RELATIVISTIC FORCES IN SPECIAL AND GENERAL RELATIVITY

Adrian Sfarti

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By

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This book is dedicated to my mentors: Jana Rancu, Eliza Haseganu, Elena Kreindler-Wexler, Mikayel Sarian, Marius Preda, Alexandru Fransua, Ovidiu Lupas, Laurentiu Lupas, Paul Cristea and Vlad Ionescu.

CONTENTS

PREFACE

The notion of force has a very important meaning in physics, from very early one we learn how to describe the particle trajectories by solving the equation $m\frac{d^2\mathbf{r}}{dt^2}=\mathbf{F}$ in the framework of Newtonian mechanics. Things get a lot more complicated at high speeds, in the framework of special relativity, where the equation of motion takes a much more complicated form of $m\frac{d}{dt}(\frac{\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}})=\mathbf{F}$ where $\mathbf{u}=\frac{d\mathbf{r}}{dt}$. We dedicate the first one third of the book to these cases by studying different forms of the equations of motion as a result of the different expressions for the force $\mathbf{F}$. Much effort is dedicated to the case of the general Lorentz force, $\mathbf{F}=q(\mathbf{v}\times\mathbf{B}+\mathbf{E})$ that intervenes so often in the design of particle accelerators. We present a few new derivations for Thomas precession and Thomas Wigner rotation as well as applications to the Compton effect. As we will see later on in the book, the situation is even more complicated in the case of the fictitious forces (d'Alembert, centrifugal, Coriolis and Euler) that appear only in non-inertial frames (accelerated linearly, uniformly rotating and in accelerated rotation). It is interesting to note that the equations of motion in this case fall out directly from the double integration with respect to time of the fictitious accelerations. The second third of this book is dedicated to these forces. The last third deals with forces in a roundabout way, since in General Relativity gravitation is not a force, so, we solve the equations of motion by deriving the Euler-Lagrange equations directly from the different metrics (Schwarzschild, Reissner-Nordstrom).

Adrian Sfarti, 2021

Biographical Note

Mr Sfarti received his PhD from the Polytechnic Institute of Bucharest, Romania, and has now accumulated over 30 years of teaching and research experience. Dr Sfarti was a Professor from the Industry at the University of California Berkeley between 1989 and 2004. He has published over 50 research papers and has 32 patents awarded.

COVARIANT TREATMENT OF COLLISIONS IN PARTICLE PHYSICS

Synopsis

The use of relativistic frame invariants is very well established, especially when it comes to the energy-momentum. In the current paper we show how the conservation of the energy and momentum applies to collisions of particles moving at relativistic speeds, like the ones encountered in nuclear accelerators. We derive the equations for two main types of collisions: elastic and inelastic. The starting point in both cases is the well known theorems of conservation of total energy and conservation of momentum for isolated systems [1-3]. The covariance, once proven, becomes a very useful tool due to the fact that researchers can use any inertial frame in solving the particle collision problems, thus greatly simplifying the solutions.

1. Fundamental notions

You should know by now the definition of proper time:

$d\tau = dt\sqrt{1-(u/c)^2}$ where u is the **coordinate speed** and t is the **coordinate time**. Coordinate time is the time measured by a clock in an arbitrary inertial frame. Proper time is the time measured on a clock commoving with the observer. The **coordinate velocity** is defined as a 3-vector:

$$\mathbf{u}=(dx/dt,dy/dt,dz/dt) \quad (1.1)$$

Now, **proper velocity**, by contrast, is a 4-vector defined as:

$$\mathbf{U}=(dx/d\tau,dy/d\tau,dz/d\tau, d(ct)/d\tau) \quad (1.2)$$

It is easy to show that:

$$\mathbf{U} = \gamma(u)(\mathbf{u}, c)$$
$$\gamma(u) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{1.3}$$

Since τ can be viewed as a proper parameter of a wordline in 4 space, it follows that, by the way it was defined, **U** is the tangent to that wordline.

Further, we can now define the **proper acceleration:**

$$\mathbf{A} = d\mathbf{U} / d\tau \tag{1.4}$$

We can show that:

$$\mathbf{A} = \gamma(u)\frac{d\mathbf{U}}{dt} = \gamma(u)(\mathbf{u}\frac{d\gamma(u)}{dt} + \mathbf{a}\gamma(u), c\frac{d\gamma(u)}{dt}) \tag{1.5}$$

where **a**=d**u**/dt is the **coordinate acceleration**

We also know that in the proper frame of the particle (the frame commoving with the particle) u=0 so, in the proper frame:

That is, the proper acceleration coincides with the coordinate acceleration in the proper frame of the particle. Thus, **A**=0, if and only if **a**=0. By contrast, **U** can never be equal to zero based on its definition. Based on the definitions of velocity, we can define the 3- and the 4-momentum respectively, as:

$$\mathbf{p} = \gamma(u)m\mathbf{u}$$
$$\mathbf{P} = m\mathbf{U} \tag{1.7}$$

Based on the above definitions, we can define the 3- and the 4-force as:

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = m\frac{d(\gamma(u)\mathbf{u})}{dt}$$
$$\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = m\frac{d\mathbf{U}}{d\tau} = m\mathbf{A} \tag{1.8}$$

Note that the derivatives are taken with respect to different times, coordinate for 3-force and proper for 4-force. Sometimes we see the 3-force defined as:

$$\mathbf{f} = \frac{d\mathbf{p}}{d\tau} = m\frac{d(\gamma(u)\mathbf{u})}{d\tau} \tag{1.9}$$

2. Introduction, Inelastic Collisions

In the present chapter we demonstrate that the equations of conservation have a covariant form, that is, they have the same form in all inertial frames. This conclusion is far from obvious since it needs to be proven mathematically. The existent [2,3,8] literature on the subject does not prove the covariance but rather assumes it from the start. We move from simple to complex, from inelastic collisions to elastic ones.

Consider two particles of proper masses m_1 and $m_2$traveling at speeds $u_1$and u_2 with respect to frame F. The particles collide and travel as **one** body at speed u after collision. The equations of conservation of momentum and energy in frame F are [1-3]:

$$\gamma(u_1)m_1u_1 + \gamma(u_2)m_2u_2 = \gamma(u)mu \tag{2.1}$$

$$\gamma(u_1)m_1c^2 + \gamma(u_2)m_2c^2 = \gamma(u)mc^2$$

where $\gamma(u_i)m_iu_i$ represent the momenta before collision, $\gamma(u_i)m_ic^2$ represent the energies before collision, $\gamma(u)mu$ represents the momentum after collision and $\gamma(u)mc^2$ represents the energy after collision. Obviously, (2.2) can be rewritten as:

$$\gamma(u_1)m_1 + \gamma(u_2)m_2 = \gamma(u)m \tag{2.3}$$

where $\gamma(u)$ is a shorthand for $\frac{1}{\sqrt{1-(\frac{u}{c})^2}}$

In other words, while rest mass is not conserved, relativistic mass is. Expression (2.3) will come in handy later on. The question is, are the equations (2.1) and (2.2) frame invariant? Is the equation of conservation of energy and momentum frame invariant? The reason it is important to settle this question is that we always prefer equations that are frame-invariant [4], due to not only their intrinsic elegance but also due to the fact that we may need to switch frames in order to be able to solve the particle trajectories [4] easier. Let a frame F' be another inertial frame moving with speed V with respect to F. Substituting:

$$\gamma(u_i) = \gamma(u'_i)\gamma(V)(1+\frac{u'_i V}{c^2})$$

$$u_i = \frac{u'_i + V}{1+\frac{u'_i V}{c^2}}$$

$$\gamma(u_i)u_i = \gamma(u'_i)\gamma(V)(u'_i + V) \tag{2.4}$$

into (2.1) we obtain:

$$\gamma(u'_1)m_1(u'_1+V) + \gamma(u'_2)m_2(u'_2+V) = \gamma(u')(u'+V)m \tag{2.5}$$

that is:

$$\begin{aligned}&\gamma(u'_1)m_1u'_1 + \gamma(u'_2)m_2u'_2 = \\ &= \gamma(u')u'm + V(\gamma(u')m - \gamma(u'_1)m_1 - \gamma(u'_2)m_2)\end{aligned} \tag{2.6}$$

$$\gamma(u')m - \gamma(u'_1)m_1 - \gamma(u'_2)m_2 =$$
$$= \frac{V}{c^2}(\gamma(u'_1)m_1u'_1 + \gamma(u'_2)m_2u'_2 - \gamma(u')u'm) \tag{2.7}$$

Substituting (2.7) back into (2.6) we obtain the final result:

$$(1-\frac{V^2}{c^2})(\gamma(u'_1)m_1u'_1 + \gamma(u'_2)m_2u'_2 - \gamma(u')u'm) = 0 \tag{2.8}$$

In other words:

$$\gamma(u'_1)m_1u'_1 + \gamma(u'_2)m_2u'_2 = \gamma(u')u'm \tag{2.9}$$

So, the equation of conservation of momentum is frame invariant. Substituting (2.9) in (2.6) we obtain that:

$$\gamma(u')m - \gamma(u'_1)m_1 - \gamma(u'_2)m_2 = 0 \tag{2.10}$$

i.e., the conservation of energy is frame invariant as well. The fact that both momentum and total energy conservation equations are frame invariant gives researchers the option to write the equations in whatever frame makes the calculations easier to perform [4]. Often the importance of covariance of the conservation of energy- momentum is ignored or underestimated, due to the fact that neither the energy nor the momentum is covariant as explained in [5]. The use of relativistic frame invariants is very well established, especially when it comes to the energy-momentum. Most traditional treatments use this particular invariant in order to calculate the "equivalent mass" of a system or, the "mass added to a system". The systems under evaluations are a most general hybrid made up of both massive particles and photons. One question that arises is what happens for the case when the direction of the boost is different from the one of the particle trajectory. To answer this question we will study a simplified case when the boost is oriented along the x-axis and the particle collision is along the y-axis, that is: $\mathbf{u_i} = (0, u_{i,y}, 0)_{i=1,2}$. In other words, we must substitute $u_i = u_{i,y}$ with $i = 1, 2$ in (2.1)-(2.3):

$$\gamma(u_{1,y})m_1u_{1,y} + \gamma(u_{2,y})m_2u_{2,y} = \gamma(u_y)mu_y \tag{2.11}$$

obtaining the equation of momentum conservation, while the equation for energy conservation becomes:

$$\gamma(u_{1,y})m_1 + \gamma(u_{2,y})m_2 = \gamma(u_y)m \tag{2.12}$$

On the other hand, (2.4) becomes:

$$0 = u_{i,x} = \frac{u'_{i,x} + V}{1 + \frac{u'_{i,x}V}{c^2}}$$

$$u_{i,y} = \frac{\frac{u'_{i,y}}{\gamma(V)}}{1 + \frac{u'_{i,x}V}{c^2}} = \frac{\frac{u'_{i,y}}{\gamma(V)}}{1 - \frac{V^2}{c^2}} = u'_{i,y}\,\gamma(V)$$

$$0 = u_{i,z} = \frac{\frac{u'_{i,y}}{\gamma(V)}}{1 - \frac{V^2}{c^2}} = u'_{i,z}\,\gamma(V)$$

$$\gamma(u_{i,y}) = \frac{1}{\gamma(V)\sqrt{1 - \frac{V^2 + u'^2_{i,y}}{c^2}}}$$

$$\gamma(u_{i,y})u_{i,y} = \frac{u'_{i,y}}{\sqrt{1 - \frac{V^2 + u'^2_{i,y}}{c^2}}} \tag{2.13}$$

where V is the relative speed between frames F and F'. Substituting (2.13) into (2.11)-(2.12) we obtain a very interesting result:

$$\frac{m_1 u'_{1,y}}{\sqrt{1-\frac{V^2+u'^2_{1,y}}{c^2}}}+\frac{m_2 u'_{2,y}}{\sqrt{1-\frac{V^2+u'^2_{2,y}}{c^2}}}=\frac{m u'_y}{\sqrt{1-\frac{V^2+u'^2_y}{c^2}}} \quad (2.14)$$

$$\frac{m_1}{\sqrt{1-\frac{V^2+u'^2_{1,y}}{c^2}}}+\frac{m_2}{\sqrt{1-\frac{V^2+u'^2_{2,y}}{c^2}}}=\frac{m}{\sqrt{1-\frac{V^2+u'^2_y}{c^2}}} \quad (2.15)$$

The equations are not as elegant as (2.11)-(2.12). There is a very profound lesson resulting from this very simple exercise, the covariance of the equations of conservation for energy-momentum is not a **given**, it needs to **be established**. A judicious choice of frames of reference, like in the beginning of the paragraph, results into one (elegant) covariant expression, while choosing a frame orthogonal onto the direction of collision results into a **different-looking**, not as elegant, **still-covariant** expression. In both cases the problem reduces to solving a system of non-linear equations of the form:

$$\gamma(u_1)m_1u_1+\gamma(u_2)m_2u_2=\gamma(u)mu \quad (2.16)$$

$$\gamma(u_1)m_1+\gamma(u_2)m_2=\gamma(u)m \quad (2.17)$$

that, fortunately, has a very nice solution for both the speed of the resulting particle and its rest mass:

$$u=\frac{\gamma(u_1)m_1u_1+\gamma(u_2)m_2u_2}{\gamma(u_1)m_1+\gamma(u_2)m_2}$$
$$m=\frac{\gamma(u_1)m_1+\gamma(u_2)m_2}{\gamma(u)} \quad (2.18)$$

Or, written in frame F’:

$$u'_y = \frac{\dfrac{m_1 u'_{1,y}}{\sqrt{1-\dfrac{V^2+u'^2_{1,y}}{c^2}}} + \dfrac{m_2 u'_{2,y}}{\sqrt{1-\dfrac{V^2+u'^2_{2,y}}{c^2}}}}{\dfrac{m_1}{\sqrt{1-\dfrac{V^2+u'^2_{1,y}}{c^2}}} + \dfrac{m_2}{\sqrt{1-\dfrac{V^2+u'^2_{2,y}}{c^2}}}} \tag{2.19}$$

A second question that often arises is: "what happens for collisions between particles at non-zero angles"? The answer is very simple, we only need to project equations (2.1),(2.3) thrice, once for each axis of coordinates:

$$\gamma(u_1)m_1u_{1,w} + \gamma(u_2)m_2u_{2,w} = \gamma(u)mu_w$$
$$w = \{x, y, z\} \tag{2.20}$$

$$\gamma(u_1)m_1 + \gamma(u_2)m_2 = \gamma(u)m \tag{2.21}$$

Note that the projection formalism does not affect the $\gamma(u_i)_{i=1,2}$ expressions. Therefore, the proof of covariance of the equations of motion reduces trivially to the previous proof. In a frame S' boosted in the x direction with respect to the original frame S, the equations become:

$$\gamma(u'_1)m_1u'_{1,x} + \gamma(u'_2)m_2u_{2,x} = \gamma(u')mu'_x \tag{2.22}$$

$$\gamma(u'_1)m_1 + \gamma(u'_2)m_2 = \gamma(u')m \tag{2.23}$$

$$\frac{m_1 u'_{1,w}}{\sqrt{1-\dfrac{V^2+u'^2_{1,w}}{c^2}}} + \frac{m_2 u'_{2,w}}{\sqrt{1-\dfrac{V^2+u'^2_{2,w}}{c^2}}} = \frac{mu'_w}{\sqrt{1-\dfrac{V^2+u'^2_{w}}{c^2}}}$$

$$\frac{m_1}{\sqrt{1-\dfrac{V^2+u'^2_{1,w}}{c^2}}} + \frac{m_2}{\sqrt{1-\dfrac{V^2+u'^2_{2,w}}{c^2}}} = \frac{m}{\sqrt{1-\dfrac{V^2+u'^2_{w}}{c^2}}} \tag{2.24}$$

where $w=\{y,z\}$ .So, what about the four-vector formalism? It is well known that four-vectors provide a "shorthand" way of expressing the same information as three-vectors, so recasting the above equations in the four-vector formalism does not add any information, nor does it simplify the proofs[8]. Using (2.13) we can re-write the energy-momentum four vector as:

$$\mathbf{p}_\mathbf{i}^{'}=\left(\frac{-m_iV}{\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}},\frac{m_iu_{i,y}^{'}}{\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}},0,\frac{m_ic^2}{\sqrt{1-\frac{V^2+u_{i,y}^{'2}}{c^2}}}\right)_{i=1,2}$$

$$\mathbf{p}^{'}=\left(\frac{-mV}{\sqrt{1-\frac{V^2+u_{y}^{'2}}{c^2}}},\frac{mu_{y}^{'}}{\sqrt{1-\frac{V^2+u_{y}^{'2}}{c^2}}},0,\frac{mc^2}{\sqrt{1-\frac{V^2+u_{y}^{'2}}{c^2}}}\right) \quad (2.25)$$

Armed with the above, we can write the covariant form of the energy conservation theorems in a much more concise form[7]:

$$\sum_{i=1}^{2}\mathbf{p}_\mathbf{i}^{'}=\mathbf{p}^{'}$$
$$\sum_{i=1}^{2}\mathbf{p}_\mathbf{i}=\mathbf{p} \quad (2.26)$$

Nevertheless, if we want to derive any measurable information, like the speed of the particle after collision or its mass, we need to go back to the three-vector formulas (2.18)-(2.19).

3. Elastic Collisions

Let's consider now a more complicated case, the case of elastic collisions. After collision the particles have different speeds from each other:

$$\gamma(u_1)m_1u_1+\gamma(u_2)m_2u_2=\gamma(U_1)m_1U_1+\gamma(U_2)m_2U_2 \quad (3.1)$$

$$\gamma(u_1)m_1 + \gamma(u_2)m_2 = \gamma(U_1)m_1 + \gamma(U_2)m_2 \tag{3.2}$$

$\gamma(u_i)m_i u_i$ represent the momenta before collision, $\gamma(u_i)m_i c^2$ represent the energies before collision, $\gamma(U_i)m_i U_i$ represents the momenta after collision and $\gamma(U_i)m_i c^2$ represent the energies after collision. Inserting (2.4) into (3.1) we obtain:

$$\begin{aligned}&\gamma(u'_1)m_1(u'_1+V)+\gamma(u'_2)m_2(u'_2+V)=\\&=\gamma(U'_1)m_1(U'_1+V)+\gamma(U'_2)m_2(U'_2+V)\end{aligned} \tag{3.3}$$

where V is the relative speed between frames F and F'. After isolating the terms in V:

$$\begin{aligned}&\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2=\\&=\gamma(U'_1)m_1U'_1+\gamma(U'_2)m_2U'_2+V(\gamma(U'_1)m_1+\\&+\gamma(U'_2)m_2-\gamma(u'_1)m_1-\gamma(u'_2)m_2)\end{aligned} \tag{3.4}$$

Inserting (2.4) into (3.2):

$$\begin{aligned}&\gamma(U'_1)m_1+\gamma(U'_2)m_2-\gamma(u'_1)m_1-\gamma(u'_2)m_2=\\&=\frac{V}{c^2}(\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2-\\&-\gamma(U'_1)U'_1\,m_1-\gamma(U'_2)U'_2\,m_2)\end{aligned} \tag{3.5}$$

Substitute the right hand side of (3.5) into the right hand side of (3.4):

$$\begin{aligned}&(1-\frac{V^2}{c^2})(\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2-\\&-\gamma(U'_1)m_1U'_1-\gamma(U'_2)m_2U'_2)=0\end{aligned} \tag{3.6}$$

That means that the equation of momentum conservation is frame-invariant:

$$\gamma(u'_1)m_1u'_1+\gamma(u'_2)m_2u'_2 = \\ =\gamma(U'_1)m_1U'_1+\gamma(U'_2)m_2U'_2 \tag{3.7}$$

Substituting (3.7) into (3.5) we obtain that the energy conservation equation is frame invariant:

$$\gamma(U'_1)m_1+\gamma(U'_2)m_2=\gamma(u'_1)m_1+\gamma(u'_2)m_2 \tag{3.8}$$

The fact that both momentum and total energy conservation equations are frame invariant gives researchers the option to write the equations in whatever frame makes the calculations easier to perform.

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CONSERVATION LAWS FOR PLASMA SYSTEMS

Synopsis

The use of relativistic frame invariants is very well established, especially when it comes to the energy-momentum. In the following paper we clarify the terms “conserved” vs. “frame invariant” and we explain the differences between the two concepts. Our paper is divided into three main sections. In the first section we explain the notion of frame invariance. In the second section we explain the energy-momentum conservation. We end up by giving a practical example (a hybrid plasma gas) of an open system, whereby energy and momentum are added from outside the system. We will show the interesting effects caused by adding photons to a system of massive particles. The new approach is extremely important in applications like particle accelerators where we can only work with directly measurable quantities, the kinetic energy KE and the momentum **p**.

1. Relativistic Frame Invariance

Frame-invariance is one of the most important properties in special relativity. As physicists, we try to express the laws of physics in frame invariant quantities in order to take advantage of the important property of such quantities remaining unchanged when passing from one inertial frame to another. It is well known that in relativity, the total energy (E) and the three-vector momentum (**p**) of a single particle are frame variant:

$$E = \gamma(u)mc^2 \tag{1.1}$$

$$\mathbf{p} = \gamma(u)m\mathbf{u} \tag{1.2}$$

The kinetic energy: $KE = \gamma(u)mc^2 - mc^2$ is also frame variant.

By contrast, the norm of the energy-momentum four-vector $\tilde{\mathbf{P}} = (E, \mathbf{p}c)$ is frame invariant:

$$\tilde{\mathbf{P}}.\tilde{\mathbf{P}} = E^2 - (\mathbf{p}c)^2 = m^2c^4 \quad (1.3)$$

In the present paper we will make extensive use of the frame variant quantities E and **p** as well as the frame invariant norm of the energy-momentum.

2. Transformation of Energy and Momentum between Frames

We have already shown that neither energy, nor the momentum is frame invariant, therefore it becomes interesting to derive the mathematical transformations when passing from one inertial frame to another. In the general case of arbitrary orientation between the axes of S and S' moving with the relative velocity **v**:

$$E = \gamma(v)(E' + \mathbf{p}'.\mathbf{v}) \quad (2.1)$$

$$\mathbf{p} = \mathbf{p}' + \gamma(v)((1-\gamma^{-1}(v))\mathbf{p}'.\mathbf{v} + \frac{v^2}{c^2}E')\frac{\mathbf{v}}{v^2} \quad (2.2)$$

Since the velocity **v** between S and S' is constant, by differentiating (2.1)-(2.2) we obtain:

$$dE = \gamma(v)(dE' + d\mathbf{p}'.\mathbf{v}) \quad (2.3)$$

$$d\mathbf{p} = d\mathbf{p}' + \gamma(v)((1-\gamma^{-1}(v))d\mathbf{p}'.\mathbf{v} + \frac{v^2}{c^2}dE')\frac{\mathbf{v}}{v^2} \quad (2.4)$$

Both (2.3) and (2.4) are instrumental in the computations involved in the next section.

3. The Theorems of Energy-Momentum Conservation for Closed Systems of Massive Particles

Let the total energy of a system of particles with arbitrarily distributed velocities $\mathbf{v}_i$ in a frame of reference S be:

$$E = c^2 \Sigma \gamma_i m_i \tag{3.1}$$

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{v_i^{\,2}}{c^2}}}$$

The total momentum in frame S is:

$$\mathbf{p} = \Sigma \gamma_i m_i \mathbf{v}_i \tag{3.2}$$

Let us calculate:

$$E^2 - (\mathbf{p}c)^2 = c^4 (\Sigma \gamma_i m_i)^2 - c^2 \Sigma (\gamma_i \gamma_j m_i m_j \mathbf{v}_i \mathbf{v}_j) \tag{3.3}$$

We can always find M and **V** such that:

$$E = c^2 \Sigma \gamma_i m_i = c^2 \gamma(V) M \tag{3.4}$$

$$\mathbf{p} = \Sigma \gamma_i m_i \mathbf{v}_i = \gamma(V) M \mathbf{V} \tag{3.5}$$

so $$E^2 - (\mathbf{p}c)^2 = M^2 c^4 \tag{3.6}$$

is clearly invariant. Obviously from (3.1),(3.2),(3.4) and (3.5) we obtain:

$$\mathbf{V} = \frac{\Sigma \gamma_i m_i \mathbf{v}_i}{\Sigma \gamma_i m_i} \tag{3.7}$$

$$M = \frac{\Sigma \gamma_i m_i}{\gamma(V)} \tag{3.8}$$

Expression (3.8) provides the relativistic equivalent mass of the system of massive particles while (3.7) represents the average relativistic speed. In classical mechanics energy and momentum conservation are independent of each other. Not so in relativity, courtesy of expression (3.6). Differentiating (3.6) we obtain:

$$EdE - c^2\mathbf{p}d\mathbf{p} = c^4 MdM \tag{3.9}$$

A closed system is defined by dM=0, or its equivalent:

$$EdE - c^2\mathbf{p}d\mathbf{p} = 0 \tag{3.10}$$

Theorem1: A closed system that exhibits conservation of three-momentum **p** will also exhibit conservation of energy.

Proof: $d\mathbf{p} = 0 => dE = 0$ (3.11)

Theorem2: If energy is conserved with respect to an inertial frame S', then it is conserved with respect to any other inertial frame S.

Proof: We start with:

$$dE = \gamma(v)(dE' + d\mathbf{p}'.\mathbf{v}) \tag{3.12}$$

From (3.10) we infer that $d\mathbf{p}' = 0 => dE' = 0 => dE = 0$ so

$$dE' = 0 => dE = 0 \tag{3.13}$$

Theorem3: A closed system that exhibits conservation of energy will exhibit conservation of momentum.

Proof: From theorem2 we obtain

$$dE = 0 => dE' = 0 => \mathbf{v}.d\mathbf{p}' = 0 => d\mathbf{p}' = 0\;. \tag{3.14}$$

Theorem4: If momentum is conserved with respect to an inertial frame S', then it is conserved with respect to any other inertial frame S.

Proof: We start with:

$$d\mathbf{p} = d\mathbf{p}' + \gamma(v)((1-\gamma^{-1}(v))d\mathbf{p}'.\mathbf{v} + \frac{v^2}{c^2}dE')\frac{\mathbf{v}}{v^2} \tag{3.15}$$

We already know that $d\mathbf{p}' = 0 => dE' = 0$ so (3.15) implies immediately that $d\mathbf{p}' = 0 => d\mathbf{p} = 0$.

Theorem5: If the four-vector momentum is conserved then the total energy and the total three-vector momentum are also conserved:

Proof:

If $d\sum\tilde{\mathbf{P}} = 0$ (3.16)

then:

$d\sum E = 0$ **and** $d\sum\mathbf{p} = 0$ (3.18)

Consequence: since $d\sum\mathbf{p} = 0$ it follows trivially that $\frac{d}{dt}\sum\mathbf{p} = 0$, that is:

$$\sum\mathbf{f} = 0 \quad (3.19)$$

4. Open Systems: Hybrid Plasma Gasses Composed of a Mix of Massless and Massive Particles

Imagine that we add a photon to the system of massive particles described in the previous paragraph. Such hybrid systems made up of photons injected into plasma form the object of statistical [8] or of kinematic treatments [9]. By contrast, we will show a relativistic-invariant based treatment, similar to the one shown in [7] while using the theory developed in the preceding paragraphs. Obviously, since the system is not closed, the energy and momentum will vary due to the addition of the photon to the existent system. To fix the ideas, let’s assume that we add a photon of energy e and momentum **p** to a system of massive particles of total energy E and total three-vector momentum **P**. This is a common application in the study of plasma systems where electromagnetic energy is injected gradually. By using (3.3) we can derive a very interesting result. Let us calculate:

$$(\Sigma E)^2 - c^2(\Sigma\mathbf{p})^2 = (E+e)^2 - c^2(\mathbf{P}+\mathbf{p})(\mathbf{P}+\mathbf{p}) =$$
$$= E^2 - (cP)^2 + 2(Ee - c^2\mathbf{P}\mathbf{p}) \quad (4.1)$$

$$\mathbf{P}\mathbf{p}_{max} = Pp \quad (4.2)$$

$$(Ee - c^2\mathbf{P}\mathbf{p})_{min} = Epc - c^2\mathrm{Pp} = \mathrm{pc}(\mathrm{E}\text{-}\mathrm{Pc}) \geq 0 \quad (4.3)$$

$$(E+e)^2 - c^2(\mathbf{P}+\mathbf{p})(\mathbf{P}+\mathbf{p}) \geq E^2 - (cP)^2 \quad (4.4)$$

The above shows that the addition of the photon results into an increase of the value of the expression (4.1).

Adding a system of photons having the total energy Ee and the total three-vector momentum $\Sigma\,\mathbf{p}$ to the system of massive particles produces an interesting situation:

$$(E+\Sigma e)^2 - c^2(\mathbf{P}+\Sigma\mathbf{p})(\mathbf{P}+\Sigma\mathbf{p}) =$$
$$= E^2 + 2E\Sigma e + (\Sigma e)^2 - (cP)^2 - c^2\Sigma p^2 - 2c^2\mathbf{P}\Sigma\mathbf{p} =$$
$$= E^2 - (cP)^2 + 2(E\Sigma e - c^2\mathbf{P}\Sigma\mathbf{p}) \quad (4.5)$$

$$(\mathbf{P}\Sigma\mathbf{p})_{max} = P\Sigma p \quad (4.6)$$

$$(E\Sigma e - c^2\mathbf{P}\Sigma\mathbf{p})_{min} = E\Sigma e - c^2 P\Sigma p = c(E - Pc)\Sigma p \geq 0 \quad (4.7)$$

$$(E+\Sigma e)^2 - c^2(\mathbf{P}+\Sigma\mathbf{p})(\mathbf{P}+\Sigma\mathbf{p}) \geq E^2 - (cP)^2 \quad (4.8)$$

In other words, the addition of photons to a system of particles always results into an increase of the expression evaluated by (4.5). Finally, from the above formalism we can easily compute [6] the “equivalent mass” of the system as a function of its directly measurable kinetic energy KE and the three-vector total momentum **P**:

$$M = \frac{(\mathbf{P}c)^2 - KE^2}{2KE * c^2} \tag{4.9}$$

From (4.9) it follows that when photons are injected, the system mass can also be expressed as a function of directly measurable quantities like the total kinetic energy KE and its total momentum **P**. The equivalent mass variation for such an open system as a function of the variation of the total kinetic energy d(KE) and the variation of total momentum dP (also as a scalar) is:

$$dM = \frac{P}{KE} dP - \frac{(Pc)^2 + KE^2}{2KE^2 c^2} d(KE) \tag{4.10}$$

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Practical and Theoretical Methods for Determining the Trajectories for Particles Involved in Elastic Collisions

Synopsis

The present chapter shows how to use the conservation of the energy-momentum in order to determine the trajectories of two particles after they are subjected to an elastic collision. While the problem is studied in existent literature, there are severe limitations in the solutions, like the fact that the solution only determines the angle between the particles after the collision and not their exact trajectories. This is not very satisfactory when it comes to setting up experiments aimed at verifying the theoretical predictions. In the following paper, we will show how to obtain a much more detailed fix of the trajectories of the particles post collision by determining their exact angles with respect to the trajectory of the particles before the collision. The new approach is extremely important in applications like particle accelerators where we can only work with directly measurable quantities, the kinetic energy KE and the momentum **p**.

1. Elastic collision of two arbitrary mass particles

Consider two particles of rest masses m_1 and m_2. In the most general case $m_1 \neq m_2$. The case $m_1 = m_2$ is well represented in literature [1,2] and we will show later on how our solution reduces in the limit to the existent ones. It is well known that in relativity, the total energy (E) and the three-vector momentum (**p**) of a system of particles involved in a collision are conserved [3]:

$$E_1 + E_2 = E_3 + E_4 \qquad (1.1)$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$$

In the above, $E_1, E_2, \mathbf{p}_1, \mathbf{p}_2$ are the total energies / momenta of the two particles before the collision, $E_3, E_4, \mathbf{p}_3, \mathbf{p}_4$ are their total energies / momenta after collision.

$$
\begin{aligned}
&E_i = \gamma(v_i) m_i c^2, i = 1,2,3,4 \\
&\mathbf{p}_i = \gamma(v_i) m_i \mathbf{v}_i \\
&E_i^2 - (\mathrm{p_i c})^2 = (m_i c^2)^2 \\
&\gamma(v_i) = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}}
\end{aligned}
\tag{1.2}
$$

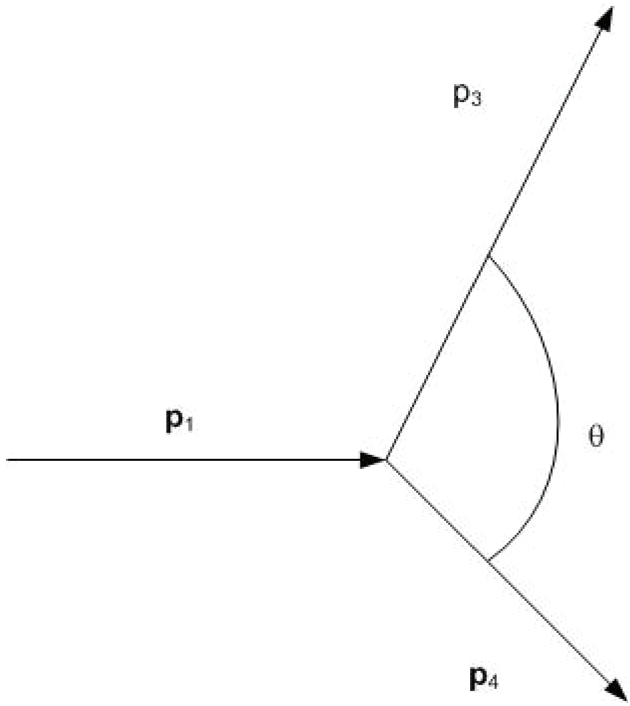

Fig. 1 The collision of two particles

Since the conservation of energy-momentum is frame invariant, we can choose to solve the system of equations (1.1) in any frame of reference [4,5]. We choose the frame commoving with particle 2 before collision, described by:

$$\mathbf{p}_2 = 0$$
$$E_2 = m_2 c^2 \tag{1.3}$$

To the above, we need to add the fact that rest mass, by virtue of being invariant, results in:

$$m_1 = m_3$$
$$m_2 = m_4 \tag{1.4}$$

In the rest of the paper, we will refer to the particle associated with $\mathbf{p}_1$ as the "bullet" and the particle associated with $\mathbf{p}_2$ as the "target". The frame of reference commoving with the "target" is **the** natural choice for the collisions where the "target" particle is at rest with respect to the lab. We have now all the information necessary to determine the angle made by the two particles after the collision has taken place. From:

$$E_1 + E_2 = E_3 + E_4 \tag{1.5}$$
$$\mathbf{p}_1 = \mathbf{p}_3 + \mathbf{p}_4$$

we obtain:

$$p_1^2 = p_3^2 + p_4^2 + 2 p_3 p_4 \cos\theta$$
$$E_1^2 = E_2^2 + E_3^2 + E_4^2 + 2E_3E_4 - 2E_2E_3 - 2E_2E_4 \tag{1.6}$$

where θ is the angle between the two particles after the collision has taken place, see Fig. 1.

Subtracting the first equation multiplied by c^2 from the second one, we obtain, after reduction of like terms and factorization the angle θ:

$$\cos\theta = \frac{(E_3 - m_2c^2)(E_4 - m_2c^2)}{p_3p_4c^2} =$$

$$= \frac{(E_3 - m_2c^2)(E_4 - m_2c^2)}{c^2\sqrt{E_3^2 - (m_1c^2)^2}\sqrt{E_4^2 - (m_2c^2)^2}} \tag{1.7}$$

The intent of the method is to express the angle only as a function of easily measurable scalars, like the total energies E_3, E_4. The above information, though accurate, is disappointing from the point of view of an experimental physicist attempting to reconcile the theoretical angle predicted by (1.7) with experiment. For several reasons:

(a) We can only determine that $\mathbf{p}_1$ lies in the plane formed by $\mathbf{p}_3,\mathbf{p}_4$ forming a "fork" that can be rotated in any fashion around $\mathbf{p}_1$. This annoying effect precludes the use of cloud chambers in doing any measurements since there is no way of determining the true angle between $\mathbf{p}_3,\mathbf{p}_4$ due to the projection effect [6]-[8].

(b) After collision, we cannot determine whether the "bullet" has the momentum $\mathbf{p}_3$ or $\mathbf{p}_4$ nor can we determine whether the "target" has the momentum $\mathbf{p}_4$ or $\mathbf{p}_3$. All we know that that one particle has momentum $\mathbf{p}_3$ and the other one has momentum $\mathbf{p}_4$. In addition to the above shortcomings reference [1] limits itself to only treating collisions of particles of equal mass. In addition, it requires that the angle of the direction of the support of the vectors $\mathbf{p}_3,\mathbf{p}_4$ with the vector $\mathbf{p}_1$ is known in the center of momentum frame, a highly unrealistic expectation for an experimentalist who has only lab frame information. Reference [2], while attempting to deal with unequal mass of "bullet" and "target", not only cannot resolve the individual angles after collision but it also makes the fatal error of assuming that the total energies of the two particles after collision, as computed in the center of momentum frame, are equal. This is obviously untrue, since $\mathbf{p}_3 + \mathbf{p}_4 = 0$ does not imply $E_3 = E_4$ **unless** the two particles have equal rest mass. Both references [1] and [2] treat the collision in the center of momentum frame. While this may be a good choice for simplifying the equations, it is a terrible choice for the experimental physicist since the physical measurements take place in the frame of the lab, so the angles of collision as predicted by the equations written in the center of momentum frame are different from the ones measured in the lab frame.

2. Improvements to the existent theory

We have already shown that the formalism describing particle collision, though exact, has severe practical and theoretical limitations. The theoretical framework can be made to yield additional information, like the **individual** angles made by $\mathbf{p}_3, \mathbf{p}_4$ with $\mathbf{p}_1$ as it can be seen in Fig.2.

In order to get this information, we need to rewrite the momentum conservation in (1.5) on a component basis, by decomposing $\mathbf{p}_3, \mathbf{p}_4$ into components parallel and respectively perpendicular on $\mathbf{p}_1$:

$$\mathrm{p}_1 = \mathrm{p}_3 \cos\alpha + \mathrm{p}_4 \cos\beta \tag{2.1}$$

$$0 = \mathrm{p}_3 \sin\alpha - \mathrm{p}_4 \sin\beta \tag{2.2}$$

where α, β are the angles made by $\mathbf{p}_3, \mathbf{p}_4$ respectively with $\mathbf{p}_1$. In this approach we can have $m_1 = m_3, m_2 = m_4$ or $m_1 = m_4, m_2 = m_3$. We do not know, nor do we care which way since we are not using the explicit expressions of momentum as a function of rest mass. Actually, rest mass never appears in the improved formalism.

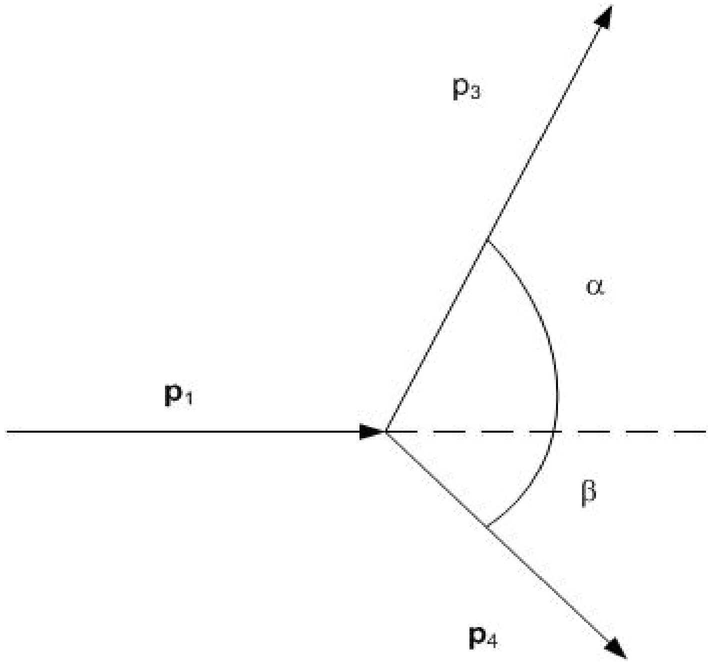

Fig. 2 Respective angles of the two particles after collision

The system produces the solutions:

$$\cos\alpha = \frac{p_1^2 + p_3^2 - p_4^2}{2\,\mathrm{p}_1\mathrm{p}_3}$$

$$\cos\beta = \frac{p_1^2 + p_4^2 - p_3^2}{2\,\mathrm{p}_1\mathrm{p}_4} \qquad (2.3)$$

We can easily show that the above solutions are valid since:

$$\frac{p_1^2 + p_3^2 - p_4^2}{2\,p_1 p_3} \le 1$$

$$\frac{p_1^2 + p_4^2 - p_3^2}{2\,p_1 p_4} \le 1 \qquad (2.4)$$

There is also an elegant way of verifying that $\alpha + \beta = \theta$, as shown below:

$$\begin{aligned} p_1^2 &= p_3^2 \cos^2\alpha + p_4^2 \cos^2\beta + 2 p_3 p_4 \cos\alpha \cos\beta = \\ &= p_3^2 + p_4^2 + 2 p_3 p_4 \cos\alpha \cos\beta - 2 p_3 \; p_4 \sin\alpha \sin\beta = \\ &= p_3^2 + p_4^2 + 2 p_3 p_4 \cos(\alpha + \beta) \end{aligned} \qquad (2.5)$$

Comparing (2.5) with (1.6) we obtain that $\alpha + \beta = \theta$.

Since the individual angles α, β are dependent on the momenta after collision, $\mathbf{p}_3, \mathbf{p}_4$ we are forced to try to detect the particles in the whole disc determined by the intersection of the cone made by $\mathbf{p}_3, \mathbf{p}_4$ with the plane (or hemisphere) past the collision point. We have seen earlier that it isn't possible to figure whether the "bullet" or the "target" is associated with $\mathbf{p}_3$ due to the symmetry of the formalism described be equations (2.1)-(2.2). Nevertheless we can still provide some insight into the physics post the collision. From (2.3) we learn that:

$$\cos\alpha - \cos\beta = \frac{(p_3 - p_4)((p_3 + p_4)^2 - p_1^2)}{2\,p_1 p_3 p_4} \qquad (2.6)$$

In other words, the particle with the smaller deflection angle post collision is the particle exhibiting the larger momentum. The above description generalizes easily for the case when both particles are moving in the lab frame, as seen in Fig.3:

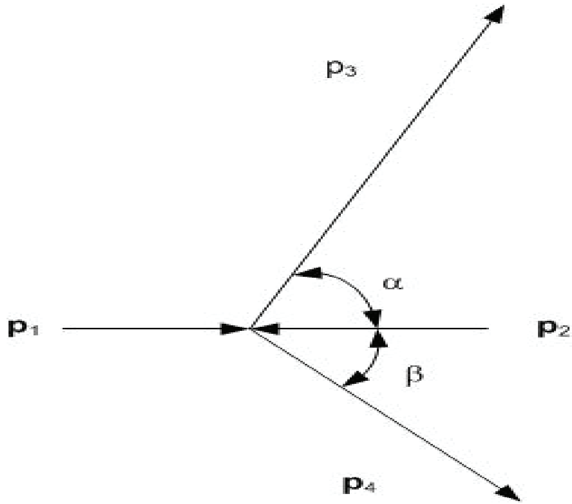

Fig. 3. Both particles in motion in the lab frame

In this case we need to simply replace p_1 with $p_1 + p_2$ in (2.1)

$$p_1 + p_2 = p_3 \cos\alpha + p_4 \cos\beta \tag{2.7}$$

$$0 = p_3 \sin\alpha - p_4 \sin\beta \tag{2.8}$$

The angles of the particles after collision are obtained therefore by making the same substitution in (2.3):

$$\cos\alpha = \frac{(p_1 + p_2)^2 + p_3^2 - p_4^2}{2(p_1 + p_2)\,p_3}$$

$$\cos\beta = \frac{(p_1 + p_2)^2 + p_4^2 - p_3^2}{2(p_1 + p_2)\,p_4} \tag{2.9}$$

The simplicity of the generalization illustrates the power of the solution.

3. Discussion

For $m_1 = m_2$ we recover the well known results from literature. Indeed, making $m_1 = m_2$ in (1.7) we obtain:

$$\cos\theta = \frac{(\gamma(v_3)-1)(\gamma(v_4)-1)}{\gamma(v_3)\gamma(v_4)\beta(v_3)\beta(v_4)} =$$

$$= \frac{(1-\sqrt{1-\beta^2(v_3)})(1-\sqrt{1-\beta^2(v_4)})}{\beta(v_3)\beta(v_4)} \tag{3.1}$$

$$\beta_i = \frac{v_i}{c}$$

In the low speed approximation, the above reduces (via a trivial Taylor expansion) to:

$$\cos\theta = \frac{\frac{1}{2}\beta^2(v_3)\frac{1}{2}\beta^2(v_4)}{\beta(v_3)\beta(v_4)} = \frac{\beta(v_3)\beta(v_4)}{4} \tag{3.2}$$

We can see that at very low speeds, $\cos\theta$ approaches zero but it is never equal to the prediction of the Newtonian dynamics which is **exactly zero**.

The reason for the discrepancy between the relativistic approximation (3.2) and Newtonian predictions is the fact that the Newtonian mechanics assume conservation of kinetic, rather than total energy:

$$\frac{p_1^2}{2m_1} = \frac{p_3^2}{2m_1} + \frac{p_4^2}{2m_2} \tag{3.3}$$

$$\mathbf{p}_1 = \mathbf{p}_3 + \mathbf{p}_4$$

So,

$$\frac{p_3^2}{2m_1}+\frac{p_4^2}{2m_2}=\frac{p_3^2+p_4^2+2p_3p_4\cos\theta}{2m_1} \tag{3.4}$$

resulting into:

$$\cos\theta=\frac{p_4}{2p_3}\frac{m_1-m_2}{m_2} \tag{3.5}$$

We can see that not even Newtonian dynamics predicts $\theta=\frac{\pi}{2}$ **unless** $m_1=m_2$.

Intuition would also let us believe that $m_1=m_2$ implies $\alpha=\beta$. Is our intuition correct? Let's check out the facts. The equations of conservation of energy and momentum are:

$$\begin{aligned}&\gamma_1\mathbf{v}_1=\gamma_3\mathbf{v}_3+\gamma_4\mathbf{v}_4\\&\gamma_1=\gamma_3+\gamma_4\end{aligned} \tag{3.6}$$

On the other hand:

$$\cos\alpha=\frac{p_1^2+p_3^2-p_4^2}{2\,\mathrm{p}_1\mathrm{p}_3}=\frac{\gamma_1^2v_1^2+\gamma_3^2v_3^2-\gamma_4^2v_4^2}{2\gamma_1\gamma_3v_1v_3} \tag{3.7}$$

From the conservation of momentum equation (3.6) we get:

$$\begin{aligned}&\gamma_1^2v_1^2=\gamma_3^2v_3^2+\gamma_4^2v_4^2+2\gamma_3\gamma_4v_1v_3\cos\theta=\\&=\gamma_3^2v_3^2+\gamma_4^2v_4^2+2c^2(\gamma_3-1)(\gamma_4-1)\end{aligned} \tag{3.8}$$

Substituting (3.8) and (3.6) in (3.7) we obtain:

$$\cos\alpha=\frac{\beta_3^2\gamma_3^2+2(\gamma_3-1)(\gamma_4-1)}{2\beta_1\beta_3\gamma_3(\gamma_3+\gamma_4)} \tag{3.9}$$

In a similar way, we get:

$$\cos\beta = \frac{\beta_4^2\gamma_4^2 + 2(\gamma_3 - 1)(\gamma_4 - 1)}{2\beta_1\beta_4\gamma_4(\gamma_3 + \gamma_4)} \qquad (3.10)$$

So, our intuition would lead us astray, $\alpha \neq \beta$.

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Relativistic Reflection Laws for Arbitrary Direction Boosts

Synopsis

In a previous paper [1] we have investigated the reflection of massive particles from moving mirrors. The adoption of the formalism based on the energy-momentum allowed us to derive the most general set of formulas, valid for both massive and, in the limit, also for massless particles (photons). In the present paper we are extending the formalism to the case of arbitrary direction boosts.

1. Generalization for arbitrary direction of motion between frames

In a previous paper [1] we have investigated the reflection of massive particles from moving mirrors (fig.1) in the particular case of a boost in the x direction. The most general case is when $\mathbf{V}$ has an arbitrary direction, so the S' x-axis is no longer aligned with the S x-axis. In this case we need to use the general Lorentz transforms, in matrix form. In this case we consider a boost in an arbitrary direction $\boldsymbol{\beta} = \frac{\mathbf{V}}{c}$ resulting into the transformation between frames S and S' [2]:

$$E' = \gamma(V)E + \gamma c(\mathbf{p}.\boldsymbol{\beta})$$

$$\mathbf{p'} = \mathbf{p} + \boldsymbol{\beta}[\frac{\gamma(V)E}{c} + \frac{(\gamma(V)-1)(\mathbf{p}.\boldsymbol{\beta})}{\beta^2}]$$

$$\gamma(V) = \frac{1}{\sqrt{1-\beta^2}}$$

$$E = \gamma(u)mc^2$$

$$\mathbf{p} = \gamma(u)m\mathbf{u} \tag{1.1}$$

In (1.1) $\mathbf{u}$ is the velocity of the particle in frame S. From (1.1) we obtain the components of the momentum in frame S' (given that $p_z = 0$):

$$p_x^{'} = p_x + \frac{V_x}{c^2}[\gamma E + \frac{(\gamma(V)-1)(p_x V_x + p_y V_y)}{\beta^2}]$$

$$p_y^{'} = p_y + \frac{V_y}{c^2}[\gamma E + \frac{(\gamma(V)-1)(p_x V_x + p_y V_y)}{\beta^2}]$$

$$p_z^{'} = \frac{V_z}{c^2}[\gamma E + \frac{(\gamma(V)-1)(p_x V_x + p_y V_y)}{\beta^2}]$$

$$p' = \sqrt{\mathbf{p'}.\mathbf{p'}} = \gamma(V)(p + \mathbf{p}.\boldsymbol{\beta}) \tag{1.2}$$

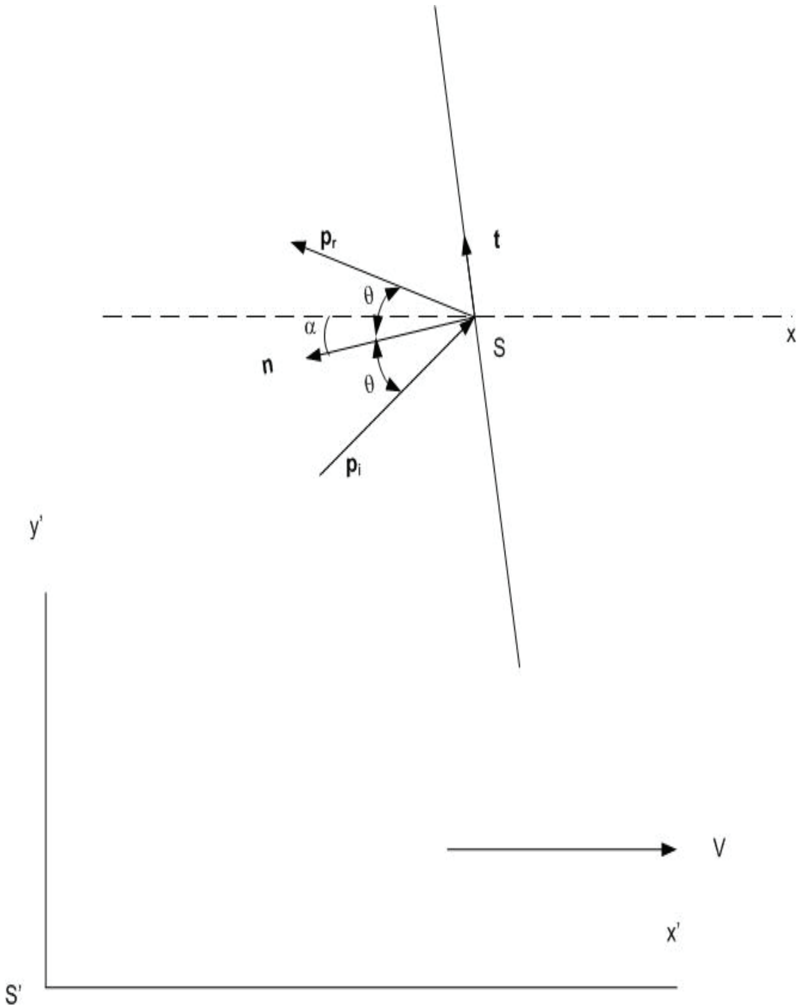

Fig. 1 Reflection of a massive particle off a moving mirror

Please note that in frame S', the momentum has a non-null component in the z direction but this fact does not affect the calculation of the cosines of the angles of incidence and reflection. It does show an interesting effect, though. Unlike the particular boost described earlier in reference [1], for the case of the general boost (1.1) we have $p'_{iz} \neq p_{iz}$ and $p'_{rz} \neq p_{rz}$. Even more interestingly, $p'_{iz} \neq p'_{rz} \neq 0$ so the plane formed by the incident and reflected directions appears rotated around the normal to the mirror in S'. The last step is to calculate the angle:

$$\cos\theta' = \frac{p'_x}{p'} \tag{1.3}$$

By substituting in (1.3) the following:

$$\begin{aligned} V_x &= V \\ V_y &= 0 \\ E &= pc \\ \boldsymbol{\beta}.\mathbf{p} &= \beta p_x \end{aligned} \tag{1.4}$$

we obtain:

$$\begin{aligned} p' &= \gamma(p + \beta p_x) \\ p'_x &= \gamma(p_x + \beta p) \end{aligned} \tag{1.5}$$

allowing us to recover the well known light aberration formula:

$$\cos\theta' = \frac{\cos\theta + \beta}{1 + \beta\cos\theta} \tag{1.6}$$

The above gives us the key to the derivation of the most general light aberration formula:

$$\cos\theta' = \frac{p_x'}{p'} = \frac{p_x + \frac{V_x}{c^2}[\gamma(V)pc + \frac{(\gamma(V)-1)(p_x V_x + p_y V_y)}{\beta^2}]}{\gamma(V)(p + \mathbf{p}.\boldsymbol{\beta})} =$$

$$= \frac{\cos\theta + \frac{V_x}{c}[\gamma(V) + \frac{\gamma(V)-1}{\beta^2}\frac{\mathbf{p}}{p}.\boldsymbol{\beta}]}{\gamma(V)(1 + \frac{\mathbf{p}}{p}.\boldsymbol{\beta})} =$$

$$= \frac{\cos\theta + \frac{V_x}{c}[\gamma(V) + \frac{\gamma(V)-1}{\beta^2}\mathbf{k}.\boldsymbol{\beta}]}{\gamma(V)(1 + \mathbf{k}.\boldsymbol{\beta})} \quad (1.7)$$

where $\mathbf{k}$ represents the light wave (unit) vector. Again, a quick sanity check allows us to recover the simpler case described in literature by observing that, in the particular case of a boost in the x direction $\mathbf{k}.\boldsymbol{\beta} = \beta\cos\theta$ and $\frac{V_x}{c} = \beta$. Substituting in (1.7) we recover (1.6). It is interesting to see that from the first transform (1.3) we can derive the general equation of the Doppler effect:

$$hf' = \gamma hf + \gamma c\frac{hf}{c}(\mathbf{k}.\boldsymbol{\beta})$$
$$f' = f\gamma(1 + \mathbf{k}.\boldsymbol{\beta}) \quad (1.8)$$

Another useful consequence of the second transform (1.3) and (1.8) is the general transform of the light wave-vector:

$$\mathbf{k'} = \frac{\mathbf{k} + \boldsymbol{\beta}[\gamma + \frac{(\gamma-1)(\mathbf{k}.\boldsymbol{\beta})}{\beta^2}]}{\gamma(1 + \mathbf{k}.\boldsymbol{\beta})}$$

(1.9)

A different approach to solving the same problem starts with calculating the angle cosines:

$$\cos\theta_i' = \frac{\mathbf{N'}.\mathbf{v}_i'}{N' v_i'}$$

$$\cos\theta_r' = \frac{\mathbf{N'}.\mathbf{v}_r'}{N' v_r'} \tag{1.10}$$

where $\mathbf{N'}/N', \mathbf{v}_i'$ and $\mathbf{v}_r'$ are the unit normal to the mirror, the velocity of the particle in the incident direction and the velocity of the particle in the reflected direction **all measured in the lab frame, S'.**

If we are interested in the relationship of the angles between frames S and S', we will need to employ the general Lorentz transforms [2]:

$$\mathbf{v}_i' = \frac{1}{\gamma(V)(1-\frac{\mathbf{v}_i.\mathbf{V}}{c^2})}(\mathbf{v}_i + [(\gamma(V)-1)\frac{\mathbf{v}_i.\mathbf{V}}{V^2} - \gamma(V)]\mathbf{V})$$

$$\mathbf{v}_r' = \frac{1}{\gamma(V)(1-\frac{\mathbf{v}_r.\mathbf{V}}{c^2})}(\mathbf{v}_r + [(\gamma(V)-1)\frac{\mathbf{v}_r.\mathbf{V}}{V^2} - \gamma(V)]\mathbf{V})$$

$$\mathbf{N}' = \mathbf{n} + (\frac{\mathbf{n}.\mathbf{V}}{V^2})(\gamma(V)-1)\mathbf{V} \tag{1.11}$$

where $\mathbf{n}, \mathbf{v}_i$, $\mathbf{v}_r$, $\mathbf{V}$ are respectively: the unit normal to the mirror, the velocity of the particle in the incident direction, the velocity of the particle in the reflected direction and the velocity between the frames, **all measured in the mirror frame, S.** We can further consider the angles:

$$\cos\theta_i = \frac{\mathbf{n.v}_i}{v_i}$$

$$\cos\theta_r = \frac{\mathbf{n.v}_r}{v_r}$$

$$\cos\alpha = \frac{\mathbf{n.V}}{V}$$

$$\cos\beta_i = \frac{\mathbf{v_i.V}}{v_i V}$$

$$\cos\beta_r = \frac{\mathbf{v}_r\mathbf{.V}}{v_r V} \tag{1.12}$$

From (1.11) and (1.12) we can derive the aberration formula for the general case:

$$\cos\theta_i^{'} = \frac{1}{N' v_i^{'}} \frac{v_i[\cos\theta_i + (\gamma^2(V)-1)\cos\alpha\cos\beta_i] - V\gamma^2(V)\cos\alpha}{\gamma(V)(1-\frac{v_i V\cos\beta_i}{c^2})} \tag{1.13}$$

$$\cos\theta_r^{'} = \frac{1}{N' v_r^{'}} \frac{v_r[\cos\theta_r + (\gamma^2(V)-1)\cos\alpha\cos\beta_r] - V\gamma^2(V)\cos\alpha}{\gamma(V)(1-\frac{v_r V\cos\beta_r}{c^2})} \tag{1.14}$$

The subject is interesting not only to physicists designing concentrators for fascicles of massive particles and electron microscopes but also to computer scientists working in raytracing operating in the photon sector [3-8].

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THE GENERAL TRAJECTORIES OF ACCELERATED PARTICLES IN SPECIAL RELATIVITY

Synopsis

Accelerated motion in special relativity is a subject that tends to be treated under very restrictive conditions, the motion is considered to be "uniformly accelerated", meaning the motion under constant force and the treatment is in one dimension only. In the present paper we treat the general case, of non-uniform force and we extend the treatment to all three spatial dimensions. The chapter is divided into three main sections: the first section is an overview of the existent solutions for the unidimensional trajectories, the second section deals with the general, three dimensional trajectories under constant force. The third section introduces the case of non-uniform force. The case of non-uniform force is further subdivided into two sub-cases: force that is explicitly time-dependent and, the more complicated case of velocity-dependent (aka Lorentz) force. All cases teach us how to deal with increasing levels of non-linearity in the equations of motion. In each case we will show how to find fully symbolic (closed) solutions for the trajectories. The last case, of the Lorentz force, is especially interesting because it is a real life case, taken from the particle accelerator applications as in the design of velocity selectors used for particle separation. What makes it even more interesting is the fact that the solution uses a physics approach at the point where the mathematical approach hits a dead end. The subject is of interest for particle physicists as well for graduate students and teachers.

1. Introduction – the uniformly accelerated motion in one dimension, the equations of hyperbolic motion

From the very beginning, it is important to stress that special relativity is quite adequate for treating accelerated motions, as for instance in many rigorous treatments of the twin paradox [1-3]. The literature treating the case of uni-dimensional accelerated motion dates back to Max Born, to 1909 [4].

Since then, there have been quite a few works dealing with this subject, yet all seem to concentrate on the same scenario [2-8]. The trajectory of the particle is found by integrating the equation of motion:

$$\mathbf{F} = \frac{d}{dt}(m_0\gamma(v)\mathbf{v}) \tag{1.1}$$

In one dimension $\frac{d\mathbf{v}}{dt}$ and **v** have the same direction and sense, so the equation of motion reduces to [1]:

$$\gamma^3 \frac{dv}{dt} = \frac{F}{m_0} \tag{1.2}$$

Using the boundary condition $v(0) = v_0$ and using the shorthand $F/m_0 = a$ we obtain:

$$v(t) = \frac{at + v_0\gamma(v_0)}{\sqrt{1 + (\frac{at + v_0\gamma(v_0)}{c})^2}} \tag{1.3}$$

This is the general form of the expression for speed for motion under constant force in one dimension [1-8]. Similar treatments can be found in [14-17]. In the above, $F/m_0 = a$ is the Newtonian acceleration, not the relativistic one. The relativistic acceleration can be obtained from (1.3):

$$a_c = \frac{dv}{dt} = \frac{a}{(\sqrt{1 + (\frac{at}{c})^2})^3} \tag{1.4}$$

a_c stands for "coordinate acceleration" and it is obviously time-varying. So, if $F/m_0 = a$ is not the relativistic acceleration and if the relativistic coordinate acceleration a_c is not frame invariant, what are we talking about when we talk about "uniformly accelerated motion in special relativity"? In order to understand that, we need to do more work in understanding the issues. Assume that an observer located in the origin of an inertial frame Σ is measuring the **proper** time separation $d\tau$ between the events ($0,\tau_1$) and ($0,\tau_2$). An observer in a frame S moving with respect to Σ with speed u observes **both** a temporal and a spatial separation between the events according to the Lorentz transforms:

$$dx = \gamma(u)(0 - ud\tau)$$
$$dt = \gamma(u)(d\tau - 0) \qquad (1.5)$$

The above results into:

$$c^2(\frac{dt}{d\tau})^2 - (\frac{dx}{d\tau})^2 = c^2 \qquad (1.6)$$

The above is the intrinsic equation of a hyperbola. We can parameterize it as:

$$\frac{dt}{d\tau} = \cosh\phi$$
$$\frac{dx}{d\tau} = c\sinh\phi \qquad (1.7)$$

ϕ is called **rapidity**. From (1.7) we can derive the relationship between the coordinate speed and rapidity:

$$u = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = c\tanh\phi \qquad (1.8)$$

Let's consider another inertial frame S' moving with speed V with respect to S. The Lorentz transform between S and S' is:

$$dx' = \gamma(V)(dx - Vdt)$$
$$dt' = \gamma(V)(dt - \frac{Vdx}{c^2}) \quad (1.9)$$

The above becomes, in hyperbolic notation:

$$dx' = dx \cosh\Phi - cdt \sinh\Phi$$
$$dt' = dt \cosh\Phi - \frac{dx}{c} \sinh\Phi$$
$$\tanh\Phi = \frac{V}{c} \quad (1.10)$$

Inserting (1.7) into (1.10) we get:

$$\cosh\phi' = \frac{dt'}{d\tau} = \cosh(\phi - \Phi)$$
$$c \sinh\phi' = \frac{dx'}{d\tau} = c \sinh(\phi - \Phi) \quad (1.11)$$

with the immediate consequence that:

$$\phi' = \phi - \Phi \quad (1.12)$$

Expression (1.12) shows that rapidity behaves exactly like speed in Newtonian physics, i.e. it is additive or subtractive, depending on the direction of relative motion. One more thing, from (1.12):

$$\frac{d\phi'}{d\tau} = \frac{d\phi}{d\tau} \quad (1.13)$$

so, $\frac{d\phi}{d\tau}$ is frame-invariant. We use the notation α for this invariant. The coordinate acceleration becomes:

$$a_c = \frac{dv}{dt} = \frac{dv / d\tau}{dt / d\tau} = c\alpha \operatorname{sech}^3 \phi \tag{1.14}$$

For the particular case when the rapidity is null, i.e. $\phi = 0$, i.e. v=0, i.e. a frame co-moving with the object that moves at speed v wrt S we obtain the **proper** acceleration, a_p. i.e. the acceleration measured by the observer co-moving with the accelerated object:

$$a_p = c\alpha \tag{1.15}$$

It is the proper acceleration that is frame-invariant, that is all inertial observers agree on its value. If a_p is also constant we obtain a special set of equations that describe "hyperbolic motion" [7,8]:

$$t(\tau) = \frac{c}{a_p} \sinh \frac{a_p \tau}{c}$$

$$x(\tau) = \frac{c^2}{a_p} \cosh \frac{a_p \tau}{c}$$

(1.16)

When we talk about "uniformly accelerated motion" we talk about the case of constant proper acceleration. Depending on the application, either expressions (1.16) or (1.3) are used [7,8,10].

2. The general case of motion under constant force in three dimensions

In three dimensions, we need to solve the more complicated equation:

$$\frac{d(m_0\gamma(v)\mathbf{v})}{dt} = \mathbf{F} \tag{2.1}$$

In this case

$$\mathbf{F} = \mathbf{u}_x F_x + \mathbf{u}_y F_y + \mathbf{u}_z F_z$$

$$\mathbf{v} = \mathbf{u}_x v_x + \mathbf{u}_y v_y + \mathbf{u}_z v_z$$

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.2}$$

and (2.1) becomes a system of non-linear differential equations:

$$d(\gamma(v)v_u) = \frac{F_u}{m_0}dt$$

$$u = x, y, z \tag{2.3}$$

There is no treatment of the general motion in existent literature, so we are trying to fill this gap. The solution is immediate:

$$v_u(t) = \frac{(F_u / m_0)t + k_u}{\sqrt{1 + \frac{\sum_{u=x}^{z}((F_u / m_0)t + k_u)^2}{c^2}}}$$

$$u = x, y, z \tag{2.4}$$

The constants k_u are obtained from the boundary conditions $v_u(0) = v_{u0}$:

$$v_{x0} = \frac{k_x}{\sqrt{1 + \frac{k_x^2 + k_y^2 + k_z^2}{c^2}}}$$

$$v_{y0} = \frac{k_y}{\sqrt{1 + \frac{k_x^2 + k_y^2 + k_z^2}{c^2}}}$$

$$v_{z0} = \frac{k_z}{\sqrt{1 + \frac{k_x^2 + k_y^2 + k_z^2}{c^2}}} \qquad (2.5)$$

The system has the solution

$k_x = v_{x0}\gamma(v_0), k_y = v_{y0}\gamma(v_0), k_z = v_{z0}\gamma(v_0), v_0 = \sqrt{v_{x0}^2 + v_{y0}^2 + v_{z0}^2}$.

In vector form, the above can be written as:

$$\mathbf{v}(t) = \frac{(\mathbf{F} / m_0)t + \mathbf{k}}{\sqrt{1 + \frac{\sum_{u=x}^{z}((F_u / m_0)t + k_u)^2}{c^2}}}$$

$$\mathbf{k} = \mathbf{u}_x k_x + \mathbf{u}_y k_y + \mathbf{u}_z k_z \qquad (2.6)$$

It is interesting to note that the motion in any of the three principal directions is determined not only by the force applied in that direction but also by the forces applied in the other two (orthogonal) directions as it can be gleaned from (2.4). The other interesting fact is that even in the absence of the "accelerating force" F_u, the resultant motion in given direction is accelerated if the initial speed in that direction is not zero. This effect is due to the "orthogonal forces". For example, the motion in the x direction is accelerated even if $F_x = 0$:

$$v_x(t) = \frac{k_x}{\sqrt{1 + \frac{\sum_{u=x}^{z}((F_u / m_0)t + k_u)^2}{c^2}}} \tag{2.7}$$

Finally, we are the point of determining the trajectories, this is done by integrating the system of equations (2.6). Without losing any generality, let's assume a two-dimensional case. In addition, let's assume $F_x = F_y = F$. The system has the solution:

$$x(t) = x(0) + \frac{c}{4a}\{2\sqrt{2a^2t^2 + 2atk_1 + 2atk_2 + c^2 + k_1^2 + k_2^2} +$$
$$+\sqrt{2}(k_1 - k_2)\log[2c^2(2at + k_1 + k_2 +$$
$$+c\sqrt{2}\sqrt{\frac{2a^2t^2 + 2atk_1 + 2atk_2 + c^2 + k_1^2 + k_2^2}{c^2}})]\}$$

$$y(t) = y(0) + \frac{c}{4a}\{2\sqrt{2a^2t^2 + 2atk_1 + 2atk_2 + c^2 + k_1^2 + k_2^2} +$$
$$+\sqrt{2}(k_1 - k_2)\log[2c^2(2at + k_1 + k_2 +$$
$$+c\sqrt{2}\sqrt{\frac{2a^2t^2 + 2atk_1 + 2atk_2 + c^2 + k_1^2 + k_2^2}{c^2}})]\} \tag{2.8}$$

where $a = F / m_0, k_1 = \gamma(v_0)v_{x0}, k_2 = \gamma(v_0)v_{y0}$. We can see that for $v_{x0} = v_{y0}$ the motion along the two axes is identical so the particle describes a straight line trajectory, the diagonal of a square. For $v_{x0} \neq v_{y0}$ the trajectory becomes curved, due to the presence of the logarithmical term. Depending which initial speed is larger, the trajectory is "pulled" towards one axis or the other. From (2.6) we can derive the coordinate acceleration:

$$\mathbf{a}(t)=\frac{\frac{\mathbf{F}}{m_0}(1+\frac{1}{c^2}\sum_{u=x}^{z}(F_u t/m_0+k_u)^2)-\frac{1}{c^2}(\frac{\mathbf{F}t}{m_0}+\mathbf{k})\sum_{u=x}^{z}F_u/m_0(F_u t/m_0+k_u)}{(1+\frac{1}{c^2}\sum_{u=x}^{z}(F_u t/m_0+k_u)^2)^{3/2}} \quad (2.9)$$

It is interesting to note that the coordinate acceleration is not collinear with the either the force $\mathbf{F}$ nor with the coordinate speed $\mathbf{v}$ and that it has a hyperbolic dependency on both the force and the time t. If $\mathbf{k}=0$ (i.e. the initial speed of the particle is zero) then:

$$\mathbf{a}(t)=\frac{\frac{\mathbf{F}}{m_0}}{(1+\frac{1}{c^2}\sum_{u=x}^{z}(F_u t/m_0)^2)^{3/2}} \quad (2.10)$$

That is, the coordinate acceleration has the same direction as the force if the initial speeds are null.

3. The case of motion under time-variable force in three dimensions

In three dimensions, we need to solve the equation:

$$\frac{d(m_0\gamma(v)\mathbf{v})}{dt}=\mathbf{F}(t) \quad (3.1)$$

where the force is now an explicit function of time. In this case

$$\mathbf{F}=\mathbf{u}_x F_x(t)+\mathbf{u}_y F_y(t)+\mathbf{u}_z F_z(t)$$

$$\mathbf{v}=\mathbf{u}_x v_x+\mathbf{u}_y v_y+\mathbf{u}_z v_z$$

$$\gamma(v)=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad (3.2)$$

The solution can be written as:

$$\mathbf{v}(t)=\frac{\mathbf{k}+1/m_0\int_0^t \mathbf{F}dt}{\sqrt{1+\frac{\sum_{u=x}^{z}(k_u+1/m_0\int_0^t F_u dt)^2}{c^2}}}$$
(3.3)

4. The case of velocity-dependent force

We start with a "warm-up" case [9,11], one that is easier to treat, the absence of electric field: a charged particle of charge q enters a spatial domain characterized by the magnetic induction $\mathbf{B}=\mathbf{e_z}B$ with the initial velocity $\mathbf{v_0}=\mathbf{e}_x v_{0x}+\mathbf{e_y}v_{0y}$. While this case is not representative of the actual, practical cases in particle accelerators, it is nevertheless interesting in setting the stage for solving the actual, general case encountered in practice. The Lorentz force experienced by the particle is:

$$\mathbf{F}=q(\mathbf{vxB})=qB(\mathbf{e_x}v_y-\mathbf{e_y}v_x) \tag{4.1}$$

It can be proven that $|\mathbf{v}|$=constant by observing that, on one hand the time derivative of total energy W is:

$$\frac{dW}{dt}=m_0c^2\frac{d\gamma}{dt} \tag{4.2}$$

while, on the other hand:

$$\frac{dW}{dt}=\mathbf{vF}=0 \tag{4.3}$$

so $\gamma(v) = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$ =constant. In fact, $|\mathbf{v_0}| = v_0 = \sqrt{v_{0x}{}^2 + v_{0y}{}^2}$

$$\gamma(v) = \gamma(v_0) = \frac{1}{\sqrt{1-\dfrac{v_0{}^2}{c^2}}}$$

The equation of motion is given by:

$$\mathbf{F} = \frac{d(\gamma m_0 \mathbf{v})}{dt} = \gamma m_0 \frac{d\mathbf{v}}{dt} \tag{4.4}$$

Separating by components:

$$\mathrm{F_x} = \gamma m_0 \frac{d\mathrm{v_x}}{dt}$$

$$\mathrm{F_y} = \gamma m_0 \frac{d\mathrm{v_y}}{dt}$$

$$\mathrm{F_z} = \gamma m_0 \frac{d\mathrm{v_z}}{dt} \tag{4.5}$$

giving:

$$qBv_y = \gamma m_0 \frac{d\mathrm{v_x}}{dt}$$

$$-qBv_x = \gamma m_0 \frac{d\mathrm{v_y}}{dt}$$

$$0 = \gamma m_0 \frac{d\mathrm{v_z}}{dt} \tag{4.6}$$

with the initial conditions: $\mathbf{v_0} = (v_{0x}, v_{0y}, 0)$

The last equation gives $\mathrm{v}_z = 0$. We are left with solving:

$$qBv_y = \gamma m_0 \frac{d\mathrm{v}_x}{dt}$$
$$-qBv_x = \gamma m_0 \frac{d\mathrm{v}_y}{dt} \tag{4.7}$$

Consider the parameterization:

$$x = -r\cos(\omega t)$$
$$y = r\sin(\omega t) \tag{4.8}$$

$$v_x = \frac{dx}{dt} = r\omega\sin(\omega t)$$
$$v_y = \frac{dy}{dt} = r\omega\cos(\omega t) \tag{4.9}$$

The above allows us to draw the very important conclusion that $|\mathbf{v}| = \sqrt{{v_x}^2 + {v_y}^2} = r\omega$ that is the speed is constant along the circular trajectory, $|\mathbf{v}| = v_0$.

$$\frac{dv_x}{dt} = r\omega^2\cos(\omega t) = \omega v_y$$
$$\frac{dv_y}{dt} = -r\omega^2\sin(\omega t) = -\omega v_x \tag{4.10}$$

Substituting (4.10) into (4.7) and remembering that $|\mathbf{v}| = v_0$ implies $\gamma(v) = \gamma(v_0)$:

$$qBv_y = \gamma(v_0)m_0\omega v_y$$
$$-qBv_x = -\gamma(v_0)m_0\omega v_x \tag{4.11}$$

Either equation gives the pulsation:

$$\omega = \frac{qB}{\gamma(v_0)m_0} \tag{4.12}$$

On one hand, $|\mathbf{v}| = \sqrt{{v_x}^2 + {v_y}^2} = r\omega$ meaning that the speed is constant along the circular trajectory and the radius of the circular trajectory is:

$$r = \frac{\gamma(v_0)m_0v_0}{qB} \tag{4.13}$$

We are now ready to tackle the general case, where particles are accelerated by **both** a magnetic **and** an electric field. This is the case in particle separators where particles are accelerated linearly by the electric field and circularly by the magnetic one. The acceleration by the electric field is necessary in order to bring the particles to the relativistic speed v_0, with respect to the lab. We will see that the case of the general Lorentz force $\mathbf{F} = q(\mathbf{vxB} + \mathbf{E})$ requires a much more complicated formalism for finding the particle trajectory. In order to tackle this issue we will need to resort to transforming the problem to a simpler one, as outlined in the next section.

5. A Different Approach

Assume an inertial frame S’ moving at speed V with respect to the frame S along the x axis. The Lorentz transforms between S and S’ are [9]:

$$x' = \gamma(V)(x - Vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(V)(t - \frac{Vx}{c^2}) \tag{5.1}$$

The correspondent of the 3-vectors

$$\mathbf{B} = (0, 0, B)$$
$$\mathbf{E} = (0, E, 0) \tag{5.2}$$

are:

$$\mathbf{E}' = \gamma(V)(\mathbf{E} + \mathbf{VxB})$$
$$\mathbf{B}' = \gamma(V)(\mathbf{B} - \frac{\mathbf{VxE}}{c^2}) \tag{5.3}$$

in the direction perpendicular to **V=**(V,0,0). For the direction parallel with **V**, the transformed values are:

$$\mathbf{E}' = \mathbf{E}$$
$$\mathbf{B}' = \mathbf{B} \tag{5.4}$$

Separating by components and using

$$\mathbf{VxB} = -\mathbf{e_y} VB$$
$$\mathbf{VxE} = \mathbf{e}_z VE \tag{5.5}$$

we obtain:

$$E'_x = E_x = 0$$
$$B'_x = B_x = 0$$
$$E'_y = \gamma(V)(E_y - VB) = \gamma(V)(E - VB)$$
$$E'_z = \gamma(V)E_z = 0$$
$$B'_y = \gamma(V)B_y = 0$$
$$B'_z = \gamma(V)(B_z - \frac{VE_y}{c^2}) = \gamma(V)(B - \frac{VE}{c^2}) \tag{5.6}$$

a. If $\frac{E}{B} < c$ by choosing $V = \frac{E}{B}$ makes $E'_y = 0$ reducing the problem to solving the system of equations:

$$\gamma(v')m_0 \frac{dv'_x}{dt'} - qB'v'_y = 0 \tag{5.7}$$

$$\gamma(v')m_0 \frac{dv'_y}{dt'} + qB'v'_x = 0 \tag{5.8}$$

The initial conditions can be chosen without any loss of generality (x(0),y(0),(z(0))=(0,0,0). The initial conditions for velocity in frame S $\mathbf{v_0} = (0, v_0, 0)$, transform in S' into:

$$v'_{0x} = \frac{v_{0x} - V}{1 - \frac{v_{0x}V}{c^2}} = -V$$

$$v'_{0y} = \frac{v_{0y}}{\gamma(V)(1 - \frac{v_{0x}V}{c^2})} = \frac{v_0}{\gamma(V)}$$

$$v'_{0z} = \frac{v_{0z}}{\gamma(V)(1 - \frac{v_{0x}V}{c^2})} = 0$$

$$v'_0 = \sqrt{V^2 + \frac{v_0^{\ 2}}{\gamma^2(V)}} = \sqrt{(\frac{E}{B})^2 + v_0^{\ 2}(1 - (\frac{E}{Bc})^2)} \tag{5.9}$$

The solution of the above system is:

$$x' = -r\cos(\omega t')$$
$$y' = r\sin(\omega t')$$
$$z' = const \tag{5.10}$$

$$\omega = \frac{qB'}{\gamma(v'_0)m_0} \tag{5.11}$$

$$r = \frac{\gamma(v'_0)m_0 v'_0}{qB'} \tag{5.12}$$

$$B' = B'_z = \gamma(V)(B - \frac{VE}{c^2}) = \gamma(\frac{E}{B})(B - \frac{E^2}{Bc^2}) \tag{5.13}$$

Now, we use the inverse Lorentz transforms:

$$x = \gamma(V)(x' + Vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(V)(t' + \frac{Vx'}{c^2}) \quad (5.14)$$

and we obtain the final equation of motion in frame S:

$$x = \gamma(\frac{E}{B})(\frac{E}{B}t' - r\cos(\omega t'))$$
$$y = r\sin(\omega t')$$
$$z = const$$
$$t = \gamma(\frac{E}{B})(t' - \frac{E}{Bc^2}r\cos(\omega t')) \quad (5.15)$$

b. If $\frac{E}{B} > c$ we can choose $V = \frac{Bc^2}{E}$ making $B'_z = 0$. The system of equations in S' becomes:

$$\gamma(v')m_0 \frac{dv'_x}{dt'} = 0 \quad (5.16)$$

$$\gamma^3(v')m_0 \frac{dv'_y}{dt'} - qE' = 0 \quad (5.17)$$

$$E' = E'_y = \gamma(V)(E_y - VB) = \gamma(V)(E - \frac{B^2c^2}{E}) \quad (5.18)$$

The first equation produces: $v'_x = v'_{0x}$ so
$x'(t') = v'_{0x}t' + x'_0 = -Vt' + x'_0$

The second equation becomes, after variable separation:

$$\frac{dv'_y}{\left(\sqrt{1-\frac{v'^2_{0x}}{c^2}-\frac{v'^2_y}{c^2}}\right)^3}=\frac{qE'}{m_0}dt' \tag{5.19}$$

with the solution:

$$\frac{v'_y}{\left(1-\frac{v'^2_{0x}}{c^2}\right)\sqrt{1-\frac{v'^2_{0x}}{c^2}-\frac{v'^2_y}{c^2}}}=\frac{qE'}{m_0}t'+k \tag{5.20}$$

an algebraic equation in v'_y with the solution:

$$v'_y=\frac{a(bt'+k)}{\sqrt{1+(bt'+k)^2}} \tag{5.21}$$

$$a=\sqrt{c^2-v'^2_{0x}}=\sqrt{c^2-V^2}=\frac{c}{\gamma(V)} \tag{5.22}$$

$$b=\frac{\left(1-\frac{v'^2_{0x}}{c^2}\right)}{c}\frac{qE'}{m_0}=\frac{c^2-V^2}{c^3}\frac{qE'}{m_0}=$$

$$=\frac{q}{m_0 c\gamma(V)}\left(E-\frac{B^2c^2}{E}\right) \tag{5.23}$$

Integrating (5.21), we obtain:

$$y'(t')=\frac{a}{b}\sqrt{1+(bt'+k)^2}+y'_0 \tag{5.24}$$

Now, we use the inverse Lorentz transforms:

$$x = \gamma(V)(x' + Vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(V)(t' + \frac{Vx'}{c^2}) \tag{5.25}$$

and we obtain the final equation of motion in frame S:

$$x = \gamma(V)x'_0 = \gamma^2(V)(x_0 - Vt)$$
$$y = \frac{a}{b}\sqrt{1+(bt'+k)^2} = \frac{a}{b}\sqrt{1+(b\gamma(V)t+k)^2}$$
$$z = const \tag{5.26}$$

This is the general solution for the case of a point charge moving in static uniform perpendicular **E** and **B** fields, with arbitrary initial conditions. The methodology described finds applications in particle accelerators, and particle separators [12,13].

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THE PARADOXICAL CASE OF FORCE-ACCELERATION TRANSFORMATION IN RELATIVITY

Synopsis

During one of my recent classes, an interesting question, never heard before, was posed by one of the students: "How come that the relativistic acceleration transformation transforms zero acceleration into zero acceleration but transforms zero force into non-zero force?" In the current note I will explain this apparent paradox. The proof is not trivial and, to my best knowledge, cannot be found in the literature.

1. The Strange Case of Misleading Newtonian Intuition

Let S and S' be two frames in inertial motion with respect to each other with the velocity V aligned with the x axis. A particle of rest mass m moves with arbitrary velocity $\mathbf{u} = (u_x, u_y, u_z)$ and arbitrary acceleration $\mathbf{a} = (a_x, a_y, a_z)$, as measured in frame S. The force applied on the particle in frame S is $\mathbf{F} = (F_x, F_y, F_z)$. In frame S', the acceleration experienced by the particle is [1-3]:

$$a_x^{'} = \frac{a_x}{\gamma^3(V)(1+\frac{u_x V}{c^2})^3}$$

$$\gamma(V) = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (1.1)$$

Throughout this paper will use the notation established by Tolman [1]. So, for $a_x = 0$ it follows that $a'_x = 0$. On the other hand, the force experienced by the particle in frame S' is [1-3]:

$$F'_x = F_x + \frac{u_y V}{c^2 + u_x V} F_y + \frac{u_z V}{c^2 + u_x V} F_z \tag{1.2}$$

So, for $F_x = 0$, $F'_x \neq 0$. This seems very puzzling, since our intuition would expect that null acceleration in one frame would result in null force in **that** frame but this is not the case for frame S'. In other words, our (Newtonian) intuition tells us that $a'_x = 0 \Rightarrow F'_x = 0$ but this is **not** the case. In order to understand what is really going on we need to remember that in relativity $\mathbf{F} \neq m\mathbf{a}$ but rather $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ where $\mathbf{p} = \gamma(u)m\mathbf{u}$. Therefore:

$$p_x = \gamma(u) m u_x \tag{1.3}$$

implies:

$$F_x = \frac{d}{dt}\gamma(u) m u_x =$$
$$= m\gamma^3(u)[(1 - \frac{u_y^2 + u_z^2}{c^2})a_x + \frac{u_x u_y}{c^2} a_y + \frac{u_x u_z}{c^2} a_z] \tag{1.4}$$

There are two possibilities:

a) $u_x = 0$ (so the mass is stationary in S at all times)

This means $a_x = 0$ and $p_x = 0$. $a_x = 0$ implies $a'_x = 0$ and $p_x = 0$ implies $F_x = 0$. According to (1.2) $F'_x \neq 0$. This is explained by the fact that:

$$F_x' = \frac{d}{dt'}\gamma(u')mu_x' =$$
$$= m\gamma^3(u')[(1-\frac{u_y'^2+u_z'^2}{c^2})a_x' + \frac{u_x'}{c^2}(u_y'a_y' + u_z'a_z')] \qquad (1.5)$$

We know that:

$$u_x' = V$$
$$u_y' = u_y / \gamma(V)$$
$$u_z' = u_z / \gamma(V) \qquad (1.6)$$

We also know that $a_y' \neq 0, a_z' \neq 0$. Inserting this into (1.5) provides the (non-trivial) reason why $F_x' \neq 0$. Of course, someone bent on arguing will claim that it is still possible for $F_x' = 0$ **provided** that:

$$F_y u_y = -F_z u_z \qquad (1.7)$$

but this a **particular** situation, **not** the general case. For general F_y, F_z:

$$F_x' = \frac{u_y V}{c^2}F_y + \frac{u_z V}{c^2}F_z \qquad (1.8)$$

or, expressed differently:

$$F_x' = m\gamma^3(u')\frac{u_x'}{c^2}(u_y'a_y' + u_z'a_z') \qquad (1.9)$$

One could argue again that $F_x' = 0$ if $u_y'a_y' = -u_z'a_z'$ but this is just a **particular** case, not the general one.

b) $u_x \neq 0$

In this case both $F_x \neq 0$ (by virtue of $p_x \neq 0$) and $F_x^{'} \neq 0$ (by virtue of either (1.2) or (1.5)). Expressions (1.4) and (1.5) demonstrate that the dependency of the force component aligned with one axis (x, in our example) on the accelerations aligned with the transverse axes (y and z, in our example) is an intrinsic effect, not an artifact of the coordinate transformation, as expression (1.2) would lead us to believe.

2. What About the Transverse Forces?

Given the symmetry of the problem it is sufficient to study only the case of the acceleration and force in one direction, for example the y-axis. The transformation formulas are [1,2]:

$$a_y^{'} = \frac{a_y}{\gamma^2(V)(1+\frac{u_x V}{c^2})^2} - \frac{a_x}{\gamma^2(V)(1+\frac{u_x V}{c^2})^3}\frac{u_y V}{c^2} \quad (2.1)$$

$$F_y^{'} = \frac{F_y}{\gamma^2(V)(1+\frac{u_x V}{c^2})} \quad (2.2)$$

In this case, we observe a similar disproof of our Newtonian intuition, $F_y = 0 \Rightarrow F_y^{'} = 0$ does not imply that $a_y = 0 \Rightarrow a_y^{'} = 0$. The reason is similar, $a_y^{'}$ is not only a function of a_y but also a function of a_x.

3. Conclusion

Starting from an apparent paradox that illustrates a discrepancy between the transformation of force and acceleration in special relativity, we have explained the fact that there is no paradox whatsoever. In order to understand what is really going on we need to remember that in relativity $\mathbf{F} \neq m\mathbf{a}$ but rather $\mathbf{F} = \frac{d\mathbf{p}}{dt}$

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LORENTZ COVARIANT FORMULATION FOR THE LAWS OF PHYSICS – PARTICLE AND ELASTIC BODY DYNAMICS

Synopsis

An equation representing a law of physics is said to be Lorentz covariant if it can be written in terms of Lorentz covariant quantities. The key property of Lorentz covariant equations is that if they hold in one inertial frame, then they hold in any inertial frame. This condition is a requirement according to the principle of relativity, that is, all non-gravitational laws must make the same predictions for identical experiments taking place at the same space-time event in two different inertial frames of reference. For example, covariant formulation of the laws of electromagnetism is well covered in literature [1-3]. Our paper is concerned with a more challenging chapter of physics, namely: the dynamics of charged particles subjected to the Lorentz force inside particle accelerators and the correct derivation of their equations of motion and, ultimately, of their trajectories. As we will see, the reason for the increased challenge is the presence of accelerated motion. While accelerated motion is part of standard special relativity, as reflected in numerous references [5-6], the covariant treatment of the laws of physics can be challenging, especially when it comes to dynamics [7-8]. The accelerated motion of particles in particle accelerators is such an example. We divided the presentation into three main sections: introduction for a simple case, the analysis of complex motion under electromagnetic Lorentz force in particle accelerators and we end with a covariant formulation for the Hooke law.

1. One-Dimensional Case for Particle Accelerated Motion under Constant Force

We will start with the case that prompted the idea of rewriting the laws of physics in covariant form:

$$\frac{dp}{dt} = F \quad \text{where} \quad F = m_0 \frac{d^2x}{dt^2} \tag{1.1}$$

where $\frac{dp}{dt}$ is the variation of momentum with respect to time under the application of force F. In one dimension, the relativistic momentum is simply $p = m_0\gamma(u)u$, so:

$$\frac{dp}{dt} = m_0\gamma^3(u)\frac{du}{dt} \tag{1.2}$$

where m_0 is the particle rest mass, u is its speed with respect to same inertial (arbitrary) frame of reference and $\gamma(u) = \frac{1}{\sqrt{1-(u/c)^2}}$. In another inertial frame of reference, S', moving with speed V along the common x-axis:

$$p' = m_0\gamma(u')u' \tag{1.3}$$

$$u' = \frac{u-V}{1-\frac{uV}{c^2}} \tag{1.4}$$

$$\gamma(u') = \gamma(u)\gamma(V)(1-\frac{uV}{c^2}) \tag{1.5}$$

So, it follows that:

$$\frac{dp'}{dt'} = m_0\gamma^3(u)\frac{du}{dt} \tag{1.6}$$

Interestingly, $\frac{dp'}{dt'} = \frac{dp}{dt}$. While p is frame variant, it turns out that for the one dimensional case $\frac{dp}{dt}$ is frame invariant. Another interesting fact is that in frame S Newton's law is $\frac{dp}{dt} = m_0\gamma^3(u)\frac{d^2x}{dt^2}$ then in frame S' the law has the same form: $\frac{dp'}{dt'} = m_0\gamma^3(u')\frac{d^2u'}{dt'^2}$.

This is an immediate consequence of the fact that:

$$\frac{d^2x'}{dt'^2} = \frac{du'}{dt'} = \frac{du'}{dt}\frac{dt}{dt'} =$$

$$= \frac{1}{\gamma(V)(1-\frac{uV}{c^2})}\frac{d}{dt}(\frac{u-V}{1-\frac{uV}{c^2}}) = \frac{\frac{du}{dt}}{\gamma^3(V)(1-\frac{uV}{c^2})^3} \tag{1.7}$$

so:

$$\gamma^3(u')\frac{d^2x'}{dt'^2} = \gamma^3(u)\frac{d^2x}{dt^2} \tag{1.8}$$

We can conclude that, at relativistic speeds, Newton law $\frac{dp}{dt} = m_0\frac{d^2x}{dt^2}$ must be replaced by $\frac{dp}{dt} = m_0\gamma^3(u)\frac{d^2x}{dt^2}$ in order to be covariant. That is, the Newtonian expression of force $F = m_0\frac{d^2x}{dt^2}$ needs to be replaced with its relativistic counterpart $F = m_0\gamma^3(u)\dfrac{d^2x}{dt^2}$. At low speeds $\gamma(u) \approx 1$ so the two expressions become indistinguishable but at relativistic speeds this is no longer the case. The above prompted Minkowski to reformulate Newton's law as:

$$\frac{d\mathbf{p}}{d\tau} = \mathbf{F_M} \tag{1.9}$$

where $d\tau = \gamma(u)dt$ is called "proper time interval" and $\mathbf{F_M} = \gamma(u)\mathbf{F}$ is called the "Minkowski force". So, $\dfrac{d\mathbf{p}}{d\tau} = \mathbf{F_M}$ is identical with $\dfrac{d\mathbf{p}}{dt} = \mathbf{F}$. The introduction of the Minkowski force gives us the idea to introduce yet a new construct [5]:

$$\tilde{\mathbf{F}} = (\mathbf{F}_M, (\mathbf{F}_M.\mathbf{u})) = \gamma(u)(\mathbf{F}, (\mathbf{F}.\mathbf{u})) \tag{1.10}$$

We also introduce the 4-vector $\tilde{\mathbf{p}} = (\mathbf{p}, E)$ where E is the total energy. The 4-vector introduced above is an extension to the 3-vector $\mathbf{p} = (p_x, p_y, p_z)$. The 4-vector $\tilde{\mathbf{p}}$ transforms exactly the same way [5] as (x, y, z, ct):

$$\mathbf{p} = \mathbf{p'} + \gamma(V)\frac{\mathbf{V}}{V^2}[(\mathbf{V}\bullet\mathbf{p'})\frac{\gamma(V)-1}{\gamma(V)} + \frac{V^2}{c^2}E']$$
$$E = \gamma(V)(E' + (\mathbf{V}\bullet\mathbf{p'})) \tag{1.11}$$

In the following section we will show that $\tilde{\mathbf{F}} = \gamma(u)(\mathbf{F}, (\mathbf{F}.\mathbf{u}))$ is also a 4-vector. In order to prove that, we will show that $\gamma(u)(\mathbf{F}.\mathbf{u})$ transforms like E and $\gamma(u)\mathbf{F}$ transforms like $\mathbf{p}$. This will complete the proof that $\tilde{\mathbf{F}} = \gamma(u)(\mathbf{F}, (\mathbf{F}.\mathbf{u}))$ is covariant.

2. The Covariance of $\tilde{\mathbf{F}} = \gamma(u)(\mathbf{F}, (\mathbf{F}.\mathbf{u}))$

In references [7-8] we have solved the problem of the complex motion of charged particles in the electromagnetic field of the particle accelerators by using the 3-vector formalism, in this paragraph we demonstrate how the problem is solved via the use of 4-vectors. The problem reduces to demonstrating the covariance of the motion law:

$$\frac{d\tilde{\mathbf{p}}}{d\tau} = \tilde{\mathbf{F}}$$

whereby $\tilde{\mathbf{F}} = (\gamma(v)q(\mathbf{vxB}), \gamma(v)q(\mathbf{vxB}).\mathbf{v}/c) = (\gamma(v)q(\mathbf{vxB}), 0)$.

We start by proving that

$\tilde{\mathbf{F}} = (\gamma(v)q(\mathbf{vxB}), \gamma(v)q(\mathbf{vxB}).\mathbf{v}/c) = (\gamma(v)q(\mathbf{vxB}), 0)$ is a 4-vector and we finish by showing the covariance of the laws of motion and by deriving the equations of motion.

We start with the Lorentz transforms between the frames S and S' moving at the relative speed V :

$$x' = \gamma(V)(x - Vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(V)(t - \frac{Vx}{c^2}) \tag{2.1}$$

The correspondent of the 3-vectors

$$\mathbf{B} = (0,0,B)$$
$$\mathbf{E} = (0,0,0) \tag{2.2}$$

are in S':

$$\mathbf{E}_{\perp}' = \gamma(V)(\mathbf{E} + \mathbf{VxB}) = \gamma(V)\mathbf{VxB}$$
$$\mathbf{B}_{\perp}' = \gamma(V)(\mathbf{B} - \frac{\mathbf{VxE}}{c^2}) = \gamma(V)\mathbf{B} \tag{2.3}$$

in the direction perpendicular to the velocity **V=**(V,0,0). For the direction parallel with **V**, the transformed values are:

$$\mathbf{E}_{\parallel}' = \mathbf{E}_{\parallel} = 0$$
$$\mathbf{B}_{\parallel}' = \mathbf{B}_{\parallel} = 0 \tag{2.4}$$

$$\mathbf{F} = q(\mathbf{vxB}) = q\begin{bmatrix} v_y B_z \\ -v_x B_z \\ 0 \end{bmatrix} = qB\begin{bmatrix} v_y \\ -v_x \\ 0 \end{bmatrix}$$
$$\tilde{\mathbf{F}} = \gamma(v)qB\begin{bmatrix} v_y \\ -v_x \\ 0 \\ 0 \end{bmatrix} \tag{2.5}$$

Thus, the Lorentz force in frame S' is obtained through transforming **F**:

$$\mathbf{F}' = \begin{bmatrix} F_x - \dfrac{v_y V / c^2}{1 - \dfrac{v_x V}{c^2}} F_y - \dfrac{v_z V / c^2}{1 - \dfrac{v_x V}{c^2}} F_z \\ \dfrac{\gamma^{-1}(V)}{1 - \dfrac{v_x V}{c^2}} F_y \\ \dfrac{\gamma^{-1}(V)}{1 - \dfrac{v_x V}{c^2}} F_z \end{bmatrix} = qB\begin{bmatrix} \dfrac{v_y}{1 - \dfrac{v_x V}{c^2}} \\ \dfrac{-v_x \gamma^{-1}(V)}{1 - \dfrac{v_x V}{c^2}} \\ 0 \end{bmatrix} \tag{2.6}$$

Given the above, we can demonstrate that

$\tilde{\mathbf{F}} = (\gamma(v)q(\mathbf{vxB}), \gamma(v)q(\mathbf{vxB}).\mathbf{v}/c) = (\gamma(v)q(\mathbf{vxB}), 0)$ is indeed a 4-vector. In frame S':

$$\tilde{\mathbf{F}}' = \gamma(v')(\mathbf{F}', \mathbf{F}'.\mathbf{v}'/c) = \gamma(v')qB \begin{bmatrix} \dfrac{v_y}{1-\frac{v_x V}{c^2}} \\ \dfrac{-v_x\gamma^{-1}(V)}{1-\frac{v_x V}{c^2}} \\ 0 \\ \dfrac{(v_x - V)v_y - v_x v_y \gamma^{-2}(V)}{c(1-\frac{v_x V}{c^2})^2} \end{bmatrix} =$$

$$0qB\gamma(v) \begin{bmatrix} v_y\gamma(V) \\ -v_x \\ 0 \\ -\gamma(V)\dfrac{Vv_y}{c} \end{bmatrix}$$

(2.7)

Thus, on a component by component basis:

$$\gamma(V)(\tilde{F}_x - \frac{V}{c}\tilde{F}_w) = \gamma(V)\tilde{F}_x = \gamma(V)\gamma(v)qBv_y = \tilde{F}_x'$$

$$\tilde{F}_y' = \tilde{F}_y$$

$$\tilde{F}_z' = \tilde{F}_z = 0$$

$$\gamma(V)(\tilde{F}_w - \frac{V}{c}\tilde{F}_x) = -\gamma(V)\frac{V}{c}\tilde{F}_x = -\gamma(V)\frac{V}{c}qB\gamma(v)v_y = \tilde{F}_w' \quad (2.8)$$

so, indeed $\tilde{\mathbf{F}} = (\gamma(v)q(\mathbf{vxB}), 0)$ is a 4-vector.

We are now ready to demonstrate the covariance of the motion laws. In frame S':

$$\mathbf{p'} = m_0\gamma(v')\mathbf{v'} \tag{2.9}$$

$$p_x' = \gamma(v)\gamma(V)(1-\frac{v_xV}{c^2})m_0v_x' = \gamma(v)\gamma(V)m_0(v_x - V) \tag{2.10}$$

$$\frac{dp_x'}{d\tau} = \gamma(v)\gamma(V)m_0 \frac{\frac{dv_x}{dt}\sqrt{1-\frac{v^2}{c^2}} + \frac{v_x - V}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^2}{c^2}\frac{dv}{dt}}{1-\frac{v^2}{c^2}} \tag{2.11}$$

This is a very ugly expression, luckily we are “rescued” by the fact that, for the motion in an uniform B field, $\frac{dv}{dt} = 0$. Indeed, the time derivative of total energy W is:

$$\frac{dW}{dt} = m_0c^2\frac{d\gamma}{dt} \tag{2.12}$$

while, on the other hand:

$$\frac{dW}{dt} = \mathbf{v.F} = q\mathbf{v.(vxB)} = 0 \tag{2.13}$$

so $\gamma(v) = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$ =constant.

Thus, the expression (2.11) reduces to a much nicer one:

$$\frac{dp_x'}{d\tau} = \gamma^2(v)\gamma(V)m_0\frac{dv_x}{dt} \tag{2.14}$$

Since $\dfrac{dp_x'}{d\tau} = F_{M,x}$ it follows that:

$$\gamma^2(v)\gamma(V)m_0\frac{dv_x}{dt} = \gamma(v)\gamma(V)qBv_y \tag{2.15}$$

That is:

$$\gamma(v)m_0\frac{dv_x}{dt} = qBv_y \tag{2.16}$$

that is, exactly the equation of motion in frame S.

For the y-component:

$$p_y' = \gamma(v)\gamma(V)(1-\frac{v_xV}{c^2})m_0v_y' = \gamma(v)m_0v_y \tag{2.17}$$

$\dfrac{dp_y'}{d\tau} = \gamma(v)m_0\dfrac{d(v_y\gamma(v))}{dt} = \gamma^2(v)m_0\dfrac{dv_y}{dt}$ due to the fact that $\gamma(v) = \text{constant}$. Thus, the equation of motion in the y direction is:

$$\gamma^2(v)m_0\frac{dv_y}{dt} = -\gamma(v)qBv_x \tag{2.18}$$

After further simplification, we recover the equation of motion in frame S:

$$\gamma(v)m_0 \frac{dv_y}{dt} = -qBv_x \tag{2.19}$$

Conclusion: the equation of motion in frame S' is identical to the equation of motion in frame S, the formulation $\frac{d\tilde{\mathbf{p}}}{d\tau} = \tilde{\mathbf{F}}$ for $\tilde{\mathbf{F}} = (\gamma(v)q(\mathbf{vxB}), 0)$ is covariant and the equations of motion are exactly the same as the ones discovered in [7-8].

3. The Motion of Charged Particles Subjected Only to Electrostatic Force F=qE

In order to examine the case of the electrostatic force we can assume that the electrostatic field is aligned with the x-axis without any loss of generality:

$$\mathbf{E} = (E, 0, 0) \tag{3.1}$$

Thus, the three-force is:

$$\mathbf{F} = (qE, 0, 0) \tag{3.2}$$

In frame S', moving at speed V with respect to S:

$$\begin{aligned} E'_x &= E_x = E \\ E'_y &= \gamma(V)E_y = 0 \\ E'_z &= \gamma(V)E_z = 0 \end{aligned} \tag{3.3}$$

Therefore:

$$\mathbf{F}' = \begin{bmatrix} qE'_x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} qE \\ 0 \\ 0 \end{bmatrix} \tag{3.4}$$

We can prove now that

$\tilde{\mathbf{F}} = \gamma(v)(q\mathbf{E}, q\mathbf{E.v}/c) = \gamma(v)(qE, 0, 0, qEv_x / c)$ is a 4-vector. In frame S':

$$\tilde{\mathbf{F}}' = \gamma(v')(\mathbf{F}', \mathbf{F}'.\mathbf{v}'/c) =$$

$$= qE\gamma(v)\gamma(V)(1-\frac{v_x V}{c^2})\begin{bmatrix} 1 \\ 0 \\ 0 \\ \dfrac{v_x - V}{c(1-\frac{v_x V}{c^2})} \end{bmatrix} \tag{3.5}$$

$$\gamma(V)(\tilde{F}_x - \frac{V}{c}\tilde{F}_w) = \gamma(V)(\gamma(v)qE - \frac{V}{c}\gamma(v)qEv_x / c) =$$

$$= \gamma(V)\gamma(v)qE(1-\frac{Vv_x}{c^2}) = \tilde{F}_x'$$

$$\tilde{F}_y' = \tilde{F}_y = 0$$

$$\tilde{F}_z' = \tilde{F}_z = 0$$

$$\gamma(V)(\tilde{F}_w - \frac{V}{c}\tilde{F}_x) = \gamma(V)(\gamma(v)qEv_x / c - \frac{V}{c}qE\gamma(v)) =$$

$$= \gamma(V)\gamma(v)qE\frac{v_x - V}{c} = \tilde{F}_w' \tag{3.6}$$

We are now ready to prove the covariance of the laws of motion.

$$\frac{dp_x'}{d\tau} = \gamma(v)\gamma(V)m_0(\gamma(v)\frac{dv_x}{dt} + \frac{v(v_x - V)}{c^2}\frac{dv}{dt}\gamma^3(v)) \tag{3.7}$$

Since $\mathbf{v} = (v_x, 0, 0)$ it follows that:

$$\frac{dp_x'}{d\tau} = \gamma^2(v)\gamma(V)m_0 \frac{dv_x}{dt}(1 + \frac{v(v_x - V)}{c^2}\gamma^2(v)) =$$
$$= \gamma^4(v)\gamma(V)m_0 \frac{dv_x}{dt}(1 - \frac{v_x V}{c^2}) \tag{3.8}$$

Since $\frac{dp_x'}{d\tau} = \mathbf{F}'_M$ it follows that:

$$\gamma(v)\gamma(V)(1 - \frac{v_x V}{c^2})qE = \gamma^4(v)\gamma(V)m_0 \frac{dv_x}{dt}(1 - \frac{v_x V}{c^2}) \tag{3.9}$$

resulting into:

$$qE = \gamma^3(v)m_0 \frac{dv_x}{dt} = \gamma^3(v)m_0 \frac{dv}{dt} \tag{3.10}$$

In frame S:

$$p = m_0\gamma(v)v \tag{3.11}$$

$$\frac{dp}{d\tau} = \frac{dt}{d\tau}\frac{dp}{dt} = \gamma(v)\frac{d}{dt}(\gamma(v)m_0 v) = \gamma^4(v)m_0 \frac{dv}{dt} \tag{3.12}$$

while the Minkowski force is:

$$F_M = \gamma(v)qE \tag{3.13}$$

so, in S:

$$qE = \gamma^3(v)m_0 \frac{dv}{dt} \tag{3.14}$$

Thus, the equation has the same exact form in both S and S', that is, the motion law is Lorentz covariant, the formulation $\frac{d\tilde{\mathbf{p}}}{d\tau}=\tilde{\mathbf{F}}$ for $\tilde{\mathbf{F}}=(\gamma(v)q\mathbf{E},\gamma(v)q\mathbf{E}.\mathbf{v}/c)$ is covariant and the equations of motion are exactly the same as the ones discovered in [7-8].

4. The Motion of Charged Particles Under the General Lorentz Force F=q(vxB+E). The General Case of Particle Accelerators.

If the conditions:

$$\mathbf{B}=(0,0,B)$$
$$\mathbf{E}=(E,0,0) \tag{4.1}$$

or

$$\mathbf{B}=(0,0,B)$$
$$\mathbf{E}=(0,E,0) \tag{4.2}$$

are fulfilled the case reduces at the superposition of the forces q**vxB** and q**E**, where **E** and **B** are orthogonal, so the general Lorentz force maintains covariance. This is obvious since the sum of two four vectors (q**vxB** and q**E** in the above case) is also a four-vector.

For the case of the more general situation:

$$\mathbf{B}=(0,0,B)$$
$$\mathbf{E}=(E_x,E_y,E_z) \tag{4.3}$$

whereby **E** and **B** are no longer orthogonal we make good use of the fact that this particular case can be reduced to (4.1)-(4.2), so the covariance holds.

Arriving at this point we can produce an alternative proof by showing that the three-vector **density** of the Lorentz force $\mathbf{f}_L = \rho(\mathbf{E} + \mathbf{v x B}) = \rho\mathbf{E} + \mathbf{j}_{\mathbf{conv}}\mathbf{xB}$ forms a four-vector $(\mathbf{f}_{\mathrm{L}}, f_4)$ where:

$$f_j = \sum_{k=1}^{4} F_{jk} s_k, \; j = 1,2,3,4 \tag{4.4}$$

In (4.4) $(\mathbf{j}_{\mathbf{conv}}, s_4) = (j_x, j_y, j_z, ic\rho)$ is the four-vector convection current density, and

$$F_{jk} = \begin{bmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -B_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_x & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{bmatrix} \tag{4.5}$$

is the electromagnetic field tensor. When passing from a frame S to another frame S' moving with relative speed V the tensor components transform as follows:

$$\begin{aligned} &B_x = B_x', E_x = E_x' \\ &B_y = \gamma(V)(B_y' - \frac{V}{c^2} E_z') \\ &B_z = \gamma(V)(B_z' + \frac{V}{c^2} E_y') \\ &E_y = \gamma(V)(E_y' + VB_z') \\ &E_z = \gamma(V)(E_z' - VB_y') \end{aligned} \tag{4.6}$$

and the four-vector components transform as:

$$\begin{aligned} j_x &= \gamma(V)(j_x' + \rho' V) \\ j_y &= j_y' \\ j_z &= j_z' \\ \rho &= \gamma(V)(\rho' + \frac{V}{c^2} j_x') \end{aligned} \tag{4.7}$$

Inserting (4.6) and (4.7) in (4.4) we obtain:

$$\begin{aligned} f_1 &= \gamma(V)(-B_y' j_z' + B_z' j_y' + \rho' E_x' + \sum E_x' j_x') \\ f_2 &= -B_z' j_x' + B_x' j_z' + \rho' E_y' = f_2' \\ f_3 &= -B_x' j_y' + B_y' j_x' + \rho' E_z' = f_3' \\ f_4 &= \frac{i}{c} \sum E_x j_x = \\ &= \frac{i}{c} \gamma(V)(\sum E_x' j_x' + V(-B_y' j_z' + B_z' j_y' + \rho' E_x')) \end{aligned} \tag{4.8}$$

On the other hand, according to (4.4), in frame S':

$$\begin{aligned} f_1' &= -B_y' j_z' + B_z' j_y' + \rho' E_x' \\ f_4' &= \frac{i}{c} \sum E_x' j_x' \end{aligned} \tag{4.9}$$

Introducing the notation $\varphi_4' = \sum E_x' j_x'$ we obtain immediately:

$$\begin{aligned} \gamma(V)(f_1' + \frac{V}{c^2} \varphi_4') &= f_1 \\ \gamma(V)(\varphi_4' + V f_1') &= f_4 \end{aligned} \tag{4.10}$$

so $(\mathbf{f}_L, f_4)$ is indeed a four-vector, thus, we have proven the covariance of the formulation $\frac{d\tilde{\mathbf{p}}}{d\tau} = \tilde{\mathbf{F}}$ for the most general case. The 4-vector formalism produces the same results as the 3-vector formalism described in references [7-8] with the advantage of being easier to use by requiring fewer steps in the derivation of the equations of motion.

5. Covariant Formulation of Hooke's Law

In 1981, Gron has shown [4] that it isn't always possible to use the Newtonian expressions of force in order to derive covariant formulations of the laws of physics in relativity. For a detailed discussion of the Gron paper, see Appendix A. The correct way is to introduce the concept of motion under stress; under the effects of the stress force $F_\mu{}'$ the end of the object will move with the velocity $\mathbf{v} = (v_x, 0, 0)$ (in the one-dimension case) as measured in the proper frame J'. We revert to our original convention of labeling the proper frame with S and the observer frame with S' while their relative speed is V (aligned with the x-axis for the one-dimension case). Assuming the above definition of motion under stress we introduce the four-vector stress differently:

$$L_\mu = (\gamma(v)l_x, l_y, l_z, \gamma(v)l_x \frac{v}{c}) \tag{5.1}$$

In the ideal case of a one dimension problem there is no deformation in the directions orthogonal to x, so we can assume that $(l_y, l_z) = (0,0)$. Therefore, in the proper frame S, the Hooke force takes the form:

$$F_\mu = k(\gamma(v)l_x, 0, 0, \gamma(v)l_x \frac{v}{c}) \tag{5.2}$$

that is, there are no force components in the y and z. The equation $\frac{dp}{d\tau} = F_\mu$ reduces to:

$$kl_x = \gamma^3(v) m_0 \frac{dv}{dt} \tag{5.3}$$

In the observer frame S', the Hooke force is defined as:

$$F_\mu' = k'(\gamma(v')l_x', 0, 0, \gamma(v')l_x' \frac{v'}{c}) \tag{5.4}$$

In the proper frame S the factor k is connected to the Young modulus E, the original cross-sectional area A_0 is through which the force is applied; and the original length of the object, L_0.

$$k = \frac{EA_0}{L_0} \tag{5.5}$$

where the Young modulus, E is defined as:

$$E = \frac{fL_0}{A_0 \Delta L} \tag{5.6}$$

The Young modulus is not frame invariant. Indeed, for a force applied along the x-axis, given that $(f_y, f_z) = (0,0)$:

$$\begin{aligned} f_x' &= f_x \\ \Delta L_x' &= \Delta L_x / \gamma(V) \end{aligned} \tag{5.7}$$

so,

$$E' = \gamma(V)E \tag{5.8}$$

meaning that

$$k' = \gamma(V)k \tag{5.9}$$

From (5.4) and (5.9) we obtain immediately:

$$\gamma(v')(k'l_x') = \gamma(v)\gamma(V)(1-\frac{vV}{c^2})(kl_x) \tag{5.10}$$

Combining (5.10) with (3.7) we obtain immediately:

$$kl_x = \gamma^3(v)m_0\frac{dv}{dt} \tag{5.11}$$

meaning that $\frac{dp'}{d\tau} = F_\mu'$, so, the formulation given by (5.1), (5.2) is indeed covariant.

By comparing (5.2) with the Minkowski definition [5-6] of four-force, $F_\mu = \gamma(v)(f_x, f_y, f_z, f_x\frac{v}{c})$ we obtain the corrected components of the four-force:

$$f_x = kl_x$$
$$f_y = 0$$
$$f_z = 0 \tag{5.12}$$

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Appendix A

In his paper, Gron starts by defining the four-strain in the proper frame J' as:

$$L_\mu{}' = (l_x{}', l_y{}', l_z{}', 0) = (x' - X', y' - Y', z' - Z', 0) \quad \text{(A.1)}$$

where $X'_\mu (\mu = 1,2,3,4)$ is the equilibrium position of the front end of the body when the body is stress-free. The position vector $x'_\mu (\mu = 1,2,3,4)$ represents the position of the front end of the body relative to its back end when the body is stressed. In a frame J moving with the relative speed u with respect to J', the stress is:

$$L_\mu = (\gamma(u) l_x{}', l_y{}', l_z{}', \gamma(u) l_x{}' \frac{u}{c}) \quad \text{(A.2)}$$

Since length measurement is defined as the process of marking both ends of the measured object simultaneously in the frame of the observer (J) it follows that:

$$\begin{aligned} l_x &= \gamma^{-1}(u) l_x{}' \\ l_y &= l_y{}' \\ l_z &= l_z{}' \end{aligned} \quad \text{(A.3)}$$

Substituting the above in the expression for the stress in frame J, L_μ, Gron obtains:

$$L_\mu = (\gamma^2(u)l_x, l_y, l_z, \gamma^2(u)l_x \frac{u}{c}) \tag{A.4}$$

If F_μ is the four-force acting on a body that induces the strain L_μ in the said body, then, the expression of the Hooke force is:

$$F_\mu = kL_\mu \tag{A.5}$$

At this point in his paper, Gron makes the mistake of equating F_μ with $\gamma(u)(f_x, f_y, f_z, f_x \frac{u}{c})$. This error results into a formulation of the Hooke law that can be proven to be non-covariant in just a few simple computations by using the mechanism developed in paragraphs 1-4 above. The relative speed between frames, u, should not occur in the definition of the four-force, see for example references [5-8], the definition is a function of the particle speed with respect to an inertially commoving frame, not of the relative speed between two frames of reference (see also paragraph 1 in the current paper).

Coordinate Time Hyperbolic Motion Treatment for Bell's Spaceship Experiment

Synopsis

In Bell's "spaceship" experiment [1], two spaceships that are initially at rest in some common inertial reference frame, are connected by a taut string. At time zero in the common inertial frame, both spaceships start accelerating, with a constant *proper acceleration* **a** as measured by an on-board accelerometer. Question: **when** does the string break as expressed as a function of coordinate time? For simplicity, throughout the paper, all objects (string, rockets) are considered as being Born-rigid, thus neglecting the very minor effects on the length of the objects during the accelerated motion. We have provided earlier [9] a treatment expressed in terms of proper time τ, in the current chapter we will show a different approach in terms of coordinate time, t .

Analysis in using the equations of hyperbolic motion as function of coordinate time

Bell's paradox is easily understood if we start by looking at the situation in the instantaneously co-moving inertial rest frame (ICIRF) of the rear spaceship at a given instant in time. In that frame at that time the rear spaceship is at rest, and at the same time the front spaceship has nonzero velocity moving forward. Now do this for the ICIRF of any small portion of the string, and each spaceship is moving away from the portion in question. So from the standpoint of the string, each small region of the string must stretch and eventually break when its elastic limit is exceeded. In [9] we have shown a treatment expressed in terms of proper time. In the current paper we will show a different treatment, expressed in terms of coordinate time. The two treatments are different both physically and mathematically.

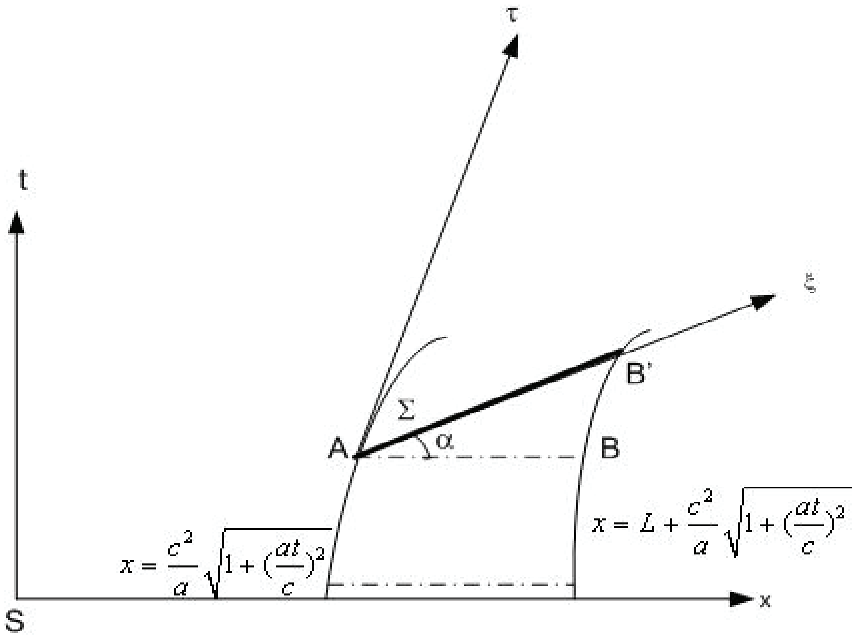

Fig. 1. Minkowski Diagram for Bell's Spaceship Experiment

The two rockets A and B describe the hyperbolic trajectories (see fig.1):

$$x_{leading} = L + \frac{c^2}{a}\sqrt{1+(\frac{at}{c})^2}$$
$$x_{trailing} = \frac{c^2}{a}\sqrt{1+(\frac{at}{c})^2} \qquad (1)$$

Let's consider the instantaneous co-moving frame Σ (an inertial frame with the origin attached to the trailing rocket) at an arbitrary coordinate time t_A. The ξ axis makes an angle α with the x axis where:

$$\tan(\alpha) = \frac{v(t_A)}{c} = \frac{\frac{at_A}{c}}{\sqrt{1+(\frac{at_A}{c})^2}} \tag{2}$$

The ξ axis represents a line of simultaneity in frame Σ. If we want to determine the distance between the two rockets at coordinate time t_A as measured in Σ, we have to intersect the ξ axis with the trajectory of the leading rocket, that is, we have to solve for t the system of equations:

$$(x - x_A) * \tan(\alpha) = c(t - t_A)$$
$$x = L + \frac{c^2}{a}\sqrt{1+(\frac{at}{c})^2} \tag{3}$$

where:

$$x_A = \frac{c^2}{a}\sqrt{1+(\frac{at_A}{c})^2} \tag{4}$$

The system reduces to a simple equation degree 2 in t that has a positive root $t_{B'} > t_A$:

$$\sqrt{1+(\frac{at}{c})^2} = \frac{t}{t_A}\sqrt{1+(\frac{at_A}{c})^2} - \frac{La}{c^2} \tag{5}$$

with the solution:

$$t_{B'} = t_A \frac{La}{c^2}(\sqrt{1+(\frac{at_A}{c})^2} + \sqrt{(\frac{c^2}{La})^2 + (\frac{at_A}{c})^2}) \tag{6}$$

From (6) it is obvious that $t_{B'} > t_A$. Once we find $t_{B'}$ we can easily find the coordinate (in frame S) of the leading rocket:

$$x_{B'} = L + \frac{c^2}{a}\sqrt{1+(\frac{at_{B'}}{c})^2} \quad (7)$$

We can now apply a Lorentz transform between the launcher frame S and the co-moving frame Σ in order to get the distance between the rockets calculated in frame Σ:

$$L' = \xi_{B'} = \gamma(v(t_A))(x_{B'} - vt_{B'}) \quad (8)$$

where:

$$\gamma(v(t_A)) = \sqrt{1+(\frac{at_A}{c})^2} \quad (9)$$

and:

$$x_{B'} - vt_{B'} = L + \frac{c^2}{a}\sqrt{1+(\frac{at_{B'}}{c})^2} - \frac{at_A t_{B'}}{\sqrt{1+(\frac{at_A}{c})^2}} \quad (10)$$

We can show easily that the function:

$$f(t_{B'}) = \frac{c^2}{a}\sqrt{1+(\frac{at_{B'}}{c})^2} - \frac{at_A t_{B'}}{\sqrt{1+(\frac{at_A}{c})^2}} > 0$$

(11)

since:

$$\frac{df}{dt_{B'}} = a(\frac{t_{B'}}{\sqrt{1+(\frac{at_{B'}}{c})^2}} - \frac{t_A}{\sqrt{1+(\frac{at_A}{c})^2}}) > 0 \quad (12)$$

for any $t_{B'} > t_A$. This is due to the fact that the function

$$g(t) = \frac{t}{\sqrt{1+(\frac{at}{c})^2}}$$

is monotonically increasing. Thus, we can conclude that the distance between the rockets in the co-moving frame is larger than their distance L as measured in the launcher frame:

$$\xi_{B'} = \sqrt{1+(\frac{at_A}{c})^2}\,(L + \frac{c^2}{a}\sqrt{1+(\frac{at_{B'}}{c})^2} - \frac{at_A t_{B'}}{\sqrt{1+(\frac{at_A}{c})^2}}) > L \tag{13}$$

Since t_A has been chosen arbitrarily and since $t_{B'}$ is a function of T(t) as described by equation (5), we can also write the distance between the two rockets as a function of the coordinate time t, as measured in the co-moving frame Σ, as a more general formula:

$$L'(t) = \sqrt{1+(\frac{at}{c})^2}\,(L + \frac{c^2}{a}\sqrt{1+(\frac{aT(t)}{c})^2} - \frac{atT(t)}{\sqrt{1+(\frac{at}{c})^2}}) \tag{14}$$

The distance between rockets increases with the coordinate time t so the string will get stretched until it breaks.

When does the string break?

The calculation of the time when the string breaks requires that we take into consideration that each infinitesimal element of the string moves at a different speed as viewed from frame Σ, as we have shown in the previous section. So, each infinitesimal element will stretch by a different amount. The formalism built in the prior section will be very useful in calculating the amount of stretching.

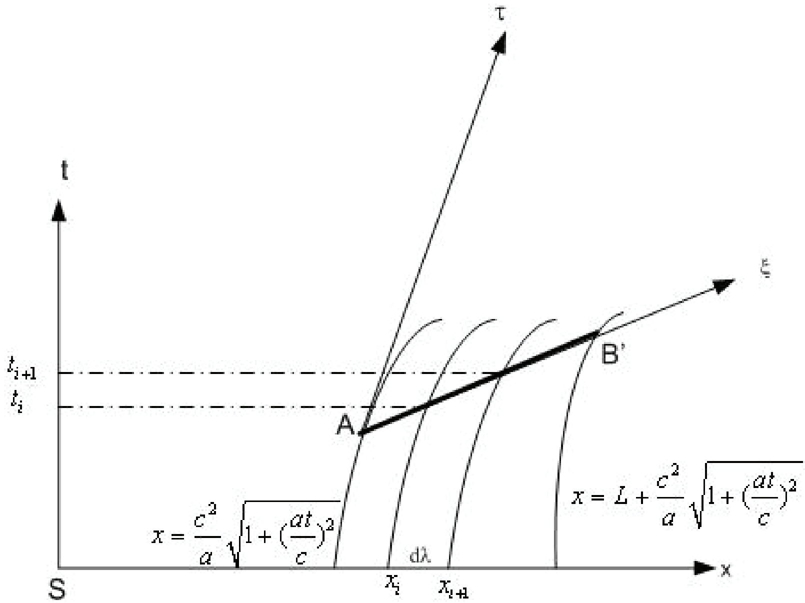

Fig. 2. Stretch Calculation

Consider an infinitesimal element of length dl as viewed from frame S, its endpoints will describe the hyperbolas:

$$x_i = \lambda + \frac{c^2}{a}\sqrt{1+(\frac{at}{c})^2}$$

$$x_{i+1} = \lambda + d\lambda + \frac{c^2}{a}\sqrt{1+(\frac{at}{c})^2} \qquad (15)$$

According to (6) the ξ axis intersects the two hyperbolas at:

$$t_i = \lambda p$$

$$t_{i+1} = (\lambda + d\lambda)p$$

$$p = \frac{at_A}{c^2}(\sqrt{1+(\frac{at_A}{c})^2} + \sqrt{(\frac{c^2}{La})^2 + (\frac{at_A}{c})^2}) \qquad (16)$$

Substituting (16) into (15) we obtain:

$$x_{i+1} - x_i = d\lambda + \frac{c^2}{a}\left(\sqrt{1 + \left(\frac{at_{i+1}}{c}\right)^2} - \sqrt{1 + \left(\frac{at_i}{c}\right)^2}\right) \tag{17}$$

So, the infinitesimal element is stretched by the amount:

$$\frac{c^2}{a}\left(\sqrt{1 + \left(\frac{at_{i+1}}{c}\right)^2} - \sqrt{1 + \left(\frac{at_i}{c}\right)^2}\right) \approx \frac{a}{2}\left({t_{i+1}}^2 - {t_i}^2\right) \tag{18}$$

Substituting (16) into (18) we obtain:

$$\frac{a}{2}\left({t_{i+1}}^2 - {t_i}^2\right) = \frac{ap^2}{2}\left((\lambda + d\lambda)^2 - \lambda^2\right) \approx ap^2 \lambda d\lambda \tag{19}$$

The total stretch is obtained by integrating (19):

$$\int_0^L ap^2 \lambda d\lambda = \frac{ap^2 L^2}{2} \tag{20}$$

Since t_A has been taken arbitrarily it means that, in general, p is a function of t:

$$p = \frac{at}{c^2}\left(\sqrt{1 + \left(\frac{at}{c}\right)^2} + \sqrt{\left(\frac{c^2}{La}\right)^2 + \left(\frac{at}{c}\right)^2}\right) \tag{21}$$

Material science teaches us that for any string of given length L, cross-section area A and given tensile strength, there is a limit, δL, beyond which it will break if stretched. The time of string breaking will be given by solving the equation (22) function of coordinate time, t:

$$\frac{ap^2 L^2}{2} = \delta L \tag{22}$$

We have shown the realistic computations of the string stretching as a function of the coordinate time.

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ROTATING FRAMES EXPERIMENT FOR DIRECT DETERMINATION OF RELATIVISTIC LENGTH CONTRACTION

Synopsis

The subject of length contraction appears very early in textbooks [1] as well as in various forms in papers dedicated to teaching relativity [2-6]. Unfortunately, no experiment has been possible to date due to the fact that the predicted effects are very small In the current paper we extend the treatment to the case of the arbitrary orientation between two inertial frames. We use the information relative to the way length contraction occurs in rotating frames combined with the discovery that light speed is locally isotropic in order to construct the special relativity theory for the Michelson-Morley experiment executed in uniformly rotating frames and we demonstrate how the experiment can be viewed as an **indirect** proof of length contraction. By contrast, the immediately following section describes the experimental setup for a **direct** measurement of length contraction, a relativistic property that has eluded so far experimental verification. We show how modern technology, in combination with the theory advanced by our paper, make this experiment feasible.

1. Length contraction in arbitrary-oriented inertial frames

The vector approach allows for generalizing the length contraction formulas to the case when the axis of the inertial frames S and S' in relative motion with the velocity **v** with respect to each other (see fig.1).

$$\mathbf{r}' = \mathbf{r} + \mathbf{v}\left(\frac{\gamma - 1}{v^2}\mathbf{r}.\mathbf{v} - \gamma t\right) \tag{1.1}$$

$$t' = \gamma\left(t - \frac{\mathbf{r}.\mathbf{v}}{c^2}\right) \tag{1.2}$$

where $\mathbf{r}'$ is the positional vector of an arbitrary point in S'.

In frame S' we need to mark both ends of the rod simultaneously in order to perform the length measurement, so $\Delta t' = 0$.Thus:

$$\Delta t = \frac{\Delta \mathbf{r}.\mathbf{v}}{c^2} \tag{1.3}$$

$$\Delta \mathbf{r}' = \Delta \mathbf{r} + \mathbf{v}\left(\frac{\gamma - 1}{v^2}\Delta \mathbf{r}.\mathbf{v} - \gamma \Delta t\right) = \Delta \mathbf{r} + \mathbf{v}\left(\frac{\gamma - 1}{v^2} - \frac{\gamma}{c^2}\right)\Delta \mathbf{r}.\mathbf{v} \tag{1.4}$$

where $\Delta \mathbf{r}'$ is a vector connecting two arbitrary points in S'.

$$\Delta \mathbf{r}' = \Delta \mathbf{r} + \mathbf{v}\frac{\gamma^{-1} - 1}{v^2}\Delta \mathbf{r}.\mathbf{v} \tag{1.5}$$

We can decompose $\Delta \mathbf{r}$ into $\Delta \mathbf{r}_{\|} \parallel \mathbf{v}$ and $\Delta \mathbf{r}_{\perp} \perp \mathbf{v}$:

$$\Delta \mathbf{r}' = \Delta \mathbf{r}_{\|} + \Delta \mathbf{r}_{\perp} + \mathbf{v}\frac{\gamma^{-1} - 1}{v^2} v \Delta r_{\|} =$$

$$= \Delta \mathbf{r}_{\|} + \Delta \mathbf{r}_{\perp} + (\gamma^{-1} - 1)\frac{\mathbf{v}}{v}\Delta r_{\|} = \Delta \mathbf{r}_{\perp} + \frac{\Delta \mathbf{r}_{\|}}{\gamma} \tag{1.6}$$

Thus:

$$\Delta r'^2 = \Delta r_{\perp}{}^2 + \frac{\Delta r_{\|}^2}{\gamma^2} \tag{1.7}$$

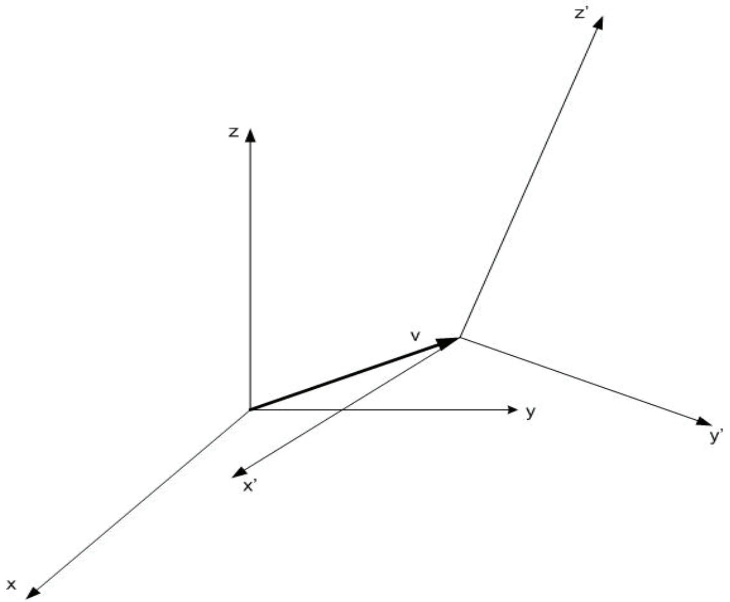

Fig. 1 Arbitrary frames with misaligned axes

For an arbitrary orientation of a rod of proper length L located in the frame S, $r_{\|} = L\cos\alpha, r_{\perp} = L\sin\alpha$, so (1.7) can be rewritten as

$$L' = L\sqrt{1-(\frac{v\cos\alpha}{c})^2} \tag{1.8}$$

To conclude the section, we compute the time dilation as simply:

$$\Delta t' = \gamma(\Delta t - \frac{\Delta\mathbf{r}.\mathbf{v}}{c^2}) = \gamma(\Delta t - 0) = \gamma\Delta t \tag{1.9}$$

Could it be that we can derive similarly elegant formulas for the case of rotating frames?

2. Length contraction in uniformly rotating frames

In this paragraph, we become even more ambitious by considering that the frame of reference S' is rotating with a constant angular speed describing a circle with the origin coincident with the origin of the inertial frame S. The coordinate transformations between S and S' are given in[7,8].

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\gamma\omega t' & \gamma\sin\gamma\omega t' \\ -\sin\gamma\omega t' & \gamma\cos\gamma\omega t' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + R\begin{bmatrix} \cos\gamma\omega t' - 1 \\ -\sin\gamma\omega t' \end{bmatrix} \quad (2.1)$$

$$z = z' \quad (2.2)$$

$$t = \gamma\left(t' - \frac{\omega R y'}{c^2}\right) \quad (2.3)$$

where $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{\omega^2 R^2}{c^2}}}$

Consider a rod of length L' situated at x'=y'=0, in frame S'. From the perspective of an observer in the inertial frame S we need to mark both ends of the rod simultaneously, so $dt = 0$. For simplicity, the measurement is executed at $t = 0$. This implies immediately that $t' = 0$. Thus:

$$dx = dx'\cos\gamma\omega t' - \gamma\omega x'\sin\gamma\omega t' dt' + \gamma dy'\sin\gamma\omega t' +$$
$$+ y'\gamma^2\omega\cos\gamma\omega t' dt' - R\gamma\omega\sin\gamma\omega t' dt'$$
$$dy = -dx'\sin\gamma\omega t' - \gamma\omega x' t'\cos\gamma\omega t' dt' +$$
$$+ \gamma dy'\cos\gamma\omega t' - y'\gamma^2\omega\sin\gamma\omega t' dt' - R\gamma\omega\cos\gamma\omega t' dt'$$
$$dt' = \frac{R\omega}{c^2} dy' \quad (2.4)$$

For $x' = y' = t' = 0$:

$$dx = dx'$$

$$dy = \gamma dy' - R\gamma\omega dt' = \gamma dy'(1 - \frac{R^2\omega^2}{c^2}) = \frac{dy'}{\gamma}$$

(2.5)

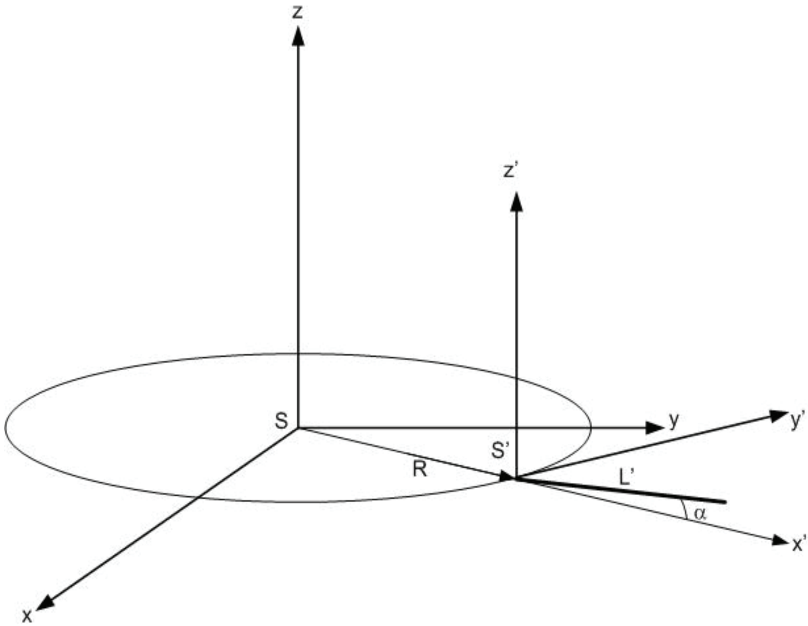

Fig. 2 Rotating frames

For an arbitrary orientation of the rod in the rotating frame S', $dx' = L'\cos\alpha, dy' = L'\sin\alpha$, where L' is the proper length. Therefore, the length contraction formula for rotating frames is:

$$L = L'\sqrt{1 - (\frac{\omega R \sin\alpha}{c})^2} \qquad (2.6)$$

If we consider two orthogonal rods of lengths L_1' and L_2' in S' (like the arms of the Kennedy-Thorndike interferometer), their lengths as measured from S are respectively:

$$L_1 = L_1'\sqrt{1-(\frac{\omega R \sin\alpha}{c})^2} \tag{2.7}$$

$$L_2 = L_2'\sqrt{1-(\frac{\omega R \cos\alpha}{c})^2} \tag{2.8}$$

The formula for time dilation in uniformly rotating frames is:

$$\Delta t = \gamma(\Delta t' - \frac{\omega R \Delta y'}{c^2})|_{\Delta y'=0} = \gamma \Delta t' \tag{2.9}$$

Interestingly enough, the solution to the problem retains its original elegance.

3. The Michelson-Morley experiment

Textbooks present the Michelson-Morley experiment from the perspective of an inertial frame. In reality, the experiment is executed in the rotating frame of the lab anchored to the **rotating** Earth, so we will use the derivation from the previous sections in order to derive the predictions of special relativity for a uniformly rotating lab. All the calculations will be done from the perspective of the inertial frame S. We must first answer the question of light isotropy in rotating frames. We start with the fact that in the **inertial** frame S, light propagates with speed c:

$$dx^2 + dy^2 + dz^2 - (cdt)^2 = 0 \tag{3.1}$$

We will use this information in order to calculate the light speed in the rotating frame S' via an approach that we have used in a prior paper [9]. For $x' = y' = 0$, we start with the infinitesimal quantities:

$$dx = dx'\cos\gamma\omega t' + \gamma dy'\sin\gamma\omega t' - R\gamma\omega\sin\gamma\omega t' dt'$$
$$dy = -dx'\sin\gamma\omega t' + \gamma dy'\cos\gamma\omega t' - R\gamma\omega\cos\gamma\omega t' dt'$$
$$dz = dz'$$
$$dt = \gamma(dt' - \frac{R\omega}{c^2}dy') \tag{3.2}$$

Substituting into (3.1), we obtain:

$$dx^2 + dy^2 + dz^2 - (cdt)^2 =$$
$$= dx'^2 + \gamma^2(1 - \frac{\omega^2 R^2}{c^2})dy'^2 + dz'^2 - c^2\gamma^2(1 - \frac{\omega^2 R^2}{c^2})dt'^2 =$$
$$= dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 \tag{3.3}$$

In other words, the light speed measured in the rotating frame S' in **a small vicinity** of the origin is also c. This result, though not surprising, is by no means trivial, since it required the computations shown above. Now we have all the tools to attack the explanation of the Michelson-Morley experiment as viewed from the rotating frame of the lab.

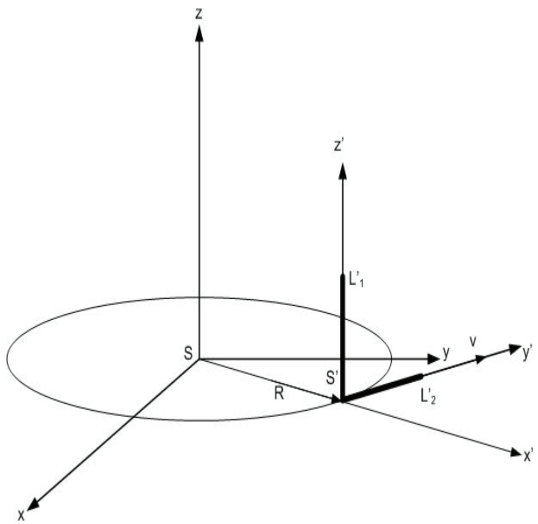

Fig. 3 The Michelson-Morley experiment in rotating frames

An observer at rest in the inertial frame S views the Michelson-Morley experiment [10] as follows: the light travels with the speed c along the rod of length L_2 , parallel with v. Because the mirror at the end of the rod recedes with the speed +v in one direction of light propagation and with speed –v in the other direction, the observer in S views the roundtrip time as:

$$t_2 = \frac{L_2}{c+v} + \frac{L_2}{c-v} = \gamma^2 \frac{2L_2}{c} \tag{3.4}$$

The rod perpendicular on v has the length L_1. In frame S, the observer sees $(ct)^2=L_1^2+(vt)^2$ from where we obtain that the roundtrip time as seen from S is:

$$t_1 = \gamma \frac{2L_1}{c} = \gamma \frac{2L_1'}{c} \tag{3.5}$$

According to the previous section, as seen from S, $L_1=L_1'$ and $L_2=L_2'/\gamma$

Therefore:

$$t_2 = \gamma \frac{2L_2'}{c} \tag{3.6}$$

From the perspective of frame S,

$$t_2 - t_1 = \gamma \frac{2(L_2' - L_1')}{c} \tag{3.7}$$

We know that an observer in S will report the time delta of an experiment performed in S' dilated by the factor γ. From (3.7) and $L_1'=L_2'$ we get that, in S'

$$t_2' - t_1' = \frac{2(L_2' - L_1')}{c} = 0 \tag{3.8}$$

Expression (3.8) explains the null result of the Michelson-Morley experiment as judged from the rotating lab frame S'.

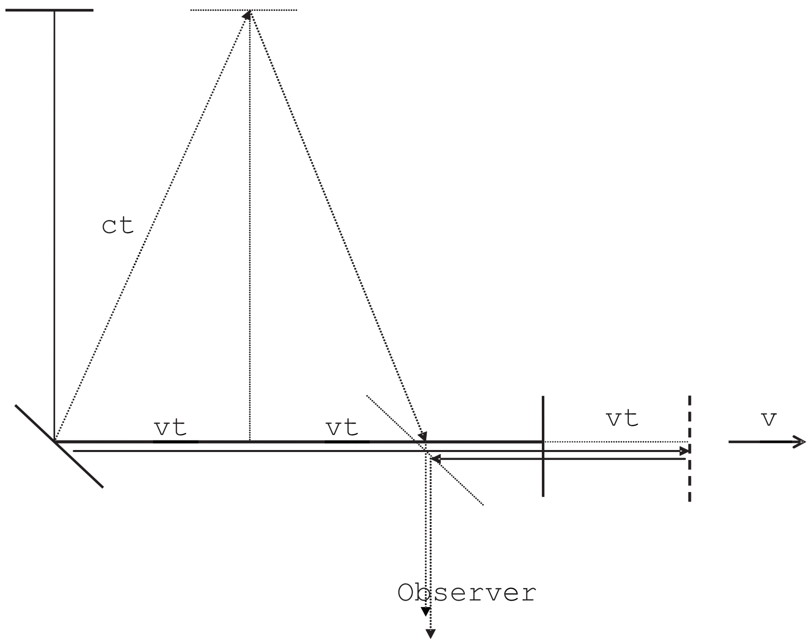

Fig. 4 The MichelsonMorley experiment

It is interesting to note that the above theoretical explanation can be construed as an indirect experimental proof for length contraction.

4. Direct measurement of length contraction

We have seen in the previous section how the Michelson-Morley experiment can be viewed as an **indirect** proof of length contraction. Based on the theory developed in section 2, we construct the experimental setup for **direct** measurement of length contraction. A small, electrically charged, steel ball of radius of r=1mm is moving at speed v=.03c in the magnetic field of a particle accelerator along a circular trajectory; it is well known that the angular velocity ω is constant. The advent of the Large Hadron Collider (LHC) has made such speed realizable. A laser beam is aimed from the center of the trajectory towards a photon counter placed outside the ball trajectory (see fig.5). In every revolution, in the absence of any length contraction, the ball would interrupt the reception of the laser beam for a time:

$$t = \frac{2r'}{\omega R} \tag{4.1}$$

where r' is the proper radius of the sphere. According to (2.6) the length contraction will affect the measurement by an amount:

$$\Delta t = \frac{2r'}{\omega R}(1 - \sqrt{1 - \frac{\omega^2 R^2}{c^2}}) \approx r'\frac{\omega R}{c^2} \tag{4.2}$$

A perfectly equivalent way of looking at the above derivation is to start from:

$$t = \gamma(t' - \frac{\omega R y'}{c^2}) \tag{4.3}$$

The extremes of the sphere diameter pass through the point $(x, y) = (0, R)$ twice, once at $t_1' = 0$ and a second time at $t_2' = \frac{2r'}{\omega R}$. Substituting into (4.3) we obtain:

$$dt = \gamma(dt' - \frac{\omega R dy'}{c^2}) = \gamma(\frac{2r'}{\omega R} - \frac{\omega R * 2r'}{c^2}) =$$
$$= \frac{2\gamma r'}{\omega R}(1 - \frac{\omega^2 R^2}{c^2}) = \frac{2r'}{\omega R}\sqrt{1 - \frac{\omega^2 R^2}{c^2}} \tag{4.4}$$

so, the measurement will be affected exactly by the amount predicted by (4.2). The time the light beam is shut off from the observer is shortened due to the length contraction of the "shutter", that is, due to the relativistic contraction of the sphere diameter. At $\omega R = 0.03c$ the effects are $dt = 2\text{ps}$ and $\Delta t \approx .1\text{ps}$. Until recently such time intervals were not measurable since they would require a counter of the order of 10Thz. Modern advances in the semiconductor technology [11] have made counters of the order of 40Thz possible. Using such a high speed counter variations of the order of .03ps can be detected so we can use speeds as low as v=.01c for our experiment.

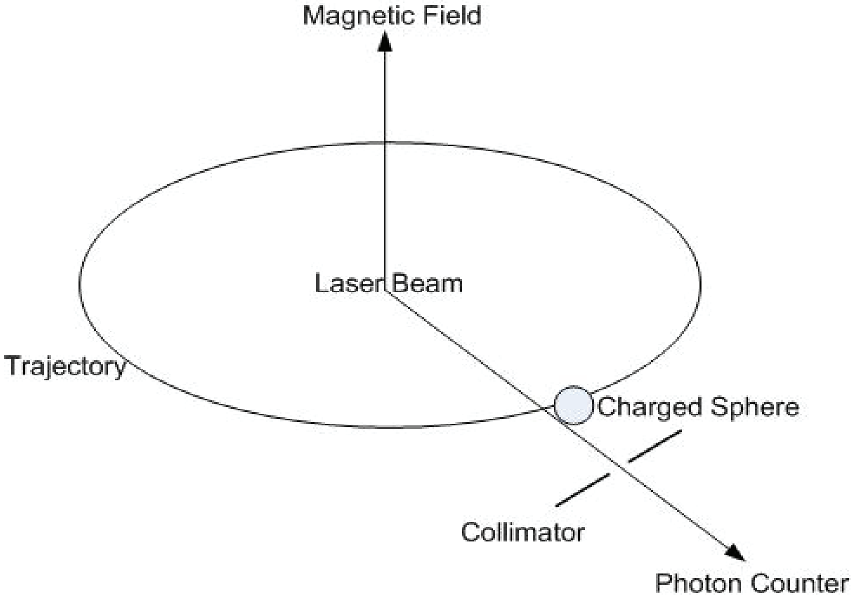

Fig. 5 Direct measurement of length contraction

At such "pedestrian" speeds of v=.01c, the Abraham-Lorentz force $\mathbf{F}_{\mathbf{rad}} = -\frac{q^2}{6\pi\varepsilon_o c^3}\frac{d\mathbf{a}}{dt}$ due to the emission of electromagnetic radiation is negligible by comparison to the Lorentz force. For example, in the case of executing the experiment at LHC, it is easy to show that at v=0.03c, $q=10^{-2}$ Coulomb and for a trajectory of a 4km radius the Lorentz-Abraham force evaluates to a meager $1.4*10^{-6}$N.

The judicious choice of $\beta = 0.03$ ensures that there is no radiated electromagnetic energy to interfere with the laser beam used for measurement. Indeed, at such low values for β the Larmor formula for electrical field:

$$\mathbf{E}_{\mathrm{rad}} = \frac{q}{4\pi\varepsilon_0 c^2}\left(\frac{\mathbf{R}}{R}\times\left(\mathbf{R}\times\frac{d\mathbf{v}}{dt}\right)\right) \tag{4.5}$$

produces $\mathbf{E}_{\text{rad}} = 0$ for circular motion. This results into a null Poynting vector, $\mathbf{S}_{\text{rad}}$. Any spurious source of background radiation is blocked by a precisely drilled collimator placed between the laser and the photon counter. The other important observation is that whole experiment lasts a few revolutions, that is, a few microseconds, thus the diurnal variation in the Earth angular velocity will not affect the experiment in any form. The third observation is that the experiment, due to its periodic nature, produces naturally multiple sets of data for error analysis. This is a major advantage of the rotational motion versus any experiments that would employ a single pass linear motion. The recent combination between giant particle accelerators and very high frequency counters has made direct measurement of length contraction perfectly realizable. The repetitive nature of the rotational motion allows multiple passes at measurement with a perfect reproducibility. Control over the speed of the steel sphere allows different measurements of different amounts of contraction. Obviously, higher speeds will allow for a bigger effect to be observed. On the other hand, further advancements in the technology of high speed counters will allow the use of lower speeds, making the experiment less expensive to reproduce.

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SPECIAL RELATIVITY EXPERIMENTS EXPLAINED FROM THE PERSPECTIVE OF THE ROTATING FRAME

Synopsis

In this chapter we present an explanation of several fundamental tests of special relativity from the perspective of the frame co-moving with a rotating observer. The solution is of great interest for real time applications because Earth-bound laboratories are inertial only in approximation. We present the derivation of the Sagnac, Michelson-Morley, Kennedy-Thorndike and the Hammar experiments as viewed from the Earth-bound uniformly rotating frame or, as in the case of the Mossbauer rotor experiments, from the perspective of the rotating device. An entire section is dedicated to length/time measurement and to clock synchronization and another one to the Doppler effect and aberration on uniformly rotating platforms. The current paper brings new information in the following areas:

- new approach for clock synchronization on a rotating platform
- new approach for length measurement in rotating frames
- new explanation of the Doppler effect and of the Mossbauer rotor experiment
- new explanation of the Kennedy Thorndike experiment

The main thrust of the paper is to give a consistent explanation of various tests of special relativity as judged from the perspective of the rotating frame of the experimental setup. In addition, we correct certain misconceptions relative to clock synchronization and length measurement that have survived a long time in the specialty literature. A special chapter is dedicated to the derivation of the Doppler effect and of aberration in rotating frames. It is shown that such derivation is far from being trivial.

1. Introduction

Real life applications include accelerating and rotating frames more often than the idealized case of inertial frames. Our daily experiments happen in the laboratories attached to the rotating, continuously accelerating Earth. Usually, such experiments are explained from the perspective of an external, inertial frame because special relativity in rotating frames is viewed as more complicated. In the present paper, we will construct a straightforward explanation by applying the formalisms developed in previous work [1-7].

2. Light Speed in Uniformly Rotating Frames

In an inertial frame K the coordinates are (T, R, Φ, Z) . In a frame K' rotating with respect the inertial frame, the coordinates are (t, r, ϕ, z) . The angular speed of rotation between the two frames is Ω . The transformation between the frames is [8]:

$$T = t$$
$$R = r$$
$$\Phi = \phi + \Omega t$$
$$Z = z$$
(2.1)

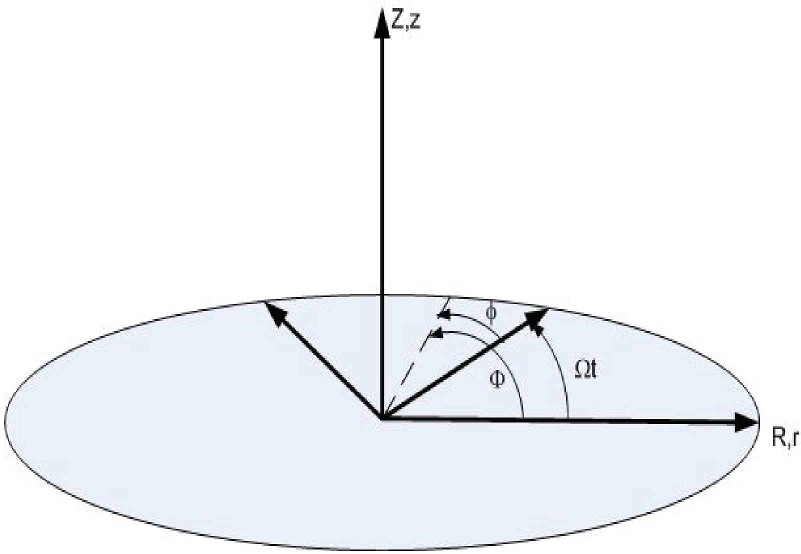

Fig. 1. Rotating frames of reference

The line element in the inertial frame is:

$$dS^2 = c^2 dT^2 - (dR^2 + R^2 d\Phi^2 + dZ^2) \tag{2.2}$$

The positive term represents the time displacement and the negative term represents the total distance displacement. The line element expressed in the rotating frame coordinates according to (2.1) is:

$$dS^2 = (c^2 - \Omega^2 r^2)dt^2 - dr^2 - 2\Omega r^2 d\phi dt - r^2 d\phi^2 - dz^2 \tag{2.3}$$

We know that light follows null geodesics, so:

$$0 = (c^2 - \Omega^2 r^2)dt^2 - dr^2 - 2\Omega r^2 d\phi dt - r^2 d\phi^2 - dz^2 \tag{2.4}$$

From (2.4) we obtain the expression of light speed in the direction collinear to the tangent to the circle described by the origin of the rotating frame by making $dr = dz = 0$:

$$0 = (c^2 - \Omega^2 r^2)dt^2 - 2\Omega r^2 d\phi dt - r^2 d\phi^2 \tag{2.5}$$

or:

$$0 = (c^2 - \Omega^2 r^2) - 2\Omega r^2 \frac{d\phi}{dt} - r^2 (\frac{d\phi}{dt})^2 \tag{2.6}$$

$$\frac{d\phi}{dt} = -\Omega \pm \frac{c}{r} \tag{2.7}$$

We are interested in the proper light speed in the rotating frame, $r\frac{d\phi}{d\tau}$, so we need to obtain the relationship between proper and coordinate time in the rotating frame. We obtain this by nulling the distance displacement in the rotating frame, $dr = dz = d\phi = 0$ such that there is only (proper) time displacement $cd\tau$ (see Fig.2):

$$c^2 d\tau^2 = (c^2 - \Omega^2 r^2) dt^2 \tag{2.8}$$

i.e.

$$d\tau = dt\sqrt{1 - \frac{\Omega^2 r^2}{c^2}} \tag{2.9}$$

Therefore:

$$v_{proper\_\tan gent} = r\frac{d\phi}{d\tau} = r\frac{d\phi}{dt}\frac{dt}{d\tau} = \frac{-\Omega r \pm c}{\sqrt{1 - \frac{\Omega^2 r^2}{c^2}}} \tag{2.10}$$

The proper light speed in the radial direction is:

$$v_{proper\_r} = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} \tag{2.11}$$

and can be calculated by making $dz = d\phi = 0$ in (2.1):

$$0 = (c^2 - \Omega^2 r^2) dt^2 - dr^2 \tag{2.12}$$

$$v_{proper\_r} = \pm c \tag{2.13}$$

The proper light speed in z direction is:

$$v_{proper\_z} = \frac{dz}{d\tau} = \frac{dz}{dt}\frac{dt}{d\tau} \tag{2.14}$$

and can be calculated by making $dr = d\phi = 0$ in (2.1):

$$0 = (c^2 - \Omega^2 r^2) dt^2 - dz^2 \tag{2.15}$$

$$v_{proper\_z} = \pm c \tag{2.16}$$

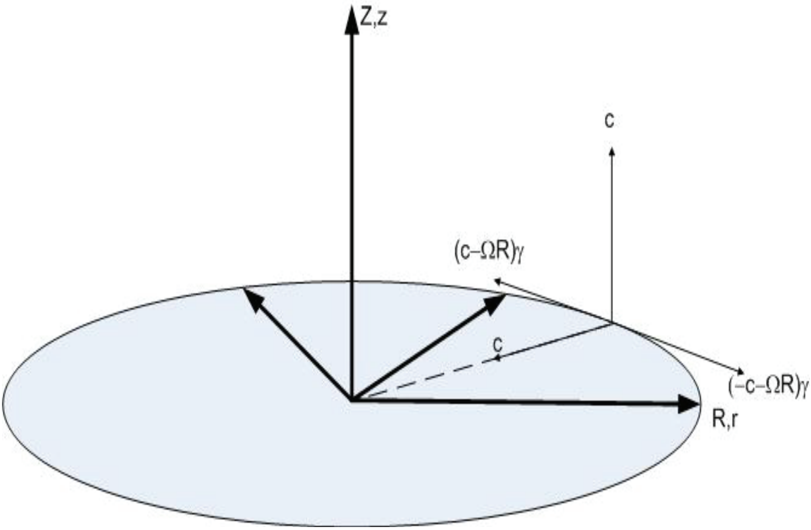

Fig. 2. Light speed in the rotating frame ($\gamma = \frac{1}{\sqrt{1-\frac{r^2\Omega^2}{c^2}}}$)

3. New Approach in Length Measurement and Clock Synchronization on Uniformly Rotating Platforms

Length measurement on a rotating platform can be somewhat challenging in terms of complexity of the physics involved. It needs to be stressed that we are interested in measuring distances between points at rest in the rotating frame K' as opposed to measuring distances between points at rest with respect to the inertial frame K as attempted from frame K'. This represents a massive difference from [30], section 82 where the authors are interested in calculating the perimeter of a circle at rest in K as being measured with a contracted yardstick at rest in K'. We conduct our experiments in the rotating frame, therefore we are interested in length measurements executed in the rotating frame. In this section we present a general formalism for length measurement in the rotating frame K' that builds on the previous section. Consider two points: A and B situated close enough on the periphery of the rotating platform such that a light signal sent from A to B and back can be approximated with being tangent to the circle passing through the two points (see Fig.3).

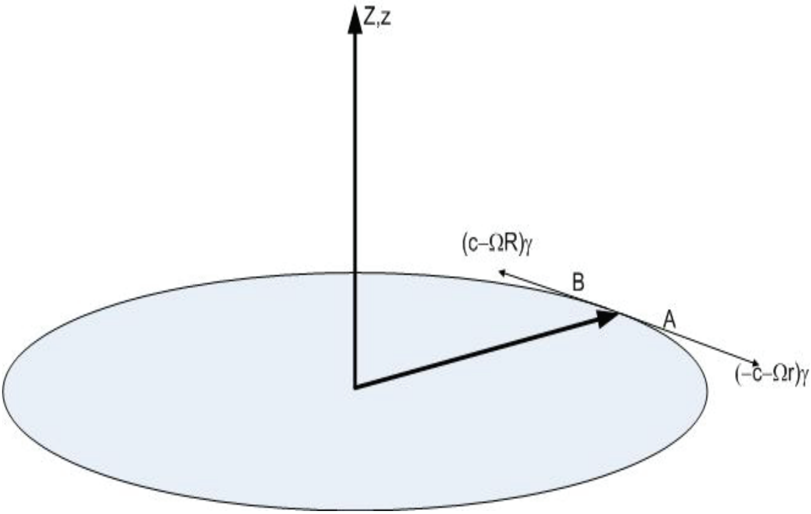

Fig. 3 Length Measurement on a rotating platform

Then, the light speed from A to B is $c_{AB} = \dfrac{c - \Omega R}{\sqrt{1 - \dfrac{\Omega^2 r^2}{c^2}}}$ and the light speed from B to A is $c_{BA} = \dfrac{c + \Omega R}{\sqrt{1 - \dfrac{\Omega^2 r^2}{c^2}}}$. The transit time for the light between A and B and back is obtained by solving with respect to time the equation (2.4) with the results:

$$|dt_{AB}| = \frac{r^2 \Omega d\phi + \sqrt{(r^2 \Omega d\phi)^2 + (c^2 - r^2\Omega^2)(dr^2 + dz^2 + r^2 d\phi^2)}}{c^2 - r^2\Omega^2}$$

$$|dt_{BA}| = \frac{-r^2 \Omega d\phi + \sqrt{(r^2 \Omega d\phi)^2 + (c^2 - r^2\Omega^2)(dr^2 + dz^2 + r^2 d\phi^2)}}{c^2 - r^2\Omega^2} \quad (3.1)$$

We needed to take the absolute values in order to get positive numbers for the time intervals. We should also note that we chose $|dt_{AB}|>|dt_{BA}|$ since the light front originating in A "chases" point B. What we really need for calculating the proper distance between A and B is the proper times:

$$\tau_{AB} = |dt_{AB}|\sqrt{1-\frac{r^2\Omega^2}{c^2}}$$
$$\tau_{BA} = |dt_{BA}|\sqrt{1-\frac{r^2\Omega^2}{c^2}} \tag{3.2}$$

The proper distance between A and B is:

$$dl = \frac{c_{AB}\tau_{AB}+c_{BA}\tau_{BA}}{2} =$$
$$= \left(\sqrt{dr^2+dz^2+\frac{r^2c^2d\phi^2}{c^2-r^2\Omega^2}} - \frac{r^3\Omega^2 d\phi}{c^2\sqrt{1-\frac{r^2\Omega^2}{c^2}}}\right)\frac{1}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}} \tag{3.3}$$

On planar circle $dz = dr = 0$ so (3.3) reduces to:

$$dl = rd\phi \tag{3.4}$$

So, the observer co-moving with the rotating frame measures the perimeter of a circle of radius r to be $2\pi r$ .It is worth noticing that formulas (3.3)-(3.4) disagree with the formulas derived at the end of chapter 89 in [30]. We can trace the error in [30] from the fact that the authors mix the isotropic light speed, c , measured in the inertial frame with the proper time measured in the rotating frame, such that their formula (84.6) in [30] is equivalent to:

$$dl = \frac{c\tau_{AB}+c\tau_{BA}}{2} = \sqrt{dr^2+dz^2+\frac{r^2c^2d\phi^2}{c^2-r^2\Omega^2}} \tag{3.5}$$

The error in (84.6) leads to the subsequent error:

$$dl = \frac{rd\phi}{\sqrt{1 - \frac{r^2\Omega^2}{c^2}}} \tag{3.6}$$

Amazingly enough, this error has "survived" four editions of this otherwise excellent book. The length measurement in the radial direction starts simply by assuming $dz = d\phi = 0$ into (3.1), therefore:

$$| dt_{AB} |=| dt_{BA} |= \frac{dr}{\sqrt{c^2 - r^2\Omega^2}} \tag{3.7}$$

This means that there is no length contraction in the radial direction:

$$dl = \frac{c\tau_{AB} + c\tau_{BA}}{2} = dr \tag{3.8}$$

The length measurement in the z direction starts simply by assuming $dr = d\phi = 0$ into (3.1), therefore:

$$| dt_{AB} |=| dt_{BA} |= \frac{dr}{\sqrt{c^2 - r^2\Omega^2}} \tag{3.9}$$

This means that there is no length contraction in any direction orthogonal to the motion. In Einstein clock synchronization a light signal is bounced from point A to point B and back to A. The clock at point A is reset at the beginning of the light roundtrip. When the light signals reaches B, the clock at B is reset to zero. When the light signal reaches A again, the clock at A is set to half the total elapsed time, thus clocks A and B show the same exact time, equal to the distance between A and B divided by the isotropic light speed, c. The method works because in inertial frames light speed is isotropic, we have seen that this is not the case on a rotating platform. Nevertheless, the Einstein method is used routinely on the rotating Earth. How much error is induced by the light speed anisotropy? The error is given by:

$$\varepsilon = \frac{\tau_{AB} + \tau_{BA}}{2} - \tau_{BA} = \frac{\tau_{AB} - \tau_{BA}}{2} \tag{3.10}$$

Inserting (3.1) into (3.10) we obtain:

$$\varepsilon = \frac{r^2 \Omega d\phi}{c^2 \sqrt{1-\frac{r^2\Omega^2}{c^2}}} \tag{3.11}$$

If we attempted to blindly synchronize clocks on the circumference of a circle through the above method, we will find that the total accumulated error to be equal to $\frac{2\pi r^2\Omega}{c^2\sqrt{1-\frac{r^2\Omega^2}{c^2}}}$.

Looking at the issue in a different way, a 10m distance between A and B corresponds to $d\phi = 10^{-6}\text{rad}$. Since the maximum value for the tangential speed is $r\Omega = 460\text{m/s}$ and $\Omega = 7.29*10^{-5}\text{rad/s}$ it follows from (3.11) that the error is of the order of $35*10^{-15}\text{s}$. This is a very small number but we can do better than that, we can make the error to be zero [10] if we have access to the center of rotation. By placing a mirror in the center of the platform and aligning its normal with the bisector of the angle AOB we can synchronize the clocks in A and B perfectly since the elapsed times for a light ray to bounce on the path AOB is equal to the time on the return path BOA due to the fact that light speed is equal to c along the radius of the circle as per (2.13). See Fig. 4

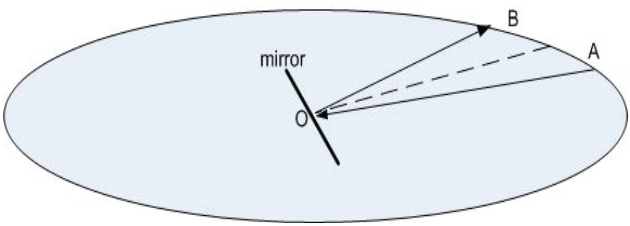

Fig. 4 Perfect synchronization on a rotating platform

The above scheme needs a slight correction: while the light travels from A to O, the platform and the attached mirror, rotate in the direction from A to B by a very small angle:

$$\phi_A = \Omega\frac{r}{c}\sqrt{1-\frac{r^2\Omega^2}{c^2}} \tag{3.12}$$

By the time the light has reached B, the angle between the initial position of A and the final position of B has opened up by:

$$\phi_B = 2\Omega\frac{r}{c}\sqrt{1-\frac{r^2\Omega^2}{c^2}} \tag{3.13}$$

The above means the angle AOB "widens" by ϕ_B so the mirror would need to be angled by ϕ_A in the direction of rotation. Luckily, on the return path from B to O to A, the reverse happens, the angle BOA "narrows" by ϕ_B so, on average, we do not have to do any correction. This is important because we couldn't do the correction in first place since it is of the order of v/c .The method can be generalized to synchronizing any number of clocks on the periphery of a rotating platform. The idea is to send an electromagnetic signal from A to B containing a digital encoding of the time τ_A on clock A. Upon receiving the signal, clock B gets reset to:

$$\tau_B = \tau_A + \frac{2r}{c} \tag{3.14}$$

After that, the mirror is rotated by the appropriate angle and a signal encoding the current (updated) value τ_B' is sent to C and the process continues.

4. The Sagnac Experiment Explained from the Perspective of the Rotating Platform

The Sagnac experiment [11-13] is usually explained from the perspective of an inertial frame since the mathematical formalism is simpler from that perspective. In this section, we will use the results derived in the previous section in order to produce an equally straightforward explanation. Based on the prior results, the observer co-moving with the rotating frame measures the perimeter of a circle of radius r to be $2\pi r$, the time difference between the clockwise and counterclockwise light fronts is calculated as:

$$\Delta\tau = 2\pi r\left(\frac{1}{\frac{c-r\Omega}{\sqrt{1-\frac{r^2\Omega^2}{c^2}}}} - \frac{1}{\frac{c+r\Omega}{\sqrt{1-\frac{r^2\Omega^2}{c^2}}}}\right) = \frac{4\pi r^2\Omega}{c^2\sqrt{1-\frac{r^2\Omega^2}{c^2}}} \tag{4.1}$$

The phase difference is:

$$\Delta\phi = \Omega\Delta\tau = \frac{4\pi r^2\Omega^2}{c^2\sqrt{1-\frac{r^2\Omega^2}{c^2}}} = \frac{4\pi R^2\Omega^2}{c^2\sqrt{1-\frac{r^2\Omega^2}{c^2}}} \tag{4.2}$$

It is interesting to notice that the phase difference is frame invariant.

5. The Michelson-Morley Experiment Explained from the Perspective of a Rotating Earth-Bound Frame

We can now explain the null result of the Michelson Morley experiment [14-22] in the rotating frame of the lab co-rotating with the Earth. The elapsed time in the direction of motion is:

$$\tau_{\parallel} = \frac{dl}{\frac{c-r\Omega}{\sqrt{1-\frac{r^2\Omega^2}{c^2}}}} + \frac{dl}{\frac{c+r\Omega}{\sqrt{1-\frac{r^2\Omega^2}{c^2}}}} = \frac{2dl}{c\sqrt{1-\frac{r^2\Omega^2}{c^2}}} \tag{5.1}$$

The elapsed time in the z direction is:

$$\tau_{\perp} = \frac{2dl}{v_z} = \frac{2dl}{c\sqrt{1-\frac{r^2\Omega^2}{c^2}}} \tag{5.2}$$

So, there is no fringe shift predicted in any frame, be it rotating or inertial.

6. New Explanation of the Kennedy-Thorndike Experiment as Viewed from the Perspective of a Rotating Earth-Bound Frame

The Kennedy-Thorndike experiment [23] exploits the fact that the Earth bound laboratory has a variable speed $v(t)$ due to the combined effect of Earth rotation around its axis and Earth revolution around the Sun. The laboratory speed $v(t)$ has contributions from the revolution of the Earth with respect to the Sun-centered frame, $v_e = 30km/s$ and Earth's daily rotation v_d so:

$$v(t) = v_e \sin[\Omega_y (t-t_0)]\cos\Phi_E + v_d \sin[\Omega_d (t+t_d)]\cos\Phi_A \tag{6.1}$$

For example, at Berkeley (latitude 37o 52'18" N), $v_d = 0.355km/s$. $\Phi_A \approx 8^o$ is the angle between the equatorial plane and the velocity of the sun. $\Phi_E \approx 6^o$ is the declination between the plane of Earth's orbit and the velocity of the Sun, $2\pi/\Omega_y = 1yr$, $2\pi/\Omega_d =$ 1 sidereal day, t_0 and t_d are determined by the phase and start date of the measurement, respectively. When one adds the fact that the arms of the Kennedy-Thorndike interferometer are unequal one obtains a difference between the light roundtrip time in the "longitudinal" arm of length dL_L (arm parallel with the direction of motion) and the "transverse" arm (arm perpendicular on the direction of motion) dL_T, as viewed from an inertial frame centered in the Sun:

$$\Delta T = T_L - T_T = \\ = dL_L \sqrt{1-\frac{v^2}{c^2}} \left(\frac{1}{c+v} + \frac{1}{c-v}\right) - \frac{2dL_T}{\sqrt{c^2-v^2}} = \frac{2(dL_L - dL_T)}{c\sqrt{1-\frac{v^2}{c^2}}} \tag{6.2}$$

The difference in travelled distance by light is:

$$\Delta L = c\Delta T = \frac{2(dL_L - dL_T)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6.3}$$

The amount of fringe shift observed between two different positions, A and B of the lab with respect to the Sun-centered inertial frame, is:

$$\Delta N_{AB} = \frac{2(dL_L - dL_T)}{\lambda_A \sqrt{1-\frac{v_A^2}{c^2}}} - \frac{2(dL_L - dL_T)}{\lambda_B \sqrt{1-\frac{v_B^2}{c^2}}} \tag{6.4}$$

But:

$$\lambda_A \sqrt{1-\frac{v_A^2}{c^2}} = \lambda_B \sqrt{1-\frac{v_B^2}{c^2}} = \lambda_{proper} \tag{6.5}$$

meaning that:

$$\Delta N_{AB} = 0 \tag{6.6}$$

in agreement with the experimental measurements.

Explaining the above results in the rotating frame of the lab is simple, given the fact that the wavelength is no longer variable:

$$\Delta t = t_L - t_T = dl_L\left(\frac{1}{\frac{c+r\Omega}{\sqrt{1-\frac{r^2\Omega^2}{c^2}}}} + \frac{1}{\frac{c-r\Omega}{\sqrt{1-\frac{r^2\Omega^2}{c^2}}}}\right) - \frac{2dl_T}{c\sqrt{1-\frac{r^2\Omega^2}{c^2}}} =$$

$$= \frac{2(dl_L - dl_T)}{c\sqrt{1-\frac{r^2\Omega^2}{c^2}}} \tag{6.7}$$

$$\Delta n_{AB} = \frac{2(dl_L - dl_T)}{\lambda\sqrt{1-\frac{r^2\Omega^2}{c^2}}} - \frac{2(dl_L - dl_T)}{\lambda\sqrt{1-\frac{r^2\Omega^2}{c^2}}} = 0 \tag{6.8}$$

7. The Doppler Effect in Rotating Frames and its Application to the Mossbauer Rotor Experiment

Let's examine the Doppler effect and light beam aberration. This is a critical issue in the explanation of the Mossbauer rotor experiments. Let f_{em} be the frequency emitted by the radiation source and let F_{obs} be the observed frequency. Let Ψ be the phase of the radiation wave as measured in the inertial frame and let ψ be the phase in the rotating frame.

$$\Psi = F_{obs}\left(T - \frac{K_x X + K_y Y}{c}\right) \tag{7.1}$$

$$\psi = f_{em}(t - \frac{k_x x + k_y y}{c}) = f_{em}(\tau\gamma - \frac{k_x x + k_y y}{c}) =$$
$$= f_{em}(\gamma T - \frac{k_x x + k_y y}{c}) =$$
$$= \gamma f_{em}(T - \frac{k_x / \gamma}{c} x - \frac{k_y / \gamma}{c} y) \quad : \quad (7.2)$$

$$x = r\cos\phi = X\cos\Omega t + Y\sin\Omega t$$
$$y = r\sin\phi = -X\sin\Omega t + Y\cos\Omega t$$
(7.3)

$$\psi = \gamma f_{em}(T - \frac{k_x \cos\Omega T - k_y \sin\Omega T}{\gamma c} X -$$
$$- \frac{k_y \cos\Omega T + k_x \sin\Omega T}{\gamma c} Y) \quad : \quad (7.4)$$

From the wave phase invariance $\Psi = \psi$ it would be naive to claim:

$$F_{obs} = \gamma f_{em}$$
(7.5)

$$K_x = (k_x \cos\Omega T - k_y \sin\Omega T) / \gamma$$
$$K_y = (k_y \cos\Omega T + k_x \sin\Omega T) / \gamma \quad (7.6)$$

Such a conclusion is incorrect because expression (7.4) depends on T both explicitly and implicitly. Not all is lost if we consider just the particular case when the axes of the two systems of coordinates align, as in the case of the Mossbauer rotor experiments. This happens when

$T = \frac{2n\pi}{\Omega}, n = 0,1,2,....$. With that, (7.4) becomes:

$$\psi = \gamma f_{em}(\frac{2\pi}{\Omega}n - \frac{k_x}{\gamma c}X - \frac{k_y}{\gamma c}Y) \tag{7.7}$$

On the other hand, (7.1) becomes:

$$\Psi = F_{obs}(\frac{2\pi}{\Omega}n - \frac{K_x X + K_y Y}{c}) \tag{7.8}$$

From the frame-invariance of the phase, by comparing (2.24) and (2.25) we obtain:

$$F_{obs} = \gamma f_{em} \tag{7.9}$$

$$K_x = \frac{k_x}{\gamma} \tag{7.10}$$

$$K_y = \frac{k_y}{\gamma} \tag{7.11}$$

There is no aberration since the vectors (K_x, K_y) and (k_x, k_y) are collinear. An excellent confirmation of the relativistic Doppler effect was achieved by the Mössbauer rotor experiment [24]. Gamma rays are sent from a source in the middle of a rotating disk (see fig.5) to an absorber at the rim and a stationary counter is placed beyond the absorber. The characteristic resonance absorption frequency of the moving absorber at the rim should decrease due to time dilation, so the transmission of gamma rays through the absorber increases, which is subsequently measured by the stationary counter beyond the absorber. The maximal deviation from time dilation was 10^{-5}. Such experiments were performed by Hay *et al.* [25,26], Champeney *et al.*[27,28] and by Kündig [29]. According to (7.9) the measured frequency is $F_{obs} = \gamma f_{em}$ .Conversely, if the positions of the source and the absorber are swapped, the formalism developed above predicts the measured frequency to be:

$$F_{obs} = \frac{f_{em}}{\gamma} \tag{7.12}$$

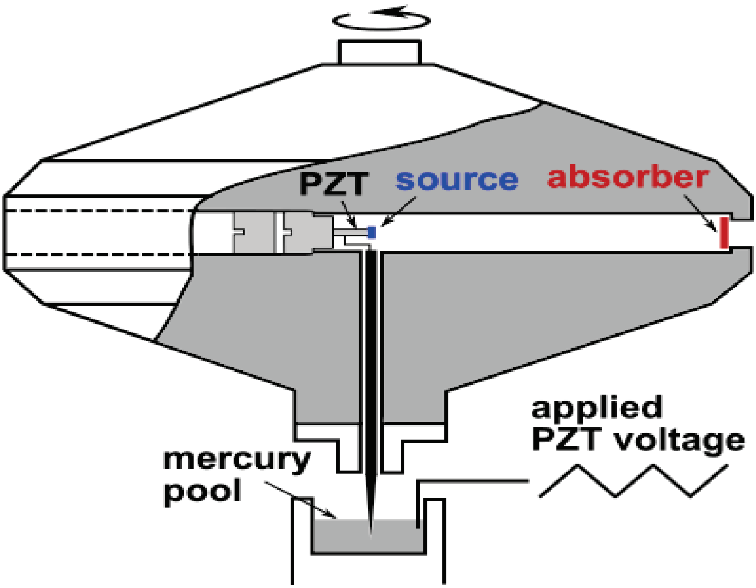

Fig. 5. The Mossbauer rotor experiment

The Mossbauer rotor experiments are the most precise measurements of the transverse Doppler effect.

8. Hammar Experiment Explained from the Perspective of of a Rotating Earth-Bound Frame

In the following, all calculations explaining the outcome of the Hammar experiment [31,32] are made from the point of view of the rotating Earth-bound frame and all employ the theory of special relativity in rotating frames. The clockwise (t_{CW}) and counterclockwise (t_{CCW}) time of light propagation are (see fig.6):

$$t_{CW} = \frac{AB}{\dfrac{c+\Omega r}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}}} + t_{BC} + \frac{CD}{\dfrac{c-\Omega r}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}}} + t_{DA}$$

$$t_{CCW} = t_{AD} + \frac{DC}{\dfrac{c+\Omega r}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}}} + t_{CB} + \frac{BA}{\dfrac{c-\Omega r}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}}} \quad (8.1)$$

In (8.1) AB, CD DC and DA are the light paths and $t_{BC}, t_{DA}, t_{AD}, t_{CB}$ are the times necessary to traverse the paths BC, DA, AD and CB respectively and "c" is the speed of light in vacuum.

Obviously, the times in the arms moving perpendicular on the "aether wind" do not depend on the traversal sense:

$$t_{DA} = t_{AD}$$
$$t_{BC} = t_{CB}$$
$$AB = BA = CD = DC = L \quad (8.2)$$

So, the time differential between the clockwise and the counterclockwise paths is:

$$\Delta t = t_{CW} - t_{CCW} = \frac{\dfrac{AB}{c+\Omega r}}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}} + \frac{\dfrac{CD}{c-\Omega r}}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}} - \frac{\dfrac{DC}{c+\Omega r}}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}} - \frac{\dfrac{BA}{c-\Omega r}}{\sqrt{1-\dfrac{\Omega^2 r^2}{c^2}}} = 0 \tag{8.3}$$

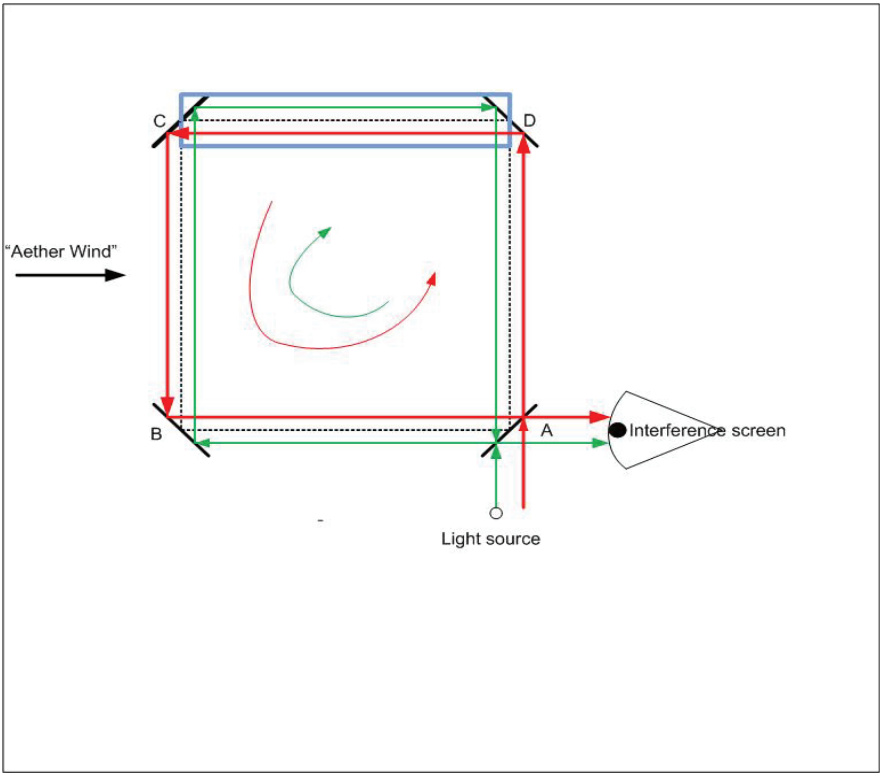

Fig. 6 Instrument motion with shielded arm moving parallel to the "aether wind"

The light source as well as the screen where interference occurs between the two light beams is located in point "A". Also, a half-silvered mirror is used as a light splitter.

If we assumed by absurd that the light speed in the shielded arm CD is $c' \neq c$ then:

$$\Delta t = t_{CW} - t_{CCW} = \frac{AB}{\frac{c+\Omega r}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}}} + \frac{CD}{\frac{c'-\Omega r}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}}} - \frac{DC}{\frac{c'+\Omega r}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}}} - \frac{BA}{\frac{c-\Omega r}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}}} = \\
= 2\Omega r L\sqrt{1-\frac{\Omega^2 r^2}{c^2}}\frac{c'^2 - c^2}{(c^2-\Omega^2 r^2)(c'^2-\Omega^2 r^2)} \quad (8.4)$$

Experiment shows $\Delta t = 0$ meaning that our starting assumption that the light speed in the shielded arm CD is $c' \neq c$ is false.

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The Relativistic Trajectory of a Charged Particle in the Magnetic Field of an Infinite Length Current Carrying Wire

Synopsis

We show how to produce the closed form solution for the motion of a charged particle in the magnetic field of an infinitely long, current-carrying wire in the relativistic range, thus extending the results produced recently [1].

Introduction

Extensive treatment of the trajectories of charged particles moving at non-relativistic speeds in a magnetic field abound in scientific literature [2-15]. An exhaustive relativistic treatment for the case of arbitrary stationary electromagnetic fields can be found in [16]. Treatments for constant magnetic fields are common in literature, the magnetic field of a wire is not constant, making the problem significantly more difficult to solve. A solution for the non-constant magnetic field has only been produced recently [1] and only for the non-relativistic regime. In the current paper we are providing the general, fully relativistic solution to the case of a charged particle in the magnetic field of an infinitely long, current-carrying wire. Such a field is obviously non uniform, thus making the problem somewhat challenging.

The Solution

In what follows we use the exact notation used in [1]. In the cylindrical coordinate system (s, φ, z), an infinitely long wire is located on the z-axis. The magnetic field produced by the current of intensity I is:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}} \tag{1}$$

The Lorentz force exerted on a particle of charge q and mass m entering the magnetic field at initial velocity $\mathbf{v_0}$ and subsequently moving with the instantaneous velocity $\mathbf{v}$ is:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{2}$$

Because the Lorentz force is perpendicular on the particle velocity, the speed of the particle is constant $v = v_0$ [16].

On the other hand, the relativistic force exerted on the particle of mass m moving with instantaneous velocity $\mathbf{v}$ is [16]:

$$\mathbf{F} = m\frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{m}{\sqrt{1-\frac{v_0^2}{c^2}}}\frac{d\mathbf{v}}{dt} = m\gamma_0\frac{d\mathbf{v}}{dt} \tag{3}$$

From (1),(2) and (3) we obtain:

$$\frac{dv_s}{dt} = -\frac{q\mu_0 I}{2\pi m\gamma_0 s}v_z = -\frac{v_L}{\gamma_0 s}v_z \tag{4a}$$

$$\frac{dv_\varphi}{dt} = 0 \tag{4b}$$

$$\frac{dv_z}{dt} = \frac{v_L}{\gamma_0 s}v_s \tag{4c}$$

where $v_L = \frac{q\mu_0 I}{2\pi m}$ is the Larmor speed. The above equations correspond exactly to equations (3a-3c) from reference [1] with the right hand term scaled by the constant γ_0 . The general case, valid at any speeds, including the relativistic ones, is solved by reducing the problem to one that has already been solved by simply scaling the Larmor speed by a constant. The effect is scaling variables u_ρ, u_ζ [1] by the same constant. As a consequence, equations (4a-4b) in [1] are unchanged. One criticism that has been brought against the above solution is that it ignores the radiation reaction. Let's examine the criticism, the radiated energy is:

$$\Delta E = \frac{4\pi e^2 \beta^3}{3r} (\frac{E}{mc^2})^4 \tag{5}$$

For a 500Mev synchrotron of 1m radius the above amounts to an energy drop per cycle $\Delta E = 10^4 \text{ev}$ for the case of an electron, a totally insignificant amount. If, instead of electrons we consider protons the energy drop through radiation is $\Delta E = \frac{10^4}{10^{12}} \text{ev}$. So, a proton of 500Mev energy loses 10^{-8}ev per cycle. This is why, the radiation reaction has been ignored in the solution.

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The Compton Scattering Experiment – the Electron Point of View

Synopsis

Arthur Compton developed a theory of the intensity of X-ray reflection from crystals as a means of studying the arrangement of electrons and atoms, and in 1918 he started a study of X-ray scattering. This led, in 1922, to his discovery of the increase of wavelength of X-rays due to scattering of the incident radiation by free electrons [1, 2]. This effect, nowadays known as the *Compton effect*, which illustrates the particle concept of electromagnetic radiation, was afterwards substantiated by C. T. R. Wilson who, in his cloud chamber, could show the presence of the tracks of the recoil electrons. The original Compton papers deal extensively with the physics of the scattered photons while there is no treatment of the physics of the recoiling electrons. In the current chapter we develop a comprehensive treatment of the "electron point of view", i.e. we treat the dynamics of the recoiling electrons. We proceed by extending the treatment to a much tougher case, the case of inverse Compton scattering, i.e. the scattering of the electrons impacting photons. We conclude with treating the Thomson scattering as a limit case of Compton scattering.

1. Electron-photon collision

The Compton scattering is an example of elastic scattering of light by an electron moving at relativistic speeds, where the wavelength of the scattered light is different from that of the incident radiation. The energy of the X ray photon is much larger than the binding energy of the electron, so the electron can be treated as being free. Light must behave as if it consists of particles, if we are to explain low-intensity Compton scattering. Because the momentum-energy of a system must be conserved, it is not generally possible for the electron simply to move in the direction of the incident photon but rather the two particles move at (different) angles with respect to the trajectory of the incident photon. In Compton's experiment a photon (X-ray) with frequency f collides with an electron in an atom, which is treated as being at rest. A new photon emerges at angle θ and frequency f' while the electron recoils at an angle ϕ after the collision (see fig.1).

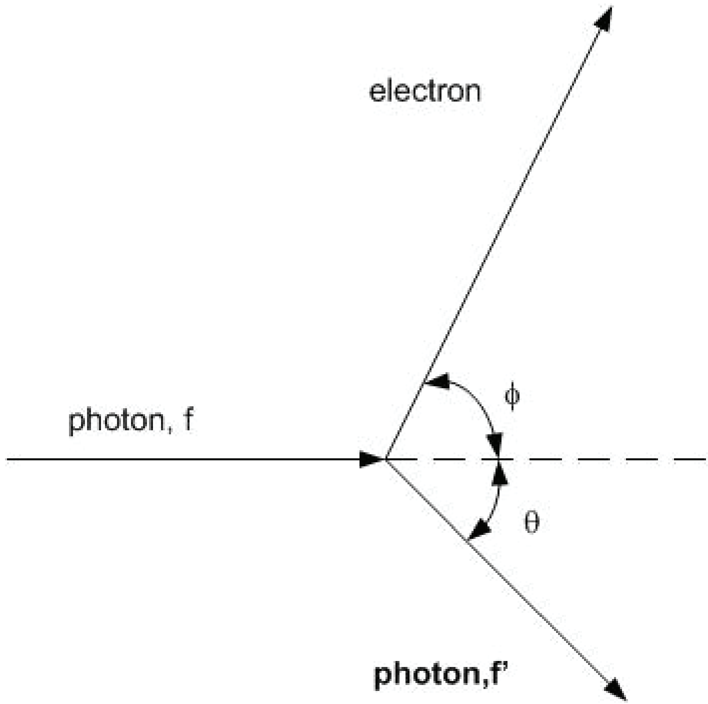

Fig. 1 Sketch of the Compton Setup

It is well known that in relativity, the total energy (E) and the three-vector momentum (**p**) of a system of particles involved in a collision are conserved [3]. While Compton preferred to express the photon energy in terms of wavelengths, it turns out that the formulas will be easier to work with when expressed in terms of frequency. The conservation of total energy is:

$$m_0c^2 + hf = \gamma m_0c^2 + hf' \tag{1.1}$$

where m_0 is the electron rest mass, f, f' are the photon frequencies before and after collision with the electron and h is Planck's constant. Referring to fig. 1 the conservation of momentum along the axis of impact is:

$$\frac{hf}{c} = \frac{hf'}{c}\cos\theta + \gamma m_0 v \cos\phi \tag{1.2}$$

The conservation of momentum along the axis perpendicular to the axis of impact is:

$$0 = \frac{hf'}{c}\sin\theta - \gamma m_0 v \sin\phi \tag{1.3}$$

From (1.1) we obtain immediately:

$$\gamma = 1 + \frac{h(f-f')}{m_0 c^2} \tag{1.4}$$

With the notation $\Delta f = f - f'$ (1.4) yields the speed of the electron after the photon impact:

$$v = c\frac{\sqrt{(\frac{h\Delta f}{m_0 c^2})^2 + 2\frac{h\Delta f}{m_0 c^2}}}{1 + \frac{h\Delta f}{m_0 c^2}}$$

$$\gamma v = c\sqrt{(\frac{h\Delta f}{m_0 c^2})^2 + 2\frac{h\Delta f}{m_0 c^2}} \tag{1.5}$$

It is very tempting to try to simplify the expression for speed via a Taylor expansion but, from the experimental data, we know that $h\Delta f$ is of the same order as $m_0 c^2$, so a Taylor expansion would introduce large errors. Anyways, expression (1.5) is exact, so we are better served by using it as is.

Let's re-write (1.2) and (1.3) as:

$$\gamma m_0 v\cos\phi = \frac{hf}{c} - \frac{hf'}{c}\cos\theta \tag{1.6}$$

$$\gamma m_0 v\sin\phi = \frac{hf'}{c}\sin\theta \tag{1.7}$$

Squaring (1.6) , (1.7) and adding them together, we obtain:

$$(\gamma m_0 v)^2 = \frac{h^2}{c^2}(f^2 + f'^2 - 2ff'\cos\theta) \tag{1.8}$$

Combining (1.4),(1.5) and (1.8) we get the frequency of the re-emitted photon as a function of the photon impacting the electron:

$$f' = \frac{f}{1 + \frac{hf}{m_0c^2}(1-\cos\theta)} \tag{1.9}$$

or:

$$\Delta f = \frac{\frac{hf^2}{m_0c^2}(1-\cos\theta)}{1 + \frac{hf}{m_0c^2}(1-\cos\theta)} \tag{1.10}$$

It is easy to show that (1.10) is identical with Compton's formula:

$$\lambda - \lambda' = \frac{c}{f} - \frac{c}{f'} = -c\frac{f-f'}{ff'} = -\frac{h}{m_0c}(1-\cos\theta) \tag{1.11}$$

or:

$$\lambda' = \lambda + \frac{h}{m_0 c}(1 - \cos\theta) \qquad (1.12)$$

This is all nice except that θ is unknown. In order to get around that, Compton relied on the fact that the photons are scattered in different directions and made measurements in all directions (see fig.2).

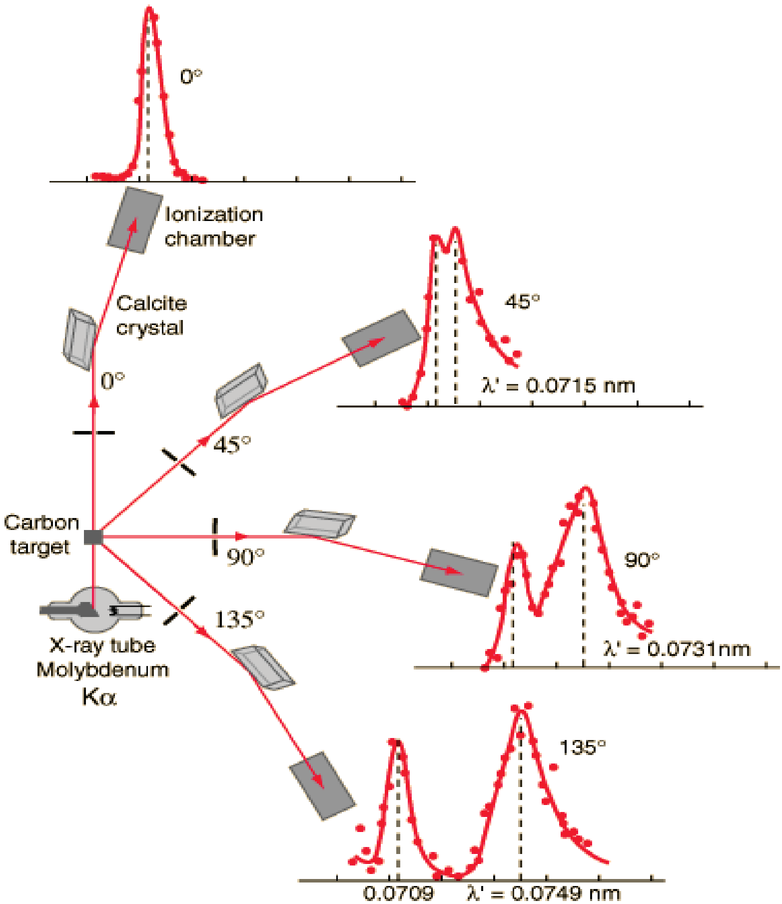

Fig. 2 The Compton experimental setup at various angles of photon scattering, θ

Notice that there are two peaks, one that shows very little wavelength shift, due to scattering from the inner electrons that are tightly bound and cannot move freely and a second peak, which shows the Compton effect, due to the "free" electrons located on the outer energy band.

Now that we know the speed of the electron after the collision with the photon, we can also try to find its trajectory. From (1.7) and (1.9) we obtain:

$$\sin\phi = \frac{\frac{hf'}{c}\sin\theta}{\gamma m_0 v} =$$

$$= \frac{hf\sin\theta}{(1+\frac{hf}{m_0c^2}(1-\cos\theta))\sqrt{(h\Delta f)^2 + 2m_0c^2h\Delta f}} \qquad (1.13)$$

2. The electron kinematics and dynamics

As pointed out earlier, the photon deflection angle θ is variable. Compton measured it for different photon trajectories. We can do the same analysis for the speed v and the deflection angle ϕ of the electron:

$\theta = 0$	$\theta = \pi / 2$	$\theta = \pi$
$v = 0$	$v = c\frac{\frac{hf}{m_0c^2}\sqrt{2+(\frac{hf}{m_0c^2})}}{1+(\frac{hf}{m_0c^2})^2}$	$v = c\frac{2(\frac{hf}{m_0c^2})^2\sqrt{1+\frac{m_0c^2(m_0c^2+2}{h^2f^2}}}{1+2hf/(m_0c^2)+2(hf/m_0}$
$\sin\phi = 0$	$\sin\phi = \frac{1}{(1+\frac{hf}{m_0c^2})\sqrt{2+(\frac{hf}{m_0c}}}$	$\sin\phi = 0$

Table 1 The electron kinematics

We can ask ourselves one more question: "What is the average force acting upon the electron?". We can figure the average force by noticing that the work exerted by such force is equal to the total energy variation:

$$\int_0^r F(x)dx = dE = hdf \tag{2.1}$$

So, over a distance $r > 0$ measured from the collision point, the average force F_m exerts a work $h\Delta f$. This means that:

$$F_m = \frac{h\Delta f}{r} = \frac{1}{r}\frac{\frac{h^2 f^2}{m_0 c^2}(1-\cos\theta)}{1+\frac{hf}{m_0 c^2}(1-\cos\theta)} \tag{2.2}$$

The electron energy after the collision with the photon is:

$$E = m_0 c^2 + h\Delta f = m_0 c^2 + \frac{\frac{h^2 f^2}{m_0 c^2}(1-\cos\theta)}{1+\frac{hf}{m_0 c^2}(1-\cos\theta)} \tag{2.3}$$

$\theta = 0$	$\theta = \pi / 2$	$\theta = \pi$
$F_m = 0$	$F_m = \frac{1}{r}\frac{\frac{h^2 f^2}{m_0 c^2}}{1+\frac{hf}{m_0 c^2}}$	$F_m = \frac{1}{r}\frac{\frac{2h^2 f^2}{m_0 c^2}}{1+\frac{2hf}{m_0 c^2}}$
$E = m_0 c^2$	$E = m_0 c^2 + \frac{\frac{h^2 f^2}{m_0 c^2}}{1+\frac{hf}{m_0 c^2}}$	$E = m_0 c^2 + \frac{\frac{2h^2 f^2}{m_0 c^2}}{1+\frac{2hf}{m_0 c^2}}$

Table 2 The electron dynamics

3. Inverse Compton scattering

Inverse Compton scattering [4,5] involves the scattering of low energy photons to high energies by ultra-relativistic electrons so that the photons gain and the electrons lose energy. The equations of energy-momentum conservation are more complicated in this case:

$$\gamma(v)m_0c^2 + hf = \gamma(v')m_0c^2 + hf' \tag{3.1}$$

$$\gamma(v)m_0v + \frac{hf}{c}\cos\theta = \frac{hf'}{c}\cos\theta' + \gamma(v')m_0v'\cos\phi' \tag{3.2}$$

$$\frac{hf}{c}\sin\theta = \frac{hf'}{c}\sin\theta' - \gamma(v')m_0v'\sin\phi' \tag{3.3}$$

where v, v' are the speeds of the electron before and after collision, ϕ, ϕ' are the angles of the electron trajectory, before and after collision, θ, θ' are the angles of the photon trajectory, before and after collision, f, f' are the frequencies of the photon before and after collision, all measured in the lab frame. At first glance, the above set of equations is much tougher to solve. We can simplify them though by doing the calculations in the frame commoving with the electron before collision and by assuming, without any loss of generality that, in the lab frame, $\theta = 0$:

$$m_0c^2 + hf = \gamma(v')m_0c^2 + hf' \tag{3.4}$$

$$\frac{hf}{c} = \frac{hf'}{c}\cos\theta' + \gamma(v')m_0v'\cos\phi' \tag{3.5}$$

$$0 = \frac{hf'}{c}\sin\theta' - \gamma(v')m_0v'\sin\phi' \tag{3.6}$$

Now, v' is the speed of the electron after collision, ϕ, ϕ' are the angles of the electron trajectory, before and after collision, θ, θ' are the angles of the photon trajectory, before and after collision, f, f' are the frequencies of the photon before and after collision, all measured in the frame commoving with the electron before collision. This frame moves with speed v with respect to the lab. Equation (3.4) yields immediately:

$$\gamma(v') = 1 + \frac{h(f - f')}{m_0 c^2} = 1 + \frac{h\Delta f}{m_0 c^2} \tag{3.7}$$

Therefore:

$$v' = c\frac{\sqrt{(\frac{h\Delta f}{m_0 c^2})^2 + 2\frac{h\Delta f}{m_0 c^2}}}{1 + \frac{h\Delta f}{m_0 c^2}}$$

$$\gamma(v')v' = c\sqrt{(\frac{h\Delta f}{m_0 c^2})^2 + 2\frac{h\Delta f}{m_0 c^2}} \tag{3.8}$$

Equation (3.6) can now be used to predict the angle of the emerging photon as a function of the angle made by the electron after the collision:

$$\sin\theta' = \frac{\gamma(v')m_0 v'\sin\phi'}{\frac{hf'}{c}} = \frac{\sqrt{(h\Delta f)^2 + 2m_0 c^2 h\Delta f}}{hf'}\sin\phi' \tag{3.9}$$

There are two remaining problems with (3.9):

- *The formula is dependent on the frequency f' of the photon after collision (in the frame of the electron), we need the formula expressed in terms of the frequency f of the photon before the collision.*

This is accomplished by solving for f' the system of equations resulting from the momentum conservation:

$$\frac{hf'}{c}\cos\theta' = \frac{hf}{c} - \gamma(v')m_0 v'\cos\phi'$$

$$\frac{hf'}{c}\sin\theta' = \gamma(v')m_0 v'\sin\phi' \tag{3.10}$$

Squaring the two equations and adding them gives the equation degree 4 in f':

$$2hfm_0\cos\phi'\sqrt{\left(\frac{h(f-f')}{m_0c^2}\right)^2 + 2\frac{h(f-f')}{m_0c^2}} =$$

$$= \left(\frac{hf}{c}\right)^2 - \left(\frac{hf'}{c}\right)^2 + \left(\frac{h(f-f')}{c}\right)^2 + 2m_0h(f-f') \tag{3.11}$$

- θ', ϕ', f *are referenced to the frame of the electron before collision, we need the variables referenced to the frame of the lab*

We resolve this other issue by use of the relativistic aberration formulas:

$$\cos\phi' = \frac{\cos\phi'_{lab} - \frac{v}{c}}{1 - \frac{v}{c}\cos\phi'_{lab}}$$

$$\cos\theta'_{lab} = \frac{\cos\theta' + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta'} \tag{3.12}$$

The primes signify that the variables are after collision. Finally, the photon frequencies before collision are connected by the well known relativistic Doppler formulas.

$$f = f_{lab}\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \tag{3.13}$$

While the inverse Compton scattering turned out to be much tougher than the standard Compton scattering, the "blueprint" presented in the previous section applied perfectly.

4. Thomson scattering

Thomson scattering [6] is a limit case of Compton scattering for $hf \ll m_0 c^2$. In this case we can apply the Taylor expansion that we cautioned against in the section on Compton scattering. The exact formula (1.10) yields the approximation:

$$\Delta f = \frac{hf^2}{m_0 c^2}(1-\cos\theta) \tag{4.1}$$

The exact formula (1.13) yields the very nice approximation:

$$\sin\phi = \frac{hf\sin\theta}{\sqrt{2m_0c^2h\Delta f}} = \frac{\sin\theta}{\sqrt{2(1-\cos\theta)}} = \cos\frac{\theta}{2} \tag{4.2}$$

For $\theta = 0$ $\Delta f = 0$ and $\phi = \pi/2$, that is, the scattered photons have maximum energy at a right angle with the direction of impact. This conclusion confirms the approach of considering the Thomson scattering as a particular case of Compton scattering.

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THE LORENTZ FORCE IN ACCELERATED FRAMES

Synopsis

In the current chapter we present a generalization of the transforms of the electromagnetic field from the frame co-moving with an accelerated particle into an inertial frame of reference. The solution is of great interest for real time applications, because earth-bound laboratories are inertial only in approximation. We conclude by deriving the general form of the relativistic Lorentz force, the relativistic Doppler effect and the relativistic aberration formulas for the case of accelerated motion.

1. Introduction

Real life applications include accelerating and rotating frames more often than the idealized case of inertial frames. Our daily experiments happen in the laboratories attached to the rotating, continuously accelerating Earth. Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-12]. In a recent pair of papers, [13-14], we have presented the equations of electrodynamics in an accelerated /rotating frame as viewed from the point of view of the inertial frame of the laboratory. In the current paper, we are presenting the equations of electrodynamics in an accelerated frame as viewed from the accelerated frame. There is also great interest in producing a general solution that deals with arbitrary orientation of acceleration in the case of rectilinear motion., so we produced the equations for the general case as well. The main idea of this paper is to generate a standard blueprint for a general solution that gives equivalent of the Lorentz transforms for the case of the transforms between an inertial frame and an accelerated frame.

2. Accelerated Rectilinear Motion – the Transforms of the Electromagnetic Field

2.1. Transforms between the Accelerated Frame and the Inertial Frame

In this section we will derive the transforms between the accelerated frame and the inertial frame for the electromagnetic tensor. Let S represent an inertial system of coordinates and $S'(\tau)$ an accelerated one. According to reference [1] the transformation for the particular case of accelerated motion along the x-axis from $S'(\tau)$ into S is:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = Phy\_rectilinear \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} + \begin{bmatrix} \frac{c^2}{g}(\cosh\frac{g\tau}{c} - 1) \\ 0 \\ 0 \\ \frac{c}{g}\sinh\frac{g\tau}{c} \end{bmatrix} \qquad (2.1)$$

where "c" is the speed of light in vacuum, "g" is the proper acceleration, τ is the proper time and

$$Phy\_rectilinear = \begin{bmatrix} \cosh\frac{g\tau}{c} & 0 & 0 & c\sinh\frac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{c}\sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c} \end{bmatrix} \qquad (2.2)$$

The electromagnetic potential, by virtue of being a 4-vector transforms the same way:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \\ c\varphi \end{pmatrix} = Phy\_rectilinear \begin{pmatrix} A_x^{'} \\ A_y^{'} \\ A_z^{'} \\ c\varphi' \end{pmatrix} \tag{2.3}$$

In the inertial frame, the differential Maxwell equations in vacuum, in the absence of electric charge, are [1]:

$$-E_x = \frac{\partial A_x}{\partial t} + c^2 \frac{\partial \varphi}{\partial x} \tag{2.4}$$

$$-E_y = \frac{\partial A_y}{\partial t} + c^2 \frac{\partial \varphi}{\partial y} \tag{2.5}$$

$$-E_z = \frac{\partial A_z}{\partial t} + c^2 \frac{\partial \varphi}{\partial z} \tag{2.6}$$

$$\mathbf{B} = curl\mathbf{A} \tag{2.7}$$

Let's start with (2.4):

$$\frac{\partial A_x}{\partial t} = \frac{\partial A_x^{'}}{\partial t}\cosh\frac{g\tau}{c} + \frac{\partial \varphi^{'}}{\partial t} c\sinh\frac{g\tau}{c}$$

$$\frac{\partial A_x^{'}}{\partial t} = \frac{\partial A_x^{'}}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial A_x^{'}}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial A_x^{'}}{\partial x'}(-c\sinh\frac{g\tau}{c}) + \frac{\partial A_x^{'}}{\partial t'}\cosh\frac{g\tau}{c}$$

$$\frac{\partial \varphi^{'}}{\partial t} = \frac{\partial \varphi^{'}}{\partial x'}(-c\sinh\frac{g\tau}{c}) + \frac{\partial \varphi^{'}}{\partial t'}\cosh\frac{g\tau}{c}$$

$$\frac{\partial A_x}{\partial t} = \frac{\partial A_x^{'}}{\partial x'}(-c\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c}) + \frac{\partial A_x^{'}}{\partial t'}\cosh^2\frac{g\tau}{c} +$$

$$+\frac{\partial \varphi^{'}}{\partial x'}(-c^2\sinh^2\frac{g\tau}{c}) + \frac{\partial \varphi^{'}}{\partial t'} c\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c} \tag{2.8}$$

In a similar manner:

$$\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}(\frac{A_x^{'}}{c}\sin\frac{g\tau}{c}+\varphi'\cos\frac{g\tau}{c})$$

$$\frac{\partial A_x^{'}}{\partial x}=\frac{\partial A_x^{'}}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial A_x^{'}}{\partial t'}\frac{\partial t'}{\partial x}=\frac{\partial A_x^{'}}{\partial x'}\cos\frac{g\tau}{c}+\frac{\partial A_x^{'}}{\partial t'}\frac{-\sin\frac{g\tau}{c}}{c}$$

$$\frac{\partial\varphi'}{\partial x}=\frac{\partial\varphi'}{\partial x'}\cos\frac{g\tau}{c}+\frac{\partial\varphi'}{\partial t'}\frac{-\sin\frac{g\tau}{c}}{c}$$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial A_x^{'}}{\partial x'}\frac{\sin\frac{g\tau}{c}\cos\frac{g\tau}{c}}{c}-\frac{\partial A_x^{'}}{\partial t'}\frac{\sin^2\frac{g\tau}{c}}{c^2}+$$

$$+\frac{\partial\varphi'}{\partial x'}\cos^2\frac{g\tau}{c}-\frac{\partial\varphi'}{\partial t'}\frac{\sin\frac{g\tau}{c}\cos\frac{g\tau}{c}}{c} \qquad (2.9)$$

Substitute (2.8),(2.9) into (2.4):

$$-E_x=\frac{\partial A_x^{'}}{\partial t'}+c^2\frac{\partial\varphi'}{\partial x'}$$

$$E_x=E_x^{'} \qquad (2.10)$$

Moving on to (2.5)

$$\frac{\partial A_y}{\partial t} = \frac{\partial A_y^{'}}{\partial t} = \frac{\partial A_y^{'}}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial A_x^{'}}{\partial t'}\frac{\partial t'}{\partial t} =$$

$$= \frac{\partial A_y^{'}}{\partial x'}(-c\sinh\frac{g\tau}{c}) + \frac{\partial A_y^{'}}{\partial t'}\cosh\frac{g\tau}{c}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y'}(\frac{A_x^{'}}{c}\sinh\frac{g\tau}{c} + \varphi'\cosh\frac{g\tau}{c}) =$$

$$= \frac{\partial A_x^{'}}{\partial y'}\frac{\sinh\frac{g\tau}{c}}{c} + \frac{\partial \varphi'}{\partial y'}\cosh\frac{g\tau}{c} \tag{2.11}$$

Substitute (2.11) into (2.5):

$$E_y = -(\frac{\partial A_y^{'}}{\partial t'} + c^2\frac{\partial \varphi'}{\partial y'})\cosh\frac{g\tau}{c} + \left(\frac{\partial A_y^{'}}{\partial x'} - \frac{\partial A_x^{'}}{\partial y'}\right)c\sinh\frac{g\tau}{c}$$

$$E_y^{'} = -(\frac{\partial A_y^{'}}{\partial t'} + c^2\frac{\partial \varphi'}{\partial y'})$$

$$E_y = E_y^{'}\cosh\frac{g\tau}{c} + \left(\frac{\partial A_y^{'}}{\partial x'} - \frac{\partial A_x^{'}}{\partial y'}\right)c\sinh\frac{g\tau}{c} \tag{2.12}$$

In a similar manner we obtain from (2.6):

$$E_z = E_z^{'}\cosh\frac{g\tau}{c} + \left(\frac{\partial A_z^{'}}{\partial x'} - \frac{\partial A_x^{'}}{\partial z'}\right)c\sinh\frac{g\tau}{c} \tag{2.13}$$

From (2.7) we obtain:

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z^{'}}{\partial y'} - \frac{\partial A_y^{'}}{\partial z'} = B_x^{'} \tag{2.14}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial A_y^{'}}{\partial x} - \frac{\partial A_x}{\partial y'} = (\frac{\partial A_y^{'}}{\partial x'} - \frac{\partial A_x^{'}}{\partial y'})\cosh\frac{g\tau}{c} -$$

$$-\frac{\partial A_y^{'}}{\partial t'}\frac{\sinh\frac{g\tau}{c}}{c} -$$

$$-\frac{\partial \varphi^{'}}{\partial y'}c\sinh\frac{g\tau}{c} = (\frac{\partial A_y^{'}}{\partial x'} - \frac{\partial A_x^{'}}{\partial y'})\cosh\frac{g\tau}{c} -$$

$$-\frac{\sinh\frac{g\tau}{c}}{c}(\frac{\partial A_y^{'}}{\partial t'} + c^2\frac{\partial \varphi^{'}}{\partial y'}) =$$

$$= B_z^{'}\cosh\frac{g\tau}{c} + E_y^{'}\frac{\sinh\frac{g\tau}{c}}{c} \tag{2.15}$$

$$B_z^{'} = \frac{\partial A_y^{'}}{\partial x'} - \frac{\partial A_x^{'}}{\partial y'} \tag{2.16}$$

Therefore:

$$E_y = E_y^{'}\cosh\frac{g\tau}{c} + B_z^{'}c\sinh\frac{g\tau}{c} \tag{2.17}$$

In a similar manner we obtain:

$$B_y = B_y^{'}\cosh\frac{g\tau}{c} - E_z^{'}\frac{\sinh\frac{g\tau}{c}}{c} \tag{2.18}$$

$$B_y^{'} = \frac{\partial A_x^{'}}{\partial z'} - \frac{\partial A_z^{'}}{\partial x'} \tag{2.19}$$

$$E_z = E_z^{'} \cosh\frac{g\tau}{c} - B_y^{'} c \sinh\frac{g\tau}{c} \tag{2.20}$$

Putting everything together:

$$E_x = E_x^{'}$$

$$E_y = E_y^{'} \cosh\frac{g\tau}{c} + B_z^{'} c \sinh\frac{g\tau}{c}$$

$$E_z = E_z^{'} \cosh\frac{g\tau}{c} - B_y^{'} c \sinh\frac{g\tau}{c}$$

$$B_x = B_x^{'}$$

$$B_y = B_y^{'} \cosh\frac{g\tau}{c} - E_z^{'} \frac{\sinh\frac{g\tau}{c}}{c}$$

$$B_z = B_z^{'} \cosh\frac{g\tau}{c} + E_y^{'} \frac{\sinh\frac{g\tau}{c}}{c} \tag{2.21}$$

Notice the resemblance with the standard Lorentz transforms in [1], for example:

$$E_x = E_x^{'}$$

$$E_y = \gamma(E_y^{'} + V\frac{H_z^{'}}{c})$$

$$E_z = \gamma(E_z^{'} - V\frac{H_y^{'}}{c})$$

$$H_x = H_x^{'}$$

$$H_y = \gamma(H_y^{'} - V\frac{E_z^{'}}{c})$$

$$H_z = \gamma(H_z^{'} + V\frac{E_y^{'}}{c}) \qquad (2.22)$$

2.2 Consequences

2.2.1. Maxwell laws in accelerated frame

$$\mathbf{B}' = curl\mathbf{A}'$$

$$\mathbf{E}' = -\frac{\partial \mathbf{A}'}{\partial t'} - \nabla \varphi' \qquad (2.23)$$

The above follows immediately from (2.14), (2.15) and (2.19).

2.2.2. The gauge invariance condition in a uniformly accelerated frame

$$div\mathbf{A}' + \frac{\partial \varphi'}{\partial t'} = div\mathbf{A} + \frac{\partial \varphi}{\partial t} = 0 \qquad (2.24)$$

$$div\mathbf{A}' = \frac{\partial A_x^{'}}{\partial x'} + \frac{\partial A_y^{'}}{\partial y'} + \frac{\partial A_z^{'}}{\partial z'} = \frac{\partial A_x^{'}}{\partial x'} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (2.25)$$

$$\frac{\partial A_x^{'}}{\partial x'}=\frac{\partial A_x}{\partial x}\cosh^2\frac{g\tau}{c}+\frac{\partial A_x}{\partial t}\frac{\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c}}{c}-$$
$$-\frac{\partial\varphi}{\partial x}c\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c}-\frac{\partial\varphi}{\partial t}\sinh^2\frac{g\tau}{c} \quad (2.26)$$

$$\frac{\partial\varphi'}{\partial t'}=-\frac{\partial A_x}{\partial x}\sinh^2\frac{g\tau}{c}-\frac{\partial A_x}{\partial t}\frac{\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c}}{c}+$$
$$+\frac{\partial\varphi}{\partial x}c\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c}+\frac{\partial\varphi}{\partial t}\cosh^2\frac{g\tau}{c} \quad (2.27)$$

$$\frac{\partial A_x^{'}}{\partial x'}+\frac{\partial\varphi'}{\partial t'}=\frac{\partial A_x}{\partial x}+\frac{\partial\varphi}{\partial t} \quad (2.28)$$

Equality (2.28) results into:

$$div\mathbf{A}'+\frac{\partial\varphi'}{\partial t'}=div\mathbf{A}+\frac{\partial\varphi}{\partial t}=0 \quad (2.29)$$

Equalities (2.23) result into Maxwell's wave equations having the same exact form in the accelerated frame as the equations in the inertial frames with the immediate consequence that light speed in vacuum in a uniformly accelerated frame is "c". Indeed, (2.23) results into:

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}'}{\partial t'^2}-\nabla^2\mathbf{E}'=0$$
$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}'}{\partial t'^2}-\nabla^2\mathbf{B}'=0 \quad (2.30)$$

The above means that electromagnetic waves propagate in the accelerated frame, in vacuum, at the same speed as they propagate in inertial frames. We will re-derive this interesting result in a different way, alongside several other interesting consequences, in the next section.

2.2.3. The Lorentz Force in an Accelerated Frame

In the inertial frame, the Lorentz force has the expression:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{v} = \mathrm{v}_x \mathbf{e}_x$$

$$\mathbf{F} = q(\mathrm{E}_x \mathbf{e}_x + \mathrm{E}_y \mathbf{e}_y + \mathrm{E}_z \mathbf{e}_z + \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \mathrm{v}_x & 0 & 0 \\ \mathrm{B}_x & \mathrm{B}_y & \mathrm{B}_z \end{bmatrix}) =$$

$$= q(\mathrm{E}_x \mathbf{e}_x + (\mathrm{E}_y - \mathrm{v}_x \mathrm{B}_z)\mathbf{e}_y + (\mathrm{E}_z + \mathrm{v}_x \mathrm{B}_y)\mathbf{e}_z) \tag{2.31}$$

We know that:

$$dx = dx' \cosh \frac{g\tau}{c} + cdt' \sinh \frac{g\tau}{c}$$

$$dt = \frac{1}{c} \sinh \frac{g\tau}{c} dx' + dt' \cosh \frac{g\tau}{c}$$

$$v_x = \frac{dx}{dt} = \frac{\frac{dx'}{dt'} \cosh \frac{g\tau}{c} + c \sinh \frac{g\tau}{c}}{\frac{1}{c} \sinh \frac{g\tau}{c} \frac{dx'}{dt'} + \cosh \frac{g\tau}{c}} =$$

$$= \frac{v_x' \cosh \frac{g\tau}{c} + c \sinh \frac{g\tau}{c}}{\frac{v_x'}{c} \sinh \frac{g\tau}{c} + \cosh \frac{g\tau}{c}} \tag{2.32}$$

The formula:

$$v_x = \frac{v_x' \cosh \frac{g\tau}{c} + c \sinh \frac{g\tau}{c}}{\frac{v_x'}{c} \sinh \frac{g\tau}{c} + \cosh \frac{g\tau}{c}} \tag{2.33}$$

ties the speed v_x' of the particle in the accelerated frame to its measured speed in the inertial frame v_x. Using the above, we obtain:

$$\mathrm{E_y\text{-}v}_x\mathrm{B}_z = E_y' \cosh\frac{g\tau}{c} + B_z' c \sinh\frac{g\tau}{c} -$$

$$-(\frac{E_y'}{c}\sinh\frac{g\tau}{c} + B_z'\cosh\frac{g\tau}{c})\frac{v_x'\cosh\frac{g\tau}{c} + c\sinh\frac{g\tau}{c}}{\frac{v_x'}{c}\sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}} =$$

$$= \frac{E_y' - v_x' B_z'}{\frac{v_x'}{c}\sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}} \tag{2.34}$$

Similarly:

$$\mathrm{E}_z + \mathrm{v}_x\mathrm{B}_y = \frac{E_z' + v_x' B_y'}{\frac{v_x'}{c}\sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}} \tag{2.35}$$

So, we can write:

$$\mathbf{F}_\mathbf{P}' = \mathbf{F}_\mathbf{P} = qE_x\mathbf{e}_x$$

$$\mathbf{F}_\perp' = \frac{\mathbf{F}_\perp}{\frac{v_x'}{c}\sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}} =$$

$$= q\frac{(E_y' - v_x' B_z')\mathbf{e}_y + (E_z' + v_x' B_y')\mathbf{e}_z}{\frac{v_x'}{c}\sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}} \tag{2.36}$$

Expressions (2.36) represent the transformation of the Lorentz force between the inertial and the accelerated frame.

2.2.4. Bremsstrahlung

Bremsstrahlung is the electromagnetic radiation produced by the *deceleration* of a charged particle. The moving particle loses kinetic energy, which is converted into a photon. This it is the process of producing the energy radiation. Bremsstrahlung has a continuous spectrum, which becomes more intense and whose peak intensity shifts toward higher frequencies as the change of the energy of the decelerated particles increases. The term is frequently used in the more narrow sense of radiation from electrons (from whatever source) slowing in matter. In astrophysics, bremsstrahlung refers to radiation emitted from zones of the universe characterized by a high concentration of plasma. The radiation in this case is created by charged particles that are free; i.e., not part of an ion, atom or molecule, both before and after the deflection (*acceleration*) that caused the emission. In any case, the total radiated power is given by [15]

$$P = \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}(\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\vec{\beta} = \frac{\vec{v}}{c}$$

$$\dot{\vec{\beta}} = \frac{\vec{g}}{c} \qquad (2.37)$$

In the case where velocity is parallel to acceleration (for example, linear motion), the formula simplifies to [15]:

$$P = \frac{q^2 g^2 \gamma^6}{6\pi\varepsilon_0 c^3} \qquad (2.38)$$

For the case of acceleration perpendicular to the velocity (as in the case of synchrotrons), the formula simplifies to:

$$P = \frac{q^2 g^2 \gamma^4}{6\pi\varepsilon_0 c^3} \tag{2.39}$$

In either case, we do not observe any significant X-rays radiated from the free electrons in the Earth atmosphere due to several factors:

- the speed of the electrons is low (γ is small, very close to unity),
- the deceleration "g" is very small due to the absence of fields or matter that could affect the free electrons
- the electron density per unit of volume is small
- the presence of c^{-3}

By contrast, this is not the case in astrophysics. where we have observed **significant** emission from certain galaxies' intra-cluster medium due to thermal bremsstrahlung. This radiation is in the energy range of X-rays and can be easily observed with space-based telescopes such as Chandra X-ray Observatory, XMM-Newton, ROSAT, ASCA, EXOSAT, Suzaku, RHESSI and future missions like IXO [16] and Astro-H [17]. The reasons for observing such effects are:

- much higher charge density
- much larger speeds and accelerations (due to the presence of strong magnetic fields)

In addition to the changes in frequency (energy), we also observe light polarization effects, due to the presence of the magnetic fields mentioned above.

3. Planar Wave Transformation and Speed of Light in a Uniformly Accelerated Frame

In this section we apply the formalism derived in the previous paragraph in order to obtain the transform of a planar wave. Assume that a planar wave is propagating along the y' axis in the accelerated frame $S'(\tau)$. The wave has the electric component $\mathbf{E}_x^{'}$ and the magnetic component $\mathbf{B}_z^{'}$ along the x' and z' axes, respectively. The components equations are (see fig.1):

$$\mathbf{E}_x^{'} = E_{0x}^{'} \cos(\omega' t' - k_y^{'} y' + \varphi')\mathbf{e}_x$$
$$\mathbf{B}_z^{'} = B_{0z}^{'} \cos(\omega' t' - k_y^{'} y' + \varphi')\mathbf{e}_z \tag{3.1}$$

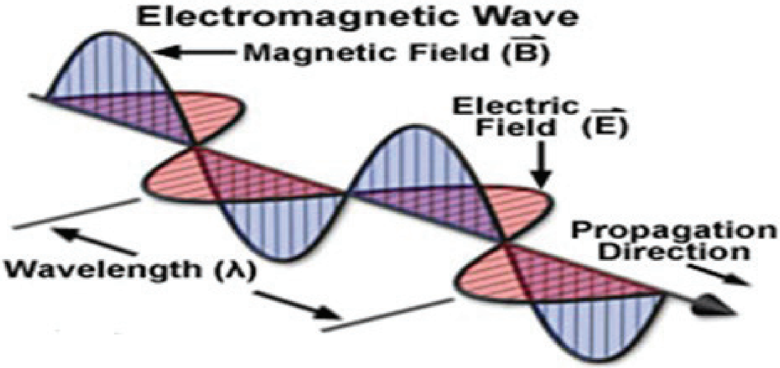

Fig. 1. The planar electromagnetic wave

Inverting transforms (2.1) we obtain:

$$y' = y$$
$$t' = -\frac{x}{c}\sinh\frac{g\tau}{c} + t\cosh\frac{g\tau}{c} - \frac{a}{c}\sinh\frac{g\tau}{c} + b\cosh\frac{g\tau}{c}$$
$$a = \frac{c^2}{g}(\cosh\frac{g\tau}{c} - 1)$$
$$b = \frac{c}{g}\sinh\frac{g\tau}{c} \tag{3.2}$$

Substituting (3.2) into (3.1) we obtain:

$$E_x^{'} = E_{0x}^{'} \cos[(\omega' \cosh \frac{g\tau}{c})t - (\frac{\omega'}{c} \sinh \frac{g\tau}{c})x -$$
$$-k_y^{'} y + \varphi' + \omega'(b \cosh \frac{g\tau}{c} - \frac{a}{c} \sinh \frac{g\tau}{c})] \quad (3.3)$$

On the other hand, in frame S, the wave equation is:

$$\mathbf{E}_x = E_{0x} \cos(\omega t - k_x x - k_y y - k_z z + \varphi)\mathbf{e}_x \quad (3.4)$$

Since $\mathbf{E}_x = \mathbf{E}_x^{'}$ it follows that:

$$E_{0x}^{'} = E_{0x}$$
$$\omega = \omega' \cosh \frac{g\tau}{c}$$
$$k_x = \frac{\omega'}{c} \sinh \frac{g\tau}{c} = \frac{\omega}{c} \tanh \frac{g\tau}{c}$$
$$k_y = k_y^{'}$$
$$k_z = 0$$
$$\varphi = \varphi' + \omega'(b \cosh \frac{g\tau}{c} - \frac{a}{c} \sinh \frac{g\tau}{c}) \quad (3.5)$$

The formula $\omega = \omega' \cosh \frac{g\tau}{c}$ represents the Doppler effect due to acceleration. We see that the pulsation decreases in time by the factor $\cosh \frac{g\tau}{c}$ in a frame that is uniformly accelerated in the same direction of the propagation of the electromagnetic wave.

From (3.8) we obtain:

$$\frac{\omega^2}{c^2} = k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}\tanh^2\frac{g\tau}{c} + k_y^{'2} \tag{3.6}$$

Therefore, in frame $S'(\tau)$ the wave vector is:

$$k' = k_y^{'} = \frac{\omega}{c\cosh\frac{g\tau}{c}} \tag{3.7}$$

We can now calculate the phase light speed in the accelerated frame:

$$v_p^{'} = \frac{\omega'}{k'} = \frac{\omega}{k} = c \tag{3.8}$$

So, the light speed in the accelerated frame equals the light speed in the inertial frame, c.

We can now proceed to calculating the amplitude and the phase transformation between the inertial and the accelerated frame:

$$\varphi = \varphi' + \omega'(b\cosh\frac{g\tau}{c} - \frac{a}{c}\sinh\frac{g\tau}{c}) =$$
$$= \varphi' + \frac{\omega c}{g}\tanh\frac{g\tau}{c}$$
$$\varphi' = \varphi - \frac{\omega c}{g}\tanh\frac{g\tau}{c} \tag{3.9}$$

The magnetic field component transforms as:

$$B_{0z} = B_{0z}^{'}\cosh\frac{g\tau}{c} \tag{3.10}$$

So, the absolute value of the Poynting vector transforms as:

$$S = E_{0x}B_{0z} = S'\cosh\frac{g\tau}{c} \tag{3.11}$$

So the electromagnetic flux in the accelerated frame decreases by with respect to the flux in the inertial frame. Finally, we can calculate the aberration in frame S induced by the acceleration:

$$\cos\theta = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} = \tanh\frac{g\tau}{c} \qquad (3.12)$$

4. General Case of Uniform Acceleration in an Arbitrary Direction

In a prior paper we have shown [12] that the particular transformation (2.1) can be generalized for the case of arbitrary direction constant acceleration $\mathbf{g} = (g_x, g_y, g_z)$ to:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = (Tr^{-1} * Phy\_rectilinear * Tr)\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} + N \qquad (4.1)$$

where:

$$Tr = Rot(\mathbf{e}_z)_{-90^0} * Rot(\mathbf{e}_y)_{90^0-\varphi} * Rot_y \qquad (4.2)$$

Introducing the triplet $(a,b,c) = (-\frac{g_z}{g}, 0, \frac{g_x}{g})$ the following expressions hold:

$$Rot_y = \begin{bmatrix} c & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4.3)$$

$$Rot(\mathbf{e}_y)_{90^0-\varphi} = \begin{bmatrix} \cos(90^0-\varphi) & 0 & -\sin(90^0-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(90^0-\varphi) & 0 & \cos(90^0-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sin\varphi & 0 & -\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.4}$$

$Rot(\mathbf{e}_y)_{90^0-\varphi} * Rot_y$ aligns $\mathbf{g}$ with $\mathbf{e}_y$. The second step is comprised by another rotation around the z-axis by -90^0 that aligns $\mathbf{g}$ with $\mathbf{e}_x$ (fig.2):

$$Rot(\mathbf{e}_z)_{-90^0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$

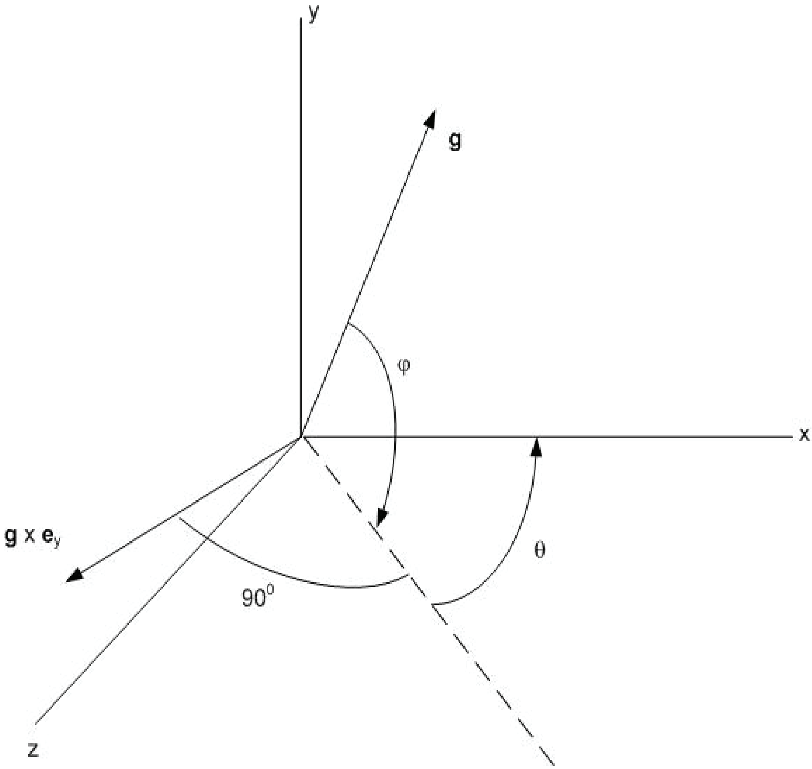

Fig. 2 General acceleration

Tr^{-1} reverses all the effects of Tr. Expression (4.1) gives the solution for the general case, of arbitrary acceleration direction. The net effect is that the derivative operators become more complicated:

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t'} \tag{4.6}$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x'} \tag{4.7}$$

$$\begin{bmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} & \frac{\partial x}{\partial t'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} & \frac{\partial y}{\partial t'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} & \frac{\partial z}{\partial t'} \\ \frac{\partial t}{\partial x'} & \frac{\partial t}{\partial y'} & \frac{\partial t}{\partial z'} & \frac{\partial t}{\partial t'} \end{bmatrix} = Tr^{-1} * Phy\_rectilinear * Tr \qquad (4.8)$$

Using (4.8) in (4.6), (4.7) gives the general forms of the transforms (2.21) for the electromagnetic field tensor. Next, we will show a very nice way of getting the general transforms. We start by writing (2.21) in the form:

$$\begin{bmatrix} E_x & 0 & 0 & 0 \\ * & \frac{E_y}{c} & B_z & * \\ B_y & * & * & \frac{E_z}{c} \\ 0 & 0 & 0 & B_x \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\frac{g\tau}{c} & \sinh\frac{g\tau}{c} & 0 \\ 0 & -\sinh\frac{g\tau}{c} & \cosh\frac{g\tau}{c} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E'_x & 0 & 0 & 0 \\ \frac{E'_z}{c} & \frac{E'_y}{c} & B'_z & B'_y \\ B'_y & B'_z & \frac{E'_y}{c} & \frac{E'_z}{c} \\ 0 & 0 & 0 & B'_x \end{bmatrix} \qquad (4.9)$$

The elements marked with asterisks represent entities without any meaning. We do not care about them. Then, the matrix for the general transform is simply:

$$Tr^{-1} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\frac{g\tau}{c} & \sinh\frac{g\tau}{c} & 0 \\ 0 & -\sinh\frac{g\tau}{c} & \cosh\frac{g\tau}{c} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * Tr \tag{4.10}$$

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Relativistic Dynamics and Electrodynamics in Uniformly Accelerated and in Uniformly Rotating Frames - the General Expressions for the Electromagnetic 4-Vector Potential

Synopsis

In the current chapter we present a generalization of the transforms from the frame co-moving with an accelerated particle, for either rectilinear or circular motion, into an inertial frame of reference. The solution is of great interest for real time applications, because earth-bound laboratories are inertial only in approximation, The motivation is that the real life applications include accelerating and rotating frames with arbitrary orientation more often than the idealized case of inertial frames; our daily experiments happen in the laboratories attached to the rotating Earth. The chapter is divided into two main sections, the first section deals with dynamics, i.e. forces, the second section deals with electromagnetism, i.e. electromagnetic potentials.

1. Introduction

Real life applications include accelerating and rotating frames more often than the idealized case of inertial frames. Our daily experiments happen in the laboratories attached to the rotating Earth. Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-13], [15]. There is great interest in producing a general solution that deals with arbitrary orientation of acceleration in the case of rectilinear motion and for arbitrary direction of uniform angular velocity.

The main idea of this chapter is to generate a standard blueprint for a general solution. The blueprint relies on transforming the problem geometrically in the "canonical reference frame" of [1], followed by the application of the physical transforms derived for such "canonical" orientations [1-7] and ending with the application of the inverse geometrical transformation:

$$Geometry\_Transform->Physics\_Transform->Inverse\_Geometry\_Transform \quad (1.1)$$

We conclude our paper with a practical application of deriving the formula of the Lorentz force in a uniformly rotating frame.

2. Dynamics in Accelerated Rectilinear Motion

Let S represent an inertial system of coordinates and $S'(\tau)$ an accelerated one. Moller [1] considers a particular case where a particle moves with acceleration $\mathbf{g}=(g,0,0)$ aligned with the x-axis. According to reference [1] the transformation for the particular case from $S'(\tau)$ into S is:

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \mathbf{Phy\_rectilinear} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \quad (2.1)$$

where:

$$\mathbf{Phy\_rectilinear} = \begin{bmatrix} \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c} \end{bmatrix} \quad (2.2)$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \frac{\mathbf{Phy\_rectilinear}\frac{d}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} + \frac{d\mathbf{Phy\_rectilinear}}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{dt}{dt'}} =$$

$$= \frac{\mathbf{Phy\_rectilinear}\frac{d}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} + \frac{d\mathbf{Phy\_rectilinear}}{d\tau}\frac{d\tau}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{dt}{dt'}} \tag{2.3}$$

Therefore:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\mathbf{Phy\_rectilinear}\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} + \frac{d\mathbf{Phy\_rectilinear}}{d\tau}\frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{v_x'}{c}\sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c} + \frac{\frac{g}{c}}{\sqrt{1+(\frac{gt'}{c})^2}}(\frac{x'}{c}\cosh\frac{g\tau}{c} + t'\sinh\frac{g\tau}{c})} \tag{2.4}$$

where:

$$\frac{d\mathbf{Phy\_rectilinear}}{d\tau}=\frac{g}{c}\begin{bmatrix}\sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c}\end{bmatrix} \quad (2.5)$$

The speed measured in the inertial frame depends both on the speed and the position measured in the accelerated frame. If we consider the case of the accelerated frame commoving with the object (particle) under study, then:

$$\begin{pmatrix}x'\\ y'\\ z'\\ ct'\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ ct'\end{pmatrix} \quad (2.6)$$

and:

$$\begin{pmatrix}v_x\\ v_y\\ v_z\\ c\end{pmatrix}=\frac{\mathbf{Phy\_rectilinear}\begin{pmatrix}v_x'\\ v_y'\\ v_z'\\ c\end{pmatrix}+\frac{d\mathbf{Phy\_rectilinear}}{d\tau}\frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\begin{pmatrix}0\\ 0\\ 0\\ ct'\end{pmatrix}}{\frac{v_x'}{c}\sinh\frac{g\tau}{c}+\cosh\frac{g\tau}{c}+\frac{\frac{gt'}{c}}{\sqrt{1+(\frac{gt'}{c})^2}}\sinh\frac{g\tau}{c}} \quad (2.7)$$

In the following section we generalize his derivation for the arbitrary case $\mathbf{g}=(g_x,g_y,g_z)$ for obtaining the general four-space coordinate transformations that take us from $S'(\tau)$ into S. Expressed in polar coordinates, the acceleration has the form:

$$g_x = g\cos\theta\cos\varphi$$
$$g_z = g\sin\theta\cos\varphi$$
$$g_y = g\sin\varphi$$
$$\varphi = \arcsin\frac{g_y}{g}$$
$$\theta = \arctan\frac{g_z}{g_x}$$

(2.8)

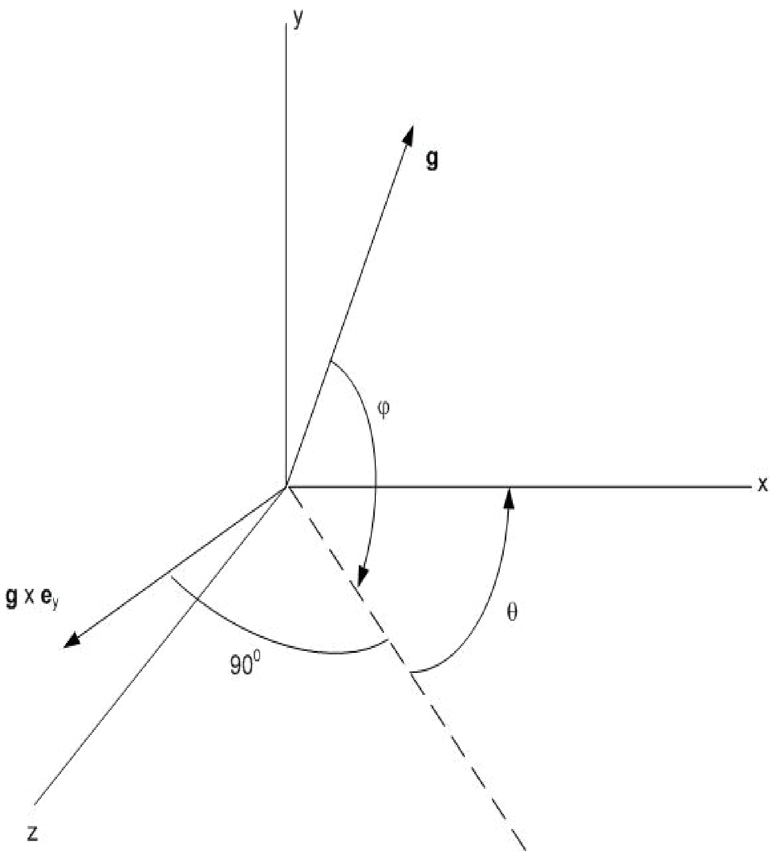

Fig. 1 Arbitrary direction rectilinear accelerated motion

The first step rotates the unit vector of acceleration $\mathbf{g}$ by $90^0-\varphi$ around the axis the vector cross-product $\mathbf{g}\times\mathbf{e}_y=-\frac{g_z}{g}\mathbf{e}_x+\frac{g_x}{g}\mathbf{e}_z$ such $\mathbf{g}$ gets aligned with the y-axis (see Fig.1). For this purpose, we will introduce the triplet $(a,b,c)=(-\frac{g_z}{g},0,\frac{g_x}{g})$. The following expressions hold [14]:

$$\mathbf{Rot_y}=\begin{bmatrix} c & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

$$\mathbf{Rot(e_y)_{90^0\text{-}\varphi}}=\begin{bmatrix} \cos(90^0-\varphi) & 0 & -\sin(90^0-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(90^0-\varphi) & 0 & \cos(90^0-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=$$

$$=\begin{bmatrix} \sin\varphi & 0 & -\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.10}$$

$\mathbf{Rot(e_y)_{90^0\text{-}\varphi}*Rot_y}$ aligns $\mathbf{g}$ with $\mathbf{e}_y$. The second step is comprised by another rotation around the z-axis by -90^0 that aligns $\mathbf{g}$ with $\mathbf{e}_x$:

$$\mathbf{Rot(e_z)_{-90^0}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.11}$$

Putting it all together:

$$\mathbf{Rr = Rot(e_x)_{90^0} * Rot(e_y)_{90^0-\varphi} * Rot_y} \tag{2.12}$$

The general coordinate transformation between S and $S'(\tau)$ is:

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \mathbf{A} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \tag{2.13}$$

where:

$$\mathbf{Rr^{-1} * Phy\_rectilinear * Rr} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \mathbf{A} \tag{2.14}$$

The general velocity transformation is therefore:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} =$$

$$= \frac{\mathbf{A}\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}} \frac{d\mathbf{A}}{d\tau}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau})} \tag{2.15}$$

where:

$$\frac{d\mathbf{A}}{d\tau} = \mathbf{Rr}^{-1} * \frac{d\mathbf{Phy\_rectilinear}}{d\tau} * \mathbf{Rr} \tag{2.16}$$

The inverse transform is:

$$\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} =$$

$$= \frac{\mathbf{A}^{-1}\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}} \frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} v_x \\ v_y \\ v_z \\ ct \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}}(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{2.17}$$

where:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \mathbf{A}^{-1} = \mathbf{Rr}^{\mathbf{-1}} * \mathbf{Phy\_rectilinear}^{-1} * \mathbf{Rr} \tag{2.18}$$

If we consider the case of the accelerated frame commoving with the object (particle) under study, then:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct' \end{pmatrix} \tag{2.19}$$

and (2.17) simplifies to:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} v_x^{'} \\ v_y^{'} \\ v_z^{'} \\ c \end{pmatrix} + \frac{d\mathbf{A}}{d\tau}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{ct'}{\sqrt{1+(\frac{gt'}{c})^2}} \end{pmatrix}}{\frac{v_x^{'}}{c}a_{41} + \frac{v_y^{'}}{c}a_{42} + \frac{v_z^{'}}{c}a_{43} + a_{44} + \dfrac{t'\frac{da_{44}}{d\tau}}{\sqrt{1+(\frac{gt'}{c})^2}}} \tag{2.20}$$

The coordinate acceleration can be derived immediately as:

$$\begin{pmatrix} a_x \\ a_y \\ a_z \\ 0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\frac{d}{dt'}\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix}}{\frac{dt}{dt'}} \tag{2.21}$$

The energy-momentum transforms the same way as the 4-coordinates (2.13) by virtue of being a 4-vector:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix} = \mathbf{A}\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ E'/c \end{pmatrix} \tag{2.22}$$

Therefore:

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ E'/c \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau})} \tag{2.23}$$

The inverse transform, from the inertial frame S into the accelerated frame S', is:

$$\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}}\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}}(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{2.24}$$

If we consider the case of the accelerated frame commoving with the object (particle) under study, then:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct' \end{pmatrix} \tag{2.25}$$

and:

$$\begin{pmatrix} p_x^{'} \\ p_y^{'} \\ p_z^{'} \\ E'/c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_0c^2/c \end{pmatrix} \tag{2.26}$$

In this case:

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ 0 \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} 0 \\ 0 \\ 0 \\ m_0 c \end{pmatrix}}{a_{44} + \frac{t'\frac{da_{44}}{d\tau}}{\sqrt{1+(\frac{gt'}{c})^2}}} \tag{2.27}$$

3. Dynamics in Uniform Angular Velocity Rotation

In this section we discuss the case of the particle moving in an arbitrary plane, with the normal given by the constant angular velocity $\boldsymbol{\omega}(a,b,c)$ (see Fig.2). According to Moller [1], the simpler case when $\boldsymbol{\omega}$ is aligned with the z-axis produces the transformation between the rotating frame $S'(\tau)$ attached to the particle and an inertial, non-rotating frame S attached to the center of rotation:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{Phy\_rotation}\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{3.1}$$

where:

$$\mathbf{Phy\_rotation} = \begin{bmatrix} \cos\alpha\cos\beta+\gamma\sin\alpha\sin\beta & \sin\alpha\cos\beta-\gamma\cos\alpha\sin\beta & 0 & -\frac{u\gamma}{c}\sin\beta \\ \cos\alpha\sin\beta-\gamma\sin\alpha\cos\beta & \sin\alpha\sin\beta+\gamma\cos\alpha\cos\beta & 0 & \frac{u\gamma}{c}\cos\beta \\ 0 & 0 & 1 & 0 \\ \frac{u\gamma}{c}\sin\alpha & -\frac{u\gamma}{c}\cos\alpha & 0 & \gamma \end{bmatrix} \tag{3.2}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$
$$u = r\omega$$
$$\alpha = \omega\gamma\tau$$
$$\beta = \omega\gamma^2\tau \tag{3.3}$$

The general case is treated by transforming the problem into the particular case treated in [1] through a transformation into the "canonical case", followed by an application of the transformation from the accelerated frame into the inertial frame, ending with the inverse of the first transformation, as shown below:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = (\mathbf{Rr^{-1} * Phy\_rotation * Rr}) \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{3.4}$$

$$\mathbf{Rr = Rot(e_x)_{90^0} * Rot(e_y)_{90^0 - \varphi} * Rot_y} \tag{3.5}$$

$\mathbf{Rot(e_y)_{90^0 - \varphi} * Rot_y}$ aligns $\mathbf{g}$ with $\mathbf{e}_y$. The second step is comprised by another rotation around the x-axis by 90^0 that aligns $\boldsymbol{\omega}$ with $\mathbf{e}_z$:

$$\mathbf{Rot(e_x)_{90^0}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.6}$$

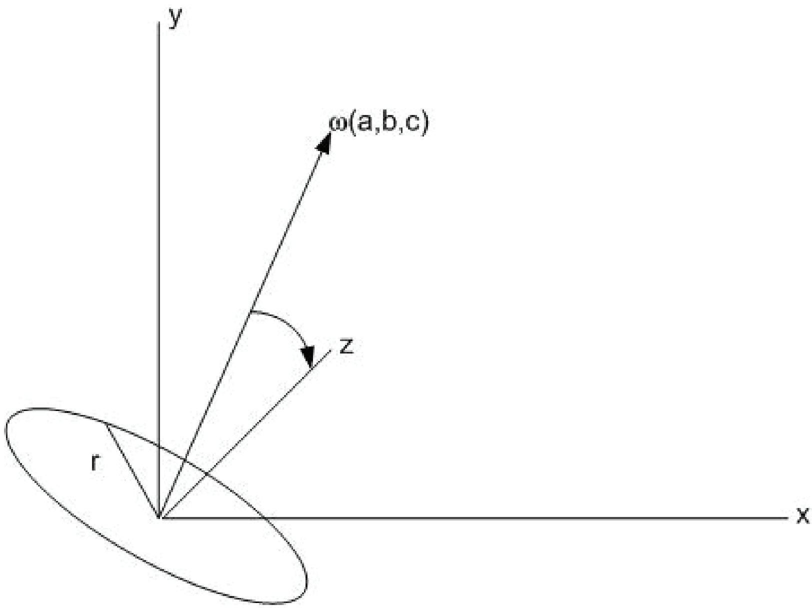

Fig. 2 Uniform rotation with arbitrary direction of angular velocity

Expression (3.4) gives the solution for the general case, of arbitrary angular velocity direction.

The general velocity transformation is:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau})}$$

(3.7)

The general force transformation is:

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ E'/c \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau})} \tag{3.8}$$

where:

$$\mathbf{A} = \mathbf{Rr}^{-1} * \mathbf{Phy\_rotation} * \mathbf{Rr} \tag{3.9}$$

The reverse transformation is:

$$\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{3.10}$$

where:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \mathbf{A}^{-1} = \mathbf{Rr}^{-1} * \mathbf{Phy\_rotation}^{-1} * \mathbf{Rr} \tag{3.11}$$

We will use (3.10) in the next section, an application that determines the expression of the Lorentz force in a rotating frame.

4. Application-The Expression of the Lorentz Force in a Uniformly Rotating Frame

Assume that we have a particle of charge q and mass m moving in the x-y plane under the influence of a magnetic field $\mathbf{B}$ aligned with the z axis. We know that in the frame of the lab, the expression of the Lorentz force acting on the particle is:

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} = q\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\omega\cos(\omega t) & r\omega\sin(\omega t) & 0 \\ 0 & 0 & B \end{bmatrix} \tag{4.1}$$

We would like to find out the expression of the force in the frame co-rotating with the charged particle. For this purpose we will resort to (3.10)

$$\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \sqrt{1-(\frac{r\omega}{c})^2}(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{4.2}$$

We know from [15,16] that:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \\ 0 \\ t \end{pmatrix} \tag{4.3}$$

$$v_x = -r\omega\sin(\omega t)$$
$$v_y = r\omega\cos(\omega t)$$
$$v_z = 0 \tag{4.4}$$

$$p_x = -\frac{mr\omega\sin(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}}$$

$$p_y = \frac{mr\omega\cos(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}}$$

$$p_z = 0$$

$$E = \frac{mc^2}{\sqrt{1-(\frac{r\omega}{c})^2}} \tag{4.5}$$

$$F_x = qBr\omega\sin(\omega t)$$

$$F_y = -qBr\omega\cos(\omega t)$$

$$F_z = 0 \tag{4.6}$$

Substituting (4.3-4.6) into (4.2) we obtain:

$$\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} qBr\omega\sin(\omega t) \\ -qBr\omega\cos(\omega t) \\ 0 \\ 0 \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} -\frac{mr\omega\sin(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}} \\ \frac{mr\omega\cos(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}} \\ 0 \\ \frac{mc}{\sqrt{1-(\frac{r\omega}{c})^2}} \end{pmatrix}}{\frac{-r\omega\sin(\omega t)}{c}b_{41} + \frac{r\omega\cos(\omega t)}{c}b_{42} + b_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(r\cos(\omega t)\frac{db_{41}}{cd\tau} + r\sin(\omega t)\frac{db_{42}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{4.7}$$

To the above, we need to add [16] the fact that:

$$\omega = \frac{qB}{\gamma(v_0)m}$$

$$r = \frac{\gamma(v_0)mv_0}{qB} \tag{4.8}$$

In (4.8) v_0 is the initial speed of the particle at $t=0$. Armed with that (4.7) gets the simpler form:

$$\begin{pmatrix} F'_x \\ F'_y \\ F'_z \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} qBv_0\sin(\frac{qBt}{\gamma(v_0)m}) \\ -qBv_0\cos(\frac{qBt}{\gamma(v_0)m}) \\ 0 \\ 0 \end{pmatrix} + \frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} -mv_0\sin(\frac{qBt}{\gamma(v_0)m}) \\ mv_0\cos(\frac{qBt}{\gamma(v_0)m}) \\ 0 \\ mc \end{pmatrix}}{\frac{v_0 b_{42}\cos\frac{qBt}{\gamma(v_0)m}}{c} - \frac{v_0 b_{41}\sin\frac{qBt}{\gamma(v_0)m}}{c} + b_{44} + \frac{mv_0}{qBc}\left(\frac{db_{41}}{d\tau}\cos\frac{qBt}{\gamma(v_0)m} + \frac{db_{42}}{d\tau}\sin\frac{qBt}{\gamma(v_0)m} + \frac{qBct}{\gamma(v_0)mv_0}\frac{db_{44}}{d\tau}\right)} \tag{4.9}$$

5. Electrodynamics in Uniformly Accelerated Frames and in Uniformly Rotating Frames – the General Expressions for the Electromagnetic 4-Vector Potential

Previously [17,18] we have dealt with the case of the transformation of Maxwell equations for the case of uniformly accelerated frames and uniformly rotating frames in arbitrary directions. The formalism derived in this paper, section 2, allows us to get a general transformation between the inertial frame S and S' and the inverse. The electromagnetic potential transforms the same way as the 4-coordinates (2.13) by virtue of being a 4-vector:

$$\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\mathbf{A}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix} \tag{5.1}$$

where $\mathbf{A}$ is given by (2.14) for uniformly accelerated frames and by (3.9) for uniformly rotating frames.

In order to transform Maxwell equations between the frames, we need the **partial** derivatives with respect to x, y, z, t. We will show how to calculate two of them, as a blueprint.

$$\frac{\partial}{\partial t}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\mathbf{A}\frac{\partial}{\partial t}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\frac{\mathbf{A}\dfrac{\partial}{\partial t'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{\dfrac{\partial t}{\partial t'}}=\frac{\mathbf{A}\dfrac{\partial}{\partial t'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{a_{44}}$$

$$\frac{\partial}{\partial x}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\mathbf{A}\frac{\partial}{\partial x}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\frac{\mathbf{A}\dfrac{\partial}{\partial x'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{\dfrac{\partial x}{\partial x'}}=\frac{\mathbf{A}\dfrac{\partial}{\partial x'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{a_{11}} \tag{5.2}$$

The inverse transforms are:

$$\frac{\partial}{\partial t'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\mathbf{A}^{-1}\frac{\partial}{\partial t'}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial t}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{\frac{\partial t'}{\partial t}}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial t}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{b_{44}}$$

$$\frac{\partial}{\partial x'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\mathbf{A}^{-1}\frac{\partial}{\partial x'}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial x}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{\frac{\partial x'}{\partial x}}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial x}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{b_{11}} \quad (5.3)$$

By using the above generalized transforms we can transform the electric and magnetic vectors, thus obtaining the general transformations for the Maxwell equations as seen in [17,18]

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Relativistic Dynamics and Electrodynamics in Uniformly Accelerated and Uniformly Rotating Frames - the General Expressions for the Electromagnetic 4-Vector Potential

Synopsis

In the current chapter we present a generalization of the transforms from the frame co-moving with an accelerated particle, for either rectilinear or circular motion, into an inertial frame of reference. The solution is of great interest for real time applications, because earth-bound laboratories are inertial only in approximation, The motivation is that the real life applications include accelerating and rotating frames with arbitrary orientation more often than the idealized case of inertial frames; our daily experiments happen in the laboratories attached to the rotating Earth. The chapter is divided into two main sections, the first section deals with dynamics, i.e. forces and the second section deals with electromagnetism, i.e. electromagnetic potentials.

1. Introduction

Real life applications include accelerating and rotating frames more often than the idealized case of inertial frames. Our daily experiments happen in the laboratories attached to the rotating Earth. Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-13], [15]. There is great interest in producing a general solution that deals with arbitrary orientation of acceleration in the case of rectilinear motion and for arbitrary direction of uniform angular velocity.

The main idea of this paper is to generate a standard blueprint for a general solution. The blueprint relies on transforming the problem geometrically in the “canonical reference frame” of [1], followed by the application of the physical transforms derived for such “canonical” orientations [1-7] and ending with the application of the inverse geometrical transformation:

$$Geometry\_Transform -> Physics\_Transform -> Inverse\_Geometry\_Transform \tag{1.1}$$

We conclude our paper with a practical application of deriving the formula of the Lorentz force in a uniformly rotating frame.

2. Dynamics in Accelerated Rectilinear Motion

Let S represent an inertial system of coordinates and $S'(\tau)$ an accelerated one. Moller [1] considers a particular case where a particle moves with acceleration $\mathbf{g} = (g, 0, 0)$ aligned with the x-axis. According to reference [1] the transformation for the particular case from $S'(\tau)$ into S is:

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \mathbf{Phy\_rectilinear} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \tag{2.1}$$

where:

$$\mathbf{Phy\_rectilinear} = \begin{bmatrix} \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c} \end{bmatrix} \tag{2.2}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \frac{\textbf{Phy\_rectilinear}\frac{d}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} + \frac{d\textbf{Phy\_rectilinear}}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{dt}{dt'}} =$$

$$= \frac{\textbf{Phy\_rectilinear}\frac{d}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} + \frac{d\textbf{Phy\_rectilinear}}{d\tau}\frac{d\tau}{dt'}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{dt}{dt'}} \tag{2.3}$$

Therefore:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\textbf{Phy\_rectilinear}\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} + \frac{d\textbf{Phy\_rectilinear}}{d\tau}\frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{v_x'}{c}\sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c} + \frac{\frac{g}{c}}{\sqrt{1+(\frac{gt'}{c})^2}}\left(\frac{x'}{c}\cosh\frac{g\tau}{c} + t'\sinh\frac{g\tau}{c}\right)} \tag{2.4}$$

where:

$$\frac{d\mathbf{Phy\_rectilinear}}{d\tau}=\frac{g}{c}\begin{bmatrix}\sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c}\end{bmatrix} \tag{2.5}$$

The speed measured in the inertial frame depends both on the speed and the position measured in the accelerated frame. If we consider the case of the accelerated frame commoving with the object (particle) under study, then:

$$\begin{pmatrix}x'\\ y'\\ z'\\ ct'\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ ct'\end{pmatrix} \tag{2.6}$$

and:

$$\begin{pmatrix}v_x\\ v_y\\ v_z\\ c\end{pmatrix}=\frac{\mathbf{Phy\_rectilinear}\begin{pmatrix}v_x'\\ v_y'\\ v_z'\\ c\end{pmatrix}+\frac{d\mathbf{Phy\_rectilinear}}{d\tau}\frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\begin{pmatrix}0\\ 0\\ 0\\ ct'\end{pmatrix}}{\frac{v_x'}{c}\sinh\frac{g\tau}{c}+\cosh\frac{g\tau}{c}+\frac{\frac{gt'}{c}}{\sqrt{1+(\frac{gt'}{c})^2}}\sinh\frac{g\tau}{c}} \tag{2.7}$$

In the following section we generalize his derivation for the arbitrary case $\mathbf{g}=(g_x,g_y,g_z)$ for obtaining the general four-space coordinate transformations that take us from $S'(\tau)$ into S. Expressed in polar coordinates, the acceleration has the form:

$$g_x = g\cos\theta\cos\varphi$$
$$g_z = g\sin\theta\cos\varphi$$
$$g_y = g\sin\varphi$$
$$\varphi = \arcsin\frac{g_y}{g}$$
$$\theta = \arctan\frac{g_z}{g_x} \quad (2.8)$$

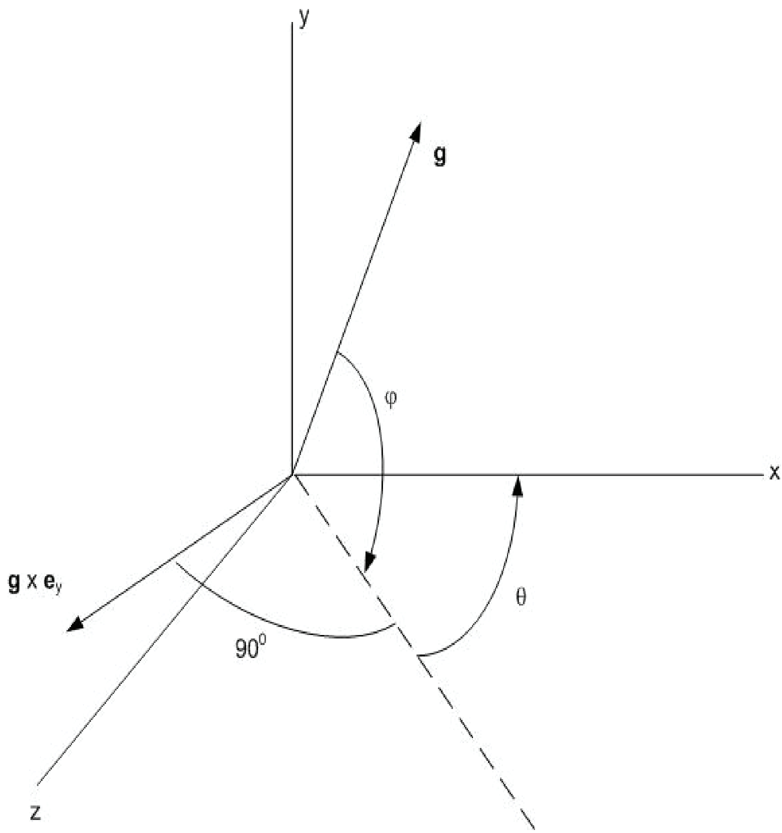

Fig. 1 Arbitrary direction rectilinear accelerated motion

The first step rotates the unit vector of acceleration $\mathbf{g}$ by $90^0 - \varphi$ around the axis the vector cross-product $\mathbf{g}\times\mathbf{e}_y = -\frac{g_z}{g}\mathbf{e}_x + \frac{g_x}{g}\mathbf{e}_z$ such $\mathbf{g}$ gets aligned with the y-axis (see Fig.1). For this purpose, we will introduce the triplet $(a,b,c) = (-\frac{g_z}{g}, 0, \frac{g_x}{g})$. The following expressions hold [14]:

$$\mathbf{Rot}_{\mathbf{y}} = \begin{bmatrix} c & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

$$\mathbf{Rot(e_y)}_{\mathbf{90^0\text{-}\varphi}} = \begin{bmatrix} \cos(90^0-\varphi) & 0 & -\sin(90^0-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(90^0-\varphi) & 0 & \cos(90^0-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sin\varphi & 0 & -\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.10}$$

$\mathbf{Rot(e_y)}_{\mathbf{90^0\text{-}\varphi}} * \mathbf{Rot}_{\mathbf{y}}$ aligns $\mathbf{g}$ with $\mathbf{e}_y$. The second step is comprised by another rotation around the z-axis by -90^0 that aligns $\mathbf{g}$ with $\mathbf{e}_x$:

$$\mathbf{Rot(e_z)}_{-90^0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.11)$$

Putting it all together:

$$\mathbf{Rr = Rot(e_x)_{90^0} * Rot(e_y)_{90^0\text{-}\varphi} * Rot_y} \quad (2.12)$$

The general coordinate transformation between S and $S'(\tau)$ is:

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \mathbf{A} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \quad (2.13)$$

where:

$$\mathbf{Rr^{-1} * Phy\_rectilinear * Rr} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \mathbf{A} \quad (2.14)$$

The general velocity transformation is therefore:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}} \frac{d\mathbf{A}}{d\tau} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\left(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau}\right)} \quad (2.15)$$

where:

$$\frac{d\mathbf{A}}{d\tau} = \mathbf{Rr}^{-1} * \frac{d\mathbf{Phy\_rectilinear}}{d\tau} * \mathbf{Rr} \tag{2.16}$$

The inverse transform is:

$$\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}}\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} v_x \\ v_y \\ v_z \\ ct \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}}(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{2.17}$$

where:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \mathbf{A}^{-1} = \mathbf{Rr}^{-1} * \mathbf{Phy\_rectilinear}^{-1} * \mathbf{Rr} \tag{2.18}$$

If we consider the case of the accelerated frame commoving with the object (particle) under study, then:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct' \end{pmatrix} \tag{2.19}$$

and (2.17) simplifies to:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} + \frac{d\mathbf{A}}{d\tau}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{ct'}{\sqrt{1+(\frac{gt'}{c})^2}} \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \dfrac{t'\frac{da_{44}}{d\tau}}{\sqrt{1+(\frac{gt'}{c})^2}}} \tag{2.20}$$

The coordinate acceleration can be derived immediately as:

$$\begin{pmatrix} a_x \\ a_y \\ a_z \\ 0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\frac{d}{dt'}\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix}}{\frac{dt}{dt'}} \tag{2.21}$$

The energy-momentum transforms the same way as the 4-coordinates (2.13) by virtue of being a 4-vector:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix} = \mathbf{A}\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ E'/c \end{pmatrix} \tag{2.22}$$

Therefore:

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} p_x^{'} \\ p_y^{'} \\ p_z^{'} \\ E'/c \end{pmatrix}}{\frac{v_x^{'}}{c}a_{41} + \frac{v_y^{'}}{c}a_{42} + \frac{v_z^{'}}{c}a_{43} + a_{44} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau})} \tag{2.23}$$

The inverse transform, from the inertial frame S into the accelerated frame S', is:

$$\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}}\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \frac{1}{\sqrt{1+(\frac{gt}{c})^2}}(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{2.24}$$

If we consider the case of the accelerated frame commoving with the object (particle) under study, then:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct' \end{pmatrix} \tag{2.25}$$

and:

$$\begin{pmatrix} p_x^{'} \\ p_y^{'} \\ p_z^{'} \\ E'/c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_0c^2/c \end{pmatrix} \tag{2.26}$$

In this case:

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ 0 \end{pmatrix} + \frac{1}{\sqrt{1+(\frac{gt'}{c})^2}}\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} 0 \\ 0 \\ 0 \\ m_0c \end{pmatrix}}{a_{44} + \frac{t'\frac{da_{44}}{d\tau}}{\sqrt{1+(\frac{gt'}{c})^2}}} \tag{2.27}$$

3. Dynamics in Uniform Angular Velocity Rotation

In this section we discuss the case of the particle moving in an arbitrary plane, with the normal given by the constant angular velocity $\boldsymbol{\omega}(a,b,c)$ (see Fig.2). According to Moller [1], the simpler case when $\boldsymbol{\omega}$ is aligned with the z-axis produces the transformation between the rotating frame $S'(\tau)$ attached to the particle and an inertial, non-rotating frame S attached to the center of rotation:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{Phy\_rotation} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{3.1}$$

where:

$$\mathbf{Phy\_rotation} = \begin{bmatrix} \cos\alpha\cos\beta + \gamma\sin\alpha\sin\beta & \sin\alpha\cos\beta - \gamma\cos\alpha\sin\beta & 0 & -\frac{u\gamma}{c}\sin\beta \\ \cos\alpha\sin\beta - \gamma\sin\alpha\cos\beta & \sin\alpha\sin\beta + \gamma\cos\alpha\cos\beta & 0 & \frac{u\gamma}{c}\cos\beta \\ 0 & 0 & 1 & 0 \\ \frac{u\gamma}{c}\sin\alpha & -\frac{u\gamma}{c}\cos\alpha & 0 & \gamma \end{bmatrix} \tag{3.2}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$
$$u = r\omega$$
$$\alpha = \omega\gamma\tau$$
$$\beta = \omega\gamma^2\tau \tag{3.3}$$

The general case is treated by transforming the problem into the particular case treated in [1] through a transformation into the "canonical case", followed by an application of the transformation from the accelerated frame into the inertial frame, ending with the inverse of the first transformation, as shown below:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = (\mathbf{Rr}^{-1} * \mathbf{Phy\_rotation} * \mathbf{Rr}) \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \quad (3.4)$$

$$\mathbf{Rr} = \mathbf{Rot}(\mathbf{e}_x)_{90^0} * \mathbf{Rot}(\mathbf{e}_y)_{90^0-\varphi} * \mathbf{Rot}_y \quad (3.5)$$

$\mathbf{Rot}(\mathbf{e}_y)_{90^0-\varphi} * \mathbf{Rot}_y$ aligns $\mathbf{g}$ with $\mathbf{e}_y$. The second step is comprised by another rotation around the x-axis by 90^0 that aligns $\boldsymbol{\omega}$ with $\mathbf{e}_z$:

$$\mathbf{Rot}(\mathbf{e}_x)_{90^0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.6)$$

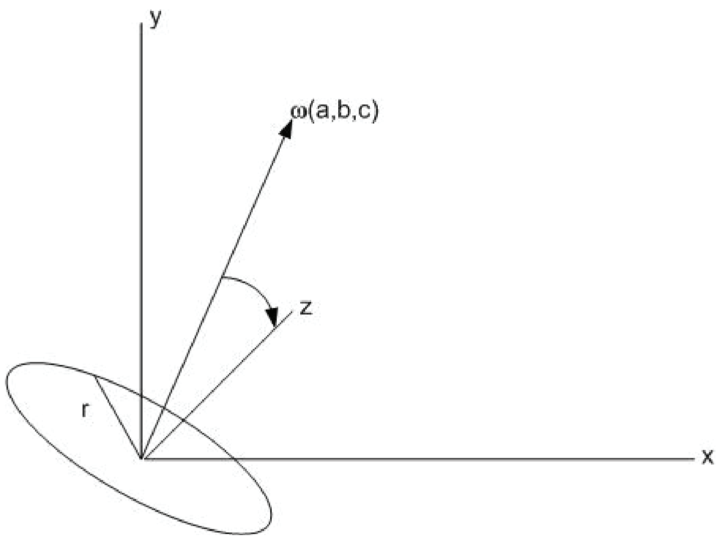

Fig. 2 Uniform rotation with arbitrary direction of angular velocity

Expression (3.4) gives the solution for the general case, of arbitrary angular velocity direction.

The general velocity transformation is:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ c \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau})} \quad (3.7)$$

The general force transformation is:

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} = \frac{\mathbf{A}\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}}{d\tau}\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ E'/c \end{pmatrix}}{\frac{v_x'}{c}a_{41} + \frac{v_y'}{c}a_{42} + \frac{v_z'}{c}a_{43} + a_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(x'\frac{da_{41}}{cd\tau} + y'\frac{da_{42}}{cd\tau} + z'\frac{da_{43}}{cd\tau} + t'\frac{da_{44}}{d\tau})} \quad (3.8)$$

where:

$$\mathbf{A} = \mathbf{Rr}^{-1} * \mathbf{Phy\_rotation} * \mathbf{Rr} \quad (3.9)$$

The reverse transformation is:

$$\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \quad (3.10)$$

where:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \mathbf{A}^{-1} = \mathbf{Rr^{-1} * Phy\_rotation}^{-1} \mathbf{* Rr} \quad (3.11)$$

We will use (3.10) in the next section, an application that determines the expression of the Lorentz force in a rotating frame.

4. Application-The Expression of the Lorentz Force in a Uniformly Rotating Frame

Assume that we have a particle of charge q and mass m moving in the x-y plane under the influence of a magnetic field $\mathbf{B}$ aligned with the z axis. We know that in the frame of the lab, the expression of the Lorentz force acting on the particle is:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\omega\cos(\omega t) & r\omega\sin(\omega t) & 0 \\ 0 & 0 & B \end{bmatrix} \quad (4.1)$$

We would like to find out the expression of the force in the frame co-rotating with the charged particle. For this purpose we will resort to (3.10)

$$\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{dt} \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}}{\frac{v_x}{c}b_{41} + \frac{v_y}{c}b_{42} + \frac{v_z}{c}b_{43} + b_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(x\frac{db_{41}}{cd\tau} + y\frac{db_{42}}{cd\tau} + z\frac{db_{43}}{cd\tau} + t\frac{db_{44}}{d\tau})} \quad (4.2)$$

We know from [15,16] that:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \\ 0 \\ t \end{pmatrix} \tag{4.3}$$

$$\begin{aligned} v_x &= -r\omega\sin(\omega t) \\ v_y &= r\omega\cos(\omega t) \\ v_z &= 0 \end{aligned} \tag{4.4}$$

$$\begin{aligned} p_x &= -\frac{mr\omega\sin(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}} \\ p_y &= \frac{mr\omega\cos(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}} \\ p_z &= 0 \\ E &= \frac{mc^2}{\sqrt{1-(\frac{r\omega}{c})^2}} \end{aligned} \tag{4.5}$$

$$\begin{aligned} F_x &= qBr\omega\sin(\omega t) \\ F_y &= -qBr\omega\cos(\omega t) \\ F_z &= 0 \end{aligned} \tag{4.6}$$

Substituting (4.3-4.6) into (4.2) we obtain:

$$\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} = \frac{\mathbf{A}^{-1}\begin{pmatrix} qBr\omega\sin(\omega t) \\ -qBr\omega\cos(\omega t) \\ 0 \\ 0 \end{pmatrix} + \sqrt{1-(\frac{r\omega}{c})^2}\,\frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} -\frac{mr\omega\sin(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}} \\ \frac{mr\omega\cos(\omega t)}{\sqrt{1-(\frac{r\omega}{c})^2}} \\ 0 \\ \frac{mc}{\sqrt{1-(\frac{r\omega}{c})^2}} \end{pmatrix}}{\frac{-r\omega\sin(\omega t)}{c}b_{41} + \frac{r\omega\cos(\omega t)}{c}b_{42} + b_{44} + \sqrt{1-(\frac{r\omega}{c})^2}\,(r\cos(\omega t)\frac{db_{41}}{cd\tau} + r\sin(\omega t)\frac{db_{42}}{cd\tau} + t\frac{db_{44}}{d\tau})} \tag{4.7}$$

To the above, we need to add [16] the fact that:

$$\omega = \frac{qB}{\gamma(v_0)m}$$
$$r = \frac{\gamma(v_0)mv_0}{qB} \tag{4.8}$$

In (4.8) v_0 is the initial speed of the particle at $t = 0$. Armed with that (4.7) gets the simpler form:

$$\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ \frac{1}{c}\frac{dE'}{dt'} \end{pmatrix} =$$

$$= \frac{\mathbf{A}^{-1}\begin{pmatrix} qBv_0 \sin(\frac{qBt}{\gamma(v_0)m}) \\ -qBv_0 \cos(\frac{qBt}{\gamma(v_0)m}) \\ 0 \\ 0 \end{pmatrix} + \frac{d\mathbf{A}^{-1}}{d\tau}\begin{pmatrix} -mv_0 \sin(\frac{qBt}{\gamma(v_0)m}) \\ mv_0 \cos(\frac{qBt}{\gamma(v_0)m}) \\ 0 \\ mc \end{pmatrix}}{\frac{v_0 b_{42} \cos\frac{qBt}{\gamma(v_0)m}}{c} - \frac{v_0 b_{41} \sin\frac{qBt}{\gamma(v_0)m}}{c} + b_{44} + \frac{mv_0}{qBc}\left(\frac{db_{41}}{d\tau}\cos\frac{qBt}{\gamma(v_0)m} + \frac{db_{42}}{d\tau}\sin\frac{qBt}{\gamma(v_0)m} + \frac{qBct}{\gamma(v_0)mv_0}\frac{db_{44}}{d\tau}\right)} \quad (4.9)$$

5. Electrodynamics in Uniformly Accelerated Frames and in Uniformly Rotating Frames – the General Expressions for the Electromagnetic 4-Vector Potential

Previously [17,18] we have dealt with the case of the transformation of Maxwell equations for the case of uniformly accelerated frames and uniformly rotating frames in arbitrary directions. The formalism derived in this paper, section 2, allows us to get a general transformation between the inertial frame S and S' and the inverse. The electromagnetic potential transforms the same way as the 4-coordinates (2.13) by virtue of being a 4-vector:

$$\begin{pmatrix} \Phi_x \\ \Phi_y \\ \Phi_z \\ c\phi \end{pmatrix} = \mathbf{A}\begin{pmatrix} \Phi_x^{'} \\ \Phi_y^{'} \\ \Phi_z^{'} \\ c\phi' \end{pmatrix} \quad (5.1)$$

where $\mathbf{A}$ is given by (2.14) for uniformly accelerated frames and by (3.9) for uniformly rotating frames.

In order to transform Maxwell equations between the frames, we need the **partial** derivatives with respect to x, y, z, t. We will show how to calculate two of them, as a blueprint.

$$\frac{\partial}{\partial t}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\mathbf{A}\frac{\partial}{\partial t}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\frac{\mathbf{A}\frac{\partial}{\partial t'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{\frac{\partial t}{\partial t'}}=\frac{\mathbf{A}\frac{\partial}{\partial t'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{a_{44}}$$

$$\frac{\partial}{\partial x}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\mathbf{A}\frac{\partial}{\partial x}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\frac{\mathbf{A}\frac{\partial}{\partial x'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{\frac{\partial x}{\partial x'}}=\frac{\mathbf{A}\frac{\partial}{\partial x'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}}{a_{11}} \tag{5.2}$$

The inverse transforms are:

$$\frac{\partial}{\partial t'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\mathbf{A}^{-1}\frac{\partial}{\partial t'}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial t}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{\frac{\partial t'}{\partial t}}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial t}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{b_{44}}$$

$$\frac{\partial}{\partial x'}\begin{pmatrix}\Phi'_x\\ \Phi'_y\\ \Phi'_z\\ c\phi'\end{pmatrix}=\mathbf{A}^{-1}\frac{\partial}{\partial x'}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial x}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{\frac{\partial x'}{\partial x}}=\frac{\mathbf{A}^{-1}\frac{\partial}{\partial x}\begin{pmatrix}\Phi_x\\ \Phi_y\\ \Phi_z\\ c\phi\end{pmatrix}}{b_{11}} \tag{5.3}$$

By using the above generalized transforms we can transform the electric and magnetic vectors, thus obtaining the general transformations for the Maxwell equations as seen in [17,18]

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Generalization of Coordinate Transformations between Accelerated and Inertial Frames – General Formulas of Thomas Precession

Synopsis

In the current chapter we present a generalization of the transforms from the frame co-moving with an accelerated particle, for either rectilinear or circular motion, into an inertial frame of reference. The solution is of great interest for real time applications, because earth-bound laboratories are inertial only in approximation. Though prior solutions exist, they are restricted to the particular cases of directions of motion aligned with the coordinate axes. It is our intent to produce a blueprint for generalizing the solutions to the arbitrary directions of motion. The motivation is that the real life applications include accelerating and rotating frames with arbitrary orientation more often than the idealized case of inertial frames. Our daily experiments happen in the laboratories attached to the rotating Earth. We conclude by deriving the general form of Thomas precession as an immediate application of arbitrary orientation of the axis of rotation with respect to the measuring frame.

1. Introduction

Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-13], [15]. There is great interest in producing a general solution that deals with arbitrary orientation of acceleration in the case of rectilinear motion and for arbitrary direction of uniform angular velocity.

The main idea of this paper is to generate a standard blueprint for a general solution. The blueprint relies on transforming the problem geometrically in the “canonical reference frame” of [1], followed by the application of the physical transforms derived for such “canonical” orientations [1-7] and ending with the application of the inverse geometrical transformation:

$$Geometry\_Transform -> Physics\_Transform -> Inverse\_Geometry\_Transform \quad (1.1)$$

2. Accelerated Rectilinear Motion

Let S represent an inertial system of coordinates and $S'(\tau)$ an accelerated one. Moller [1] considers a particular case where a particle moves with acceleration $\mathbf{g} = (g, 0, 0)$ aligned with the x-axis. In the following section we will generalize his derivation for the arbitrary case $\mathbf{g} = (g_x, g_y, g_z)$ for obtaining the general four-space coordinate transformations that take us from $S'(\tau)$ into S. Expressed in polar coordinates, the acceleration has the form:

$$\begin{aligned} g_x &= g \cos\theta \cos\varphi \\ g_z &= g \sin\theta \cos\varphi \\ g_y &= g \sin\varphi \\ \varphi &= \arcsin \frac{g_y}{g} \\ \theta &= \arctan \frac{g_z}{g_x} \end{aligned} \quad (2.1)$$

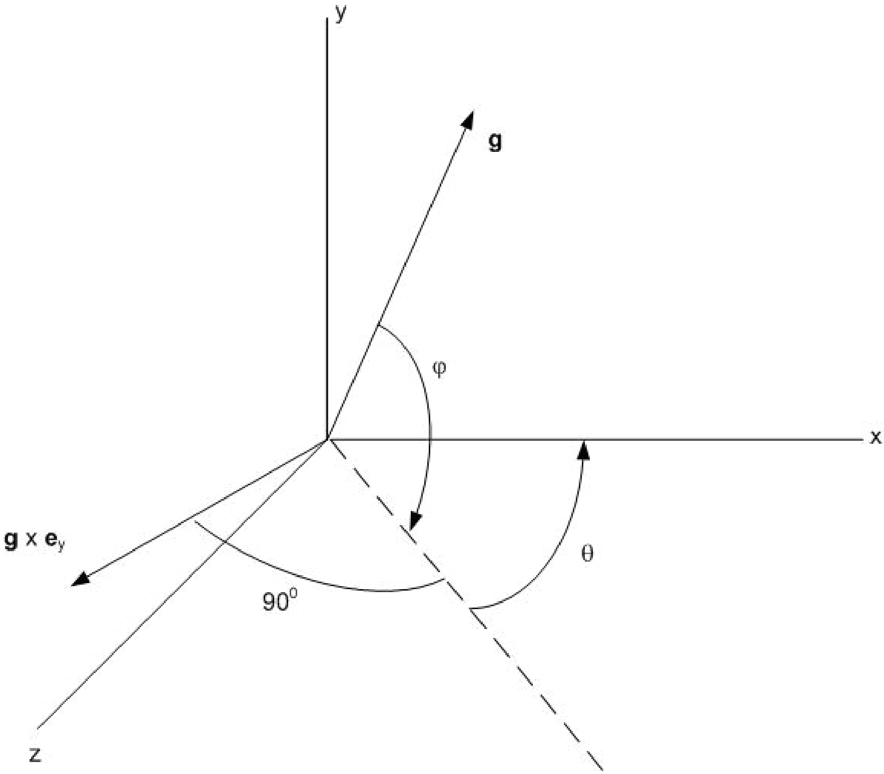

Fig. 1 Arbitrary direction rectilinear accelerated motion

According to reference [1] the transformation for the particular case from $S'(\tau)$ into S is:

$$(x, y, z, ct) = (x', y', z', c't')\mathbf{Phy\_rectilinear} \qquad (2.2)$$

where:

$$\textbf{Phy\_rectilinear} = \begin{bmatrix} \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c} \end{bmatrix} \tag{2.3}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ cT \end{pmatrix} = \textbf{Phy\_rectilinear} \begin{pmatrix} x' \\ y' \\ z' \\ c't' \end{pmatrix} \tag{2.4}$$

$$\begin{pmatrix} cp_x \\ cp_y \\ cp_z \\ E \end{pmatrix} = \textbf{Phy\_rectilinear} \begin{pmatrix} c'p_x' \\ c'p_y' \\ c'p_z' \\ E' \end{pmatrix} \tag{2.5}$$

$$\begin{pmatrix} k_x \\ k_y \\ k_z \\ 1 \end{pmatrix} h\lambda = \textbf{Phy\_rectilinear} \begin{pmatrix} k_x' \\ k_y' \\ k_z' \\ 1 \end{pmatrix} h\lambda' \tag{2.6}$$

$$\begin{pmatrix} k_x \\ k_y \\ k_z \\ 1 \end{pmatrix} \lambda = \begin{bmatrix} \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c} \end{bmatrix} \begin{pmatrix} k_x^{'} \\ k_y^{'} \\ k_z^{'} \\ 1 \end{pmatrix} \lambda' \tag{2.7}$$

$$k_x \lambda = (k_x^{'} \cosh\frac{g\tau}{c} + \sinh\frac{g\tau}{c})\lambda'$$

$$\lambda = (k_x^{'} \sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c})\lambda' \tag{2.8}$$

$$k_x = \frac{k_x^{'} \cosh\frac{g\tau}{c} + \sinh\frac{g\tau}{c}}{k_x^{'} \sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}} \tag{2.9}$$

$$k_y \lambda = k_y^{'} \lambda'$$

$$k_y = \frac{k_y^{'}}{k_x^{'} \sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}}$$

$$k_z \lambda = k_z^{'} \lambda'$$

$$k_z = \frac{k_z^{'}}{k_x^{'} \sinh\frac{g\tau}{c} + \cosh\frac{g\tau}{c}} \tag{2.10}$$

The first step rotates the unit vector of acceleration $\mathbf{g}$ by $90^0 - \varphi$ around the axis the vector cross-product $\mathbf{g} \times \mathbf{e}_y = -\frac{g_z}{g}\mathbf{e}_x + \frac{g_x}{g}\mathbf{e}_z$ such $\mathbf{g}$ gets aligned with the y-axis (see Fig.1). For this purpose, we will introduce the triplet $(a,b,c)=(-\frac{g_z}{g},0,\frac{g_x}{g})$. The following expressions hold [14]:

$$\mathbf{Rot_y} = \begin{bmatrix} c & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.11}$$

$$\mathbf{Rot(e_y)_{90^0\text{-}\varphi}} = \begin{bmatrix} \cos(90^0-\varphi) & 0 & -\sin(90^0-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(90^0-\varphi) & 0 & \cos(90^0-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sin\varphi & 0 & -\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

$\mathbf{Rot(e_y)_{90^0\text{-}\varphi} * Rot_y}$ aligns $\mathbf{g}$ with $\mathbf{e}_y$. The second step is comprised by another rotation around the z-axis by -90^0 that aligns $\mathbf{g}$ with $\mathbf{e}_x$:

$$\mathbf{Rot(e_z)_{-90^0}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

Putting it all together:

$$\mathbf{Rr = Rot(e_x)_{90^0} * Rot(e_y)_{90^0-\varphi} * Rot_y} \tag{2.14}$$

According to (1.1) the general transformation between S and $S'(\tau)$ is:

$$\mathbf{Rr^{-1} * Phy\_rectilinear * Rr} \tag{2.15}$$

$\mathbf{Rr^{-1}}$ reverses all the effects of $\mathbf{Rr}$. Expression (2.15) gives the solution for the general case, of arbitrary acceleration direction.

3. Uniform Angular Velocity Rotation

In this section we discuss the case of the particle moving in an arbitrary plane, with the normal given by the constant angular velocity $\boldsymbol{\omega}(a,b,c)$ (see Fig.2). According to Moller [1], the simpler case when $\boldsymbol{\omega}$ is aligned with the z-axis produces the transformation between the rotating frame $S'(\tau)$ attached to the particle and an inertial, non-rotating frame S attached to the center of rotation:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{Phy\_rotation} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{3.1}$$

where:

$$\mathbf{Phy\_rotation} = \begin{bmatrix} \cos\alpha\cos\beta + \gamma\sin\alpha\sin\beta & \sin\alpha\cos\beta - \gamma\cos\alpha\sin\beta & 0 & -\frac{u\gamma}{c}\sin\beta \\ \cos\alpha\sin\beta - \gamma\sin\alpha\cos\beta & \sin\alpha\sin\beta + \gamma\cos\alpha\cos\beta & 0 & \frac{u\gamma}{c}\cos\beta \\ 0 & 0 & 1 & 0 \\ \frac{u\gamma}{c}\sin\alpha & -\frac{u\gamma}{c}\cos\alpha & 0 & \gamma \end{bmatrix} \tag{3.2}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$
$$u = r\omega$$
$$\alpha = \omega\gamma\tau$$
$$\beta = \omega\gamma^2\tau \tag{3.3}$$

The general case is treated by transforming the problem into the particular case treated in [1] through a transformation into the “canonical case”, followed by an application of the transformation from the accelerated frame into the inertial frame, ending with the inverse of the first transformation, as shown below:

$$\begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} = (\mathbf{Rr}^{-1} * \mathbf{Phy\_rotation} * \mathbf{Rr}) \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{3.4}$$

$$\mathbf{Rr} = \mathbf{Rot(e_x)_{90^0} * Rot(e_y)_{90^0-\varphi} * Rot_y} \tag{3.5}$$

$\mathbf{Rot(e_y)_{90^0-\varphi} * Rot_y}$ aligns $\mathbf{g}$ with $\mathbf{e}_y$. The second step is comprised by another rotation around the x-axis by 90^0 that aligns $\boldsymbol{\omega}$ with $\mathbf{e}_z$:

$$\mathbf{Rot}(\mathbf{e}_x)_{90^0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.6}$$

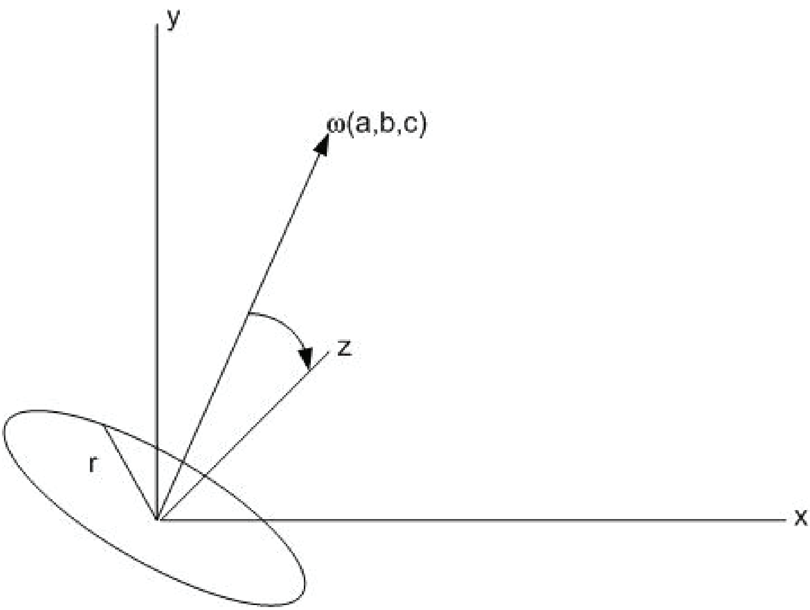

Fig. 2 Uniform rotation with arbitrary direction of angular velocity

Expression (3.4) gives the solution for the general case, of arbitrary angular velocity direction.

4. Application: The General Formulas for Thomas Precession

Moller [1] provides a very good explanation of the Thomas rotation [2-7] by comparing:

$$\mathbf{Phy\_rotation}_{\tau=0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & \frac{u\gamma}{c} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{u\gamma}{c} & 0 & \gamma \end{bmatrix} \tag{4.1}$$

with

$$\mathbf{Phy\_rotation}_{\tau=2\pi/\omega\gamma} = \begin{bmatrix} \cos\beta_1 & -\gamma\sin\beta_1 & 0 & -\frac{u\gamma}{c}\sin\beta_1 \\ \sin\beta_1 & \gamma\cos\beta_1 & 0 & \frac{u\gamma}{c}\cos\beta_1 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{u\gamma}{c} & 0 & \gamma \end{bmatrix} \tag{4.2}$$

where $\beta_1 = 2\pi(\gamma - 1)$ is the angle of rotation over the period of one revolution, $\frac{2\pi}{\omega\gamma}$. The Thomas effect is responsible for the change in the orientation of the particle spin [5].

The precession effect for the general case is therefore described by the 3x3 spatial coordinate sub-matrix of the 4x4 matrix:

$$\mathbf{Rr^{-1} * Phy\_rotation * Rr} \tag{4.3}$$

In other words, in order to get the general equations of the Thomas precession about an arbitrary axis, we will need to compare the expressions of $\mathbf{Rr^{-1} * Phy\_rotation * Rr}$ at $\tau = 2\pi / \omega\gamma$ and $\tau = 0$. In order to evaluate the effect on the coordinates, we need to evaluate only the 3x3 matrix responsible for transforming the spatial coordinates. Indeed:

$$\mathbf{Rr}^{-1} * \mathbf{Phy\_rotation}_{\tau=2\pi/\omega\gamma} * \mathbf{Rr} - \mathbf{Rr}^{-1} * \mathbf{Phy\_rotation}_{\tau=0} * \mathbf{Rr} =$$
$$= \mathbf{Rr}^{-1} * (\mathbf{Phy\_rotation}_{\tau=2\pi/\omega\gamma} - \mathbf{Phy\_rotation}_{\tau=0}) * \mathbf{Rr} \quad (4.4)$$

The net effect is therefore the effect as measured for the canonical orientation of the rotation axis modified by $\mathbf{Rr}^{-1}$ and $\mathbf{Rr}$. We notice from the above that the x, y components of the system of coordinates $S'(\tau_1)$ are affected by the z component, while the z component is unchanged. In other words, the axes of coordinates of the inertial system S undergo a more complex transformation than the rotation by $\beta_1 = 2\pi(\gamma - 1)$ and this is to be expected.

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Generalization of the Thomas-Wigner Rotation to Uniformly Accelerating Boosts

Synopsis

In the current chapter we present a generalization of the transforms from the frame co-moving with an accelerated particle for uniformly accelerated motion into an inertial frame of reference. The motivation is that the real life applications include accelerating and rotating frames with arbitrary orientations more often than the idealized case of inertial frames; our daily experiments happen in Earth-bound laboratories. We use the transforms in order to generalize the Thomas-Wigner rotation to the case of uniformly accelerated boosts.

1. Introduction

Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-13], [15]. There is great interest in producing a general solution that deals with arbitrary orientation of the uniform acceleration. The main idea of this paper is to generate a standard blueprint for a general solution. The blueprint relies on transforming the problem geometrically in the "canonical reference frame" of [1], followed by the application of the physical transforms derived for such "canonical" orientations [1-7] and ending with the application of the inverse geometrical transformation:

$$Geometry\_Transform->Physics\_Transform->Inverse\_Geometry\_Transform \tag{1.1}$$

We conclude our paper with a practical application of deriving the formula of the Thomas-Wigner rotation due to the composition of accelerated boosts.

2. Dynamics in Uniformly Accelerated Frames

It is well known that the composition of two non-*collinear Lorentz boosts* results in a *Lorentz transformation* that is the composition of a boost and a rotation. This rotation is called Thomas–Wigner rotation. The rotation was discovered by *Llewellyn Thomas* in 1926 [19] and derived formally by Wigner in 1939 [20]. In this paper we extend the formalism to the case of two successive boosts due to two arbitrary-direction constant accelerations. Consider the case of a particle moving in an arbitrary plane with the normal given by the constant acceleration $\mathbf{g} = (g_x, g_y, g_z)$ (see Fig.1). According to Moller [1], the simpler case when $\mathbf{g}$ is aligned with the x-axis produces the transformation between the accelerating frame $S'(\tau)$ attached to the particle and an inertial frame S :

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \mathbf{Phy\_rectilinear} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \tag{2.1}$$

where:

$$\mathbf{Phy\_rectilinear} = \begin{bmatrix} \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c} \end{bmatrix} \tag{2.2}$$

In the following section, we generalize Moller's derivation for the arbitrary case $\mathbf{g} = (g_x, g_y, g_z)$ for obtaining the general four-space coordinate transformations that take us from $S'(\tau)$ into S . Expressed in polar coordinates, the acceleration has the form:

$$g_x = g\cos\theta\cos\varphi$$
$$g_z = g\sin\theta\cos\varphi$$
$$g_y = g\sin\varphi$$
$$\varphi = \arcsin\frac{g_y}{g}$$
$$\theta = \arctan\frac{g_z}{g_x} \qquad (2.3)$$

The general case is treated by transforming the problem into the particular case treated in [1] through a transformation into the "canonical case", followed by an application of the transformation from the uniformly rotating frame into the inertial frame, ending with the inverse of the first transformation, as shown below:

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = (\mathbf{Rr}^{-1} * \mathbf{Phy\_rectilinear} * \mathbf{Rr})\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \mathbf{A}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \qquad (2.4)$$

$$\mathbf{Rr} = \mathbf{Rot}(\mathbf{e}_z)_{-90^0} * \mathbf{Rot}(\mathbf{e}_y)_{90^0-\varphi} * \mathbf{Rot}_y \qquad (2.5)$$

$\mathbf{Rot}(\mathbf{e}_y)_{90^0-\varphi} * \mathbf{Rot}_y$ aligns $\mathbf{g}$ with $\mathbf{e}_y$ so the first step rotates $\mathbf{g}$ by $90^0 - \varphi$ around the axis the vector cross-product $\mathbf{g}\times\mathbf{e}_y = -\frac{g_z}{g}\mathbf{e}_x + \frac{g_x}{g}\mathbf{e}_z$ such $\mathbf{g}$ gets aligned with the y-axis (see Fig.1).

For this purpose, we will introduce the pair $(a,b) = (-\frac{g_z}{g}, \frac{g_x}{g})$. The following expressions hold [14]:

$$\mathbf{Rot}_{\mathrm{y}} = \begin{bmatrix} b & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6}$$

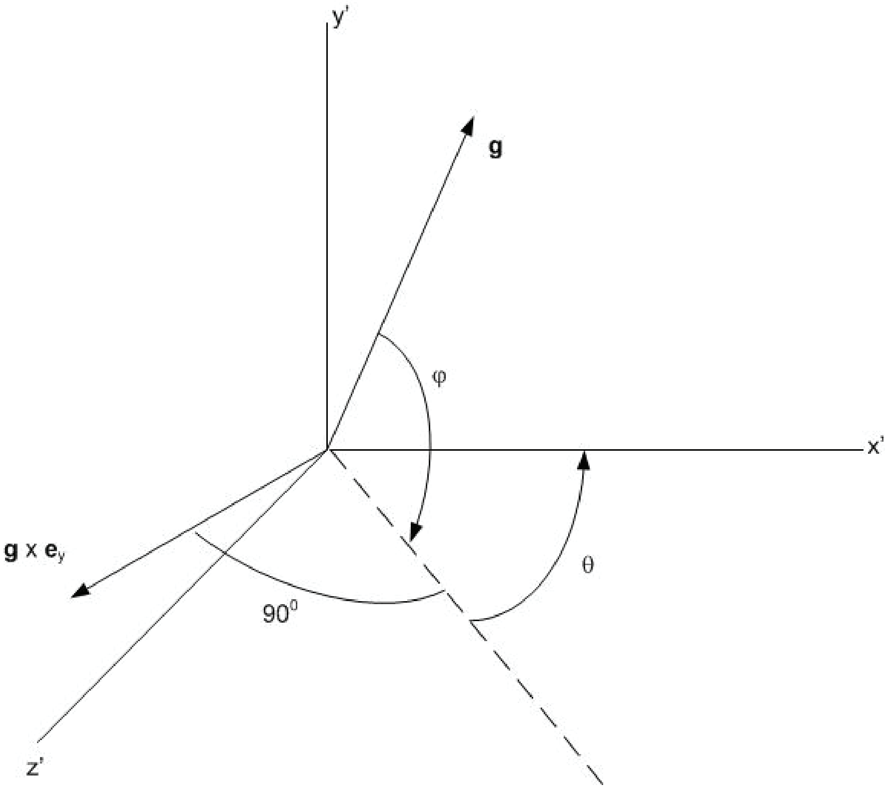

Fig. 1 Uniform rotation with arbitrary direction of angular velocity

$$\mathbf{Rot}(\mathbf{e}_y)_{90^0\text{-}\varphi} = \begin{bmatrix} \cos(90^0-\varphi) & 0 & -\sin(90^0-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(90^0-\varphi) & 0 & \cos(90^0-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sin\varphi & 0 & -\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

The second step is another rotation around the z-axis by -90^0 that aligns $\mathbf{g}$ with $\mathbf{e}_x$:

$$\mathbf{Rot}(\mathbf{e}_z)_{-90^0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

Putting it all together:

$$\mathbf{Rr} = \mathbf{Rot}(\mathbf{e}_z)_{-90^0} * \mathbf{Rot}(\mathbf{e}_y)_{90^0\text{-}\varphi} * \mathbf{Rot}_y \tag{2.9}$$

$$\mathbf{A} = \mathbf{Rr}^{-1} * \mathbf{Phy\_rectilinear} * \mathbf{Rr} \tag{2.10}$$

Let us consider a second boost given by the constant acceleration $\mathbf{g}'$. Expressed in polar coordinates, the acceleration has the form:

$$\varphi' = \arcsin \frac{g_y'}{g'}$$

$$\theta' = \arctan \frac{g_z'}{g_x'} \tag{2.11}$$

We now introduce the pair $(a', b') = (-\frac{g_z'}{g'}, \frac{g_x'}{g'})$. The following expressions hold [14]:

$$\mathbf{Rot}_y' = \begin{bmatrix} b' & 0 & -a' & 0 \\ 0 & 1 & 0 & 0 \\ a' & 0 & b' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

$$\mathbf{Rot'(e_y)_{90^0-\varphi}} = \begin{bmatrix} \sin\varphi' & 0 & -\cos\varphi' & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi' & 0 & \sin\varphi' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

$$\mathbf{Phy\_rectilinear'} = \begin{bmatrix} \cosh\frac{g'\tau}{c} & 0 & 0 & \sinh\frac{g'\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{g'\tau}{c} & 0 & 0 & \cosh\frac{g'\tau}{c} \end{bmatrix} \tag{2.14}$$

$$\mathbf{Rr' = Rot(e}_z\mathbf{)_{-90^0} * Rot'(e_y)_{90^0-\varphi} * Rot}_y' \tag{2.15}$$

$$\mathbf{A' = Rr'^{-1} * Phy\_rectilinear' * Rr'} \tag{2.16}$$

The boost in the arbitrary direction of acceleration $\boldsymbol{g}$ followed by the boost in the arbitrary direction of acceleration $\boldsymbol{g}'$ is therefore completely described by the transformation matrix:

$$\mathbf{B} = \mathbf{A'A} = \mathbf{Rr}'^{-1} * \mathbf{Phy\_rectilinear}' * \mathbf{Rr}' * \mathbf{Rr}^{-1} * \mathbf{Phy\_rectilinear} * \mathbf{Rr} \tag{2.17}$$

Formula (2.17) represents the generalization of the Thomas-Wigner rotation for the case of uniformly accelerated boosts. While Thomas rotation is a kinematic effect, the effect presented in this paper is a dynamic effect, the rotation is due to the changes in the direction of the acceleration. If the accelerations $\mathbf{g}, \mathbf{g}'$ are collinear, then:

$$\mathbf{Rr}' = \mathbf{Rr} \tag{2.18}$$

$$\mathbf{B} = \mathbf{A'A} = \mathbf{Rr}^{-1} * \mathbf{Phy\_rectilinear}' * \mathbf{Phy\_rectilinear} * \mathbf{Rr} \tag{2.19}$$

$$\mathbf{Phy\_rectilinear}' * \mathbf{Phy\_rectilinear} =$$

$$= \begin{bmatrix} \cosh\frac{(g+g')\tau}{c} & 0 & 0 & \sinh\frac{(g+g')\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{(g+g')\tau}{c} & 0 & 0 & \cosh\frac{(g+g')\tau}{c} \end{bmatrix} \tag{2.20}$$

Expression (2.20) serves as a quick sanity check for the formalism as we can see that the accelerations add algebraically.

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THE RELATIVISTIC CYCLOTRON RADIATION IN THE CIRCULAR ROTATING FRAME OF THE MOVING HEAVY PARTICLE

Synopsis

In the current chapter we tackle the task of determining the formula for the cyclotron radiation as measured from a frame co-moving with the particle being accelerated. In the case of cyclotrons, as opposed to synchrotrons, the magnetic field is constant, resulting into spiral trajectories for light particle, like electrons and into circular trajectories for heavier particles, like protons, as we will demonstrate in the current paper. This due to the fact that the braking force is a very small percentage of the accelerating (Lorentz) force, as will be shown later in our paper. These proofs have never been attempted before owing to the difficulty of dealing with rotating frames. The chapter is divided into two main sections, the first section deals with cyclotron radiation measured in the inertial frame of the lab, the second section deals with cyclotron radiation as measured in a frame co-rotating with the particle along a circular path, at a uniform speed.

1. Introduction - Bremsstrahlung

Bremsstrahlung is the *electromagnetic radiation* produced by the *deceleration* of a charged particle. The moving particle loses *kinetic energy*, which is converted into a *photon*, it is the process of producing the energy radiation [1]:

$$p = \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}(\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\vec{\beta} = \frac{\vec{v}}{c}$$

$$\dot{\vec{\beta}} = \frac{\vec{a}}{c}$$

$$\vec{a} = \frac{d\vec{v}}{d\tau} \tag{1.1}$$

For the case of acceleration perpendicular to the velocity (as in the case of synchrotrons), the formula simplifies to:

$$p = \frac{q^2 a^2 \gamma^4}{6\pi\varepsilon_0 c^3} \tag{1.2}$$

where p is the power measured in the frame of the lab. In the current paper we will make the attempt of finding the power as expressed in the frame co-moving with the particle.

2. Kinematics in Uniform Angular Velocity Rotation

In this section we introduce all the fundamental notions that will help discussing the case of the particle moving in an arbitrary plane, with the normal given by the constant angular velocity $\boldsymbol{\omega}(a,b,c)$, as in the case of a charged particle in circular motion in a synchrotron. According to Moller [2], the simpler case when $\boldsymbol{\omega}$ is aligned with the z-axis produces the transformation between the rotating frame $S'(\tau)$ attached to the particle and an inertial, non-rotating frame S attached to the center of rotation [2-7], [15],[16] (see fig.1):

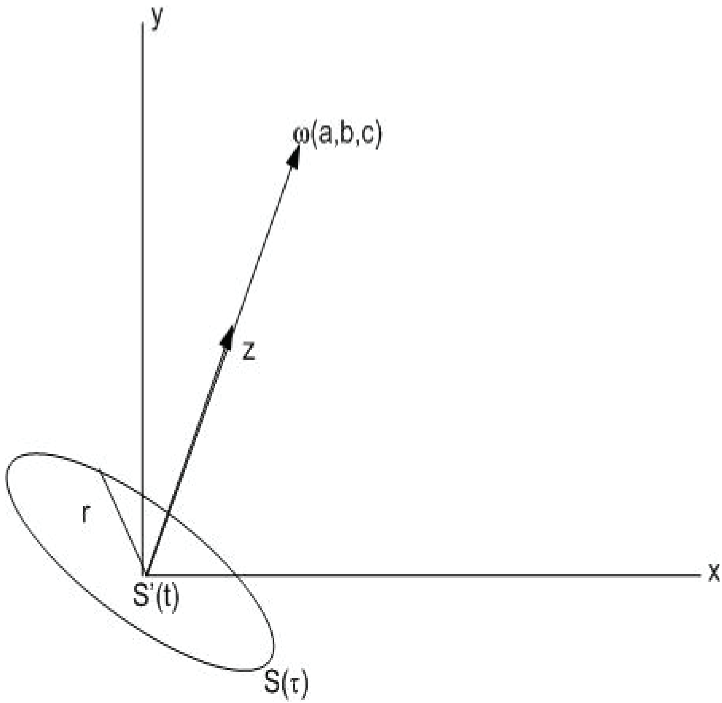

Fig. 1. Relationship between rotating and inertial frames

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \mathbf{A} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \tag{2.1}$$

where [7]:

$$\mathbf{A}=\begin{bmatrix} \cos\alpha\cos\beta+\gamma\sin\alpha\sin\beta & \sin\alpha\cos\beta-\gamma\cos\alpha\sin\beta & 0 & -\frac{u\gamma}{c}\sin\beta \\ \cos\alpha\sin\beta-\gamma\sin\alpha\cos\beta & \sin\alpha\sin\beta+\gamma\cos\alpha\cos\beta & 0 & \frac{u\gamma}{c}\cos\beta \\ 0 & 0 & 1 & 0 \\ \frac{u\gamma}{c}\sin\alpha & -\frac{u\gamma}{c}\cos\alpha & 0 & \gamma \end{bmatrix}$$

$$\mathbf{A}^{-1}=\begin{bmatrix} \cos\alpha\cos\beta+\gamma\sin\alpha\sin\beta & \cos\alpha\sin\beta-\gamma\sin\alpha\cos\beta & 0 & -\frac{u\gamma}{c}\sin\alpha \\ \sin\alpha s\cos\beta-\gamma\cos\alpha\sin\beta & \sin\alpha\sin\beta+\gamma\cos\alpha\cos\beta & 0 & \frac{u\gamma}{c}\cos\alpha \\ 0 & 0 & 1 & 0 \\ -\frac{u\gamma}{c}\sin\beta & \frac{u\gamma}{c}\cos\beta & 0 & \gamma \end{bmatrix} \quad (2.2)$$

$$\gamma=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u=r\omega$$

$$\alpha=\omega\gamma\tau$$

$$\beta=\omega\gamma^2\tau \quad (2.3)$$

3. Bremsstrahlung in a Uniformly Rotating Frame

Assume that we have a particle of charge q and mass m moving in the x-y plane under the influence of a constant magnetic field $\mathbf{B}$ aligned with the z axis. The magnetic field is the only field present since the particle is to have a circular motion [4,8]. We know that in the frame of the lab, the expression of the Lorentz force acting on the particle is [8]:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\omega\cos(\omega t) & r\omega\sin(\omega t) & 0 \\ 0 & 0 & B \end{bmatrix} \tag{3.1}$$

We would like to find out the expression of the force in the frame co-rotating with the charged particle. For this purpose we will resort to the fact [9] that four-force transforms like four-coordinate (2.1)

$$\begin{pmatrix} \gamma'(u')F_x' \\ \gamma'(u')F_y' \\ \gamma'(u')F_z' \\ \gamma'(u')\dfrac{\mathbf{F'.u'}}{c} \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} \gamma(u)F_x \\ \gamma(u)F_y \\ \gamma(u)F_z \\ \gamma(u)\dfrac{\mathbf{F.u}}{c} \end{pmatrix} \tag{3.2}$$

The term $\gamma(u)\dfrac{\mathbf{F.u}}{c}$ represents the power imparted by the magnetic field to the particle measured in the lab frame (divided by c) while the term $\gamma'(u')\dfrac{\mathbf{F'.u'}}{c}$ represents the power imparted by the magnetic field to the particle measured in the frame commoving with the particle (divided by c). Transformation (3.2) gives the general formulas for transforming four-force (proper force) in rotating frames.

We know from [8] that:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \\ 0 \\ t \end{pmatrix} \tag{3.3}$$

$$u_x = -r\omega \sin \omega t$$
$$u_y = r\omega \cos \omega t$$
$$u_z = 0$$
$$u = \sqrt{u_x^2 + u_y^2 + u_z^2} = r\omega = u_0$$
$$\gamma(u) = \gamma(u_0) \tag{3.4}$$

$$F_x = -qBr\omega \sin \omega t$$
$$F_y = qBr\omega \cos \omega t$$
$$F_z = 0 \tag{3.5}$$

Substituting (3.4), (3.5) into (3.2) we obtain:

$$\begin{pmatrix} \gamma'(u')F_x' \\ \gamma'(u')F_y' \\ \gamma'(u')F_z' \\ P'/c \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} \gamma(u_0)F_x \\ \gamma(u_0)F_y \\ \gamma(u_0)F_z \\ \gamma(u_0)\dfrac{\mathbf{F.u}}{c} \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} \gamma(u_0)qBu_0 \sin \omega t \\ -\gamma(u_0)qBu_0 \cos \omega t \\ 0 \\ \gamma(u_0)\dfrac{qBu_0^2}{c} \end{pmatrix} \tag{3.6}$$

Therefore:

$$\begin{pmatrix} \gamma'(u')F_x' \\ \gamma'(u')F_y' \\ \gamma'(u')F_z' \\ P'/c \end{pmatrix} =$$

$$= \begin{bmatrix} \cos\omega t \cos\gamma\omega t + \gamma \sin\omega t \sin\gamma\omega t & \sin\omega t \cos\gamma\omega t - \gamma \cos\omega t \sin\gamma\omega t & 0 & -\dfrac{u_0\gamma \sin\gamma\omega t}{c} \\ \cos\omega t \sin\gamma\omega t - \gamma \sin\omega t \cos\gamma\omega t & \sin\omega t \sin\gamma\omega t + \gamma \cos\omega t \cos\gamma\omega t & 0 & \dfrac{u_0\gamma \cos\gamma\omega t}{c} \\ 0 & 0 & 1 & 0 \\ -\dfrac{u_0\gamma \sin\omega t}{c} & \dfrac{u_0\gamma \cos\omega t}{c} & 0 & \gamma \end{bmatrix} \begin{pmatrix} \gamma qBu_0 \sin\omega t \\ -\gamma qBu_0 \cos\omega t \\ 0 \\ \gamma\dfrac{qBu_0^2}{c} \end{pmatrix} \tag{3.7}$$

The final formula for the power imparted by the magnetic field to the particle measured in the frame commoving with the particle is:

$$P' = 2qB\gamma^2(u_0)u_0^2 \tag{3.8}$$

Expression (3.7) provides us with the expression of the force acting on the particle as measured in the frame of the particle (the proper force). For example:

$$\overset{:}{F}_x' = \gamma'(u')F_x' = \gamma^2(u_0)qBu_0(1-\frac{u_0^2}{c^2})\sin\gamma\omega t = qBu_0 \sin\gamma\omega t$$

$$\overset{:}{F}_y' = \gamma'(u')F_y' = -\gamma^2(u_0)qBu_0(1-\frac{u_0^2}{c^2})\cos\gamma\omega t = qBu_0 \cos\gamma\omega t$$

$$\overset{:}{F}_z' = 0 \tag{3.9}$$

The term $\gamma^2(u_0)qBu_0$ in (3.9) represents the non-fictitious component, the active Lorentz force while the term $-\gamma^2(u_0)qBu_0(\frac{u_0^2}{c^2})$ represents the fictitious component, the centrifugal "force" due to the calculations being done in the (uniformly) rotating frame.

We are now ready to derive the radiated power. From [4] we know that:

$$\omega = \frac{qB}{\gamma(v_0)m} \tag{3.10}$$

Substituting (3.4),(3.10) into (1.2) we obtain:

$$p = \frac{q^2a^2\gamma^4}{6\pi\varepsilon_0 c^3} == \frac{q^2B^2}{\gamma^2(u_0)m^2}\frac{q^2\gamma^6(u_0)u_0^2}{6\pi\varepsilon_0 c^3} = \frac{q^4\gamma^4(u_0)u_0^2B^2}{6\pi\varepsilon_0 c^3 m^2} \tag{3.11}$$

The radiated power in this case is a constant that depends on the mass of the particle, m , its charge, q , its initial speed of injection into the synchrotron,

u_0 and the magnitude of the magnetic field B. The constancy is due to the fact that $r\omega = u_0$. The braking force due to radiation acts in direct opposition to the direction of motion (direction of **u**):

$$f_x = +\frac{p}{u_0}\sin\omega t$$
$$f_y = -\frac{p}{u_0}\cos\omega t$$
$$f_z = 0$$
$$\frac{f_x}{f_y} = -\tan\omega t \tag{3.12}$$

We know that:

$$\begin{pmatrix} \gamma'(u')f_x' \\ \gamma'(u')f_y' \\ \gamma'(u')f_z' \\ p'/c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} \gamma(u_0)f_x \\ \gamma(u_0)f_y \\ \gamma(u_0)f_z \\ \gamma(u_0)\dfrac{\mathbf{f.u}}{c} \end{pmatrix} =$$

$$= \mathbf{A}^{-1}\begin{pmatrix} \gamma(u_0)\dfrac{p}{u_0}\sin\omega t \\ -\gamma(u_0)\dfrac{p}{u_0}\cos\omega t \\ 0 \\ \gamma(u_0)\dfrac{p}{c} \end{pmatrix} = \gamma(u_0)p\mathbf{A}^{-1}\begin{pmatrix} \dfrac{\sin\omega t}{u_0} \\ -\dfrac{\cos\omega t}{u_0} \\ 0 \\ \dfrac{1}{c} \end{pmatrix} \tag{3.13}$$

Therefore:

$$\begin{pmatrix} \gamma'(u')f_x' \\ \gamma'(u')f_y' \\ \gamma'(u')f_z' \\ p'/c \end{pmatrix} =$$

$$= \gamma(u_0)p \begin{bmatrix} \cos\omega t\cos\gamma\omega t + \gamma\sin\omega t\sin\gamma\omega t & \sin\omega t\cos\gamma\omega t - \gamma\cos\omega t\sin\gamma\omega t & 0 & -\frac{u_0\gamma\sin\gamma\omega t}{c} \\ \cos\omega t\sin\gamma\omega t - \gamma\sin\omega t\cos\gamma\omega t & \sin\omega t\sin\gamma\omega t + \gamma\cos\omega t\cos\gamma\omega t & 0 & \frac{u_0\gamma\cos\gamma\omega t}{c} \\ 0 & 0 & 1 & 0 \\ -\frac{u_0\gamma\sin\omega t}{c} & \frac{u_0\gamma\cos\omega t}{c} & 0 & \gamma \end{bmatrix} \begin{pmatrix} \frac{\sin\omega t}{u_0} \\ -\frac{\cos\omega t}{u_0} \\ 0 \\ \frac{1}{c} \end{pmatrix} \quad (3.14)$$

From (3.14) we get the radiated power measured in the frame co-moving with the particle:

$$p' = 0 \quad (3.15)$$

This should come as no surprise since the particle is not accelerating in its own frame of reference. We also get the braking force due to radiation:

$$\overset{:}{f_x'} = \gamma'(u')f_x' = -\gamma^2(u_0)\frac{p}{u_0}(1-\frac{u_0^2}{c^2})\sin\gamma\omega t = -\frac{p}{u_0}\sin\gamma\omega t$$

$$\overset{:}{f_y'} = \gamma'(u')f_y' = \gamma^2(u_0)\frac{p}{u_0}(1-\frac{u_0^2}{c^2})\cos\gamma\omega t = \frac{p}{u_0}\cos\gamma\omega t$$

$$\overset{:}{f_z'} = 0$$

$$\frac{\overset{:}{f_x'}}{\overset{:}{f_y'}} = -\tan\gamma\omega t \quad (3.16)$$

Notice the similarity between expressions (3.16) and (3.13). Also notice how the ratio between the force components in the x and y directions has been increased by the contribution of the gamma factor. We can see that the particle experiences a braking force in the frame co-moving with it. This is extremely important for practical reasons: despite of the absence of radiating power as measured in the co-moving frame, the particle is still being slowed down, independent of the frame of reference used for calculations, thus requiring external energy to be imparted in order to maintain its speed [10-14].

4. Braking force as a percentage of the Lorentz force

The total radiated power goes as m^{-4}, which accounts for why electrons lose energy due to bremsstrahlung radiation much more rapidly than heavier charged particles (e.g., muons, protons, alpha particles). This is the reason why the TeV energy electron-positron colliders cannot use circular tunnels (requiring constant acceleration), while a proton-proton colliders (such as LHC) can utilize circular tunnels (and, consequently, the acting Lorentz force is dependent only on the magnetic field, as explained in (3.1)). The electrons lose energy due to bremsstrahlung at a rate $(m_p / m_e)^4 \approx 10^{13}$ times higher than protons do [10-14].

Another way of looking at the issue is by calculating the ratio between the braking force and the active (Lorentz) force. From (3.12) we know that the braking force is:

$$f = \frac{p}{u_0} = \frac{q^4 \gamma^4 (u_0) u_0 B^2}{6\pi\varepsilon_0 c^3 m^2} \tag{4.1}$$

The Lorentz force is:

$$F = qBu_0 \tag{4.2}$$

Therefore, their ratio is:

$$r = \frac{f}{F} = \frac{q^3 \gamma^4 (u_0) B}{6\pi\varepsilon_0 c^3 m^2} \tag{4.3}$$

When comparing the ratios for the cases of an electron vs. a proton, for the same conditions in terms of magnetic field and initial particle injection sped we find out that:

$$\frac{r_e}{r_p} = (\frac{m_p}{m_e})^2 \tag{4.4}$$

The reaction force in the case of an electron is much larger (about 10^7) than that the one for a proton for the case of equal particle accelerating Lorentz forces. We could to the above calculations in the frame co-rotating with the particle. From (3.9) we obtain:

$$\overset{:}{F'} = \gamma^2(u_0)qBu_0(1+\frac{u_0^2}{c^2}) \tag{4.5}$$

From (3.16) we obtain:

$$\overset{:}{f'} = \frac{p}{u_0} \tag{4.6}$$

The interesting result is that we obtain the same exact result as the one calculated in the frame of the lab, the ratio of forces depends only on the inverse ratio of masses, as shown in (4.4). Even more interestingly, the braking force is the same in both frames.

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Relativistic d'Alembert Force in Uniformly Accelerating Frames

Synopsis

In this chapter we present a generalization of the transforms from the frame co-moving with an accelerated particle for uniformly accelerated motion into an inertial frame of reference. The solution is of great interest for real time applications because earth-bound laboratories are inertial only in approximation. The motivation is that the real life applications include accelerating and rotating frames with arbitrary orientations more often than the idealized case of inertial frames; our daily experiments happen in the laboratories attached to the rotating Earth. The chapter is divided into two main sections, the first section deals with the theory of the dynamics, i.e. forces and the second section deals with the application of the theory to the derivation of the relativistic d'Alembert force occurring in the accelerating frame. We will show that there is not only a fictitious force that emerges in the accelerating frame but also a fictitious d'Alembert power.

1. Introduction

Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-13], [15]. There is great interest in producing a general solution that deals with arbitrary orientation the uniform acceleration. The main idea of this chapter is to generate a standard blueprint for a general solution. The blueprint relies on transforming the problem geometrically in the "canonical reference frame" of [1], followed by the application of the physical transforms derived for such "canonical" orientations [1-7] and ending with the application of the inverse geometrical transformation:

$$Geometry\_Transform -> Physics\_Transform -> Inverse\_Geometry\_Transform \tag{1.1}$$

We conclude our paper with a practical application of deriving the formula of the d'Alembert force in a uniformly rotating frame. We will show that there is not only a fictitious force that emerges in the accelerating frame but also a fictitious d'Alembert power.

2. Dynamics in Uniformly Accelerated Frames

In this section we discuss the case of the particle moving in an arbitrary plane with the normal given by the constant acceleration $\mathbf{g}=(g_x,g_y,g_z)$ (see Fig.1). According to Moller [1], the simpler case when $\mathbf{g}$ is aligned with the x-axis produces the transformation between the accelerating frame $S'(\tau)$ attached to the particle and an inertial frame S :

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \mathbf{Phy\_rectilinear} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \tag{2.1}$$

where:

$$\mathbf{Phy\_rectilinear} = \begin{bmatrix} \cosh\frac{g\tau}{c} & 0 & 0 & \sinh\frac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\frac{g\tau}{c} & 0 & 0 & \cosh\frac{g\tau}{c} \end{bmatrix} \tag{2.2}$$

In the following section we generalize his derivation for the arbitrary case $\mathbf{g}=(g_x,g_y,g_z)$ for obtaining the general four-space coordinate transformations that take us from $S'(\tau)$ into S . Expressed in polar coordinates, the acceleration has the form:

$$g_x = g\cos\theta\cos\varphi$$
$$g_z = g\sin\theta\cos\varphi$$
$$g_y = g\sin\varphi$$
$$\varphi = \arcsin\frac{g_y}{g}$$
$$\theta = \arctan\frac{g_z}{g_x} \qquad (2.3)$$

The general case is treated by transforming the problem into the particular case treated in [1] through a transformation into the "canonical case", followed by an application of the transformation from the uniformly rotating frame into the inertial frame, ending with the inverse of the first transformation, as shown below:

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = (\mathbf{Rr^{-1} * Phy\_rectilinear * Rr})\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \mathbf{A}\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \qquad (2.4)$$

$$\mathbf{Rr = Rot(e_{\mathit{z}})_{-90^0} * Rot(e_y)_{90^0\text{-}\varphi} * Rot_y} \qquad (2.5)$$

$\mathbf{Rot(e_y)_{90^0\text{-}\varphi} * Rot_y}$ aligns $\mathbf{g}$ with $\mathbf{e}_y$ so the first step rotates $\mathbf{g}$ by $90^0-\varphi$ around the axis the vector cross-product $\mathbf{g}\times\mathbf{e}_y = -\frac{g_z}{g}\mathbf{e}_x + \frac{g_x}{g}\mathbf{e}_z$ such $\mathbf{g}$ gets aligned with the y-axis (see Fig.1).

For this purpose, we will introduce the pair $(a,b) = (-\frac{g_z}{g}, \frac{g_x}{g})$. The following expressions hold [14]:

$$\mathbf{Rot}_y = \begin{bmatrix} b & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6}$$

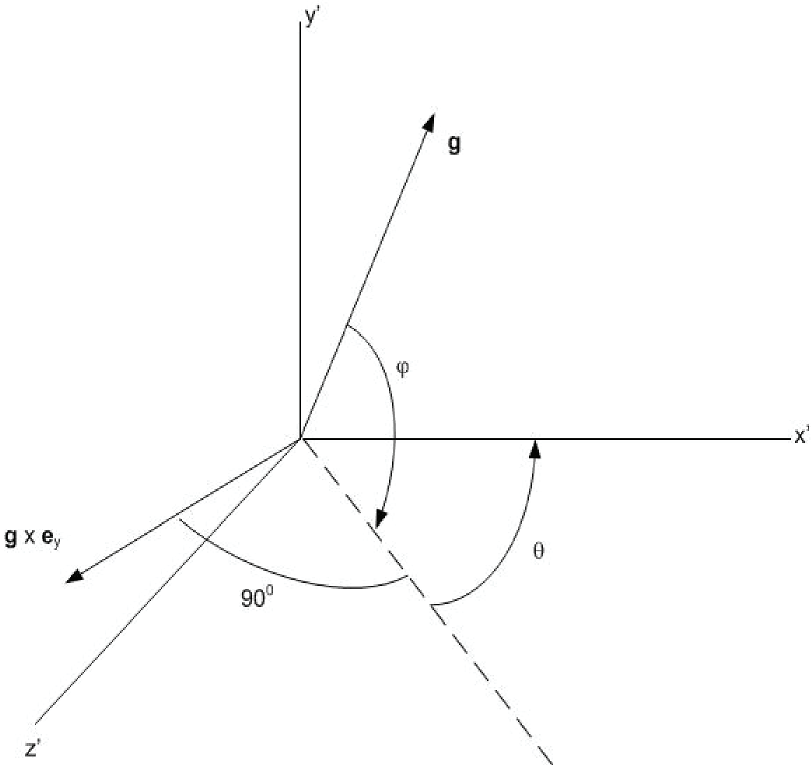

Fig.1 Uniform rotation with arbitrary direction of angular velocity

$$\mathbf{Rot(e_y)_{90^0\text{-}\varphi}} = \begin{bmatrix} \cos(90^0-\varphi) & 0 & -\sin(90^0-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(90^0-\varphi) & 0 & \cos(90^0-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sin\varphi & 0 & -\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

The second step is another rotation around the z-axis by -90^0 that aligns $\mathbf{g}$ with $\mathbf{e}_x$:

$$\mathbf{Rot(e_{\mathit{z}})_{-90^0}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

Putting it all together:

$$\mathbf{Rr = Rot(e_{\mathit{z}})_{-90^0} * Rot(e_y)_{90^0\text{-}\varphi} * Rot_y} \tag{2.9}$$

$$\mathbf{A = Rr^{-1} * Phy\_rectilinear * Rr} \tag{2.10}$$

The general proper velocity transformation is:

$$\begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ \gamma c \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} v_x \\ v_y \\ v_z \\ \gamma c \end{pmatrix} + \frac{d\mathbf{A}^{-1}}{d\tau} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

$$\gamma = \cosh \frac{g\tau}{c} \tag{2.11}$$

The energy-momentum transforms the same way as the 4-coordinates (2.13) by virtue of being a 4-vector:

$$\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ E'/c \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix} \tag{2.12}$$

Therefore the general force transformation is:

$$\begin{pmatrix} F_x' \\ F_y' \\ F_z' \\ \frac{1}{c}\frac{dE'}{d\tau} \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} F_x \\ F_y \\ F_z \\ \frac{1}{c}\frac{dE}{d\tau} \end{pmatrix} + \frac{d\mathbf{A}^{-1}}{d\tau} \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix} \tag{2.13}$$

We will use (2.13) in the next section, the application used for determining the expression of the d'Alembert force in an accelerating frame.

3. Application-The Expression of the d'Alembert Force in a Uniformly Accelerating Frame

Assume we have a particle of mass m moving inertial in the x'-y' plane in frame S'. When the laws of dynamics are transformed from an inertial frame to an accelerating frame of reference, fictitious forces, such as the d'Alembert force appear. We would like to find out the expression of the d'Alembert force exerted on the particle as measured in frame S'. In frame the inertial frame S the following hold true:

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} v_{0x}t + x_0 \\ v_{0y}t + y_0 \\ v_{0z}t + z_0 \\ ct \end{pmatrix} \tag{3.1}$$

$$\mathbf{v} = \mathbf{v_0} \tag{3.2}$$

$\mathbf{v_0}$ is the initial velocity of the particle at $t = 0$.

$$\mathbf{p} = \frac{m\mathbf{v_0}}{\sqrt{1-(\frac{v_0^2}{c})^2}}$$

$$E = \frac{mc^2}{\sqrt{1-(\frac{v_0^2}{c})^2}} \tag{3.3}$$

$$\mathbf{F} = 0 \tag{3.4}$$

Substituting (3.1-3.4) into (2.13) we obtain:

$$\begin{pmatrix} F_x^{'} \\ F_y^{'} \\ F_z^{'} \\ \frac{1}{c}\frac{dE'}{d\tau} \end{pmatrix} = \frac{m}{\sqrt{1-(\frac{v_0}{c})^2}} \frac{d\mathbf{A}^{-1}}{d\tau} \begin{pmatrix} v_{0x} \\ v_{0y} \\ v_{0z} \\ c \end{pmatrix} = \gamma(v_0) m \frac{d\mathbf{A}^{-1}}{d\tau} \mathbf{v_0} \tag{3.5}$$

It is interesting to see that the acceleration introduces not only a fictitious force but also a fictitious power, according to (3.5).

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Relativistic Fictitious Forces from the Perspective of the Accelerated Rotating Platform

Synopsis

In this chapter we present the expression of the relativistic fictitious forces as measured in the frame of a non-uniformly rotating platform. The solution is of great interest for real time applications because earth-bound laboratories are inertial only in approximation. The motivation is that the real life applications include accelerating and rotating frames more often than the idealized case of inertial frames; our daily experiments happen in the laboratories attached to the rotating Earth. The accelerations play an important role in centrifuges ramping up to speed. We will provide a straightforward method of deriving the fictitious forces arising in the rotating frame in their relativistic form. We are also correcting the expression of the Euler force in its classical (non-relativistic) form by correcting an error in its derivation that has persisted for centuries.

1. Dynamics in Uniform Angular Velocity Rotation

Consider an inertial frame K; the coordinates are (x, y, z, t). In a frame K' rotating with respect the inertial frame, the coordinates are (x', y', z', t'). The angular speed of rotation between the two frames is ω and it is assumed to be constant. The transformation between the frames is [1]:

$$\begin{aligned} t &= t' \\ x &= x'\cos\omega t' - y'\sin\omega t' \\ y &= x'\sin\omega t' + y'\cos\omega t' \\ z &= z' \end{aligned} \tag{1.1}$$

Let's assume that in the inertial frame K a particle moves with the constant velocity $\mathbf{v}$. Because $t = t'$ we can write, by differentiating (1.1) with respect to the proper time τ :

$$\begin{aligned} v_x &= \frac{dx}{d\tau} = \frac{dx'}{d\tau}\cos\omega t - \omega x'\frac{dt}{d\tau}\sin\omega t - \frac{dy'}{d\tau}\sin\omega t - \omega y'\frac{dt}{d\tau}\cos\omega t \\ v_y &= \frac{dy}{d\tau} = \frac{dx'}{d\tau}\sin\omega t' + \omega x'\frac{dt}{d\tau}\cos\omega t + \frac{dy'}{d\tau}\cos\omega t - \omega y'\frac{dt}{d\tau}\sin\omega t \end{aligned} \tag{1.2}$$

Since $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \text{constant}$ (1.2) can be re-written as:

$$\begin{aligned} v_x &= \frac{dx'}{d\tau}\cos\omega t - \omega x'\gamma\sin\omega t - \frac{dy'}{d\tau}\sin\omega t - \omega y'\gamma\cos\omega t \\ v_y &= \frac{dx'}{d\tau}\sin\omega t' + \omega x'\gamma\cos\omega t + \frac{dy'}{d\tau}\cos\omega t - \omega y'\gamma\sin\omega t \end{aligned} \tag{1.3}$$

Differentiating (1.3) with respect to the proper time we obtain the expressions of the proper accelerations:

$$\begin{aligned} 0 &= \frac{d^2x'}{d\tau^2}\cos\omega t - \frac{d^2y'}{d\tau^2}\sin\omega t - 2\omega\gamma\frac{dx'}{d\tau}\sin\omega t - 2\omega\gamma\frac{dy'}{d\tau}\cos\omega t + \\ &+\omega^2 y'\gamma\sin\omega t - \omega^2 x'\gamma\cos\omega t \\ 0 &= \frac{d^2x'}{d\tau^2}\sin\omega t + \frac{d^2y'}{d\tau^2}\cos\omega t + 2\omega\gamma\frac{dx'}{d\tau}\cos\omega t - 2\omega\gamma\frac{dy'}{d\tau}\sin\omega t - \\ &-\omega^2 y'\gamma\cos\omega t - \omega^2 x'\gamma\sin\omega t \end{aligned} \tag{1.4}$$

We can now form a linear system of equations:

$$\frac{d^2x'}{d\tau^2}\cos\omega t-\frac{d^2y'}{d\tau^2}\sin\omega t=2\omega\gamma\frac{dx'}{d\tau}\sin\omega t+2\omega\gamma\frac{dy'}{d\tau}\cos\omega t-$$
$$-\omega^2y'\gamma\sin\omega t+\omega^2x'\gamma\cos\omega t$$
$$\frac{d^2x'}{d\tau^2}\sin\omega t+\frac{d^2y'}{d\tau^2}\cos\omega t=-2\omega\gamma\frac{dx'}{d\tau}\cos\omega t+2\omega\gamma\frac{dy'}{d\tau}\sin\omega t+$$
$$+\omega^2y'\gamma\cos\omega t+\omega^2x'\gamma\sin\omega t \tag{1.5}$$

The system (1.5) has the solution:

$$\frac{d^2x'}{d\tau^2}=2\gamma\omega\frac{dy'}{d\tau}+\gamma\omega^2x'$$
$$\frac{d^2y'}{d\tau^2}=-2\gamma\omega\frac{dx'}{d\tau}+\gamma\omega^2y' \tag{1.6}$$

We recognize immediately the relativistic Coriolis acceleration, $\mathbf{a}_{Coriolis}=(2\gamma\omega\frac{dy'}{d\tau},-2\gamma\omega\frac{dx'}{d\tau})$ and the relativistic centrifugal acceleration, $\mathbf{a}_{centifugal}=(\gamma\omega^2x',\gamma\omega^2y')$ [2-12].

2. The Case of Non-Uniform Angular Speed, the Emergence of the Euler force

In a previous paper [13] we concerned ourselves with the uniform angular speed. What happens if $\boldsymbol{\omega}$ varies in time? This gives rise to the so called Euler force. At relativistic speeds, the formula for the Euler force can be calculated by revisiting (1.2) in the previous section:

$$v_x = \frac{dx'}{d\tau}\cos\omega t - \omega x'\frac{dt}{d\tau}\sin\omega t - \frac{dy'}{d\tau}\sin\omega t -$$

$$-\omega y'\frac{dt}{d\tau}\cos\omega t + \frac{d}{d\omega}(x'\cos\omega t - y'\sin\omega t)\frac{d\omega}{d\tau} =$$

$$= \frac{dx'}{d\tau}\cos\omega t - \omega x'\frac{dt}{d\tau}\sin\omega t - \frac{dy'}{d\tau}\sin\omega t - \omega y'\frac{dt}{d\tau}\cos\omega t -$$

$$-(x't\sin\omega t + y't\cos\omega t)\frac{d\omega}{d\tau}$$

$$v_y = \frac{dx'}{d\tau}\sin\omega t' + \omega x'\frac{dt}{d\tau}\cos\omega t + \frac{dy'}{d\tau}\cos\omega t -$$

$$-\omega y'\frac{dt}{d\tau}\sin\omega t + \frac{d}{d\omega}(x'\sin\omega t + y'\cos\omega t)\frac{d\omega}{d\tau} =$$

$$= \frac{dx'}{d\tau}\sin\omega t' + \omega x'\frac{dt}{d\tau}\cos\omega t + \frac{dy'}{d\tau}\cos\omega t - \omega y'\frac{dt}{d\tau}\sin\omega t +$$

$$+(x't\cos\omega t - y't\sin\omega t)\frac{d\omega}{d\tau} \tag{2.1}$$

In order to get all three fictitious accelerations (Coriolis, centrifugal and Euler) we will need to differentiate (2.1) with respect to the proper time and to form the linear system in $(\frac{d^2x'}{d\tau^2}, \frac{d^2y'}{d\tau^2})$. We will concentrate only on the Euler component:

$$\frac{d^2x'}{d\tau^2}\cos\omega t - \frac{d^2y'}{d\tau^2}\sin\omega t = (\frac{dx'}{d\tau}t\sin\omega t + x'\gamma\sin\omega t + x't\omega\gamma\cos\omega t +$$

$$+\frac{dy'}{d\tau}t\cos\omega t + y'\gamma\cos\omega t - y't\omega\gamma\sin\omega t)\frac{d\omega}{d\tau} + (x't\sin\omega t + y't\cos\omega t)\frac{d^2\omega}{d\tau^2}$$

$$\frac{d^2x'}{d\tau^2}\sin\omega t + \frac{d^2y'}{d\tau^2}\cos\omega t = \ldots\ldots - (x't\cos\omega t - y't\sin\omega t)\frac{d^2\omega}{d\tau^2} \tag{2.2}$$

One very interesting conclusion can be drawn from (2.2): the second order derivative of the angular speed, if it exists, will manifest itself in the Euler acceleration. Indeed:

$$\frac{d^2x'}{d\tau^2} = ... + y't\frac{d^2\omega}{d\tau^2}$$
$$\frac{d^2y'}{d\tau^2} = ... - x't\frac{d^2\omega}{d\tau^2} \tag{2.3}$$

We can trace the error that has persisted over 240 years to the use of the fact that the time derivatives of the unit vectors in the rotating frames can be expressed as:

$$\frac{d\mathbf{u}}{dt} = \boldsymbol{\omega} \times \mathbf{u} \tag{2.4}$$

But (2.4) is true only in the case of constant angular speed, it is obviously not true for variable angular speed. Indeed, let's consider, for example:

$$\mathbf{u}_\mathbf{x} = (\cos\omega t, -\sin\omega t) \tag{2.5}$$

Then:

$$\frac{d\mathbf{u}_\mathbf{x}}{dt} = (\omega + t\frac{d\omega}{dt})(-\sin\omega t, -\cos\omega t) \tag{2.6}$$

In the next section we will be providing a rigorous formalism that addresses all the derivations of fictitious accelerations and forces and produces the complete, exact expressions for both the classical and the relativistic case.

3. A More General Formalism, Matrix Based

In this section we will present a matrix-based approach to deriving the compact, exact expressions for both fictitious acceleration and fictitious forces in the relativistic sector. We start by rewriting (1.1) as:

$$\mathbf{r} = R\mathbf{r}'$$
$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{3.1}$$

Then:

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \dot{R}\mathbf{r}' + R\dot{\mathbf{r}}' = \dot{R}R^{-1}\mathbf{r} + R\dot{\mathbf{r}}' = \dot{R}R^{T}\mathbf{r} + R\dot{\mathbf{r}}' \tag{3.2}$$

The overdot represents differentiation with respect to the coordinate time, t.

It follows immediately that:

$$\dot{\mathbf{r}}' = R^{T}(\dot{\mathbf{r}} - \dot{R}R^{T}\mathbf{r}) \tag{3.3}$$

After some elementary algebraic manipulation, we get:

$$\dot{\mathbf{r}}' = R^{T}\dot{\mathbf{r}} - (\omega + t\dot{\omega})Q\mathbf{r}$$
$$Q = \begin{bmatrix} \sin\omega t & \cos\omega t \\ -\cos\omega t & \sin\omega t \end{bmatrix} \tag{3.4}$$

The resulting fictitious acceleration in the rotating frame is:

$$\ddot{\mathbf{r}}' = \dot{R}^{T}\dot{\mathbf{r}} + R^{T}\ddot{\mathbf{r}} - (2\dot{\omega} + \ddot{\omega})Q\mathbf{r} - (\omega + t\dot{\omega})\dot{Q}\mathbf{r} - (\omega + t\dot{\omega})Q\dot{\mathbf{r}} =$$
$$= \dot{R}^{T}\dot{\mathbf{r}} - (2\dot{\omega} + \ddot{\omega})Q\mathbf{r} - (\omega + t\dot{\omega})\dot{Q}\mathbf{r} - (\omega + t\dot{\omega})Q\dot{\mathbf{r}} \tag{3.5}$$

After some more manipulation, noting that

$$\dot{Q} = (\omega + t\dot{\omega})R^T$$

$$\dot{R}^T = -(\omega + t\dot{\omega})Q \tag{3.6}$$

we obtain the final expression for the classical fictitious acceleration:

$$\ddot{\mathbf{r}}' = -2(\omega + t\dot{\omega})Q\mathbf{v} - [(\omega + t\dot{\omega})^2 R^T + (2\dot{\omega} + \ddot{\omega})Q]\mathbf{r} \tag{3.7}$$

One last step is to re-order (3.7) after the powers of ω and its derivatives:

$$\ddot{\mathbf{r}}' = -\omega^2 R^T\mathbf{r} - 2(\omega + t\dot{\omega})Q\mathbf{v} - [(2\dot{\omega} + \ddot{\omega})Q + (t\dot{\omega})^2 R^T +$$
$$+2t\omega\dot{\omega}R^T]\mathbf{r} \tag{3.8}$$

Separating the components, we obtain:

$$\mathbf{a}'_{centrifugal} = -\omega^2 R^T\mathbf{r}$$
$$\mathbf{a}'_{coriolis} = -2(\omega + t\dot{\omega})Q\mathbf{v}$$
$$\mathbf{a}'_{euler} = -[(2\dot{\omega} + \ddot{\omega})Q + (t\dot{\omega})^2 R^T + 2t\omega\dot{\omega}R^T]\mathbf{r} \tag{3.9}$$

It is interesting to note that not only the Euler acceleration looks different but so does the Coriolis acceleration, all due to the fact that the angular speed is variable. A quick sanity check recovers the classical case for $\omega = \text{constant}$:

$$\mathbf{a}'_{centrifugal} = -\omega^2 R^T\mathbf{r}$$
$$\mathbf{a}'_{coriolis} = -2\omega Q\mathbf{v}$$
$$\mathbf{a}'_{euler} = 0 \tag{3.10}$$

We now move to the derivations of the fictitious accelerations and forces for the relativistic domain. In order to do that we will replace the time differential with respect to coordinate time with the differential with respect to proper time:

$$\frac{d}{d\tau} = \frac{d}{dt}\frac{dt}{d\tau} = \gamma(v)\frac{d}{dt}$$

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \text{constant} \tag{3.11}$$

Expression (3.3) becomes:

$$\frac{d\mathbf{r}'}{d\tau} = R^T\frac{d\mathbf{r}}{d\tau} - R^T\frac{dR}{d\tau}R^T\mathbf{r} \tag{3.12}$$

Now:

$$\frac{dQ}{d\tau} = (\gamma(v)\omega + t\dot{\omega})R^T$$

$$\frac{dR^T}{d\tau} = -(\gamma(v)\omega + t\dot{\omega})Q \tag{3.13}$$

Armed with the above, (3.8) becomes:

$$\frac{d^2\mathbf{r}'}{d\tau^2} = -\gamma^2(v)\omega^2R^T\mathbf{r} - 2(\gamma(v)\omega + t\dot{\omega})Q\mathbf{v} + [(2\gamma(v)\dot{\omega} + t\ddot{\omega})Q -$$
$$-(t\dot{\omega})^2R^T - 2\gamma(v)t\omega\dot{\omega}R^T]\mathbf{r} \tag{3.14}$$

The overdot represents differentiation of the with respect to proper time. For $\omega = \text{constant}$:

$$\frac{d^2\mathbf{r}'}{d\tau^2} = -\gamma^2(v)\omega^2R^T\mathbf{r} - 2\gamma(v)\omega Q\mathbf{v} \tag{3.15}$$

The fictitious accelerations are:

$$\mathbf{a}'_{centrifugal} = -\gamma^2(v)\omega^2 R^T \mathbf{r}$$

$$\mathbf{a}'_{coriolis} = -2(\gamma(v)\omega + t\dot{\omega})Q\mathbf{v}$$

$$\mathbf{a}'_{euler} = -[(2\gamma(v)\dot{\omega} + \ddot{\omega}t)Q + (t\dot{\omega})^2 R^T + 2\gamma(v)t\omega\dot{\omega}R^T]\mathbf{r} \quad (3.16)$$

In order to calculate the relativistic fictitious force we need to start with the relativistic proper momentum:

$$\mathbf{p}' = \frac{m\mathbf{v}'}{\sqrt{1-\frac{v'^2}{c^2}}}$$

$$\mathbf{v}' = \frac{d\mathbf{r}'}{d\tau} \quad (3.17)$$

The proper force is:

$$\mathbf{F}' = \frac{d\mathbf{p}'}{d\tau} = m\frac{d}{d\tau}\frac{\mathbf{v}'}{\sqrt{1-\frac{v'^2}{c^2}}} = m\frac{1}{(\sqrt{1-\frac{v'^2}{c^2}})^3}\frac{d^2\mathbf{r}'}{d\tau^2} =$$

$$= m\gamma^3(v')\frac{d^2\mathbf{r}'}{d\tau^2} \quad (3.18)$$

From (3.18) and (3.16) we obtain:

$$\mathbf{F}'_{centrifugal} = -m\gamma^2(v)\gamma^3(v')\omega^2 R^T \mathbf{r}$$

$$\mathbf{F}'_{coriolis} = -2m\gamma^3(v')(\gamma(v)\omega + t\dot{\omega})Q\mathbf{v}$$

$$\mathbf{F}'_{euler} = -m\gamma^3(v')[(2\gamma(v)\dot{\omega} + \ddot{\omega}t)Q + (t\dot{\omega})^2 R^T +$$

$$+2\gamma(v)t\omega\dot{\omega}R^T]\mathbf{r} \quad (3.19)$$

For $\omega = \text{constant}$ **:**

$$\mathbf{F}'_{centrifugal} = -m\gamma^2(v)\gamma^3(v')\omega^2 R^T \mathbf{r}$$
$$\mathbf{F}'_{coriolis} = -2m\gamma^3(v')\gamma(v)\omega Q\mathbf{v}$$
$$\mathbf{F}'_{euler} = 0 \tag{3.20}$$

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RELATIVISTIC FICTITIOUS FORCES IN ACCELERATED ROTATING FRAMES OF REFERENCE

Synopsis

In the current chapter we present the expression of the relativistic fictitious forces as measured in the frame of a non-uniformly rotating frame of reference. The solution is of great interest for real time applications because earth-bound laboratories are inertial only in approximation. The motivation is that the real life applications include accelerating and rotating frames more often than the idealized case of inertial frames; our daily experiments happen in the laboratories attached to the rotating Earth. The accelerations play an important role in centrifuges ramping up to speed. We will provide a straightforward method of deriving the fictitious forces arising in the rotating frame in their relativistic form. We are also correcting the expression of the Euler force in its classical (non-relativistic) form by correcting an error in its derivation that has persisted for centuries.

1. Dynamics of Accelerated Particles in a Variable Angular Velocity Rotating Frame

Consider an inertial frame K; the coordinates are (x, y, z, t). In a frame K' rotating with respect the inertial frame, the coordinates are (x', y', z', t'). The angular speed of rotation between the two frames is ω and it is assumed to be variable. The transformation between the frames is [1]:

$$
\begin{aligned}
&t = t' \\
&x = x'\cos\omega t' - y'\sin\omega t' \\
&y = x'\sin\omega t' + y'\cos\omega t' \\
&z = z'
\end{aligned}
\tag{1.1}
$$

Let's assume that in the inertial frame K a particle moves with the variable velocity $\mathbf{v}$. We set to determine the fictitious accelerations and the fictitious forces that appear in the rotating frame K'.

2. Fictitious Relativistic Accelerations and Fictitious Relativistic Forces

In this section we will present a matrix-based approach to deriving the compact, exact expressions for both fictitious acceleration and fictitious forces in the relativistic sector. We start by rewriting (1.1) as:

$$\mathbf{r} = R\mathbf{r'}$$
$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\mathbf{r'} = \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{2.1}$$

Then:

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \dot{R}\,\mathbf{r'} + R\,\dot{\mathbf{r}}' = \dot{R}\,R^{-1}\mathbf{r} + R\,\dot{\mathbf{r}}' = \dot{R}\,R^{T}\mathbf{r} + R\,\dot{\mathbf{r}}' \tag{2.2}$$

The overdot represents differentiation with respect to the coordinate time t.

It follows immediately that:

$$\dot{\mathbf{r}}' = R^{T}(\dot{\mathbf{r}} - \dot{R}\,R^{T}\mathbf{r}) \tag{2.3}$$

After some elementary algebraic manipulation, we get:

$$\dot{\mathbf{r}}' = R^{T}\,\dot{\mathbf{r}} - (\omega + t\,\dot{\omega})Q\mathbf{r}$$
$$Q = \begin{bmatrix} \sin\omega t & \cos\omega t \\ -\cos\omega t & \sin\omega t \end{bmatrix} \tag{2.4}$$

The resulting fictitious acceleration in the rotating frame is:

$$\ddot{\mathbf{r}}' = \dot{R}^T \dot{\mathbf{r}} + R^T \ddot{\mathbf{r}} - (2\dot{\omega} + \ddot{\omega})Q\mathbf{r} - (\omega + t\dot{\omega})\dot{Q}\mathbf{r} - (\omega + t\dot{\omega})Q\dot{\mathbf{r}} =$$
$$= \dot{R}^T \dot{\mathbf{r}} - (2\dot{\omega} + \ddot{\omega})Q\mathbf{r} - (\omega + t\dot{\omega})\dot{Q}\mathbf{r} - (\omega + t\dot{\omega})Q\dot{\mathbf{r}} \quad (2.5)$$

We assume the most general case, $\mathbf{v}$ being variable, so $\ddot{\mathbf{r}} \neq 0$. After some more manipulation, noting that

$$\dot{Q} = (\omega + t\dot{\omega})R^T$$
$$\dot{R}^T = -(\omega + t\dot{\omega})Q \quad (2.6)$$

we obtain the final expression for the classical fictitious acceleration:

$$\ddot{\mathbf{r}}' = R^T \ddot{\mathbf{r}} - 2(\omega + t\dot{\omega})Q\mathbf{v} - [(\omega + t\dot{\omega})^2 R^T + (2\dot{\omega} + \ddot{\omega})Q]\mathbf{r} \quad (2.7)$$

One last step is to re-order (2.7) after the powers of ω and its derivatives:

$$\ddot{\mathbf{r}}' = R^T \ddot{\mathbf{r}} - \omega^2 R^T \mathbf{r} - 2(\omega + t\dot{\omega})Q\mathbf{v} -$$
$$-[(2\dot{\omega} + \ddot{\omega})Q + (t\dot{\omega})^2 R^T + 2t\omega\dot{\omega}R^T]\mathbf{r} \quad (2.8)$$

Separating the components, we obtain:

$$\mathbf{a}'_{centrifugal} = -\omega^2 R^T \mathbf{r}$$
$$\mathbf{a}'_{coriolis} = -2(\omega + t\dot{\omega})Q\mathbf{v}$$
$$\mathbf{a}'_{euler} = -[(2\dot{\omega} + \ddot{\omega})Q + (t\dot{\omega})^2 R^T + 2t\omega\dot{\omega}R^T]\mathbf{r}$$
$$\mathbf{a}'_{real} = R^T \ddot{\mathbf{r}} \quad (2.9)$$

It is interesting to note that not only the Euler acceleration looks different but so does the Coriolis acceleration, all due to the fact that the angular speed is variable. A quick sanity check recovers the classical case for $\omega = \text{constant}$:

$$\mathbf{a}'_{centrifugal} = -\omega^2 R^T \mathbf{r}$$
$$\mathbf{a}'_{coriolis} = -2\omega Q\mathbf{v}$$
$$\mathbf{a}'_{euler} = 0$$
$$\mathbf{a}'_{real} = R^T \ddot{\mathbf{r}} \tag{2.10}$$

We now move to the derivations of the fictitious accelerations and forces for the relativistic domain. In order to do that we will replace the time differential with respect to coordinate time with the differential with respect to proper time:

$$\frac{d}{d\tau} = \frac{d}{dt}\frac{dt}{d\tau} = \gamma(v)\frac{d}{dt}$$
$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \text{variable} \tag{2.11}$$

Expression (2.3) becomes:

$$\frac{d\mathbf{r}'}{d\tau} = R^T \frac{d\mathbf{r}}{d\tau} - R^T \frac{dR}{d\tau} R^T \mathbf{r} \tag{2.12}$$

Now:

$$\frac{dQ}{d\tau} = (\gamma(v)\omega + t\dot{\omega})R^T$$
$$\frac{dR^T}{d\tau} = -(\gamma(v)\omega + t\dot{\omega})Q \tag{2.13}$$

Armed with the above, (2.8) becomes:

$$\frac{d^2\mathbf{r}'}{d\tau^2} = \frac{d^2\mathbf{r}}{d\tau^2} - \gamma^2(v)\omega^2 R^T\mathbf{r} - 2(\gamma(v)\omega + t\dot{\omega})Q\mathbf{v} -$$
$$-[(2\gamma(v)\dot{\omega} + \ddot{\omega}t + \dot{\gamma}(v)\omega)Q + 2\gamma(v)t\omega\dot{\omega}R^T]\mathbf{r}$$
(2.14)

The fictitious accelerations are:

$$\mathbf{a}'_{centrifugal} = -\gamma^2(v)\omega^2 R^T\mathbf{r}$$
$$\mathbf{a}'_{coriolis} = -2(\gamma(v)\omega + t\dot{\omega})Q\mathbf{v}$$
$$\mathbf{a}'_{euler} = -[(2\gamma(v)\dot{\omega} + \ddot{\omega}t + \dot{\gamma}(v)\omega)Q + 2\gamma(v)t\omega\dot{\omega}R^T]\mathbf{r} \quad (2.15)$$

There is an extra term in the expression of the Euler acceleration due to the fact that the particle is moving with variable velocity in the inertial frame of the lab:

$$\dot{\gamma}(v) = \frac{d\gamma}{dv}\frac{dv}{d\tau} = \frac{\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dv}{d\tau} = \frac{\frac{1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}v\frac{dv}{d\tau} =$$
$$= \frac{\frac{1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\mathbf{v}\frac{d\mathbf{v}}{d\tau} = \frac{\frac{1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\mathbf{r}}{d\tau}\frac{d^2\mathbf{r}}{d\tau^2}$$
(2.16)

In order to calculate the relativistic fictitious force we need to start with the relativistic proper momentum:

$$\mathbf{p'} = \frac{m\mathbf{v'}}{\sqrt{1 - \frac{v'^2}{c^2}}}$$

$$\mathbf{v'} = \frac{d\mathbf{r'}}{d\tau} \tag{2.17}$$

The proper force is:

$$\mathbf{F'} = \frac{d\mathbf{p'}}{d\tau} = m\frac{d}{d\tau}\frac{\mathbf{v'}}{\sqrt{1-\frac{v'^2}{c^2}}} = m\frac{1}{(\sqrt{1-\frac{v'^2}{c^2}})^3}\frac{d^2\mathbf{r'}}{d\tau^2} =$$

$$= m\gamma^3(v')\frac{d^2\mathbf{r'}}{d\tau^2} \tag{2.18}$$

From (2.18) and (2.15) we obtain:

$$\mathbf{F}'_{centrifugal} = -m\gamma^2(v)\gamma^3(v')\omega^2 R^T\mathbf{r}$$

$$\mathbf{F}'_{coriolis} = -2m\gamma^3(v')(\gamma(v)\omega + t\dot{\omega})Q\mathbf{v}$$

$$\mathbf{F}'_{euler} = -m\gamma^3(v')[(2\gamma(v)\dot{\omega} + \ddot{\omega}t + \gamma(\dot{v})\omega)Q +$$

$$+2\gamma(v)t\omega\dot{\omega}R^T]\mathbf{r} \tag{2.19}$$

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WAS GALILEO RIGHT?

Synopsis

An important scientific debate took place regarding falling bodies hundreds of years ago, and it still warrants close examination. Galileo argued that in a vacuum all bodies fall at the same rate relative to the earth, independent of their mass. As we shall see, the problem is more subtle than meets the eye -- even in vacuum. In principle the results of a free fall experiment depend on whether falling masses are sequential or concurrent, whether they fall side by side or diametrically opposed. In the current paper we will present both the classical mechanics treatment and the general relativity one. In the case of classical mechanics, we start from the basic equations of motion. On the other hand, the determination of particle equations of motion in gravitational fields in general relativity is done routinely via the use of covariant derivatives. Since the geodesic equations based on covariant derivatives are derived from the Euler-Lagrange equations and since the Euler-Lagrange formalism is very intuitive, easy to derive with no mistakes, there is every reason to use them even for the most complicated situations and this is exactly what we do in the second part of the current paper.

1. Classical treatment of radial motion-Time to collision

In the early 17th century, Galileo [1] made the observation: “But I, Simplicia, who have made the test, can assure you that a cannon ball weighing one or two hundred pounds, or even more, will not reach the ground by as much as a span ahead of a musket ball weighing only half a pound, provided both are dropped from a height of 200 cubits.”

Galileo argued that the slight difference in time could be ascribed to the resistance offered by the medium to the motion of the falling body. In air, feathers do fall more slowly than rocks. Galileo then made the idealization that in a medium devoid of resistance (a vacuum), all bodies will fall at the same speed. This idealization neglected the complexity of the fall of objects in media accessible to Galileo and was indeed a significant advance toward a deeper understanding of the motion of bodies. Galileo used experiments with an inclined plane to promote his view that heavy and light bodies fall equally fast. Another Italian, Galileo's contemporary, Torricelli, in his opus, *De motu gravium*, seeks to further demonstrate Galileo's principle regarding equal velocities of free fall of weights along inclined planes of equal height. We ask ourselves, "were Galileo and Torricelli right?". As we shall see in the next paragraph the answer is complex: within the experimental precision, they were right; from the point of view of a rigorous application of mechanics, they were both wrong.

In Newtonian mechanics formulation, for the case of radial motion reduces to solving the equation of motion:

$$m\frac{d^2z}{dt^2} = -\frac{GMm}{d^2} \quad (1.1)$$

where z represents the radial coordinate and d is the distance between the centers of the attracting bodies. It is interesting to note that GR and Newtonian mechanics produce exactly the same equation of motion. Equation (1.1) gives us the tool for determining when two bodies of radiuses R and r_1 and masses M and m will collide after starting from rest at locations $z(0) = D$ and respectively $Z(0) = 0$ separated by the initial distance D (see fig.1).

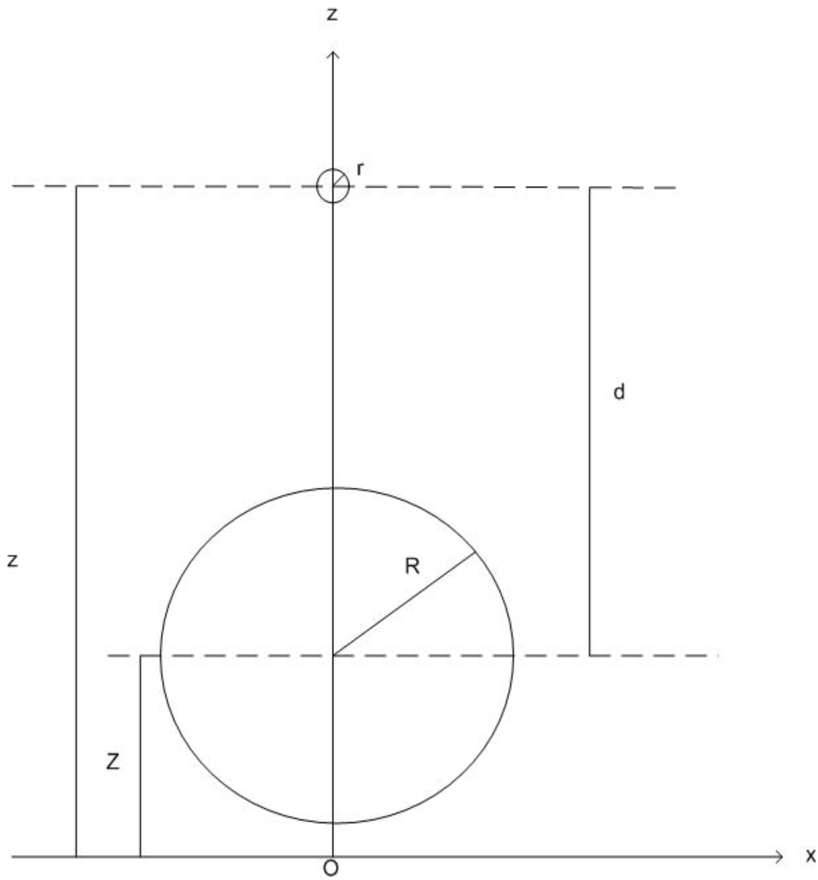

Fig. 1 Simple setup for radial motion in a gravitational field

We would need to solve the system of differential equations:

$$\frac{d^2z}{dt^2} = -\frac{GM}{(Z-z)^2}$$
$$\frac{d^2Z}{dt^2} = +\frac{Gm}{(Z-z)^2} \tag{1.2}$$

with initial conditions:

$$z(0) = D$$
$$Z(0) = 0$$
$$\frac{dZ}{dt}|_{t=0} = \frac{dz}{dt}|_{t=0} = 0 \tag{1.3}$$

$$Z - z \geq R + r$$

and find out the time when $z(t) - Z(t) = R + r_1$ (i.e., when the two masses touch) by solving a transcendental equation in t. The system gets easily reduced to a single equation by subtracting the two equations:

$$\frac{d^2(z-Z)}{dt^2} = -\frac{G(M+m)}{(z-Z)^2} \tag{1.4}$$

Equation (4) has the general solution (see Appendix):

$$t\sqrt{\frac{G(M+m)}{D}} = D(arctg\sqrt{\frac{z-Z}{D-(z-Z)}} - \sqrt{(z-Z)(D-(z-Z))}) \tag{1.5}$$

At the time of collision, $z - Z = R + r_1$ so:

$$t = \frac{D^{3/2}}{\sqrt{G(M+m)}}(arctg\sqrt{\frac{R+r_1}{D-(R+r_1)}} - \frac{\sqrt{(R+r_1)(D-(R+r_1))}}{D}) \tag{1.6}$$

The time to collision does depend on the mass of the probe, so both Galileo and Torricelli were wrong. The reason is that, while the larger gravitating body attracts the smaller one, the effect is reciprocated by the smaller one. Thus, the time to collision is dependent on both masses. It is interesting to see that the effect is dependent on the sum of masses. We could not have demonstrated the above without solving, in a rigorous manner, the equations of motion. If we ask ourselves: "how big is the effect?" then (1.6) provides the answer, the effect is of the order of $\frac{m}{2M}$. To put things in perspective, if we dropped a 1000kg mass on the Moon, the effect would be of the order of $7*10^{-21}$. This is why Galileo could not measure it, it is too small. But it is there. Let's now study a different case, the case of two test probes dropped simultaneously, side by side (see fig.2):

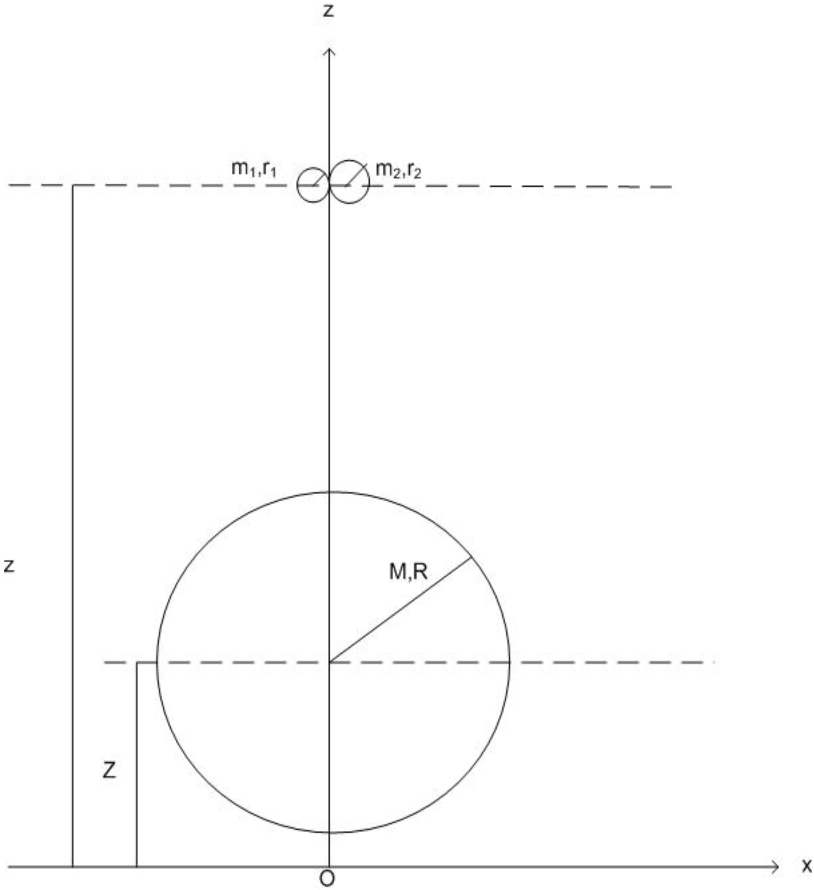

Fig. 2 Two test probes side by side, dropped simultaneously

$$m_1 \frac{d^2 z_1}{dt^2} = -\left(\frac{G m_1 M}{(Z - z_1)^2} + \frac{G m_1 m_2}{(z_2 - z_1)^2 + (r_2 + r_1)^2} \cos\alpha\right) \tag{1.7}$$

$$m_2 \frac{d^2 z_2}{dt^2} = -(\frac{Gm_2 M}{(Z-z_2)^2} + \frac{Gm_2 m_1}{(z_2-z_1)^2+(r_2+r_1)^2}\cos\alpha) \quad (1.7)$$

$$M \frac{d^2 Z}{dt^2} = +(\frac{GMm_1}{(Z-z_1)^2} + \frac{GMm_2}{(Z-z_2)^2})$$

$\cos\alpha = \frac{|z_2-z_1|}{\sqrt{(z_2-z_1)^2+(r_2+r_1)^2}}$, α being the angle of the central attraction force with the z-axis.

The initial conditions are:

$$z_1(0) = z_2(0) = D$$
$$Z(0) = 0$$
$$\frac{dZ}{dt}\Big|_{t=0} = \frac{dz_i}{dt}\Big|_{t=0} = 0 \quad (1.8)$$

The above results into a complicated system:

$$\frac{d^2 z_1}{dt^2} = -\frac{GM}{(Z-z_1)^2} - \frac{Gm_2|z_2-z_1|}{(\sqrt{(z_2-z_1)^2+(r_2+r_1)^2})^3} \quad (1.9)$$

$$\frac{d^2 z_2}{dt^2} = -\frac{GM}{(Z-z_2)^2} - \frac{Gm_1|z_2-z_1|}{(\sqrt{(z_2-z_1)^2+(r_2+r_1)^2})^3} \quad (1.9)$$

$$\frac{d^2 Z}{dt^2} = \frac{Gm_1}{(Z-z_1)^2} + \frac{Gm_2}{(Z-z_2)^2}$$

While the above system may be very difficult to solve, we can still glean a very important physical property, the above system tells us that the two test probes will hit the Earth simultaneously. Indeed, subtracting the first two equations:

$$\frac{d^2(z_1 - z_2)}{dt^2} = \frac{GM}{(Z - z_2)^2} - \frac{GM}{(Z - z_1)^2} - \frac{G(m_2 - m_1)\,|\,z_2 - z_1\,|}{(\sqrt{(z_2 - z_1)^2 + (r_2 + r_1)^2})^3} \quad (1.10)$$

We easily verify that $z_1(t) - z_2(t) = 0$ is a solution. Using the observation the above system can be solved much easier since it simplifies to:

$$\frac{d^2 z}{dt^2} = -\frac{GM}{(Z - z)^2} \quad (1.11)$$

$$\frac{d^2 Z}{dt^2} = \frac{G(m_1 + m_2)}{(Z - z)^2}$$

Subtracting the first equation from the second one we obtain:

$$\frac{d^2(Z - z)}{dt^2} = \frac{G(M + m_1 + m_2)}{(Z - z)^2} \quad (1.12)$$

with the initial conditions:

$$z(0) = D$$
$$Z(0) = 0$$
$$\frac{dZ}{dt}\Big|_{t=0} = \frac{dz}{dt}\Big|_{t=0} = 0 \quad (1.13)$$

We need to find out the time when $Z(t) - z(t) = R + r_i$ (i.e., when the two masses touch):

$$t_i = \frac{D^{3/2}}{\sqrt{G(M + m_1 + m_2)}}\left(arctg\sqrt{\frac{R + r_i}{D - (R + r_i)}} - \frac{\sqrt{(R + r_i)(D - (R + r_i))}}{D}\right) \quad (1.14)$$

If the test probes have equal radiuses, their times to collisions will be equal.

On the other hand, if the particles start simultaneously, diametrically opposed, the equations of motion are simpler (see situation depicted in fig. 3):

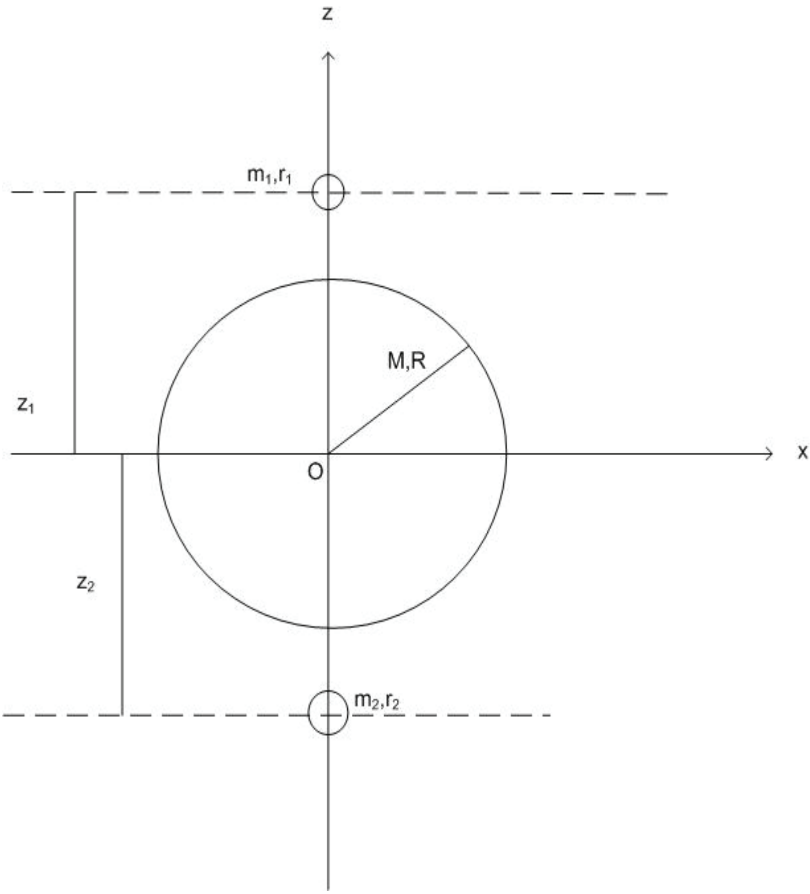

Fig. 3 Two test probes dropped simultaneously, diametrically opposed

$$m_1 \frac{d^2 z_1}{dt^2} = -(\frac{Gm_1 M}{(Z-z_1)^2} + \frac{Gm_1 m_2}{(z_2 - z_1)^2})$$

$$m_2 \frac{d^2 z_2}{dt^2} = +(\frac{Gm_2 M}{(Z-z_2)^2} + \frac{Gm_2 m_1}{(z_2 - z_1)^2}) \quad (1.15)$$

$$M \frac{d^2 Z}{dt^2} = \frac{GMm_1}{(Z-z_1)^2} - \frac{GMm_2}{(Z-z_2)^2}$$

The initial conditions are:

$$z_1(0) = D$$

$$z_2(0) = -D$$

$$Z(0) = 0$$

$$\frac{dZ}{dt}\Big|_{t=0} = \frac{dz_i}{dt}\Big|_{t=0} = 0 \quad (1.16)$$

After some simplifications, equations (1.15) become:

$$\frac{d^2 z_1}{dt^2} = -\frac{GM}{(Z-z_1)^2} - \frac{Gm_2}{(z_2 - z_1)^2}$$

$$\frac{d^2 z_2}{dt^2} = \frac{GM}{(Z-z_2)^2} + \frac{Gm_1}{(z_2 - z_1)^2} \quad (1.17)$$

$$\frac{d^2 Z}{dt^2} = \frac{Gm_1}{(Z-z_1)^2} - \frac{Gm_2}{(Z-z_2)^2}$$

Though the equations of motion are simpler, our chances of solving system (1.17) are next to nil, at least symbolically. In this case we have means of determining which object collides first with the Earth. Nevertheless, we observe that by adding the three equations (1.17) we obtain an interesting relationship:

$$\frac{d^2}{dt^2}(MZ + m_1 z_1 + m_2 z_2) = 0 \tag{1.18}$$

Given the initial conditions, (18) results immediately into:

$$MZ + m_1 z_1 + m_2 z_2 = D(m_1 - m_2) \tag{1.19}$$

The physical interpretation of the above is that the two test probes and the Earth all move in such a fashion that their center of mass is stationary:

$$Z_{COM} = \frac{MZ(t) + m_1 z_1(t) + m_2 z_2(t)}{M + m_1 + m_2} = \frac{D(m_1 - m_2)}{M + m_1 + m_2} \tag{1.20}$$

The above means that the test probes and the Earth must move towards the COM such that they all reach it at the same instant or the COM would move, which is not allowed as per (1.20). This gives us an idea: if $M >> m_1, m_2$ then, as per (1.20), $Z_{COM} \approx 0$, i.e. the center of mass of the system coincides with the initial position of the Earth and does not move. Therefore, we can make $Z(t) = 0$ in (1.17) such that the equations (1.17) simplify to:

$$\frac{d^2 z_1}{dt^2} = -\frac{GM}{z_1^{\,2}} - \frac{Gm_2}{(z_2 - z_1)^2}$$

$$\frac{d^2 z_2}{dt^2} = \frac{GM}{z_2^{\,2}} + \frac{Gm_1}{(z_2 - z_1)^2} \tag{1.21}$$

$$\frac{m_1}{z_1^{\,2}} = \frac{m_2}{z_2^{\,2}}$$

Substituting $z_2 = -z_1\sqrt{\frac{m_2}{m_1}}$ into the first equation we obtain a form that we already know how to solve:

$$\frac{d^2 z_1}{dt^2} = -\frac{G(M + \frac{m_1 m_2}{(\sqrt{m_1} + \sqrt{m_2})^2})}{z_1^{\ 2}} \tag{1.22}$$

The time to collision for the first test probe is:

$$t_1 = \frac{D^{3/2}}{\sqrt{G(M + \frac{m_1 m_2}{(\sqrt{m_1} + \sqrt{m_2})^2})}} (arctg \sqrt{\frac{R + r_1}{D - (R + r_1)}} - \frac{\sqrt{(R + r_1)(D - (R + r_1))}}{D}) \tag{1.23}$$

By symmetry, the time to collision for the second probe is:

$$t_2 = \frac{D^{3/2}}{\sqrt{G(M + \frac{m_1 m_2}{(\sqrt{m_1} + \sqrt{m_2})^2})}} (arctg \sqrt{\frac{R + r_2}{D - (R + r_2)}} - \frac{\sqrt{(R + r_2)(D - (R + r_2))}}{D}) \tag{1.24}$$

If the two test probes have identical radii, $r_1 = r_2$ they will hit the Earth simultaneously if dropped simultaneously from the same height above the Earth, diametrically opposed. The reason for this is that, by making the Earth the (stationary) center of mass, the heavier test probe cannot draw the Earth towards itself as in the previous example. This seems to contradict our earlier point that the two test probes must hit the Earth simultaneously since the COM is stationary. The contradiction is only apparent since, when drawing that conclusion, we have neglected a possible difference in the radii of the two test probes.

2. The GR treatment of the problem using the lagrangian method

While radial motion is the easiest type of motion to describe in natural language, it turns out that its equations are far from trivial [3]. We will show how to derive the equations of motion via a very accessible approach, requiring only elementary calculus and lagrangian mechanics. In order to find the equations of motion we start with the "reduced" Schwarzschild metric for the particular case of absence of rotation ($d\theta = d\varphi = 0$):

$$ds^2=\alpha dt^2-\frac{1}{\alpha}dr^2$$

$$\alpha=1-\frac{r_s}{r} \tag{2.1}$$

where $r_s=\frac{2GM}{c^2}$ is the Schwarzschild radius. For example, the Schwarzschild radius of the Earth is 9 millimeters. From the metric we obtain:

a) the lagrangian [2]

$$L=\alpha\,{\frac{dt}{ds^2}}^{2}-\frac{1}{\alpha}\frac{dr^2}{ds^2} \tag{2.2}$$

b) from the lagrangian we obtain the Euler-Lagrange system of equations:

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right)-\frac{\partial L}{\partial t}=0$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0 \tag{2.3}$$

and, respectively:

$$\frac{d}{ds}\left(\alpha\frac{dt}{ds}\right)=0$$

$$\alpha\frac{dt}{ds}=k$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=\frac{d}{ds}\left(\frac{-2\dot{r}}{\alpha}\right)-\dot{t}^2\frac{d\alpha}{dr}+\dot{r}^2\frac{d}{dr}\left(\frac{1}{\alpha}\right)=$$

$$=2\left(-\frac{\ddot{r}}{\alpha}+\dot{r}^2\frac{\frac{d\alpha}{dr}}{\alpha^2}\right)-\dot{r}^2\frac{\frac{d\alpha}{dr}}{\alpha^2}-\dot{t}^2\frac{d\alpha}{dr}=-2\frac{\ddot{r}}{\alpha}+\dot{r}^2\frac{\frac{d\alpha}{dr}}{\alpha^2}-\dot{t}^2\frac{d\alpha}{dr} \tag{2.4}$$

The over-dots signify derivative with respect to s. From the metric (2.1) we obtain:

$$\alpha(\frac{dt}{ds})^2 = 1 + \frac{1}{\alpha}(\frac{dr}{ds})^2 \tag{2.5}$$

Substituting (2.5) into (2.4) we obtain

c) the equation of motion:

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{d\alpha}{dr} = 0 \tag{2.6}$$

with the solution

$$s\sqrt{\frac{r_s/2}{D}} = Darctg\sqrt{\frac{r}{D-r}} - \sqrt{r(D-r)} \tag{2.7}$$

where $D = r(0)$, exactly like in the classical case described in the previous paragraph. From (2.7) and the condition $r = R + r_1$ we obtain the time to collision:

$$\tau = \frac{D^{3/2}}{\sqrt{GM}}(arctg\sqrt{\frac{R+r_1}{D-(R+r_1)}} - \frac{\sqrt{(R+r_1)(D-(R+r_1))}}{D}) \tag{2.8}$$

Comparing the GR solution with the classical Newtonian solution we observe that the GR solution does not depend on the mass of the test probe, so there is a slight disagreement, of the order of $\frac{m}{M}$ between the classical and the contemporary theory. This can be explained easily by remembering that, in GR, the test probes have negligible mass, so the answer in (2.8) is given for the case $m = 0$. This completely reconciles the Newtonian theory with GR.

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Appendix

$$\frac{d^2z}{dt^2}=\frac{d}{dt}(\frac{dz}{dt})=\frac{d}{dz}(\frac{dz}{dt})\frac{dz}{dt}=\frac{d}{dz}(\frac{dt}{dz})^{-1}(\frac{dt}{dz})^{-1}=$$

$$=-(\frac{dt}{dz})^{-2}\frac{d^2t}{dz^2}(\frac{dt}{dz})^{-1}=-(\frac{dt}{dz})^{-3}\frac{d^2t}{dz^2}=\frac{1}{2}\frac{d}{dz}(\frac{dt}{dz})^{-2} \quad \text{(A1)}$$

Applying the above, equation (28) becomes:

$$\frac{d}{dz}(\frac{dt}{dz})^{-2}=-\frac{k}{z^2} \quad \text{(A2)}$$

With the notation $y=(\frac{dt}{dz})^{-2}$ equation (A2) becomes:

$$\frac{dy}{dz}=-\frac{k}{z^2} \quad \text{(A3)}$$

with the immediate solution:

$$y=\frac{k}{z}-\frac{k}{z_0} \quad \text{(A4)}$$

where $z_0=z(0)$. On the other hand, $y=(\frac{dz}{dt})^2$, so (A4) reduces to:

$$\frac{dz}{dt} = \sqrt{\frac{k}{z} - \frac{k}{z_0}} \tag{A5}$$

Finally, we are now ready to obtain the equation of motion by solving (A5) through variable separation:

$$\frac{dz}{\sqrt{\frac{k}{z} - \frac{k}{z_0}}} = dt \tag{A6}$$

(A6) has the immediate solution:

$$t\sqrt{\frac{k}{z_0}} = z_0 arctg\sqrt{\frac{z}{z_0 - z}} - \sqrt{z(z_0 - z)} \tag{A7}$$

The Two Test Probe Chase

Synopsis

In the current chapter we examine the "chase" between two test probes dropped radially and simultaneously from two different heights. We know that in the case of constant gravitational acceleration their separation is invariant and equal to their initial separation. Is this true for the realistic case of variable gravitational acceleration? We will present two working examples, one based on Newtonian mechanics, the other using the machinery of general relativity. The same methodology is used in both cases.

1. Introduction-The Newtonian approach

Assume that two test probes are dropped simultaneously, radially from the radial distances H and, respectively, h with $H > h$. Since the body at a lower height (distance) from the center of the gravity source (mass M) "feels" larger gravitational acceleration and the more distant body accelerates at a lower rate, it is evident that the separation between the two bodies will increase. The general relativistic situation for weak field approximation must agree with the Newtonian solution, as will be shown in the second half of the paper.

The equation of motion describing the motion of each test probe is [1,2]

$$\frac{d^2r}{dt^2} = -\frac{m}{r^2}$$
$$m = GM \tag{1.1}$$

In (1.1) M is the mass of the attracting body. We are also considering an ideal case with no atmospheric friction and negligible gravitational attraction between the two test probes. From (1.1) we can see that the acceleration increases as the radial coordinate decreases. The solution of the equation (1.1) is:

$$\frac{dr}{dt} = \sqrt{\frac{2m}{r} - \frac{2m}{r_0}} \tag{1.2}$$

where $r_0 = r(0)$.

From (1.2) we note that the speed increases as the radial distance decreases. We are wondering how this observation reflects on the variation with time of the separation between the two test probes. For the rest of the paper we will use the same methodology for solving this problem, We start by observing that the time varies with the radial coordinate according to:

$$\frac{dt}{dr} = \frac{1}{\sqrt{\frac{2m}{r} - \frac{2m}{r_0}}} \tag{1.3}$$

The time the test probe falling from r_0 reaches the arbitrary location at radial coordinate r is:

$$t = \sqrt{\frac{r_0}{2m}}\left(r_0 \arctan\sqrt{\frac{r}{r_0 - r}} - \sqrt{r(r_0 - r)}\right) \tag{1.4}$$

The variation of the temporal separation Δ with the radial coordinate is derived trivially from (1.4):

$$\frac{d\Delta}{dr} = \frac{1}{\sqrt{\frac{2m}{r} - \frac{2m}{H}}} - \frac{1}{\sqrt{\frac{2m}{r} - \frac{2m}{h}}} \tag{1.5}$$

Since $H > h > r$:

$$\frac{1}{\sqrt{\frac{2m}{r} - \frac{2m}{H}}} < \frac{1}{\sqrt{\frac{2m}{r} - \frac{2m}{h}}} \tag{1.6}$$

It follows that $\frac{\mathrm{d}\Delta}{\mathrm{d}r} < 0$ so the function $\Delta(H,h,r) = t(h,r) - T(H,r)$ is monotonically decreasing with respect to r. Thus when r decreases (the two test probes falling “down”), $\Delta(H,h,r)$ will increase. Hence, the time separation between the two test probes **increases.**

2. The GR approach

We consider the radial fall towards a non-rotating, non-charged gravitating body of mass M. The space surrounding such a body is described by the Schwarzschild metric [3-6]. In GR, the coordinate speed for a particle starting at radial coordinate r_0 outside the event horizon is given by a slightly more complicated formula (2.1) (see references [1,2]) rather than the one shown in (1.2):

$$\frac{\mathrm{d}r}{\mathrm{d}t} = (1 - \frac{r_s}{r})\sqrt{\frac{\frac{r_s}{r} - \frac{r_s}{r_0}}{1 - \frac{r_s}{r_0}}}$$

$$r_s = \frac{2GM}{c^2} \tag{2.1}$$

The coordinate speed is a function of the initial radial coordinate r_0, the current radial coordinate r and the Schwarzschild radius r_s. Exactly as in the Newtonian case we will use the variation of coordinate time with the radial coordinate:

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{1 - \frac{r_s}{r}} \frac{\sqrt{1 - \frac{r_s}{r_0}}}{\sqrt{\frac{r_s}{r} - \frac{r_s}{r_0}}} \tag{2.2}$$

The variation of the temporal separation with the radial coordinate is derived from (2.2):

$$\frac{\mathrm{d}\Delta}{\mathrm{d}r}=\frac{1}{1-\frac{r_s}{r}}\left(\frac{\sqrt{1-\frac{r_s}{H}}}{\sqrt{\frac{r_s}{r}-\frac{r_s}{H}}}-\frac{\sqrt{1-\frac{r_s}{h}}}{\sqrt{\frac{r_s}{r}-\frac{r_s}{h}}}\right) \tag{2.3}$$

Simple algebra shows that $\frac{\mathrm{d}\Delta}{\mathrm{d}r}<0$ because $r>r_s$. Therefore when r decreases (the two test probes falling), $\Delta(H,h,r)$ increases. Hence, the time separation between the two test probes increases. This is totally expected since GR in the weak field approximation and Newtonian mechanics need to agree on the outcome. The agreement is no longer valid in the case of strong gravity since additional terms will occur in the case of strong gravitational fields.

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Application of the Euler Lagrange Formalism for Determining the Equation of Motion in the Case of Radial Fall into a Non-Rotating, Charged Black Hole

Synopsis

In this chapter we set to accomplish two things: determine the equation of motion for an uncharged test probe falling radially into a charged, non-rotating black hole and determine the relationship between coordinate acceleration and coordinate speed. The chapter is concerned only what happens outside the event horizon, since we are using only the external Reissner-Nordstrom equations in the derivations. What happens inside the event horizon (the presence of a wormhole connecting the black hole to a white hole) is not the purvey of this paper.

1. Introduction

We will present a method based on the Lagrangian for the derivation of the equation of motion of an uncharged test probe falling radially into a charged, non-rotating black hole. In a prior paper [1] we have derived the solution for the case of non-rotating, non-charged black holes. In the following, we are extending the derivation to the case of charged black holes. While we could have started from the geodesic equation, the derivation based on the Euler-Lagrange equations is more intuitive and less prone to error. The Reissner-Nordstrom metric for the particular case of absence of rotation ($d\theta = d\varphi = 0$) is [2,6-8]:

$$ds^2 = \alpha dt^2 - \frac{1}{\alpha} dr^2$$

$$\alpha = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \qquad (1.1)$$

where $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius , $r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4}$ where G Is the universal attraction constant, Q is the black hole charge , r is the radial coordinate, c is the speed of light in vacuum and ε_0 is the vacuum electric permittivity. From the metric we obtain, as shown in [1,2]:

a) the Lagrangian

$$L = \alpha \frac{dt}{ds^2}{}^{2} - \frac{1}{\alpha} \frac{dr^2}{ds^2} \qquad (1.2)$$

The Lagrangian (2) is obtained following an idea by Rindler [3], whereby one replaces the parameter t with the arc length s along the solution curve, provided that that curve isn't null. This allows replacement of the Lagrangian $\mathbb{L} = \sqrt{g_{ij} \dot{x}_i \dot{x}_j}$ with its square $L = g_{ij} \dot{x}_i \dot{x}_j$ where the overdot represents now derivative with respect to the arc length s. Rindler proves that the Euler-Lagrange equations for $\mathbb{L} = \sqrt{g_{ij} \dot{x}_i \dot{x}_j}$ are equivalent to those for $L = g_{ij} \dot{x}_i \dot{x}_j$ [3].

b) from the Lagrangian we obtain the Euler-Lagrange system of equations [1,2]:

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{1.3}$$

and, respectively:

$$\frac{d}{ds}\left(\alpha \frac{dt}{ds}\right) = 0$$

$$\alpha \frac{dt}{ds} = k$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{ds}\left(\frac{-2\dot{r}}{\alpha}\right) - \dot{t}^2 \frac{d\alpha}{dr} + \dot{r}^2 \frac{d}{dr}\left(\frac{1}{\alpha}\right) =$$

$$= 2\left(-\frac{\ddot{r}}{\alpha} + \dot{r}^2 \frac{\frac{d\alpha}{dr}}{\alpha^2}\right) - \dot{r}^2 \frac{\frac{d\alpha}{dr}}{\alpha^2} - \dot{t}^2 \frac{d\alpha}{dr} = -2\frac{\ddot{r}}{\alpha} + \dot{r}^2 \frac{\frac{d\alpha}{dr}}{\alpha^2} - \dot{t}^2 \frac{d\alpha}{dr} \tag{1.4}$$

The over-dots signify derivative with respect to s . From the metric (1.1) we obtain:

$$\alpha\left(\frac{dt}{ds}\right)^2 = 1 + \frac{1}{\alpha}\left(\frac{dr}{ds}\right)^2 \tag{1.5}$$

Substituting (1.5) into (1.4) we obtain

c) the equation of motion:

$$\frac{d^2 r}{ds^2} + \frac{1}{2}\frac{d\alpha}{dr} = 0 \tag{1.6}$$

From (1.1) we obtain:

$$\left(\frac{dr}{ds}\right)^2 = \alpha^2\left(\frac{dt}{ds}\right)^2 - \alpha$$

$$\frac{dr}{ds} = \sqrt{k^2 - \alpha} \tag{1.7}$$

In (1.7) k can be determined by setting the condition that the coordinate (or proper) speed is zero when the particle is dropped from radial distance r_0 towards the mass M:

$$0 = \frac{dr}{ds} = \sqrt{k^2 - \alpha(r_0)}$$

$$k = \sqrt{\alpha(r_0)} \tag{1.8}$$

Therefore proper speed is:

$$\frac{dr}{ds} = \sqrt{\alpha(r_0) - \alpha(r)} = \sqrt{\frac{r_s}{r} - \frac{r_Q^2}{r^2} - \frac{r_s}{r_0} + \frac{r_Q^2}{r_0^2}} \tag{1.9}$$

Finally, the equation of motion is:

$$\frac{d^2r}{ds^2} = \frac{-\frac{d\alpha}{ds}}{2\sqrt{k^2 - \alpha}} = -\frac{1}{2\sqrt{k^2 - \alpha}}\frac{d\alpha}{dr}\frac{dr}{ds} = -\frac{1}{2}\left(\frac{r_s}{r^2} - \frac{2r_Q^2}{r^3}\right) \tag{1.10}$$

Equation (1.10) is the equation of motion expressed in terms of the proper acceleration $\frac{d^2r}{ds^2}$. A quick comparison with the results [1,2,5] for non-charged, non-rotating black holes shows that the right hand side of the equation of motion changes from $-\frac{1}{2}\frac{r_s}{r^2}$ to $-\frac{1}{2}\left(\frac{r_s}{r^2} - \frac{2r_Q^2}{r^3}\right)$. This change is perfectly intuitive if we consider that in both cases, the equation of motion [1,2,5], expressed in terms of α is given by (1.6). It is interesting to note that the charge of the black hole contributes an acceleration that is inversely proportional to the cube of the radial distance and it is of opposite sense to the standard gravitational acceleration. This contribution exists even though the test probe is uncharged.

2. Discussion

Although charged black holes with $r_Q \ll r_S$ are similar to the Schwarzschild black hole, it is known that they have two horizons: the event horizon and an internal Cauchy horizon. As with the Schwarzschild metric, the event horizons for the spacetime are located where the metric component α diverges; that is, where:

$$0 = \alpha = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \tag{2.1}$$

The above equation has two solutions, each corresponding to one horizon:

$$r = 0.5(r_s \pm \sqrt{r_s^2 - 4r_Q^2}) \tag{2.2}$$

Black holes with $2r_Q > r_S$ are believed not to exist in nature because they would contain a naked singularity, in our paper we will consider only the physically realistic case $2r_Q \le r_s < r$ which corresponds to:

$$0 < \alpha < 1 \tag{2.3}$$

In other words, our paper deals only with the realistic case of radial fall outside the external event horizon of a physically realizable charged hole, one that would not contain a naked singularity.

3. The dependency between coordinate acceleration and coordinate speed in Reissner-Nordstrom coordinates

In this section we determine the relationship between coordinate acceleration and coordinate speed. Using (1.1) and (1.3) the coordinate speed is:

$$\left(\frac{dr}{dt}\right)^2 = \alpha^2 - \alpha\left(\frac{ds}{dt}\right)^2$$

$$\frac{dr}{dt} = \sqrt{\alpha^2 - \alpha\left(\frac{ds}{dt}\right)^2} = \sqrt{\alpha^2 - \frac{\alpha^3}{k^2}} = \alpha(r)\sqrt{1 - \frac{\alpha(r)}{\alpha(r_0)}} \quad (3.1)$$

From (2.1) we get the coordinate acceleration:

$$\frac{d^2r}{dt^2} = \frac{d}{ds}\left(\frac{dr}{dt}\right)\frac{ds}{dt} = \frac{2\alpha - \frac{3\alpha^2}{k^2}}{2\sqrt{\alpha^2 - \frac{\alpha^3}{k^2}}}\frac{d\alpha}{ds}\frac{\alpha}{k} =$$

$$= \frac{\alpha}{k}\frac{2\alpha - \frac{3\alpha^2}{k^2}}{2\frac{\alpha}{k}\sqrt{k^2 - \alpha}}\left(\frac{r_s}{r^2} - \frac{2r_Q^2}{r^3}\right)\frac{dr}{ds} =$$

$$= \frac{2\alpha - \frac{3\alpha^2}{k^2}}{2}\left(\frac{r_s}{r^2} - \frac{2r_Q^2}{r^3}\right) \quad (3.2)$$

Eliminating k between (3.1) and (3.2) we obtain:

$$\frac{d^2r}{dt^2} = \frac{1}{2}\left(\frac{3}{\alpha}\left(\frac{dr}{dt}\right)^2 - \alpha\right)\left(\frac{r_s}{r^2} - \frac{2r_Q^2}{r^3}\right) \quad (3.3)$$

So, the relationship between coordinate acceleration and coordinate speed in the case of radial motion into a charged, non-rotating black hole is:

$$\frac{d^2r}{dt^2}-\frac{3\left(\frac{r_s}{r^2}-\frac{2r_Q^2}{r^3}\right)}{2\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)}\left(\frac{dr}{dt}\right)^2+\frac{1}{2}\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)\left(\frac{r_s}{r^2}-\frac{2r_Q^2}{r^3}\right)=0 \tag{3.4}$$

A quick sanity check shows that for zero charge ($r_Q = 0$) we recover the equation for the Schwarzschild case developed in [1]. An alternative expression for the coordinate acceleration as a function of the radial coordinate can be obtained from (3.1) and (3.3):

$$\frac{d^2r}{dt^2}=\alpha(r)\left(1-\frac{3\alpha(r)}{2\alpha(r_0)}\right)\left(\frac{r_s}{r^2}-\frac{2r_Q^2}{r^3}\right) \tag{3.5}$$

As per (2.3):

$$2r_Q \le r_s < r$$
$$0 < \alpha < 1 \tag{3.6}$$

Another useful formula can be derived from (3.1):

$$\frac{dt}{dr}=\frac{1}{\alpha(r)\sqrt{1-\frac{\alpha(r)}{\alpha(r_0)}}} \tag{3.7}$$

Formula (3.7) proves essential in determining the radial separation for two test probes falling into a black hole [5]. Elementary algebra shows that under conditions (3.6) the expression:

$$1-\frac{\alpha(r)}{\alpha(r_0)}>0 \tag{3.8}$$

is always positive, so the equation (3.7) has physically realizable solutions.

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THE TWO TEST PROBE CHASE TO THE EVENT HORIZON OF A CHARGED, NON-ROTATING BLACK HOLE

Synopsis

In the current chapter we examine the "chase" between two test probes dropped radially and simultaneously from two different heights towards a charged, non-rotating black hole. We have developed a consistent methodology that uses the coordinate speed in order to derive the variation of coordinate time with the Schwarzschild radial coordinate. This allows us to determine how the time differential between the two falling test probes varies with respect to the Schwarzschild radial coordinate.

1. The GR approach to the problem

Assume that two test probes are dropped simultaneously, from the radial distances H and, respectively, h with $H > h$. We consider that the fall is radial and that it is directed towards a non-rotating, charged gravitating body of mass M. The space surrounding such a body is described by the Reissner-Nordstrom metric [3-6]. In a prior paper [7] we have studied the case of radial fall towards a non-charged, non-rotating black hole. We have found that the distance between the two test probes increases with time. Will this be the case in the case of a charged black hole? In GR, the coordinate speed for a particle starting at radial coordinate r_0 outside the event horizon is given by the formula [1,2]:

$$\frac{dr}{dt} = \alpha(r)\sqrt{1-\frac{\alpha(r)}{\alpha(r_0)}} \tag{1.1}$$

where:

$$\alpha = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \tag{1.2}$$

and $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius , $r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4}$ where G Is the universal attraction constant, Q is the black hole charge , r is the radial coordinate, c is the speed of light in vacuum and ε_0 is the vacuum electric permittivity. We will use the variation of coordinate time with the radial coordinate:

$$\frac{dt}{dr} = \frac{1}{1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}} \frac{\sqrt{1 - \frac{r_s}{r_0} + \frac{r_Q^2}{r_0^2}}}{\sqrt{\frac{r_s}{r} - \frac{r_s}{r_0} - (\frac{r_Q^2}{r^2} - \frac{r_Q^2}{r_0^2})}} \tag{1.3}$$

The variation of the temporal separation $\Delta(H,h,r) = T(H,r) - t(h,r)$ with the radial coordinate is derived trivially from (1.3):

$$\frac{d\Delta}{dr} = \frac{1}{1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}} \left(\frac{\sqrt{1 - \frac{r_s}{H} + \frac{r_Q^2}{H^2}}}{\sqrt{\frac{r_s}{r} - \frac{r_s}{H} - (\frac{r_Q^2}{r^2} - \frac{r_Q^2}{H^2})}} - \frac{\sqrt{1 - \frac{r_s}{h} + \frac{r_Q^2}{h^2}}}{\sqrt{\frac{r_s}{r} - \frac{r_s}{h} - (\frac{r_Q^2}{r^2} - \frac{r_Q^2}{h^2})}} \right) \tag{1.4}$$

Simple algebra shows that $\frac{d\Delta}{dr} < 0$ if $r_s > r_Q^2(\frac{1}{h} + \frac{1}{H})$. If, on the other hand $r_s < r_Q^2(\frac{1}{h} + \frac{1}{H})$ then $\frac{d\Delta}{dr} > 0$. This additional condition is interesting, since we have found that the temporal distance increases **unconditionally** in the case of Schwarzschild black holes [7].

Therefore when r decreases (the two test probes falling), and $r_s > r_Q^2(\frac{1}{h}+\frac{1}{H})$, $\Delta(H,h,r)$ increases. Hence, the time separation between the two test probes increases. This case is similar with the one of uncharged (Schwarzschild) black holes.

When $r_s < r_Q^2(\frac{1}{h}+\frac{1}{H})$, $\Delta(H,h,r)$ decreases. Hence, the time separation between the two test probes decreases.

When $r_s = r_Q^2(\frac{1}{h}+\frac{1}{H})$, $\Delta(H,h,r)=0$. Hence, the time separation between the two test probes remains constant and equal to $H-h$. The last two cases are dissimilar to the case of uncharged (Schwarzschild) black holes.

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EULER-LAGRANGE SOLUTION FOR CALCULATING PARTICLE ORBITS IN GRAVITATIONAL FIELDS

Synopsis

The determination of particle equations of motion in gravitational fields in general relativity is done routinely via the use of covariant derivatives. Since the geodesic equations based on covariant derivatives are derived from the Euler-Lagrange equations in first place and since the Euler-Lagrange formalism is very intuitive, easy to derive with no mistakes, there is every reason to use them even for the most complicated situations. In the current paper we will show the application of the lagrangian equations for various scenarios in general relativity. A special paragraph is dedicated to radial motion. Radial motion is given less attention in textbooks than orbital motion, perhaps because solving the equations of motion is more difficult than the case of orbital motion (definitely more difficult than circular orbits).

1. Introduction: the Lagrangian method applied to radial motion

The pedagogical approach permeating through the paper is straightforward: derive the lagrangian from the metric, derive the Euler-Lagrange equations from the lagrangian and solve them. In the concluding paragraphs, four novel applications of the lagrangian method are presented. Firstly, we show the application for deriving the advancement of the perihelion of not only Mercury but also for Venus and Earth in a novel way by combining perturbation theory with the lagrangian approach. Secondly, we show how to calculate the length of a rod while in radial motion. While the paper is constructed around the case of gravitational fields described by the Schwarzschild metric, we demonstrate how to extend the algorithms to other metrics, like Reissner-Nordstrom or Kerr, for example, therefore we show an application in the concluding paragraph. In the cases of Reissner-Nordstrom or Kerr metrics, the lagrangian method has a definite advantage since the Christoffel symbols are much more difficult to calculate than in the case of the Schwarzschild metric. We chose the case of the Reissner-Nordstrom metric because it is encountered in literature much less than the Schwarzschild solution and because finding the equations of motion for objects falling into or gravitating around a charged black hole is considerably more difficult than in the case of the Schwarzschild solution. We will show how to use the lagrangian approach in solving this problem and we will even solve the difficult problem of calculating the perihelion advancement for objects describing arbitrary orbits. We conclude by deriving the trajectories of light in the vicinity of a charged black hole. While radial motion is the easiest type of motion to describe in natural language, it turns out that its equations are far from trivial. In order to find the equations of motion we start with the Schwarzschild metric for the particular case of absence of rotation ($d\theta = d\varphi = 0$):

$$ds^2 = \alpha dt^2 - \frac{1}{\alpha} dr^2$$
$$\alpha = 1 - \frac{2m}{r} = 1 - \frac{r_s}{r} \tag{1}$$

where $m = \frac{GM}{c^2} << 1$ and $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius. For example, the Schwarzschild radius of the Earth is only 9 millimeters. From the metric we obtain:

a) the lagrangian

$$L = \alpha \frac{dt}{ds^2}^{2} - \frac{1}{\alpha}\frac{dr^2}{ds^2} \tag{2}$$

b) from the lagrangian we obtain the Euler-Lagrange system of equations [1,5]:

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right)-\frac{\partial L}{\partial t}=0$$
$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0 \qquad (3)$$

and, respectively:

$$\frac{d}{ds}\left(\alpha\frac{dt}{ds}\right)=0$$
$$\alpha\frac{dt}{ds}=k$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=\frac{d}{ds}\left(\frac{-2\dot{r}}{\alpha}\right)-\dot{t}^2\frac{d\alpha}{dr}+\dot{r}^2\frac{d}{dr}\left(\frac{1}{\alpha}\right)=$$
$$=2\left(-\frac{\ddot{r}}{\alpha}+\dot{r}^2\frac{\frac{d\alpha}{dr}}{\alpha^2}\right)-\dot{r}^2\frac{\frac{d\alpha}{dr}}{\alpha^2}-\dot{t}^2\frac{d\alpha}{dr}=-2\frac{\ddot{r}}{\alpha}+\left(\frac{\dot{r}^2}{\alpha^2}-\dot{t}^2\right)\frac{d\alpha}{dr} \qquad (4)$$

The over-dots signify derivative with respect to s. From the metric (1) we obtain:

$$\alpha\left(\frac{dt}{ds}\right)^2=1+\frac{1}{\alpha}\left(\frac{dr}{ds}\right)^2 \qquad (5)$$

Substituting (5) into (4) we obtain

c) the equation of motion:

$$\frac{d^2r}{ds^2}+\frac{1}{2}\frac{d\alpha}{dr}=0 \qquad (6)$$

that is,

$$\frac{d^2r}{ds^2} = -\frac{m}{r^2} \tag{7}$$

From (7) we can see that the acceleration increases as the radial coordinate decreases. In order to solve (7) we will need to resort to a lemma.

Lemma:

$$\frac{d^2r}{ds^2} = \frac{1}{2}\frac{d}{dr}(\frac{ds}{dr})^{-2} \tag{8}$$

Proof:

$$\frac{d^2r}{ds^2} = \frac{d}{ds}(\frac{dr}{ds}) = \frac{d}{dr}(\frac{dr}{ds})\frac{dr}{ds} = \frac{d}{dr}(\frac{ds}{dr})^{-1}(\frac{ds}{dr})^{-1} =$$
$$= -(\frac{ds}{dr})^{-2}\frac{d^2s}{dr^2}(\frac{ds}{dr})^{-1} = -(\frac{ds}{dr})^{-3}\frac{d^2s}{dr^2} = \frac{1}{2}\frac{d}{dr}(\frac{ds}{dr})^{-2} \tag{9}$$

Applying the lemma, equation (7) becomes:

$$\frac{d}{dr}(\frac{ds}{dr})^{-2} = -\frac{2m}{r^2} \tag{10}$$

With the notation $y = (\frac{ds}{dr})^{-2}$ equation (10) becomes:

$$\frac{dy}{dr} = -\frac{2m}{r^2} \tag{11}$$

with the immediate solution:

$$y = \frac{2m}{r} - \frac{2m}{r_0} \tag{12}$$

where $r_0 = r(0)$. On the other hand, $y = (\frac{dr}{ds})^2$, so (12) reduces to:

$$\frac{dr}{ds} = \sqrt{\frac{2m}{r} - \frac{2m}{r_0}} \qquad (13)$$

From (13) we can see that the proper speed increases as the radial distance decreases. Finally, we are ready to obtain the equation of motion by solving (13) through variable separation:

$$s\sqrt{\frac{2m}{r_0}} = r_0 arctg\sqrt{\frac{r}{r_0 - r}} - \sqrt{r(r_0 - r)} \qquad (14)$$

Unfortunately, expression (14) is a transcendental equation in r, so we cannot obtain r as a symbolic function of the proper time s. Yet, as we will see later in this paper, the information is very valuable in solving other classes of problems.

2. A different approach for radial motion

We can determine the proper and coordinate speed for radial motion with a slightly different approach. From (1)

$$(\frac{dr}{ds})^2 = \alpha^2 (\frac{dt}{ds})^2 - \alpha \qquad (15)$$

$$\frac{dr}{ds} = \sqrt{k^2 - \alpha} \qquad (16)$$

$$\frac{d^2 r}{ds^2} = \frac{-\frac{d\alpha}{ds}}{2\sqrt{k^2 - \alpha}} = \frac{-1}{2\sqrt{k^2 - \alpha}} \frac{2m}{r^2} \frac{dr}{ds} = -\frac{m}{r^2} \qquad (17)$$

Using (1) and (3) the coordinate speed is:

$$(\frac{dr}{dt})^2 = \alpha^2 - \alpha(\frac{ds}{dt})^2$$

$$\frac{dr}{dt} = \sqrt{\alpha^2 - \alpha(\frac{ds}{dt})^2} = \sqrt{\alpha^2 - \frac{\alpha^3}{k^2}} \tag{18}$$

From (18) we get the coordinate acceleration:

$$\frac{d^2 r}{dt^2} = \frac{d}{ds}(\frac{dr}{dt})\frac{ds}{dt} = \frac{2\alpha - \frac{3\alpha^2}{k^2}}{2\sqrt{\alpha^2 - \frac{\alpha^3}{k^2}}}\frac{d\alpha}{ds}\frac{\alpha}{k} =$$

$$= \frac{\alpha}{k}\frac{2\alpha - \frac{3\alpha^2}{k^2}}{2\frac{\alpha}{k}\sqrt{k^2 - \alpha}}\frac{2m}{r^2}\frac{dr}{ds} =$$

$$= \frac{2\alpha - \frac{3\alpha^2}{k^2}}{\sqrt{k^2 - \alpha}}\frac{m}{r^2}\sqrt{k^2 - \alpha} = \frac{m}{r^2}\alpha(2 - \frac{3\alpha}{k^2}) \tag{19}$$

k can be determined by setting the condition that the coordinate (or proper) speed is zero when the particle is dropped from radial distance r_0 towards the mass M:

$$0 = \frac{dr}{ds} = \sqrt{k^2 - \alpha(r_0)}$$

$$k = \sqrt{\alpha(r_0)} = \sqrt{1 - \frac{2m}{r_0}} \tag{20}$$

or:

$$0 = \frac{dr}{dt} = \sqrt{\alpha^2(r_0) - \frac{\alpha^3(r_0)}{k^2}}$$

$$k = \sqrt{\alpha(r_0)} \tag{21}$$

Given (21) the coordinate acceleration becomes:

$$a = \frac{d^2r}{dt^2} = \frac{m}{r^2}\alpha(2 - \frac{3\alpha}{k^2}) = \frac{m}{r^2}\alpha(2 - \frac{3\alpha(r)}{\alpha(r_0)}) =$$

$$= -\frac{m}{r^2}(1 - \frac{2m}{r})(3\frac{1-\frac{2m}{r}}{1-\frac{2m}{r_0}} - 2) \tag{22}$$

If the particles is dropped from infinity (22) becomes:

$$a = \frac{d^2r}{dt^2} = -\frac{m}{r^2}(1 - \frac{2m}{r})(1 - \frac{6m}{r}) \tag{22a}$$

The proper speed (16) is:

$$\frac{dr}{ds} = \sqrt{k^2 - \alpha} = \sqrt{\alpha(r_0) - \alpha(r)} = \sqrt{\frac{2m}{r} - \frac{2m}{r_0}} \tag{23}$$

We can see that we have re-derived expression (13) through the new method.

Finally, the coordinate speed (in units of c=1) is:

$$\frac{dr}{dt} = \sqrt{\alpha^2 - \frac{\alpha^3}{k^2}} = \alpha\sqrt{1 - \frac{\alpha(r)}{\alpha(r_0)}} = (1 - \frac{2m}{r})\sqrt{1 - \frac{1-\frac{2m}{r}}{1-\frac{2m}{r_0}}} \tag{24}$$

3. Classical treatment of unidimensional radial motion

The same problem, in Newtonian formulation, for the case of unidimensional radial motion reduces to the equation of motion:

$$m\frac{d^2r}{dt^2}=-\frac{GMm}{r^2} \tag{25}$$

It is interesting to note that GR and Newtonian mechanics produce exactly the same equation of motion. Equation (25) gives us the tool for determining when two bodies of radiuses r_1 and r_2 and masses M and m will collide after starting from rest at locations $x_1(0)$ and respectively $x_2(0)$ separated by the initial distance $D=x_1(0)-x_2(0)$. We would need to solve the system of differential equations:

$$\frac{d^2x_1}{dt^2}=-\frac{GM}{(x_1-x_2)^2}$$
$$\frac{d^2x_2}{dt^2}=+\frac{Gm}{(x_1-x_2)^2} \tag{26}$$

with initial conditions:

$$x_1(0)=D$$
$$x_2(0)=0$$
$$\frac{dx_1}{dt}\Big|_{t=0}=\frac{dx_2}{dt}\Big|_{t=0}=0 \tag{27}$$

and find out the time when $x_1-x_2=r_1+r_2$ (i.e., when the two masses touch) by solving a transcendental equation in t. The system gets easily reduced to a single equation by subtracting the two equations:

$$\frac{d^2(x_1 - x_2)}{dt^2} = -\frac{G(M+m)}{(x_1 - x_2)^2} \tag{28}$$

From (7), we know that equation (28) has the general solution:

$$t\sqrt{\frac{2G(M+m)}{D}} = D * arctg\sqrt{\frac{x_1 - x_2}{D-(x_1 - x_2)}} - \sqrt{(x_1 - x_2)(D-(x_1 - x_2))} \tag{29}$$

At the time of collision, $x_1 - x_2 = r_1 + r_2$ so:

$$t = \frac{D^{3/2}}{\sqrt{2G(M+m)}}(arctg\sqrt{\frac{r_1 + r_2}{D-(r_1 + r_2)}} - \frac{\sqrt{(r_1 + r_2)(D-(r_1 + r_2))}}{D}) \tag{30}$$

Now, we can see that the transcendental equation (15) proved instrumental in finding the "time to collision" for the unidimensional classical problem.

4. Generalization to arbitrary planar orbits

In the case of arbitrary planar orbits characterized by constant θ (that is, $d\theta = 0$) we start with the Schwarzschild metric:

$$ds^2 = \alpha dt^2 - \frac{1}{\alpha}dr^2 - (rd\varphi)^2 \tag{31}$$

The lagrangian associated with the metric (31) is:

$$L = \alpha \frac{dt}{ds^2}^2 - \frac{1}{\alpha}\frac{dr^2}{ds^2} - r^2 \frac{d\varphi^2}{ds^2} \tag{32}$$

The lagrangian (32) is the generalization for the more particular lagrangian (2). Likewise, the generalization for the Euler-Lagrange equations is:

$$-2\frac{d}{ds}\left(\frac{\dot{r}}{\alpha}\right)-\dot{t}^2\frac{d\alpha}{dr}+\dot{r}^2\frac{d}{dr}\left(\frac{1}{\alpha}\right)+2r\dot{\varphi}^2=$$

$$=-2\frac{\ddot{r}}{\alpha}+\left(\frac{\dot{r}^2}{\alpha^2}-\dot{t}^2\right)\frac{d\alpha}{dr}+2r\dot{\varphi}^2=0 \qquad (33)$$

$$\alpha\frac{dt}{ds}=k \qquad (34)$$

$$\frac{d}{ds}(r^2\dot{\varphi})=0$$

$$r^2\dot{\varphi}=h \qquad (35)$$

From the general equation of motion (33) we can obtain interesting particular cases.

a. For circular orbits, $r=R, \dot{r}=0$ so:

$$-\dot{t}^2\frac{d\alpha}{dr}+2r\dot{\varphi}^2=0 \qquad (36)$$

meaning that:

$$\left(\frac{dt}{ds}\right)^2\frac{2m}{r^2}=2r\left(\frac{d\varphi}{ds}\right)^2 \qquad (37)$$

$$\frac{d\varphi}{dt}=\sqrt{\frac{m}{r^3}} \qquad (38)$$

Inserting (38) back into the metric (31) we obtain:

$$ds^2=\left(1-\frac{3m}{r}\right)dt^2 \qquad (39)$$

with the immediate consequence:

$$\frac{d\varphi}{ds} = \sqrt{\frac{m}{r^3}} \frac{1}{\sqrt{1 - \frac{3m}{r}}} \tag{40}$$

Thus, we have recovered a well-known equation of the mechanics describing circular orbits.

b. For radial orbits, $d\varphi = 0$, so, the Euler-Lagrange (33) reduces to:

$$-2\frac{d}{ds}(\frac{\dot{r}}{\alpha}) - \dot{t}^2 \frac{d\alpha}{dr} + \dot{r}^2 \frac{d}{dr}(\frac{1}{\alpha}) = -2\frac{\ddot{r}}{\alpha} + (\frac{\dot{r}^2}{\alpha^2} - \dot{t}^2)\frac{d\alpha}{dr} = 0 \tag{41}$$

If we add to the above the fact that the metric (31) reduces to:

$$ds^2 = \alpha dt^2 - \frac{1}{\alpha} dr^2 \tag{42}$$

From (41) and (42) we obtain the equation of motion:

$$-2\ddot{r} - \frac{2m}{r^2} = 0 \tag{43}$$

that is, we recovered equation (7).

c. For arbitrary planar orbits, the Euler-Lagrange equation is given by (33). Coupled with the general metric (31) the equation reduces to:

$$\ddot{r} = -\frac{m}{r^2} + (r - 3m)(\dot{\varphi})^2 \tag{44}$$

The above is very interesting since it allows recovering the previous answers to both the radial and circular orbits situations. Indeed, $\dot{\varphi} = 0$

implies $\ddot{r} = -\frac{m}{r^2}$ and $\dot{r} = 0$ implies $\frac{d\varphi}{ds} = \sqrt{\frac{m}{r^3}}\frac{1}{\sqrt{1-\frac{3m}{r}}}$. Thus, we have recovered equation (40)

5. The derivation of the advancement of Mercury perihelion via perturbation theory

In this paragraph we will combine the lagrangian approach with perturbation theory in producing a novel solution to the advancement of the perihelion of not only Mercury, but also for Venus, Earth and Mars. Using the Euler-Lagrange equation (35) equation (44) can be simplified to:

$$\ddot{r} + \frac{m}{r^2} = \frac{h^2}{r^3} - 3m\frac{h^2}{r^4} \tag{45}$$

Using the substitution:

$$u(\varphi) = \frac{1}{r(\varphi)} \tag{46}$$

with the immediate consequence:

$$\dot{r} = -\frac{\dot{u}}{u^2} = -r^2\frac{du}{d\varphi}\frac{d\varphi}{ds} = -h\frac{du}{d\varphi}$$
$$\ddot{r} = -h\frac{d^2u}{d\varphi^2}\frac{d\varphi}{ds} = -h^2u^2\frac{d^2u}{d\varphi^2} \tag{47}$$

Equation (45) transforms into:

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 3mu^2 \tag{48}$$

Equation (48) is nothing but the Kepler's first law from Newtonian mechanics:

$$\frac{d^2u}{d\varphi^2}+u=\frac{m}{h^2} \tag{49}$$

with the added relativistic perturbation of $+3mu^2$. Now, we know the solution for (49) is:

$$u(\varphi)=\frac{m}{h^2}(1-e\cos\varphi) \tag{50}$$

or, expressed in terms of $r=r(\varphi)$:

$$r(\varphi)=\frac{r_c}{1-e\cos\varphi} \tag{51}$$

where for $e<1$ (51) represents the parametric equation of an ellipse in which case $r_c=\frac{h^2}{m}=r(\frac{\pi}{2})$ is the radial distance from the focus to the ellipse. Armed with the solution for classical mechanics equation (49) we can now attempt to solve the GR equation (48) by applying perturbation theory. An appropriate solution is:

$$u(\varphi)=\frac{m}{h^2}(1-e\cos(\Omega\varphi)) \tag{52}$$

In order for (52) to be a solution for (48) it must satisfy the condition:

$$\frac{3m^2}{h^2}(1+e^2\cos^2(\Omega\varphi))-\frac{6m^2}{h^2}e\cos(\Omega\varphi)=$$
$$=-e(1-\Omega^2)\cos(\Omega\varphi) \tag{53}$$

Since:

$$\frac{m^2}{h^2} = \frac{m}{h^2 / m} = \frac{r_s / 2}{r_c} << 1 \tag{54}$$

it follows that (53) is satisfied if:

$$\frac{6m^2}{h^2} = 1 - \Omega^2 \tag{55}$$

that is,

$$\Omega = \sqrt{1 - \frac{6m^2}{h^2}} \approx 1 - \frac{3m^2}{h^2} \tag{56}$$

Rindler [1] produces a similar explanation but his derivation relies on a less rigorous series of multiple approximations. Thus, the solution for the GR equation (48) is:

$$u(\varphi) = \frac{m}{h^2}(1 - e\cos((1 - \frac{3m^2}{h^2})\varphi)) \tag{57}$$

$$r(\varphi) = \frac{r_c}{1 - e\cos((1 - \frac{3m^2}{h^2})\varphi)} \tag{58}$$

Solution (58) agrees with the Newtonian solution for $m = 0$, thus giving us a high level of confidence that it is correct. When $0 < \varphi < 2\pi$, $0 < (1 - \frac{3m^2}{h^2})\varphi < 2\pi - \frac{6\pi m^2}{h^2}$, that is the orbit “misses” its closure [2] by $\frac{6\pi m^2}{h^2}$ per revolution, resulting into a precession phenomenon seen in figure 1:

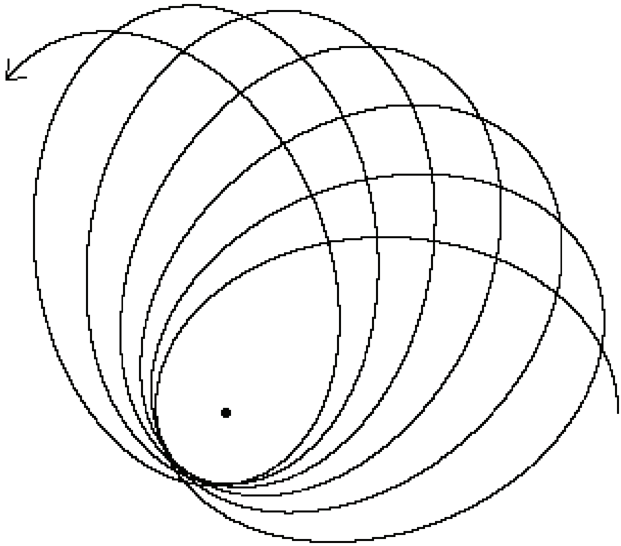

Figure 1. Orbit Precession

The precession per revolution is a direct function of $\frac{m^2}{h^2} = \frac{r_s}{2r_c}$ while the overall observed precession per century is a function of the number of revolutions per century. From table 1 we can see that physics "conspires" in such a fashion that, for our solar system, Mercury is by far the best candidate for observing the precession given that it has not only the highest precession per revolution but also the largest number of revolutions per century.

Planet	r_c (10^6km)	Precession per revolution	Revolutions per century	Precession per century (arcsec)
Mercury	55.443	0.1034	414.9378	42.9195
Venus	108.1947	0.0530	162.6016	8.6186
Earth	149.5568	0.03835	100	3.8335
Mars	225.9289	0.0254	53.1915	1.3502

Table 1

Over the centuries, the astronomers have observed that the precession of Mercury perihelion is actually a much larger number (5600 arcsecond/century). Of the total 5600 arcsecond/century, 5557 can be accounted for by Newtonian mechanics, leaving the balance of 43 arcsecond/century to be explained by the disparity between the Newtonian equation (49) and its relativistic counterpart (48). It was Einstein genius that explained[1] the disparity between the Newtonian calculations and the observed values. Given the advancements in modern measuring devices, today we can not only account for the advancement of Mercury perihelion but also for the advancements for Venus and Earth [3-4].

6. Application to calculating the length of a rod in radial fall

Let's assume that we are asked to find the length of a rod of proper length L as calculated from the perspective of a distant Schwarzschild observer. Now, our observer has read reference [6] and understands that there are several ways of operationally determining the length of an object in motion.

So, the observer decides to drop the rod from r_0 and he decides to set a "trap" at location $r_1 < r_0$. By calculating the time interval Δt between the leading end of the rod and the trailing end of the rod passing through the "trap" set at r_1 and by knowing the coordinate speed at the same point, our observer can determine the length of the moving rod. We will assume throughout this chapter that the rod is Born-rigid, so it is not distorted by tidal forces, that is, all its points travel at the same speed. The coordinate speed is variable along the trajectory and, at location r_1, according to what we derived in equation (13) it is:

$$v|_{r=r_1} = (1-\frac{2m}{r_1})\sqrt{1-\frac{1-\frac{2m}{r_1}}{1-\frac{2m}{r_0}}} \tag{59}$$

The time for the leading end of the rod to reach location r_1 is:

$$t_{lead} = \int_{r_1}^{r_0} dt \tag{60}$$

while the time for the trailing end of the rod is:

$$t_{trail} = \int_{r_1}^{r_0+L} dt \tag{61}$$

Thus, the elapsed time for the rod to pass through the "speed trap" at location r_1 is:

$$\Delta t = \int_{r_1}^{r_0+L} dt - \int_{r_1}^{r_0} dt = \int_{r_0}^{r_0+L} dt \tag{62}$$

From (14) we also know that:

$$dt = \frac{dr}{(1-\frac{2m}{r})\sqrt{1-\frac{1-\frac{2m}{r}}{1-\frac{2m}{r_0}}}} \tag{63}$$

We are now ready to calculate the length of the rod, as it passes through r_1:

$$\Delta l = v\Delta t = (1-\frac{2m}{r_1})\sqrt{\frac{1}{r_1}-\frac{1}{r_0}} \int_{r_0}^{r_0+L} \frac{dr}{(1-\frac{2m}{r})\sqrt{\frac{1}{r}-\frac{1}{r_0}}} \tag{64}$$

The above represents one operational way of determining the length of the falling rod. Now, the astute observer, who has read Geroch's book [6], may decide to apply a different operational definition in determining the rod's length, such as marking both ends of the rod at the same coordinate time. So, the observer decides to find out where the trailing end of the rod is when the leading end has reached r_1. Assume that this is at the radial location r, where:

$$t_{trail} = \int_{r}^{r_0+L} dt \tag{65}$$

In this case, the coordinate length of the rod is $r - r_1$ where r is the solution of the integral equation:

$$t_{trail} = t_{lead}$$
$$\int_{r}^{r_0+L} dt = \int_{r_1}^{r_0} dt \tag{66}$$

The above equation can be further simplified in two steps. Firstly, we reduce it to:

$$\int_{r_1}^{r} dt = \int_{r_0}^{r_0+L} dt \tag{67}$$

Now, the RHS is a constant, independent of r and the LHS is a polynomial in r. In the second step, we notice that, for $r, r_1 >> 2m$:

$$\frac{1}{(1-\frac{2m}{r})\sqrt{\frac{2m}{r}-\frac{2m}{r_0}}} \approx (1+\frac{2m}{r})\sqrt{\frac{r}{2m}} = \sqrt{\frac{r}{2m}} + \sqrt{\frac{2m}{r}} \tag{68}$$

Thus, equation (67) reduces to a simple algebraic equation in r:

$$\sqrt{2mr} + \frac{r}{3}\sqrt{\frac{r}{2m}} = \sqrt{2mr_1} + \frac{r_1}{3}\sqrt{\frac{r_1}{2m}} - \sqrt{2mr_0} - \frac{r_0}{3}\sqrt{\frac{r_0}{2m}} + \sqrt{2m(r_0+L)} + \frac{r_0+L}{3}\sqrt{\frac{r_0+L}{2m}} \tag{69}$$

We presented just two different modes of determining the length of a moving rod, the reader can decide on his/her own operational way of determining the length since several more ways can be found in literature[6].

7. Charged black holes

In (1916) Reissner [7] and in 1918 Nordström [8] derived independently the metric that represents the static solution to the Einstein field equations in empty space, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body of mass M and charge Q. Finding the equations of motion for objects falling into or gravitating around a charged black hole is considerably more difficult than in the case of the Schwarzschild solution. We will show how to use the lagrangian approach in solving this problem and we will even solve the difficult problem of calculating the perihelion advancement for objects describing arbitrary orbits. The Reissner-Nordstrom metric is given by:

$$ds^2 = \alpha' dt^2 - \frac{1}{\alpha'} dr^2 - r^2 d\varphi^2 \tag{70}$$

where:

$$\alpha' = 1 - \frac{2m}{r} + \frac{r_Q^2}{r^2} \tag{71}$$

$$r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4} \tag{72}$$

The Euler-Lagrange equations are:

$$-2\frac{d}{ds}\left(\frac{\dot{r}}{\alpha'}\right) - \dot{t}^2 \frac{d\alpha'}{dr} + \dot{r}^2 \frac{d}{dr}\left(\frac{1}{\alpha'}\right) + 2r\dot{\varphi}^2 = 0$$

$$\alpha \frac{dt}{ds} = k \tag{73}$$

$$r^2 \dot{\varphi} = h$$

a. For circular orbits we obtain:

$$\frac{d\varphi}{dt} = \sqrt{\frac{m}{r^3} - \frac{r_Q^2}{r^4}} \tag{74}$$

Inserting (74) back into the metric (70) we obtain:

$$ds^2 = (1 - \frac{3m}{r} + \frac{2r_Q^2}{r^2})dt^2 \tag{75}$$

with the immediate consequence:

$$\frac{d\varphi}{ds} = \sqrt{\frac{m}{r^3} - \frac{r_Q^2}{r^4}} \frac{1}{\sqrt{1 - \frac{3m}{r} + \frac{r_Q^2}{r^2}}} \tag{76}$$

Thus we obtained a very elegant result showing that circular orbits for charged black holes can be obtained by applying a charge-dependent correction to the solution for neutral black holes.

b. For radial orbits, the Euler-Lagrange equation reduces to:

$$-2\frac{\ddot{r}}{\alpha'} + (\frac{\dot{r}^2}{\alpha'^2} - \dot{t}^2)\frac{d\alpha'}{dr} = 0 \tag{77}$$

If we add to the above the fact that the metric (70) reduces to:

$$ds^2 = \alpha' dt^2 - \frac{1}{\alpha'}dr^2 \tag{78}$$

From (41) and (42) we obtain the equation of motion:

$$-2\ddot{r} - \frac{da'}{dr} = -2\ddot{r} - \frac{2m}{r^2} + \frac{2r_Q^2}{r^3} = 0 \tag{79}$$

that is, we recovered equation (7) with the charge-related perturbation $\frac{2r_Q^2}{r^3}$.

c. For arbitrary planar orbits the equation reduces to:

$$\ddot{r} = -\frac{m}{r^2} + \frac{r_Q^2}{r^3} + (1 - \frac{3m}{r} + \frac{2r_Q^2}{r^2}) r (\dot{\varphi})^2 \tag{80}$$

The computation for the advancement of the perihelion becomes more complicated since the starting point is now the equation:

$$\ddot{r} + \frac{m}{r^2} - \frac{r_Q^2}{r^3} = \frac{h^2}{r^3} - 3m\frac{h^2}{r^4} + \frac{2h^2 r_Q^2}{r^5} \tag{81}$$

Using again the substitution: $u(\varphi) = \frac{1}{r(\varphi)}$ and neglecting the term in r^5 we obtain:

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 3mu^2 - \frac{r_Q^2}{h^2}u \tag{82}$$

an equation very similar to (48). We can proceed by considering the equation:

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} \tag{83}$$

with the perturbation $3mu^2 - \frac{r_Q^2}{h^2}u$ .As an alternative, we can start from:

$$\frac{d^2u}{d\varphi^2} + (1 + \frac{r_Q^2}{h^2})u = \frac{m}{h^2} \tag{84}$$

with the known solution:

$$u(\varphi) = \frac{1}{\sqrt{1+\frac{r_Q^2}{h^2}}}\frac{m}{h^2} + e\cos(\varphi\sqrt{1+\frac{r_Q^2}{h^2}}) \tag{85}$$

and apply the perturbation $3mu^2$. Either way, the perturbation approach that we developed in (52)-(56) bears fruit since the computation of the perihelion advancement becomes a simple algebraic exercise.

8. Light bending by charged black holes

Light bending can be calculated starting from the fact that the light path is null:

$$0 = \alpha dt^2 - \frac{1}{\alpha}dr^2 - r^2 d\varphi^2 \tag{86}$$

so:

$$dr^2 = \alpha^2 dt^2 - \alpha r^2 d\varphi^2 \tag{87}$$

To the above we add the two Euler-Lagrange equations (34)(35):

$$\alpha \dot{t} = k \tag{88}$$

$$r^2 \dot{\varphi} = h \tag{89}$$

Combining (87) with (88)(89) we obtain immediately:

$$(\frac{dr}{ds})^2 = k^2 - h^2\frac{\alpha}{r^2} \tag{90}$$

Differentiating (90) with respect to r we obtain a simpler expression:

$$2\frac{d^2r}{ds^2}=\frac{h^2}{r^3}(2\alpha - r\frac{d\alpha}{dr}) \tag{91}$$

For the case of uncharged black holes $\alpha = 1-\frac{2m}{r}$ so (91), using the notation $u(\varphi)=\frac{1}{r(\varphi)}$, reduces to[1]:

$$\frac{d^2u}{d\varphi^2}+u=3mu^2 \tag{92}$$

For the case of charged black holes $\alpha = 1-\frac{2m}{r}+\frac{r_Q^2}{r^2}$ and the equation becomes:

$$\frac{d^2u}{d\varphi^2}+u=3mu^2-2r_Q^2u^3 \tag{93}$$

The solution for (91) is the superposition of the solutions for (90) and the solution for:

$$\frac{d^2u}{d\varphi^2}+u=-2r_Q^2u^3 \tag{94}$$

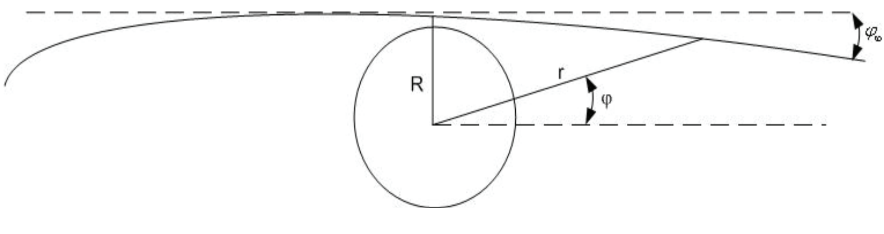

Figure 2. Light bending by charged black holes

The solution for equation (90) is [1]:

$$u = C\sin\varphi + \frac{3mC^2}{2}(1+\frac{\cos 2\varphi}{3}) \tag{95}$$

where $C = \frac{1}{R}$ and R is the effective radius (see fig.2). The solution for (94) is:

$$u = -\frac{3A}{8}\varphi\cos\varphi + \frac{A}{32}\sin 3\varphi \tag{96}$$

where $A = -2r_Q^2C^3$. When $r \to \infty, u \to 0$ and $\varphi \to \varphi_\infty$ so:

$$0 = C\varphi_\infty + 2mC^2 + \frac{9r_Q^2C^3}{16}\varphi_\infty \tag{97}$$

resulting into:

$$\varphi_\infty \approx -\frac{2m}{R}(1-\frac{9r_Q^2 / R^2}{16}) \tag{98}$$

The total deflection angle is

$$\vartheta = 2|\varphi_\infty| \approx \frac{4m}{R}(1-\frac{9r_Q^2 / R^2}{16}) \tag{99}$$

Comparing (97) with the deflection by an uncharged black hole [1], we can conclude that the charge contributes the additional effect $-\frac{4m}{R}\frac{9r_Q^2 / R^2}{16}$.

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“BEYOND GRAVITOELECTROMAGNETISM: CRITICAL SPEED IN GRAVITATIONAL MOTION” BY B. MASHHOON, IJMP D, 14, 12, (2005) PP. 2025-2037, A REVIEW AND A REBUTTAL

Synopsis

In this chapter we will present a rebuttal of a recent paper by Mashhoon published in “, IJMP D, **14**,12, (2005) pp. 2025-2037. We will rebut the idea put forward by the author that there is such a thing as a “critical speed” v_c during the fall of a test particle where the “*the gravitational attraction turns to repulsion”*. The conclusion drawn by Mashhoon is unphysical, there is no such thing as a **gravitational repulsion.** The root of the error can be found in the author basing his derivation on **coordinate** acceleration. The correct analysis should have used **proper** acceleration. We demonstrate that, contrary to the author’s claims, there is no such thing as gravitational “repulsion”.

Analysis and disproof of Mashhoon’s conclusions via the Euler-Lagrange formalism

Exactly as in [2], in order to find the equations of motion of a test particle moving radially in a gravitational field we start with the Schwarzschild metric for the particular case of absence of rotation ($d\theta = d\varphi = 0$). Throughout this note we will use the formalism and the results developed in [2]. We start with the simplified metric [3]:

$$ds^2 = (1-\frac{r_s}{r})dt^2 - \frac{1}{1-\frac{r_s}{r}}dr^2 \tag{1}$$

where r_s is the Schwarzschild radius. From the metric we obtained [2] the proper acceleration:

$$\frac{d^2r}{ds^2} = -\frac{r_s}{2r^2} \tag{2}$$

From (2) we can see that the acceleration increases monotonically as the radial coordinate decreases. The proper speed for a test particle dropped from infinity is derived[2] by integrating (3):

$$\frac{dr}{ds} = \sqrt{\frac{r_s}{r}} \tag{3}$$

From (3) we can see that the proper speed increases monotonically as well as the radial coordinate decreases. This is in line with our knowledge derived from Newtonian mechanics. For the test particle dropped from infinity the coordinate acceleration is[2]:

$$\frac{d^2r}{dt^2} = -\frac{r_s}{2r^2}(1-\frac{r_s}{r})(1-\frac{3r_s}{r}) \tag{4}$$

while the corresponding coordinate speed is:

$$\frac{dr}{dt} = (1-\frac{r_s}{r})\sqrt{\frac{r_s}{r}} \tag{5}$$

At $r = 3r_s$ the coordinate speed reaches a maximum:

$$\frac{dr}{dt}\Big|_{r=3r_S} = \frac{2}{3\sqrt{3}} \tag{6}$$

The corresponding proper speed for $r = 3r_s$ is:

$$\frac{dr}{ds}\Big|_{r=3r_S} = \frac{1}{\sqrt{3}} = v_c \tag{7}$$

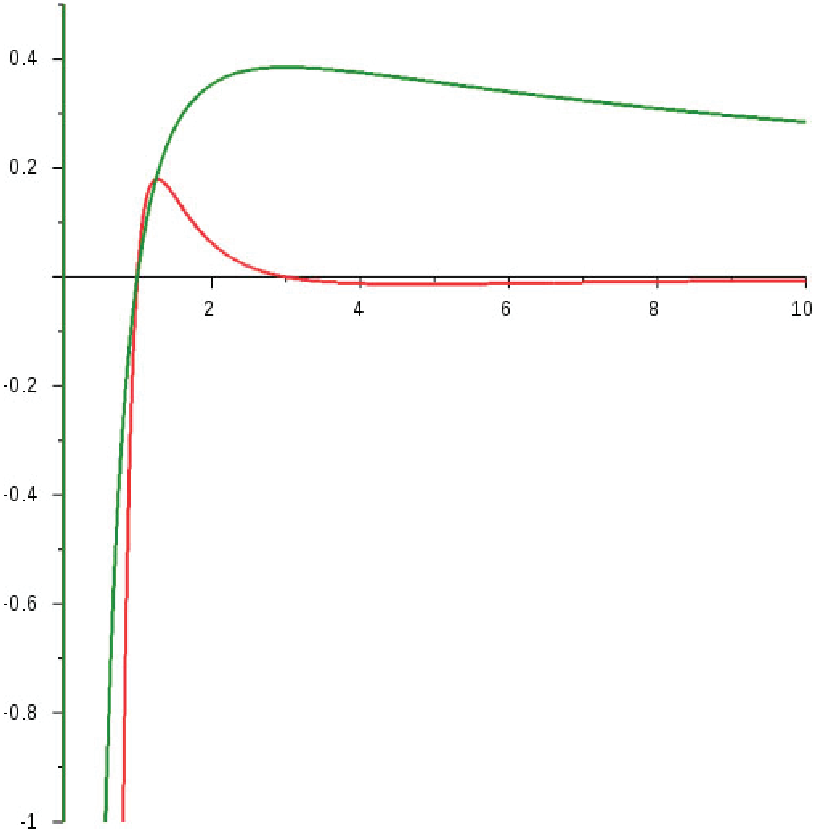

Fig. 1 Coordinate acceleration (in red) and coordinate speed (in green)

From (4) we see that the coordinate acceleration changes sign, from positive to negative, at $r = 3r_s$. The radial coordinate $r = 3r_s$ coincides with the point where the proper speed equals what Mashhoon calls “critical speed” $v_c = 1/\sqrt{3}$. The problem with Mashhoon’s paper is that the author is drawing his conclusions based on the change of sign of **coordinate** acceleration instead of analyzing the behavior of **proper** acceleration. The coordinate acceleration and coordinate speed are not meaningful from a physical point of view, only the proper acceleration and proper speed are. Therefore, contrary to Mashhoon's conclusions[1], there is no such thing as *"….for motion with* $v < v_c$*, we have the standard attractive force of gravity familiar from Newtonian physics, while for* $v = v_c$*, the particle experiences no force and for* $v > v_c$ *the gravitational attraction turns to repulsion"* **since there is no change of sign in the proper acceleration whatsoever.**

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