and satisfies $[\psi]_\infty := \sup_{1\leqslant p<+\infty} \|\psi\|_p < \infty$. Let us note that B^∞_{A} is a complete seminormed space as well as that the spaces B^p_{A} for $1 \leqslant p \leqslant +\infty$ are not in general Fréchet spaces since they are not separated in general. The main features of these spaces are:

(i) The Gelfand transform $\mathcal{G} : \mathrm{A} \to C(\Delta(\mathrm{A}))$ can be continuously extended to a unique continuous linear mapping, denoted by the same symbol, of B^p_{A} into $L^p(\Delta(\mathrm{A}))$, which induces an isometric isomorphism $\mathcal{G}_1$ of $B^p_{\mathrm{A}}/\mathcal{N} := \mathcal{B}^p_{\mathrm{A}}$ onto $L^p(\Delta(\mathrm{A}))$, where $\mathcal{N} := \{u \in B^p_{\mathrm{A}} : \mathcal{G}(u) = 0\}$. Here, $\Delta(\mathrm{A})$ denotes the spectrum of A.
(ii) The mean value M defined initially on A, extends by continuity to a positive continuous linear form, denoted by the same symbol, on B^p_{A} satisfying $M(u(\cdot)) = M(u(\cdot + a))$ for all $a \in \mathbb{R}^n$ and $\|u\|_p = [M(|u|^p)]^{1/p}$.

In [1031], the author has used the theory of strongly continuous n-parameter groups to build a framework for solving random homogenization problems. The starting point is the fact that the expression

$$T(x) : \mathcal{B}^p_{\mathrm{A}} \to \mathcal{B}^p_{\mathrm{A}},\ T(x)(u + \mathcal{N}) := u(\cdot + x) + \mathcal{N} \quad \text{for } u \in B^p_{\mathrm{A}}$$

defines an n-parameter group of isometries $(T(x))_{x\in\mathbb{R}^n}$. A compactness result for Young measures in the algebras with mean is also proved and an important achievement in the study of the homogenization problem associated with a stochastic Ladyzhenskaya model for incompressible viscous flow is presented. Introverted algebras with mean value and their applications have been analyzed in [1030]. We start by recalling that the spectrum $\Delta(\mathrm{A})$ is topologized with the usual Gelfand topology, which is the relative weak$*$ topology induced on $\Delta(\mathrm{A})$ by $\sigma(\mathrm{A}', \mathrm{A})$; as is well known, $\Delta(\mathrm{A})$ is a compact topological space. If A is the algebra of almost periodic functions, then we know that $\Delta(\mathrm{A})$ is a compact topological abelian group; furthermore, if A is the algebra of periodic functions, then we know that $\Delta(\mathrm{A})$ is topologically homeomorphic to the n-dimensional torus $\mathbb{T}^n$.

In [1031], the author has investigated many intriguing questions about the properties of topological space $\Delta(\mathrm{A})$ when A is a general algebra with mean value (almost nothing has been known before 2014 with regards to this issue). Among many other clarifications, the author has proved the following:

(i) If the algebra A is introverted (see [777] for the notion), then $\Delta(\mathrm{A})$ is a compact topological semigroup; furthermore, if the multiplication defined on $\Delta(\mathrm{A})$ is jointly continuous, then $\Delta(\mathrm{A})$ is a compact topological group.
(ii) If the algebra A is introverted, then A is a subalgebra of the weakly almost periodic functions; furthermore, if the multiplication in $\Delta(\mathrm{A})$ is jointly continuous, then A is a subalgebra of the almost periodic functions.
(iii) If the algebra A is introverted, then the kernel $K(\Delta(\mathrm{A}))$ of $\Delta(\mathrm{A})$ is a compact topological group, and the mean value on A can be identified as the Haar integral over $K(\Delta(\mathrm{A}))$.

As an application, the author has examined the homogenization of the following parameterized Wilson–Cowan model with delay

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} = -u_\varepsilon(x + a, t) + \int_{\mathbb{R}^n} K^\varepsilon(x - \xi) f\big(\xi/\varepsilon, u_\varepsilon(\xi, t)\big)\, d\xi \quad \text{in } \mathbb{R}^n_T \equiv \mathbb{R}^n \times (0, T);$$

$$u_\varepsilon(x, 0) = u^0(x), \quad x \in \mathbb{R}^n.$$

Here, $a \in \mathbb{R}^n$ is fixed, $u_\varepsilon(\cdot,\cdot)$ denotes the electrical activity level field, $f(\cdot)$ the firing rate function and K^ε the connectivity kernel. The author has assumed that $K \in \mathcal{K}(\mathbb{R}^n : \mathrm{A})$, where A is an introverted algebra with mean value on $\mathbb{R}^n$, is nonnegative and $\int_{\mathbb{R}^n} K^\varepsilon(x)\, dx \leqslant 1$ as well as that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a nonnegative Carathéodory function satisfying certain conditions.

We refer the reader to [247, 248, 397, 555, 812–814, 868, 898, 964, 1029] for many other nontrivial applications to the partial differential equations established by J. L. Woukeng and his collaborators.

Almost automorphic functions on semi-topological groups

The first systematic study of almost automorphic functions on topological groups was conducted by W. A. Veech in [993, 994] (see also the papers [870] by A. Reich and [976] by R. Terras). Following P. Milnes [772], who considered only the scalar-valued case, we say that a continuous function $f : G \to Y$, where G is a (semi-)topological group, is almost automorphic if and only if for any sequence (n_i') in G there exists a subsequence (n_i) of (n_i') such that the joint limit $\lim_{i,j} f(n_i n_j^{-1} t) = f(t)$ exists for all $t \in G$. It is clear that $\mathbb{R}^n \times X$ is a semi-topological group as well as that the notion of $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphy can be extended in this rather general framework.

In this section, we will briefly explain the main ideas and results about almost automorphic functions on semi-topological groups established by P. Milnes in [772]; we will also remind the reader of some known results about almost automorphic functions on topological groups obtained by other authors (there is a vast literature about topological groups and their generalizations; we will only refer the reader to the recent book [789] edited by S. A. Morris and the references cited therein).

Let G be a topological space which is also a multiplicative group. Then we say that G is a semi-topological space if and only if the mappings $s \mapsto st$ and $s \mapsto ts$ from G into G are continuous for all $t \in G$; furthermore, G is called a topological group if, in addition to the above, the mapping $(s,t) \mapsto st^{-1}$ from $G \times G$ into G is continuous. By $\mathcal{J}$ we denote the topology on G and by $C_b(G : Y)$ we denote the space of all bounded continuous functions $f : G \to Y$ equipped with the sup-norm $\|\cdot\|_\infty$. We say that:

(i) a subset D of a semi-topological group G is left relatively dense if and only if there exists a finite set of elements $\{s_i : 1 \leqslant i \leqslant N\}$ in G such that $G \subseteq \bigcup_{i=1}^N (s_i D)$;

(ii) a topological group G is totally bounded if and only if for every non-empty neighborhood V in G we have the existence of a finite set of elements $\{s_i : 1 \leqslant i \leqslant N\}$ in G such that $G \subseteq \bigcup_{i=1}^N (s_i V)$.

For any $s \in G$, the left (right) translate f_s (f^s) of f is defined through $f_s(\cdot) := f(s\cdot)$ $(f_s(\cdot) := f(\cdot s))$. A subspace C of $C_b(G : Y)$ is called translation invariant if and only if f_s and f^s belong to C for every $f \in C$. If $f : G \to Y$ and $g : G \to Y$ are given functions and $(\alpha_i)_{i\in I}$, resp. $(n_i)_{i\in\mathbb{N}}$, is a net in G, resp. a sequence in G, then we write $T_\alpha f = g$ if and only if the net of left translations f_{α_i}, resp. f_{n_i}, converges pointwise on G. The right uniformly continuous subspace $RUC_b(G : Y)$ of $C_b(G : Y)$ is defined as the set of all functions $f \in C_b(G : Y)$ such that $\|f^{\alpha_i} - f^s\|$ tends to zero whenever $(\alpha_i)_{i\in I}$ is a net in G converging to $s \in G$; the left continuous subspace $LUC_b(G : Y)$ of $C_b(G : Y)$ is defined similarly.

Definition 9.0.24. Let G be a semi-topological group. Then we say that a continuous function $f : G \to Y$ is left almost automorphic if and only if every net $\alpha' \subseteq G$ has a subnet $\alpha \subseteq G$ such that $T_\alpha f = g$ and $T_{\alpha^{-1}} g = f$, where $\alpha^{-1} = (\alpha_i^{-1})$; the notion of right almost automorphy is introduced similarly, with the analogous conditions involving right translates. By $\mathrm{LAA}(G : Y)$ and $\mathrm{RAA}(G : Y)$ we denote the family of all left almost automorphic functions on G and the right almost automorphic functions on G, respectively.

A function $f \in C_b(G : Y)$ is called almost periodic if and only if the set of all left translations $\{f_s : s \in G\}$ is relatively compact in $C_b(G : Y)$. Any almost periodic function $f \in C_b(G : Y)$ is left almost automorphic and satisfies the requirement that the convergence in $T_\alpha f = g$ is uniform on G, along with the convergence in $T_{\alpha^{-1}} g = f$. We know that $\mathrm{LAA}(G : Y)$ and $\mathrm{RAA}(G : Y)$ are translation invariant spaces and that the limit $T_\alpha f = g$ need not be continuous on G.

Suppose, for the time being, that $Y = \mathbb{C}$. Then we know that, if G is a Hausdorff topological group that is complete in a left invariant metric or locally compact and $f \in C_b(G : \mathbb{C})$, then we can always find a net $(\alpha_i)_{i \in I}$ such that $T_\alpha f = g$ is discontinuous on G if and only if $f \notin RUC_b(G : \mathbb{C})$. In what follows, it will be said that the Bohr topology B on a semi-topological group G is that topology which has the property that a subbase of B-neighborhoods of a point $s \in G$ forms the sets $\{t \in G : |f(t) - f(s)| < \varepsilon\}$, where $f : G \to \mathbb{C}$ is almost periodic and $\varepsilon > 0$; a function $f : G \to \mathbb{C}$ is said to be Bohr continuous if and only if the function $f(\cdot)$ is continuous for the Bohr topology. Due to [772, Theorem 8], a necessary and sufficient condition for a topological group G to be totally bounded is that every continuous complex-valued function on G is Bohr continuous.

For the scalar-valued functions, [772, Theorem 13] states that for any continuous function $f : G \to \mathbb{C}$, where G is a semi-topological group, the following conditions are mutually equivalent:

1. (2.) $f(\cdot)$ is left (right) almost automorphic.
3. $f(\cdot)$ is Bohr continuous.
4. For every $\varepsilon > 0$ and for every finite set $N \subseteq G$, there exists a left relatively dense subset $D \subseteq G \ni D^{-1}D \subseteq \{s \in G : \sup_{r,t \in N} |f(rst) - f(rt)| < \varepsilon\}$.
5. (6.) For every $\varepsilon > 0$ and $t \in G$, there exists a left relatively dense subset $D \subseteq G \ni D^{-1}D \subseteq \{s \in G : \sup_{r,t \in N} |f(ts) - f(t)| < \varepsilon\}$ ($D \subseteq G \ni D^{-1}D \subseteq \{s \in G : \sup_{r,t \in N} |f(st) - f(t)| < \varepsilon\}$).
7. For every net $\alpha \subseteq G$, there exists a subnet $\alpha \subseteq G$ such that the joint limit $\lim_{i,j} f(s\alpha_i\alpha_j^{-1}t) = f(st)$ for all $s,\ t \in G$.
8. (9.) For every net $\alpha \subseteq G$, there exists a subnet $\alpha \subseteq G$ such that the joint limit $\lim_{i,j} f(\alpha_i\alpha_j^{-1}t) = f(t)$ for all $t \in G$ ($\lim_{i,j} f(t\alpha_i\alpha_j^{-1}) = f(t)$ for all $t \in G$).
10. For every sequence $\mathbf{n}' \subseteq G$, there exists a subnet $\mathbf{n} \subseteq G$ such that the joint limit $\lim_{i,j} f(sn_in_j^{-1}t) = f(st)$ for all $s,\ t \in G$
11. (12.) For every sequence $\mathbf{n}' \subseteq G$, there exists a subnet $\mathbf{n} \subseteq G$ such that the joint limit $\lim_{i,j} f(n_in_j^{-1}t) = f(t)$ for all $t \in G$ ($\lim_{i,j} f(tn_in_j^{-1}) = f(t)$ for all $t \in G$).

Although it would be very unpleasant to clarify the validity or non-validity of the above conditions for the vector-valued functions $f : G \to Y$, especially for those Banach spaces Y which are not separable (see e. g., the proof of [772, Theorem 10]), we would like to note that some equivalence relations clarified above hold for the vector-valued functions $f : G \to Y$ on topological groups G. For example, B. Basit has proved, in [121, Theorem 1.2], that a bounded continuous function $f : G \to Y$ is almost automorphic if and only if $f(\cdot)$ is Levitan almost periodic (see [121, Definition 1.1]), which immediately implies the equivalence of [1. (2.)] and [8. (9.)] in this framework. Keeping this in mind, it seems reasonable to further explore the following notion (more details will appear somewhere else; see also the section regarding Weyl multi-dimensional almost automorphic functions, where this approach has been essentially followed).

Definition 9.0.25. Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function as well as that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ we have $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then we say that $F(\cdot;\cdot)$ is:
(i) jointly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ such that

$$\lim_{(l,m)\to+\infty} F\big(\mathbf{t} - (b_{k_l}^1, \ldots, b_{k_l}^n) + (b_{k_m}^1, \ldots, b_{k_m}^n); x\big) = F(\mathbf{t}; x), \tag{9.17}$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$;
(ii) jointly $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ such that (9.17) holds pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that for each $x \in B$ the convergence in (9.17) is uniform in $\mathbf{t}$ for any set of the collection $W_{B,(\mathbf{b}_k)}(x)$;
(iii) jointly $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ such that (9.17) holds pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that the convergence in (9.17) is uniform in $(\mathbf{t}; x)$ for any set of the collection $\mathrm{P}_{B,(\mathbf{b}_k)}$.

Arguing as above, it can be simply shown that any $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function $F : \mathbb{R}^n \times X \to Y$ is jointly $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic with $\mathrm{P}_{\mathcal{B},\mathrm{R}} = \{\{\mathbb{R}^n \times B\}\}$. We also have the following.

Proposition 9.0.26. *Suppose that $F : \mathbb{R}^n \to Y$ is a c-uniformly recurrent function, where the sequence (τ_k) satisfies $\lim_{k\to+\infty} |\tau_k| = +\infty$ and*

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in\mathbb{R}^n} \big\|F(\mathbf{t} + \tau_k) - F(\mathbf{t})\big\|_Y = 0.$$

Let R *denote the collection consisting of the sequence* (τ_k) *and all its subsequences. Then the function* $F(\cdot)$ *is jointly* $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$*-multi-almost automorphic with* P_{R} *being the singleton* $\{\mathbb{R}^n\}$.

Proof. Let (τ'_k) be any subsequence of (τ_k). Then we have

$$\lim_{k\to+\infty}\sup_{\mathbf{t}\in\mathbb{R}^n}\|F(\mathbf{t}-\tau'_k;x)-c^{-1}F(\mathbf{t};x)\|_Y=0.$$

The final conclusion simply follows from the above estimates, the corresponding definition of joint $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-multi-almost automorphy and the decomposition:

$$\begin{aligned}
&\sup_{\mathbf{t}\in\mathbb{R}^n}\|F(\mathbf{t}-\tau'_l+\tau'_m;x)-F(\mathbf{t};x)\|_Y\\
&\quad\leqslant\sup_{\mathbf{t}\in\mathbb{R}^n}\|F(\mathbf{t}-\tau'_l+\tau'_m;x)-cF(\mathbf{t}-\tau'_l;x)\|_Y+\sup_{\mathbf{t}\in\mathbb{R}^n}\|cF(\mathbf{t}-\tau'_l;x)-F(\mathbf{t};x)\|_Y\\
&\quad=\sup_{\mathbf{t}\in\mathbb{R}^n}\|F(\mathbf{t}-\tau'_l+\tau'_m;x)-cF(\mathbf{t}-\tau'_l;x)\|_Y+|c|\sup_{\mathbf{t}\in\mathbb{R}^n}\|F(\mathbf{t}-\tau'_l;x)-c^{-1}F(\mathbf{t};x)\|_Y.
\end{aligned}$$ □

Furthermore, we can similarly introduce and analyze the notions of joint $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy, joint $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy and joint $(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy (see Definition 9.0.25).

The results on approximations of almost automorphic functions, proved by W. A. Veech [993, 994] on topological groups, continue to hold on semi-topological groups without any essential changes. For example, by [772, Theorem 18], we know that a continuous function $f : G \to Y$ is almost automorphic if and only if there exists a uniformly bounded sequence (f_k) of almost periodic functions $f_k : G \to \mathbb{C}$ $(k \in \mathbb{N})$ such that, for every $s \in G$ and $\varepsilon > 0$, we have the existence of a Bohr neighborhood V of s and an integer $k_0 \in \mathbb{N}$ such that, for very integer $k \geqslant k_0$, we have $|f_k(t)-f(t)| < \varepsilon$ for all $t \in V$. See also [993, Subsection 6.2] for some elementary facts regarding analytic almost automorphic functions defined on the additive group of integers $\mathbb{Z}$.

The complete characterization of those semi-topological groups for which the equality $\mathrm{BUC}(G : \mathbb{C}) = \mathrm{AA}(G : \mathbb{C})$ holds is given in [772, Theorem 23]. In [772, Theorem 25], P. Milnes has shown that, if G is arbitrary semi-topological group and $f : G \to Y$ is almost automorphic, then $f(\cdot)$ is almost periodic if and only if $T_\alpha f \in \mathrm{AA}(G : \mathbb{C})$ whenever it exists, extending thus a result of W. A. Veech known on topological groups before that. It is also worth noting that R. Terras [976] has constructed an almost automorphic function $f : \mathbb{Z} \to \mathbb{R}$ for which the limit

$$\lim_{N\to+\infty}\frac{1}{2N+1}\sum_{i=-N}^{N}f(i)$$

does not exist. It is well known that this example can be transferred to the continuous setting as well as that there exists an almost automorphic function $f : \mathbb{R} \to \mathbb{R}$ such that the limit

$$\mathcal{M}(f) := \lim_{t\to+\infty} \frac{1}{2t} \int_{-t}^{t} f(s)\, ds$$

does not exist.

Concerning the notion of almost automorphy and the notion of almost periodicity for functions defined on (semi-)topological groups, it should be noted that some definitions for introducing these notions do not require a priori the continuity or measurability of the function $f : G \to Y$ under consideration; see the research articles by H. W. Davies [332] and W. A. Veech [995] for some results obtained in this direction.

Concerning differences of almost periodic and almost automorphic functions defined on topological groups, with values in general locally convex spaces, we refer the reader to the research articles [123] by B. Basit, M. Emam and [382] by S. D. Dimitrova, D. B. Dimitrov. Mention should also be made of Ref. [831] by Y. Péraire.

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