Post-Newtonian Hydrodynamics

Theory and Applications

Gilberto Medeiros Kremer

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To Maria Rachel

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PREFACE

This book is about the post-Newtonian theory, a method of successive approximations of Einstein's field equations in powers of the light speed. This method was proposed in 1938 by Einstein, Infeld and Hoffmann[1] and in 1965 the first post-Newtonian hydrodynamic equations for a perfect fluid were derived by Chandrasekhar.[2] Nowadays the post-Newtonian theory is still a field of investigation by many researches.

The aim of this book is to present the post-Newtonian theory and some applications in a self-contained manner. The development of the theory follows the works of Chandrasekhar and its collaborators and the book by Weinberg.[3] For another different approach and applications of the post-Newtonian theory

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[1] A. Einstein, L. Infeld and B. Hoffmann, The gravitational equations and the problem of motion, *Ann. of Math.* **39**, 65 (1938).

[2] S. Chandrasekhar, The post-Newtonian equations of hydrodynamics in general relativity, *Ap. J.* **142**, 1488 (1965).

[3] S. Weinberg, *Gravitation and cosmology. Principles and applications of the theory of relativity* (Wiley, New York, 1972).

the reader is referred to the book by Poisson and Will.[4]

The book is organized as follows. In the first Chapter an overview of the non-relativistic and relativistic Boltzmann equation with the corresponding transfer and balance equations are introduced. The particle four-flow and the energy-momentum tensor are calculated with the equilibrium Maxwell-Jüttner distribution function and it is shown that the equilibrium condition of the Boltzmann equation in gravitational fields leads to Tolman and Klein laws.

In Chapter two the first post-Newtonian approximation of Einstein's field equations is determined from Chandrasekhar and Weinberg methods, which introduce different gauge conditions and equivalent gravitational potentials. The post-Newtonian balance equations for an Eulerian and non-perfect fluids are obtained and the Brans-Dicke theory in the post-Newtonian approximation is developed. Other subjects of this chapter include the analysis of the gravitational potentials, the conservation laws and the virial theorem in the post-Newtonian approximation.

The second post-Newtonian approximation is the subject of Chapter three, where new gravitational potentials come out from Einstein's field equations. The Eulerian balance equations are determined and the conservation laws are investigated in this approximation.

In Chapter four the first and second post-Newtonian approximations of the Boltzmann equation and of the Maxwell-Jüttner

[4]E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*, (Cambridge UP, Cambridge, 2014).

distribution function are derived. From a transfer equation of the post-Newtonian Boltzmann equations the Eulerian balance equations for perfect gases are obtained for the two approximations. Furthermore, the post-Newtonian Jeans equations for stationary spherically symmetrical and axisymmetrical self-gravitating systems are derived.

The aim of Chapter five is the search for polytropic solutions of the post-Newtonian Lane-Emden equation for some stars like the Sun, white and brown dwarfs, red giants and neutron stars. The post-Newtonian solutions are compared with the ones that come out from the Newtonian Lane-Emden equation.

In Chapter six the problem of spherically symmetrical accretion is investigated where the Bernoulli equation and the critical values of the flow fields are determined in the post-Newtonian approximation. The solutions of the post-Newtonian Bernoulli equation are compared with the ones that follow from the Bernoulli equations of a relativistic theory and its weak field approximation.

The Jeans instability from the hydrodynamic equations is the subject of Chapter seven. Here the Newtonian Jeans instability is investigated for a non-expanding and expanding Universe. The post-Newtonian Jeans instability are obtained from the mass density and momentum density balance equations in the first and second approximations.

The aim of Chapter eight is to study Jeans instability within the framework of the Boltzmann equation. For the Newtonian and post-Newtonian Boltzmann equations two approaches are used to obtain the dispersion relation which leads to the Jeans instability. In one of them the perturbed distribution function is left unspecified while in the other the perturbed distribution function is written in terms of the summational invariants of the Boltzmann equation. The determination of Jeans instability for an expanding Universe and for a BGK model of the Boltzmann equation – where collision between the particles are taken into account – are also examined.

In the last chapter it is investigated the rotation curves of galaxies within the post-Newtonian framework and the solution of Jeans equation for stationary spherically symmetrical self-gravitating systems.

The notations used in this book are: Greek indices take the values 0,1,2,3 and Latin indices the values 1,2,3. The semicolon denotes the covariant differentiation, the indices of Cartesian tensors will be written as subscripts, the summation convention over repeated indices will be assumed and the partial differentiation will be denoted by $\partial/\partial x^i$.

It is expected that this book can be helpful not only as a text for advanced courses but also as a reference for physicists, astrophysicists and applied mathematicians who are interested in the post-Newtonian theory and its applications.

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Itajaí, Brazil
July 2021

CHAPTER 1

THE BOLTZMANN EQUATION: AN OVERVIEW

In this chapter an outline of the Boltzmann equation is presented. The non-relativistic Boltzmann equation is based on the book [1] while the relativistic one on the book [2]. For more details and references on non-relativistic and relativistic Boltzmann equation the reader should consult these two books and the references therein.

1.1 Non-relativistic Boltzmann equation

The Boltzmann equation is a non-linear integro-differential equation for the space-time evolution of the one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ in the phase space spanned by the space coordinates $\mathbf{x}$ and velocity $\mathbf{v}$ of the particles. The one-particle distribution function is such that $dN = f(\mathbf{x}, \mathbf{v}, t)d^3xd^3v$ gives at time t the number of particles in the volume element d^3x about $\mathbf{x}$ and with velocities in a range d^3v about $\mathbf{v}$. In the non-relativistic kinetic theory of monatomic gases the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial v_i} = \int \big[f(\mathbf{x}, \mathbf{v}'_*, t) f(\mathbf{x}, \mathbf{v}', t) - f(\mathbf{x}, \mathbf{v}_*, t) f(\mathbf{x}, \mathbf{v}, t)\big] g\, \sigma\, d\Omega\, d^3v_*. \tag{1.1}$$

Here $\mathbf{F}$ is a force per unit mass which acts on the particles and do not depend on its velocities. The right-hand side is a consequence of the so-called *Stoßzahlansatz* which considers only binary collisions of two beams of particles which before collision have velocities $(\mathbf{v}, \mathbf{v}_*)$ and after collision $(\mathbf{v}', \mathbf{v}'_*)$. Furthermore, $g = |\mathbf{v}_* - \mathbf{v}|$ is a relative velocity, σ a collision differential cross section and $d\Omega$ an element of solid angle of the scattered particles. In the binary collision the momentum and energy conservation laws hold

$$m\mathbf{v} + m\mathbf{v}_* = m\mathbf{v}' + m\mathbf{v}'_*, \quad \frac{1}{2}mv^2 + \frac{1}{2}mv_*^2 = \frac{1}{2}mv'^2 + \frac{1}{2}mv'^2_*, \tag{1.2}$$

where m is the particle rest mass.

In the kinetic theory of gases the macroscopic fields are given in terms of integrals over the microscopic quantities of the particles multiplied by the one-particle distribution function. The microscopic quantities mass m, momentum $m\mathbf{v}$ and energy $mv^2/2$ of a particle imply the macroscopic fields of mass density ρ, momentum density $\rho\mathbf{V}$ and energy density ρu of the gas defined by

$$\rho(\mathbf{x},t) = \int mf(\mathbf{x},\mathbf{v},t)d^3v, \ \rho\mathbf{V}(\mathbf{x},t) = \int m\mathbf{v}f(\mathbf{x},\mathbf{v},t)d^3v, \quad (1.3)$$

$$\rho u(\mathbf{x},t) = \int \frac{m}{2}v^2 f(\mathbf{x},\mathbf{v},t)d^3v. \quad (1.4)$$

The energy density can be decomposed into a sum of a kinetic energy density $\rho V^2/2$ and an internal energy density $\rho\varepsilon$ by introducing the peculiar velocity $\mathcal{V}_i = v_i - V_i$ which is the difference of the particle velocity $\mathbf{v}$ and the hydrodynamic velocity $\mathbf{V}$. Hence we have

$$\rho u = \frac{1}{2}\rho V^2 + \rho\varepsilon, \quad \text{where} \quad \rho\varepsilon = \int \frac{1}{2}m\mathcal{V}^2 f(\mathbf{x},\mathbf{v},t)d^3v. \quad (1.5)$$

Note that $\int \mathcal{V}_i f d^3v = 0$.

An important quantity in the kinetic theory of gases is the so-called summational invariant ψ defined by the relationship $\psi + \psi_* = \psi' + \psi'_*$. It is easy to see that the mass m, the momentum $m\mathbf{v}$ and the energy $mv^2/2$ of a particle are summational invariants. One important consequence is that the representation of the summational invariant as a sum of mass, momentum and energy of a particle leads to the determination of the one-particle distribution function at equilibrium. Indeed, the equilibrium is characterized when the collision term of the Boltzmann equation (1.1) vanishes, i.e., at equilibrium the number of particles entering in the phase space volume is equal to those that leaving it. In this sense $f(\mathbf{x}, \mathbf{v}'_*, t)f(\mathbf{x}, \mathbf{v}', t) = f(\mathbf{x}, \mathbf{v}_*, t)f(\mathbf{x}, \mathbf{v}, t)$ implying that $\ln f(\mathbf{x}, \mathbf{v}, t)$ is a summation invariant so that at equilibrium the one-particle distribution function becomes the Maxwellian distribution function

$$f = \frac{\rho}{m}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left[-\frac{m\mathcal{V}^2}{2kT}\right], \tag{1.6}$$

where the absolute temperature T is related with the specific internal energy by $\varepsilon = 3kT/2m$ with k denoting the Boltzmann constant.

The derivation of hydrodynamic equations from a transfer equation for arbitrary macroscopic quantities which are associated with mean values of microscopic quantities is an old subject in the literature of kinetic theory of gases which goes back to the work of Maxwell in 1867 [3]. In 1911 Enskog [4] determined from the Boltzmann equation a general transfer equation for an arbitrary function of the space-time and particle velocity where the hydrodynamic equations could be obtained. The starting point for the knowledge of the so-called Maxwell-Enskog transfer equation follows from the multiplication of the Boltzmann equation (1.1) by an arbitrary function of the space-time coordinates and particle velocity $\Psi(\mathbf{x}, \mathbf{v}, t)$ and subsequent integration of the resulting equation over all values of the particle velocity components d^3v. Hence it follows the Maxwell-Enskog transfer equation

$$\int \Psi \left[\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial v_i} \right] d^3 v = \frac{\partial}{\partial t} \int \Psi f d^3 v$$
$$+ \frac{\partial}{\partial x_i} \int \Psi v_i f d^3 v + \underline{\int \frac{\partial \Psi f F_i}{\partial v_i} d^3 v}$$
$$- \int \left[\frac{\partial \Psi}{\partial t} + v_i \frac{\partial \Psi}{\partial x_i} + F_i \frac{\partial \Psi}{\partial v_i} \right] f d^3 v$$
$$= \frac{1}{4} \int \left[\Psi + \Psi_* - \Psi' - \Psi'_* \right] \left[f'_* f' - f_* f \right] g \, \sigma \, d\Omega \, d^3 v_* d^3 v. \quad (1.7)$$

In the above equation the underlined term vanishes since it can be converted by the use of the divergence theorem into an integral over a surface situated far away in the velocity space where the distribution function tends to zero. Its right-hand side follows by considering the symmetry properties of the collision operator of the Boltzmann equation where it was introduced the abbreviations $f'_* \equiv f(\mathbf{x}, \mathbf{v}'_*, t)$, $f \equiv f(\mathbf{x}, \mathbf{v}, t)$ and so on. Note that the right-hand side of the transfer equation vanishes if Ψ is a summational invariant, i.e., for $\Psi \equiv \psi$.

The balance equations for the fields of mass density ρ, momentum density $\rho \mathbf{V}$ and energy density ρu are obtained from the transfer equation (1.7) by choosing Ψ equal to the mass m, momentum $m\mathbf{v}$ and energy $mv^2/2$ of the particles. Hence, it follows respectively

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_i}{\partial x_i} = 0, \quad (1.8)$$

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial(\rho V_i V_j + p_{ij})}{\partial x_j} = -\rho \frac{\partial \phi}{\partial x_i}, \tag{1.9}$$

$$\frac{\partial \left[\rho\left(\varepsilon + \frac{V^2}{2}\right)\right]}{\partial t} + \frac{\partial \left[\rho\left(\varepsilon + \frac{V^2}{2}\right)V_i + q_i + p_{ij}V_j\right]}{\partial x_i} = -\rho \frac{\partial \phi}{\partial x_i} V_i. \tag{1.10}$$

In the above equations we have identified the force per unit mass $\mathbf{F}$ as the gravitational field $\mathbf{g} = -\nabla\phi$ where ϕ is the Newtonian gravitational potential, which is related with the mass density ρ and the universal gravitational constant G through the Poisson equation $\nabla^2\phi = 4\pi G\rho$. Furthermore, it was introduced the pressure tensor p_{ij} and the heat flux vector q_i which are given in terms of the one-particle distribution function by

$$p_{ij} = \int m\mathcal{V}_i\mathcal{V}_j f d^3v, \qquad q_i = \int \frac{1}{2} m\mathcal{V}^2\mathcal{V}_i f d^3v. \tag{1.11}$$

The pressure is the trace of the pressure tensor $p = p_{rr}/3$ and for perfect gases it is related to the specific internal energy by $p = 2\rho\varepsilon/3 = \rho kT/m$.

If we eliminate the time derivative of the hydrodynamic velocity $\mathbf{V}$ from the balance equation for the energy density (1.10) by using the momentum density balance equation (1.9) we get the internal energy density balance equation

$$\frac{\partial \rho\varepsilon}{\partial t} + \frac{\partial(\rho\varepsilon V_i + q_i)}{\partial x_i} + p_{ij}\frac{\partial V_i}{\partial x_i} = 0. \tag{1.12}$$

1.2 Boltzmann equation in special relativity

In special relativity it is considered that a gas particle of rest mass m is characterized by the space-time coordinates $(x^\alpha) = (ct, \mathbf{x})$ and momentum four-vector $(p^\alpha) = (p^0, \mathbf{p})$. From the constraint that the length of the momentum four-vector is equal to mc, its time component p^0 is given in terms of the spatial components $\mathbf{p}$ by $p^0 = \sqrt{|\mathbf{p}|^2 + m^2c^2}$.

The one-particle distribution function $f(x^\alpha, p^\alpha) = f(\mathbf{x}, \mathbf{p}, t)$ is defined in terms of the space-time and momentum coordinates so that the number of particles in the volume element d^3x about $\mathbf{x}$ and with momenta in a range d^3p about $\mathbf{p}$ at time t is given by $dN = f(\mathbf{x}, \mathbf{p}, t)d^3x\, d^3p$.

In order to know if the one-particle distribution function is a scalar invariant we have to know if d^3xd^3p is a scalar invariant, because the number of particles in a volume element is indeed a scalar invariant due to fact that all observers will count the same number of particles.

We consider two inertial systems which transform according a homogeneous Lorentz group in a Minkowski space-time and whose components of the metric tensor are diag$(1, -1, -1, -1)$. The volume elements $d^4x = d^4x'$ and $d^4p = d^4p'$ are scalar invariants. If we choose the primed frame of reference as a rest frame where $\mathbf{p}' = \mathbf{0}$, we have that d^3x' is the proper volume whose transformation law is

$$d^3x = \sqrt{1 - v^2/c^2}d^3x'. \tag{1.13}$$

The transformation law for p^0 and d^3p – by taking into account the primed frame as a rest frame where $\mathbf{p}' = \mathbf{0}$ – are

$$p^0 = \frac{1}{\sqrt{1-v^2/c^2}} p'^0, \qquad \frac{d^3p'}{p'_0} = \frac{d^3p}{p_0}. \tag{1.14}$$

In a Minkowski space-time $p_0 = p^0$ hence from the above equations we have that $d^3x d^3p = d^3x' d^3p'$ is a scalar invariant and as a consequence the one-particle distribution function is also a scalar invariant. Note that d^3p/p_0 is a scalar invariant.

In the phase space spanned by the space coordinates $\mathbf{x}$ and momentum $\mathbf{p}$ of the particles the space-time evolution of the one-particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is given by the Boltzmann equation

$$p^\mu \frac{\partial f}{\partial x^\mu} = \int \Big[f(\mathbf{x}, \mathbf{p}'_*, t) f(\mathbf{x}, \mathbf{p}', t) - f(\mathbf{x}, \mathbf{p}_*, t) f(\mathbf{x}, \mathbf{p}, t) \Big] F \, \sigma \, d\Omega \frac{d^3p_*}{p_{*0}}. \tag{1.15}$$

The right-hand side of the above equation represents the collision term which takes into account the binary collision of two beams of particles which before collision have momenta $(\mathbf{p}, \mathbf{p}_*)$ and after collision $(\mathbf{p}', \mathbf{p}'_*)$. The relative velocity here is given by the invariant flux

$$F = \frac{p^0 p^0_*}{c} \sqrt{(\mathbf{v} - \mathbf{v}_*)^2 - \frac{1}{c^2}(\mathbf{v} \times \mathbf{v}_*)^2} = \sqrt{(p^\alpha_* p_\alpha)^2 - m^4 c^4}. \tag{1.16}$$

Furthermore, σ is the invariant differential cross-section and $d\Omega$ the solid angle element. At collision the energy-momentum conservation law holds

$$p^\mu + p_*^\mu = p'^\mu + p_*'^\mu, \tag{1.17}$$

which is a summational invariant.

The transfer equation for an arbitrary function $\Psi(x^\mu, p^\mu)$ is obtained from the multiplication of the Boltzmann equation (1.15) by $\Psi(x^\mu, p^\mu)$ and integration of the resulting equation with respect to d^3p/p_0, yielding

$$\frac{\partial}{\partial x^\mu}\int \Psi p^\mu f \frac{d^3p}{p_0} - \int \frac{\partial \Psi}{\partial x^\mu} p^\mu f \frac{d^3p}{p_0}$$
$$= \frac{1}{4}\int \left[\Psi + \Psi_* - \Psi' - \Psi_*'\right]\left[f_*' f' - f_* f\right] F\,\sigma\, d\Omega \frac{d^3p_*}{p_{*0}}\frac{d^3p}{p_0}, \tag{1.18}$$

where the right-hand side follows from the symmetry properties of the collision operator of the Boltzmann equation. Here it was introduced the abbreviations $f_*' \equiv f(\mathbf{x}, \mathbf{p}_*', t)$, $f \equiv f(\mathbf{x}, \mathbf{p}, t)$ and so on.

The equilibrium state is attained when the right-hand side of Boltzmann equation (1.15) vanishes so that $\ln f(\mathbf{x}, \mathbf{p}, t)$ is a summational invariant and the one-particle distribution function at equilibrium becomes the Maxwell-Jüttner distribution function

$$f(\mathbf{x}, \mathbf{p}, t) = \frac{n}{4\pi m^2 ckTK_2(\zeta)} \exp\left(-\frac{p^\mu U_\mu}{kT}\right). \tag{1.19}$$

Here n is the particle number density, U_μ the hydrodynamic four-velocity – such that $U_\mu U^\mu = c^2$ – and $K_2(\zeta)$ the modified Bessel function of second kind defined by

$$K_n(\zeta) = \left(\frac{\zeta}{2}\right)^n \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_1^\infty e^{-\zeta y}(y^2-1)^{n-\frac{1}{2}} dy. \quad (1.20)$$

The relativistic parameter $\zeta = mc^2/kT$ is the ratio of the rest energy of the gas particle mc^2 and the thermal energy of the gas kT. In the non-relativistic limiting case $\zeta \gg 1$ while in the ultra-relativistic limiting case $\zeta \ll 1$.

The macroscopic fields of particle four-flow N^μ and energy-momentum tensor $T^{\mu\nu}$ are defined in terms of the one-particle distribution function as

$$N^\mu = \int cp^\mu f(\mathbf{x},\mathbf{p},t)\frac{d^3p}{p_0}, \; T^{\mu\nu} = \int cp^\mu p^\nu f(\mathbf{x},\mathbf{p},t)\frac{d^3p}{p_0}. (1.21)$$

The balance equations for the macroscopic fields are obtained from the transfer equation (1.18) by choosing $\Psi = c$ and $\Psi = cp^\mu$, yielding

$$\frac{\partial}{\partial x^\mu}\int cp^\mu f\frac{d^3p}{p_0} = 0, \quad \Rightarrow \quad \partial_\mu N^\mu = 0, \quad (1.22)$$

$$\frac{\partial}{\partial x^\nu}\int cp^\mu p^\nu f\frac{d^3p}{p_0} = 0, \quad \Rightarrow \quad \partial_\nu T^{\mu\nu} = 0. \quad (1.23)$$

Let us determine the equilibrium values of the particle four-flow N^μ and energy-momentum tensor $T^{\mu\nu}$ from the Maxwell-Jüttner distribution function. We choose a local Lorentz frame where the spatial components of the hydrodynamic four-velocity vanishes, i.e., $U^\mu = (c, \mathbf{0})$ and write the particle four-flow as

$$N^\mu = \int cp^\mu f \frac{d^3p}{p_0} = -\frac{cn}{4\pi m^2 ckTK_2(\zeta)} \frac{\partial}{\partial \mathcal{U}_\mu} \int e^{-(p^\mu \mathcal{U}_\mu)} \frac{d^3p}{p_0}, \tag{1.24}$$

where we have introduced $\mathcal{U}_\mu = U_\mu/kT$ which obeys the relationships

$$\mathcal{U}^\mu \mathcal{U}_\mu = \frac{\zeta^2}{(mc)^2}, \qquad \frac{\partial \zeta}{\partial \mathcal{U}_\mu} = \frac{(mc)^2}{\zeta} \mathcal{U}^\mu = mU^\mu. \tag{1.25}$$

In a local Lorentz frame we can use spherical coordinates to write

$$d^3p = |\mathbf{p}|^2 \sin\theta d|\mathbf{p}| d\theta d\varphi, \tag{1.26}$$

where the range of the angles are $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. Furthermore we change the integration variable and introduce a new variable y such that

$$p_0 = mcy, \quad |\mathbf{p}|^2 = p_0^2 - m^2c^2 = m^2c^2(y^2-1), \tag{1.27}$$

$$\frac{d|\mathbf{p}|}{p_0} = \frac{dy}{\sqrt{y^2-1}}. \tag{1.28}$$

Hence by considering that the integrals over the angles θ and φ furnish 4π, (1.24) becomes

$$\begin{aligned} N^\mu &= -\frac{\zeta n}{mK_2(\zeta)} \frac{\partial}{\partial \mathcal{U}_\mu} \int e^{-\zeta y} \sqrt{y^2-1} dy \\ &= -\frac{\zeta n}{mK_2(\zeta)} \frac{\partial K_1(\zeta)/\zeta}{\partial \mathcal{U}_\mu} = nU^\mu. \end{aligned} \tag{1.29}$$

The evaluation of the energy-momentum tensor proceeds in the same way

$$T^{\mu\nu} = \int cp^\mu p^\nu f \frac{d^3p}{p_0} = \frac{\zeta n}{mK_2(\zeta)} \frac{\partial^2 K_1(\zeta)/\zeta}{\partial \mathcal{U}_\mu \partial \mathcal{U}_\nu}$$
$$= (\epsilon + p)\frac{U^\mu U^\nu}{c^2} - pg^{\mu\nu}, \tag{1.30}$$

Here $g^{\mu\nu}$ is the Minkowski metric tensor. The energy density ϵ and the hydrostatic pressure p are given by

$$\epsilon = \rho c^2 \left(\frac{K_3(\zeta)}{K_2(\zeta)} - \frac{1}{\zeta}\right), \qquad p = nkT. \tag{1.31}$$

In the above equations it was used the recurrence relation for the modified Bessel function of second kind

$$\frac{d}{d\zeta}\left(\frac{K_n(\zeta)}{\zeta^n}\right) = -\frac{K_{n+1}}{\zeta^n}. \tag{1.32}$$

The energy density has the following values in the non-relativistic $\zeta \gg 1$ and ultra-relativistic $\zeta \ll 1$ limiting cases

$$\epsilon = \rho c^2 \left(1 + \frac{3kT}{2mc^2}\right), \qquad \text{for} \qquad \zeta \gg 1, \tag{1.33}$$
$$\epsilon = 3nkT = 3p, \qquad \text{for} \qquad \zeta \ll 1, \tag{1.34}$$

by using the asymptotic expressions for the modified Bessel function of the second kind given in the Appendix.

Another quantity that is very important in the analysis of the Boltzmann equation is the entropy. In a relativistic theory the entropy four-flow is given in terms of the one-particle distribution function by

$$S^\mu = -k \int cf \ln f \, p^\mu \frac{d^3p}{p_0}. \tag{1.35}$$

If we choose $\Psi = -kc \ln f$ in the transfer equation (1.18) we get the balance equation for the entropy four-flow

$$\frac{\partial}{\partial x^\mu} \int (-kc \ln f) p^\mu f \frac{d^3p}{p_0} = -kc \int \frac{\partial f}{\partial x^\mu} p^\mu f \frac{d^3p}{p_0}$$
$$+ \frac{kc}{4} \int \left[\ln \frac{f' f'_*}{f f_*} \right] \left[\frac{f' f'_*}{f f_*} - 1 \right] f_* f \, F \, \sigma \, d\Omega \frac{d^3p_*}{p_{*0}} \frac{d^3p}{p_0}. \tag{1.36}$$

The first term in the right-hand side of the above equation vanishes, since it can be identified as the multiplication of the Boltzmann equation (1.15) by kc, integration over all values of $\frac{d^3p}{p_0}$ and considering the symmetry properties of the collision operator. The second term is non-negative thanks to the relationship $(x-1)\ln x \geq 0$ which is valid for all $x > 0$. Hence the entropy four-flow balance equation reduces to

$$\partial_\mu S^\mu \geq 0. \tag{1.37}$$

The equilibrium entropy four-flow can be obtained from the insertion of the Maxwell-Jüttner distribution function (1.19) into its definition (1.35) and integration of the resulting equation, yielding

$$S^\mu = k \int f p^\mu e^{-\frac{p^\mu U_\mu}{kT}} \left\{ \frac{p^\nu U_\nu}{kT} - \ln \left[\frac{n}{4\pi m^2 ckT K_2(\zeta)} \right] \right\} \frac{d^3p}{p_0}$$

$$
\begin{aligned}
&= \frac{T^{\mu\nu}U_\nu}{T} + k\ln\left[\frac{4\pi m^2ckTK_2(\zeta)}{n}\right]N^\mu \\
&= n\left\{k\ln\left[\frac{4\pi m^2ckTK_2(\zeta)}{n}\right] + \frac{\epsilon}{nT}\right\}U^\mu. \qquad (1.38)
\end{aligned}
$$

thanks to (1.29) and (1.30). The entropy per particle s is related to the equilibrium value of the entropy four-flow written as $S^\mu = nsU^\mu$.

The Gibbs function per particle is identified with the chemical potential μ and defined by

$$\mu = \frac{\epsilon}{n} - Ts + \frac{p}{n} = kT\left\{\ln\left[\frac{n}{4\pi m^2ckTK_2(\zeta)}\right] + 1\right\}. \qquad (1.39)$$

From this last result we can rewrite the Maxwell-Jüttner distribution function (1.19) as

$$f = \exp\left[\frac{\mu}{kT} - 1 - \frac{p_\mu U^\mu}{kT}\right]. \qquad (1.40)$$

1.3 Boltzmann equation in gravitational fields

In order to write the number of particles in terms of the one-particle distribution function we have to know the transformations of the volume elements d^3x and d^3p in a Riemannian space. These transformations read

$$p^0\sqrt{-g}d^3x = p'^0\sqrt{-g'}d^3x', \qquad \sqrt{-g}\frac{d^3p}{p_0} = \sqrt{-g'}\frac{d^3p'}{p'_0}, \quad (1.41)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$.

Hence in a Riemannian space the one-particle distribution function is the scalar invariant $f(\mathbf{x}, \mathbf{p}, t)$ such that

$$dN = f(\mathbf{x}, \mathbf{p}, t)\, p^0 \sqrt{-g}\, d^3x \sqrt{-g}\, \frac{d^3p}{p_0}, \tag{1.42}$$

gives the number of particle world lines that crosses the hypersurface element represented by the three-dimensional space on the surface x^0 = constant and with momentum four-vector contained in the cell d^3p/p_0 of the mass-shell. In a Minkowski space $\sqrt{-g} = 1$, $p_0 = p^0$ and $dN = f(\mathbf{x}, \mathbf{p}, t) d^3x d^3p$.

In the presence of a gravitational field the left-hand side of the Boltzmann equation should be modified. For that end we shall write the one-particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$ as $f(x^\mu(\tau^*), p^i(\tau^*))$ where $\tau^* = \tau/m$ is an affine parameter along the world line of a particle of rest mass m and τ denotes the proper time. The variation of the one-particle distribution function with respect to the affine parameter τ^* reads

$$\frac{df(x^\mu(\tau^*), p^i(\tau^*))}{d\tau^*} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\tau^*} + \frac{\partial f}{\partial p^i} \frac{dp^i}{d\tau^*}. \tag{1.43}$$

Now from the equation of motion of a particle in the presence of a gravitational field

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0, \tag{1.44}$$

rewritten as

$$\frac{dp^i}{d\tau^*} = -\Gamma^i_{\mu\nu} p^\mu p^\nu, \qquad \text{where} \qquad p^\mu = \frac{dx^\mu}{d\tau^*}, \tag{1.45}$$

it follows that (1.43) becomes

$$\frac{df(x^\mu(\tau^*), p^i(\tau^*))}{d\tau^*} = p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i}. \tag{1.46}$$

Hence, the left-hand side of the Boltzmann equation is replaced by

$$p^\mu \frac{\partial f}{\partial x^\mu} \longrightarrow p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i}, \tag{1.47}$$

while in its right-hand side we should replace the invariant element $d^3 p_*/p_{*0}$ by $\sqrt{-g} d^3 p_*/p_{*0}$. Therefore the Boltzmann equation in the presence of a gravitational field reads

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = \int (f'_* f' - f_* f)\; F\, \sigma\, d\Omega \sqrt{-g} \frac{d^3 p_*}{p_{*0}}. \tag{1.48}$$

Another expression for the Boltzmann equation in gravitational fields is obtained when the mass-shell condition $p_\mu p^\nu = m^2 c^2$ is not taken into account. First we note that

$$\frac{\partial f(x^\mu, p^i)}{\partial x^\mu} = \frac{\partial f(x^\mu, p^\mu)}{\partial x^\mu} + \frac{\partial f(x^\mu, p^\mu)}{\partial p^0} \frac{\partial p^0}{\partial x^\mu}, \tag{1.49}$$

$$\frac{\partial f(x^\mu, p^i)}{\partial p^i} = \frac{\partial f(x^\mu, p^\mu)}{\partial p^i} + \frac{\partial f(x^\mu, p^\mu)}{\partial p^0} \frac{\partial p^0}{\partial p^i}, \tag{1.50}$$

while from the mass-shell condition $p_\mu p^\nu = m^2 c^2$ and (1.45) it follows that

$$\frac{\partial p^0}{\partial x^\mu} = -\frac{1}{p_0} p^\nu p_\kappa \Gamma^\kappa_{\mu\nu}, \qquad \frac{\partial p^0}{\partial p^i} = -\frac{p_i}{p_0}. \tag{1.51}$$

Hence by taking into account (1.49) – (1.51) the Boltzmann equation (1.48) can be rewritten as

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^\sigma} = \int (f'_* f' - f_* f) \; F \, \sigma \, d\Omega \sqrt{-g} \frac{d^3 p_*}{p_{*0}}. \tag{1.52}$$

The particle four-flow N^μ and the energy-momentum tensor $T^{\mu\nu}$ are defined in terms of the one-particle distribution function by

$$N^\mu = \int c p^\mu f(\mathbf{x}, \mathbf{p}, t) \sqrt{-g} \frac{d^3 p}{p_0}, \tag{1.53}$$

$$T^{\mu\nu} = \int c p^\mu p^\nu f(\mathbf{x}, \mathbf{p}, t) \sqrt{-g} \frac{d^3 p}{p_0}. \tag{1.54}$$

To obtain the balance equations for the particle four-flow and energy-momentum tensor we need to know a relationship that follows from the Liouville theorem in a seven-dimensional phase space spanned by the coordinates (x^μ, p^i).

In a Riemannian space d^4x is a scalar density of weight -1 whose invariant volume element is $\sqrt{-g} d^4 x = \sqrt{-g'} d^4 x'$. Let us consider a seven-dimensional phase space spanned by the coordinates (x^μ, p^i) where the invariant volume element is given by $d\mathcal{F} = \sqrt{-g} d^4 x \sqrt{-g} \frac{d^3 p}{p_0}$. In this phase space we introduce a seven-dimensional momentum p^A and a corresponding seven-dimensional gradient $\partial/(\partial x^A)$ defined by

$$(p^A) = \left(\frac{dx^\mu}{d\tau^*}, \frac{dp^i}{d\tau^*} \right) \equiv \left(p^\mu, -\Gamma^i_{\mu\nu} p^\mu p^\nu \right), \tag{1.55}$$

$$\left(\frac{\partial}{\partial x^A}\right) = \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial p^i}\right). \tag{1.56}$$

According to the Liouville theorem the density of the points in the phase space is constant along the trajectories in the phase space, which means that the density of the points in the phase space moves like an incompressible fluid. By identifying $-g/p_0$ as the density of the points in the phase space spanned by (x^μ, p^i) the Liouville theorem implies that the divergence of $-gp^A/p_0$ must vanish, i.e.

$$\frac{\partial}{\partial x^A}\left(\frac{-g}{p_0}p^A\right) = \frac{\partial}{\partial x^\mu}\left(\frac{-g}{p_0}p^\mu\right) + \frac{\partial}{\partial p^i}\left(\frac{g}{p_0}\Gamma^i_{\mu\nu}p^\mu p^\nu\right) = 0. \tag{1.57}$$

The balance equation for the particle four-flow is obtained from the multiplication of the Boltzmann equation (1.48) by c and the integration of the resulting equation over the invariant volume element $d\mathcal{F} = \sqrt{-g}d^4x\sqrt{-g}\frac{d^3p}{p_0}$, yielding

$$\int c\left\{p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu}p^\mu p^\nu \frac{\partial f}{\partial p^i}\right\}\left(\frac{-g}{p_0}\right) d^3p d^4x$$
$$= \int \left\{ \frac{\partial}{\partial x^\mu}\left[cp^\mu f\left(\frac{-g}{p_0}\right)\right] \right.$$
$$\left. \underline{-\frac{\partial}{\partial p^i}\left[cf\left(\frac{-g}{p_0}\right)\Gamma^i_{\mu\nu}p^\mu p^\nu\right]}\right\} d^3p d^4x = 0, \tag{1.58}$$

thanks to (1.57) and to the vanishing of the right-hand side of the Boltzmann equation for all summational invariant. The underlined term vanishes, since the volume integral in the momentum space can be transformed into an integral over an infinitely far surface where the one-particle distribution function tends to zero. The remaining integral above can be rewritten as

$$\begin{aligned}
&\int \frac{\partial}{\partial x^\mu}\left[cp^\mu f\left(\frac{-g}{p_0}\right)\right] d^3p d^4x \\
&\quad = \int \frac{\partial}{\partial x^\mu}\left\{\sqrt{-g}\left[\int cp^\mu f\sqrt{-g}\frac{d^3p}{p_0}\right]\right\}d^4x \\
&\quad = \int \left\{\frac{\partial}{\partial x^\mu}\left[\int cp^\mu f\sqrt{-g}\frac{d^3p}{p_0}\right]\right. \\
&\quad \left. + \frac{\partial \ln\sqrt{-g}}{\partial x^\mu}\int cp^\mu f\sqrt{-g}\frac{d^3p}{p_0}\right\}\sqrt{-g}d^4x \\
&\quad = \int \left[\int cp^\mu f\sqrt{-g}\frac{d^3p}{p_0}\right]_{;\mu}\sqrt{-g}d^4x \\
&\quad = \int N^\mu{}_{;\mu}\sqrt{-g}d^4x = 0,
\end{aligned} \tag{1.59}$$

where the following relationships were used

$$\frac{\partial \ln\sqrt{-g}}{\partial x^\mu} = \Gamma^\nu{}_{\mu\nu}, \qquad A^\mu{}_{;\mu} = \frac{\partial A^\mu}{\partial x^\mu} + \Gamma^\mu{}_{\mu\nu}A^\nu. \tag{1.60}$$

Now by considering that the integration over $\sqrt{-g}d^4x$ is arbitrary, the integrand of the above equation must vanish and we find the balance equation for the particle four flow, namely

$$N^\mu{}_{;\mu} = 0. \tag{1.61}$$

We shall determine the energy-momentum tensor balance equation in two steps, in the first one the time component of the momentum four-vector p^μ is considered and in the other its spatial components. Let us begin by multiplying the Boltzmann (1.48) with $p^0(-g/p_0)d^3pd^4x$ and the integration of the resulting equation

$$\int cp^0\left\{p^\mu\frac{\partial f}{\partial x^\mu}-\Gamma^i_{\mu\nu}p^\mu p^\nu\frac{\partial f}{\partial p^i}\right\}\left(-\frac{g}{p_0}\right)d^3pd^4x$$
$$=\int\left\{\frac{\partial}{\partial x^\mu}\left[\int cp^0p^\mu f\sqrt{-g}\frac{d^3p}{p_0}\right]\right.$$
$$+\frac{\partial\ln\sqrt{-g}}{\partial x^\mu}\int cp^0p^\mu f\sqrt{-g}\frac{d^3p}{p_0}$$
$$\left.-\int cf\left[\frac{\partial p^0}{\partial x^\mu}p^\mu-\frac{\partial p^0}{\partial p^i}\Gamma^i_{\mu\nu}p^\mu p^\nu\right]\sqrt{-g}\frac{d^3p}{p_0}\right\}\sqrt{-g}d^4x$$
$$\underline{-\int\frac{\partial}{\partial p^i}\left[cp^0f\left(-\frac{g}{p_0}\right)\Gamma^i_{\mu\nu}p^\mu p^\nu\right]}d^3pd^4x=0. \quad (1.62)$$

If we consider that the underlined term vanishes, use (1.51) and the fact that the integration over $\sqrt{-g}d^4x$ is arbitrary so that the integrand of the remaining equation is zero, we get from the above equation that

$$\frac{\partial T^{0\mu}}{\partial x^\mu}+\Gamma^\nu{}_{\mu\nu}T^{0\mu}+\Gamma^0{}_{\mu\nu}T^{\mu\nu}=T^{0\mu}{}_{;\mu}=0. \quad (1.63)$$

Following the same methodology for the spatial components of the momentum four-vector and multiplying the Boltzmann equation (1.48) with $p^i(-g/p_0)d^3pd^4x$ and integrating the resulting equation we get

$$\int cp^i\left\{p^\mu\frac{\partial f}{\partial x^\mu}-\Gamma^j_{\mu\nu}p^\mu p^\nu\frac{\partial f}{\partial p^j}\right\}\left(-\frac{g}{p_0}\right)d^3pd^4x$$
$$=\int\left\{\frac{\partial}{\partial x^\mu}\left[\int cp^ip^\mu f\sqrt{-g}\frac{d^3p}{p_0}\right]\right.$$
$$+\frac{\partial\ln\sqrt{-g}}{\partial x^\mu}\int cp^ip^\mu f\sqrt{-g}\frac{d^3p}{p_0}$$
$$\left.+\int cf\frac{\partial p^i}{\partial p^j}\Gamma^j_{\mu\nu}p^\mu p^\nu\sqrt{-g}\frac{d^3p}{p_0}\right\}\sqrt{-g}d^4x$$
$$-\underline{\int\frac{\partial}{\partial p^i}\left[cp^0f\left(-\frac{g}{p_0}\right)\Gamma^i_{\mu\nu}p^\mu p^\nu\right]}d^3pd^4x=0. \tag{1.64}$$

The underlined term above vanishes and note that x^μ and p^i are independent variables. The above equation leads to

$$\frac{\partial T^{i\mu}}{\partial x^\mu}+\Gamma^\nu{}_{\mu\nu}T^{i\mu}+\Gamma^i{}_{\mu\nu}T^{\mu\nu}=T^{i\mu}{}_{;\mu}=0. \tag{1.65}$$

Hence by collecting the two above results (1.63) and (1.65), the balance equation for the energy-momentum tensor is

$$T^{\nu\mu}{}_{;\mu}=0. \tag{1.66}$$

Previously it was pointed out that the right-hand side of Boltzmann's equation (1.48) vanishes identically at equilibrium when the one-particle distribution function is given by (1.19) or (1.40), which is the Maxwell-Jüttner distribution function. We shall determine the restrictions dictated by the left-hand side of (1.48) when the one-particle distribution function is the Maxwell-Jüttner one. If we insert (1.40) into the left-hand side of the the Boltzmann equation (1.48) we get the momentum four-vector polynomial equation

$$p^\nu \partial_\nu \left[\frac{\mu}{kT}\right] - \frac{1}{2} p^\mu p^\nu \left\{ \left[\frac{U_\mu}{kT}\right]_{;\nu} + \left[\frac{U_\nu}{kT}\right]_{;\mu} \right\} = 0. \qquad (1.67)$$

The above equation is valid for all values of p^μ so that the coefficients of the polynomial equation must vanish, yielding

$$\partial_\nu \left[\frac{\mu}{kT}\right] = 0, \qquad \left[\frac{U_\mu}{kT}\right]_{;\nu} + \left[\frac{U_\nu}{kT}\right]_{;\mu} = 0. \qquad (1.68)$$

Here it was assumed that the particles have non-vanishing rest mass.

Let us first analyze $(1.68)_2$ which is the so-called Killing equation and U_ν/kT is a (timelike) Killing vector. We rewrite the Killing equation as

$$U_{\mu;\nu} + U_{\nu;\mu} - \frac{1}{T}\left(U_\mu \partial_\nu T + U_\nu \partial_\mu T\right) = 0, \qquad (1.69)$$

and perform the projections with respect to $U^\mu U^\nu$ and U^ν, yielding

$$\dot{T} \equiv U^\mu \partial_\mu T = 0, \quad \text{and} \quad \dot{U}_\mu \equiv U^\nu U_{\mu;\nu} = \frac{c^2}{T} \partial_\mu T, \qquad (1.70)$$

respectively. The interpretation of equations (1.70) is: in equilibrium a gas must have a stationary temperature and its acceleration must be counterbalanced by a spatial temperature gradient. Note that the condition $(1.70)_2$ is not compatible with a geodesic fluid motion which would require $\dot{U}^\mu = 0$.

We consider a fluid at rest where the spatial components of the four-velocity vanish so that (see (2.6))

$$(U^\mu) = \left(\frac{c}{\sqrt{g_{00}}}, \mathbf{0}\right). \tag{1.71}$$

The existence of a time-like Killing vector corresponds to a stationary metric, where the acceleration term becomes

$$\begin{aligned}\dot{U}^\mu &= U^\nu U^\mu{}_{;\nu} = U^0\left(\frac{\partial U^\mu}{\partial x^0} + \Gamma^\mu{}_{00}U^0\right)\\ &= \frac{c^2}{g_{00}}\Gamma^\mu_{00} = -c^2 g^{\mu\nu}\partial_\nu \ln\sqrt{g_{00}}.\end{aligned} \tag{1.72}$$

Here we have used (2.10) and neglected all time derivatives, since we are dealing with a stationary metric. Now from $(1.70)_2$ and (1.72) we have

$$c^2 g^{\mu\nu}\partial_\nu\left[\ln\left(\sqrt{g_{00}}\,T\right)\right] = 0, \tag{1.73}$$

which implies Tolman's law [5, 6]

$$\sqrt{g_{00}}\,T = \text{constant}\,. \tag{1.74}$$

From Tolman's law and the equilibrium condition $(1.68)_1$ follows Klein's law [7]

$$\sqrt{g_{00}}\,\mu = \text{constant}\,. \tag{1.75}$$

We note that both laws were obtained from the equilibrium conditions applied to the Maxwell-Jüttner distribution function, but they can also be derived on purely thermodynamics grounds as in Tolman and Klein's original papers.

Appendix

The asymptotic expansion of $K_n(\zeta)$ for large values of ζ, i.e. $\zeta \gg 1$, is given by (see [8, 9])

$$K_n(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\frac{1}{e^\zeta}\Big[1 + \frac{4n^2-1}{8\zeta} + \frac{(4n^2-1)(4n^2-9)}{2!(8\zeta)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8\zeta)^3} + \dots\Big], \quad (1.76)$$

while for small values of ζ, i.e. $\zeta \ll 1$, it reads [8, 9]

$$K_n(\zeta) = \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k \frac{(n-k-1)!}{k!\left(\frac{\zeta}{2}\right)^{n-2k}} + (-1)^{n+1}\sum_{k=0}^{\infty}\frac{\left(\frac{\zeta}{2}\right)^{n+2k}}{k!(n+k)!} \times\Big[\ln\frac{\zeta}{2} - \frac{1}{2}\psi(k+1) - \frac{1}{2}\psi(n+k+1)\Big]. \quad (1.77)$$

Above the function $\psi(n)$ is defined in terms of Euler's constant $\gamma = 0.577\,215\,664\dots$ by

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}, \qquad \psi(1) = -\gamma. \quad (1.78)$$

References

[1] G. M. Kremer, *An introduction to the Boltzmann equation and transport processes in gases* (Springer, Berlin, 2010).

[2] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann equation: theory and applications* (Birkhäuser, Basel, 2002).

[3] J. C. Maxwell, On the dynamical theory of gases, *Phil. Trans. R. Soc. London* **157**, 49 (1867).

[4] D. Enskog, Bermerkungen zu einer Fundamentalgleichung in der kinetischen Gastheorie, *Phys. Z.* **12**, 534 (1911).

[5] R. C. Tolman, On the weight of heat and thermal equilibrium in general relativity, *Phys. Rev.* **35**, 904 (1930).

[6] R. C. Tolman and P. Ehrenfest, Temperature equilibrium in a static gravitational field, *Phys. Rev.* **36**, 1791 (1930).

[7] O. Klein, On the thermodynamical equilibrium of fluids in gravitational fields, *Rev. Mod. Phys.* **21**, 531 (1949).

[8] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions* (Dover, New York , 1968).

[9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products* 5th ed. (Academic Press, San Diego, 1994).

CHAPTER 2

FIRST POST-NEWTONIAN APPROXIMATION

In this chapter the first post-Newtonian approximation of Einstein's field equations is derived following the Chandrasekhar and Weinberg methods and the corresponding Poisson equations and Eulerian hydrodynamic equations are determined. The first post-Newtonian approximation of the Brans-Dicke theory is analysed and the hydrodynamic equations for non-perfect fluids are obtained. The gravitational potentials and conservation laws in the first post-Newtonian approximation are also discussed.

2.1 Preliminaries

We start with the general expression for the line element ds in terms of the metric tensor $g_{\mu\nu}$, namely

$$\begin{aligned} ds^2 = c^2 d\tau^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{00}(dx^0)^2 + 2g_{0i}dx^0 dx^i + g_{ij}dx^i dx^j, \end{aligned} \qquad (2.1)$$

where τ is the proper time and $dx^0 = cdt$.

If we introduce the spatial metric tensor

$$\gamma_{ij} = -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}}, \qquad (2.2)$$

the line element (2.1) can be rewritten as

$$\begin{aligned} ds^2 = c^2 d\tau^2 = g_{00}(cdt)^2 &+ 2g_{0i}dx^0 dx^i \\ &+ \left(\frac{g_{0i}g_{0j}}{g_{00}} - \gamma_{ij}\right) dx^i dx^j. \end{aligned} \qquad (2.3)$$

From the above expression we can derive a relationship between the time t and the proper time τ differentials through the division of (2.3) by $(cdt)^2$ and the introduction of the velocity and speed defined by

$$V^i = \frac{dx^i}{dt}, \qquad V = \sqrt{\gamma_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}. \qquad (2.4)$$

Hence it follows that

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{g_{00}\left(1 + \frac{g_{0i}V^i}{g_{00}c}\right)^2 - \frac{V^2}{c^2}}}. \qquad (2.5)$$

Note that in a Minkowski space-time $g_{00} = 1$, $g_{0i} = 0$ and $\gamma = 1/\sqrt{1 - V^2/c^2}$ reduces to the Lorentz factor of special relativity.

The contravariant components of the four-velocity $U^\mu = dx^\mu/d\tau$ are given as functions of the velocity by

$$(U^\mu) = \left(\frac{dx^0}{d\tau} = \gamma c, \frac{dx^i}{d\tau} = \gamma V^i\right), \tag{2.6}$$

while the covariant components read

$$(U_\mu) = (g_{\mu\nu}U^\nu) = \gamma\left(cg_{00} + g_{0i}V^i, cg_{0i} + g_{ij}V^j\right). \tag{2.7}$$

It is straightforward to obtain from (2.6) and (2.7) that $U^\mu U_\mu = c^2$.

A macroscopic description of a relativistic fluid is based on the balance equations of particle four-flow N^μ and energy-momentum tensor $T^{\mu\nu}$, namely

$$N^\mu{}_{;\mu} = \frac{\partial N^\mu}{\partial x^\mu} + \Gamma^\mu{}_{\mu\lambda}N^\lambda = 0, \tag{2.8}$$

$$T^{\mu\nu}{}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu{}_{\nu\lambda}T^{\lambda\nu} + \Gamma^\nu{}_{\nu\lambda}T^{\mu\lambda} = 0, \tag{2.9}$$

where the semicolon denotes the covariant derivative and $\Gamma^\sigma{}_{\mu\nu}$ are the Christoffel symbols

$$\Gamma^\sigma{}_{\mu\nu} = \frac{g^{\sigma\tau}}{2}\left(\frac{\partial g_{\mu\tau}}{\partial x^\nu} + \frac{\partial g_{\tau\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\tau}\right). \tag{2.10}$$

A perfect fluid is characterized by the absence of dissipative effects like viscous stresses and heat conduction. For a perfect fluid the particle four-flow and the energy-momentum tensor are represented as

$$N^\mu = nU^\mu, \qquad T^{\mu\nu} = (\epsilon + p)\frac{U^\mu U^\nu}{c^2} - pg^{\mu\nu}. \tag{2.11}$$

Here n denotes the particle number density of the relativistic fluid, p and ϵ its pressure and energy density, respectively. The energy density has two parts $\epsilon = \rho c^2(1 + \varepsilon/c^2)$ one associated with the mass density $\rho = mn$ and another to the internal energy density $\rho\varepsilon$. The specific internal energy for a non-relativistic perfect fluid is given by $\varepsilon = c_v T$, where c_v is the specific heat at constant volume and T the absolute temperature. For monatomic gases $c_v = 3k/2m$ with k denoting Boltzmann constant and m the rest mass of a fluid particle.

The connection between the space-time geometry and the matter content inside it is governed by Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{2.12}$$

where $G = 6.674\times10^{-11}$ $\mathrm{m}^3/(\mathrm{s}^2\mathrm{kg})$ is the universal gravitational constant.

The Ricci tensor

$$R_{\mu\nu} = R^\tau{}_{\mu\tau\nu} = \frac{\partial\Gamma^\tau{}_{\mu\tau}}{\partial x^\nu} - \frac{\partial\Gamma^\tau{}_{\mu\nu}}{\partial x^\tau} + \Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\nu\sigma} - \Gamma^\sigma{}_{\mu\nu}\Gamma^\tau{}_{\sigma\tau}, \tag{2.13}$$

is a contraction of the Riemann curvature tensor (or Riemann–Christoffel tensor)

$$R^\tau{}_{\mu\sigma\nu} = \frac{\partial\Gamma^\tau{}_{\mu\sigma}}{\partial x^\nu} - \frac{\partial\Gamma^\tau{}_{\mu\nu}}{\partial x^\sigma} + \Gamma^\tau{}_{\nu\epsilon}\Gamma^\epsilon{}_{\mu\sigma} - \Gamma^\tau{}_{\sigma\epsilon}\Gamma^\epsilon{}_{\mu\nu}, \tag{2.14}$$

and a contraction of the Ricci tensor $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature (or Ricci scalar).

An alternative form of the Ricci tensor is given in terms of second derivatives of the metric tensor

$$R_{\mu\nu} = \frac{g^{\sigma\tau}}{2}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma x^\tau} + \frac{\partial^2 g_{\sigma\tau}}{\partial x^\mu x^\nu} - \frac{\partial^2 g_{\sigma\nu}}{\partial x^\mu x^\tau} - \frac{\partial^2 g_{\mu\tau}}{\partial x^\sigma x^\nu}\right) + g^{\sigma\tau} g_{\kappa\eta}\left(\Gamma^\kappa{}_{\sigma\tau}\Gamma^\eta{}_{\mu\nu} - \Gamma^\kappa{}_{\sigma\nu}\Gamma^\eta{}_{\mu\tau}\right). \quad (2.15)$$

Equivalently Einstein's field equations may be written as

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}T^\sigma{}_\sigma g_{\mu\nu}\right) = -\frac{8\pi G}{c^4}\mathfrak{T}_{\mu\nu}, \quad (2.16)$$

where $T^\sigma{}_\sigma = g^{\sigma\kappa}T_{\kappa\sigma}$ is the trace of the energy-momentum tensor.

2.2 The first post-Newtonian approximation

The post-Newtonian theory is a method of successive approximations in $1/c^2$ powers for the determination of the components of the metric tensor from Einstein's field equations which was proposed by Einstein, Infeld and Hoffmann [1] in 1938. In this method Einstein's field equations (2.16) of $\mathcal{O}(c^{-n})$ – order can be written as

$$\overset{n}{R}_{\mu\nu} = -\frac{8\pi G}{c^4}\overset{n-2}{\mathfrak{T}}_{\mu\nu}. \quad (2.17)$$

Hence from the knowledge of the energy-momentum tensor in the $(n-2)$th-order the Ricci tensor and consequently the metric tensor in the nth-order can be determined.

From the knowledge of the metric tensor components in a Minkowski space-time $g_{00} = 1$, $g_{ij} = -\delta_{ij}$ and $g_{0i} = 0$ we can split the contravariant and covariant components of the metric tensor as

$$g_{00} = 1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \overset{6}{g}_{00} + \mathcal{O}(c^{-8}), \tag{2.18}$$

$$g^{00} = 1 + \overset{2}{g}{}^{00} + \overset{4}{g}{}^{00} + \overset{6}{g}{}^{00} + \mathcal{O}(c^{-8}), \tag{2.19}$$

$$g_{ij} = -\delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \mathcal{O}(c^{-6}), \quad g_{0i} = \overset{3}{g}_{0i} + \overset{5}{g}_{0i} + \mathcal{O}(c^{-7}), \tag{2.20}$$

$$g^{ij} = -\delta_{ij} + \overset{2}{g}{}^{ij} + \overset{4}{g}{}^{ij} + \mathcal{O}(c^{-6}), \quad g^{0i} = \overset{3}{g}{}^{0i} + \overset{5}{g}{}^{0i} + \mathcal{O}(c^{-7}), \tag{2.21}$$

where $\overset{n}{g}_{\mu\nu}$ and $\overset{n}{g}{}^{\mu\nu}$ denote the metric tensor components of order $\mathcal{O}(c^{-n})$.

The relationships between the covariant and contravariant components of the metric tensor can be found from $g^{\mu\sigma}g_{\sigma\nu} = \delta^{\nu}_{\mu}$, which implies that

$$g^{0\sigma}g_{0\sigma} = g^{00}g_{00} + g^{0i}g_{0i} = 1, \tag{2.22}$$

$$g^{0\sigma}g_{i\sigma} = g^{00}g_{i0} + g^{0j}g_{ij} = 0, \tag{2.23}$$

$$g^{i\sigma}g_{j\sigma} = g^{i0}g_{j0} + g^{ik}g_{jk} = \delta^{i}_{j}. \tag{2.24}$$

The above equations with the representations (2.18) – (2.21) become

$$1 + \overset{2}{g}_{00} + \overset{2}{g}{}^{00} + \mathcal{O}(c^{-4}) = 1, \tag{2.25}$$

$$\overset{3}{g}_{0i} - \overset{3}{g}{}^{0j}\delta_{ij} + \mathcal{O}(c^{-4}) = 0, \tag{2.26}$$

$$\delta^i_j - \delta_{ik}\overset{2}{g}_{jk} - \delta_{jk}\overset{2}{g}{}^{ik} + \mathcal{O}(c^{-4}) = \delta^i_j, \tag{2.27}$$

so that we can infer that: $\overset{2}{g}{}^{00} = -\overset{2}{g}_{00}$, $\overset{2}{g}{}^{ij} = -\overset{2}{g}_{ij}$ and $\overset{3}{g}{}^{0i} = \overset{3}{g}_{0i}$.

On the basis of (2.18) – (2.21) the components of the Christoffel symbols (2.10) can be split in orders $\mathcal{O}(c^{-n})$ as

(i) $\Gamma^0{}_{00} = \overset{3}{\Gamma}{}^0{}_{00} + \overset{5}{\Gamma}{}^0{}_{00} + \mathcal{O}(c^{-7})$, where

$$\overset{3}{\Gamma}{}^0{}_{00} = \frac{1}{2c}\frac{\partial \overset{2}{g}_{00}}{\partial t}, \quad \overset{5}{\Gamma}{}^0{}_{00} = \frac{1}{2c}\frac{\partial \overset{4}{g}_{00}}{\partial t} + \frac{\overset{2}{g}{}^{00}}{2c}\frac{\partial \overset{2}{g}_{00}}{\partial t} - \frac{\overset{3}{g}{}^{0i}}{2c}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}; \tag{2.28}$$

(ii) $\Gamma^0{}_{0i} = \overset{2}{\Gamma}{}^0{}_{0i} + \overset{4}{\Gamma}{}^0{}_{0i} + \mathcal{O}(c^{-6})$, where

$$\overset{2}{\Gamma}{}^0{}_{0i} = \frac{1}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}, \qquad \overset{4}{\Gamma}{}^0{}_{0i} = \frac{1}{2}\frac{\partial \overset{4}{g}_{00}}{\partial x^i} + \frac{\overset{2}{g}{}^{00}}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}; \tag{2.29}$$

(iii) $\Gamma^0{}_{ij} = \overset{3}{\Gamma}{}^0{}_{ij} + \mathcal{O}(c^{-5})$, where

$$\overset{3}{\Gamma}{}^0{}_{ij} = \frac{1}{2}\left(\frac{\partial \overset{3}{g}_{0i}}{\partial x^j} + \frac{\partial \overset{3}{g}_{0j}}{\partial x^i} - \frac{1}{c}\frac{\partial \overset{2}{g}_{ij}}{\partial t}\right); \tag{2.30}$$

(iv) $\Gamma^i{}_{00} = \overset{2}{\Gamma}{}^i{}_{00} + \overset{4}{\Gamma}{}^i{}_{00} + \mathcal{O}(c^{-6})$, where

$$\overset{2}{\Gamma}{}^i{}_{00} = \frac{1}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}, \quad \overset{4}{\Gamma}{}^i{}_{00} = \frac{1}{2}\frac{\partial \overset{4}{g}_{00}}{\partial x^i} - \frac{\overset{2}{g}{}^{ij}}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^j} - \frac{1}{c}\frac{\partial \overset{3}{g}_{0i}}{\partial t}; \tag{2.31}$$

(v) $\Gamma^i{}_{0j} = \overset{3}{\Gamma}{}^i{}_{0j} + \mathcal{O}(c^{-5})$, where

$$\overset{3}{\Gamma}{}^i{}_{0j} = -\frac{1}{2}\left(\frac{1}{c}\frac{\partial \overset{2}{g}_{ij}}{\partial t} + \frac{\partial \overset{3}{g}_{0i}}{\partial x^j} - \frac{\partial \overset{3}{g}_{0j}}{\partial x^i}\right); \tag{2.32}$$

(vi) $\Gamma^i{}_{jk} = \overset{2}{\Gamma}{}^i{}_{jk} + \mathcal{O}(c^{-4})$, where

$$\overset{2}{\Gamma}{}^i{}_{jk} = -\frac{1}{2}\left(\frac{\partial \overset{2}{g}_{ij}}{\partial x^k} + \frac{\partial \overset{2}{g}_{ik}}{\partial x^j} - \frac{\partial \overset{2}{g}_{jk}}{\partial x^i}\right). \tag{2.33}$$

The slip of the Ricci tensor in orders $\mathcal{O}\left(c^{-n}\right)$ is based on its definition (2.13) or (2.15) and on the splits of the Christoffel symbols or of the metric tensor, so that we can write

$$R_{00} = \overset{2}{R}_{00} + \overset{4}{R}_{00} + \mathcal{O}(c^{-6}), \tag{2.34}$$

$$R_{ij} = \overset{2}{R}_{ij} + \overset{4}{R}_{ij} + \mathcal{O}(c^{-6}), \tag{2.35}$$

$$R_{0i} = \overset{3}{R}_{0i} + \overset{5}{R}_{0i} + \mathcal{O}(c^{-7}). \tag{2.36}$$

Let us determine explicitly the components of the Ricci tensor in terms of the derivatives of the metric tensor. We begin with the time component of the Ricci tensor that can be written thanks to (2.13) as

$$\begin{aligned} R_{00} = {} & \frac{\partial \Gamma^i{}_{0i}}{\partial x^0} - \frac{\partial \Gamma^i{}_{00}}{\partial x^i} + \Gamma^i{}_{00}\Gamma^0{}_{0i} \\ & + \Gamma^i{}_{0j}\Gamma^j{}_{0i} - \Gamma^0{}_{00}\Gamma^i{}_{0i} - \Gamma^i{}_{00}\Gamma^j{}_{ij}, \end{aligned} \tag{2.37}$$

so that its second and fourth order are given by

$$\overset{2}{R}_{00} = -\frac{\partial \overset{2}{\Gamma}{}^{i}{}_{00}}{\partial x^i}, \quad \overset{4}{R}_{00} = \frac{1}{c}\frac{\partial \overset{3}{\Gamma}{}^{i}{}_{0i}}{\partial t} - \frac{\partial \overset{4}{\Gamma}{}^{i}{}_{00}}{\partial x^i} + \overset{2}{\Gamma}{}^{i}{}_{00}\overset{2}{\Gamma}{}^{0}{}_{0i} - \overset{2}{\Gamma}{}^{i}{}_{00}\overset{2}{\Gamma}{}^{j}{}_{ij}. \tag{2.38}$$

Now by using the above relationships for the Christoffel symbols we get

$$\overset{2}{R}_{00} = -\frac{1}{2}\nabla^2 \overset{2}{g}_{00}, \tag{2.39}$$

$$\overset{4}{R}_{00} = -\frac{1}{2}\nabla^2 \overset{4}{g}_{00} - \frac{1}{2c^2}\frac{\partial^2 \overset{2}{g}_{ii}}{\partial t^2} + \frac{1}{c}\frac{\partial^2 \overset{3}{g}_{0i}}{\partial t \partial x^i} + \frac{\overset{2}{g}{}^{ij}}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j}$$
$$+\frac{1}{4}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}\frac{\partial \overset{2}{g}_{00}}{\partial x^i} + \frac{1}{4}\frac{\partial \overset{2}{g}_{jj}}{\partial x^i}\frac{\partial \overset{2}{g}_{00}}{\partial x^i} - \frac{1}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}\frac{\partial \overset{2}{g}_{ij}}{\partial x^j}. \tag{2.40}$$

In the same way the spatial components of the Ricci tensor read

$$\overset{2}{R}_{ij} = \frac{\partial \overset{2}{\Gamma}{}^{0}{}_{i0}}{\partial x^j} + \frac{\partial \overset{2}{\Gamma}{}^{k}{}_{ik}}{\partial x^j} - \frac{\partial \overset{2}{\Gamma}{}^{k}{}_{ij}}{\partial x^k} = -\frac{1}{2}\nabla^2 \overset{2}{g}_{ij} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j}$$
$$-\frac{1}{2}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^j \partial x^k} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{jk}}{\partial x^i \partial x^k}, \tag{2.41}$$

while its space-time components become

$$\overset{3}{R}_{i0} = \frac{1}{c}\frac{\partial \overset{2}{\Gamma}{}^{j}{}_{ij}}{\partial t} - \frac{\partial \overset{3}{\Gamma}{}^{j}{}_{0i}}{\partial x^j} = -\frac{1}{2}\nabla^2 \overset{3}{g}_{0i} + \frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{ij}}{\partial t \partial x^j}$$
$$-\frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial t \partial x^i} + \frac{1}{2}\frac{\partial^2 \overset{3}{g}_{0j}}{\partial x^i \partial x^j}. \tag{2.42}$$

For the determination of the components of the metric tensor from Einstein's field equations (2.17) one has to known how the energy-momentum tensor is split in orders of $1/c^n$. For that end we note that in the non-relativistic case the contravariant components of the four-velocity (2.6) reduce to $(U^\mu) = (c, V_i)$, so that the components of the energy-momentum tensor $(2.11)_2$ become $T^{00} = \rho c^2$, $T^{ij} = \rho V_i V_j + p\delta_{ij}$ and $T^{0i} = \rho c V_i$. Hence on the basis of the expression of T^{00} we may infer that T^{ij} and T^{0i} are of orders $\mathcal{O}(c^{-2})$ and $\mathcal{O}(c^{-1})$, respectively, and we may write

$$T^{00} = \overset{0}{T}{}^{00} + \overset{2}{T}{}^{00} + \mathcal{O}(c^{-4}), \tag{2.43}$$

$$T^{ij} = \overset{2}{T}{}^{ij} + \overset{4}{T}{}^{ij} + \mathcal{O}(c^{-6}), \tag{2.44}$$

$$T^{0i} = \overset{1}{T}{}^{0i} + \overset{3}{T}{}^{0i} + \mathcal{O}(c^{-5}). \tag{2.45}$$

Hence we can split components of the tensor $\mathfrak{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}T^\sigma{}_\sigma g_{\mu\nu}$ as

$$\mathfrak{T}^{00} = \overset{0}{\mathfrak{T}}{}^{00} + \overset{2}{\mathfrak{T}}{}^{00} + \mathcal{O}(c^{-4}), \tag{2.46}$$

$$\mathfrak{T}^{ij} = \overset{0}{\mathfrak{T}}{}^{ij} + \overset{2}{\mathfrak{T}}{}^{ij} + \mathcal{O}(c^{-4}), \tag{2.47}$$

$$\mathfrak{T}^{0i} = \overset{1}{\mathfrak{T}}{}^{0i} + \overset{3}{\mathfrak{T}}{}^{0i} + \mathcal{O}(c^{-5}). \tag{2.48}$$

The trace of the energy-momentum tensor up to $1/c^2$ order is given by

$$T^\sigma{}_\sigma = g_{00}T^{00} + 2g_{0i}T^{0i} + g_{ij}T^{ij} = \overset{0}{T}{}^{00} + \overset{2}{g}_{00}\overset{0}{T}{}^{00} + \overset{2}{T}{}^{00} - \overset{2}{T}{}^{ii}, \tag{2.49}$$

so that in the first orders the tensor components of $\mathfrak{T}_{\mu\nu} = g_{\mu\sigma}g_{\nu\tau}\left(T^{\sigma\tau} - T^{\kappa}{}_{\kappa}g^{\sigma\tau}/2\right)$ read

$$\overset{0}{\mathfrak{T}}_{00} = \frac{1}{2}\overset{0}{T}{}^{00} = \frac{1}{2}\rho c^2, \qquad \overset{0}{\mathfrak{T}}_{ij} = \frac{1}{2}\overset{0}{T}{}^{00}\delta_{ij} = \frac{1}{2}\rho c^2\delta_{ij}, \tag{2.50}$$

$$\overset{1}{\mathfrak{T}}_{0i} = -\overset{1}{T}{}^{0i} = -\rho c V_i, \; \overset{2}{\mathfrak{T}}_{00} = \frac{1}{2}\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii} + 2\overset{2}{g}_{00}\overset{0}{T}{}^{00}\right). \tag{2.51}$$

In the non-relativistic case the components of the particle four-flow are $N^0 = nc$ and $N^i = nV_i$ so that we can split the time and space components of the particle four-flow as

$$N^0 = \overset{0}{N}{}^0 + \overset{2}{N}{}^0 + \mathcal{O}(c^{-4}), \qquad N^i = \overset{1}{N}{}^i + \overset{3}{N}{}^i + \mathcal{O}(c^{-5}). \tag{2.52}$$

We proceed to determine the components of the metric tensor from Einstein's field equations (2.17) by following two different methods, one will be based on the book by Weinberg [2] and the other on the paper by Chandrasekhar [3].

2.3 The solution of Einstein's field equations

2.3.1 The Weinberg method

The expressions for the components of the Ricci tensor $\overset{4}{R}_{00}$, $\overset{2}{R}_{ij}$ and $\overset{3}{R}_{0i}$, given by (2.40), (2.41) and (2.42), respectively, can be simplified through the use of the so-called harmonic coordinate conditions, which refer to the gauge conditions $g^{\mu\nu}\Gamma^{\tau}{}_{\mu\nu} = 0$.

From these gauge conditions it follows that $g^{\mu\nu}\Gamma^0{}_{\mu\nu}$ and $g^{\mu\nu}\Gamma^i{}_{\mu\nu}$ up to order $\mathcal{O}\left(c^{-3}\right)$ become

$$\frac{1}{2c}\frac{\partial \overset{2}{g}_{00}}{\partial t}+\frac{1}{2c}\frac{\partial \overset{2}{g}_{kk}}{\partial t}-\frac{\partial \overset{3}{g}_{0k}}{\partial x^k}=0, \qquad \frac{1}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}+\frac{\partial \overset{2}{g}_{ik}}{\partial x^k}-\frac{1}{2}\frac{\partial \overset{2}{g}_{kk}}{\partial x^i}=0. \tag{2.53}$$

The two above relationships were introduced by Einstein, Infeld and Hoffmann [1] in 1938 and referred as coordinate conditions.

From the differentiation of the expressions (2.53) with respect to the space and time coordinates it follows that

$$\frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{00}}{\partial t\partial x^i}+\frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial t\partial x^i}-\frac{\partial^2 \overset{3}{g}_{0k}}{\partial x^k\partial x^i}=0, \tag{2.54}$$

$$\frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{00}}{\partial t^2}+\frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial t^2}-\frac{\partial^2 \overset{3}{g}_{0k}}{\partial x^k\partial t}=0, \tag{2.55}$$

$$\frac{1}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i\partial x^j}+\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^k\partial x^j}-\frac{1}{2}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i\partial x^j}=0, \tag{2.56}$$

$$\frac{1}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i\partial t}+\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^k\partial t}-\frac{1}{2}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i\partial t}=0. \tag{2.57}$$

Now the elimination of $(\partial^2 \overset{2}{g}_{00}/\partial x^i\partial t)$ from (2.57) by the use of (2.54) leads to

$$\frac{1}{c}\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^k\partial t}-\frac{1}{c}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i\partial t}+\frac{\partial^2 \overset{3}{g}_{0k}}{\partial x^i\partial x^k}=0. \tag{2.58}$$

The sum of (2.56) with the same equation where the indexes are interchanged $i \leftrightarrow j$ yields

$$\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i\partial x^j}+\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^k\partial x^j}+\frac{\partial^2 \overset{2}{g}_{jk}}{\partial x^k\partial x^i}-\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i\partial x^j}=0. \tag{2.59}$$

With the above results it is possible to reduce the expressions for the components of the Ricci tensor. We begin with the time component (2.40) which by the use of $(2.53)_2$ and (2.55) reduces to

$$\overset{4}{R}_{00} = -\frac{1}{2}\nabla^2 \overset{4}{g}_{00} + \frac{1}{2c^2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial t^2} + \frac{\overset{2}{g}{}^{ij}}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}. \quad (2.60)$$

Furthermore, the spatial components of the Ricci tensor (2.41) and space-time components (2.42) become

$$\overset{2}{R}_{ij} = -\frac{1}{2}\nabla^2 \overset{2}{g}_{ij}, \qquad \overset{3}{R}_{i0} = -\frac{1}{2}\nabla^2 \overset{3}{g}_{0i}, \quad (2.61)$$

thanks to (2.59) and (2.58), respectively.

Now we are ready to obtain the expressions for the metric tensor components in terms of gravitational potentials. We begin by introducing the Poisson equation $\nabla^2\phi = 4\pi G\rho$, which relates the Newtonian gravitational potential ϕ with the mass density ρ and universal gravitational constant G.

Let us analyze the time component of Einstein's field equations (2.17) which together with the Ricci tensor (2.39), the energy-momentum tensor component $(2.50)_1$ and the Poisson equation lead to the following relationship

$$\overset{2}{R}_{00} = -\frac{1}{2}\nabla^2 \overset{2}{g}_{00} = -\frac{8\pi G}{c^4}\overset{0}{\mathfrak{T}}_{00} = -\frac{4\pi G\rho}{c^2} = -\frac{\nabla^2\phi}{c^2}. \quad (2.62)$$

Here we may identify the time component of the metric tensor with the Newtonian gravitational potential $\overset{2}{g}_{00} = 2\phi/c^2$. The solution of the Poisson equation for a Newtonian gravitational potential that vanishes at infinity is given by

$$\phi(\mathbf{x},t) = -G\int_V \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{2.63}$$

where the integration is extended over the entire volume V occupied by the fluid and $d^3x' = dx_1'dx_2'dx_3'$ denotes a volume element.

The spatial components of Einstein's field equations (2.17) together with the Ricci tensor $(2.61)_1$, the energy-momentum tensor components $(2.50)_2$ and the Poisson equation imply that

$$\overset{2}{R}_{ij} = -\frac{1}{2}\nabla^2\overset{2}{g}_{ij} = -\frac{8\pi G}{c^4}\overset{0}{\mathfrak{T}}_{ij} = -\frac{4\pi G\rho}{c^2}\delta_{ij} = -\frac{\nabla^2\phi}{c^2}\delta_{ij}, \tag{2.64}$$

so that the spatial components of the metric tensor are also identified with the Newtonian gravitational potential, namely $\overset{2}{g}_{ij} = (2\phi/c^2)\delta_{ij}$.

The identification of the space-time components of the metric tensor follows the same methodology by using Einstein's field equation (2.17), the Ricci tensor $(2.61)_2$ and the energy-momentum tensor components $(2.51)_1$:

$$\overset{3}{R}_{i0} = -\frac{1}{2}\nabla^2\overset{3}{g}_{0i} = -\frac{8\pi G}{c^4}\overset{1}{\mathfrak{T}}_{0i} = \frac{8\pi G}{c^4}\rho c V_i. \tag{2.65}$$

Here we identify the space-time component of the metric tensor with the vector gravitational potential ξ_i through

$$\overset{3}{g}_{0i} = -\frac{\xi_i}{c^3}, \qquad \text{so that} \qquad \nabla^2\xi_i = 16\pi G\rho V_i. \tag{2.66}$$

Furthermore, the solution of (2.65) for the vector gravitational potential ξ_i which vanishes at infinity is given by

$$\xi_i(\mathbf{x}, t) = -4G \int_V \frac{\rho(\mathbf{x}', t) V_i(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3 x'. \tag{2.67}$$

The last identification is the time component of the metric tensor of order $\mathcal{O}(c^{-4})$. Here we use Einstein's field equations (2.17) together with the Ricci tensor (2.60) and the energy-momentum tensor $(2.51)_2$ to obtain

$$\begin{aligned} \overset{4}{R}_{00} &= -\frac{1}{2}\nabla^2 \overset{4}{g}_{00} + \frac{1}{c^4}\left[\frac{\partial^2 \phi}{\partial t^2} - 2\phi\nabla^2\phi + 2\left(\nabla\phi\right)^2\right] \\ &= -\frac{8\pi G}{c^4}\overset{2}{\mathfrak{T}}_{00} = -\frac{4\pi G}{c^4}\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii} + 4\phi\rho\right), \end{aligned} \tag{2.68}$$

thanks to the following relationships $\overset{2}{g}_{00} = 2\phi/c^2$, $\overset{2}{g}{}^{ij} = -\overset{2}{g}_{ij} = -(2\phi/c^2)\delta_{ij}$ and $\overset{0}{T}{}^{00} = \rho c^2$. Now by considering the Poisson equation and the identity $\nabla^2\phi^2 = 2\phi\nabla^2\phi + 2\left(\nabla\phi\right)^2$ the above equation reduces to

$$\frac{1}{2}\nabla^2\left(\overset{4}{g}_{00} - 2\frac{\phi^2}{c^4}\right) = \frac{4\pi G}{c^4}\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}\right) + \frac{1}{c^4}\frac{\partial^2\phi}{\partial t^2}. \tag{2.69}$$

From the above equation we may identify $\overset{4}{g}_{00}$ with another scalar gravitational potential ψ through the relation

$$\overset{4}{g}_{00} = \frac{2}{c^4}(\psi + \phi^2), \quad \text{so that} \quad \nabla^2\psi = 4\pi G\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}\right) + \frac{\partial^2\phi}{\partial t^2}. \tag{2.70}$$

Since ϕ and $\overset{4}{g}_{00}$ vanish at infinity, the solution of (2.69) is given by

$$\psi(\mathbf{x},t) = -\int_V \left[G\left(\overset{2}{T}{}^{00}(\mathbf{x}',t) + \overset{2}{T}{}^{ii}(\mathbf{x}',t)\right) + \frac{1}{4\pi}\frac{\partial^2\phi(\mathbf{x}',t)}{\partial t^2}\right]\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}. \tag{2.71}$$

Once the components of the metric tensor as functions of gravitational potentials are known we can investigate the restrictions imposed by the harmonic coordinate conditions (2.53). While the first condition imposes that the gravitational potentials must obey the relation

$$\frac{1}{c^2}\left[4\frac{\partial\phi}{\partial t} + \frac{\partial\xi_i}{\partial x^i}\right] = 0, \tag{2.72}$$

the second one is identically zero.

It is interesting to note that the Laplacian of (2.72) together with the Newtonian Poisson equation $\nabla^2\phi = 4\pi G\rho$ and $(2.66)_2$ leads to

$$0 = \frac{1}{c^2}\left[4\frac{\partial\nabla^2\phi}{\partial t} + \frac{\partial\nabla^2\xi_i}{\partial x^i}\right] = \frac{16\pi G}{c^2}\left[\frac{\partial\rho}{\partial t} + \frac{\partial\rho V_i}{\partial x^i}\right], \tag{2.73}$$

where its right-hand side represents the Newtonian continuity equation, which is valid at $\mathcal{O}\left(c^{-2}\right)$ post-Newtonian level.

2.3.2 Explicit expressions for the components

Here we shall give the final expressions for the components of the metric tensor, Christoffel symbols, four-velocity, particle four-flow and energy-momentum tensor in the first post-Newtonian approximation with respect to gravitational potentials determined in the Weinberg's method.

Metric tensor and Christoffel symbols components

The expressions for the components of the metric tensor in the first post-Newtonian approximation read

$$g_{00} = 1 + \frac{2\phi}{c^2} + \frac{2}{c^4}\left(\phi^2 + \psi\right) + \mathcal{O}(c^{-6}), \tag{2.74}$$

$$g^{00} = 1 - \frac{2\phi}{c^2} + \frac{2}{c^4}\left(\phi^2 - \psi\right) + \mathcal{O}(c^{-6}), \tag{2.75}$$

$$g_{0i} = g^{0i} = -\frac{1}{c^3}\xi_i + \mathcal{O}(c^{-5}), \tag{2.76}$$

$$g_{ij} = -\left(1 - \frac{2\phi}{c^2}\right)\delta_{ij} + \mathcal{O}(c^{-4}), \tag{2.77}$$

$$g^{ij} = -\left(1 + \frac{2\phi}{c^2}\right)\delta_{ij} + \mathcal{O}(c^{-4}). \tag{2.78}$$

The substitution of the components of the metric tensor (2.74) – (2.77) into the Christoffel symbols (2.28) – (2.33) lead to the following expressions

$$\overset{3}{\Gamma}{}^0{}_{00} = \frac{1}{c^3}\frac{\partial\phi}{\partial t}, \qquad \overset{5}{\Gamma}{}^0{}_{00} = \frac{1}{c^5}\left(\frac{\partial\psi}{\partial t} + \xi_i\frac{\partial\phi}{\partial x^i}\right), \tag{2.79}$$

$$\overset{2}{\Gamma}{}^0{}_{0i} = \frac{1}{c^2}\frac{\partial\phi}{\partial x^i}, \qquad \overset{4}{\Gamma}{}^0{}_{0i} = \frac{1}{c^4}\frac{\partial\psi}{\partial x^i}, \tag{2.80}$$

$$\overset{3}{\Gamma}{}^0{}_{ij} = -\frac{1}{2c^3}\left(\frac{\partial\xi_i}{\partial x^j} + \frac{\partial\xi_j}{\partial x^i} + 2\frac{\partial\phi}{\partial t}\delta_{ij}\right), \tag{2.81}$$

$$\overset{2}{\Gamma}{}^i{}_{00} = \frac{1}{c^2}\frac{\partial\phi}{\partial x^i},\ \overset{4}{\Gamma}{}^i{}_{00} = \frac{1}{c^4}\left(2\frac{\partial\phi^2}{\partial x^i} + \frac{\partial\psi}{\partial x^i} + \frac{\partial\xi_i}{\partial t}\right), \tag{2.82}$$

$$\overset{3}{\Gamma}{}^i{}_{0j} = \frac{1}{2c^3}\left(\frac{\partial\xi_i}{\partial x^j} - \frac{\partial\xi_j}{\partial x^i} - 2\frac{\partial\phi}{\partial t}\delta_{ij}\right), \tag{2.83}$$

$$\overset{2}{\Gamma}{}^i{}_{jk} = \frac{1}{c^2}\left(\frac{\partial\phi}{\partial x^i}\delta_{jk} - \frac{\partial\phi}{\partial x^j}\delta_{ik} - \frac{\partial\phi}{\partial x^k}\delta_{ij}\right). \tag{2.84}$$

Four-velocity components

In order to determine the components of the four-velocity given by $(U^\mu) = \left(\gamma c, \gamma V^i\right)$ in first post-Newtonian approximation we note that from (2.1) and (2.74) – (2.77) we can write

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{1}{\gamma^2} = g_{00} + \frac{2}{c}g_{0i}V^i + \frac{1}{c^2}g_{ij}V^iV^j$$
$$= 1 + \frac{1}{c^2}\left(2\phi - V^2\right) + \frac{2}{c^4}\left(\psi + \phi^2 + \phi V^2 - \xi_i V^i\right). \tag{2.85}$$

If we use the approximation $1/\sqrt{1+x} \approx 1 - x/2 + 3x^2/8$ we obtain from the above equation the following expression for γ:

$$\gamma = 1 + \frac{1}{c^2}\left(\frac{V^2}{2} - \phi\right) + \frac{1}{c^4}\left(\frac{3V^4}{8} - \frac{5\phi V^2}{2} + \frac{\phi^2}{2} - \psi + \xi_i V^i\right). \tag{2.86}$$

Hence the expressions for the contravariant and covariant four-velocity components up to order $\mathcal{O}(c^{-4})$ read

$$U^0 = c\Bigg[1 + \frac{1}{c^2}\left(\frac{V^2}{2} - \phi\right) + \frac{1}{c^4}\Bigg(\frac{3V^4}{8} - \frac{5\phi V^2}{2}$$

$$+\frac{\phi^2}{2} - \psi + \xi_i V_i\Big)\Big], \tag{2.87}$$

$$U^i = V_i\Big[1 + \frac{1}{c^2}\Big(\frac{V^2}{2} - \phi\Big) + \frac{1}{c^4}\Big(\frac{3V^4}{8} - \frac{5\phi V^2}{2}$$

$$+\frac{\phi^2}{2} - \psi + \xi_i V_i\Big)\Big], \tag{2.88}$$

$$U_0 = c\Big[1 + \frac{1}{c^2}\Big(\frac{V^2}{2} + \phi\Big) + \frac{1}{c^4}\Big(\frac{3V^4}{8}$$

$$-\frac{3\phi V^2}{2} + \frac{\phi^2}{2} + \psi\Big)\Big], \tag{2.89}$$

$$U_i = -V_i\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} - 3\phi\right)\right] - \frac{\xi_i}{c^2}. \tag{2.90}$$

Particle four-flow and energy-momentum tensor components

The particle four-flow components (2.52) can now be determined from the knowledge of the four-vector components (2.87) and (2.88). Up to order $\mathcal{O}\left(c^{-2}\right)$ the components read

$$\overset{0}{N}{}^0 = nc, \qquad \overset{2}{N}{}^0 = \frac{n}{c}\left(\frac{V^2}{2} - \phi\right), \tag{2.91}$$

$$\overset{1}{N}{}^i = nV_i, \qquad \overset{3}{N}{}^i = \frac{nV_i}{c^2}\left(\frac{V^2}{2} - \phi\right). \tag{2.92}$$

In the same way it follows the energy-momentum tensor components (2.43) – (2.45) for a perfect fluid, namely

$$\overset{0}{T}{}^{00} = \rho c^2, \quad \overset{2}{T}{}^{00} = \rho\left(V^2 + \varepsilon - 2\phi\right), \quad \overset{1}{T}{}^{i0} = \rho c V_i, \quad (2.93)$$

$$\overset{3}{T}{}^{i0} = \frac{\rho V_i}{c}\left(V^2 - 2\phi + \varepsilon + \frac{p}{\rho}\right), \quad \overset{2}{T}{}^{ij} = \rho V_i V_j + p\delta_{ij}, \quad (2.94)$$

$$\overset{4}{T}{}^{ij} = \frac{\rho V_i V_j}{c^2}\left(V^2 - 2\phi + \varepsilon + \frac{p}{\rho}\right) + \frac{2\phi p}{c^2}\delta_{ij}, \quad (2.95)$$

$$\overset{4}{T}{}^{00} = \frac{\rho}{c^2}\left[V^2\left(V^2 + \varepsilon + \frac{p}{\rho} - 6\phi\right) - 2\varepsilon\phi + 2\xi_i V_i + 2\phi^2 - 2\psi\right]. \quad (2.96)$$

Here we note that for the correspondence of (2.93) – (2.95) with the equations *(9.8.4) – (9.8.6)* and *(9.8.11) – (9.8.13)* of Weinberg [2] one has to introduce the speed of light c and identify the mass density ρ of that work with $\epsilon = \rho c^2(1 + \varepsilon/c^2)$ and the vector potential ζ_i with ξ_i.

Furthermore the expressions for the tensor $\mathfrak{T}_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}T^\sigma{}_\sigma/2$ read

$$\mathfrak{T}_{00} = \frac{\rho c^2}{2} + \rho\left(V^2 + \phi + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right) + \frac{\rho}{c^2}\Big[V^4 + V^2\left(\varepsilon + \frac{p}{\rho} - 2\phi\right) + \phi\left(\varepsilon + 3\frac{p}{\rho}\right) + \phi^2 + \psi\Big], \quad (2.97)$$

$$\mathfrak{T}_{ij} = \frac{\rho c^2}{2}\delta_{ij} + \rho\left[V_i V_j + \frac{1}{2}\left(\varepsilon - \frac{p}{\rho} - 2\phi\right)\delta_{ij}\right], \quad (2.98)$$

$$\mathfrak{T}_{0i} = -\rho c V_i - \frac{\rho}{c}\left[V_i\left(V^2 + \varepsilon + \frac{p}{\rho} - 2\phi\right) + \frac{\xi_i}{2}\right]. \quad (2.99)$$

By considering the expressions for the energy-momentum tensor components, the Poisson equation (2.70) for the scalar gravitational potential ψ can be written as

$$\nabla^2\psi = 8\pi G\rho\left(V^2 - \phi + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right) + \frac{\partial^2\phi}{\partial t^2}, \qquad (2.100)$$

thanks to (2.93) and (2.94).

2.3.3 The Chandrasekhar method

Here we shall adopt the Chandrasekhar notation for the gravitational potentials and later we will identify the connection of these potentials with those obtained from the Weinberg method of the last section.

We begin to investigate the time component of Einstein's field equations (2.17) corresponding to the Ricci tensor (2.39) and energy-momentum tensor $(2.50)_1$, namely

$$\overset{2}{R}_{00} = -\frac{1}{2}\nabla^2\overset{2}{g}_{00} = -\frac{8\pi G}{c^4}\overset{0}{\mathfrak{T}}_{00} = -\frac{4\pi G\rho}{c^2} = \frac{\nabla^2 U}{c^2}. \qquad (2.101)$$

Here the Newtonian gravitational potential has an opposite sign with respect to the one of the last section, i.e., $U = -\phi$ and the Poisson equation in this case is written as $\nabla^2 U = -4\pi G\rho$. From the above equation it follows that the time component of the metric tensor is $\overset{2}{g}_{00} = -2U/c^2$.

The spatial component of Einstein's field equations (2.17) for the Ricci tensor (2.41) and energy-momentum tensor $(2.50)_2$

reads

$$\overset{2}{R}_{ij} = -\frac{1}{2}\nabla^2 \overset{2}{g}_{ij} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^j \partial x^k} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{jk}}{\partial x^i \partial x^k} = -\frac{8\pi G}{c^4}\overset{0}{\mathfrak{T}}_{ij} = -\frac{4\pi G\rho}{c^2}\delta_{ij} = \frac{\nabla^2 U}{c^2}\delta_{ij}. \quad (2.102)$$

It is easy to verify that this equation is satisfied for $\overset{2}{g}_{ij} = -(2U/c^2)\delta_{ij}$ together with $\overset{2}{g}_{00} = -2U/c^2$.

Up to now the only difference in the expressions for the components of the metric tensor in the two descriptions is the opposite sign in the Newtonian gravitational potential. A more subtle difference will appear when the others components of the metric tensor will be determined, since the gauge condition adopted by Chandrasekhar is

$$\frac{1}{2c}\frac{\partial \overset{2}{g}_{kk}}{\partial t} = \frac{\partial \overset{3}{g}_{0i}}{\partial x^i}. \quad (2.103)$$

Let us analyze the space-time component of Einstein's field equations (2.17) together with the Ricci tensor (2.42) and the energy-momentum tensor $(2.51)_1$, namely

$$\overset{3}{R}_{i0} = -\frac{1}{2}\nabla^2 \overset{3}{g}_{0i} + \frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{ij}}{\partial t \partial x^j} - \frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial t \partial x^i} + \frac{1}{2}\frac{\partial^2 \overset{3}{g}_{0j}}{\partial x^i \partial x^j} = -\frac{1}{2}\nabla^2 \overset{3}{g}_{0i} + \frac{1}{2c}\frac{\partial^2 \overset{2}{g}_{ij}}{\partial t \partial x^j} - \frac{1}{4c}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial t \partial x^i} = -\frac{1}{2}\nabla^2 \overset{3}{g}_{0i} + \frac{1}{2c^3}\frac{\partial^2 U}{\partial t \partial x^i} = -\frac{8\pi G}{c^4}\overset{1}{\mathfrak{T}}_{0i} = \frac{8\pi G}{c^4}\rho c V_i, \quad (2.104)$$

thanks to the gauge condition (2.103) and the expressions for the components of the metric tensor $\overset{2}{g}_{00} = -2U/c^2$ and $\overset{2}{g}_{ij} = -(2U/c^2)\delta_{ij}$. Now we define $\overset{3}{g}_{0i}$ in terms of a vector U_i and a scalar χ gravitational potential through the expression

$$\overset{3}{g}_{0i} = \frac{1}{c^3}\left(4U_i - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}\right), \tag{2.105}$$

where the gravitational potentials satisfy the equations

$$\nabla^2 U_i = -4\pi G\rho V_i, \qquad \nabla^2\chi = -2U. \tag{2.106}$$

Lastly we get that the time component of Einstein's field equations (2.17) together with the Ricci tensor (2.40) and the energy-momentum tensor $(2.51)_2$ lead to

$$\begin{aligned}\overset{4}{R}_{00} &= -\frac{1}{2}\nabla^2\overset{4}{g}_{00} - \frac{1}{2c^2}\frac{\partial^2\overset{2}{g}_{ii}}{\partial t^2} + \frac{1}{c}\frac{\partial^2\overset{3}{g}_{0i}}{\partial t\partial x^i} + \frac{\overset{2}{g^{ij}}}{2}\frac{\partial^2\overset{2}{g}_{00}}{\partial x^i\partial x^j}\\ &\quad + \frac{1}{4}\frac{\partial\overset{2}{g}_{00}}{\partial x^i}\frac{\partial\overset{2}{g}_{00}}{\partial x^i} + \frac{1}{4}\frac{\partial\overset{2}{g}_{jj}}{\partial x^i}\frac{\partial\overset{2}{g}_{00}}{\partial x^i} - \frac{1}{2}\frac{\partial\overset{2}{g}_{00}}{\partial x^i}\frac{\partial\overset{2}{g}_{ij}}{\partial x^j}\\ &= -\frac{1}{2}\nabla^2\overset{4}{g}_{00} - \frac{2}{c^4}\left[U\nabla^2 U - (\nabla U)^2\right] = -\frac{8\pi G}{c^4}\overset{2}{\mathfrak{T}}_{00}\\ &= -\frac{4\pi G}{c^4}\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii} - 4\rho U\right).\end{aligned} \tag{2.107}$$

Here we have used the gauge condition (2.103) and the components of the metric tensor $\overset{2}{g}_{00} = -2U/c^2$ and $\overset{2}{g^{ij}} = -\overset{2}{g}_{ij} = (2U/c^2)\delta_{ij}$.

Equation (2.107) can be simplified by considering again the following identity $\nabla^2 U^2 = 2U\nabla^2 U + 2(\nabla U)^2$ and the Poisson equation $\nabla^2 U = -4\pi G\rho$, yielding

$$\nabla^2 \left(-\frac{1}{2}\overset{4}{g}_{00} + \frac{U^2}{c^4}\right) = -\frac{4\pi G}{c^4}\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}\right). \qquad (2.108)$$

From the above equation we identify $\overset{4}{g}_{00}$ with another scalar gravitational potential Φ such that

$$\overset{4}{g}_{00} = \frac{2}{c^4}\left(U^2 - 2\Phi\right). \qquad (2.109)$$

The new scalar gravitational potential satisfies the equation

$$\nabla^2\Phi = -2\pi G\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}\right) = -4\pi G\rho\varphi. \qquad (2.110)$$

where φ represents the following abbreviation introduced in [3].

$$\varphi = \left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right). \qquad (2.111)$$

Here we can also investigate the restrictions imposed by the gauge condition (2.103) on the gravitational potentials. If we insert (2.105) and $\overset{2}{g}_{ij} = -(2U/c^2)\delta_{ij}$ into (2.103) and use $(2.106)_2$ we get

$$0 = \frac{3}{c^2}\frac{\partial U}{\partial t} + \frac{1}{c^2}\left(4\frac{\partial U_i}{\partial x^i} - \frac{1}{2}\frac{\partial \nabla^2\chi}{\partial t}\right) = \frac{4}{c^2}\left(\frac{\partial U}{\partial t} + \frac{\partial U_i}{\partial x^i}\right). \quad (2.112)$$

This equation leads also to the Newtonian continuity equation at $\mathcal{O}\left(c^{-2}\right)$ post-Newtonian level. Indeed, by taking the Laplacian of (2.112) and using the Poisson equation $\nabla^2 U = -4\pi G\rho$ and $(2.106)_1$ we get

$$\frac{1}{c^2}\nabla^2\left(\frac{\partial U}{\partial t}+\frac{\partial U_i}{\partial x_i}\right)=-\frac{4\pi G}{c^2}\left[\frac{\partial \rho}{\partial t}+\frac{\partial \rho V_i}{\partial x^i}\right]=0. \quad (2.113)$$

As it was pointed out the correspondence of the Newtonian gravitational potentials in the two methods is given by $\phi = -U$, while other relationships between the gravitational potentials follow from the comparison of (2.66) with (2.105) and (2.70) with (2.109), yielding

$$\xi_i = -4U_i + \frac{1}{2}\frac{\partial^2 \chi}{\partial t \partial x^i}, \qquad \psi = -2\Phi. \quad (2.114)$$

For the determination of the components of the metric tensor, Christoffel symbols, four-velocity, particle four-vector and energy momentum tensor in terms of the gravitational potentials of Chandrasekhar's method it is enough to use the identifications $\phi = -U$ and (2.114) in the expressions (2.74) – (2.99). Here we collect the Poisson equations in the Chandrasekhar method:

$$\nabla^2 U = -4\pi G\rho, \quad \nabla^2\Phi = -4\pi G\rho\left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right), (2.115)$$

$$\nabla^2 \Pi_i = -16\pi G\rho V_i + \frac{\partial^2 U}{\partial t \partial x^i}, (2.116)$$

where we have introduced the vector gravitational potential $\Pi_i = -\xi_i$.

Note that the gauge condition (2.112) can be also expressed as

$$4\left(\frac{\partial U}{\partial t}+\frac{\partial U_i}{\partial x^i}\right)=3\frac{\partial U}{\partial t}+\frac{\partial \Pi_i}{\partial x^i}=0. \qquad (2.117)$$

2.4 Hydrodynamic equations for an Eulerian fluid

The first post-Newtonian approximation for the Eulerian hydrodynamic equations are obtained from the balance laws for the particle four-flow (2.8) and energy-momentum tensor (2.9) by considering the expressions for the particle four-flow and energy-momentum tensor given by (2.91) – (2.96), which refer to a perfect fluid where dissipative effects are neglected.

We begin by writing the particle four-flow balance law (2.8) up to the order $\mathcal{O}(c^{-4})$, namely

$$\underline{\frac{\partial \overset{0}{N}{}^0}{\partial x^0}+\frac{\partial \overset{1}{N}{}^i}{\partial x^i}}+\frac{\partial \overset{2}{N}{}^0}{\partial x^0}+\frac{\partial \overset{3}{N}{}^i}{\partial x^i}+\left(\overset{3}{\Gamma}{}^0{}_{00}+\overset{3}{\Gamma}{}^j{}_{0j}\right)\overset{0}{N}{}^0 +\left(\overset{2}{\Gamma}{}^0{}_{i0}+\overset{2}{\Gamma}{}^j{}_{ij}\right)\overset{1}{N}{}^i=0. \qquad (2.118)$$

From the underlined terms of (2.118) together with $(2.91)_1$ and $(2.92)_1$ we get the Newtonian continuity equation

$$\frac{\partial n}{\partial t}+\frac{\partial nV_i}{\partial x^i}=0, \qquad \text{or} \qquad \frac{\partial \rho}{\partial t}+\frac{\partial \rho V_i}{\partial x^i}=0, \qquad (2.119)$$

for the particle number density n and for the mass density $\rho = mn$, where m denotes the rest mass of a fluid particle.

Now from the full equation (2.118) together with (2.91) and (2.92) it follows that

$$\frac{\partial n}{\partial t} + \frac{\partial n V_i}{\partial x^i} + \frac{1}{c^2}\left\{\frac{\partial}{\partial t}\left[n\left(\frac{V^2}{2} - \phi\right)\right]\right.$$
$$\left. + \frac{\partial}{\partial x^i}\left[nV_i\left(\frac{V^2}{2} - \phi\right)\right] - 2n\frac{\partial \phi}{\partial t} - 2nV_i\frac{\partial \phi}{\partial x^i}\right\} = 0. \quad (2.120)$$

Here we note that the two last terms of the above equation can be written as

$$-\frac{2}{c^2}\left\{n\frac{\partial \phi}{\partial t} + nV_i\frac{\partial \phi}{\partial x^i}\right\} = -\frac{2}{c^2}\left\{\frac{\partial n\phi}{\partial t} + \frac{\partial n V_i \phi}{\partial x^i}\right.$$
$$\left. -\phi\left[\frac{\partial n}{\partial t} + \frac{\partial n V_i}{\partial x^i}\right]\right\} = -\frac{2}{c^2}\left\{\frac{\partial n\phi}{\partial t} + \frac{\partial n V_i \phi}{\partial x^i}\right\} + \mathcal{O}(c^{-4}), \quad (2.121)$$

thanks to the continuity equation (2.119). Hence, (2.120) with (2.121) and $\rho = mn$ reduces to

$$\frac{\partial \rho_*}{\partial t} + \frac{\partial \rho_* V_i}{\partial x^i} = 0, \quad (2.122)$$

which is a continuity equation for the mass density ρ_* in the first post-Newtonian approximation

$$\rho_* = \rho\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} - 3\phi\right)\right]. \quad (2.123)$$

The notation for the mass density ρ_* was introduced by Fock [6] and (2.122) corresponds to eq. *(117)* of Chandrasekhar [3].

The time component of the energy-momentum tensor balance law (2.9) up to the order $\mathcal{O}(c^{-3})$ is

$$\underline{\frac{\partial \overset{0}{T}{}^{00}}{\partial x^0} + \frac{\partial \overset{1}{T}{}^{0i}}{\partial x^i}} + \frac{\partial \overset{2}{T}{}^{00}}{\partial x^0} + \frac{\partial \overset{3}{T}{}^{0i}}{\partial x^i} + \left(2\overset{3}{\Gamma}{}^0{}_{00} + \overset{3}{\Gamma}{}^j{}_{0j}\right)\overset{0}{T}{}^{00} + \left(3\overset{2}{\Gamma}{}^0{}_{i0} + \overset{2}{\Gamma}{}^j{}_{ij}\right)\overset{1}{T}{}^{0i} = 0. \tag{2.124}$$

The underlined terms of (2.124) together with (2.93) imply again in the continuity equation for the mass-energy density $(2.119)_2$, while the full equation (2.124) together with (2.93) and (2.94) leads to

$$\frac{\partial}{\partial t}\left\{\rho\left[1+\frac{1}{c^2}\left(V^2+\varepsilon-2\phi\right)\right]\right\} + \frac{\partial}{\partial x^i}\left\{\rho V_i\left[1+\frac{1}{c^2}\left(V^2 +\varepsilon-2\phi+\frac{p}{\rho}\right)\right]\right\} = \frac{\rho}{c^2}\frac{\partial \phi}{\partial t}. \tag{2.125}$$

If we identify $\epsilon = \rho c^2(1+\varepsilon/c^2)$ with ρ the above expression corresponds to eq. *(9.8.14)* of Weinberg [2]. On the other hand by introducing the mass-energy density σ defined by

$$\sigma = \rho\left[1+\frac{1}{c^2}\left(V^2+\varepsilon-2\phi+\frac{p}{\rho}\right)\right], \tag{2.126}$$

the mass-energy density hydrodynamic equation (2.125) can be written as

$$\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma V_i}{\partial x^i} = \frac{1}{c^2}\left(\rho\frac{\partial \phi}{\partial t} + \frac{\partial p}{\partial t}\right), \tag{2.127}$$

which corresponds to equation *(64)* of Chandrasekhar [3].

Up to order $\mathcal{O}(c^{-4})$ the equation for the spatial components of the energy-momentum tensor (2.9) read

$$\underline{\frac{\partial \overset{1}{T}{}^{0i}}{\partial x^0} + \frac{\partial \overset{2}{T}{}^{ij}}{\partial x^j} + \overset{2}{\Gamma}{}^i{}_{00}\overset{0}{T}{}^{00}} + \frac{\partial \overset{3}{T}{}^{0i}}{\partial x^0} + \frac{\partial \overset{4}{T}{}^{ij}}{\partial x^j} + \overset{4}{\Gamma}{}^i{}_{00}\overset{0}{T}{}^{00}$$
$$+\left[\left(\overset{2}{\Gamma}{}^0{}_{0j} + \overset{2}{\Gamma}{}^l{}_{jl}\right)\delta_{ik} + \overset{2}{\Gamma}{}^i{}_{jk}\right]\overset{2}{T}{}^{kj} + \overset{2}{\Gamma}{}^i{}_{00}\overset{2}{T}{}^{00}$$
$$+\left[2\overset{3}{\Gamma}{}^i{}_{0j} + \left(\overset{3}{\Gamma}{}^0{}_{00} + \overset{3}{\Gamma}{}^k{}_{0k}\right)\delta_{ij}\right]\overset{1}{T}{}^{0j} = 0. \qquad (2.128)$$

The underlined terms in the above equation together with (2.93) and (2.94) leads to the Newtonian momentum density hydrodynamic equation

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial\left(\rho V_i V_j + p\delta_{ij}\right)}{\partial x_j} + \rho\frac{\partial \phi}{\partial x^i} = 0. \qquad (2.129)$$

Furthermore, from the full expression of (2.128) with (2.93) – (2.95) we get

$$\frac{\partial}{\partial t}\left\{\rho V_i\left[1 + \frac{1}{c^2}\left(V^2 - 2\phi + \varepsilon + \frac{p}{\rho}\right)\right]\right\}$$
$$+\frac{\partial}{\partial x^j}\left\{\rho V_i V_j\left[1 + \frac{1}{c^2}\left(V^2 + \varepsilon - 2\phi + \frac{p}{\rho}\right)\right]\right\}$$
$$+\frac{\partial}{\partial x^i}\left[p\left(1 + \frac{2\phi}{c^2}\right)\right] + \rho\frac{\partial}{\partial x^i}\left[\phi + \frac{1}{c^2}\left(2\phi^2 + \psi\right)\right]$$
$$+\frac{\rho}{c^2}\frac{\partial \xi_i}{\partial t} - \frac{p}{c^2}\frac{\partial \phi}{\partial x^i} - 4\frac{\rho}{c^2}V_i\left(\frac{\partial \phi}{\partial t} + V_j\frac{\partial \phi}{\partial x^j}\right)$$

$$+\frac{\rho}{c^2}V_j\left(\frac{\partial\xi_i}{\partial x^j}-\frac{\partial\xi_j}{\partial x^i}\right)+\frac{\rho}{c^2}\left(2V^2+\varepsilon-2\phi\right)\frac{\partial\phi}{\partial x^i}=0, \quad (2.130)$$

which without the term ε corresponds to equation *(9.8.15)* of Weinberg [2]. This equation can be rewritten as

$$\frac{\partial\sigma V_i}{\partial t}+\frac{\partial\sigma V_iV_j}{\partial x^j}+\frac{\partial}{\partial x^i}\left[p\left(1-\frac{2\phi}{c^2}\right)\right]$$
$$+\rho\frac{\partial\phi}{\partial x^i}\left[1+\frac{2}{c^2}\left(V^2-\phi+\frac{\varepsilon}{2}+\frac{3p}{2\rho}\right)\right]+\frac{\rho}{c^2}\frac{\partial\psi}{\partial x^i}$$
$$-\frac{4\rho}{c^2}\frac{d}{dt}\left(\phi V_i-\frac{\xi_i}{4}\right)-\frac{\rho}{c^2}V_j\frac{\partial\xi_j}{\partial x^i}=0, \quad (2.131)$$

by using the definition (2.126) of σ, introducing the material time derivative $d/dt=\partial/\partial t+V_i\partial/\partial x^i$ and employing the Newtonian continuity equation $(2.119)_2$ and the Newtonian momentum hydrodynamic equation (2.129) in the terms of order $1/c^2$. The expression (2.131) corresponds to equation *(68)* of Chandrasekhar [3] if we identify ε with Π and Chandrasekhar's gravitational potentials U, Φ, U_i and χ with

$$\phi=-U, \qquad \psi\equiv-2\Phi, \qquad \xi_i=-4U_i+\frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}. \quad (2.132)$$

The momentum density hydrodynamic equation (2.131) can be rewritten as

$$\rho\frac{dV_i}{dt}+\frac{\partial p}{\partial x^i}\left[1-\frac{1}{c^2}\left(V^2-4\phi+\varepsilon+\frac{p}{\rho}\right)\right]$$
$$+\rho\frac{\partial\phi}{\partial x^i}\left[1+\frac{1}{c^2}\left(V^2+4\phi\right)\right]+\frac{\rho}{c^2}\left[\frac{\partial\psi}{\partial x^i}+\frac{d\xi_i}{dt}\right.$$

$$-V_j\frac{\partial \xi_j}{\partial x^i}+V_i\left(\frac{1}{\rho}\frac{\partial p}{\partial t}+\frac{\partial \phi}{\partial t}-4\frac{d\phi}{dt}\right)\Bigg]=0, \quad (2.133)$$

by taking into account the definition of σ given by (2.126) and the mass-energy hydrodynamic equation (2.127). If the terms of order $1/c^2$ are neglected the above equation reduces to the Newtonian momentum density hydrodynamic equation (2.129).

The hydrodynamic equation for the total energy density which is a sum of the internal $\rho\varepsilon$ and kinetic $\rho V^2/2$ energy densities can be obtained by subtracting (2.122) from (2.127), yielding

$$\frac{1}{c^2}\Bigg\{\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x^i}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)V_i\right]+p\frac{\partial V_i}{\partial x^i}$$
$$+V_i\left(\rho\frac{\partial \phi}{\partial x^i}+\frac{\partial p}{\partial x^i}\right)+\phi\underline{\left(\frac{\partial \rho}{\partial t}+\frac{\partial \rho V_i}{\partial x^i}\right)}\Bigg\}=0. \quad (2.134)$$

If we consider the Newtonian continuity equation (2.119) the underlined term vanishes and (2.134) reduces to the well-known Newtonian hydrodynamic equation for the total energy density, namely

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x^i}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)V_i\right]$$
$$+p\frac{\partial V_i}{\partial x^i}+V_i\left(\rho\frac{\partial \phi}{\partial x^i}+\frac{\partial p}{\partial x^i}\right)=0. \qquad (2.135)$$

Note that the post-Newtonian contributions do not show up in this equation. For the determination of the first post-Newtonian contributions to the total energy density we have to go further and determine the second post-Newtonian approximation. This will be the subject of the next chapter.

The hydrodynamic equation for the internal energy density of an Eulerian fluid follows from the elimination of the time derivative of the hydrodynamic velocity from (2.135) by using the Newtonian momentum density hydrodynamic equation (2.129), yielding

$$\rho\frac{d\varepsilon}{dt}+p\frac{\partial V_i}{\partial x^i}=0. \tag{2.136}$$

We call attention to the fact that all hydrodynamic equations of this section refer to Eulerian fluids, where viscous and heat-conducting effects are not taken into account. In Section 2.6 these dissipative effects will be considered.

2.5 Brans-Dicke post-Newtonian approximation

2.5.1 Brans-Dicke theory

The Brans-Dicke theory [8] is a scalar-tensor theory, where the gravitational constant is not considered as a constant but connected with an average value of a scalar field coupled with the mass density of the universe. The scalar field can vary from place to place with time and is based on Mach's principle that the inertial masses of the particles are not constants but represent interactions with the mass distribution of the Universe.

According to the discussion of Sciama [9] about Mach's principle and the dimensional analysis proposed by Dicke [10] the gravitational constant is related to the mass distribution of a uniform expanding Universe by $GM/Rc^2 \sim 1$, where M and R refer to the mass and radius of the observable Universe. The approximated values for the mass and radius of the observable Universe are $M \sim 10^{53}$ kg and $R \sim 10^{26}$ m, and the relationship $GM/Rc^2 \sim 1$ implies that $G \sim 9 \times 10^{-11}$ Nm^2/kg^2, which is very close to its present value of $G = 6.67 \times 10^{-11}$ Nm^2/kg^2.

The starting point of the Brans-Dicke theory is the Einstein-Hilbert action

$$\delta \int \left[R + \frac{16\pi G}{c^4} \mathcal{L}_M \right] \sqrt{-g} d^4x = 0, \qquad (2.137)$$

where $\mathcal{L}_M$ is the Lagrangian of the matter field. On the basis of (2.137) the following action was proposed in [8]

$$\delta \int \left[\phi R + \frac{16\pi}{c^4} \mathcal{L}_M + \frac{\omega}{\phi} \phi^{,\mu} \phi_{,\mu} \right] \sqrt{-g} d^4x = 0, \qquad (2.138)$$

where the scalar field ϕ plays the role of the inverse of the gravitational constant $1/G$ and it is assumed that the gravitational constant is a function of the scalar field. In the action (2.138) the term $\omega \phi^{,\mu} \phi_{,\mu} / \phi$ is the Lagrangian density of the scalar field and ω denotes a dimensionless coupling constant.

In order to get the field equations from the action (2.138) we start by taking the variation with respect to the scalar field ϕ, yielding

$$\int \left[R\delta\phi + 2\frac{\omega}{\phi} \delta\phi_{,\mu} \phi^{,\mu} - \frac{\omega}{\phi^2} \phi^{,\mu} \phi_{,\mu} \delta\phi \right] \sqrt{-g} d^4x = 0. \qquad (2.139)$$

The second term within the above brackets can be written as

$$2\frac{\omega}{\phi}\delta\phi_{,\mu}\phi^{,\mu}\sqrt{-g} = 2\omega\Bigg\{\left(\frac{\delta\phi}{\phi}\phi^{,\mu}\sqrt{-g}\right)_{,\mu}$$
$$+\frac{\delta\phi}{\phi}\sqrt{-g}\left(\frac{1}{\phi}\phi^{,\mu}\phi_{,\mu} - \phi^{,\mu}{}_{,\mu} - \phi^{,\mu}(\ln\sqrt{-g})_{,\mu}\right)\Bigg\}. \tag{2.140}$$

The first term within the above braces drops out, since it can be converted by the use of Gauss theorem into an integral over the hypersurface of the four-volume, where the variation of the field ϕ vanishes at the boundary. Hence (2.140) can be written as

$$2\frac{\omega}{\phi}\delta\phi_{,\mu}\phi^{,\mu}\sqrt{-g} = 2\omega\frac{\delta\phi}{\phi}\sqrt{-g}\left(\frac{1}{\phi}\phi^{,\mu}\phi_{,\mu} - \Box\phi\right), \tag{2.141}$$

where we have introduced the covariant d'Alembertian $\Box$ defined by

$$\Box\phi = \frac{1}{\sqrt{-g}}\left[\sqrt{-g}\phi^{,\mu}\right]_{,\mu} = \phi^{,\mu}{}_{;\mu} = \phi^{,\mu}{}_{,\mu}$$
$$+\phi^{,\mu}(\ln\sqrt{-g})_{,\mu} = \phi^{,\mu}{}_{,\mu} + \Gamma^{\nu}{}_{\nu\mu}\phi^{,\mu}. \tag{2.142}$$

The insertion of (2.141) into (2.139) leads to

$$\int\delta\phi\left[R - 2\frac{\omega}{\phi}\Box\phi + \frac{\omega}{\phi^2}\phi^{,\mu}\phi_{,\mu}\right]\sqrt{-g}d^4x = 0. \tag{2.143}$$

Now the condition for a stationary action implies that the term within the brackets must vanish and we have

$$2\omega\Box\phi = \phi R + \frac{\omega}{\phi}\phi^{,\mu}\phi_{,\mu}. \tag{2.144}$$

As was pointed out in [8] the expression (2.144) is a wave equation for ϕ where the right-hand terms act as sources for the generation of the ϕ waves.

The variation of the action (2.138) with respect to the metric tensor $g^{\mu\nu}$ is more involved, since we have to write the term $\phi R = \phi g^{\mu\nu} R_{\mu\nu}$ and take into account the expression (2.14) of the Ricci tensor, namely

$$
\begin{aligned}
&\delta \int \left[\phi \left(R_{\mu\nu} + \frac{\omega}{\phi^2}\phi_{,\mu}\phi_{,\nu}\right) g^{\mu\nu} + \frac{16\pi}{c^4}\mathcal{L}_M\right] \sqrt{-g} d^4x \\
&= \delta \int \left\{\phi g^{\mu\nu}\left[\Gamma^{\sigma}{}_{\mu\sigma,\nu} - \Gamma^{\sigma}{}_{\mu\nu,\sigma} + \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\rho}\right.\right. \\
&\left.\left. -\Gamma^{\sigma}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu} - \frac{\omega}{\phi^2}\phi_{,\mu}\phi_{,\nu}\right] + \frac{16\pi}{c^4}\mathcal{L}_M\right\}\sqrt{-g} d^4x \\
&= \delta \int \left\{\underline{\left(\phi g^{\mu\nu}\Gamma^{\sigma}{}_{\mu\sigma}\sqrt{-g}\right)_{,\nu}} - \underline{\left(\phi g^{\mu\nu}\Gamma^{\sigma}{}_{\mu\nu}\sqrt{-g}\right)_{,\sigma}}\right. \\
&+g^{\mu\nu}\sqrt{-g}\left[\phi\left(\Gamma^{\sigma}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\rho}\right) + \Gamma^{\sigma}{}_{\mu\nu}\phi_{,\sigma}\right. \\
&\left.\left.-\Gamma^{\sigma}{}_{\mu\sigma}\phi_{,\nu} + \frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu}\right] + \frac{16\pi}{c^4}\mathcal{L}_M\sqrt{-g}\right\} d^4x = 0. \quad (2.145)
\end{aligned}
$$

The two underlined terms in the above equation drop out if we use Gauss theorem to convert them into integrals over the hypersurface of the four volume where the variation of the fields vanish. Here we follow Dirac [11] and analyze separately the next two terms of the last equality. We begin with the variation of the first term of (2.145) which can be transformed into

$$\begin{aligned}
\delta\left(g^{\mu\nu}\sqrt{-g}\Gamma^{\sigma}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu}\right) &= \Gamma^{\rho}{}_{\mu\nu}\delta\Big(g^{\mu\nu}\sqrt{-g}\Gamma^{\sigma}{}_{\sigma\rho}\Big) \\
+g^{\mu\nu}\sqrt{-g}\Gamma^{\sigma}{}_{\sigma\rho}\delta\Gamma^{\rho}{}_{\mu\nu} &= \Gamma^{\rho}{}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}{}_{,\rho}\right) \\
+\Gamma^{\sigma}{}_{\sigma\rho}\delta\left(g^{\mu\nu}\sqrt{-g}\Gamma^{\rho}{}_{\mu\nu}\right) &- \Gamma^{\sigma}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right) \\
&= \Gamma^{\rho}{}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}{}_{,\rho}\right) - \Gamma^{\sigma}{}_{\sigma\rho}\delta\left(g^{\rho\nu}\sqrt{-g}\right)_{,\nu} \\
&-\Gamma^{\sigma}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right), \qquad (2.146)
\end{aligned}$$

where we have used the relationships (2.332) and (2.334) of the Appendix. The variation of the second term of (2.145) by taking into account (2.334) can be written as

$$\begin{aligned}
\delta\left(g^{\mu\nu}\sqrt{-g}\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma}\right) &= 2g^{\mu\nu}\sqrt{-g}\Gamma^{\rho}{}_{\sigma\nu}\delta\Gamma^{\sigma}{}_{\mu\rho} \\
+\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma}\delta\left(g^{\mu\nu}\sqrt{-g}\right) &= 2\delta\left(g^{\mu\nu}\sqrt{-g}\Gamma^{\sigma}{}_{\mu\rho}\right)\Gamma^{\rho}{}_{\sigma\nu} \\
-\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma}\delta\left(g^{\mu\nu}\sqrt{-g}\right) &= -\delta\left(g^{\sigma\nu}{}_{,\rho}\sqrt{-g}\right)\Gamma^{\rho}{}_{\sigma\nu} \\
&-\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma}\delta\left(g^{\mu\nu}\sqrt{-g}\right). \qquad (2.147)
\end{aligned}$$

The variation of the third and fourth terms of (2.145) can be transformed into

$$\begin{aligned}
&\delta\left(\Gamma^{\sigma}{}_{\mu\nu}g^{\mu\nu}\sqrt{-g}\right)\phi_{,\sigma} - \delta\left(\Gamma^{\sigma}{}_{\mu\sigma}g^{\mu\nu}\sqrt{-g}\right)\phi_{,\nu} \\
&= -\delta\Big[\Big(g^{\mu\sigma}\sqrt{-g}\Big)_{,\mu}\phi_{,\sigma}\Big] - \delta\Big(\phi^{,\mu}\sqrt{-g}_{,\mu}\Big) = \delta\Big(\sqrt{-g}\Big)\phi^{,\sigma}{}_{,\sigma} \\
&-2\delta\Big(\phi^{\mu}\sqrt{-g}\Big)_{,\mu} + \delta\Big(g^{\mu\nu}\sqrt{-g}\Big)\phi_{,\mu,\nu}, \qquad (2.148)
\end{aligned}$$

where the relationships (2.332) and (2.334) of the Appendix were taken into account. The variation of the last term of (2.145) is

$$\delta\left(\frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}\sqrt{-g}\right)=\frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu}\delta\left(g^{\mu\nu}\sqrt{-g}\right). \qquad (2.149)$$

Now by collecting all terms (2.146) – (2.149) the action (2.145) can be written as

$$\int\left\{\underline{\left(\phi\Gamma^{\rho}{}_{\mu\nu}\delta\sqrt{-g}g^{\mu\nu}\right)_{,\rho}}-\underline{\left(\phi\Gamma^{\rho}{}_{\rho\mu}\delta\sqrt{-g}g^{\mu\nu}\right)_{,\nu}}\right.$$
$$-\underline{2\delta\left(\phi^{\mu}\sqrt{-g}\right)_{,\mu}}+\left[\phi R_{\mu\nu}+\phi_{,\mu,\nu}-\Gamma^{\rho}{}_{\mu\nu}\phi_{,\rho}\right.$$
$$\left.+\frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu}\right]\delta\left(\sqrt{-g}g^{\mu\nu}\right)+\left(\phi^{,\sigma}{}_{,\sigma}+\Gamma^{\rho}{}_{\rho\mu}\phi^{,\mu}\right)\delta\left(\sqrt{-g}\right)$$
$$\left.+\frac{16\pi}{c^{4}}\frac{\delta\mathcal{L}_{M}\sqrt{-g}}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\right\}d^{4}x=0, \qquad (2.150)$$

where we have introduced the expression for the Ricci tensor $R_{\mu\nu}$ given by (2.14) and rearranged the terms in order to get the two first perfect differentials. The underlined terms drop out, since again we can use Gauss theorem to transform them into integrals over the hypersurface of the four volume where the variation of the fields vanish.

By taking into account the variations (2.337) and (2.338) in the Appendix the action obtained from (2.150) becomes

$$\int\phi\left\{\left[R_{\mu\nu}+\frac{\omega}{\phi^{2}}\phi_{,\mu}\phi_{,\nu}+\frac{1}{\phi}\phi_{,\mu;\nu}\right]\left(\frac{1}{2}g^{\mu\nu}g^{\sigma\tau}-g^{\mu\sigma}g^{\nu\tau}\right)\right.$$
$$\left.+\frac{1}{2\phi}\phi^{,\mu}{}_{;\mu}g^{\sigma\tau}+\frac{16\pi}{c^{4}\phi\sqrt{-g}}\frac{\delta\mathcal{L}_{M}\sqrt{-g}}{\delta g_{\sigma\tau}}\right\}\delta g_{\sigma\tau}\sqrt{-g}d^{4}x=0, \qquad (2.151)$$

where we have introduced the covariant derivatives

$$\phi_{,\mu;\nu} = \phi_{,\mu,\nu} - \Gamma^{\rho}{}_{\mu\nu}\phi_{,\rho}, \qquad \phi^{,\mu}{}_{;\mu} = \phi^{,\mu}{}_{,\mu} + \Gamma^{\rho}{}_{\rho\mu}\phi^{,\mu}. \tag{2.152}$$

The expression within the braces of (2.151) must vanish due to the stationary condition of the action and we get the modification of Einstein's field equations proposed by Brans-Dicke [8]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi}{c^4\phi}T_{\mu\nu} - \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\phi_{,\sigma}\phi^{,\sigma}g_{\mu\nu}\right)$$
$$-\frac{1}{\phi}\left(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi\right), \tag{2.153}$$

where the energy-momentum tensor of the matter field is defined by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M\sqrt{-g}}{\delta g_{\mu\nu}}. \tag{2.154}$$

The covariant divergence of the energy-momentum tensor of the matter field vanishes, which can be seen by the multiplication of the Brans-Dicke equations (2.153) by ϕ and taking the covariant divergence of the resulting equation, yielding

$$\phi\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)_{;\nu} + \phi_{,\nu}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) = -\frac{8\pi}{c^4}T^{\mu\nu}{}_{;\nu}$$
$$+\frac{\omega}{2\phi^2}\phi^{,\mu}\phi^{,\nu}\phi_{,\nu} - \frac{\omega}{\phi}\phi^{,\mu}\phi^{,\nu}{}_{;\nu} - \phi^{,\mu;\nu}{}_{;\nu} + \phi^{,\nu}{}_{;\nu}{}^{;\mu}. \tag{2.155}$$

By using the Bianchi identity

$$\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)_{;\nu} = 0, \tag{2.156}$$

the property of the curvature tensor

$$R^{\mu\nu}\phi_{\nu} = \phi^{,\nu}{}_{;\nu}{}^{;\mu} - \phi^{\nu;\mu}{}_{;\nu}, \tag{2.157}$$

and the relationship that follows from (2.144)

$$\frac{1}{2}\phi^{,\mu}R = \phi^{,\mu}\left(\frac{\omega}{\phi}\phi^{\nu}{}_{;\nu} - \frac{\omega}{2\phi^2}\phi^{,\nu}\phi_{,\nu}\right), \tag{2.158}$$

it follows from (2.155) the vanishing covariant divergence of the energy-momentum tensor of the matter field, namely $T^{\mu\nu}{}_{;\nu} = 0$.

If we take the trace of the Brans-Dicke equations (2.153) it follows

$$R = \frac{8\pi}{c^4\phi}T - \frac{\omega}{\phi^2}\phi^{,\mu}\phi_{,\mu} - \frac{3}{\phi}\Box\phi, \tag{2.159}$$

where $T = T^{\mu}{}_{\mu}$ is the trace of the energy-momentum tensor of the matter field. Now by making use of (2.144) to eliminate the scalar curvature R a new wave equation emerges

$$\Box\phi = \frac{8\pi}{(3+2\omega)c^4}T. \tag{2.160}$$

Another expression for the Brans-Dicke field equations is obtained from the elimination of the scalar curvature from (2.153) by the use of (2.159), yielding

$$R_{\mu\nu} = -\frac{8\pi}{c^4\phi}\left[T_{\mu\nu} - \frac{1+\omega}{3+2\omega}Tg_{\mu\nu}\right] - \frac{\omega}{\phi^2}\phi_{,\mu}\phi_{,\nu} - \frac{\phi_{,\mu;\nu}}{\phi}. \tag{2.161}$$

2.5.2 Post-Newtonian Brans-Dicke theory

The post-Newtonian hydrodynamic equations in the theory of Brans-Dicke were determined by Nutku [12] by following the method of Chandrasekhar described in Section 2.3.3. In his book Weinberg [2] analyzed also the Brans-Dicke theory within the post-Newtonian approximation. Here we shall follow Weinberg's method described in Section 2.3.1, but instead of using the harmonic coordinate conditions we shall use another gauge condition proposed by Brans and Dicke [8], which will be introduced bellow.

In order to distinguish the Brans-Dicke scalar field ϕ from the Newtonian gravitational potential ϕ of Section 2.3.1, we follow Weinberg [2] and write the Brans-Dicke scalar field as $\phi = (1+\zeta)/\mathcal{G}$, where ζ is a new scalar field and $\mathcal{G}$ a constant of order of the gravitational constant G. In terms of the scalar field ζ the wave equation (2.160) and the Brans-Dicke (2.161) equations can be rewritten as

$$\Box\zeta = \frac{8\pi\mathcal{G}}{(3+2\omega)c^4}T, \tag{2.162}$$

$$R_{\mu\nu} + \frac{\omega\zeta_{,\mu}\zeta_{,\nu}}{(1+\zeta)^2} + \frac{\zeta_{,\mu;\nu}}{1+\zeta} = -\frac{8\pi\mathcal{G}}{c^4(1+\zeta)}\mathfrak{T}_{\mu\nu}. \tag{2.163}$$

Here the energy-momentum tensor of the matter field $\mathfrak{T}_{\mu\nu}$ is defined by

$$\mathfrak{T}_{\mu\nu} = T_{\mu\nu} - \frac{1+\omega}{3+2\omega}Tg_{\mu\nu}. \tag{2.164}$$

First we note that the lowest order of the trace of the energy-momentum tensor of the matter field is $T = \overset{0}{T}{}^{00} = \frac{1}{2}\rho c^2$, so that we infer from (2.162) that the expansion of ζ should be given by

$$\zeta = \overset{2}{\zeta} + \overset{4}{\zeta} + \mathcal{O}\left(c^{-6}\right). \tag{2.165}$$

The determination of $\overset{2}{\zeta}$ follows from (2.162) which reduces to

$$\nabla^2 \overset{2}{\zeta} = -\frac{8\pi\mathcal{G}}{(3+2\omega)c^4}\overset{0}{T}{}^{00} = -\frac{8\pi\mathcal{G}}{(3+2\omega)c^2}\rho. \tag{2.166}$$

The components of the energy-momentum tensor of the matter field $\mathfrak{T}_{\mu\nu}$ in the first orders read

$$\overset{0}{\mathfrak{T}}_{00} = \frac{2+\omega}{3+2\omega}\rho c^2,\ \overset{0}{\mathfrak{T}}_{ij} = \frac{1+\omega}{3+2\omega}\rho c^2\delta_{ij},\ \overset{1}{\mathfrak{T}}_{0i} = -\rho c V_i, \tag{2.167}$$

$$\overset{2}{\mathfrak{T}}_{00} = \frac{2+\omega}{3+2\omega}\overset{2}{T}{}^{00} + \frac{1+\omega}{3+2\omega}\overset{2}{T}{}^{ii} + \frac{4+2\omega}{3+2\omega}\overset{2}{g}_{00}\overset{0}{T}{}^{00}, \tag{2.168}$$

thanks to (2.164) and (2.49).

To solve the Brans-Dicke field equations (2.163) we need to know the product $\zeta_{,\mu}\zeta_{,\nu}$ and the covariant derivative $\zeta_{,\mu;\nu}$ in the first post-Newtonian approximation. The product of the components are of orders

$$\overset{2}{\zeta}_{,0}\overset{2}{\zeta}_{,0} = \mathcal{O}\left(c^{-6}\right),\ \overset{2}{\zeta}_{,0}\overset{2}{\zeta}_{,i} = \mathcal{O}\left(c^{-5}\right),\ \overset{2}{\zeta}_{,i}\overset{2}{\zeta}_{,j} = \mathcal{O}\left(c^{-4}\right), \tag{2.169}$$

and they do not contribute to the Brans-Dicke field equations (2.163) in the first post-Newtonian approximation. The post-

Newtonian orders of the covariant derivative components read

$$\overset{2}{\zeta}_{,0;0} = \frac{1}{c^2}\frac{\partial^2 \overset{2}{\zeta}}{\partial t^2} - \overset{2}{\Gamma}{}^i{}_{00}\frac{\partial \overset{2}{\zeta}}{\partial x^i}, \quad \overset{2}{\zeta}_{,0;i} = \frac{1}{c}\frac{\partial^2 \overset{2}{\zeta}}{\partial t \partial x^i}, \tag{2.170}$$

$$\overset{2}{\zeta}_{,i;j} = \frac{\partial^2 \overset{2}{\zeta}}{\partial x^i \partial x^j} - \overset{2}{\Gamma}{}^k{}_{ij}\frac{\partial \overset{2}{\zeta}}{\partial x^k}. \tag{2.171}$$

Now we can write the components of the Brans-Dicke field equations (2.163) in the post-Newtonian approximation by making use of the expressions for the Ricci tensor (2.39) – (2.42) for the energy-momentum tensor components (2.167) and (2.168) and for the covariant derivatives (2.170) and (2.171). They read

$$-\frac{1}{2}\nabla^2 \overset{2}{g}_{00} = -\frac{8\pi\mathcal{G}}{c^4}\overset{0}{\mathfrak{T}}_{00} = -\frac{8\pi\mathcal{G}}{c^2}\frac{2+\omega}{3+2\omega}\rho, \tag{2.172}$$

$$\begin{aligned}
&-\frac{1}{2}\nabla^2 \overset{4}{g}_{00} - \frac{1}{2c^2}\frac{\partial^2 \overset{2}{g}_{ii}}{\partial t^2} + \frac{1}{c}\frac{\partial^2 \overset{3}{g}_{0i}}{\partial t \partial x^i} + \frac{\overset{2}{g}{}^{ij}}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j} \\
&\quad + \frac{1}{4}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}\frac{\partial \overset{2}{g}_{00}}{\partial x^i} + \frac{1}{4}\frac{\partial \overset{2}{g}_{jj}}{\partial x^i}\frac{\partial \overset{2}{g}_{00}}{\partial x^i} - \frac{1}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}\frac{\partial \overset{2}{g}_{ij}}{\partial x^j} \\
&\quad + \frac{1}{c^2}\frac{\partial^2 \overset{2}{\zeta}}{\partial t^2} - \overset{2}{\Gamma}{}^i{}_{00}\frac{\partial \overset{2}{\zeta}}{\partial x^i} = -\frac{8\pi\mathcal{G}}{c^4}\overset{2}{\mathfrak{T}}_{00} = -\frac{8\pi\mathcal{G}}{c^4}\left[\frac{2+\omega}{3+2\omega}\overset{2}{T}{}^{00}\right. \\
&\quad \left. + \frac{1+\omega}{3+2\omega}\overset{2}{T}{}^{ii} + \frac{3+\omega}{3+2\omega}\overset{2}{g}_{00}\overset{0}{T}{}^{00}\right],
\end{aligned} \tag{2.173}$$

$$\begin{aligned}
&-\frac{1}{2}\nabla^2 \overset{2}{g}_{ij} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^j \partial x^k} \\
&\quad + \frac{1}{2}\frac{\partial^2 \overset{2}{g}_{jk}}{\partial x^i \partial x^k} + \frac{\partial^2 \overset{2}{\zeta}}{\partial x^i \partial x^j} - \overset{2}{\Gamma}{}^k{}_{ij}\frac{\partial \overset{2}{\zeta}}{\partial x^k}
\end{aligned}$$

$$= -\frac{8\pi\mathcal{G}}{c^4}\overset{0}{\mathfrak{T}}_{ij} = -\frac{8\pi\mathcal{G}}{c^2}\frac{1+\omega}{3+2\omega}\rho\delta_{ij}, \tag{2.174}$$

$$-\frac{1}{2}\nabla^2\overset{3}{g}_{0i} + \frac{1}{2c}\frac{\partial^2\overset{2}{g}_{ij}}{\partial t\partial x^j} - \frac{1}{2c}\frac{\partial^2\overset{2}{g}_{kk}}{\partial t\partial x^i} + \frac{1}{2}\frac{\partial^2\overset{3}{g}_{0j}}{\partial x^i\partial x^j} + \frac{1}{c}\frac{\partial^2\overset{2}{\zeta}}{\partial t\partial x^i}$$
$$= -\frac{8\pi\mathcal{G}}{c^4}\overset{1}{\mathfrak{T}}_{0i} = \frac{8\pi\mathcal{G}}{c^3}\rho V_i. \tag{2.175}$$

As was previously pointed out, in the Brans-Dicke theory [8] the harmonic coordinate conditions are modified and written as $g^{\mu\nu}\Gamma^{\tau}{}_{\mu\nu} = \partial\zeta/\partial x_\tau$, which are now the new gauge conditions. The time and space components of the gauge conditions up to order $\mathcal{O}\left(c^{-3}\right)$ are

$$\frac{1}{2c}\frac{\partial\overset{2}{g}_{00}}{\partial t} + \frac{1}{2c}\frac{\partial\overset{2}{g}_{kk}}{\partial t} - \frac{\partial\overset{3}{g}_{0k}}{\partial x^k} = \frac{1}{c}\frac{\partial\overset{2}{\zeta}}{\partial t}, \tag{2.176}$$

$$\frac{1}{2}\frac{\partial\overset{2}{g}_{kk}}{\partial x^i} - \frac{1}{2}\frac{\partial\overset{2}{g}_{00}}{\partial x^i} - \frac{\partial\overset{2}{g}_{ik}}{\partial x^k} = \frac{\partial\overset{2}{\zeta}}{\partial x^i}. \tag{2.177}$$

The derivatives of the above expressions with respect to the space and time coordinates become

$$\frac{1}{2c}\frac{\partial^2\overset{2}{g}_{00}}{\partial t\partial x^i} + \frac{1}{2c}\frac{\partial^2\overset{2}{g}_{kk}}{\partial t\partial x^i} - \frac{\partial^2\overset{3}{g}_{0k}}{\partial x^k\partial x^i} = \frac{1}{c}\frac{\partial^2\overset{2}{\zeta}}{\partial t\partial x^i}, \tag{2.178}$$

$$\frac{1}{2c}\frac{\partial^2\overset{2}{g}_{00}}{\partial t^2} + \frac{1}{2c}\frac{\partial^2\overset{2}{g}_{kk}}{\partial t^2} - \frac{\partial^2\overset{3}{g}_{0k}}{\partial x^k\partial t} = \frac{1}{c}\frac{\partial^2\overset{2}{\zeta}}{\partial t^2}, \tag{2.179}$$

$$\frac{1}{2}\frac{\partial^2\overset{2}{g}_{kk}}{\partial x^i\partial x^j} - \frac{1}{2}\frac{\partial^2\overset{2}{g}_{00}}{\partial x^i\partial x^j} - \frac{\partial^2\overset{2}{g}_{ik}}{\partial x^k\partial x^j} = \frac{\partial^2\overset{2}{\zeta}}{\partial x^i\partial x^j}, \tag{2.180}$$

$$\frac{1}{2}\frac{\partial^2\overset{2}{g}_{kk}}{\partial x^i\partial t} - \frac{1}{2}\frac{\partial^2\overset{2}{g}_{00}}{\partial x^i\partial t} - \frac{\partial^2\overset{2}{g}_{ik}}{\partial x^k\partial t} = \frac{\partial^2\overset{2}{\zeta}}{\partial x^i\partial t}. \tag{2.181}$$

Now we eliminate $(\partial^2 \overset{2}{g}_{00}/\partial x^i \partial t)$ from (2.178) by using (2.181) and get

$$\frac{1}{c}\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i \partial t} - \frac{1}{c}\frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^k \partial t} - \frac{\partial^2 \overset{3}{g}_{0k}}{\partial x^i \partial x^k} = \frac{2}{c}\frac{\partial^2 \overset{2}{\zeta}}{\partial t \partial x^i}, \tag{2.182}$$

furthermore the sum of (2.180) with the same equation where the indexes are interchanged $i \leftrightarrow j$ leads to

$$\frac{\partial^2 \overset{2}{g}_{kk}}{\partial x^i \partial x^j} - \frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j} - \frac{\partial^2 \overset{2}{g}_{ik}}{\partial x^k \partial x^j} - \frac{\partial^2 \overset{2}{g}_{jk}}{\partial x^k \partial x^i} = 2\frac{\partial^2 \overset{2}{\zeta}}{\partial x^i \partial x^j}. \tag{2.183}$$

If we take into account (2.177), (2.179), (2.182) and (2.183) the Brans-Dicke field equations (2.173), (2.174) and (2.175) reduce to

$$-\frac{1}{2}\nabla^2 \overset{4}{g}_{00} + \frac{1}{2c^2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial t^2} + \frac{\overset{2}{g}{}^{ij}}{2}\frac{\partial^2 \overset{2}{g}_{00}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}\frac{\partial \overset{2}{g}_{00}}{\partial x^i}$$
$$= -\frac{8\pi\mathcal{G}}{c^4}\left[\frac{2+\omega}{3+2\omega}\overset{2}{T}{}^{00} + \frac{1+\omega}{3+2\omega}\overset{2}{T}{}^{ii} + \frac{4+2\omega}{3+2\omega}\overset{2}{g}_{00}\overset{0}{T}{}^{00}\right], \tag{2.184}$$

$$\frac{1}{2}\nabla^2 \overset{2}{g}_{ij} = \frac{8\pi\mathcal{G}}{c^2}\frac{1+\omega}{3+2\omega}\rho\delta_{ij}, \tag{2.185}$$

$$\frac{1}{2}\nabla^2 \overset{3}{g}_{0i} = -\frac{8\pi\mathcal{G}}{c^3}\rho V_i. \tag{2.186}$$

Let us find the solutions of (2.172) and (2.184) – (2.186) for the components of the metric tensor. We begin with the determination of $\overset{2}{g}_{00}$ from (2.172) which can be rewritten as

$$\frac{1}{2}\nabla^2 \overset{2}{g}_{00} = \frac{8\pi\mathcal{G}}{c^2}\frac{2+\omega}{3+2\omega}\rho = \frac{4\pi\mathbb{G}}{c^2}\rho, \tag{2.187}$$

where we have introduced a new gravitational constant

$$\mathbb{G} = \frac{4 + 2\omega}{3 + 2\omega}\mathcal{G}. \tag{2.188}$$

Hence one may identify from (2.187) the usual relationship between $\overset{2}{g}_{00}$ and the Newtonian potential ϕ through

$$\overset{2}{g}_{00} = \frac{2\phi}{c^2}, \qquad \text{so that} \qquad \nabla^2\phi = 4\pi\mathbb{G}\rho. \tag{2.189}$$

For the determination of $\overset{2}{g}_{ij}$ we write (2.185) as

$$\frac{1}{2}\nabla^2\overset{2}{g}_{ij} = \frac{1 + \omega}{2 + \omega}\frac{\nabla^2\phi}{c^2}\delta_{ij}, \text{ so that } \overset{2}{g}_{ij} = \frac{2 + 2\omega}{2 + \omega}\frac{\phi}{c^2}\delta_{ij}. \tag{2.190}$$

The equation (2.186) for $\overset{3}{g}_{0i}$ can be written as

$$\nabla^2\overset{3}{g}_{0i} = -\frac{16\pi\mathbb{G}}{c^3}\frac{3 + 2\omega}{4 + 2\omega}\rho V_i, \tag{2.191}$$

where we can identify a vector gravitational potential ξ_i by

$$\overset{3}{g}_{0i} = -\frac{3 + 2\omega}{4 + 2\omega}\frac{\xi_i}{c^3} \qquad \text{so that} \qquad \nabla^2\xi_i = 16\pi\mathbb{G}\rho V_i. \tag{2.192}$$

The determination of $\overset{4}{g}_{00}$ is more involved. We insert the values of $\overset{2}{g}_{00} = 2\phi/c^2$, $\overset{0}{T}{}^{00} = \rho c^2$ into (2.184), use the identity $\nabla^2\phi^2 = 2\phi\nabla^2\phi + 2(\nabla\phi)^2$ and after some rearrangements we arrive at

$$\frac{1}{2}\nabla^2\left(\overset{4}{g}_{00} - \frac{2\phi^2}{c^4}\right) = \frac{4\pi\mathbb{G}}{c^4}\frac{3 + 2\omega}{2 + \omega}\left[\frac{4 + 2\omega}{3 + 2\omega}\overset{2}{T}{}^{00} + \frac{2 + 2\omega}{3 + 2\omega}\overset{2}{T}{}^{ii} + \frac{2\phi\rho}{3 + 2\omega}\right] + \frac{1}{c^4}\frac{\partial^2\phi}{\partial t^2}. \tag{2.193}$$

Here we identify the scalar gravitational potential ψ through

$$\overset{4}{g}_{00} = \frac{2}{c^4}\left[\frac{3+2\omega}{4+2\omega}\psi + \phi^2\right], \qquad \text{so that} \tag{2.194}$$

$$\nabla^2\psi = 4\pi\mathbb{G}\left[\frac{4+2\omega}{3+2\omega}\overset{2}{T}{}^{00} + \frac{2+2\omega}{3+2\omega}\overset{2}{T}{}^{ii} + \frac{2\phi\rho}{3+2\omega}\right] + \frac{1}{c^4}\frac{\partial^2\phi}{\partial t^2}. \tag{2.195}$$

The last identification refers to the scalar field $\overset{2}{\zeta}$ from (2.166), which can be rewritten, thanks to (2.188) and (2.189), as

$$\nabla^2\overset{2}{\zeta} = -\frac{1}{(2+\omega)c^2}\nabla^2\phi, \quad \text{so that} \quad \overset{2}{\zeta} = -\frac{\phi}{(2+\omega)c^2}. \tag{2.196}$$

From the investigation of the gauge condition (2.176) it follows that

$$0 = \frac{1}{c^3}\frac{3+2\omega}{2+\omega}\left[4\frac{\partial\nabla^2\phi}{\partial t} + \frac{\partial\nabla^2\xi_i}{\partial x^i}\right] = \frac{16\pi\mathbb{G}}{c^3}\frac{3+2\omega}{2+\omega}\left[\frac{\partial\rho}{\partial t} + \frac{\partial\rho V_i}{\partial x^i}\right], \tag{2.197}$$

which is equivalent to (2.73) and in its right-hand side the Newtonian continuity equation shows up, which is valid at $\mathcal{O}\left(c^{-3}\right)$ post-Newtonian level. Furthermore the gauge condition (2.177) is identically verified.

Explicit expressions for the components

As in Section 2.3.2 we shall give here the final expressions for the components of the metric tensor, Christoffel symbols, four-velocity, particle four-vector and energy momentum tensor in the first post-Newtonian approximation of the Brans-Dicke theory.

The metric tensor components in the first post-Newtonian approximation read

$$g_{00} = 1 + \frac{2\phi}{c^2} + \frac{2}{c^4}\left(\frac{3+2\omega}{4+2\omega}\psi + \phi^2\right) + \mathcal{O}(c^{-6}), \quad (2.198)$$

$$g_{0i} = -\frac{3+2\omega}{4+2\omega}\frac{\xi_i}{c^3} + \mathcal{O}(c^{-5}), \quad (2.199)$$

$$g_{ij} = -\left(1 - \frac{2+2\omega}{2+\omega}\frac{\phi}{c^2}\right)\delta_{ij} + \mathcal{O}(c^{-4}). \quad (2.200)$$

The Christoffel symbols (2.28) – (2.33) obtained from the substitution of the components of the metric tensor (2.198) – (2.200) become

$$\overset{3}{\Gamma}{}^0{}_{00} = \frac{1}{c^3}\frac{\partial\phi}{\partial t}, \qquad \overset{5}{\Gamma}{}^0{}_{00} = \frac{1}{c^5}\frac{3+2\omega}{4+2\omega}\left(\frac{\partial\psi}{\partial t} + \xi_i\frac{\partial\phi}{\partial x^i}\right), \quad (2.201)$$

$$\overset{2}{\Gamma}{}^0{}_{0i} = \frac{1}{c^2}\frac{\partial\phi}{\partial x^i}, \qquad \overset{4}{\Gamma}{}^0{}_{0i} = \frac{1}{c^4}\frac{3+2\omega}{4+2\omega}\frac{\partial\psi}{\partial x^i}, \quad (2.202)$$

$$\overset{3}{\Gamma}{}^0{}_{ij} = -\frac{1}{2c^3}\left[\frac{3+2\omega}{4+2\omega}\left(\frac{\partial\xi_i}{\partial x^j} + \frac{\partial\xi_j}{\partial x^i}\right) + \frac{2+2\omega}{2+\omega}\frac{\partial\phi}{\partial t}\delta_{ij}\right], \quad (2.203)$$

$$\overset{2}{\Gamma}{}^i{}_{00} = \frac{1}{c^2}\frac{\partial\phi}{\partial x^i}, \qquad \overset{4}{\Gamma}{}^i{}_{00} = \frac{1}{c^4}\frac{3+2\omega}{4+2\omega}\left(2\frac{\partial\phi^2}{\partial x^i} + \frac{\partial\psi}{\partial x^i} + \frac{\partial\xi_i}{\partial t}\right), \quad (2.204)$$

$$\overset{3}{\Gamma}{}^i{}_{0j} = \frac{1}{2c^3}\left[\frac{3+2\omega}{4+2\omega}\left(\frac{\partial\xi_i}{\partial x^j} - \frac{\partial\xi_j}{\partial x^i}\right) - \frac{2+2\omega}{2+\omega}\frac{\partial\phi}{\partial t}\delta_{ij}\right], \quad (2.205)$$

$$\overset{2}{\Gamma}{}^{i}{}_{jk} = \frac{1}{c^2}\frac{1+\omega}{2+\omega}\left(\frac{\partial\phi}{\partial x^i}\delta_{jk} - \frac{\partial\phi}{\partial x^j}\delta_{ik} - \frac{\partial\phi}{\partial x^k}\delta_{ij}\right). \tag{2.206}$$

The expressions for the four-velocity components up to order $\mathcal{O}(c^{-4})$ are

$$U^0 = c\left\{1 + \frac{1}{c^2}\left(\frac{V^2}{2} - \phi\right) + \frac{1}{c^4}\left[\frac{3V^4}{8} + \frac{\phi^2}{2} - \frac{8+5\omega}{4+2\omega}\phi V^2 - \frac{3+2\omega}{4+2\omega}\left(\psi - \xi_i V_i\right)\right]\right\}, \tag{2.207}$$

and $U^i = U^0 V_i / c$.

The particle four-flow components up to order $\mathcal{O}\left(c^{-2}\right)$ are the same as the ones given in (2.91) and (2.92), namely

$$\overset{0}{N}{}^0 = nc, \qquad \overset{2}{N}{}^0 = \frac{n}{c}\left(\frac{V^2}{2} - \phi\right), \tag{2.208}$$

$$\overset{1}{N}{}^i = nV_i, \qquad \overset{3}{N}{}^i = \frac{nV_i}{c^2}\left(\frac{V^2}{2} - \phi\right), \tag{2.209}$$

while the energy-momentum tensor components read

$$\overset{0}{T}{}^{00} = \rho c^2, \quad \overset{2}{T}{}^{00} = \rho\left(V^2 + \varepsilon - 2\phi\right), \quad \overset{1}{T}{}^{i0} = \rho c V_i, \tag{2.210}$$

$$\overset{3}{T}{}^{i0} = \frac{\rho V_i}{c}\left(V^2 - 2\phi + \varepsilon + \frac{p}{\rho}\right), \; \overset{2}{T}{}^{ij} = \rho V_i V_j + p\delta_{ij}, \tag{2.211}$$

$$\overset{4}{T}{}^{ij} = \frac{\rho V_i V_j}{c^2}\left(V^2 - 2\phi + \varepsilon + \frac{p}{\rho}\right) + \frac{2+2\omega}{2+\omega}\frac{\phi p}{c^2}\delta_{ij}. \tag{2.212}$$

Note that the if we compare (2.210) – (2.212) with (2.93) – (2.95) we infer that the only difference is the last term in (2.212).

The Poisson equation (2.195) for ψ can be rewritten as

$$\nabla^2\psi = 8\pi\mathbb{G}\rho\left(V^2 - \phi + \frac{2+\omega}{3+2\omega}\varepsilon + \frac{3+3\omega}{3+2\omega}\frac{p}{\rho}\right) + \frac{\partial^2\phi}{\partial t^2}, \quad (2.213)$$

by taking into account (2.210) and (2.211).

2.5.3 Hydrodynamic equations for an Eulerian fluid

For the determination of the continuity equation in the post-Newtonian Brans-Dicke theory we make use of the particle four-flow balance law (2.118) together with the the particle four-flow components (2.208), (2.209) and the components of the Cristoffel symbols (2.201) – (2.206), yielding

$$\frac{\partial\left\{n\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}-\phi\right)\right]\right\}}{\partial t} + \frac{\partial\left\{nV_i\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}-\phi\right)\right]\right\}}{\partial x^i}$$
$$+\frac{n}{c^2}\frac{1+2\omega}{2+\omega}\left(\frac{\partial\phi}{\partial t} + V_i\frac{\partial\phi}{\partial t}\right) = 0. \quad (2.214)$$

Now making use of the relationship (2.121) we get the continuity equation

$$\frac{\partial\rho_*}{\partial t} + \frac{\partial\rho_* V_i}{\partial x^i} = 0, \quad (2.215)$$

for the mass density $\rho*$ defined by

$$\rho_* = \rho\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} - \frac{3(1+\omega)}{2+\omega}\phi\right)\right]. \quad (2.216)$$

The hydrodynamic equation for the mass-energy density is obtained from the time component of the energy-momentum tensor equation (2.124) together with the representations of its components (2.210) – (2.212) and the components of the Cristoffel symbols (2.201) – (2.206). The final result is

$$\frac{\partial\sigma}{\partial t}+\frac{\partial\sigma V_i}{\partial x^i}=\frac{1}{c^2}\left[\rho\frac{\partial\phi}{\partial t}+\frac{\partial p}{\partial t}\right]-\frac{3\rho}{c^2(2+\omega)}\frac{d\phi}{dt}, \tag{2.217}$$

where the mass-energy density σ is defined by (2.126), which we reproduce here

$$\sigma=\rho\left[1+\frac{1}{c^2}\left(V^2+\varepsilon-2\phi+\frac{p}{\rho}\right)\right]. \tag{2.218}$$

From the equation for the spatial components of the energy-momentum tensor (2.128) together with the representations of its components (2.210) – (2.212) and the components of the Cristoffel symbols (2.201) – (2.206) it follows after some rearrangements the hydrodynamic equation for the momentum density

$$\begin{aligned}\frac{\partial\sigma V_i}{\partial t}+\frac{\partial\sigma V_iV_j}{\partial x^j}+\frac{\partial}{\partial x^i}\left[p\left(1-\frac{1+2\omega}{2+\omega}\frac{\phi}{c^2}\right)\right]+\rho\frac{\partial\phi}{\partial x^i}\Bigg[1\\
+\frac{1}{c^2}\left(\frac{3+2\omega}{2+\omega}V^2-\frac{1+2\omega}{2+\omega}\phi+\varepsilon+\frac{3+3\omega}{2+\omega}\frac{p}{\rho}\right)\Bigg]\\
-\frac{3+4\omega}{2+\omega}\frac{\rho}{c^2}\frac{d\phi V_i}{dt}+\frac{3+2\omega}{4+2\omega}\frac{\rho}{c^2}\left[\frac{\partial\psi}{\partial x^i}-V_j\frac{\partial\xi_j}{\partial x^i}+\frac{d\xi_i}{dt}\right]=0.\end{aligned} \tag{2.219}$$

Equations (2.215) – (2.219) are the same as those derived in the work of Nutku [12] if we made the following identifications $\varepsilon = \Pi$, $\phi = -U$ and

$$\frac{3+2\omega}{4+2\omega}\psi = -2\Phi, \qquad \frac{3+2\omega}{4+2\omega}\xi_i = -\frac{6+4\omega}{2+\omega}U_i + \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}. \tag{2.220}$$

Another expression for the hydrodynamic equation of the momentum density can be obtained by substituting the mass-energy density (2.218) into (2.219) and after some rearrangements, yields

$$\rho\frac{dV_i}{dt} + \rho\frac{\partial\phi}{\partial x^i}\left[1 + \frac{1}{c^2}\left(\frac{1+\omega}{2+\omega}V^2 + \frac{6+4\omega}{2+\omega}\phi\right)\right] + \frac{\partial p}{\partial x^i}\left[1 - \frac{1}{c^2}\left(V^2 + \frac{p}{\rho} - \frac{6+4\omega}{2+\omega}\phi + \varepsilon\right)\right] + \frac{3+2\omega}{4+2\omega}\frac{\rho}{c^2}\left[\frac{\partial\psi}{\partial x^i} + \frac{d\xi_i}{dt} - V_j\frac{\partial\xi_j}{\partial x^i}\right] + \frac{\rho V_i}{c^2}\left(\frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{\partial\phi}{\partial t} - \frac{6+4\omega}{2+\omega}\frac{d\phi}{dt}\right) = 0. \tag{2.221}$$

By subtracting (2.215) from (2.217) it follows the hydrodynamic equation for the total energy density which is a sum of the internal $\rho\varepsilon$ and kinetic $\rho V^2/2$ energy densities, namely

$$\frac{1}{c^2}\left\{\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2} + \varepsilon\right)\right] + \frac{\partial}{\partial x^i}\left[\rho\left(\frac{V^2}{2} + \varepsilon\right)V_i\right] + p\frac{\partial V_i}{\partial x^i} + V_i\left(\rho\frac{\partial\phi}{\partial x^i} + \frac{\partial p}{\partial x^i}\right) + \frac{\omega-1}{2+\omega}\phi\underline{\left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho V_i}{\partial x^i}\right)}\right\} = 0. \tag{2.222}$$

Now by using the Newtonian continuity equation (2.119) for the underlined term we get the Newtonian hydrodynamic equation (2.135) for the total energy density.

2.6 Non-perfect fluid hydrodynamic equations

Up to now we have considered a perfect relativistic gas dictated by the constitutive relations (2.11) for the particle four-flow and energy-momentum tensor. The question which arises refers to the post-Newtonian expressions for the hydrodynamic equations for the mass-density and momentum density for a viscous and heat-conducting fluid where shear stresses and heat flux are taken into account.

Before we analyze this subject we shall introduce a projector that will be useful to interpret physically the components of the energy-momentum tensor. From the definition of the four-velocity of the fluid U^μ we introduce a symmetric tensor (see e.g. [13])

$$\Delta^{\mu\nu} = g^{\mu\nu} - \frac{1}{c^2}U^\mu U^\nu, \tag{2.223}$$

that projects an arbitrary four-vector into another four-vector perpendicular to U^μ since $\Delta^{\mu\nu}U_\nu = 0$. The tensor $\Delta^{\mu\nu}$ is called a projector and it has the properties

$$\Delta^{\mu\nu}\Delta_{\nu\sigma} = \Delta^\mu{}_\sigma, \quad \Delta^\mu{}_\nu\Delta^{\nu\sigma} = \Delta^{\mu\sigma}, \quad \Delta^\mu{}_\mu = 3. \tag{2.224}$$

In a local Minkowski rest frame where $U^\mu = (c, \mathbf{0})$ the projector has the form: $\Delta^{\mu\nu} = \mathrm{diag}(0, -1, -1, -1)$. The post-

Newtonian approximation of the components of the projector are obtained from (2.223) together with (2.74) – (2.77), (2.87) and (2.88) and read

$$\Delta^{00} = -\frac{V^2}{c^2} + \mathcal{O}(c^{-4}), \quad \Delta^{0i} = -\frac{V_i}{c} + \mathcal{O}(c^{-3}), \tag{2.225}$$

$$\Delta^{ij} = -\left(1 + \frac{2\phi}{c^2}\right)\delta_{ij} - \frac{V_iV_j}{c^2} + \mathcal{O}(c^{-4}). \tag{2.226}$$

If A^μ is a four-vector and $A^{\mu\nu}$ a tensor, then

$$A^{(\mu)} = \Delta^\mu{}_\nu A^\nu, \tag{2.227}$$

$$A^{(\mu\nu)} = \frac{1}{2}\left(\Delta^\mu{}_\sigma\Delta^\nu{}_\tau + \Delta^\nu{}_\sigma\Delta^\mu{}_\tau\right)A^{\sigma\tau}, \tag{2.228}$$

$$A^{[\mu\nu]} = \frac{1}{2}\left(\Delta^\mu{}_\sigma\Delta^\nu{}_\tau - \Delta^\nu{}_\sigma\Delta^\mu{}_\tau\right)A^{\sigma\tau}, \tag{2.229}$$

represent a four-vector, a symmetric tensor and an antisymmetric tensor that have only the spatial components in a local Minkowski rest frame. Furthermore,

$$A^{\langle\mu\nu\rangle} = A^{(\mu\nu)} - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}A^{(\sigma\tau)}, \tag{2.230}$$

is a symmetric traceless tensor where the projection $\Delta_{\mu\nu}A^{\langle\mu\nu\rangle} = 0$ and $g_{\mu\nu}A^{\langle\mu\nu\rangle} = 0$ hold.

In order to identify the relativistic non-equilibrium quantities with the non-relativistic ones it is useful to introduce decompositions of the particle four-flow and energy-momentum tensor with respect to the four-velocity U^μ. The most usual decomposition is due to Eckart [14] where the particle four-flow N^μ and the energy-momentum tensor $T^{\mu\nu}$ for a viscous heat conducting fluid are written as:

$$N^\mu = nU^\mu, \tag{2.231}$$

$$T^{\mu\nu} = p^{\langle\mu\nu\rangle} - (p+\varpi)\,\Delta^{\mu\nu} + \frac{\epsilon}{c^2}U^\mu U^\nu + \frac{1}{c^2}\Big(U^\mu q^{(\nu)} + U^\nu q^{(\mu)}\Big). \tag{2.232}$$

Note that the decomposition of the particle four-flow is the same as the one for a perfect fluid $(2.11)_1$. The above decompositions define the quantities n, $p^{\langle\mu\nu\rangle}$, p, ϖ, $q^{(\mu)}$ and ϵ, which are identified as:

$$n = \frac{1}{c^2}N^\mu U_\mu \text{– particle number density,} \tag{2.233}$$

$$p^{\langle\mu\nu\rangle} = \left(\Delta^\mu{}_\sigma \Delta^\nu{}_\tau - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right)T^{\sigma\tau} \text{– pressure deviator,} \tag{2.234}$$

$$(p+\varpi) = -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} \text{– hydrostatic + dynamic pressures,} \tag{2.235}$$

$$q^{(\mu)} = \Delta^\mu{}_\nu U_\sigma T^{\nu\sigma} \text{ – heat flux,} \tag{2.236}$$

$$\epsilon = \frac{1}{c^2}U_\mu T^{\mu\nu} U_\nu \text{ – energy density.} \tag{2.237}$$

The dynamic pressure is the non-equilibrium part of the trace of the energy-momentum tensor, since the hydrostatic pressure p and the energy density ϵ refer to equilibrium quantities. The constitutive equation for the dynamic pressure is proportional to the velocity divergent and the coefficient of proportionality is the volume viscosity which is of order $1/c^4$ (see e.g. [13]). Hence the dynamic pressure will not be considered in the following analysis.

We begin by writing the spatial components of the pressure deviator as

$$p^{\langle ij\rangle} = \mathfrak{p}_{ij} + \frac{1}{2c^2}\left(\mathfrak{p}_{ik}V_kV_j + \mathfrak{p}_{jk}V_kV_i\right), \text{ with } \mathfrak{p}_{ij} = p_{ij} - \frac{p_{rr}}{3}\delta_{ij} \tag{2.238}$$

denoting the non-relativistic pressure deviator. Due to the relationship $U_\mu p^{\langle\mu\nu\rangle} = g_{\nu\sigma}U^\sigma p^{\langle\mu\nu\rangle} = 0$ we have that

$$\left(g_{00}U^0 + g_{0j}U^j\right)p^{\langle 0i\rangle} + \left(g_{0j}U^0 + g_{jk}U^k\right)p^{\langle ij\rangle} = 0, \tag{2.239}$$

$$\left(g_{00}U^0 + g_{0j}U^j\right)p^{\langle 00\rangle} + \left(g_{0j}U^0 + g_{jk}U^k\right)p^{\langle 0j\rangle} = 0, \tag{2.240}$$

which imply the time and space-time components of the pressure deviator

$$p^{\langle 00\rangle} = \mathfrak{p}_{ij}\frac{V_iV_j}{c^2} + \mathcal{O}(c^{-4}), \qquad p^{\langle 0i\rangle} = \mathfrak{p}_{ij}\frac{V_j}{c} + \mathcal{O}(c^{-3}). \tag{2.241}$$

Note that with the representations given above the trace of the pressure deviator $p^{\langle\mu\nu\rangle}$ vanish. Indeed

$$g_{\mu\nu}p^{\langle\mu\nu\rangle} = g_{00}p^{\langle 00\rangle} + 2g_{0i}p^{\langle 0i\rangle} + g_{ij}p^{\langle ij\rangle} = 0. \tag{2.242}$$

A similar result follows from the condition $\Delta_{\mu\nu}p^{\langle\mu\nu\rangle}$.

We write the spatial components of the heat flux as

$$q^{(i)} = \mathfrak{q}_i, \tag{2.243}$$

where $\mathfrak{q}_i$ is the non-relativistic heat flux vector. By using the relationship $U_\mu q^{(\mu)} = g_{\mu\nu} U^\nu q^{(\mu)} = 0$ we get the time component of the heat flux

$$q^{(0)} = \mathfrak{q}_i \frac{V_i}{c} + \mathcal{O}(c^{-3}). \tag{2.244}$$

Up to order $\mathcal{O}\left(c^{-2}\right)$ the expressions for the components of the pressure deviator and heat flux are

$$p^{\langle ij \rangle} = \mathfrak{p}_{ij} + \frac{1}{2c^2}\left(\mathfrak{p}_{ik} V_k V_j + \mathfrak{p}_{jk} V_k V_i\right), \qquad p^{\langle 00 \rangle} = \mathfrak{p}_{ij} \frac{V_i V_j}{c^2}, \tag{2.245}$$

$$p^{\langle 0i \rangle} = \mathfrak{p}_{ij} \frac{V_j}{c}, \qquad q^{(i)} = \mathfrak{q}_i, \qquad q^{(0)} = \mathfrak{q}_i \frac{V_i}{c}. \tag{2.246}$$

In the non-relativistic limiting case the above expressions reduce to

$$p^{\langle ij \rangle} = \mathfrak{p}_{ij}, \quad p^{\langle 00 \rangle} = 0, \quad p^{\langle 0i \rangle} = 0, \quad q^{(i)} = \mathfrak{q}_i, \quad q^{(0)} = 0. \tag{2.247}$$

We note that the pressure deviator and the heat flux vector must vanish at equilibrium, i.e., $\mathfrak{p}_{ij}|_E = 0$ and $\mathfrak{q}_i|_E = 0$.

Now the different orders of the energy-momentum tensor components can be identified from (2.232) together with (2.225), (2.226), (2.245) and (2.246). They read

$$\overset{0}{T}{}^{00} = \rho c^2, \quad \overset{2}{T}{}^{00} = \rho\left(V^2 + \varepsilon - 2\phi\right), \tag{2.248}$$

$$\overset{1}{T}{}^{i0} = \rho c V_i, \quad \overset{3}{T}{}^{i0} = \rho\left(V^2 - 2\phi + \varepsilon + \frac{p}{\rho}\right)\frac{V_i}{c} + \frac{\mathfrak{p}_{ij} V_j}{c} + \frac{\mathfrak{q}_i}{c}, \tag{2.249}$$

$$\overset{2}{T}{}^{ij} = \rho V_i V_j + p\delta_{ij} + \mathfrak{p}_{ij}, \tag{2.250}$$

$$\overset{4}{T}{}^{ij} = \rho\left(V^2 - 2\phi + \varepsilon + \frac{p}{\rho}\right)\frac{V_iV_j}{c^2} + \frac{2\phi p}{c^2}\delta_{ij} + \frac{1}{2c^2}\left(\mathfrak{p}_{ik}V_kV_j + \mathfrak{p}_{jk}V_kV_i\right) + \frac{1}{c^2}\left(\mathfrak{q}_iV_j + \mathfrak{q}_jV_i\right). \tag{2.251}$$

It is important to call attention that the introduction of non-equilibrium quantities related with the pressure deviator and heat flux does not change the determination of the components of the metric tensor from Einstein's field equations in the first post-Newtonian approximation. This can be verified since the components of the energy-momentum tensor which appear in Einstein's field equations to compute $\overset{2}{g}_{00}$, $\overset{4}{g}_{00}$, $\overset{2}{g}_{ij}$ and $\overset{3}{g}_{0i}$ are $\overset{0}{\mathfrak{T}}_{00}$, $\overset{2}{\mathfrak{T}}_{00}$, $\overset{0}{\mathfrak{T}}_{ij}$ and $\overset{1}{\mathfrak{T}}_{0i}$, respectively, and in none of these quantities neither the pressure deviator $\mathfrak{p}_{ij}$ nor the heat flux $\mathfrak{q}_i$ show up.

As it was previously pointed out the expression for the particle four-flow in the Eckart decomposition is the same as that of a perfect fluid. Hence, its hydrodynamic equation is the same as that of a perfect fluid, i.e. the continuity equation (2.122) for the mass density ρ_* holds.

The hydrodynamic equation for the mass-energy density follows from time component of the balance law for the energy-momentum tensor (2.124), yielding

$$\frac{\partial\sigma}{\partial t} + \frac{\partial\sigma V_i}{\partial x^i} - \frac{1}{c^2}\left(\rho\frac{\partial\phi}{\partial t} + \frac{\partial p}{\partial t}\right) + \frac{1}{c^2}\left(\frac{\partial\mathfrak{p}_{ij}V_j}{\partial x^i} + \frac{\partial\mathfrak{q}_i}{\partial x^i}\right) = 0. \tag{2.252}$$

Here we note the new contributions of the pressure deviator $\mathfrak{p}_{ij}$ and heat flux $\mathfrak{q}_i$ when this equation is compared with the corresponding one for the perfect fluid (2.127).

From the balance equation for the spatial components of the energy-momentum tensor (2.128) we obtain the hydrodynamic equation for the momentum density:

$$\frac{\partial \sigma V_i}{\partial t} + \frac{\partial \sigma V_i V_j}{\partial x^j} + \frac{\partial}{\partial x^i}\left[p\left(1 + \frac{2\phi}{c^2}\right)\right] + \frac{\partial \mathfrak{p}_{ij}}{\partial x^j}$$
$$+\rho\frac{\partial \phi}{\partial x^i}\left[1 + \frac{1}{c^2}\left(2V^2 + \varepsilon + 2\phi - \frac{p}{\rho}\right)\right] - 4\frac{\mathfrak{p}_{ij}}{c^2}\frac{\partial \phi}{\partial x^j}$$
$$+\frac{\rho}{c^2}V_j\left(\frac{\partial \xi_i}{\partial x^j} - \frac{\partial \xi_j}{\partial x^i}\right) - 4\frac{\rho}{c^2}V_i\left(\frac{\partial \phi}{\partial t} + V_j\frac{\partial \phi}{\partial x^j}\right)$$
$$+\frac{\rho}{c^2}\left(\frac{\partial \psi}{\partial x^i} + \frac{\partial \xi_i}{\partial t}\right) + \frac{1}{c^2}\frac{\partial\left(\mathfrak{p}_{ij}V_j + \mathfrak{q}_i\right)}{\partial t}\frac{\partial \phi}{\partial x^i}$$
$$+\frac{1}{c^2}\frac{\partial}{\partial x^j}\left[\mathfrak{q}_i V_j + \mathfrak{q}_j V_i + \frac{\left(\mathfrak{p}_{ik}V_j + \mathfrak{p}_{jk}V_i\right)V_k}{2}\right] = 0. \quad (2.253)$$

Without the dissipative terms $\mathfrak{p}_{ik}$ and $\mathfrak{q}_i$ the above equation reduces to (2.131).

The momentum density hydrodynamic equation corresponding to the Eulerian equation (2.133) is:

$$\rho\frac{dV_i}{dt} + \frac{\partial\left(\mathfrak{p}_{ij} + p\delta_{ij}\right)}{\partial x^j}\left[1 - \frac{1}{c^2}\left(V^2 - 4\phi + \varepsilon + \frac{p}{\rho}\right)\right]$$
$$+\rho\frac{\partial \phi}{\partial x^i}\left[1 + \frac{1}{c^2}\left(V^2 + 4\phi\right)\right] + \frac{\rho}{c^2}\left[\frac{\partial \psi}{\partial x^i} + \frac{d\xi_i}{dt} - V_j\frac{\partial \xi_j}{\partial x^i}\right.$$

$$+V_i\left(\frac{1}{\rho}\frac{\partial p}{\partial t}+\frac{\partial \phi}{\partial t}-4\frac{d\phi}{dt}\right)\bigg]+\frac{1}{c^2}\frac{\partial\left(\mathfrak{p}_{ij}V_j+\mathfrak{q}_i\right)}{\partial t}$$
$$+\frac{1}{c^2}\frac{\partial}{\partial x^j}\left[\mathfrak{q}_iV_j+\mathfrak{q}_jV_i+\frac{1}{2}\left(\mathfrak{p}_{ik}V_j+\mathfrak{p}_{jk}V_i\right)V_k-4\phi\mathfrak{p}_{ij}\right]$$
$$+2\frac{\phi}{c^2}\frac{\partial \mathfrak{p}_{ij}}{\partial x^j}-\frac{V_i}{c^2}\left(\frac{\partial \mathfrak{p}_{jk}V_k}{\partial x^j}+\frac{\partial \mathfrak{q}_j}{\partial x^j}\right)=0. \quad (2.254)$$

Here the definition of σ given in (2.126) and the mass-energy hydrodynamic equation (2.252) were used.

The Newtonian momentum density hydrodynamic equation follows from (2.254) by neglecting all terms of $\mathcal{O}\left(c^{-2}\right)$ order, yielding

$$\rho\frac{dV_i}{dt}+\frac{\partial\left(\mathfrak{p}_{ij}+p\delta_{ij}\right)}{\partial x^j}+\rho\frac{\partial \phi}{\partial x^i}=0. \quad (2.255)$$

By subtracting the mass density equation (2.122) from the mass-energy equation (2.252) we get the total energy density hydrodynamic equation

$$\frac{1}{c^2}\left\{\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x^i}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)V_i\right]+\rho V_i\frac{\partial \phi}{\partial x^i}\right.$$
$$\left.+\frac{\partial\left[\mathfrak{p}_{ij}V_j+\mathfrak{q}_i+pV_i\right]}{\partial x^i}-\phi\underline{\left(\frac{\partial \rho}{\partial t}+\frac{\partial \rho V_i}{\partial x^i}\right)}\right\}=0, \quad (2.256)$$

which corresponds to (2.134). If we use the Newtonian continuity equation for the underlined term, (2.256) reduces to the Newtonian total energy density hydrodynamic equation for a viscous and heat conducting fluid

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x^i}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)V_i\right]+\rho V_i\frac{\partial \phi}{\partial x^i}$$
$$+\frac{\partial\left[\mathfrak{p}_{ij}V_j+\mathfrak{q}_i+pV_i\right]}{\partial x^i}=0, \quad (2.257)$$

As was pointed out in the Section 2.4 – where the Eulerian hydrodynamic equations were introduced – in order to get the first post-Newtonian corrections to the total energy density hydrodynamic equation we must go further to the second post-Newtonian approximation.

The hydrodynamic equation for the internal energy density is obtained from the elimination from (2.256) of the time derivative of the hydrodynamic velocity by using the Newtonian momentum density hydrodynamic equation (2.255), yielding

$$\rho\frac{d\varepsilon}{dt}+\frac{\partial \mathfrak{q}_i}{\partial x^i}+\left[\mathfrak{p}_{ij}+p\delta_{ij}\right]\frac{\partial V_i}{\partial x^j}=0. \quad (2.258)$$

This equation refers to the Newtonian internal energy density hydrodynamic equation for a viscous and heat-conducting fluid and it corresponds to (2.136) of the Eulerian fluid.

2.7 The gravitational potentials

In this section we shall express the gravitational potentials – defined in the method of Chandrasekhar – in terms of integrals over the entire volume V occupied by the fluid. The first one is the Newtonian gravitational potential U which is solution of the Poisson equation $\nabla^2 U = -4\pi G\rho$,

$$U(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{2.259}$$

Here and in the following expressions we shall not specify the time dependence of the fields. The above equation corresponds to (2.63) with the identification $U = -\phi$.

The equation for the vector gravitational potential U_i, given by $(2.106)_1$, has the corresponding integral solution

$$U_i(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')V_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{2.260}$$

while the integral solution of the scalar gravitational potential Φ, which satisfies (2.110) is

$$\Phi(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')\varphi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \text{where} \quad \varphi = V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}. \tag{2.261}$$

According to the work of Chandrasekhar and Lebovitz [7] the scalar gravitational potential χ introduced in (2.105) is a super-potential, since it obeys the equation $\nabla^4\chi = 8\pi G\rho$. We proceed to derive this equation on the basis of this paper. To begin with we introduce the gravitational potential symmetric tensor

$$\mathfrak{U}_{ij}(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')(x_i - x_i')(x_j - x_j')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x', \tag{2.262}$$

and another vector gravitational potential defined by

$$D_i(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')x'_i}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{2.263}$$

Note that the trace of the gravitational potential tensor (2.262) is the Newtonian gravitational potential

$$U(\mathbf{x}) = \mathfrak{U}_{ii}(\mathbf{x}) = G\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{2.264}$$

If we differentiate the vector gravitational potential (2.263) and the Newtonian gravitational potential (2.264) with respect to x_j we get respectively

$$\frac{\partial D_i}{\partial x_j} = -G\int_V \frac{\rho(\mathbf{x}')x'_i(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x', \tag{2.265}$$

$$\frac{\partial U}{\partial x_j} = -G\int_V \frac{\rho(\mathbf{x}')(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'. \tag{2.266}$$

Now if we subtract from (2.265) the equation (2.266) multiplied by x_i we obtain the gravitational potential tensor, namely

$$\frac{\partial D_i}{\partial x_j} - x_i\frac{\partial U}{\partial x_j} = G\int_V \frac{\rho(\mathbf{x}')(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3x' = \mathfrak{U}_{ij}. \tag{2.267}$$

Since the gravitational potential tensor is a symmetric tensor we can write from (2.267) that

$$\frac{\partial D_i}{\partial x_j} - x_i\frac{\partial U}{\partial x_j} = \frac{\partial D_j}{\partial x_i} - x_j\frac{\partial U}{\partial x_i}, \tag{2.268}$$

which in terms of a rotational of a vector reduces to

$$\nabla \times \mathbf{D} = \nabla U \times \mathbf{x} = \nabla \times (U\mathbf{x})\,. \tag{2.269}$$

From the above equation we infer that $\mathbf{D}$ is given by $U\mathbf{x}$ plus a gradient of a scalar field χ (say), i.e.,

$$D_i = Ux_i + \frac{\partial \chi}{\partial x_i}. \tag{2.270}$$

The divergence of the above equation with respect to x_i leads to

$$\frac{\partial D_i}{\partial x_i} = 3U + x_i \frac{\partial U}{\partial x_i} + \nabla^2 \chi. \tag{2.271}$$

If we use the trace of (2.267), namely

$$\frac{\partial D_i}{\partial x_i} - x_i \frac{\partial U}{\partial x_i} = U, \tag{2.272}$$

equation (2.271) reduces to $(2.106)_2$, i.e., $\nabla^2\chi = -2U$. This last equation together with the Poisson equation $\nabla^2 U = -4\pi G\rho$ implies the equation for the super-potential of the gravitational field $\nabla^4\chi = 8\pi G\rho$.

Another expression for the gravitational potential tensor $\mathfrak{U}_{ij}$ can be obtained from (2.267) and (2.270), yielding

$$\mathfrak{U}_{ij} = U\delta_{ij} + \frac{\partial^2 \chi}{\partial x_i \partial x_j}. \tag{2.273}$$

As it was pointed out in [7] an alternative way to obtain the relationship (2.273) is to introduce the following definition of the super-potential χ

$$\chi = -G \int_V \rho(\mathbf{x}')|\mathbf{x} - \mathbf{x}'| d^3x'. \tag{2.274}$$

The differentiation of the above equation with respect to x_i leads to

$$\frac{\partial \chi}{\partial x_i} = -G \int_V \frac{\rho(\mathbf{x}')(x_i - x'_i)}{|\mathbf{x} - \mathbf{x}'|} d^3x' = U x_i + D_i, \tag{2.275}$$

thanks to the definitions of D_i and U given by (2.263) and (2.264), respectively. Note that this equation is the same as the one given in (2.270). Furthermore, its differentiation with respect to x_j leads to (2.273), since

$$\frac{\partial^2 \chi}{\partial x_i \partial x_j} = G \int_V \frac{\rho(\mathbf{x}')(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x} - \mathbf{x}'|^3} d^3x' \\ -\delta_{ij} G \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' = \mathfrak{U}_{ij} - U\delta_{ij}. \tag{2.276}$$

In the derivation of the virial theorem the gravitational potential energy tensor $\mathfrak{W}_{ij}$ plays an important role. It is defined in terms of the gravitational potential tensor $\mathfrak{U}_{ij}$ by

$$\mathfrak{W}_{ij} = -\frac{1}{2} \int_V \rho(\mathbf{x}) \mathfrak{U}_{ij}(\mathbf{x}) d^3x \\ = -\frac{G}{2} \int_V \int_V \frac{\rho(\mathbf{x})\rho(\mathbf{x}')(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x} - \mathbf{x}'|^3} d^3x' d^3x. \tag{2.277}$$

Another expression for the gravitational potential energy tensor is given by the relation

$$\mathfrak{W}_{ij} = \int_V \rho(\mathbf{x}) x_i \frac{\partial U}{\partial x_j} d^3x = \int_V \rho(\mathbf{x}) x_j \frac{\partial U}{\partial x_i} d^3x. \qquad (2.278)$$

Indeed by considering the definition of the gravitational potential (2.264) we have

$$\begin{aligned}
&\int_V \rho(\mathbf{x}) x_i \frac{\partial U}{\partial x_j} d^3x \\
&= \int_V \rho(\mathbf{x}) x_i \left[-G \int_V \frac{\rho(\mathbf{x}')(x_j - x'_j)}{|\mathbf{x} - \mathbf{x}'|^3} d^3x' \right] d^3x \\
&= \int_V \rho(\mathbf{x}') x'_i \left[-G \int_V \frac{\rho(\mathbf{x})(x'_j - x_j)}{|\mathbf{x} - \mathbf{x}'|^3} d^3x \right] d^3x' \\
&= -\frac{G}{2} \int_V \int_V \frac{\rho(\mathbf{x})\rho(\mathbf{x}')(x_i - x'_i)(x_j - x'_j) d^3x' d^3x}{|\mathbf{x} - \mathbf{x}'|^3} = \mathfrak{W}_{ij},
\end{aligned} \qquad (2.279)$$

where the third equality was obtained by interchanging the primed and the unprimed labels.

The trace of the gravitational potential energy tensor (2.277) and (2.278) is the gravitational potential energy, namely

$$\begin{aligned}
\mathfrak{W} = \mathfrak{W}_{ii} &= -\frac{1}{2} \int_V \rho(\mathbf{x}) U(\mathbf{x}) d^3x \\
= -\frac{G}{2} \int_V \int_V \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' d^3x &= \int_V \rho(\mathbf{x}) x_i \frac{\partial U}{\partial x_i} d^3x.
\end{aligned} \qquad (2.280)$$

2.8 The conservation laws

Here we shall write the hydrodynamic equations of Section 2.4 in terms of the Chandrasekhar gravitational potentials by identifying

$$\phi = -U, \quad \psi \equiv -2\Phi, \quad \xi_i = -4U_i + \frac{1}{2}\frac{\partial^2 \chi}{\partial t \partial x^i} = -\Pi_i. \tag{2.281}$$

Note that we have introduced the vector gravitational potential $\Pi_i = -\xi_i$ in order to have the same structure of equations of [4] where it is denoted by P_i.

In the analysis of the conservation laws it is necessary to use Reynolds' transport theorem which is valid for an arbitrary scalar–, vector– or tensor–valued function $F(\mathbf{x}, t)$, namely

$$\frac{d}{dt}\int_V F(\mathbf{x},t) d^3x = \int_V \left(\frac{\partial F(\mathbf{x},t)}{\partial t} + \frac{\partial F(\mathbf{x},t)V_i}{\partial x^i}\right). \tag{2.282}$$

Conservation of total linear momentum density

The hydrodynamic equation for the momentum density (2.131) rewritten in terms of Chandrasekhar's potentials, read

$$\begin{aligned} &\frac{\partial \sigma V_i}{\partial t} + \frac{\partial \sigma V_i V_j}{\partial x^j} - \rho\frac{\partial U}{\partial x^i} + \frac{\partial}{\partial x^i}\left[p\left(1 + \frac{2U}{c^2}\right)\right] \\ &+ \frac{4\rho}{c^2}\frac{d}{dt}\left(UV_i - \frac{\Pi_i}{4}\right) - 2\frac{\rho}{c^2}\left(\varphi\frac{\partial U}{\partial x^i} + \frac{\partial \Phi}{\partial x^i}\right) \\ &+ \frac{\rho}{c^2}V_j\left(4\frac{\partial U_j}{\partial x^i} - \frac{1}{2}\frac{\partial^3 \chi}{\partial t \partial x^i \partial x^j}\right) = 0. \end{aligned} \tag{2.283}$$

Note that in the above equation we introduce the abbreviation φ defined in (2.111).

If we integrate (2.283) over the volume V occupied by the fluid we get

$$\int_V \left\{ \frac{\partial \sigma V_i}{\partial t} + \frac{\partial \sigma V_i V_j}{\partial x^j} - \rho \frac{\partial U}{\partial x^i} + \frac{\partial}{\partial x^i}\left[p\left(1 + \frac{2U}{c^2}\right)\right] + \frac{4\rho}{c^2}\frac{d}{dt}\left(UV_i - \frac{\Pi_i}{4}\right) - 2\frac{\rho}{c^2}\left(\varphi\frac{\partial U}{\partial x^i} + \frac{\partial \Phi}{\partial x^i}\right) + \frac{\rho}{c^2}V_j\left(4\frac{\partial U_j}{\partial x^i} - \frac{1}{2}\frac{\partial^3 \chi}{\partial t \partial x^i \partial x^j}\right)\right\} d^3x = 0. \qquad (2.284)$$

Below we shall analyze separately the terms in (2.283) and will enumerate them for an easy view.

(i)

$$\int_V \left\{ \frac{\partial \sigma V_i}{\partial t} + \frac{\partial \sigma V_i V_j}{\partial x^j} \right\} d^3x = \frac{d}{dt}\int_V \sigma V_i d^3x, \qquad (2.285)$$

by the use of Reynolds' transport theorem (2.282).

(ii)

$$-\int_V \rho(\mathbf{x})\frac{\partial U}{\partial x^i} d^3x = \frac{G}{2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_i - x_i'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x d^3x' + \frac{G}{2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_i' - x_i}{|\mathbf{x}-\mathbf{x}'|^3} d^3x d^3x' = 0, \quad (2.286)$$

where in the second term above we have changed the primed and the unprimed labels.

(iii)

$$\int_V \frac{\partial}{\partial x^i}\left[p\left(1+\frac{2U}{c^2}\right)\right]d^3x = 0, \tag{2.287}$$

since this integral can be converted – by using Gauss' divergence theorem – into a surface integral where the pressure vanishes on the boundary of the configuration.

(iv)

$$\frac{4}{c^2}\int_V \rho\frac{d}{dt}\left(UV_i - \frac{\Pi_i}{4}\right)d^3x = \frac{4}{c^2}\int_V \left\{\frac{\partial\rho\left(UV_i - \frac{\Pi_i}{4}\right)}{\partial t}\right.$$
$$\left. + \frac{\partial\rho V_j\left(UV_i - \frac{\Pi_i}{4}\right)}{\partial x^j} - \left(UV_i - \frac{\Pi_i}{4}\right)\underline{\left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho V_j}{\partial x^j}\right)}\right\}d^3x$$
$$= \frac{4}{c^2}\frac{d}{dt}\int_V \rho\left(UV_i - \frac{\Pi_i}{4}\right)d^3x, \tag{2.288}$$

where the underlined term vanishes, since we can use the Newtonian continuity equation at this approximation. For the last integral we have applied Reynolds' transport theorem.

(v)

$$\int_V \rho\frac{\partial\Phi}{\partial x^i}d^3x = -G\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\varphi(\mathbf{x}')\frac{x_i - x'_i}{|\mathbf{x}-\mathbf{x}'|^3}d^3x d^3x'$$
$$= -G\int_V d^3x'\rho(\mathbf{x}')\varphi(\mathbf{x}')\frac{\partial}{\partial x'^i}\int_V d^3x\frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{x}'|}$$
$$= -\int_V d^3x'\rho(\mathbf{x}')\varphi(\mathbf{x}')\frac{\partial U(\mathbf{x}')}{\partial x'^i}, \qquad \text{hence}$$
$$-\frac{2}{c^2}\int_V \rho\left(\varphi\frac{\partial U}{\partial x^i} + \frac{\partial\Phi}{\partial x^i}\right)d^3x = 0, \tag{2.289}$$

(vi)

$$\int_V \rho V_j \frac{\partial U_j}{\partial x^i} d^3x$$
$$= -\frac{G}{2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')V_j(\mathbf{x})V_j(\mathbf{x}')\frac{x_i - x_i'}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x'$$
$$-\frac{G}{2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')V_j(\mathbf{x})V_j(\mathbf{x}')\frac{x_i' - x_i}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x' = 0, \quad (2.290)$$

where the primed and the unprimed labels were changed in the second expression above.

(vii) The determination of the last integral is more involved. We begin by evaluating

$$\frac{\partial \chi}{\partial t} = -G\int_V \frac{\partial \rho(\mathbf{x}')}{\partial t}|\mathbf{x}-\mathbf{x}'|d^3x'$$
$$= G\int_V \frac{\partial \rho(\mathbf{x}')V_k(\mathbf{x}')}{\partial x'^k}|\mathbf{x}-\mathbf{x}'|d^3x'$$
$$= \underline{G\int_V \frac{\partial \rho(\mathbf{x}')V_k(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|}{\partial x'^k}d^3x'}$$
$$+G\int_V \rho(\mathbf{x}')V_k(\mathbf{x}')\frac{x_k - x_k'}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \quad (2.291)$$

For the second integral we make use of the Newtonian continuity equation (2.129), since the term which we are interested is of $\mathcal{O}\left(c^{-2}\right)$ – order. The underlined term vanishes thanks to Gauss' divergence theorem. From this equation it follows by differentiating it with respect to x^i and x^j that

$$\frac{\partial^3\chi}{\partial x^i\partial x^j\partial t} = -G\int_V \rho(\mathbf{x}')V_k(\mathbf{x}')(x_k - x'_k)\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$
$$-G\int_V \rho(\mathbf{x}')\left[V_i(\mathbf{x}')(x_j - x'_j) + V_j(\mathbf{x}')(x_i - x'_i)\right]\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}$$
$$+3G\int_V \rho(\mathbf{x}')V_k(\mathbf{x}')(x_k - x'_k)\frac{(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^5}d^3x'. \quad (2.292)$$

Now from the above equation we get

$$\int_V \rho(\mathbf{x})V_j(\mathbf{x})\frac{\partial^3\chi(\mathbf{x})}{\partial x^i\partial x^j\partial t}d^3x$$
$$= G\int_V\int_V\left\{3\frac{V_k(\mathbf{x}')(x_k - x'_k)V_j(\mathbf{x})(x_j - x'_j)(x_i - x'_i)}{|\mathbf{x}-\mathbf{x}'|^5}\right.$$
$$-\frac{V_i(\mathbf{x})V_j(\mathbf{x}')(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^3} - \frac{V_i(\mathbf{x}')V_j(\mathbf{x})(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}$$
$$\left.-\frac{V_j(\mathbf{x})V_j(\mathbf{x}')(x_i - x'_i)}{|\mathbf{x}-\mathbf{x}'|^3}\right\}\rho(\mathbf{x}')\rho(\mathbf{x})d^3xd^3x' = 0. \quad (2.293)$$

This integral vanishes by interchanging the primed with the unprimed labels. Now collecting all results above (i) – (vii) we arrive at the conservation of the total linear momentum density in the first post-Newtonian approximation, namely

$$\frac{d}{dt}\int_V\left\{\rho V_i\left[1 + \frac{1}{c^2}\left(V^2 + 6U + \varepsilon + \frac{p}{\rho}\right)\right] - \rho\Pi_i\right\}d^3x = 0, \quad (2.294)$$

where the total linear momentum density is

$$\mathfrak{P}_i = \rho V_i + \frac{\rho}{c^2}\left[V_i\left(V^2 + 6U + \varepsilon + \frac{p}{\rho}\right) - \Pi_i\right]. \qquad (2.295)$$

Conservation of total angular momentum density

For the determination of the total angular momentum density conservation we need to evaluate integrals over the volume V which are obtained from the multiplication of the hydrodynamic equation for the momentum density (2.283) with x_j, namely

$$\int_V \left\{\frac{\partial \sigma V_i}{\partial t} + \frac{\partial \sigma V_i V_j}{\partial x^j} - \rho\frac{\partial U}{\partial x^i} + \frac{\partial}{\partial x^i}\left[p\left(1 + \frac{2U}{c^2}\right)\right]\right. + \frac{4\rho}{c^2}\frac{d}{dt}\left(UV_i - \frac{\Pi_i}{4}\right) - 2\frac{\rho}{c^2}\left(\varphi\frac{\partial U}{\partial x^i} + \frac{\partial \Phi}{\partial x^i}\right) \left. + \frac{\rho}{c^2}V_j\left(4\frac{\partial U_j}{\partial x^i} - \frac{1}{2}\frac{\partial^3 \chi}{\partial t \partial x^i \partial x^j}\right)\right\} x_j d^3x = 0. \qquad (2.296)$$

We proceed to evaluate the integrals following the same methodology above.

(viii)

$$\int_V x_j \left\{\frac{\partial \sigma V_i}{\partial t} + \frac{\partial \sigma V_i V_k}{\partial x^k}\right\} d^3x = -\int_V \sigma V_i V_j d^3x + \int_V \left\{\frac{\partial \sigma V_i x_j}{\partial t} + \frac{\partial \sigma V_i x_j V_k}{\partial x^k}\right\} d^3x = -2\mathfrak{K}_{ij} + \frac{d}{dt}\int_V \sigma V_i x_j d^3x. \qquad (2.297)$$

Here Reynolds' transport theorem (2.282) was used and the kinetic energy tensor was introduced

$$\mathfrak{K}_{ij} = \frac{1}{2}\int_V \sigma V_i V_j d^3x. \tag{2.298}$$

(ix)

$$-\int_V \rho(\mathbf{x}) x_j \frac{\partial U}{\partial x^i} d^3x = -\mathfrak{W}_{ij} \tag{2.299}$$

where $\mathfrak{W}_{ij}$ is the gravitational potential energy tensor (2.277) which is symmetric.

(x)

$$\int_V x_j \frac{\partial}{\partial x^i}\left[p\left(1+\frac{2U}{c^2}\right)\right] d^3x = -\delta_{ij}\int_V \left[p\left(1+\frac{2U}{c^2}\right)\right] d^3x$$
$$+\int_V \frac{\partial}{\partial x^i}\left[x_j p\left(1+\frac{2U}{c^2}\right)\right] d^3x. \tag{2.300}$$

The last integral vanishes, since it can be converted in a surface integral by the use of Gauss' divergence theorem and the pressure vanishes on the boundary of the configuration.

(xi)

$$\frac{4}{c^2}\int_V x_j \rho \frac{d}{dt}\left(UV_i - \frac{\Pi_i}{4}\right) d^3x = \frac{4}{c^2}\int_V \left\{\frac{\partial \rho x_j \left(UV_i - \frac{\Pi_i}{4}\right)}{\partial t}\right.$$
$$+\frac{\partial \rho V_k x_j \left(UV_i - \frac{\Pi_i}{4}\right)}{\partial x^k} - \left(UV_i - \frac{\Pi_i}{4}\right) x_j \underline{\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_k}{\partial x^k}\right)}$$

$$-\rho V_j \left(UV_i - \frac{\Pi_i}{4} \right) \Bigg\} d^3x = \frac{4}{c^2} \frac{d}{dt} \int_V \rho x_j \left(UV_i - \frac{\Pi_i}{4} \right) d^3x$$
$$-\frac{4}{c^2} \int_V \rho V_j \left(UV_i - \frac{\Pi_i}{4} \right) d^3x, \quad (2.301)$$

thanks to the use of the Reynolds transport theorem (2.282) and the Newtonian approximation of the continuity equation where the underlined term vanishes. We shall write the second integral in the last equality as

$$-\frac{4}{c^2} \int_V \rho V_j \left(UV_i - \frac{\Pi_i}{4} \right) d^3x = \frac{4}{c^2} \int \rho V_j \Bigg(U_i - UV_i$$
$$- \frac{1}{8} \frac{\partial^2 \chi}{\partial t \partial x^i} \Bigg) d^3x. \quad (2.302)$$

thanks to $(2.281)_3$. According to (2.260) the first term in the integral can be written as

$$\int_V \rho V_j U_i d^3x = G \int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}')V_i(\mathbf{x}')V_j(\mathbf{x}) \frac{d^3x d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$
$$= \frac{G}{2} \int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}') \left[V_i(\mathbf{x}')V_j(\mathbf{x}) + V_i(\mathbf{x})V_j(\mathbf{x}') \right] \frac{d^3x d^3x'}{|\mathbf{x} - \mathbf{x}'|}, \quad (2.303)$$

which is a symmetric tensor. For the term in the integral (2.302) related with χ we differentiate (2.291) so that we can write it as

$$\int_V \rho V_j \frac{\partial^2 \chi}{\partial t \partial x^i} d^3x = G \int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}') \frac{V_j(\mathbf{x})V_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x d^3x'$$

$$-G\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\frac{V_j(\mathbf{x})V_k(\mathbf{x}')(x_k-x'_k)(x_i-x'_i)}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x'. \tag{2.304}$$

(xii)

$$\begin{aligned}\int_V \rho x_j\left(\varphi\frac{\partial U}{\partial x^i}+\frac{\partial\Phi}{\partial x^i}\right)d^3x &= -G\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\big[\varphi(\mathbf{x})\\ +\varphi(\mathbf{x}')\big]\frac{(x_i-x'_i)x_j}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x' &= -\frac{G}{2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\big[\varphi(\mathbf{x})\\ &+\varphi(\mathbf{x}')\big]\frac{(x_i-x'_i)(x_j-x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x'\\ &= -G\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')\varphi(\mathbf{x})\frac{(x_i-x'_i)(x_j-x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x'\\ &= -G\int_V d^3x\rho(\mathbf{x})\varphi(\mathbf{x})\int_V d^3x'\rho(\mathbf{x}')\frac{(x_i-x'_i)(x_j-x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}\\ &= -\int \rho\varphi\mathfrak{U}_{ij}d^3x = -\mathfrak{F}_{ij}. \end{aligned} \tag{2.305}$$

where $\mathfrak{U}_{ij}$ is the gravitational potential tensor (2.262). Note that $\mathfrak{F}_{ij}$ is a symmetric tensor.

(xiii)

$$\begin{aligned}\int_V \rho x_j V_k\frac{\partial U_k}{\partial x^i}d^3x = -\frac{G}{2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')V_k(\mathbf{x})V_k(\mathbf{x}')\\ \times\frac{(x_i-x'_i)(x_j-x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x',\end{aligned} \tag{2.306}$$

is also a symmetric tensor. We collect together the following terms that appear in (2.302), (2.303) and (2.306) and denote it by the symmetric tensor $\mathfrak{V}_{ij}$, namely

$$\mathfrak{V}_{ij} = \int_V \rho \left[V_j \left(U_i - UV_i \right) + x_j V_k \frac{\partial U_k}{\partial x^i} \right] d^3x. \tag{2.307}$$

(xiv) For the last integral obtained from (2.296) we incorporate the integral (2.303), use (2.293) and introduce the tensor $\mathfrak{X}_{ij}$

$$\begin{aligned}\mathfrak{X}_{ij} &= \int_V \rho(\mathbf{x}) x_j V_k(\mathbf{x}) \frac{\partial^3 \chi(\mathbf{x})}{\partial x^i \partial x^k \partial t} d^3x \\ &+ \int_V \rho(\mathbf{x}) V_j(\mathbf{x}) \frac{\partial^2 \chi(\mathbf{x})}{\partial t \partial x^i} d^3x \\ &= -G \int_V \int_V \rho(\mathbf{x}')\rho(\mathbf{x}) \Bigg\{ \frac{1}{2} \frac{V_k(\mathbf{x})V_k(\mathbf{x}')(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} \\ &+ \frac{V_k(\mathbf{x}')(x_k - x_k') \left[V_j(\mathbf{x})(x_i - x_i') + V_i(\mathbf{x})(x_j - x_j') \right]}{|\mathbf{x} - \mathbf{x}'|^3} \\ &- \frac{3}{2} \frac{V_k(\mathbf{x}')(x_k - x_k')V_l(\mathbf{x})(x_l - x_l')(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^5} \\ &- \frac{V_j(\mathbf{x})V_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \Bigg\} d^3x d^3x'.\end{aligned} \tag{2.308}$$

From the above equation we conclude that the tensor $\mathfrak{X}_{ij}$ is also symmetric.

By collecting the results (viii) – (xiv) above we get that

$$\frac{d}{dt} \int_V x_j \mathfrak{P}_i d^3x = 2\mathfrak{K}_{ij} + \mathfrak{W}_{ij} + \delta_{ij} \int_V \left[p \left(1 + \frac{2U}{c^2} \right) \right] d^3x$$

$$-\frac{1}{c^2}\left(2\mathfrak{F}_{ij}+4\mathfrak{V}_{ij}-\frac{1}{2}\mathfrak{X}_{ij}\right), \tag{2.309}$$

where $\mathfrak{P}_i$ is the total linear momentum density (2.295).

Now by taking the antisymmetric part of (2.309) it follows the conservation of the total angular momentum, which is expressed as

$$\frac{d}{dt}\int_V (x_j\mathfrak{P}_i - x_i\mathfrak{P}_j)\, d^3x = 0. \tag{2.310}$$

Conservation of total energy density

The hydrodynamic equation for the total energy density (2.134) – which is a sum of the kinetic $\rho V^2/2$ and internal $\rho\varepsilon$ energy densities – written in terms of Chandrasekhar's potentials is

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x^i}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)V_i\right]+\frac{\partial pV_i}{\partial x^i}$$
$$-\rho V_i\frac{\partial U}{\partial x^i}-U\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho V_i}{\partial x^i}\right)=0. \tag{2.311}$$

Following the same methodology above we evaluate the integrals over the volume V separately.

(xv)

$$\int_V\left\{\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)\right]+\frac{\partial}{\partial x^i}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)V_i\right]\right\}d^3x$$
$$=\frac{d}{dt}\int_V\rho\left(\frac{V^2}{2}+\varepsilon\right)d^3x, \tag{2.312}$$

thanks to Reynolds' transport theorem (2.282).

(xvi)

$$\int_V \frac{\partial p V_i}{\partial x^i} d^3x = 0, \tag{2.313}$$

by using Gauss divergence theorem and assuming that the pressure vanishes on the boundary of the configuration.

(xvii)

$$\begin{aligned}
&\int_V \rho(\mathbf{x}) V_i(\mathbf{x}) \frac{\partial U(\mathbf{x})}{\partial x^i} d^3x \\
&= -G \int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}') V_i(\mathbf{x}) \frac{x_i - x_i'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x d^3x' \\
&= -\frac{G}{2} \int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}') [V_i(\mathbf{x}) - V_i(\mathbf{x}')] \frac{x_i - x_i'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x d^3x' \\
&= \frac{G}{2} \int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}') \frac{d}{dt} \frac{1}{|\mathbf{x}-\mathbf{x}'|} d^3x d^3x' \\
&= \frac{G}{2} \int_V \rho(\mathbf{x}) d^3x \left[\frac{d}{dt} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \right. \\
&\left. - \int_V \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} \left(\frac{\partial \rho(\mathbf{x}')}{\partial t} + \frac{\partial \rho(\mathbf{x}') V_i(\mathbf{x}')}{\partial x^i} \right) \right] \\
&= -\frac{1}{2} \int_V U(\mathbf{x}') \left(\frac{\partial \rho(\mathbf{x}')}{\partial t} + \frac{\partial \rho(\mathbf{x}') V_i(\mathbf{x}')}{\partial x^i} \right) d^3x' \\
&+ \frac{1}{2} \int_V \rho(\mathbf{x}) \frac{dU(\mathbf{x})}{dt} d^3x = \frac{1}{2} \frac{d}{dt} \int_V \rho(\mathbf{x}) U(\mathbf{x}) d^3x \\
&- \int_V U(\mathbf{x}) \left(\frac{\partial \rho(\mathbf{x})}{\partial t} + \frac{\partial \rho(\mathbf{x}) V_i(\mathbf{x})}{\partial x^i} \right) d^3x.
\end{aligned} \tag{2.314}$$

Above we have applied twice the Reynolds transport theorem (2.282). Note that the second integral in the last equality is just the integral of the last term in (2.311).

By collecting the results (xv) – (xvii) follows the conservation of total energy

$$\frac{d}{dt}\int_V \rho\left(\frac{V^2}{2}+\varepsilon-\frac{U}{2}\right)d^3x = 0. \tag{2.315}$$

This is the Newtonian expression for the total energy density conservation. For the determination of the post-Newtonian contribution to the total energy density we have to go further and find the second post-Newtonian approximation. This will be the subject of the next chapter.

2.9 The post-Newtonian virial theorem

In this section we shall determine the post-Newtonian approximation of the tensor virial theorem. We begin by considering one-half of the symmetric part of (2.309), namely

$$\frac{1}{2}\frac{d}{dt}\int_V (x_j\mathfrak{P}_i + x_i\mathfrak{P}_j)d^3x = 2\mathfrak{K}_{ij} + \mathfrak{W}_{ij}$$
$$+\delta_{ij}\int_V \left[p\left(1+\frac{2U}{c^2}\right)\right]d^3x - \frac{1}{c^2}\left(2\mathfrak{F}_{ij} + 4\mathfrak{V}_{ij} - \frac{1}{2}\mathfrak{X}_{ij}\right). \tag{2.316}$$

Next we introduce the moment of inertia tensor in terms of the mass density ρ_* defined in (2.123), namely

$$\mathfrak{I}_{ij} = \int_V \rho_* x_i x_j d^3x, \tag{2.317}$$

so that its time derivative becomes

$$\begin{aligned}\frac{d\mathfrak{I}_{ij}}{dt} &= \int_V \left[\frac{\partial \rho_* x_i x_j}{\partial t} + \frac{\partial \rho_* x_i x_j V_k}{\partial x^k}\right] d^3x \\ &= \int_V \left[\frac{\partial \rho_*}{\partial t} + \frac{\partial \rho_* V_k}{\partial x^k}\right] x_i x_j d^3x + \int_V \rho_*(x_i V_j + x_j V_i) d^3x \\ &= \int_V \rho_*(x_i V_j + x_j V_i) d^3x. \end{aligned} \tag{2.318}$$

Here the Reynolds transport theorem (2.282) was used to write the first equality, moreover the first integral in the second equality vanishes due to the continuity equation (2.122). Hence we obtain from the time differentiation of (2.318) that

$$\frac{d^2\mathfrak{I}_{ij}}{dt^2} = \frac{d}{dt}\int_V \rho_*(x_i V_j + x_j V_i) d^3x. \tag{2.319}$$

The post-Newtonian virial theorem follows from (2.316) together with (2.319), yielding

$$\begin{aligned}&\frac{1}{2}\frac{d^2\mathfrak{I}_{ij}}{dt^2} + \frac{1}{c^2}\frac{d}{dt}\int_V \rho\left[\left(\frac{V^2}{2} + 3U + \varepsilon + \frac{p}{\rho}\right)(V_i x_j + V_j x_i)\right. \\ &\left.-(\Pi_i x_j + \Pi_j x_i)\right] d^3x = 2\delta_{ij}\int_V \left[p\left(1 + \frac{2U}{c^2}\right)\right] d^3x \\ &+\mathfrak{K}_{ij} + \mathfrak{W}_{ij} - \frac{1}{c^2}\left(2\mathfrak{F}_{ij} + 4\mathfrak{V}_{ij} - \frac{1}{2}\mathfrak{X}_{ij}\right), \end{aligned} \tag{2.320}$$

where the total linear momentum density (2.295) was writing as

$$\mathfrak{P}_i = \rho_* V_i + \frac{\rho}{c^2}\left[\left(\frac{V^2}{2} + 3U + \varepsilon + \frac{p}{\rho}\right) V_i - \Pi_i\right], \quad (2.321)$$

thanks to (2.123).

In the Newtonian case (2.320) reduces to the well-know expression for the tensor virial theorem

$$\frac{1}{2}\frac{d}{dt}\int_V \rho\left(x_i V_j + x_i V_j\right) d^3x = \int_V \rho V_i V_j d^3x + \delta_{ij}\int_V p d^3x$$
$$+\frac{1}{2}\int_V \rho\left(\frac{\partial U}{\partial x^i} x_j + \frac{\partial U}{\partial x^j} x_i\right) d^3x, \quad (2.322)$$

by considering the expression for the gravitational potential energy tensor $\mathfrak{W}_{ij}$ given in (2.278). Its contracted expression by taking into account (2.280) is

$$\frac{d}{dt}\int_V \rho x_i V_i d^3x = \int_V \rho\left(V^2 - \frac{U}{2} + 3p\right) d^3x. \quad (2.323)$$

The stationary version of the post-Newtonian virial theorem (2.320) reduces to

$$2\mathfrak{K}_{ij} + \mathfrak{W}_{ij} + \delta_{ij}\int_V \left[p\left(1 + \frac{2U}{c^2}\right)\right] d^3x$$
$$-\frac{1}{c^2}\left(2\mathfrak{F}_{ij} + 4\mathfrak{V}_{ij} - \frac{1}{2}\mathfrak{X}_{ij}\right) = 0. \quad (2.324)$$

The contracted version of the above equation reads

$$\int_V \rho\left(V^2 - \frac{U}{2} + 3\frac{p}{\rho}\right) d^3x + \frac{1}{c^2}\left\{\int_V \rho\left[V^2\left(V^2 + \varepsilon\right.\right.\right.$$

$$
\left. + \frac{p}{\rho} + 4U \right) - U\left(2U + \varepsilon - 3\frac{p}{\rho}\right) - \frac{7}{4}V_iU_i \Bigg] d^3x
$$

$$
- \frac{G}{4}\int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}')d^3xd^3x'
$$

$$
\left. \times \frac{V_i(\mathbf{x}')(x_i - x_i')V_j(\mathbf{x})(x_j - x_j')}{|\mathbf{x}-\mathbf{x}'|^3} \right\} = 0, \tag{2.325}
$$

thanks to the expressions of the contracted tensors

$$
2\mathfrak{K}_{ii} = \int_V \rho V^2 \left[1 + \frac{1}{c^2}\left(V^2 + 2U + \varepsilon + \frac{p}{\rho}\right)\right], \tag{2.326}
$$

$$
\mathfrak{W}_{ii} = -\frac{1}{2}\int_V \rho U d^3x, \tag{2.327}
$$

$$
\mathfrak{V}_{ii} = \frac{1}{2}\int_V \rho\left(U_iV_i - 2UV^2\right)d^3x, \tag{2.328}
$$

$$
\mathfrak{F}_{ii} = \int_V \rho U\left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right)d^3x, \tag{2.329}
$$

$$
\mathfrak{X}_{ii} = \frac{1}{2}\int_V \rho U_iV_i d^3x - \frac{G}{2}\int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}') \times \frac{V_i(\mathbf{x}')(x_i - x_i')V_j(\mathbf{x})(x_j - x_j')}{|\mathbf{x}-\mathbf{x}'|^3}d^3xd^3x'. \tag{2.330}
$$

The contacted version of the stationary version of the post-Newtonian virial theorem (2.325) corresponds to the equation *(146)* of [3].

Appendix

In this appendix we give some relationships that are used in the previous sections.

If g denotes the determinant of the metric tensor $g_{\mu\nu}$ its differentiation with respect to x^μ is

$$\frac{\partial g}{\partial x^\mu} = g_{,\mu} = g g^{\nu\sigma} g_{\nu\sigma,\mu}, \tag{2.331}$$

which is obtained from the differentiation of each component of the metric tensor $g_{\nu\sigma}$ multiplied by its cofactor $gg^{\nu\sigma}$.

The contracted Christoffel symbols are also expressed in terms of the determinant of the metric tensor as

$$\begin{aligned}\Gamma^\nu{}_{\mu\nu} &= \frac{g^{\sigma\nu}}{2}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right) \\ &= \frac{g^{\sigma\nu} g_{\sigma\nu,\mu}}{2} = \frac{g_{,\mu}}{2g} = \left(\ln\sqrt{-g}\right)_{,\mu}, \end{aligned} \tag{2.332}$$

thanks to (2.331).

From the vanishing covariant derivative of the metric tensor $g^{\mu\nu}{}_{;\sigma} = 0$ we have that

$$g^{\mu\nu}{}_{,\sigma} = -g^{\mu\tau}\Gamma^\nu{}_{\tau\sigma} - g^{\nu\tau}\Gamma^\mu{}_{\tau\sigma}, \tag{2.333}$$

from which together with (2.332) yields

$$\left(g^{\mu\nu}\sqrt{-g}\right)_{,\nu} = -g^{\nu\sigma}\Gamma^\mu{}_{\sigma\nu}\sqrt{-g}. \tag{2.334}$$

The variation of the metric tensor $g^{\mu\nu}$ is obtained from

$$\delta\left(g^{\mu\sigma} g_{\sigma\nu}\right) = \delta g^{\mu\sigma} g_{\sigma\nu} + g^{\mu\sigma}\delta g_{\sigma\nu} = \delta(\delta^\mu_\nu) = 0, \tag{2.335}$$

which by the multiplication with $g^{\nu\tau}$ implies that

$$\delta g^{\mu\tau} = -g^{\mu\sigma} g^{\nu\tau} \delta g_{\sigma\nu}. \tag{2.336}$$

On the other hand the variation of $\sqrt{-g}$ is given by

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta(-g) = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}, \tag{2.337}$$

if we use a corresponding equation to (2.331). Hence from (2.336) and (2.337) we have

$$\delta\left(g^{\mu\nu}\sqrt{-g}\right) = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}g^{\sigma\tau} - g^{\mu\sigma}g^{\nu\tau}\right)\delta g_{\sigma\tau}. \tag{2.338}$$

References

[1] A. Einstein, L. Infeld and B. Hoffmann, The gravitational equations and the problem of motion, *Ann. of Math.* **39**, 65 (1938).

[2] S. Weinberg, *Gravitation and cosmology. Principles and applications of the theory of relativity* (Wiley, New York, 1972).

[3] S. Chandrasekhar, The post-Newtonian equations of hydrodynamics in general relativity, *Ap. J.* **142**, 1488 (1965).

[4] S. Chandrasekhar, Conservation laws in general relativity and in the post-Newtonian approximations, *Ap. J.* **158**, 45 (1969).

[5] L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, 4th ed. (Pergamon Press, Oxford, 1980).

[6] V. Fock, *The theory of space time and gravitation* (Pergamon Press, London, 1959).

[7] S. Chandrasekhar and N. R. Lebovitz, On super-potentials in the theory of Newtonian gravitation, *Ap. J.* **135**, 238 (1961).

[8] C. Brans and R. H. Dicke, Mach's principle and a relativistic theory of gravitation, *Phys. Rev.* **124**, 925 (1961).

[9] D. W. Sciama, On the origin of inertia, *Monthly Notices Roy. Astron. Soc.* **113**, 34 (1953).

[10] R. H. Dicke, New research on old gravitation, *Science* **129**, 621 (1959).

[11] P. A. M. Dirac, *General theory of relativity* (Princeton University Press, Princeton, 1996)

[12] Y. Nutku, The post-Newtonian equations of hydrodynamics in the Brans-Dicke theory, *Ap. J.* **155**, 999 (1969).

[13] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann equation: theory and applications* (Birkäuser, Basel, 2002).

[14] C. Eckart, The thermodynamics of irreversible processes, III. Relativistic theory of a simple fluid, *Phys. Rev.* **58**, 919 (1940).

CHAPTER 3

SECOND POST-NEWTONIAN APPROXIMATION

The second post-Newtonian approximation to Einstein's field equations is analyzed in this chapter. The corresponding Poisson equations for the gravitational potentials and the Eulerian hydrodynamic equations are obtained in this approximation. The search for the conservation laws of the total linear momentum density and total energy density are based on the framework of general relativity conservation laws where the energy-momentum pseudo-tensor plays an important role in the determination of the conservative quantities.

3.1 Preliminaries

Apart from the knowledge of the hydrodynamic equations in the first post-Newtonian approximation it is important to know the corresponding expressions for the conserved quantities which are related with the total linear and angular momenta and total energy of the system. The expressions for these quantities were derived by Chandrasekhar in [1] by using the hydrodynamic equations supplied by the condition of isentropic flow. This condition – which is a consequence of the first law of thermodynamics $d'Q = dE + pdV = 0$ – can be obtained from (2.136) together with the continuity equation $d\rho/dt + \rho\partial V_i/\partial x^i = 0$, yielding

$$\frac{d\varepsilon}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt}. \qquad (3.1)$$

In a later paper [2] Chandrasekhar derived the conservation laws within the framework of general relativity by taking into account the symmetric energy-momentum complex of Landau and Lifshitz [3] which is a sum of the energy-momentum tensor and the energy-momentum pseudo-tensor.[1] He showed that the use of the energy-momentum complex in the first post-Newtonian approximation led to the Newtonian conservation law for the total energy of the system and argued that the first post-Newtonian conservation law for the total energy of the sys-

[1] In Section 3.7 the energy-momentum pseudo-tensor and the energy-momentum complex will be introduced.

tem should be derived from the knowledge of the second post-Newtonian approximation.

The second post-Newtonian approximation and the corresponding conservation laws were developed by Chandrasekhar and Nutku [4]. The starting point is the decompositions of the components of the metric tensor

$$g_{00} = 1 - \frac{2U}{c^2} + \frac{2}{c^4}\left(U^2 - 2\Phi\right) + \frac{\Psi_{00}}{c^6} + \mathcal{O}(c^{-8}), \quad (3.2)$$

$$g_{0i} = \frac{1}{c^3}\left(4U_i - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}\right) + \frac{\Psi_{0i}}{c^5} + \mathcal{O}(c^{-7}), \quad (3.3)$$

$$g_{ij} = -\left(1 + \frac{2U}{c^2}\right)\delta_{ij} + \frac{\Psi_{ij}}{c^4} + \mathcal{O}(c^{-6}). \quad (3.4)$$

Here new gravitational potentials Ψ_{00}, Ψ_{0i} and Ψ_{ij} were introduced and they will be determined from Einstein's field equations in the next sections.

In the previous chapter it was shown that the gravitational potentials U, U_i, χ and Φ satisfy the Poisson equations

$$\nabla^2 U = -4\pi G\rho, \qquad \nabla^2 U_i = -4\pi G\rho V_i, \qquad \nabla^2\chi = -2U, \quad (3.5)$$

$$\nabla^2\Phi = -4\pi G\rho\varphi = -4\pi G\rho\left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right). \quad (3.6)$$

We follow [4] and introduce the vector gravitational potential Π_i defined by the relationship

$$\Pi_i = 4U_i - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}, \quad (3.7)$$

and we note that $\Pi_i = -\xi_i$, which is the vector gravitational potential in Weinberg's method. Furthermore, we have from (3.5) and (3.7) that the Poisson equation for Π_i is given by

$$\nabla^2\Pi_i = -16\pi G\rho V_i + \frac{\partial^2 U}{\partial t \partial x^i}, \tag{3.8}$$

while the gauge condition (2.112) implies that

$$\frac{3}{c^2}\frac{\partial U}{\partial t} + \frac{1}{c^2}\frac{\partial \Pi_i}{\partial x_i} = 0. \tag{3.9}$$

The contra- and co-variant components of the four-velocity and of the tensor $\mathfrak{T}_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}T^{\sigma}{}_{\sigma}/2$ are given in Section 2.3.2 which we reproduce here in terms of the Chandrasekhar potentials

$$U^0 = c\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}+U\right)+\frac{1}{c^4}\left(\frac{3V^4}{8}+\frac{5UV^2}{2}+\frac{U^2}{2}+2\Phi-\Pi_i V_i\right)\right], \tag{3.10}$$

$$U^i = \frac{U^0}{c}V_i, \qquad U_i = -V_i\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}+3U\right)\right]+\frac{\Pi_i}{c^2}, \tag{3.11}$$

$$U_0 = c\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}-U\right)+\frac{1}{c^4}\left(\frac{3V^4}{8}+\frac{3UV^2}{2}+\frac{U^2}{2}-2\Phi\right)\right], \tag{3.12}$$

$$\mathfrak{T}_{00} = \frac{\rho c^2}{2} + \rho\left(V^2 - U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right) + \frac{\rho}{c^2}\left[V^4 + U^2\right.$$

$$+V^2\left(\varepsilon+\frac{p}{\rho}+2U\right)-U\left(\varepsilon+3\frac{p}{\rho}\right)-2\Phi\Bigg], \tag{3.13}$$

$$\mathfrak{T}_{ij}=\frac{\rho c^2}{2}\delta_{ij}+\rho\left[V_iV_j+\frac{1}{2}\left(\varepsilon-\frac{p}{\rho}+2U\right)\delta_{ij}\right], \tag{3.14}$$

$$\mathfrak{T}_{0i}=-\rho c V_i-\frac{\rho}{c}\left[V_i\left(V^2+\varepsilon+\frac{p}{\rho}+2U\right)-\frac{\Pi_i}{2}\right]. \tag{3.15}$$

The components of the Christoffel symbols are given in Appendix A.

In the next sections we shall determine the second post-Newtonian approximation of the metric tensor by following very close the paper of Chandrasekhar and Nutku [4].

3.2 Equation for determination Ψ_{ij}

For the determination of Ψ_{ij} we have to solve the spatial components of Einstein's field equations

$$\overset{4}{R}_{ij}=-\frac{8\pi G}{c^4}\overset{2}{\mathfrak{T}}_{ij}, \tag{3.16}$$

where the spatial components of the Ricci tensor are given by

$$\overset{4}{R}_{ij}=\frac{\partial\overset{4}{\Gamma}{}^0{}_{0i}}{\partial x^j}+\frac{\partial\overset{4}{\Gamma}{}^k{}_{ik}}{\partial x^j}-\frac{1}{c}\frac{\partial\overset{3}{\Gamma}{}^0{}_{ij}}{\partial t}-\frac{\partial\overset{4}{\Gamma}{}^k{}_{ij}}{\partial x^k}+\overset{2}{\Gamma}{}^0{}_{0i}\overset{2}{\Gamma}{}^0{}_{0j}$$
$$+\overset{2}{\Gamma}{}^k{}_{il}\overset{2}{\Gamma}{}^l{}_{kj}-\overset{2}{\Gamma}{}^k{}_{ij}\overset{2}{\Gamma}{}^0{}_{k0}-\overset{2}{\Gamma}{}^k{}_{ij}\overset{2}{\Gamma}{}^l{}_{kl}. \tag{3.17}$$

By considering the components of the Christoffel symbols given in the Appendix A the left-hand side of Einstein's field equations (3.16) reduces to

$$\overset{4}{R}_{ij} = -\frac{1}{2c^4}\left[\nabla^2\Psi_{ij} + \frac{\partial^2\Psi_{kk}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^k}\left(\frac{\partial\Psi_{ik}}{\partial x^j} + \frac{\partial\Psi_{jk}}{\partial x^i}\right)\right]$$
$$-\frac{1}{c^4}\left[2\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} - \frac{\partial^2(U^2+2\Phi)}{\partial x^i \partial x^j}\right.$$
$$\left.-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\Pi_i}{\partial x^j} + \frac{\partial\Pi_j}{\partial x^i}\right) - \delta_{ij}\left(\nabla U^2 + \frac{\partial^2 U}{\partial t^2}\right)\right]. \qquad (3.18)$$

Furthermore, from (3.14) its right-hand side becomes

$$-\frac{8\pi G}{c^4}\overset{2}{\mathfrak{T}}_{ij} = -\frac{8\pi G}{c^4}\left[V_iV_j + \frac{1}{2}\left(\varepsilon - \frac{p}{\rho} + 2U\right)\delta_{ij}\right]$$
$$+\frac{2}{c^4}\left[\underline{\nabla^2\Phi + 4\pi G\rho\left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right)}\right]\delta_{ij}$$
$$= \frac{2}{c^2}\left[\nabla^2\Phi\delta_{ij} - 4\pi G\rho\left(V_iV_j - V^2\delta_{ij} - 2\frac{p}{\rho}\delta_{ij}\right)\right]. \qquad (3.19)$$

Note that the underlined term is identically zero due to (3.6) and it has been added in order to write a more compact form of Einstein's field equations (3.16). Hence, from (3.18) and (3.19) the spatial components of Einstein's field equations (3.16) can be written as

$$\nabla^2\Psi_{ij} - \frac{\partial}{\partial x^i}\left(\frac{\partial\Psi_{jk}}{\partial x^k} - \frac{1}{2}\frac{\partial\Psi_{kk}}{\partial x^j}\right)$$
$$-\frac{\partial}{\partial x^j}\left(\frac{\partial\Psi_{ik}}{\partial x^k} - \frac{1}{2}\frac{\partial\Psi_{kk}}{\partial x^i}\right) = S_{ij}, \qquad (3.20)$$

where S_{ij} is the symmetric tensor

$$
\begin{aligned}
S_{ij} = -2\left(\delta_{ij}\nabla^2 + \frac{\partial^2}{\partial x^i \partial x^j}\right)(U^2 + 2\Phi) + 4\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} \\
-2\frac{\partial^2 U}{\partial t^2}\delta_{ij} - \frac{\partial}{\partial t}\left(\frac{\partial \Pi_i}{\partial x^j} + \frac{\partial \Pi_j}{\partial x^i}\right) \\
+16\pi G\rho\left(V_i V_j - V^2\delta_{ij} - 2\frac{p}{\rho}\delta_{ij}\right).
\end{aligned}
\tag{3.21}
$$

Equation (3.20) can be reduced to a simpler form by noting that its contraction and differentiation with respect to x^j leads respectively to

$$\nabla^2\Psi_{kk} - \frac{\partial^2 \Psi_{jk}}{\partial x^j \partial x^k} = \frac{1}{2}S_{kk}, \tag{3.22}$$

$$\frac{\partial}{\partial x^i}\left(\nabla^2\Psi_{kk} - \frac{\partial^2 \Psi_{jk}}{\partial x^j \partial x^k}\right) = \frac{\partial S_{ij}}{\partial x^j}. \tag{3.23}$$

The combination of the two above equations implies the following integrability condition for (3.20):

$$\frac{\partial}{\partial x^i}\left(S_{ij} - \frac{1}{2}S_{kk}\delta_{ij}\right) = 0. \tag{3.24}$$

Now by introducing an arbitrary vector function w_i defined by

$$w_i = \frac{\partial \Psi_{ij}}{\partial x^j} - \frac{1}{2}\frac{\partial \Psi_{jj}}{\partial x^i}, \tag{3.25}$$

we can rewrite (3.20) as

$$\nabla^2\Psi_{ij} = S_{ij} + \frac{\partial w_i}{\partial x^j} + \frac{\partial w_j}{\partial x^i}. \tag{3.26}$$

The above equation satisfies the integrability condition (3.24). Indeed, its contraction

$$\nabla^2\Psi_{kk} = S_{kk} + 2\frac{\partial w_k}{\partial x^k}, \tag{3.27}$$

together with the differentiation of the combination $\nabla^2\Psi_{ij} - \frac{1}{2}\nabla^2\Psi_{kk}\delta_{ij}$ with respect to x^j imply that

$$\frac{\partial}{\partial x^j}\left(\nabla^2\Psi_{ij} - \frac{1}{2}\nabla^2\Psi_{kk}\delta_{ij}\right) = \nabla^2 w_i, \tag{3.28}$$

which is just the Laplacian of (3.25). Hence we can consider (3.26) as the Poisson equation for the determination of the tensor gravitational potential Ψ_{ij} in terms of the arbitrary spacetime vector function w_i.

Here it is necessary to verify the validity of the integrability condition (3.24). For that end we contract S_{ij} from (3.21) and get

$$\begin{aligned} S_{kk} &= -8\nabla^2(U^2 + 2\Phi) - 32\pi G\rho\left(V^2 + 3\frac{p}{\rho}\right) \\ &\qquad -2\frac{\partial}{\partial t}\left(3\frac{\partial U}{\partial t} + \frac{\partial \Pi_i}{\partial x^i}\right) + 4\left(\frac{\partial U}{\partial x^j}\right)^2 \\ &= 32\pi G\rho\left(V^2 + 4U + \varepsilon\right) - 12\left(\frac{\partial U}{\partial x^j}\right)^2. \end{aligned} \tag{3.29}$$

Here we have used the Poisson equations for the gravitational potentials U and Φ given by $(3.5)_1$ and (3.6), respectively, and the gauge condition (3.9). Next we differentiate S_{ij} – given by (3.21) – with respect to x^j, yielding

$$\frac{\partial S_{ij}}{\partial x^j} = -4\frac{\partial}{\partial x^i}\left[\nabla^2\left(U^2+2\Phi\right)\right] + 4\left(\frac{\partial U}{\partial x^i}\nabla^2 U + \frac{\partial^2 U}{\partial x^i \partial x^j}\frac{\partial U}{\partial x^j}\right) + 16\pi G\frac{\partial}{\partial x^j}\left[\rho\left(V_iV_j - V^2\delta_{ij}\right) - 2p\delta_{ij}\right] - \frac{\partial^2}{\partial t\partial x^i}\left(2\frac{\partial U}{\partial t} + \frac{\partial \Pi_j}{\partial x^j}\right) - \frac{\partial}{\partial t}\nabla^2\Pi_i = 16\pi G\left[\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x^j} + \frac{\partial p}{\partial x^i} - \rho\frac{\partial U}{\partial x^i} + \frac{\partial \rho(V^2+4U+\varepsilon)}{\partial x^i}\right] - 12\frac{\partial^2 U}{\partial x^i\partial x^j}\frac{\partial U}{\partial x^j}, \tag{3.30}$$

where we have also taken into account the relationship $\nabla^2 U^2 = 2U\nabla^2 U + 2(\nabla U)^2$, the Poisson equation (3.8) for Π_i and the gauge condition (3.9). Now from the two last equations (3.29) and (3.30) we have

$$\frac{\partial}{\partial x^i}\left(S_{ij} - \frac{1}{2}S_{kk}\delta_{ij}\right) = 16\pi G\left[\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x^j} + \frac{\partial p}{\partial x^i} - \rho\frac{\partial U}{\partial x^i}\right]. \tag{3.31}$$

One recognizes the term within the brackets as the Newtonian hydrodynamic equation for the momentum density (2.129), which can be taken here equal to zero, since we are dealing with equations that are at most of order $\mathcal{O}\left(c^{-4}\right)$. Hence, the integrability condition (3.24) is verified.

3.3 Equation for determination Ψ_{0i}

The knowledge of Ψ_{0i} follows from the solution of Einstein's field equations

$$\overset{5}{R}_{0i} = -\frac{8\pi G}{c^4}\overset{3}{\mathfrak{T}}_{0i}, \tag{3.32}$$

where its right-hand side, thanks to (3.15), is given by

$$-\frac{8\pi G}{c^4}\overset{3}{\mathfrak{T}}_{0i} = \frac{8\pi G}{c^5}\rho\left[V_i\left(V^2+\varepsilon+\frac{p}{\rho}+2U\right)-\frac{\Pi_i}{2}\right]. \tag{3.33}$$

The space-time component of the Ricci tensor read

$$\overset{5}{R}_{0i} = \frac{\partial\overset{5}{\Gamma}{}^0{}_{00}}{\partial x^i}+\frac{\partial\overset{5}{\Gamma}{}^k{}_{0k}}{\partial x^i}-\frac{1}{c}\frac{\partial\overset{4}{\Gamma}{}^0{}_{0i}}{\partial t}-\frac{\partial\overset{5}{\Gamma}{}^j{}_{0i}}{\partial x^j}+\overset{2}{\Gamma}{}^j{}_{00}\overset{3}{\Gamma}{}^0{}_{ji}$$
$$+\overset{2}{\Gamma}{}^k{}_{ij}\overset{3}{\Gamma}{}^j{}_{0k}-\overset{2}{\Gamma}{}^0{}_{0i}\overset{3}{\Gamma}{}^j{}_{0j}-\overset{2}{\Gamma}{}^j{}_{kj}\overset{3}{\Gamma}{}^k{}_{0i}. \tag{3.34}$$

If we take into account the components of the Christoffel symbols given in the Appendix A, the Ricci tensor (3.34) becomes

$$\overset{5}{R}_{0i} = -\frac{1}{2c^5}\left[\nabla^2\Psi_{0i}-\frac{\partial^2\Psi_{0k}}{\partial x^k\partial x^i}\right]+\frac{1}{2c^5}\left[\frac{\partial w_i}{\partial t}-\frac{1}{2}\frac{\partial\Psi_{kk}}{\partial t\partial x^i}\right]$$
$$+\frac{1}{c^5}\left(4U\nabla^2U_i-\frac{\partial U}{\partial x^j}\frac{\partial\Pi_j}{\partial x^i}-5\frac{\partial U}{\partial t}\frac{\partial U}{\partial x^i}+\Pi_j\frac{\partial^2 U}{\partial x^i\partial x^j}\right), \tag{3.35}$$

where w_i is the arbitrary vector defined by (3.25).

Einstein's field equations (3.32) together with (3.33) and (3.35) lead to the following Poisson equation for the vector gravitational potential Ψ_{0i}

$$\nabla^2\Psi_{0i} = -16\pi G\rho\left[V_i\left(V^2+\varepsilon+\frac{p}{\rho}+4U\right)-\frac{\Pi_i}{2}\right]$$
$$-10\frac{\partial U}{\partial t}\frac{\partial U}{\partial x^i}-2\frac{\partial U}{\partial x^j}\frac{\partial \Pi_j}{\partial x^i}+2\Pi_j\frac{\partial^2 U}{\partial x^i\partial x^j}+\frac{\partial w_i}{\partial t}+\frac{\partial w}{\partial x^i}, \quad (3.36)$$

where w is an unspecified space-time scalar function introduced by Chandrasekhar and Nutku [4] and defined by

$$\frac{1}{c^2}w = \frac{1}{c^2}\left[\frac{\partial \Psi_{0j}}{\partial x^j}-\frac{1}{2}\frac{\partial \Psi_{jj}}{\partial t}\right]+4\underline{\left(\frac{\partial U}{\partial t}+\frac{\partial U_i}{\partial x^i}\right)}. \quad (3.37)$$

Note that the underlined term is identically zero at $1/c^2$ order according to (2.112).

At this point it is important to check if the definition of w given by (3.37) is consistent with (3.27) and (3.36), which can be verified by taking the Laplacian of (3.37). By taking into account (2.113), (3.29) and introducing $\sigma = \rho[1+(V^2+2U+\varepsilon+p/\rho)/c^2]$ we arrive – after some rearrangements – at the following expression

$$\frac{1}{c^2}\nabla^2 w = \frac{1}{c^2}\left[\frac{\partial \nabla^2\Psi_{0j}}{\partial x^j}-\frac{1}{2}\frac{\partial \nabla^2\Psi_{jj}}{\partial t}\right]+4\left(\frac{\partial \nabla^2 U}{\partial t}\right.$$
$$\left.+\frac{\partial \nabla^2 U_i}{\partial x^i}\right) = \frac{1}{c^2}\nabla^2 w - 16\pi G\left[\frac{\partial \sigma}{\partial t}+\frac{\partial \sigma V_i}{\partial x^i}\right.$$
$$\left.+\frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t}-\frac{\partial p}{\partial t}\right)\right]-\frac{32\pi G}{c^2}\left(\frac{\partial \rho}{\partial t}+\frac{\partial \rho V_i}{\partial x^i}\right). \quad (3.38)$$

Now by making use of (2.113) and the mass-energy density hydrodynamic equation in the first post-Newtonian approximation (2.127), the second and the third terms in the right-hand side of (3.37) vanish and we get the identity $\frac{1}{c^2}\nabla^2 w = \frac{1}{c^2}\nabla^2 w$, confirming the consistency of the definition of the scalar function w, which can be considered an arbitrary function in space-time.

3.4 Equation for determination Ψ_{00}

Here the time component of Einstein's field equations for the determination of Ψ_{00} is

$$\overset{6}{R}_{00} = -\frac{8\pi G}{c^4}\overset{4}{\mathfrak{T}}_{00}, \tag{3.39}$$

and its right-hand side follows from (3.13), yielding

$$-\frac{8\pi G}{c^4}\overset{4}{\mathfrak{T}}_{00} = -\frac{8\pi G}{c^6}\rho\Bigg[V^4 + V^2\left(\varepsilon + \frac{p}{\rho} + 2U\right) -U\left(\varepsilon + 3\frac{p}{\rho}\right) + U^2 - 2\Phi\Bigg]. \tag{3.40}$$

The time component of the Ricci tensor is given by

$$\overset{6}{R}_{00} = \frac{1}{c}\frac{\partial\overset{5}{\Gamma^i}_{0i}}{\partial t} - \frac{\partial\overset{6}{\Gamma^i}_{00}}{\partial x^i} + \overset{2}{\Gamma^i}_{00}\overset{4}{\Gamma^0}_{0i} + \overset{2}{\Gamma^0}_{0i}\overset{4}{\Gamma^i}_{00} +\overset{3}{\Gamma^i}_{0j}\overset{3}{\Gamma^j}_{i0} - \overset{3}{\Gamma^0}_{00}\overset{3}{\Gamma^i}_{0i} - \overset{2}{\Gamma^i}_{00}\overset{4}{\Gamma^j}_{ij} - \overset{2}{\Gamma^j}_{ij}\overset{4}{\Gamma^i}_{00}, \tag{3.41}$$

which by using the components of the Christoffel symbols given in the Appendix A, yields

$$\overset{6}{R}_{00} = -\frac{1}{2c^6}\nabla^2\Psi_{00} + \frac{1}{c^6}\frac{\partial U}{\partial x^i}\frac{\partial \Pi_i}{\partial t} - \frac{3}{c^6}\left(\frac{\partial U}{\partial t}\right)^2 + \frac{1}{c^6}\frac{\partial w}{\partial t}$$
$$+\frac{w_i}{c^6}\frac{\partial U}{\partial x^i} + \frac{1}{c^6}\frac{\partial U}{\partial x^i}\frac{\partial}{\partial x^i}\left(6\Phi - 5U^2\right) + \frac{1}{2c^6}\frac{\partial \Pi_j}{\partial x^i}\left(\frac{\partial \Pi_i}{\partial x^j}\right.$$
$$\left.-\frac{\partial \Pi_j}{\partial x^i}\right) + \frac{2U}{c^2}\nabla^2\left(U^2 - 2\Phi\right) + \frac{1}{c^6}\left(4U^2\delta_{ij} + \Psi_{ij}\right)\frac{\partial^2 U}{\partial x^i \partial x^j}. \quad (3.42)$$

From the knowledge of the left- and right-hand sides of the time component of Einstein's field equations we insert (3.40) and (3.42) into (3.39), use (3.8) and get the following Poisson equation for the scalar gravitational potential Ψ_{00}:

$$\nabla^2\Psi_{00} = 16\pi G\rho\left[V^2\left(V^2 + \varepsilon + \frac{p}{\rho} + 4U\right) - U^2 - 2\Phi\right]$$
$$+2\frac{\partial U}{\partial x^i}\frac{\partial \Pi_i}{\partial t} - 6\left(\frac{\partial U}{\partial t}\right)^2 + 2\frac{\partial w}{\partial t} + 2w_i\frac{\partial U}{\partial x^i} + 12\frac{\partial U}{\partial x^i}\frac{\partial \Phi}{\partial x^i}$$
$$-12U\left(\frac{\partial U}{\partial x^i}\right)^2 + \frac{\partial \Pi_j}{\partial x^i}\left(\frac{\partial \Pi_i}{\partial x^j} - \frac{\partial \Pi_j}{\partial x^i}\right) + 2\Psi_{ij}\frac{\partial^2 U}{\partial x^i \partial x^j}. \quad (3.43)$$

The only difference of this equation with the corresponding one of the paper by Chandrasekhar and Nutku [4] is that they introduce two new quantities Σ_{ij} and W_i through

$$\Sigma_{ij} = \Psi_{ij} - \frac{\partial W_i}{\partial x^j} - \frac{\partial W_j}{\partial x^i}, \quad (3.44)$$

where the following relationships hold $\nabla^2\Sigma_{ij} = S_{ij}$ and $\nabla^2 W_i = w_i$ and verify (3.26).

3.5 A gauge choice

As was noticed in the previous sections, the solutions of Einstein's field equations for the metric tensor components Ψ_{ij} and Ψ_{0i} comprehend the arbitrary scalar function w and the arbitrary vector function w_i. As was pointed out in [4] we have the freedom to choice a particular gauge which do not affect the choices that were already introduced in the analysis of the first post-Newtonian approximation.

The choice proposed in [4] was the vanishing values of both functions, i.e., $w = 0$ and $w_i = 0$. This gauge choice implies that we have from (3.25), (3.37) and (2.117):

$$\frac{\partial \Psi_{ij}}{\partial x^j} - \frac{1}{2}\frac{\partial \Psi_{jj}}{\partial x^i} = 0, \tag{3.45}$$

$$3\frac{\partial U}{\partial t} + \frac{\partial \Pi_i}{\partial x^i} + \frac{1}{c^2}\left[\frac{\partial \Psi_{0j}}{\partial x^j} - \frac{1}{2}\frac{\partial \Psi_{jj}}{\partial t}\right] = 0. \tag{3.46}$$

Here we write a summary of the Poisson equations in the proposed gauge for the second post-Newtonian approximation

$$\nabla^2 U = -4\pi G\rho,\ \nabla^2\Phi = -4\pi G\rho\left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right), \tag{3.47}$$

$$\nabla^2\Pi_i = -16\pi G\rho V_i + \frac{\partial^2 U}{\partial t \partial x^i}, \tag{3.48}$$

$$\nabla^2\Psi_{ij} = 16\pi G\rho\left(V_iV_j - V^2\delta_{ij} - 2\frac{p}{\rho}\delta_{ij}\right) + 4\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j}$$
$$-2\frac{\partial^2 U}{\partial t^2}\delta_{ij} - 2\left(\delta_{ij}\nabla^2 + \frac{\partial^2}{\partial x^i \partial x^j}\right)(U^2 + 2\Phi)$$

$$-\frac{\partial}{\partial t}\left(\frac{\partial \Pi_i}{\partial x^j}+\frac{\partial \Pi_j}{\partial x^i}\right), \quad (3.49)$$

$$\nabla^2\Psi_{0i} = -16\pi G\rho\left[V_i\left(V^2+\varepsilon+\frac{p}{\rho}+4U\right)-\frac{\Pi_i}{2}\right]$$
$$-10\frac{\partial U}{\partial t}\frac{\partial U}{\partial x^i}-2\frac{\partial U}{\partial x^j}\frac{\partial \Pi_j}{\partial x^i}+2\Pi_j\frac{\partial^2 U}{\partial x^i\partial x^j}, \quad (3.50)$$

$$\nabla^2\Psi_{00} = 16\pi G\rho\left[V^2\left(V^2+\varepsilon+\frac{p}{\rho}+4U\right)-U^2-2\Phi\right]$$
$$+2\frac{\partial U}{\partial x^i}\frac{\partial \Pi_i}{\partial t}-6\left(\frac{\partial U}{\partial t}\right)^2+12\frac{\partial U}{\partial x^i}\frac{\partial \Phi}{\partial x^i}-12U\left(\frac{\partial U}{\partial x^i}\right)^2$$
$$+\frac{\partial \Pi_j}{\partial x^i}\left(\frac{\partial \Pi_i}{\partial x^j}-\frac{\partial \Pi_j}{\partial x^i}\right)+2\Psi_{ij}\frac{\partial^2 U}{\partial x^i\partial x^j}. \quad (3.51)$$

Furthermore, in this gauge we have from (3.27) and (3.29)

$$\nabla^2\Psi_{kk} = 32\pi G\rho\left(V^2+4U+\varepsilon\right)-12\left(\frac{\partial U}{\partial x^j}\right)^2. \quad (3.52)$$

3.6 Hydrodynamic equations for an Eulerian fluid

To determine the hydrodynamic equations for an Eulerian fluid in the second post-Newtonian approximation, we shall need the expressions of the four-velocity, particle four-vector and energy momentum tensor in this approximation.

The components of the four-velocity $(U^\mu) = \left(\gamma c, \gamma V^i\right)$ are determined from (2.85) and reads

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{1}{\gamma^2} = g_{00} + \frac{2}{c} g_{0i} V^i + \frac{1}{c^2} g_{ij} V^i V^j$$
$$= 1 - \frac{1}{c^2}\left(V^2 + 2U\right) + \frac{2}{c^4}\left(U^2 - 2\Phi - UV^2 + \Pi_i V_i\right)$$
$$+ \frac{1}{c^6}\left(\Psi_{00} + 2\Psi_{0i} V_i + \Psi_{ij} V_i V_j\right). \tag{3.53}$$

By using the approximation $1/\sqrt{1+x} \approx 1 - x^2 + 3x^2/8 - 5x^3/16$ we find that the time component of the four-velocity becomes

$$U^0 = c\left\{1 + \frac{1}{c^2}\left(\frac{V^2}{2} + U\right) + \frac{1}{c^4}\left(\frac{3V^4}{8} + \frac{5UV^2}{2} + \frac{U^2}{2}\right.\right.$$
$$\left. + 2\Phi - \Pi_i V_i\right) + \frac{1}{c^6}\left[\frac{V^2}{4}\left(\frac{5V^4}{4} + \frac{27UV^2}{2} + 21U^2\right)\right.$$
$$+ 6\left(\frac{V^2}{2} + U\right)\left(\Phi - \frac{\Pi_i V_i}{2}\right) - \frac{U^3}{2}$$
$$\left.\left. - \frac{1}{2}\left(\Psi_{00} + 2\Psi_{0i} V_i + \Psi_{ij} V_i V_j\right)\right]\right\}, \tag{3.54}$$

while the spatial components are simply $U^i = U^0 V_i / c$.

From the knowledge of the four-velocity components it is easy to obtain the components of the particle four-flow $N^\mu = nU^\mu$ up to order $\mathcal{O}\left(c^{-4}\right)$, namely

$$\overset{0}{N}{}^0 = nc, \qquad \overset{2}{N}{}^0 = \frac{n}{c}\left(\frac{V^2}{2} + U\right), \tag{3.55}$$

$$\overset{4}{N}{}^{0} = \frac{n}{c^3}\left(\frac{3V^4}{8} + \frac{5UV^2}{2} + \frac{U^2}{2} + 2\Phi - \Pi_i V_i\right), \quad (3.56)$$

$$\overset{1}{N}{}^{i} = nV_i, \qquad \overset{3}{N}{}^{i} = \frac{nV_i}{c^2}\left(\frac{V^2}{2} + U\right), \quad (3.57)$$

$$\overset{5}{N}{}^{i} = \frac{nV_i}{c^4}\left(\frac{3V^4}{8} + \frac{5UV^2}{2} + \frac{U^2}{2} + 2\Phi - \Pi_i V_i\right). \quad (3.58)$$

The components of the energy-momentum tensor

$$T^{\mu\nu} = \rho\left[1 + \frac{1}{c^2}\left(\varepsilon + \frac{p}{\rho}\right)\right]U^\mu U^\nu - pg^{\mu\nu} \quad (3.59)$$

up to order $\mathcal{O}\left(c^{-6}\right)$ are

$$\overset{0}{T}{}^{00} = \rho c^2, \overset{2}{T}{}^{00} = \rho\left(V^2 + \varepsilon + 2U\right), \quad (3.60)$$

$$\overset{4}{T}{}^{00} = \frac{\rho}{c^2}\Bigg[V^2\left(V^2 + \varepsilon + \frac{p}{\rho} + 6U\right) + 2U\varepsilon$$
$$-2\Pi_i V_i + 2U^2 + 4\Phi\Bigg], \quad (3.61)$$

$$\overset{6}{T}{}^{00} = \frac{\rho}{c^4}\Bigg[V^6 + 10V^4U + 16U^2V^2 - 4\Pi_i V_i\big(V^2 + 2\Phi\big)$$
$$+8\Phi\big(V^2 + 2U\big) + \left(\varepsilon + \frac{p}{\rho}\right)\big(V^4 + 6V^2U - 2\Pi_i V_i\big)$$
$$+2\varepsilon(U^2 + 2\Phi^2\big) - 2\Psi_{0i}V_i - \Psi_{00} - \Psi_{ij}V_iV_j\Bigg], \quad (3.62)$$

$$\overset{1}{T}{}^{i0} = \rho c V_i, \qquad \overset{3}{T}{}^{i0} = \frac{\rho V_i}{c}\left(V^2 + 2U + \varepsilon + \frac{p}{\rho}\right), \quad (3.63)$$

$$\overset{5}{T}{}^{i0} = \frac{\rho}{c^3}\left\{V_i\left[V^4 + 6V^2U + 2U^2 + \left(\varepsilon + \frac{p}{\rho}\right)(V^2 + 2U)\right.\right.$$
$$\left.\left. +4\Phi - 2\Pi_j V_j\right] - \frac{p}{\rho}\Pi_i\right\}, \tag{3.64}$$

$$\overset{2}{T}{}^{ij} = \rho V_i V_j + p\delta_{ij}, \tag{3.65}$$

$$\overset{4}{T}{}^{ij} = \frac{\rho V_i V_j}{c^2}\left(V^2 + 2U + \varepsilon + \frac{p}{\rho}\right) - \frac{2Up}{c^2}\delta_{ij}, \tag{3.66}$$

$$\overset{6}{T}{}^{ij} = \frac{\rho}{c^4}\left\{V_i V_j\left[V^4 + 6V^2U + 2U^2 + \left(\varepsilon + \frac{p}{\rho}\right)(V^2 + 2U)\right.\right.$$
$$\left.\left. +4\Phi - 2\Pi_k V_k\right] + \frac{p}{\rho}\left(4U^2\delta_{ij} + \Psi_{ij}\right)\right\}. \tag{3.67}$$

Mass density hydrodynamic equation

The mass density hydrodynamic equation in the second post-Newtonian approximation follows from the balance equation for the particle four-flow (2.8) which up to $\mathcal{O}\left(c^{-4}\right)$ reads

$$\frac{\partial\left(\overset{0}{N}{}^0 + \overset{2}{N}{}^0 + \overset{4}{N}{}^0\right)}{\partial x^0} + \frac{\partial\left(\overset{1}{N}{}^i + \overset{3}{N}{}^i + \overset{5}{N}{}^i\right)}{\partial x^i}$$
$$+\left(\overset{2}{\Gamma}{}^0{}_{0i} + \overset{2}{\Gamma}{}^j{}_{ij}\right)\overset{3}{N}{}^i + \left(\overset{3}{\Gamma}{}^0{}_{00} + \overset{3}{\Gamma}{}^j{}_{0j} + \overset{5}{\Gamma}{}^0{}_{00} + \overset{5}{\Gamma}{}^j{}_{0j}\right)\overset{0}{N}{}^0$$
$$+\left(\overset{3}{\Gamma}{}^0{}_{00} + \overset{3}{\Gamma}{}^j{}_{0j}\right)\overset{2}{N}{}^0 + \left(\overset{2}{\Gamma}{}^0{}_{i0} + \overset{2}{\Gamma}{}^j{}_{ij} + \overset{4}{\Gamma}{}^0{}_{i0} + \overset{4}{\Gamma}{}^j{}_{ij}\right)\overset{1}{N}{}^i = 0. \tag{3.68}$$

By taking into account the components of the particle four-flow (3.55) – (3.58) and of the Christoffel symbols given in the Appendix A, we arrive at

$$\frac{\partial \mathfrak{n}}{\partial t}+\frac{\partial \mathfrak{n}V_i}{\partial x^i}-\underline{\frac{2n}{c^4}\left[\frac{\partial}{\partial t}+V_i\frac{\partial}{\partial x^i}\right]\left(\Phi+U^2+\frac{\Psi_{kk}}{4}\right)}$$
$$+\underline{2\frac{n}{c^2}\left(1+\frac{V^2}{2c^2}\right)\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)}. \qquad (3.69)$$

where we have introduced the abbreviation

$$\mathfrak{n}=n\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}+U\right)+\frac{1}{c^4}\left(\frac{3V^4}{8}+\frac{5UV^2}{2}\right.\right.$$
$$\left.\left.+\frac{U^2}{2}+2\Phi-\Pi_iV_i\right)\right]. \qquad (3.70)$$

The first underlined term in (3.69) can be rewritten as

$$-\frac{2n}{c^4}\left[\frac{\partial}{\partial t}+V_i\frac{\partial}{\partial x^i}\right]\left(\Phi+U^2+\frac{\Psi_{kk}}{4}\right)$$
$$=-\frac{2}{c^4}\frac{\partial n\left(\Phi+U^2+\frac{\Psi_{kk}}{4}\right)}{\partial t}-\frac{2}{c^4}\frac{\partial nV_i\left(\Phi+U^2+\frac{\Psi_{kk}}{4}\right)}{\partial x^i}$$
$$+\frac{2}{c^4}\left(\Phi+U^2+\frac{\Psi_{kk}}{4}\right)\underline{\left(\frac{\partial n}{\partial t}+\frac{\partial nV_i}{\partial x^i}\right)}. \qquad (3.71)$$

For the above underlined term the Newtonian continuity equation can be used so that it vanishes. The second underlined term in (3.69), by following the same methodology of the first underlined term, can be rewritten as

$$2\frac{n}{c^2}\left(1+\frac{V^2}{2c^2}\right)\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)=\frac{2}{c^2}\frac{\partial nU\left(1+\frac{V^2}{2c^2}+\frac{3U}{2c^2}\right)}{\partial t}$$
$$+\frac{2}{c^2}\frac{\partial nV_iU\left(1+\frac{V^2}{2c^2}+\frac{3U}{2c^2}\right)}{\partial x^i}, \tag{3.72}$$

by taking into account the continuity equation in the first post-Newtonian approximation (2.122), namely

$$\frac{\partial n}{\partial t}+\frac{\partial nV_i}{\partial x^i}=-\frac{1}{c^2}\left[\frac{\partial n\left(\frac{V^2}{2}+3U\right)}{\partial t}+\frac{\partial nV_i\left(\frac{V^2}{2}+3U\right)}{\partial x^i}\right]. \tag{3.73}$$

If we introduce the mass density $\rho=mn$ – where m denotes the fluid particle rest mass – and insert (3.71) and (3.72) into (3.69) it follows the continuity equation

$$\frac{\partial\widetilde{\rho}}{\partial t}+\frac{\partial\widetilde{\rho}V_i}{\partial x^i}=0, \tag{3.74}$$

for the mass density in the second post-Newtonian approximation

$$\widetilde{\rho}=\rho\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}+3U\right)+\frac{1}{c^4}\left(\frac{3}{8}V^4+\frac{7}{2}UV^2+\frac{3}{2}U^2\right.\right.$$
$$\left.\left.-\frac{1}{2}\Psi_{kk}-\Pi_iV_i\right)\right]. \tag{3.75}$$

The mass density above corresponds to eq. *(53)* of [4], which was obtained from the consideration that the volume integral of $\rho U^0\sqrt{-g}$ is constant. This expression is a consequence of the particle four-flow hydrodynamic equation, which can be expressed as

$$(mnU^\mu)_{;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial\rho\sqrt{-g}U^\mu}{\partial x^\mu}$$
$$= \frac{1}{\sqrt{-g}}\left(\frac{\partial\rho\sqrt{-g}U^0}{c\partial t} + \frac{\partial\rho\sqrt{-g}U^0V_i}{c\partial x^i}\right) = 0, \quad (3.76)$$

so that the above equation leads – by using Reynolds transport theorem (2.282) – to the mass conservation

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho U^0\sqrt{-g}d^3x = 0. \quad (3.77)$$

From the components of the metric tensor in the second post-Newtonian approximation (3.2) – (3.4) one obtains that up to order $\mathcal{O}\left(c^{-4}\right)$

$$g = -\left[1 + \frac{4U}{c^2} + \frac{1}{c^4}\left(2U^2 - 4\Phi - \Psi_{kk}\right)\right], \quad (3.78)$$

$$\sqrt{-g} = \left[1 + \frac{2U}{c^2} - \frac{1}{c^4}\left(U^2 + 2\Phi + \frac{\Psi_{kk}}{2}\right)\right], \quad (3.79)$$

which implies that $\rho U^0\sqrt{-g} = \widetilde{\rho}$.

Mass-energy density hydrodynamic equation

The determination of the hydrodynamic equation for the mass-energy density in the second post-Newtonian approximation follows from the time component of the balance equation for the energy-momentum tensor $T^{0\nu}{}_{;\nu} = 0$ in $\mathcal{O}\left(c^{-5}\right)$ – order, namely

$$\frac{\partial\left(\overset{0}{T}{}^{00}+\overset{2}{T}{}^{00}+\overset{4}{T}{}^{00}\right)}{\partial x^0}+\frac{\partial\left(\overset{1}{T}{}^{0i}+\overset{3}{T}{}^{0i}+\overset{5}{T}{}^{0i}\right)}{\partial x^i}$$
$$+\left(2\overset{3}{\Gamma}{}^0{}_{00}+\overset{3}{\Gamma}{}^j{}_{0j}\right)\overset{2}{T}{}^{00}+\left(3\overset{2}{\Gamma}{}^0{}_{i0}+\overset{2}{\Gamma}{}^j{}_{ij}\right)\overset{3}{T}{}^{0i}$$
$$+\left(2\overset{3}{\Gamma}{}^0{}_{00}+\overset{3}{\Gamma}{}^j{}_{0j}+2\overset{5}{\Gamma}{}^0{}_{00}+\overset{5}{\Gamma}{}^j{}_{0j}\right)\overset{0}{T}{}^{00}+\overset{3}{\Gamma}{}^0{}_{ij}\overset{2}{T}{}^{ij}$$
$$+\left(3\overset{2}{\Gamma}{}^0{}_{i0}+\overset{2}{\Gamma}{}^j{}_{ij}+3\overset{4}{\Gamma}{}^0{}_{i0}+\overset{4}{\Gamma}{}^j{}_{ij}\right)\overset{1}{T}{}^{0i}=0. \tag{3.80}$$

From the knowledge of the different orders of the components of the energy-momentum tensor (3.60) – (3.67) and Christoffel symbols given in the Appendix A, one can obtain from (3.80) the hydrodynamic equation for the mass-energy density in the second post-Newtonian approximation

$$\frac{\partial\widetilde{\sigma}}{\partial t}+\frac{\partial\widetilde{\sigma}V_i}{\partial x^i}+\frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t}-\frac{\partial p}{\partial t}\right)+\frac{2\rho}{c^4}\left[\varphi\frac{\partial U}{\partial t}-V_i\frac{\partial\Phi}{\partial x^i}\right.$$
$$\left.+\frac{1}{\rho}\frac{\partial pUV_i}{\partial x^i}-\frac{V_i}{2}\frac{\partial\Pi_i}{\partial t}\right]=0. \tag{3.81}$$

Here we have introduced $\varphi = V^2+U+\frac{\varepsilon}{2}+\frac{3p}{2\rho}$ from (2.111) and

$$\widetilde{\sigma}=\rho\left\{1+\frac{1}{c^2}\left(V^2+2U+\varepsilon+\frac{p}{\rho}\right)+\frac{1}{c^4}\left[V^4+6V^2U-U^2\right.\right.$$
$$\left.\left.+2U\varepsilon+V^2\left(\varepsilon+\frac{p}{\rho}\right)-\Pi_iV_i-\frac{1}{2}\Psi_{kk}\right]\right\}. \tag{3.82}$$

Note that if we consider terms up to order $\mathcal{O}\left(c^{-2}\right)$ equation (3.81) reduces to (2.127). To obtain (3.81) we have neglected the term

$$\frac{1}{c^4}\left(3U^2+4\Phi+\frac{\Psi_{kk}}{2}\right)\left[\frac{\partial\rho}{\partial t}+\frac{\partial\rho V_i}{\partial x^i}\right] \\ -\frac{\Pi_i}{c^4}\left[\frac{\partial\rho V_i}{\partial t}+\frac{\partial\rho V_i V_j}{\partial x^j}+\frac{\partial p}{\partial x^i}-\rho\frac{\partial U}{\partial x^i}\right], \quad (3.83)$$

since we can use the Newtonian continuity equation and the momentum hydrodynamic equation for the first and the second terms within the brackets, respectively.

Hydrodynamic equation for the total energy density

As in Section 2.4 we shall determine here the hydrodynamic equation for the total energy which is a sum of the internal $\rho\varepsilon$ and kinetic $\rho V^2/2$ energy densities. To begin with we subtract the continuity equation (3.74) from the mass-energy density hydrodynamic equation (3.81), yielding

$$\frac{1}{c^2}\left\{\frac{\partial}{\partial t}\left[\rho\left(\frac{V^2}{2}+\varepsilon\right)+\frac{\rho}{c^2}\left(\frac{5}{8}V^4+\frac{5}{2}V^2U-\frac{5}{2}U^2+2\varepsilon U \right.\right.\right. \\ \left.\left. +V^2\left(\varepsilon+\frac{p}{\rho}\right)\right)\right]+\frac{\partial}{\partial x^i}\left[\rho V_i\left(\frac{V^2}{2}+\varepsilon\right)+\frac{\rho V_i}{c^2}\left(\frac{5}{8}V^4\right.\right. \\ \left.\left.+\frac{5}{2}V^2U-\frac{5}{2}U^2+2\varepsilon U+V^2\left(\varepsilon+\frac{p}{\rho}\right)\right)\right]+\frac{\partial pV_i}{\partial x^i} \\ -\rho V_i\frac{\partial U}{\partial x^i}-U\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho V_i}{\partial x^i}\right)+\frac{2\rho}{c^2}\left[\varphi\frac{\partial U}{\partial t}-V_i\frac{\partial\Phi}{\partial x^i}\right.$$

$$+\frac{1}{\rho}\frac{\partial pUV_i}{\partial x^i}-\frac{V_i}{2}\frac{\partial \Pi_i}{\partial t}\Bigg]\Bigg\}=0. \tag{3.84}$$

We note that the total energy density hydrodynamic equation is of order $\mathcal{O}\left(c^{-2}\right)$, meaning that the post-Newtonian corrections to the resulting equation corresponds to the first post-Newtonian approximation.

From this equation we can obtain the total energy density conservation law by integrating (3.84) over the volume occupied by the fluid. First we note that the integral

$$\int_V\left(\frac{\partial pV_i}{\partial x^i}+\frac{2}{c^2}\frac{\partial pUV_i}{\partial x^i}\right)d^3x=\int_S p\left(1+\frac{2U}{c^2}\right)V_in_idS=0, \tag{3.85}$$

vanishes by using using Gauss divergence theorem and considering that the pressure vanishes on the boundary of the configuration. Next we take into account the previous result (2.314) and the results (3.180) and (3.181) given in the Appendix B which we reproduce here

$$\int_V \rho(\mathbf{x})V_i(\mathbf{x})\frac{\partial U(\mathbf{x})}{\partial x^i}d^3x=\frac{1}{2}\frac{d}{dt}\int_V\rho(\mathbf{x})U(\mathbf{x})d^3x$$
$$-\int_V U(\mathbf{x})\left(\frac{\partial \rho(\mathbf{x})}{\partial t}+\frac{\partial \rho(\mathbf{x})V_i(\mathbf{x})}{\partial x^i}\right)d^3x, \tag{3.86}$$

$$-\frac{1}{c^2}\int_V \rho V_i\frac{\partial \Pi_i}{\partial t}=-\frac{1}{2c^2}\frac{d}{dt}\int_V\rho V_i\Pi_i d^3x, \tag{3.87}$$

$$\int_V\rho V_i\frac{\partial \Phi}{\partial x^i}d^3x=\int\rho\varphi\frac{\partial U}{\partial t}d^3x. \tag{3.88}$$

Now by collecting the above results we arrive at the total energy density conservation law in the first post-Newtonian approximation

$$\frac{1}{c^2}\frac{d}{dt}\int_V \mathfrak{E}\, d^3x = 0, \qquad \text{where} \tag{3.89}$$

$$\mathfrak{E} = \rho\left(\frac{V^2}{2} + \varepsilon - \frac{U}{2}\right) + \frac{\rho}{c^2}\left[\frac{5}{8}V^4 + \frac{5}{2}V^2U - \frac{5}{2}U^2 + 2\varepsilon U + V^2\left(\varepsilon + \frac{p}{\rho}\right) - \frac{1}{2}\Pi_i V_i\right]. \tag{3.90}$$

The expression for the total energy density $\mathfrak{E}$ was derived in [1] by considering the isentropic flow condition and in [4] by taking into account the symmetric energy-momentum complex of Landau and Lifshitz [3]. This latter method for the derivation of the total energy density conservation law will be the subject of Section 3.7.3.

The hydrodynamic equation for the internal energy density in the first post-Newtonian approximation is obtained from the multiplication of (3.84) by c^2 and the use of the hydrodynamic equations for the mass density (2.122) and momentum density (2.131) in the first post-Newtonian approximation to eliminate the time derivative of the mass density ρ and hydrodynamic velocity V_i. After some rearrangements we get that

$$\rho\frac{d\varepsilon}{dt} + p\left(1 - \frac{5}{3}\frac{V^2}{c^2}\right)\frac{\partial V_i}{\partial x^i} - \frac{V^2}{c^2}\frac{\partial p}{\partial t} + \frac{\rho\,(2\varepsilon - 5U)}{c^2}\frac{dU}{dt} + \frac{\rho V_i}{c^2}\left(V^2 + \frac{2}{3}\varepsilon\right)\left[\frac{\partial U}{\partial x^i} - \frac{1}{\rho}\frac{\partial p}{\partial x^i}\right] = 0. \tag{3.91}$$

Without the terms in c^{-2} this equation reduces to the Newtonian internal energy density hydrodynamic equation for an Eulerian fluid.

Momentum density hydrodynamic equation

The hydrodynamic equation for the momentum density in the second post-Newtonian approximation is obtained from the spatial balance equation for the energy-momentum tensor $T^{i\nu}{}_{;\nu} = 0$, which up to the sixth order reads

$$\frac{\left(\partial \overset{1}{T}{}^{0i} + \overset{3}{T}{}^{0i} + \overset{5}{T}{}^{0i}\right)}{\partial x^0} + \frac{\partial\left(\overset{2}{T}{}^{ij} + \overset{4}{T}{}^{ij} + \overset{6}{T}{}^{ij}\right)}{\partial x^j}$$
$$+\overset{2}{\Gamma}{}^i{}_{00}\overset{4}{T}{}^{00} + \overset{6}{\Gamma}{}^i{}_{00}\overset{0}{T}{}^{00} + \left(\overset{2}{\Gamma}{}^i{}_{00} + \overset{4}{\Gamma}{}^i{}_{00}\right)\left(\overset{0}{T}{}^{00} + \overset{2}{T}{}^{00}\right)$$
$$+\left[\overset{2}{\Gamma}{}^i{}_{jk} + \left(\overset{2}{\Gamma}{}^0{}_{0j} + \overset{2}{\Gamma}{}^l{}_{jl}\right)\delta_{ik}\right]\left(\overset{2}{T}{}^{kj} + \overset{4}{T}{}^{kj}\right)$$
$$+\left[\overset{4}{\Gamma}{}^i{}_{jk} + \left(\overset{4}{\Gamma}{}^0{}_{0j} + \overset{4}{\Gamma}{}^l{}_{jl}\right)\delta_{ik}\right]\overset{2}{T}{}^{kj}$$
$$+\left[2\overset{5}{\Gamma}{}^i{}_{0j} + \left(\overset{5}{\Gamma}{}^0{}_{00} + \overset{5}{\Gamma}{}^k{}_{0k}\right)\delta_{ij}\right]\overset{1}{T}{}^{0j}$$
$$+\left[2\overset{3}{\Gamma}{}^i{}_{0j} + \left(\overset{3}{\Gamma}{}^0{}_{00} + \overset{3}{\Gamma}{}^k{}_{0k}\right)\delta_{ij}\right]\left(\overset{1}{T}{}^{0j} + \overset{3}{T}{}^{0j}\right) = 0. \quad (3.92)$$

If one insert the different orders of the energy-momentum tensor and Christoffel symbols components, one arrive after a long calculation to

$$\frac{\partial \rho \mathfrak{V}_i}{\partial t} + \frac{\partial \rho \mathfrak{V}_i V_j}{\partial x^j} + \frac{\partial p}{\partial x^i}\left[1 + \frac{2U}{c^2} - \frac{1}{c^4}\left(U^2 + 2\Phi + \frac{\Psi_{kk}}{2}\right)\right]$$
$$-\rho\frac{\partial U}{\partial x^i}\left\{1 + \frac{2}{c^2}\left(V^2 + U + \frac{\varepsilon}{2} + \frac{p}{2\rho}\right) + \frac{2}{c^4}\left[V^4 + 5UV^2\right.\right.$$
$$\left.\left. -\frac{3U^2}{2} + \Phi + (V^2 + U)\left(\varepsilon + \frac{p}{\rho}\right) - \Pi_i V_i - \frac{\Psi_{kk}}{4}\right]\right\}$$
$$+\frac{\rho}{c^2}\left(V_j\frac{\partial \Pi_j}{\partial x^i} - 2\frac{\partial \Phi}{\partial x^i}\right)\left[1 + \frac{1}{c^2}\left(V^2 + 4U + \varepsilon + \frac{p}{\rho}\right)\right]$$
$$+\frac{\rho}{2c^4}\left(\frac{\partial \Psi_{00}}{\partial x^i} + 2V_j\frac{\partial \Psi_{0j}}{\partial x^i} + V_j V_k\frac{\partial \Psi_{jk}}{\partial x^i}\right) = 0, \qquad (3.93)$$

where we have introduced the abbreviation for the momentum density

$$\rho\mathfrak{V}_i = \rho V_i\left\{1 + \frac{1}{c^2}\left(V^2 + 6U + \varepsilon + \frac{p}{\rho}\right) + \frac{1}{c^4}\left[V^4 + 10V^2U\right.\right.$$
$$\left.\left. +13U^2 + 2\Phi - 2\Pi_i V_i - \frac{\Psi_{kk}}{2} + \left(V^2 + 6U\right)\left(\varepsilon + \frac{p}{\rho}\right)\right]\right\}$$
$$-\frac{\rho}{c^2}\Pi_i\left[1 + \frac{1}{c^2}\left(V^2 + 4U + \varepsilon + \frac{p}{\rho}\right)\right] - \frac{\rho}{c^4}\left(\Psi_{0i} + \Psi_{ij}V_j\right). \qquad (3.94)$$

Expression (3.94) corresponds to eq. *(54)* of the work [4].

3.7 Conservation laws in general relativity

In this section we follow Synge [5] and introduce a symmetric array of quantities $W^{\mu\nu} = W^{\nu\mu}$ which satisfies the partial differential equations

$$\frac{\partial W^{\mu\nu}}{\partial x^\nu} = 0. \tag{3.95}$$

It is supposed that this equation must hold for all transformation of coordinates of $W^{\mu\nu}$, but neither $W^{\mu\nu}$ nor the equation (3.95) have tensorial characters.

The integration of (3.95) in an infinite three-dimensional volume by considering x^0 = constant, leads to

$$\int \frac{\partial W^{\mu\nu}}{\partial x^\nu} d^3x = \int \left(\frac{1}{c}\frac{\partial W^{\mu 0}}{\partial t} + \frac{\partial W^{\mu i}}{\partial x^i} \right) d^3x$$
$$= \int \frac{1}{c}\frac{\partial W^{\mu 0}}{\partial t} d^3x + \int W^{\mu i} n^i dS = 0, \tag{3.96}$$

where $d^3x = dx^1 dx^2 dx^3$ is the volume element. If we consider that the quantities $W^{\mu\nu}$ vanish at the surface of the infinite three-dimensional volume, we have that for x^0 = constant the following relationship hods

$$\frac{1}{c}\int \frac{\partial W^{\mu 0}}{\partial t} d^3x + \frac{1}{c}\int W^{\mu 0} V_j n_j dS = \frac{1}{c}\int \left(\frac{\partial W^{\mu 0}}{\partial t} \right.$$
$$\left. + \frac{\partial W^{\mu 0} V_j}{\partial x^j} \right) d^3x = \frac{d}{dt}\left[\frac{1}{c}\int W^{\mu 0} d^3x \right] = 0. \tag{3.97}$$

This equation implies that there exist four conservative quantities P^μ independent of $x^0 = ct$.

Another conservative quantity can be build from the derivative

$$\frac{\partial}{\partial x^\tau}\left(x^\mu W^{\nu\tau} - x^\nu W^{\mu\tau}\right) = 0, \tag{3.98}$$

which vanishes due to the symmetry of $W^{\mu\nu}$. The integration of (3.98) in an infinite three-dimensional volume leads to

$$\int \left[\frac{1}{c}\frac{\partial}{\partial t}\left(x^\mu W^{\nu 0} - x^\nu W^{\mu 0}\right) + \frac{\partial}{\partial x^i}\left(x^\mu W^{\nu i} - x^\nu W^{\mu i}\right)\right] d^3x = 0, \tag{3.99}$$

and by considering the vanishing of $W^{\mu\nu}$ at the surface of the infinite three-dimensional volume it reduces to

$$\frac{d}{dt}\left[\frac{1}{c}\int \left(x^\mu W^{\nu 0} - x^\nu W^{\mu 0}\right) d^3x\right] = 0. \tag{3.100}$$

We conclude that there exist more six conservative quantities $M^{\mu\nu} = -M^{\nu\mu}$ independent of $x^0 = ct$.

Now we have to link the conservative quantities P^μ and $M^{\mu\nu}$ with the four-momentum and four-tensor angular momentum, respectively. For that end we follow Chandrasekhar and define $W^{\mu\nu}$ in terms of the *energy-momentum complex* $\Theta^{\mu\nu} = (-g)\left(T^{\mu\nu} + t^{\mu\nu}\right)$, namely

$$W^{\mu\nu} = \frac{8\pi G}{c^4}\Theta^{\mu\nu} = \frac{8\pi G}{c^4}(-g)\left(T^{\mu\nu} + t^{\mu\nu}\right). \tag{3.101}$$

Here $t^{\mu\nu}$ is the so-called *energy-momentum pseudo-tensor* of the gravitational field.

3.7.1 Energy-momentum pseudo-tensor of the gravitational field

The aim of this section is the determination of the energy-momentum pseudo-tensor of the gravitational field and for that end we introduce the tensor

$$U^{\mu\sigma\nu\lambda} = g\left(g^{\mu\lambda}g^{\sigma\nu} - g^{\mu\nu}g^{\sigma\lambda}\right), \tag{3.102}$$

which has the same properties as the Riemann-Christoffel tensor $R_{\mu\nu\sigma\tau}$, namely

1. Symmetry: $U^{\mu\sigma\nu\lambda} = U^{\nu\lambda\mu\sigma}$,
2. Cyclicity: $U^{\mu\sigma\nu\lambda} + U^{\mu\nu\lambda\sigma} + U^{\mu\lambda\sigma\nu} = 0$,
3. Anti-symmetry: $U^{\mu\sigma\nu\lambda} = -U^{\sigma\mu\nu\lambda} = -U^{\mu\sigma\lambda\nu} = U^{\sigma\mu\lambda\nu}$.

From the symmetry and anti-symmetry properties of the tensor $U^{\mu\sigma\nu\lambda}$ it is easy to obtain the relations for its derivatives

$$\frac{\partial^2 U^{\mu\sigma\nu\lambda}}{\partial x^\sigma \partial x^\lambda} = \frac{\partial^2 U^{\nu\lambda\mu\sigma}}{\partial x^\sigma \partial x^\lambda}, \tag{3.103}$$

$$\frac{\partial^3 U^{\mu\sigma\nu\lambda}}{\partial x^\nu \partial x^\sigma \partial x^\lambda} = -\frac{\partial^3 U^{\mu\sigma\lambda\nu}}{\partial x^\nu \partial x^\sigma \partial x^\lambda} = -\frac{\partial^2 U^{\mu\sigma\nu\lambda}}{\partial x^\lambda \partial x^\sigma \partial x^\nu}. \tag{3.104}$$

We infer from (3.103) that the contracted second derivative of $U^{\mu\sigma\nu\lambda}$ is symmetric in the two remaining indices, while from (3.104) we get that its third contracted derivative vanishes. Hence, we can make the following identification

$$W^{\mu\nu} = \frac{1}{2}\frac{\partial^2 U^{\mu\sigma\nu\lambda}}{\partial x^\sigma \partial x^\lambda}, \tag{3.105}$$

since $W^{\mu\nu} = W^{\nu\mu}$ and (3.95) holds.

To determine explicitly the contracted second derivative of $U^{\mu\sigma\nu\lambda}$, we note that the covariant derivative of $\left(g^{-1}U^{\mu\sigma\nu\lambda}\right)_{;\tau}$ vanishes, thanks to the vanishing of the covariant of the metric tensor $g^{\mu\nu}{}_{;\tau} = 0$. Hence, we can write

$$0 = \left(g^{-1}U^{\mu\sigma\nu\lambda}\right)_{;\tau} = -\frac{1}{g^2}\frac{\partial g}{\partial x^\tau}U^{\mu\sigma\nu\lambda} + \frac{1}{g}\Bigg(\frac{\partial U^{\mu\sigma\nu\lambda}}{\partial x^\tau} + \Gamma^\mu{}_{\tau\epsilon}U^{\epsilon\sigma\nu\lambda} + \Gamma^\sigma{}_{\tau\epsilon}U^{\mu\epsilon\nu\lambda} + \Gamma^\nu{}_{\tau\epsilon}U^{\mu\sigma\epsilon\lambda} + \Gamma^\lambda{}_{\tau\epsilon}U^{\mu\sigma\nu\epsilon}\Bigg). \quad (3.106)$$

Moreover by using (2.331), the derivative of $U^{\mu\sigma\nu\lambda}$ with respect to x^τ becomes

$$\frac{\partial U^{\mu\sigma\nu\lambda}}{\partial x^\tau} = 2\Gamma^\epsilon{}_{\epsilon\tau}U^{\mu\sigma\nu\lambda} - \Gamma^\mu{}_{\tau\epsilon}U^{\epsilon\sigma\nu\lambda} - \Gamma^\sigma{}_{\tau\epsilon}U^{\mu\epsilon\nu\lambda} - \Gamma^\nu{}_{\tau\epsilon}U^{\mu\sigma\epsilon\lambda} - \Gamma^\lambda{}_{\tau\epsilon}U^{\mu\sigma\nu\epsilon}. \quad (3.107)$$

From the above equation it follows when $\tau = \sigma$ the contracted expression

$$\frac{\partial U^{\mu\sigma\nu\lambda}}{\partial x^\sigma} = \Gamma^\epsilon{}_{\epsilon\sigma}U^{\mu\sigma\nu\lambda} - \Gamma^\nu{}_{\sigma\epsilon}U^{\mu\sigma\epsilon\lambda} - \Gamma^\lambda{}_{\sigma\epsilon}U^{\mu\sigma\nu\epsilon}, \quad (3.108)$$

where the anti-symmetry property of $U^{\epsilon\sigma\nu\lambda} = -U^{\sigma\epsilon\nu\lambda}$ and the symmetry property of $\Gamma^\mu{}_{\sigma\epsilon} = \Gamma^\mu{}_{\epsilon\sigma}$ was used to get that $\Gamma^\mu{}_{\sigma\epsilon}U^{\epsilon\sigma\nu\lambda} = 0$.

The partial derivative of (3.108) with respect to x^λ leads to

$$\frac{\partial^2 U^{\mu\sigma\nu\lambda}}{\partial x^\sigma \partial x^\lambda} = A^{\mu\nu} + B^{\mu\nu}, \quad (3.109)$$

where $A^{\mu\nu}$ and $B^{\mu\nu}$ are given by

$$A^{\mu\nu} = \frac{\partial \Gamma^{\tau}{}_{\tau\sigma}}{\partial x^{\lambda}} U^{\mu\sigma\nu\lambda} - \frac{\partial \Gamma^{\nu}{}_{\sigma\epsilon}}{\partial x^{\lambda}} U^{\mu\sigma\epsilon\lambda} - \frac{\partial \Gamma^{\lambda}{}_{\sigma\epsilon}}{\partial x^{\lambda}} U^{\mu\sigma\nu\epsilon}, \quad (3.110)$$

$$B^{\mu\nu} = \Gamma^{\tau}{}_{\tau\sigma} \frac{\partial U^{\mu\sigma\nu\lambda}}{\partial x^{\lambda}} - \Gamma^{\nu}{}_{\sigma\epsilon} \frac{\partial U^{\mu\sigma\epsilon\lambda}}{\partial x^{\lambda}} - \Gamma^{\lambda}{}_{\sigma\epsilon} \frac{\partial U^{\mu\sigma\nu\epsilon}}{\partial x^{\lambda}}. \quad (3.111)$$

The tensor $A^{\mu\nu}$ defined in (3.110) can be expressed as

$$A^{\mu\nu} = U^{\mu\sigma\nu\lambda} \left(\frac{\partial \Gamma^{\tau}{}_{\tau\sigma}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\epsilon}{}_{\sigma\lambda}}{\partial x^{\epsilon}} \right) + \frac{1}{2} U^{\mu\sigma\epsilon\lambda} \left(\frac{\partial \Gamma^{\nu}{}_{\lambda\sigma}}{\partial x^{\epsilon}} - \frac{\partial \Gamma^{\nu}{}_{\epsilon\sigma}}{\partial x^{\lambda}} \right), \quad (3.112)$$

thanks to the use of the anti-symmetry property of $U^{\mu\sigma\nu\lambda}$ in the expression

$$-\frac{\partial \Gamma^{\nu}{}_{\sigma\epsilon}}{\partial x^{\lambda}} U^{\mu\sigma\epsilon\lambda} = \frac{\partial \Gamma^{\nu}{}_{\sigma\epsilon}}{\partial x^{\lambda}} U^{\mu\sigma\lambda\epsilon} = \frac{\partial \Gamma^{\nu}{}_{\sigma\lambda}}{\partial x^{\epsilon}} U^{\mu\sigma\epsilon\lambda}. \quad (3.113)$$

In terms of the Riemann-Christoffel (2.14) and Ricci (2.13) tensors $A^{\mu\nu}$ reads

$$\begin{aligned} A^{\mu\nu} &= U^{\mu\sigma\nu\lambda} \left(R_{\sigma\lambda} + \Gamma^{\epsilon}{}_{\sigma\lambda} \Gamma^{\tau}{}_{\tau\epsilon} - \Gamma^{\epsilon}{}_{\sigma\tau} \Gamma^{\tau}{}_{\lambda\epsilon} \right) \\ &+ \frac{1}{2} U^{\mu\sigma\epsilon\lambda} \left(R^{\nu}{}_{\sigma\lambda\epsilon} + \Gamma^{\tau}{}_{\sigma\epsilon} \Gamma^{\nu}{}_{\lambda\tau} - \Gamma^{\tau}{}_{\lambda\sigma} \Gamma^{\nu}{}_{\epsilon\tau} \right), \end{aligned} \quad (3.114)$$

which can be simplified and written as

$$\begin{aligned} A^{\mu\nu} &= 2g \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) + g \left(g^{\mu\lambda} g^{\sigma\epsilon} - g^{\mu\epsilon} g^{\sigma\lambda} \right) \Gamma^{\tau}{}_{\sigma\epsilon} \Gamma^{\nu}{}_{\lambda\tau} \\ &+ g \left(g^{\mu\lambda} g^{\sigma\nu} - g^{\mu\nu} g^{\sigma\lambda} \right) \left[\Gamma^{\epsilon}{}_{\sigma\lambda} \Gamma^{\tau}{}_{\tau\epsilon} - \Gamma^{\epsilon}{}_{\sigma\tau} \Gamma^{\tau}{}_{\lambda\epsilon} \right], \end{aligned} \quad (3.115)$$

thanks to the following relationships

$$U^{\mu\sigma\nu\lambda}R_{\sigma\lambda} = g\left(g^{\mu\lambda}g^{\sigma\nu} - g^{\mu\nu}g^{\sigma\lambda}\right)R_{\sigma\lambda} = g\left(R^{\mu\nu} - Rg^{\mu\nu}\right), \tag{3.116}$$

$$\frac{1}{2}U^{\mu\sigma\epsilon\lambda}R^{\nu}{}_{\lambda\sigma\epsilon} = \frac{1}{2}g\left(g^{\mu\lambda}g^{\sigma\epsilon} - g^{\mu\epsilon}g^{\sigma\lambda}\right)R^{\nu}{}_{\sigma\lambda\epsilon}$$
$$= \frac{1}{2}gg^{\mu\lambda}g^{\sigma\epsilon}g^{\nu\tau}R_{\sigma\tau\epsilon\lambda} + \frac{1}{2}gg^{\mu\epsilon}g^{\sigma\lambda}g^{\nu\tau}R_{\sigma\tau\lambda\epsilon} = gR^{\mu\nu}, \tag{3.117}$$

$$-\frac{1}{2}U^{\mu\sigma\epsilon\lambda}\Gamma^{\tau}{}_{\lambda\sigma}\Gamma^{\nu}{}_{\epsilon\tau} = \frac{1}{2}U^{\mu\sigma\lambda\epsilon}\Gamma^{\tau}{}_{\lambda\sigma}\Gamma^{\nu}{}_{\epsilon\tau} = \frac{1}{2}U^{\mu\sigma\epsilon\lambda}\Gamma^{\tau}{}_{\epsilon\sigma}\Gamma^{\nu}{}_{\lambda\tau}. \tag{3.118}$$

The expression of $B^{\mu\nu}$ is more involved and below we give the expressions for the three terms which appear in (3.111) by using the derivative of $U^{\mu\sigma\nu\lambda}$ given in (3.107):

$$\Gamma^{\tau}{}_{\tau\sigma}\frac{\partial U^{\mu\sigma\nu\lambda}}{\partial x^{\lambda}} = g\Gamma^{\tau}{}_{\tau\sigma}\Big[\Gamma^{\epsilon}{}_{\epsilon\lambda}\Big(g^{\mu\lambda}g^{\sigma\nu} - g^{\mu\nu}g^{\sigma\lambda}\Big)$$
$$+\Gamma^{\mu}{}_{\epsilon\lambda}\Big(g^{\sigma\lambda}g^{\epsilon\nu} - g^{\sigma\nu}g^{\epsilon\lambda}\Big) + \Gamma^{\sigma}{}_{\epsilon\lambda}\Big(g^{\mu\nu}g^{\epsilon\lambda} - g^{\mu\lambda}g^{\epsilon\nu}\Big)\Big], \tag{3.119}$$

$$-\Gamma^{\nu}{}_{\sigma\epsilon}\frac{\partial U^{\mu\sigma\epsilon\lambda}}{\partial x^{\lambda}} = g\Gamma^{\nu}{}_{\sigma\epsilon}\Big[\Gamma^{\tau}{}_{\tau\lambda}\Big(g^{\mu\epsilon}g^{\sigma\lambda} - g^{\mu\lambda}g^{\sigma\epsilon}\Big)$$
$$+\Gamma^{\mu}{}_{\tau\lambda}\Big(g^{\sigma\epsilon}g^{\tau\lambda} - g^{\sigma\lambda}g^{\epsilon\tau}\Big) + \Gamma^{\sigma}{}_{\tau\lambda}\Big(g^{\mu\lambda}g^{\epsilon\tau} - g^{\mu\epsilon}g^{\tau\lambda}\Big)\Big], \tag{3.120}$$

$$-\Gamma^{\lambda}{}_{\sigma\epsilon}\frac{\partial U^{\mu\sigma\nu\epsilon}}{\partial x^{\lambda}} = g\Gamma^{\lambda}{}_{\sigma\epsilon}\Big[2\Gamma^{\tau}{}_{\tau\lambda}\Big(g^{\mu\nu}g^{\sigma\epsilon} - g^{\mu\epsilon}g^{\sigma\nu}\Big)$$
$$+\Gamma^{\mu}{}_{\tau\lambda}\Big(g^{\nu\sigma}g^{\epsilon\tau} - g^{\nu\tau}g^{\epsilon\sigma}\Big) + \Gamma^{\nu}{}_{\tau\lambda}\Big(g^{\mu\epsilon}g^{\sigma\tau} - g^{\mu\tau}g^{\epsilon\sigma}\Big)$$
$$+2\Gamma^{\epsilon}{}_{\tau\lambda}\Big(g^{\sigma\nu}g^{\mu\tau} - g^{\mu\nu}g^{\sigma\tau}\Big)\Big]. \tag{3.121}$$

In the above equations the symmetry property of the Christoffel symbols was taken into account.

Now from (3.109), (3.115) and (3.119) – (3.121) we get

$$W^{\mu\nu} = \frac{1}{2}\frac{\partial^2 U^{\mu\sigma\nu\lambda}}{\partial x^\sigma \partial x^\lambda} = g\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) - \frac{1}{2}gV^{\mu\nu}, \quad (3.122)$$

where the symmetric quantity $V^{\mu\nu}$ is given by

$$\begin{aligned} V^{\mu\nu} = &\left(g^{\mu\nu}g^{\sigma\lambda} - g^{\mu\lambda}g^{\sigma\nu}\right)\Big[\Gamma^\tau{}_{\tau\sigma}\Gamma^\epsilon{}_{\epsilon\lambda} + \Gamma^\tau{}_{\epsilon\lambda}\Gamma^\epsilon{}_{\tau\sigma} \\ &-2\Gamma^\tau{}_{\tau\epsilon}\Gamma^\epsilon{}_{\sigma\lambda}\Big] + g^{\mu\lambda}g^{\sigma\epsilon}\Big[\Gamma^\tau{}_{\tau\lambda}\Gamma^\nu{}_{\sigma\epsilon} + \Gamma^\tau{}_{\sigma\epsilon}\Gamma^\nu{}_{\tau\lambda} - \Gamma^\tau{}_{\tau\epsilon}\Gamma^\nu{}_{\sigma\lambda} \\ &-\Gamma^\tau{}_{\sigma\lambda}\Gamma^\nu{}_{\tau\epsilon}\Big] + g^{\nu\lambda}g^{\sigma\epsilon}\Big[\Gamma^\tau{}_{\tau\lambda}\Gamma^\mu{}_{\sigma\epsilon} + \Gamma^\tau{}_{\sigma\epsilon}\Gamma^\mu{}_{\tau\lambda} - \Gamma^\tau{}_{\tau\epsilon}\Gamma^\mu{}_{\sigma\lambda} \\ &-\Gamma^\tau{}_{\sigma\lambda}\Gamma^\mu{}_{\tau\epsilon}\Big] + g^{\lambda\epsilon}g^{\sigma\tau}\Big[\Gamma^\mu{}_{\tau\lambda}\Gamma^\nu{}_{\sigma\epsilon} - \Gamma^\mu{}_{\lambda\epsilon}\Gamma^\nu{}_{\tau\sigma}\Big]. \end{aligned} \quad (3.123)$$

Here it is important to call attention that $V^{\mu\nu}$ is not a tensor.

Now by using Einstein's field equations

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}, \quad (3.124)$$

we can rewrite (3.122) as

$$W^{\mu\nu} = \frac{8\pi G}{c^4}(-g)\left(T^{\mu\nu} + \frac{c^4}{16\pi G}V^{\mu\nu}\right). \quad (3.125)$$

From the comparison of (3.125) with (3.101) we can identify the energy-momentum pseudo-tensor $t^{\mu\nu}$ with $V^{\mu\nu}$ whose expression[2] is given in (3.123):

$$t^{\mu\nu} = \frac{c^4}{16\pi G} V^{\mu\nu}. \tag{3.126}$$

According to (3.95) the ordinary divergence of the energy-momentum complex $\Theta^{\mu\nu} = (-g)\left(T^{\mu\nu} + t^{\mu\nu}\right)$ vanishes, i.e.

$$\Theta^{\mu\nu}{}_{,\nu} = \left[(-g)\left(T^{\mu\nu} + t^{\mu\nu}\right)\right]_{,\nu} = 0, \tag{3.127}$$

so that we can infer from (3.97) that

$$\int_V \Theta^{00} d^3x = \text{constant} \qquad \text{and} \qquad \int_V \Theta^{0i} d^3x = \text{constant}. \tag{3.128}$$

Below we shall determine the components of the energy-momentum complex Θ^{00} and Θ^{0i} in the first post-Newtonian approximation, which correspond to the total energy and linear momentum conservation laws, respectively.

3.7.2 The total linear momentum density conservation

We begin by determining the energy-momentum pseudo-tensor component t^{0i} from (3.123) and (3.126), yielding

$$\overset{3}{t}{}^{0i} = \frac{c^4}{16\pi G}\left[\overset{3}{\Gamma}{}^{j}{}_{0j}\left(\overset{2}{\Gamma}{}^{k}{}_{ik} - \overset{2}{\Gamma}{}^{i}{}_{kk}\right) + \overset{3}{\Gamma}{}^{i}{}_{0j}\left(\overset{2}{\Gamma}{}^{k}{}_{jk} - \overset{2}{\Gamma}{}^{j}{}_{kk}\right)\right.$$

[2] This is the same as the one given in the book of Landau and Lifshitz [3], namely eq. *(96.8)*

$$-2\overset{3}{\Gamma}{}^j{}_{0i}\overset{2}{\Gamma}{}^k{}_{jk}+\overset{3}{\Gamma}{}^0{}_{jj}\left(2\overset{2}{\Gamma}{}^0{}_{i0}-\overset{2}{\Gamma}{}^i{}_{kk}+\overset{2}{\Gamma}{}^k{}_{ik}\right)$$
$$+\left(\overset{3}{\Gamma}{}^j{}_{0k}+\overset{3}{\Gamma}{}^0{}_{jk}\right)\left(\overset{2}{\Gamma}{}^k{}_{ij}+\overset{2}{\Gamma}{}^i{}_{jk}\right)\Bigg]$$
$$=\frac{1}{4\pi Gc}\left[3\frac{\partial U}{\partial t}\frac{\partial U}{\partial x^i}+4\frac{\partial U}{\partial x^j}\frac{\partial U_j}{\partial x^i}-4\frac{\partial U}{\partial x^j}\frac{\partial U_i}{\partial x^j}\right]. \qquad (3.129)$$

All terms that appear in (3.129) must be worked in order to identify the terms that can be put in a divergence form, since their volume integral vanishes and do not contribute to $(3.128)_2$. We begin by writing the first term in (3.129) as

$$\frac{\partial U}{\partial t}\frac{\partial U}{\partial x^i}=\frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{\partial U^2}{\partial t}\right)-U\frac{\partial^2 U}{\partial t\partial x^i}, \qquad (3.130)$$

while the transformation of the second term reads

$$\frac{\partial U}{\partial x^j}\frac{\partial U_j}{\partial x^i}=\frac{\partial}{\partial x^j}\left(U\frac{\partial U_j}{\partial x^i}\right)-U\frac{\partial^2 U_j}{\partial x^i\partial x^j}$$
$$=\frac{\partial}{\partial x^j}\left(U\frac{\partial U_j}{\partial x^i}\right)+U\frac{\partial^2 U}{\partial t\partial x^i}, \qquad (3.131)$$

where we have used the relationship given in (2.112), namely $\partial U/\partial t+\partial U_i/\partial x^i=0$. For the third term in (3.129) we write it as

$$\frac{\partial U}{\partial x^j}\frac{\partial U_i}{\partial x^j}=\frac{\partial^2\left(UU_i\right)}{\partial x^j\partial x^j}-\frac{\partial}{\partial x^j}\left(U\frac{\partial U_i}{\partial x^j}\right)-U_i\nabla^2 U. \qquad (3.132)$$

We have to evaluate also the term which appears in the equations (3.130) and (3.131)

$$U\frac{\partial^2 U}{\partial t\partial x^i} = -\frac{U}{2}\frac{\partial^4\chi}{\partial t\partial x^i\partial x^j\partial x^j} = -\frac{1}{2}\frac{\partial}{\partial x^j}\left(U\frac{\partial^3\chi}{\partial t\partial x^i\partial x^j}\right)$$
$$+\frac{1}{2}\frac{\partial U}{\partial x^j}\frac{\partial^3\chi}{\partial t\partial x^i\partial x^j} = -\frac{1}{2}\frac{\partial}{\partial x^j}\left(U\frac{\partial^3\chi}{\partial t\partial x^i\partial x^j}\right)$$
$$-\frac{1}{2}\nabla^2 U\frac{\partial^2\chi}{\partial t\partial x^i} + \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial U}{\partial x^j}\frac{\partial^2\chi}{\partial t\partial x^i}\right) = -\frac{1}{2}\nabla^2 U\frac{\partial^2\chi}{\partial t\partial x^i}$$
$$+\frac{1}{2}\frac{\partial}{\partial x^j}\left[\frac{\partial}{\partial x^j}\left(\frac{\partial U}{\partial x^j}\frac{\partial^2\chi}{\partial t\partial x^i}\right) - U\frac{\partial^3\chi}{\partial t\partial x^i\partial x^j}\right], \tag{3.133}$$

where we have used the relationship $\nabla^2\chi = -2U$. Now by collecting the results (3.130) – (3.133) and using Poisson's equation $\nabla^2 U = -4\pi G\rho$, we get that the component of the energy-momentum pseudo-tensor (3.129) becomes

$$\overset{3}{t}{}^{0i} = -\frac{1}{c}\left(4U_i - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}\right) + \frac{1}{4\pi Gc}\frac{\partial}{\partial x^j}\left\{\frac{3}{2}\frac{\partial U^2}{\partial t}\delta_{ij}\right.$$
$$-U\frac{\partial^3\chi}{\partial t\partial x^i\partial x^j} + 4U\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i}\right)$$
$$\left.+\frac{\partial}{\partial x^j}\left[U\left(\frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i} - 4U_i\right)\right]\right\}. \tag{3.134}$$

The above equation can be rewritten without the divergence terms and by considering $\left(4U_i - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}\right) = \Pi_i$ as

$$\overset{3}{t}{}^{0i} = -\frac{\Pi_i}{c} + \text{divergence terms}. \tag{3.135}$$

Now we can build the component of the energy-momentum complex

$$\frac{1}{c}\Theta^{0i} = -\frac{1}{c}g\left(\overset{1}{T}{}^{0i} + \overset{3}{T}{}^{0i} + \overset{3}{t}{}^{0i}\right), \tag{3.136}$$

by taking into account (3.63), (3.78) and (3.135), yielding again the expression for the total linear momentum in the first post-Newtonian approximation (2.295):

$$\frac{1}{c}\Theta^{0i} = \rho V_i + \frac{1}{c^2}\left[V_i\left(V^2 + 6U + \varepsilon + \frac{p}{\rho}\right) - \Pi_i\right] = \mathfrak{P}_i. \tag{3.137}$$

3.7.3 The total energy density conservation

The two approximations of the energy-momentum tensor component t^{00} which follows from (3.123) and (3.126) read

$$\begin{aligned}
\overset{2}{t}{}^{00} &= \frac{c^4\delta_{ij}}{16\pi G}\left[2\overset{2}{\Gamma}{}^k{}_{ij}\overset{2}{\Gamma}{}^l{}_{lk} - \overset{2}{\Gamma}{}^k{}_{ki}\overset{2}{\Gamma}{}^l{}_{lj} - \overset{2}{\Gamma}{}^k{}_{li}\overset{2}{\Gamma}{}^l{}_{kj}\right] \\
&= -\frac{7}{8\pi G}\left(\frac{\partial U}{\partial x^i}\right)^2, \qquad (3.138) \\
\overset{4}{t}{}^{00} &= \frac{c^4}{16\pi G}\left\{\delta_{ij}\left[2\overset{2}{\Gamma}{}^k{}_{ij}\overset{4}{\Gamma}{}^l{}_{lk} + 2\overset{4}{\Gamma}{}^k{}_{ij}\overset{2}{\Gamma}{}^l{}_{lk} - 2\overset{2}{\Gamma}{}^k{}_{ki}\overset{4}{\Gamma}{}^l{}_{lj}\right.\right. \\
&\left.\left. -\overset{2}{\Gamma}{}^k{}_{li}\overset{2}{\Gamma}{}^l{}_{kj} - \overset{2}{\Gamma}{}^k{}_{li}\overset{2}{\Gamma}{}^l{}_{kj}\right] + \delta_{ij}\delta_{kj}\left[\overset{3}{\Gamma}{}^0{}_{il}\overset{3}{\Gamma}{}^0{}_{jk} - \overset{3}{\Gamma}{}^0{}_{ij}\overset{3}{\Gamma}{}^0{}_{kl}\right]\right\} \\
&= \frac{1}{16\pi Gc^2}\left\{56U\left(\frac{\partial U}{\partial x^i}\right)^2 + \left(6\frac{\partial \Psi_{jj}}{\partial x^i} - 4\frac{\partial \Psi_{ij}}{\partial x^j}\right)\frac{\partial U}{\partial x^i}\right.
\end{aligned}$$

$$+\frac{1}{2}\left[\frac{\partial\Pi_i}{\partial x^j}\left(\frac{\partial\Pi_i}{\partial x^j}+\frac{\partial\Pi_j}{\partial x^i}\right)-6\left(\frac{\partial U}{\partial t}\right)^2\right]\Bigg\}. \tag{3.139}$$

By taking into account the approximations of the components of the energy-momentum tensor (3.60) and (3.61) and of the pseudo-tensor (3.138) and (3.139) the component of the energy-momentum complex Θ^{00} reads

$$\Theta^{00} = -g\left(\overset{0}{T}{}^{00}+\overset{2}{T}{}^{00}+\overset{4}{T}{}^{00}+\overset{2}{t}{}^{00}+\overset{4}{t}{}^{00}\right) = \rho c^2$$
$$+\rho\left(V^2+6U+\varepsilon\right)+\frac{\rho}{c^2}\Bigg[V^4+10UV^2+12U^2+6U\varepsilon$$
$$+V^2\left(\varepsilon+\frac{p}{\rho}\right)-2V_i\Pi_i-\Psi_{jj}\Bigg]-\frac{7}{8\pi G}\left(\frac{\partial U}{\partial x^i}\right)^2\left(1+\frac{4U}{c^2}\right)$$
$$+\frac{1}{16\pi G c^2}\Bigg\{56U\left(\frac{\partial U}{\partial x^i}\right)^2+4\frac{\partial\Psi_{jj}}{\partial x^i}\frac{\partial U}{\partial x^i}$$
$$+\frac{1}{2}\left[\frac{\partial\Pi_i}{\partial x^j}\left(\frac{\partial\Pi_i}{\partial x^j}+\frac{\partial\Pi_j}{\partial x^i}\right)-6\left(\frac{\partial U}{\partial t}\right)^2\right]\Bigg\}. \tag{3.140}$$

Here (3.45) was used which is a consequence of the gauge choice.

As we did in the last section all terms that appear in (3.140) must be worked to identify the terms that can be put in a divergence form, due to the fact that their volume integrals vanish and do not contribute to $(3.128)_1$. Bellow we enumerate the terms:

(i)

$$\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^i}=\frac{1}{2}\frac{\partial^2 U^2}{\partial x^i\partial x^i}-U\nabla^2 U=\frac{1}{2}\frac{\partial^2 U^2}{\partial x^i\partial x^i}+4\pi G\rho U, \tag{3.141}$$

thanks to the Poisson equation $\nabla^2 U = -4\pi G\rho$,

(ii)

$$4\frac{\partial \Psi_{jj}}{\partial x^i}\frac{\partial U}{\partial x^i} = 4\frac{\partial}{\partial x^i}\left(\Psi_{jj}\frac{\partial U}{\partial x^i}\right) + 16\pi G\rho\Psi_{jj}, \tag{3.142}$$

where the Poisson equation was used.

(iii)

$$\begin{aligned}\frac{\partial \Pi_i}{\partial x^j}\frac{\partial \Pi_i}{\partial x^j} &= \frac{\partial}{\partial x^j}\left(\Pi_i\frac{\partial \Pi_i}{\partial x^j}\right) - \Pi_i\nabla^2\Pi_i = \frac{\partial}{\partial x^j}\left(\Pi_i\frac{\partial \Pi_i}{\partial x^j}\right)\\ &+16\pi G\rho V_i\Pi_i - \Pi_i\frac{\partial^2 U}{\partial t\partial x^i} = 16\pi G\rho V_i\Pi_i\\ &+\frac{\partial}{\partial x^j}\left(\Pi_i\frac{\partial \Pi_i}{\partial x^j} - \Pi_j\frac{\partial U}{\partial t}\right) + \frac{\partial \Pi_i}{\partial x^i}\frac{\partial U}{\partial t} = 16\pi G\rho V_i\Pi_i\\ &+\frac{\partial}{\partial x^j}\left(\Pi_i\frac{\partial \Pi_i}{\partial x^j} - \Pi_j\frac{\partial U}{\partial t}\right) - 3\left(\frac{\partial U}{\partial t}\right)^2,\end{aligned} \tag{3.143}$$

thanks to (3.8) and (3.9).

(iv)

$$\begin{aligned}\frac{\partial \Pi_i}{\partial x^j}\frac{\partial \Pi_j}{\partial x^i} &= \frac{\partial}{\partial x^j}\left(\Pi_i\frac{\partial \Pi_j}{\partial x^i}\right) - \Pi_i\frac{\partial^2 \Pi_j}{\partial x^i\partial x^j}\\ = \frac{\partial}{\partial x^j}\left(\Pi_i\frac{\partial \Pi_j}{\partial x^i}\right) &- \frac{\partial}{\partial x^i}\left(\Pi_i\frac{\partial \Pi_j}{\partial x^j}\right) + \frac{\partial \Pi_i}{\partial x^i}\frac{\partial \Pi_j}{\partial x^j}\\ &= \frac{\partial}{\partial x^j}\left(\Pi_i\frac{\partial \Pi_j}{\partial x^i} - \Pi_j\frac{\partial \Pi_i}{\partial x^i}\right) + 9\left(\frac{\partial U}{\partial t}\right)^2.\end{aligned} \tag{3.144}$$

Here (3.9) was used.

Now collecting all results (3.141) – (3.144) the component of the energy-momentum complex (3.140) without the divergence terms becomes

$$\Theta^{00} = \rho c^2 + \rho\left(V^2 + \frac{5}{2}U + \varepsilon\right) + \frac{\rho}{c^2}\left[V^4 + 10UV^2 + 12U^2 + 6U\varepsilon + V^2\left(\varepsilon + \frac{p}{\rho}\right) - \frac{3}{2}V_i\Pi_i\right]. \qquad (3.145)$$

This is the energy-momentum complex of the mass-energy density. If we want to have the one for the energy density we have to subtract the mass density (3.75) as we did to obtain the hydrodynamic equation for the total energy density, i.e.,

$$\mathfrak{E} = \Theta^{00} - \widetilde{\rho}c^2 = \rho\left(\frac{V^2}{2} + \varepsilon - \frac{U}{2}\right) + \frac{\rho}{c^2}\left[\frac{5}{8}V^4 + \frac{13}{2}V^2U + \frac{21}{2}U^2 + 6U\varepsilon + V^2\left(\varepsilon + \frac{p}{\rho}\right) + \frac{1}{2}\Psi_{ii} - \frac{1}{2}V_i\Pi_i\right]. \qquad (3.146)$$

At this point we have to work with the term $\frac{1}{2}\rho\Psi_{ii}$ and for that end we write

$$\frac{\rho\Psi_{jj}}{2} = -\frac{\Psi_{jj}\nabla^2 U}{8\pi G} = -\frac{1}{8\pi G}\left[\frac{\partial}{\partial x^i}\left(\Psi_{jj}\frac{\partial U}{\partial x^i}\right) - \frac{\partial \Psi_{jj}}{\partial x^i}\frac{\partial U}{\partial x^i}\right]$$
$$= -\frac{1}{8\pi G}\left[\frac{\partial}{\partial x^i}\left(\Psi_{jj}\frac{\partial U}{\partial x^i}\right) - \frac{\partial}{\partial x^i}\left(U\frac{\partial \Psi_{jj}}{\partial x^i}\right) + U\frac{\partial^2 \Psi_{jj}}{\partial x^i \partial x^i}\right]$$
$$= -\frac{1}{8\pi G}\left\{\frac{\partial}{\partial x^i}\left(\Psi_{jj}\frac{\partial U}{\partial x^i}\right) - \frac{\partial}{\partial x^i}\left(U\frac{\partial \Psi_{jj}}{\partial x^i}\right) + U\left[32\pi G(V^2 + 4U + \varepsilon) - 12\left(4\pi G\rho U + \frac{1}{2}\frac{\partial^2 U^2}{\partial x^i \partial x^i}\right)\right]\right\}, \qquad (3.147)$$

where we make use of (3.52) and (3.141). Now we have to evaluate

$$
\begin{aligned}
U\frac{\partial^2 U^2}{\partial x^i \partial x^i} &= \frac{\partial}{\partial x^i}\left(U\frac{\partial U^2}{\partial x^i}\right) - \frac{\partial U}{\partial x^i}\frac{\partial U^2}{\partial x^i} \\
&= \frac{\partial}{\partial x^i}\left(U\frac{\partial U^2}{\partial x^i}\right) - U^2\nabla^2 U - \frac{\partial}{\partial x^i}\left(U^2\frac{\partial U}{\partial x^i}\right) \\
&= \frac{\partial}{\partial x^i}\left(U\frac{\partial U^2}{\partial x^i} - U^2\frac{\partial U}{\partial x^i}\right) - 4\pi G\rho U^2. \qquad (3.148)
\end{aligned}
$$

By collecting the terms without the divergences from (3.147) and (3.148) we have that

$$
\frac{1}{2}\rho\Psi_{jj} = -\rho\left(4V^2U + 4U\varepsilon + 13U^2\right), \qquad (3.149)
$$

and the total energy density (3.146) becomes

$$
\begin{aligned}
\mathfrak{E} = \rho\left(\frac{V^2}{2} + \varepsilon - \frac{U}{2}\right) + \frac{\rho}{c^2}\bigg[\frac{5}{8}V^4 + \frac{5}{2}V^2U - \frac{5}{2}U^2 \\
+2U\varepsilon + V^2\left(\varepsilon + \frac{p}{\rho}\right) - \frac{1}{2}V_i\Pi_i\bigg]. \qquad (3.150)
\end{aligned}
$$

As it should be, this is the same expression as the one (3.90), which was found through the analysis of the hydrodynamic equation for the total energy density.

Appendix A

In this appendix we give the components of the Christoffel symbols that are used in the previous sections. From Section 2.3.2

by identifying the gravitational potentials $\phi = -U$, $\xi_i = -\Pi_i$ and $\psi = -2\Phi$ we have

$$\overset{3}{\Gamma}{}^0{}_{00} = -\frac{1}{c^3}\frac{\partial U}{\partial t}, \quad \overset{5}{\Gamma}{}^0{}_{00} = \frac{1}{c^5}\left(\Pi_i \frac{\partial U}{\partial x^i} - 2\frac{\partial \Phi}{\partial t}\right), \tag{3.151}$$

$$\overset{2}{\Gamma}{}^0{}_{0i} = -\frac{1}{c^2}\frac{\partial U}{\partial x^i}, \qquad \overset{4}{\Gamma}{}^0{}_{0i} = -\frac{2}{c^4}\frac{\partial \Phi}{\partial x^i}, \tag{3.152}$$

$$\overset{2}{\Gamma}{}^i{}_{00} = -\frac{1}{c^2}\frac{\partial U}{\partial x^i}, \; \overset{4}{\Gamma}{}^i{}_{00} = \frac{2}{c^4}\frac{\partial (U^2 - \Phi)}{\partial x^i} - \frac{1}{c^4}\frac{\partial \Pi_i}{\partial t}, \tag{3.153}$$

$$\overset{3}{\Gamma}{}^0{}_{ij} = \frac{1}{2c^3}\left(\frac{\partial \Pi_i}{\partial x^j} + \frac{\partial \Pi_j}{\partial x^i} + 2\frac{\partial U}{\partial t}\delta_{ij}\right), \tag{3.154}$$

$$\overset{2}{\Gamma}{}^i{}_{jk} = \frac{1}{c^2}\left(\frac{\partial U}{\partial x^j}\delta_{ik} + \frac{\partial U}{\partial x^k}\delta_{ij} - \frac{\partial U}{\partial x^i}\delta_{jk}\right), \tag{3.155}$$

$$\overset{3}{\Gamma}{}^j{}_{0i} = \frac{1}{2c^3}\left(\frac{\partial \Pi_i}{\partial x^j} - \frac{\partial \Pi_j}{\partial x^i} + 2\frac{\partial U}{\partial t}\delta_{ij}\right). \tag{3.156}$$

The components of the Christoffel symbols $\overset{4}{\Gamma}{}^i{}_{jk}$ and $\overset{5}{\Gamma}{}^i{}_{0j}$ are obtained from

$$\begin{aligned}\overset{4}{\Gamma}{}^i{}_{jk} = &-\frac{1}{2}\left(\frac{\partial \overset{4}{g}_{ij}}{\partial x^k} + \frac{\partial \overset{4}{g}_{ik}}{\partial x^j} - \frac{\partial \overset{4}{g}_{jk}}{\partial x^i}\right) + \frac{\overset{2}{g}{}^{il}}{2}\left(\frac{\partial \overset{2}{g}_{jl}}{\partial x^k} + \frac{\partial \overset{2}{g}_{kl}}{\partial x^j}\right.\\ &\left.-\frac{\partial \overset{2}{g}_{jk}}{\partial x^l}\right) = -\frac{1}{2c^4}\left(\frac{\partial \Psi_{ij}}{\partial x^k} + \frac{\partial \Psi_{ik}}{\partial x^j} - \frac{\partial \Psi_{jk}}{\partial x^i}\right)\\ &-\frac{1}{c^4}\left(\frac{\partial U^2}{\partial x^k}\delta_{ij} + \frac{\partial U^2}{\partial x^j}\delta_{ik} - \frac{\partial U^2}{\partial x^i}\delta_{jk}\right).\end{aligned} \tag{3.157}$$

$$\overset{5}{\Gamma}{}^{i}{}_{0j} = \frac{\overset{2}{g}{}^{0i}}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^j} - \frac{1}{2}\left(\frac{\partial \overset{5}{g}_{i0}}{\partial x^j} + \frac{1}{c}\frac{\partial \overset{4}{g}_{ij}}{\partial t} - \frac{\partial \overset{5}{g}_{j0}}{\partial x^i}\right)$$
$$+\frac{\overset{2}{g}{}^{ik}}{2}\left(\frac{\partial \overset{3}{g}_{k0}}{\partial x^j} + \frac{1}{c}\frac{\partial \overset{2}{g}_{kj}}{\partial t} - \frac{\partial \overset{3}{g}_{0j}}{\partial x^k}\right) = -\frac{1}{c^5}\Pi_i\frac{\partial U}{\partial x^j}$$
$$+\frac{U}{c^5}\left(\frac{\partial \Pi_i}{\partial x^j} - \frac{\partial \Pi_j}{\partial x^i} - 2\frac{\partial U}{\partial t}\delta_{ij}\right)$$
$$-\frac{1}{2c^5}\left(\frac{\partial \Psi_{0i}}{\partial x^j} - \frac{\partial \Psi_{0j}}{\partial x^i} + \frac{\partial \Psi_{ij}}{\partial t}\right). \tag{3.158}$$

The determination of $\overset{6}{\Gamma}{}^{i}{}_{00}$ is more involved. We begin by writing it as

$$\overset{6}{\Gamma}{}^{i}{}_{00} = \frac{\overset{3}{g}{}^{0i}}{2c}\frac{\partial \overset{2}{g}_{00}}{\partial t} + \frac{\overset{2}{g}{}^{ij}}{2}\left(\frac{2}{c}\frac{\partial \overset{3}{g}_{0j}}{\partial t} - \frac{\partial \overset{4}{g}_{00}}{\partial x^j}\right)$$
$$-\frac{\delta_{ij}}{2}\left(\frac{2}{c}\frac{\partial \overset{5}{g}_{0j}}{\partial t} - \frac{\partial \overset{6}{g}_{00}}{\partial x^j}\right) - \frac{\overset{4}{g}{}^{ij}}{2}\frac{\partial \overset{2}{g}_{00}}{\partial x^j}. \tag{3.159}$$

The known metric tensor coefficients that appear in the above equation are

$$\overset{3}{g}{}^{0i} = \overset{3}{g}_{0i} = \frac{\Pi_i}{c^3}, \quad \overset{2}{g}_{00} = -\frac{2U}{c^2}, \quad \overset{2}{g}{}^{ij} = -\overset{2}{g}_{ij} = \frac{2U}{c^2}\delta_{ij}, \tag{3.160}$$
$$\overset{4}{g}_{00} = -\frac{2}{c^4}\left(U^2 - 2\Phi\right), \qquad \overset{5}{g}_{0i} = \frac{\Psi_{0i}}{c^5}. \tag{3.161}$$

In the expression of $\overset{6}{\Gamma}{}^{i}{}_{00}$ there is one component of the metric tensor which is not known till now, namely $\overset{4}{g}{}^{ij}$. For the determination of $\overset{4}{g}{}^{ij}$ we make use of the relationship $g^{i0}g_{j0}+g^{ik}g_{jk}=\delta_{ij}$ given in (2.24), which here reduces to

$$\begin{aligned}\delta_{ij} &= \overset{3}{g}{}^{0i}\overset{3}{g}_{0j}+\left(-\delta_{ik}+\overset{2}{g}{}^{ik}+\overset{4}{g}{}^{ik}\right)\left(-\delta_{kj}+\overset{2}{g}_{kj}+\overset{4}{g}{}^{kj}\right)\\ &= \delta_{ij}-\overset{2}{g}{}^{ij}-\overset{2}{g}_{ij}+\overset{2}{g}{}^{ik}\overset{2}{g}_{kj}-\overset{4}{g}{}^{ij}-\overset{4}{g}_{ij}+\mathcal{O}\left(c^{-6}\right). \qquad (3.162)\end{aligned}$$

Now by using $\overset{2}{g}{}^{ij}=-\overset{2}{g}_{ij}=2U\delta_{ij}/c^2$ and $\overset{4}{g}_{ij}=\Psi_{ij}/c^4$ we get that

$$\overset{4}{g}{}^{ij}=-\frac{1}{c^4}\left(4U^2\delta_{ij}+\Psi_{ij}\right). \qquad (3.163)$$

By taking into account (3.160), (3.161) and (3.163) it follows that (3.159) reduces to

$$\begin{aligned}\overset{6}{\Gamma}{}^{i}{}_{00} &= \frac{1}{2c^6}\frac{\partial\Psi_{00}}{\partial x^i}-\frac{1}{c^6}\frac{\partial\Psi_{0i}}{\partial t}-\frac{\Pi_i}{c^6}\frac{\partial U}{\partial t}-\frac{4U}{c^6}\left(\frac{\partial U^2}{\partial x^i}-\frac{\partial\Phi}{\partial x^i}\right)\\ &\quad+\frac{2U}{c^6}\frac{\partial\Pi_i}{\partial t}-\frac{\Psi_{ij}}{c^6}\frac{\partial U}{\partial x^j}. \qquad (3.164)\end{aligned}$$

Appendix B

In this appendix we shall determine some integrals which are used in the determination of the total energy conservation law

(3.90). The results are based on the paper by Chandrasekhar [1]. We are interested in calculating the volume integral of

$$-\frac{\rho}{c^2}V_i\frac{\partial\Pi_i}{\partial t} = -\frac{\rho}{c^2}V_i\left(4\frac{\partial U_i}{\partial t} - \frac{1}{2}\frac{\partial^3\chi}{\partial t^2\partial x^i}\right), \tag{3.165}$$

where it was considered $\Pi_i = 4U_i - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i},$.

First we shall evaluate the term with χ in (3.165) and for that end we differentiate (2.291) with respect to x^i and get

$$\frac{\partial^2\chi}{\partial t\partial x^i} = -G\int_V \rho(\mathbf{x}')V_k(\mathbf{x}')(x_k - x'_k)(x_i - x'_i)\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}$$
$$+G\int_V \rho(\mathbf{x}')V_i(\mathbf{x}')\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} = -Z_i + U_i, \tag{3.166}$$

where U_i is the vector gravitational potential (2.260) and we have introduced the abbreviation Z_i for the second integral above. Next we write

$$\frac{\partial^3\chi}{\partial t^2\partial x^i} = \frac{d(U_i - Z_i)}{dt} - V_j\frac{\partial(U_i - Z_i)}{\partial x^j}, \tag{3.167}$$

and the two terms will be evaluated separately.

(i) The first one is the integral

$$\int_V \frac{\rho V_i}{2c^2}\frac{d(U_i - Z_i)}{dt}d^3x = \int_V \frac{\rho V_i}{2c^2}\left(\frac{\partial(U_i - Z_i)}{\partial t}\right.$$
$$\left.+V_j\frac{\partial(U_i - Z_i)}{\partial x^j}\right)d^3x = \frac{1}{2c^2}\int_V\left(\frac{\partial\rho V_i(U_i - Z_i)}{\partial t}\right.$$

$$+\frac{\partial \rho V_i V_j (U_i - Z_i)}{\partial x^j}\Bigg) d^3x - \frac{1}{2c^2}\int_V \rho(U_i - Z_i)$$
$$\times\left(\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x^j}\right) d^3x = \frac{1}{2c^2}\frac{d}{dt}\int_V \rho V_i (U_i - Z_i) d^3x$$
$$-\frac{1}{2c^2}\int \rho(U_i - Z_i)\frac{dV_i}{dt} d^3x, \tag{3.168}$$

where Reynolds transport theorem (2.282) and the Newtonian continuity equation were used, since this term is of order $1/c^2$. Now we evaluate the first term of the second integral in (3.168) by considering the definition of U_i given by (2.260)

$$\frac{1}{2c^2}\int_V \rho U_i \frac{dV_i}{dt} d^3x = \frac{1}{2c^2}\int_V \left(\frac{\partial \rho V_i U_i}{\partial t} + \frac{\partial \rho V_i V_j U_i}{\partial x^j}\right) d^3x$$
$$-\frac{1}{2c^2}\int_V \rho V_i \left(\frac{\partial U_i}{\partial t} + V_j \frac{\partial U_i}{\partial x^j}\right) d^3x = \frac{1}{2c^2}\frac{d}{dt}\int_V \rho V_i U_i d^3x$$
$$-\frac{G}{2c^2}\int_V\int_V \rho(\mathbf{x})V_i(\mathbf{x})\Bigg\{\frac{\partial}{\partial t}\left[\frac{\rho(\mathbf{x}')V_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]$$
$$+\frac{\partial}{\partial x^j}\left[\frac{\rho(\mathbf{x}')V_i(\mathbf{x}')V_j(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]\Bigg\} d^3x d^3x' = \frac{1}{2c^2}\frac{d}{dt}\int_V \rho V_i U_i d^3x$$
$$-\frac{G}{2c^2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')V_i(\mathbf{x})\frac{d}{dt}\left(\frac{V_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) d^3x d^3x'$$
$$= \frac{1}{2c^2}\frac{d}{dt}\int_V \rho V_i U_i d^3x - \frac{1}{2c^2}\int_V \rho U_i \frac{dV_i}{dt} d^3x$$
$$-\frac{G}{2c^2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')V_i(\mathbf{x})V_i(\mathbf{x}')\frac{d}{dt}\frac{1}{|\mathbf{x}-\mathbf{x}'|} d^3x d^3x'. \tag{3.169}$$

Hence from (3.168) and (3.169) we get that

$$\int_V \frac{\rho V_i}{2c^2}\frac{d(U_i - Z_i)}{dt}d^3x = \frac{1}{2c^2}\Big\{ \int \rho Z_i \frac{dV_i}{dt}d^3x$$
$$+\frac{d}{dt}\int_V \rho V_i \left(\frac{U_i}{2} - Z_i\right) d^3x + \frac{G}{2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')$$
$$\times V_i(\mathbf{x})V_i(\mathbf{x}')\frac{d}{dt}\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3xd^3x'\Big\}. \qquad (3.170)$$

(ii) For the evaluation of the second term in (3.167) we determine first the expression

$$W_i(\mathbf{x}) = V_j\frac{\partial(U_i - Z_i)}{\partial x^j} = V_j(\mathbf{x})\frac{\partial}{\partial x^j}\Big\{G\int_V \frac{\rho(\mathbf{x}')V_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$
$$-G\int_V \rho(\mathbf{x}')V_k(\mathbf{x}')(x_k - x_k')\frac{(x_i - x_i')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'\Big\}$$
$$= -G\int_V \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}\Big\{V_j(\mathbf{x})V_j(\mathbf{x}')(x_i - x_i')$$
$$+\big[V_i(\mathbf{x}')V_j(\mathbf{x}) + V_i(\mathbf{x})V_j(\mathbf{x}')\big](x_j - x_j')$$
$$-3V_j(\mathbf{x})(x_j - x_j')V_k(\mathbf{x}')(x_k - x_k')\frac{x_i - x_i'}{|\mathbf{x}-\mathbf{x}'|^2}\Big\}. \qquad (3.171)$$

Now we integrate over the volume the above expression

$$-\frac{1}{2c^2}\int_V \rho(\mathbf{x})W_i(\mathbf{x})V_i(\mathbf{x})d^3x = \frac{G}{4c^2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')$$
$$\times\Big\{\Big[V_j(\mathbf{x})V_j(\mathbf{x})V_i(\mathbf{x}') + 2V_j(\mathbf{x})V_j(\mathbf{x}')\left(V_i(\mathbf{x}) - V_i(\mathbf{x}')\right)$$

$$-V_j(\mathbf{x}')V_j(\mathbf{x}')V_i(\mathbf{x})\Big](x_i - x_i') + 3V_i(\mathbf{x}')(x_i - x_i')V_j(\mathbf{x})$$

$$\times(x_j - x_j')V_k(\mathbf{x}')\frac{(x_k - x_k')}{|\mathbf{x}-\mathbf{x}'|^2} - 3V_i(\mathbf{x})(x_i - x_i')V_j(\mathbf{x}')$$

$$\times(x_j - x_j')V_k(\mathbf{x})\frac{(x_k - x_k')}{|\mathbf{x}-\mathbf{x}'|^2}\Bigg\}\frac{d^3x d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}. \quad (3.172)$$

Note that in (3.172) we have considered the interchanging of the primed and unprimed labels. This expression can be put in another form by noting that

$$\frac{1}{4c^2}\frac{d}{dt}\int_V \rho V_i Z_i d^3x = \frac{1}{4c^2}\int_V \left[\frac{\partial \rho V_i Z_i}{\partial t} + \frac{\partial \rho V_i Z_i V_j}{\partial x^j}\right]d^3x$$

$$= \frac{1}{4c^2}\int_V \rho(\mathbf{x})\left[Z_i(\mathbf{x})\frac{dV_i(\mathbf{x})}{dt} + V_i(\mathbf{x})\frac{dZ_i(\mathbf{x})}{dt}\right]d^3x, \quad (3.173)$$

where the Reynolds transport theorem and the Newtonian continuity equation were used. The evaluation of $dZ_i(\mathbf{x})/dt$ by using its definition given in (3.166) reads

$$\frac{1}{4c^2}\frac{dZ_i(\mathbf{x})}{dt} = \frac{G}{4c^2}\frac{d}{dt}\int_V \frac{\rho(\mathbf{x}')V_k(\mathbf{x}')(x_k - x_k')(x_i - x_i')d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$= \frac{G}{4c^2}\int_V \rho(\mathbf{x}')\frac{dV_k(\mathbf{x}')}{dt}(x_k - x_k')(x_i - x_i')\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$+\frac{G}{4c^2}\int_V \rho(\mathbf{x}')\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}\Big\{V_j(\mathbf{x}')(x_j - x_j')\left[V_i(\mathbf{x}) - V_i(\mathbf{x}')\right]$$

$$+(x_i - x_i')V_j(\mathbf{x}')\left[V_j(\mathbf{x}) - V_j(\mathbf{x}')\right] - 3V_j(\mathbf{x}')(x_j - x_j')$$

$$\times(x_i - x_i')\left[V_k(\mathbf{x}) - V_k(\mathbf{x}')\right]\frac{(x_k - x_k')}{|\mathbf{x}-\mathbf{x}'|^2}\Big\}, \quad (3.174)$$

while its volume integral becomes

$$\begin{aligned}
&\frac{1}{4c^2}\int_V \rho(\mathbf{x})V_i(\mathbf{x})\frac{dZ_i(\mathbf{x})}{dt}d^3x = \frac{G}{4c^2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}') \\
&\times\frac{dV_k(\mathbf{x}')}{dt}(x_k - x'_k)V_i(\mathbf{x})(x_i - x'_i)\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|^3} \\
&+\frac{G}{4c^2}\int_V\int_V \rho(\mathbf{x})V_i(\mathbf{x})\rho(\mathbf{x}')\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|^3}\Big\{V_j(\mathbf{x}') \\
&\times(x_j - x'_j)\left[V_i(\mathbf{x}) - V_i(\mathbf{x}')\right] + (x_i - x'_i)V_j(\mathbf{x}') \\
&\times\left[V_j(\mathbf{x}) - V_j(\mathbf{x}')\right] - 3V_j(\mathbf{x}')(x_j - x'_j) \\
&\times(x_i - x'_i)\left[V_k(\mathbf{x}) - V_k(\mathbf{x}')\right]\frac{(x_k - x'_k)}{|\mathbf{x}-\mathbf{x}'|^2}\Big\}. \qquad (3.175)
\end{aligned}$$

Hence we can write (3.172) thanks to (3.173) and (3.175) as

$$\begin{aligned}
-\frac{1}{2c^2}\int_V \rho(\mathbf{x})W_i(\mathbf{x})V_i(\mathbf{x})d^3x &= \frac{1}{4c^2}\frac{d}{dt}\int_V \rho V_i Z_i d^3x \\
&-\frac{G}{4c^2}\int_V\int_V \rho(\mathbf{x})\rho(\mathbf{x}')V_i(\mathbf{x})V_i(\mathbf{x}')\frac{d}{dt}\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3xd^3x' \\
&-\frac{1}{2c^2}\int_V \rho(\mathbf{x})Z_i(\mathbf{x})\frac{dV_i(\mathbf{x})}{dt}d^3x. \qquad (3.176)
\end{aligned}$$

Now by collecting the results (3.170) and (3.176) the volume integral of (3.167), yields

$$\frac{1}{2c^2}\int_V \rho V_i\frac{\partial^3\chi}{\partial t^2\partial x^i}d^3x = \frac{1}{4c^2}\frac{d}{dt}\int_V \rho V_i\left(U_i - Z_i\right)d^3x. \quad (3.177)$$

For the determination of the volume integral of (3.165) we need also to know the following integral

$$\frac{1}{c^2}\int_V \rho V_i \frac{\partial U_i}{\partial t} d^3x = \frac{1}{c^2}\int_V \rho V_i \frac{dU_i}{dt} d^3x - \frac{1}{c^2}\int_V \rho V_i V_j \frac{\partial U_i}{\partial x_j} d^3x. \tag{3.178}$$

By following the same methodology above one finds that

$$\frac{1}{c^2}\int_V \rho V_i \frac{\partial U_i}{\partial t} d^3x = \frac{1}{2c^2}\frac{d}{dt}\int_V \rho V_i U_i d^3x. \tag{3.179}$$

Finally the volume integral of (3.165) becomes

$$-\frac{1}{c^2}\int_V \rho V_i \frac{\partial \Pi_i}{\partial t} = -\frac{1}{c^2}\frac{d}{dt}\int_V \rho V_i \left(\frac{7}{4}U_i + \frac{1}{4}Z_i\right) d^3x$$
$$= -\frac{1}{2c^2}\frac{d}{dt}\int_V \rho V_i \Pi_i d^3x, \tag{3.180}$$

thanks to (3.165) and (3.166).

Another relationship that is useful to the calculation of the total energy conservation law is obtained from

$$\int_V \rho V_i \left(\varphi \frac{\partial U}{\partial x^i} + \frac{\partial \Phi}{\partial x^i}\right) d^3x = -G\int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}')$$
$$\times \Big[\varphi(\mathbf{x}) + \varphi(\mathbf{x}')\Big] \frac{V_i(x)(x_i - x_i')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x d^3x'$$
$$= -\frac{G}{2}\int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}') \frac{[V_i(x) - V_i(\mathbf{x}')](x_i - x_i')}{|\mathbf{x}-\mathbf{x}'|^3}$$
$$\times \Big[\varphi(\mathbf{x}) + \varphi(\mathbf{x}')\Big] d^3x d^3x' = \frac{G}{2}\int_V \int_V \rho(\mathbf{x})\rho(\mathbf{x}')$$

$$
\begin{aligned}
&\times\Big[\varphi(\mathbf{x})+\varphi(\mathbf{x}')\Big]\frac{d}{dt}\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3xd^3x' \\
&= G\int_V d^3x\rho(\mathbf{x})\varphi(\mathbf{x})\int_V d^3x'\rho(\mathbf{x}')\frac{d}{dt}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \\
&= G\int_V d^3x\rho(\mathbf{x})\varphi(\mathbf{x})\frac{d}{dt}\int_V d^3x'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\
&= \int_V \rho\varphi\frac{dU}{dt}d^3x. \qquad (3.181)
\end{aligned}
$$

Hence we conclude from this equation that $\int_V \rho V_i \frac{\partial\Phi}{\partial x^i} d^3x = \int \rho\varphi\frac{\partial U}{\partial t}d^3x$.

References

[1] S. Chandrasekhar, The post-Newtonian equations of hydrodynamics in general relativity, *Ap. J.* **142**, 1488 (1965).

[2] S. Chandrasekhar, Conservation laws in general relativity and in the post-Newtonian approximations, *Ap. J.* **158**, 45 (1969).

[3] L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, 4th ed. (Pergamon Press, Oxford, 1980).

[4] S. Chandrasekhar and Y. Nutku, The second post-Newtonian equations of hydrodynamics in general relativity, *Ap. J.* **158**, 55 (1969).

[5] J. L. Synge, *Relativity: the general theory* (North-Holland, Amsterdam, 1960).

CHAPTER 4

POST-NEWTONIAN KINETIC THEORY

In this chapter a kinetic theory of relativistic gases is developed within the framework of the post-Newtonian approximation. The first post-Newtonian approximation of the Boltzmann equation for collisionless systems was first determined in the papers [1, 2]. Here the Boltzmann equation, the Maxwell-Jüttner distribution function, the particle four-flow the energy-momentum tensor and the Eulerian hydrodynamic equations are determined from a kinetic theory in the first and second post-Newtonian approximations. The derivation of the post-Newtonian Boltzmann equation follows the methodology that was outlined in the book [3] and in Section 1.3, while the determination of the post-Newtonian Maxwell-Jüttner distribution function and of the particle four-flow and energy-momentum tensor follow the work [4]. Another issue developed here are the Jeans equations in the first post-Newtonian approximation for stationary spherically symmetrical and axisymmetrical self-gravitating systems.

4.1 First post-Newtonian approximation

4.1.1 Post-Newtonian Boltzmann equation

We start by writing the equation of motion of the gas particles

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}=0, \tag{4.1}$$

and computing the acceleration which follows from this equation (see Weinberg [5])

$$\begin{aligned}\frac{d^2x^i}{d(x^0)^2}&=\left(\frac{dx^0}{d\tau}\right)^{-1}\frac{d}{d\tau}\left[\left(\frac{dx^0}{d\tau}\right)^{-1}\frac{dx^i}{d\tau}\right]\\&=\left(\frac{dx^0}{d\tau}\right)^{-2}\left[\frac{d^2x^i}{d\tau^2}-\left(\frac{dx^0}{d\tau}\right)^{-1}\frac{d^2x^0}{d\tau^2}\frac{dx^i}{d\tau}\right].\end{aligned} \tag{4.2}$$

The above equation can be rewritten by using (4.1) as

$$\frac{d^2x^i}{d(x^0)^2}=\left(\frac{dx^0}{d\tau}\right)^{-2}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left[\Gamma^0{}_{\mu\nu}\left(\frac{dx^0}{d\tau}\right)^{-1}\frac{dx^i}{d\tau}-\Gamma^i{}_{\mu\nu}\right]$$

$$= -\Gamma^i{}_{00} - \Gamma^i{}_{jk}\frac{dx^j}{dx^0}\frac{dx^k}{dx^0} - 2\Gamma^i{}_{0j}\frac{dx^j}{dx^0} + \frac{dx^i}{dx^0}\left[\Gamma^0{}_{00} + 2\Gamma^0{}_{0j}\frac{dx^j}{dx^0} + \Gamma^0{}_{jk}\frac{dx^j}{dx^0}\frac{dx^k}{dx^0}\right]. \tag{4.3}$$

If we use the expressions for the components of the Christoffel symbols (2.79) – (2.84), the post-Newtonian approximation of (4.3) up to order $\mathcal{O}(c^{-4})$ becomes

$$\frac{d^2x^i}{dt^2} = c^2\Bigg\{ -\overset{2}{\Gamma}{}^i{}_{00} - \overset{4}{\Gamma}{}^i{}_{00} - 2\frac{v_j}{c}\overset{3}{\Gamma}{}^i{}_{0j} - \frac{v_jv_k}{c^2}\overset{2}{\Gamma}{}^i{}_{jk} + \frac{v_i}{c}\Bigg[\overset{3}{\Gamma}{}^0{}_{00} + 2\frac{v_j}{c}\overset{2}{\Gamma}{}^0{}_{0j} + \underline{\frac{v_jv_k}{c^2}\overset{3}{\Gamma}{}^0{}_{jk}}\Bigg]\Bigg\} = -\frac{\partial\phi}{\partial x^i} + \frac{1}{c^2}\Bigg[4v_iv_j\frac{\partial\phi}{\partial x^j} - \frac{\partial\psi}{\partial x^i} - 4\phi\frac{\partial\phi}{\partial x^i} + 3v_i\frac{\partial\phi}{\partial t} - \frac{\partial\xi_i}{\partial t} - v_j\left(\frac{\partial\xi_i}{\partial x^j} - \frac{\partial\xi_j}{\partial x^i}\right) - v^2\frac{\partial\phi}{\partial x^i}\Bigg]. \tag{4.4}$$

Note that the underlined term above was not considered, since it is of order $\mathcal{O}(c^{-5})$.

Now we can write the one-particle distribution function $f = f(\mathbf{x}, \mathbf{v}, t)$ as $f(x^\mu(\tau), v_i(\tau))$ where τ is the proper time along the world line of the one-particle distribution function. Hence the variation of the one-particle distribution function with respect to the proper time reads

$$\frac{df(x^\mu(\tau), v_i(\tau))}{d\tau} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial f}{\partial v_i}\frac{dv_i}{d\tau} = u^\mu\frac{\partial f}{\partial x^\mu} + \frac{\partial f}{\partial v_i}\frac{dv_i}{dt}\frac{dt}{d\tau}, \tag{4.5}$$

where $u^\mu = (u^0, u^0 v_i/c)$ is the four-velocity of the gas particles.

The first term of the second equality up to the order $\mathcal{O}(c^{-2})$ is

$$u^\mu\frac{\partial f}{\partial x^\mu}=u^0\frac{\partial f}{\partial x^0}+u^i\frac{\partial f}{\partial x^i}=\left[1+\frac{1}{c^2}\left(\frac{v^2}{2}-\phi\right)\right]\left(\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial x^i}\right), \tag{4.6}$$

where we have used the corresponding expressions (2.87) and (2.88) for u^0 and u^i through the substitution of the hydrodynamic velocities U^μ and $\mathbf{V}$ by the particle velocities u^μ and $\mathbf{v}$.

The second term of the last equality in (4.5) up to the order $\mathcal{O}(c^{-2})$ becomes

$$\frac{\partial f}{\partial v_i}\frac{dv_i}{dt}\frac{dt}{d\tau}=\frac{\partial f}{\partial v_i}\Bigg\{-\frac{\partial\phi}{\partial x^i}+\frac{1}{c^2}\Bigg[4v_iv_j\frac{\partial\phi}{\partial x^j}+3v_i\frac{\partial\phi}{\partial t}$$
$$-\left(\frac{3v^2}{2}+3\phi\right)\frac{\partial\phi}{\partial x^i}-\frac{\partial\psi}{\partial x^i}-\frac{\partial\xi_i}{\partial t}-v_j\left(\frac{\partial\xi_i}{\partial x^j}-\frac{\partial\xi_j}{\partial x^i}\right)\Bigg]\Bigg\}, \tag{4.7}$$

thanks to (2.86) and (4.4).

By collecting the results (4.5) – (4.7) we get the Boltzmann equation in the first post-Newtonian approximation

$$\left[\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial x^i}\right]\left[1+\frac{1}{c^2}\left(\frac{v^2}{2}-\phi\right)\right]-\frac{\partial\phi}{\partial x^i}\frac{\partial f}{\partial v_i}$$
$$+\frac{1}{c^2}\Bigg[4v_iv_j\frac{\partial\phi}{\partial x^j}+3v_i\frac{\partial\phi}{\partial t}-\left(\frac{3v^2}{2}+3\phi\right)\frac{\partial\phi}{\partial x^i}$$
$$-\frac{\partial\psi}{\partial x^i}-\frac{\partial\xi_i}{\partial t}-v_j\left(\frac{\partial\xi_i}{\partial x^j}-\frac{\partial\xi_j}{\partial x^i}\right)\Bigg]\frac{\partial f}{\partial v_i}=\mathcal{Q}(f,f), \tag{4.8}$$

where we have introduced the collision operator of the Boltzmann equation $\mathcal{Q}(f,f)$, which refers to the binary collision of the particles and is given in terms of an integral of the product of two particle distribution functions at collision. Note that without the terms of order $\mathcal{O}\left(c^{-2}\right)$ equation (4.8) reduces to (1.1).

4.1.2 Post-Newtonian Maxwell-Jüttner distribution function

The relativistic equilibrium distribution function is the Maxwell-Jüttner distribution function (1.19) which was introduced in Section 1.2 and reads

$$f(\mathbf{x}, \mathbf{p}, t) = \frac{n}{4\pi m^2 ckTK_2(\zeta)} \exp\left(-\frac{p^\mu U_\mu}{kT}\right). \tag{4.9}$$

For the determination of the Maxwell-Jüttner distribution function in the post-Newtonian approximation we need to evaluate the exponential term in (4.9). For that end we introduce the components of the four-velocity of the gas particles $u^\mu = (u^0, u^0 v_i/c)$ which are obtained from the components of the hydrodynamic four-velocity $U^\mu = (U^0, U^0 V_i/c)$ given by (2.87) and (2.88) and replacing U^0 and V_i by u^0 and v_i, respectively. Hence, we have $u^i = u^0 v_i/c$ and

$$\frac{u^0}{c} = 1 + \frac{1}{c^2}\left(\frac{v^2}{2} - \phi\right) + \frac{1}{c^4}\left(\frac{3v^4}{8} - \frac{5v^2\phi}{2} + \frac{\phi^2}{2} - \psi + \xi_i v_i\right). \tag{4.10}$$

Next we introduce the peculiar velocity $\mathcal{V}_i = v_i - V_i$ – which is the difference of the fluid particle velocity $\mathbf{v}$ and the hydrodynamic velocity $\mathbf{V}$ – in the expression for the components of the particle velocity and get

$$\frac{g_{\mu\nu}p^\mu U^\nu}{kT} = \frac{m\big(g_{00}u^0U^0 + g_{0i}u^0U^i + g_{0i}u^iU^0 + g_{ij}u^iU^j\big)}{kT}$$
$$= \frac{m}{kT}\left\{c^2 + \frac{\mathcal{V}^2}{2} + \frac{1}{c^2}\left[\frac{3\mathcal{V}^4}{8} - 2\phi\mathcal{V}^2 + \frac{V^2\mathcal{V}^2}{2}\right.\right.$$
$$\left.\left.+\frac{(V_i\mathcal{V}_i)^2}{2} + (V_i\mathcal{V}_i)\mathcal{V}^2\right]\right\}. \tag{4.11}$$

In the above equation the components of the metric tensor (2.74) – (2.78) were introduced.

Up to $1/c^4$ order the modified Bessel function of second kind reads

$$\frac{1}{K_2(\zeta)} = \sqrt{\frac{2mc^2}{\pi kT}}\, e^{\frac{mc^2}{kT}} \left(1 - \frac{15kT}{8mc^2} + \frac{345(kT)^2}{128m^2c^4} + \dots\right). \tag{4.12}$$

thanks to the asymptotic expansion (1.76). Hence, by taking into account (4.9), (4.11) and (4.12) we get the first post-Newtonian approximation of the Maxwell-Jüttner distribution function

$$f = \frac{n}{(2\pi mkT)^{\frac{3}{2}}} e^{-\frac{m\mathcal{V}^2}{2kT}}\left\{1 - \frac{m}{kTc^2}\left[\frac{3\mathcal{V}^4}{8} + \frac{15k^2T^2}{8m^2}\right.\right.$$
$$\left.\left.-2\phi\mathcal{V}^2 + \frac{(V_i\mathcal{V}_i)^2}{2} + \frac{V^2\mathcal{V}^2}{2} + (V_i\mathcal{V}_i)\mathcal{V}^2\right]\right\}. \tag{4.13}$$

Here we have considered the terms with the factor $1/c^2$ of small order and used the approximation $e^{-x} \approx 1 - x$.

4.1.3 Post-Newtonian macroscopic fields

The expressions for the particle four-flow N^μ and for the energy-momentum tensor $T^{\mu\nu}$ in terms of the one-particle distribution function were introduced in Chapter 1 (see (1.53) and (1.54)) and their expressions as function of the particle four-velocity u^μ are

$$N^\mu = m^3 c \int u^\mu \sqrt{-g} f \frac{d^3u}{u_0}, \qquad T^{\mu\nu} = m^4 c \int u^\mu u^\nu \sqrt{-g} f \frac{d^3u}{u_0}. \tag{4.14}$$

The transformation of the differential element of integration $d^3u = du^1 du^2 du^3$ in terms of the one related with the peculiar velocity $d^3\mathcal{V} = d\mathcal{V}_1 d\mathcal{V}_2 d\mathcal{V}_3$ is given by the determinant of the Jacobian matrix

$$d^3u = |J| d^3\mathcal{V}, \qquad \text{where} \qquad J = \frac{\partial(u^1, u^2, u^2)}{\partial(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)}. \tag{4.15}$$

From the expression of u^i in terms of the peculiar velocity $\mathcal{V}_i$ one can obtain that

$$\frac{\partial u^i}{\partial \mathcal{V}_j} = \delta_{ij}\left(1 + \frac{\mathcal{V}^2 + V^2 + 2V_k\mathcal{V}_k}{2c^2} - \frac{\phi}{c^2}\right) + \frac{(V_i + \mathcal{V}_i)(V_j + \mathcal{V}_j)}{c^2}, \tag{4.16}$$

and it follows by considering terms up to the $1/c^2$ order that

$$d^3u = \left\{1 + \frac{1}{c^2}\left[\frac{5(V^2 + 2V_k\mathcal{V}_k + \mathcal{V}^2)}{2} - 3\phi\right]\right\} d^3\mathcal{V}. \tag{4.17}$$

Likewise from the expressions for the components of the metric tensor (2.74) – (2.78) we can build the relation up to the $1/c^2$

order

$$u_0 = (g_{00}u^0 + g_{0i}u^i) = g_{00}u^0\left(1 + \frac{g_{0i}u^i}{g_{00}u^0}\right) = u^0\left(1 + 2\frac{\phi}{c^2}\right). \tag{4.18}$$

Now by considering that $\sqrt{-g} = 1 - 2\phi/c^2$ we have the following relationship up to the $1/c^2$ order

$$\frac{\sqrt{-g}\,d^3u}{u_0} = \left\{1 + \frac{1}{c^2}\left[\frac{5(V^2 + 2V_k\mathcal{V}_k + \mathcal{V}^2)}{2} - 7\phi\right]\right\}\frac{d^3\mathcal{V}}{u^0}. \tag{4.19}$$

Another expression for the integration element which will be used in the next sections is

$$\frac{\sqrt{-g}\,d^3u}{u_0} = \left\{1 + \frac{1}{c^2}\left[2v^2 - 6\phi\right]\right\}\frac{d^3v}{c}, \tag{4.20}$$

thanks to (4.10). Here we have written the integration element as function of the particle velocity $\mathbf{v}$.

From the knowledge of the post-Newtonian Maxwell-Jüttner distribution function (4.13) and the element of integration (4.19) we can calculate the expressions for the particle four-flow and the energy-momentum tensor in this approximation.

The time component of the particle four-flow $(4.14)_1$ can be written as

$$N^0 = m^3c\int u^0\sqrt{-g}f\frac{d^3u}{u_0} = \int_0^\infty \frac{m^3cne^{-\frac{m\mathcal{V}^2}{2kT}}4\pi\mathcal{V}^2}{(2\pi mkT)^{\frac{3}{2}}}d\mathcal{V}$$
$$\times\left\{1 - \frac{m}{kTc^2}\left[\frac{3\mathcal{V}^4}{8} + \frac{15k^2T^2}{8m^2} - 2\phi\mathcal{V}^2 + \frac{(V_i\mathcal{V}_i)^2}{2} + \frac{V^2\mathcal{V}^2}{2}\right.\right.$$

$$+(V_i\mathcal{V}_i)\mathcal{V}^2\Bigg] + \frac{1}{c^2}\left[\frac{5(V^2+2V_k\mathcal{V}_k+\mathcal{V}^2)}{2} - 7\phi\right]\Bigg\}$$

$$= nc\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} - \phi\right)\right]. \tag{4.21}$$

Above we have introduced spherical coordinates to express the integral element $d^3\mathcal{V} = \mathcal{V}^2 \sin\theta d\theta d\varphi$, where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$ and performed the integrations in these angles whose result is 4π. Furthermore, we have used the expressions for the Gaussian integrals in the Appendix A. Note that the integrals of the odd velocities $\mathcal{V}_i$ are zero.

The expression for N^0 given by (4.21) matches with the one of the phenomenological equation (2.208).

The components of the energy-momentum tensor are obtained through integration of $(4.14)_2$ by using (4.19) and (4.13), yielding

$$T^{00} = m^4c\int u^0u^0\sqrt{-g}f\frac{d^3u}{u_0} = \int_0^\infty \frac{m^4c^2ne^{-\frac{m\mathcal{V}^2}{2kT}}4\pi\mathcal{V}^2}{(2\pi mkT)^{\frac{3}{2}}}d\mathcal{V}$$

$$\times\Bigg\{1 - \frac{m}{kTc^2}\Bigg[\frac{3\mathcal{V}^4}{8} + \frac{15k^2T^2}{8m^2} - 2\phi\mathcal{V}^2 + \frac{(V_i\mathcal{V}_i)^2}{2}$$

$$+\frac{V^2\mathcal{V}^2}{2} + (V_i\mathcal{V}_i)\mathcal{V}^2\Bigg] + \frac{1}{c^2}\left[3(V^2+2V_k\mathcal{V}_k+\mathcal{V}^2) - 8\phi\right]\Bigg\}$$

$$= \rho c^2\left[1 + \frac{1}{c^2}\left(V^2 - 2\phi + \varepsilon\right)\right], \tag{4.22}$$

$$T^{0i} = m^4c\int u^0u^i\sqrt{-g}f\frac{d^3u}{u_0} = \int_0^\infty \frac{m^4cne^{-\frac{m\mathcal{V}^2}{2kT}}4\pi\mathcal{V}^2}{(2\pi mkT)^{\frac{3}{2}}}d\mathcal{V}$$

$$\times(V_i+\mathcal{V}_i)\left\{1-\frac{m}{kTc^2}\left[\frac{3\mathcal{V}^4}{8}+\frac{15k^2T^2}{8m^2}-2\phi\mathcal{V}^2+\frac{(V_i\mathcal{V}_i)^2}{2}\right.\right.$$
$$\left.\left.+\frac{V^2\mathcal{V}^2}{2}+(V_i\mathcal{V}_i)\mathcal{V}^2\right]+\frac{1}{c^2}\left[3(V^2+2V_k\mathcal{V}_k+\mathcal{V}^2)-8\phi\right]\right\}$$
$$=\rho cV_i\left[1+\frac{1}{c^2}\left(V^2-2\phi+\varepsilon+\frac{p}{\rho}\right)\right], \tag{4.23}$$

$$T^{ij}=m^4c\int u^iu^j\sqrt{-g}f\frac{d^3u}{u_0}=\int_0^\infty\frac{m^4ne^{-\frac{m\mathcal{V}^2}{2kT}}4\pi\mathcal{V}^2}{(2\pi mkT)^{\frac{3}{2}}}d\mathcal{V}$$
$$\times(V_i+\mathcal{V}_i)(V_j+\mathcal{V}_j)\left\{1-\frac{m}{kTc^2}\left[\frac{3\mathcal{V}^4}{8}+\frac{15k^2T^2}{8m^2}\right.\right.$$
$$\left.-2\phi\mathcal{V}^2+\frac{(V_i\mathcal{V}_i)^2}{2}+\frac{V^2\mathcal{V}^2}{2}+(V_i\mathcal{V}_i)\mathcal{V}^2\right]$$
$$\left.+\frac{1}{c^2}\left[3(V^2+2V_k\mathcal{V}_k+\mathcal{V}^2)-8\phi\right]\right\}=p\left[1+\frac{2\phi}{c^2}\right]\delta_{ij}$$
$$+\rho\left[1+\frac{1}{c^2}\left(V^2-2\phi+\varepsilon+\frac{p}{\rho}\right)\right]V_iV_j. \tag{4.24}$$

The above expressions match the phenomenological ones (2.93) – (2.95), the only difference is that here the specific internal energy refers to the one of a monatomic gas $\varepsilon = 3kT/2m$. Note that above $p = nkT$ is the pressure of a perfect fluid.

4.1.4 Post-Newtonian transfer and Eulerian hydrodynamic equations

Here we follow the methodology described in Chapter 1 and multiply the Boltzmann equation (4.8) by an arbitrary function $\Psi(\mathbf{x},\mathbf{v},t)$ and integrate the resulting equation by using the invariant integration element (4.20). Hence it follows the post-Newtonian version of the Maxwell-Enskog transfer equation, namely

$$\frac{\partial}{\partial t}\int\Psi\left[1+\frac{1}{c^2}\left(\frac{5v^2}{2}-7\phi\right)\right]fd^3v+\frac{\partial}{\partial x^i}\int\Psi v_i\Bigg[1
+\frac{1}{c^2}\left(\frac{5v^2}{2}-7\phi\right)\Bigg]fd^3v-\frac{2}{c^2}\int\Psi\left[\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x^i}v_i\right]fd^3v
-\int\left[\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi}{\partial x^i}v_i\right]\left[1+\frac{1}{c^2}\left(\frac{5v^2}{2}-7\phi\right)\right]fd^3v
+\frac{\partial\phi}{\partial x^i}\int\frac{\partial\Psi}{\partial v^i}\left[1+\frac{1}{c^2}\left(\frac{7}{2}v^2-3\phi\right)\right]fd^3v
-\frac{1}{c^2}\int\frac{\partial\Psi}{\partial v^i}\Bigg[4v_iv_j\frac{\partial\phi}{\partial x^j}-v_j\left(\frac{\partial\xi_i}{\partial x^j}-\frac{\partial\xi_j}{\partial x^i}\right)+3v_i\frac{\partial\phi}{\partial t}-\frac{\partial\psi}{\partial x^i}
-\frac{\partial\xi_i}{\partial t}\Bigg]fd^3v=\int\Psi\left\{1+\frac{1}{c^2}\left[2v^2-6\phi\right]\right\}\mathcal{Q}(f,f)d^3v. \quad (4.25)$$

Likewise in Chapter 1, for the divergence term in the velocity space we used the divergence theorem and considered that the one-particle distribution function vanishes at the surface situated far away in the velocity space.

As usual in kinetic theory of gases the hydrodynamic equations for the mass, mass-energy and momentum densities are obtained from the transfer equation (4.25) by choosing appropriated values for the arbitrary function $\Psi(\mathbf{x}, \mathbf{v}, t)$.

We begin by choosing $\Psi = m^4$ in (4.25) and considering the equilibrium distribution function (4.13). By performing the integrations we get

$$\frac{\partial}{\partial t}\left\{\rho\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}-\phi\right)\right]\right\}+\frac{\partial}{\partial x^i}\left\{\rho V_i\left[1\right.\right.$$
$$\left.\left.+\frac{1}{c^2}\left(\frac{V^2}{2}-\phi\right)\right]\right\}=\frac{2\rho}{c^2}\left(\frac{\partial\phi}{\partial t}+V_i\frac{\partial\phi}{\partial x^i}\right)$$
$$=\frac{2}{c^2}\left(\frac{\partial\rho\phi}{\partial t}+\frac{\partial\rho\phi V_i}{\partial x^i}\right)-\frac{2\phi}{c^2}\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho V_i}{\partial x^i}\right). \quad (4.26)$$

Now we use the Newtonian approximation of the continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho V_i}{\partial x^i}=0, \quad (4.27)$$

for the terms of order $\mathcal{O}(c^{-2})$ in (4.26). If we introduce the mass density

$$\rho^*=\rho\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}-3\phi\right)\right], \quad (4.28)$$

we arrive at the continuity equation for the mass density ρ^* in the post-Newtonian approximation, namely

$$\frac{\partial\rho^*}{\partial t}+\frac{\partial\rho^* V_i}{\partial x^i}=0. \quad (4.29)$$

The above equations match equations (2.122) and (2.123) of the phenomenological theory.

The hydrodynamic equation for the mass-energy density is obtained by choosing $\Psi = m^4 u^0$ in (4.25) and following the same methodology, yielding

$$\frac{\partial}{\partial t}\left\{\rho\left[1+\frac{1}{c^2}\left(V^2-2\phi+\varepsilon\right)\right]\right\}+\frac{\partial}{\partial x^i}\left\{\rho V_i\left[1\right.\right.$$
$$\left.\left.+\frac{1}{c^2}\left(V^2-2\phi+\varepsilon+\frac{p}{\rho}\right)\right]\right\}-\frac{\rho}{c^2}\frac{\partial\phi}{\partial t}=0. \qquad (4.30)$$

In terms of $\sigma = \rho\left[1+\frac{1}{c^2}\left(V^2-2\phi+\varepsilon+\frac{p}{\rho}\right)\right]$, it can be written as

$$\frac{\partial\sigma}{\partial t}+\frac{\partial\sigma V_i}{\partial x^i}-\frac{1}{c^2}\left(\rho\frac{\partial\phi}{\partial t}+\frac{\partial p}{\partial t}\right)=0, \qquad (4.31)$$

which matches equation (2.127) of the phenomenological theory.

For the momentum density we choose $\Psi = m^4 u^i$ in (4.25), use the distribution function (4.13), perform the integrations and get

$$\frac{\partial}{\partial t}\left\{\rho\left[1+\frac{1}{c^2}\left(V^2-2\phi+\varepsilon+\frac{p}{\rho}\right)\right]V_i\right\}$$
$$+\frac{\partial}{\partial x^i}\left[p\left(1+\frac{2\phi}{c^2}\right)\right]-4\frac{\rho}{c^2}V_i\left(\frac{\partial\phi}{\partial t}+V_j\frac{\partial\phi}{\partial x^j}\right)$$
$$+\frac{\partial}{\partial x^j}\left\{\rho\left[1+\frac{1}{c^2}\left(V^2-2\phi+\varepsilon+\frac{p}{\rho}\right)\right]V_iV_j\right\}$$
$$+\rho\left[1+\frac{1}{c^2}\left(2V^2+2\phi+\varepsilon-\frac{p}{\rho}\right)\right]\frac{\partial\phi}{\partial x^i}+\frac{\rho}{c^2}\left(\frac{\partial\xi_i}{\partial t}+\frac{\partial\psi}{\partial x^i}\right)$$

$$+\frac{\rho}{c^2}V_j\left(\frac{\partial\xi_i}{\partial x^j}-\frac{\partial\xi_j}{\partial x^i}\right)=0. \tag{4.32}$$

By introducing σ and the material time derivative $d/dt = \partial/\partial t + V^i\partial/\partial x^i$ the above equation can be rewritten as

$$\frac{\partial\sigma V_i}{\partial t}+\frac{\partial\sigma V_iV_j}{\partial x^j}+\rho\frac{\partial\phi}{\partial x^i}\left[1+\frac{2}{c^2}\left(V^2-\phi+\frac{\varepsilon}{2}+\frac{3p}{2\rho}\right)\right]$$
$$+\frac{\partial}{\partial x^i}\left[p\left(1-\frac{2\phi}{c^2}\right)\right]-\frac{\rho}{c^2}V_j\frac{\partial\xi_j}{\partial x^i}+\frac{\rho}{c^2}\frac{\partial\psi}{\partial x^i}$$
$$-\frac{4\rho}{c^2}\frac{d}{dt}\left(V_i\phi-\frac{\xi_i}{4}\right)=0. \tag{4.33}$$

Here we have used the Newtonian expression of the momentum density hydrodynamic equation (2.129) for the terms of order $\mathcal{O}\left(c^{-2}\right)$. The above equation matches (2.131) of the phenomenological theory.

It is noteworthy to call attention to the fact that the right-hand side of the Boltzmann equation vanishes for the choices m^4, m^4u^0, m^4u^i, since mass, momentum and energy densities are conservative quantities at collision.

4.2 Second post-Newtonian approximation

In this section we shall derive the Boltzmann equation and the Maxwell-Jüttner distribution function in the second post-Newtonian approximation by using the Chandrasekhar potentials and the results of Chapter 3.

4.2.1 Post-Newtonian Boltzmann equation

We start by writing the acceleration term (4.4) and taking into account the higher order Christoffel symbols up to $\mathcal{O}\left(c^{-6}\right)$. From the expressions for the Christoffel symbols given in the Appendix A of Chapter 3 we get

$$\frac{d^2x^i}{dt^2} = c^2\Bigg\{ - \overset{2}{\Gamma}{}^i{}_{00} - \overset{4}{\Gamma}{}^i{}_{00} - \overset{6}{\Gamma}{}^i{}_{00} - 2\frac{v_j}{c}\left(\overset{3}{\Gamma}{}^i{}_{0j} + \overset{5}{\Gamma}{}^i{}_{0j}\right)$$
$$+\frac{v_i}{c}\left[\overset{3}{\Gamma}{}^0{}_{00} + \overset{5}{\Gamma}{}^0{}_{00} + 2\frac{v_j}{c}\left(\overset{2}{\Gamma}{}^0{}_{0j} + \overset{4}{\Gamma}{}^0{}_{0j}\right) + \frac{v_jv_k}{c^2}\overset{3}{\Gamma}{}^0{}_{jk}\right]$$
$$-\frac{v_jv_k}{c^2}\left(\overset{2}{\Gamma}{}^i{}_{jk} + \overset{4}{\Gamma}{}^i{}_{jk}\right)\Bigg\} = \frac{\partial U}{\partial x^i} - \frac{v_i}{c^2}\left[\frac{\partial U}{\partial t} + 2v_j\frac{\partial U}{\partial x^j}\right]$$
$$-\frac{1}{c^2}\left(1-\frac{2U}{c^2}\right)\left[2\frac{\partial\left(U^2-\Phi\right)}{\partial x^i} + \left(2v_iv_j\frac{\partial U}{\partial x^j} - v^2\frac{\partial U}{\partial x^i}\right)\right.$$
$$\left.-\frac{\partial\Pi_i}{\partial t} - v_j\left(\frac{\partial\Pi_i}{\partial x^j} - \frac{\partial\Pi_j}{\partial x^i} - 2\delta_{ij}\frac{\partial U}{\partial t}\right)\right] + \frac{v_i}{c^4}\left[v_jv_k\frac{\partial\Pi_j}{\partial x^k}\right.$$
$$\left.+v^2\frac{\partial U}{\partial t}\right] + \frac{v_i}{c^4}\left[\Pi_j\frac{\partial U}{\partial x^j} - 2\frac{\partial\Phi}{\partial t} - 4v_j\frac{\partial\Phi}{\partial x^j}\right] + \frac{v_jv_k}{2c^4}\left[2\frac{\partial\Psi_{ij}}{\partial x^k}\right.$$
$$\left.-\frac{\partial\Psi_{jk}}{\partial x^i}\right] + \frac{1}{c^4}\left[\Pi_i\frac{\partial U}{\partial t} + \frac{\partial\Psi_{i0}}{\partial t} - \frac{1}{2}\frac{\partial\Psi_{00}}{\partial x^i} + \Psi_{ij}\frac{\partial U}{\partial x^j}\right]$$
$$+\frac{2v_j}{c^4}\left[\Pi_i\frac{\partial U}{\partial x^j} + \frac{1}{2}\frac{\partial\Psi_{i0}}{\partial x^j} - \frac{1}{2}\frac{\partial\Psi_{j0}}{\partial x^i} + \frac{1}{2}\frac{\partial\Psi_{ij}}{\partial t}\right]. \tag{4.34}$$

According to (4.5) the variation of the one-particle distribution function with respect to the proper time is given by

$$\frac{df(x^\mu(\tau), v^i(\tau))}{d\tau} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial f}{\partial v_i}\frac{dv_i}{d\tau} = u^\mu \frac{\partial f}{\partial x^\mu} + \frac{\partial f}{\partial v_i}\frac{d^2x^i}{dt^2}\frac{dt}{d\tau}, \tag{4.35}$$

where the time component of the four-velocity of the gas particles in the second post-Newtonian approximation is given by

$$u^0 = c\left\{1 + \frac{1}{c^2}\left(\frac{v^2}{2} + U\right) + \frac{1}{c^4}\left(\frac{3v^4}{8} + \frac{5Uv^2}{2} + \frac{U^2}{2} + 2\Phi - \Pi_i v_i\right)\right\}, \tag{4.36}$$

which follows from (3.10) by replacing the components of the hydrodynamic velocities by the gas particles velocities. The spatial component of the four velocity of the gas particles is $u^i = u^0 v_i/c$ and $dt/d\tau = u^0/c$.

The Boltzmann equation in the second post-Newtonian approximation follows from (4.35) by taking into account (4.34) and (4.36) and reads

$$\left[\frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial v_i}\frac{\partial U}{\partial x^i}\right]\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + U\right) + \frac{1}{c^4}\left(\frac{3v^4}{8} + \frac{5v^2U}{2} + \frac{U^2}{2} + 2\Phi - \Pi_j v_j\right)\right] + \frac{1}{c^2}\frac{\partial f}{\partial v_i}\left\{\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} - U\right)\right]\left[v_j\left(\frac{\partial \Pi_i}{\partial x^j} - \frac{\partial \Pi_j}{\partial x^i}\right) - 2v_i\frac{\partial U}{\partial t} + \frac{\partial \Pi_i}{\partial t} - 2\frac{\partial\left(U^2 - \Phi\right)}{\partial x^i} - 2v_i v_j\frac{\partial U}{\partial x^j} + v^2\frac{\partial U}{\partial x^i}\right] - v_i\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + U\right)\right]\left[\frac{\partial U}{\partial t}\right.\right.$$

$$+2v_j\frac{\partial U}{\partial x^j}\bigg] + \frac{v_jv_k}{2c^2}\bigg[2\frac{\partial\Psi_{ij}}{\partial x^k} - \frac{\partial\Psi_{jk}}{\partial x^i}\bigg] + \frac{1}{c^2}\bigg[\Pi_i\frac{\partial U}{\partial t} + \frac{\partial\Psi_{i0}}{\partial t}$$
$$-\frac{1}{2}\frac{\partial\Psi_{00}}{\partial x^i} + \Psi_{ij}\frac{\partial U}{\partial x^j}\bigg] + \frac{v_i}{c^2}\bigg[v_jv_k\frac{\partial\Pi_j}{\partial x^k} + v^2\frac{\partial U}{\partial t}\bigg]$$
$$+\frac{2v_j}{c^2}\bigg[\Pi_i\frac{\partial U}{\partial x^j} + \frac{1}{2}\frac{\partial\Psi_{i0}}{\partial x^j} - \frac{1}{2}\frac{\partial\Psi_{j0}}{\partial x^i} + \frac{1}{2}\frac{\partial\Psi_{ij}}{\partial t}\bigg]$$
$$+\frac{v_i}{c^2}\bigg[\Pi_j\frac{\partial U}{\partial x^j} - 2\frac{\partial\Phi}{\partial t} - 4v_j\frac{\partial\Phi}{\partial x^j}\bigg]\bigg\} = \mathcal{Q}(f,f). \tag{4.37}$$

4.2.2 Post-Newtonian Maxwell-Jüttner distribution function

The determination of the Maxwell-Jüttner distribution function in the second post-Newtonian approximation follows the same methodology described in Section 4.2. First we compute the term in the exponential up to the order $1/c^4$, namely

$$\frac{g_{\mu\nu}p^\mu U^\nu}{kT} = \frac{m}{kT}\bigg\{c^2 + \frac{\mathcal{V}^2}{2} + \frac{1}{c^2}\bigg[2U\mathcal{V}^2 + \frac{V^2\mathcal{V}^2}{2}$$
$$+\frac{(V_i\mathcal{V}_i)^2}{2} + (V_i\mathcal{V}_i)\mathcal{V}^2 + \frac{3\mathcal{V}^4}{8}\bigg] + \frac{1}{c^4}\bigg[3U^2\mathcal{V}^2$$
$$+4UV^2\mathcal{V}^2 + 4U(V_i\mathcal{V}_i)^2 + 8U(V_i\mathcal{V}_i)\mathcal{V}^2 + 3U\mathcal{V}^4$$
$$+\frac{V^4\mathcal{V}^2}{2} + V^2(V_i\mathcal{V}_i)^2 + 2V^2(V_i\mathcal{V}_i)\mathcal{V}^2 + \frac{3V^2\mathcal{V}^4}{4}$$
$$+(V_i\mathcal{V}_i)^3 + \frac{9(V_i\mathcal{V}_i)^2\mathcal{V}^2}{4} + \frac{3(V_i\mathcal{V}_i)\mathcal{V}^4}{2} + \frac{5\mathcal{V}^6}{16} - \Pi_i\mathcal{V}_i(V_i\mathcal{V}_i)$$

$$-\Pi_i V_i \mathcal{V}^2 - \Pi_i \mathcal{V}_i \mathcal{V}^2 + 2\Phi\mathcal{V}^2 - \frac{\mathcal{V}_i\mathcal{V}_j\Psi_{ij}}{2}\Bigg]\Bigg\}, \tag{4.38}$$

thanks to (3.2) – (3.4), (3.10) and (4.36).

The equilibrium Maxwell-Jüttner distribution function in the second post-Newtonian approximation is obtained from the insertion of (4.38) into (4.9) and considering the approximation $e^{-x} \approx 1 - x + x^2/2$. Up to $(1/c^4)$–terms the resulting equation reads

$$f = \frac{n}{(2\pi mkT)^{\frac{3}{2}}} e^{-\frac{m\mathcal{V}^2}{2kT}} \Bigg\{ 1 - \frac{1}{c^2}\Bigg[\frac{15kT}{8m} + \frac{m(V_i\mathcal{V}_i)^2}{2kT}$$
$$+\frac{2mU\mathcal{V}^2}{kT} + \frac{3m\mathcal{V}^4}{8kT} + \frac{mV^2\mathcal{V}^2}{2kT} + \frac{m(V_i\mathcal{V}_i)\mathcal{V}^2}{kT}\Bigg]$$
$$+\frac{1}{c^4}\Bigg[\frac{2m^2U^2\mathcal{V}^4}{(kT)^2} + \frac{m^2UV^2\mathcal{V}^4}{(kT)^2} + \frac{3m^2U\mathcal{V}^6}{4(kT)^2} + \frac{m^2V^4\mathcal{V}^4}{8(kT)^2}$$
$$+\frac{m^2U(V_i\mathcal{V}_i)^2\mathcal{V}^2}{(kT)^2} + \frac{2m^2U(V_i\mathcal{V}_i)\mathcal{V}^4}{(kT)^2} + \frac{3m^2V^2\mathcal{V}^6}{16(kT)^2}$$
$$+\frac{m^2(V_i\mathcal{V}_i)^4}{8(kT)^2} + \frac{m^2V^2(V_i\mathcal{V}_i)^2\mathcal{V}^2}{4(kT)^2} + \frac{m^2V^2(V_i\mathcal{V}_i)\mathcal{V}^4}{2(kT)^2}$$
$$+\frac{m^2(V_i\mathcal{V}_i)^3\mathcal{V}^2}{2(kT)^2} + \frac{11m^2(V_i\mathcal{V}_i)^2\mathcal{V}^4}{16(kT)^2} + \frac{3m^2(V_i\mathcal{V}_i)\mathcal{V}^6}{8(kT)^2}$$
$$+\frac{9m^2\mathcal{V}^8}{128(kT)^2} + \frac{345(kT)^2}{128m^2} - \frac{3mU^2\mathcal{V}^2}{kT} - \frac{4mUV^2\mathcal{V}^2}{kT}$$
$$-\frac{4mU(V_i\mathcal{V}_i)^2}{kT} - \frac{8mU(V_i\mathcal{V}_i)\mathcal{V}^2}{kT} - \frac{3mU\mathcal{V}^4}{kT}$$

$$
\begin{aligned}
&-\frac{mV^4\mathcal{V}^2}{2kT}-\frac{mV^2(V_i\mathcal{V}_i)^2}{kT}-\frac{2mV^2(V_i\mathcal{V}_i)\mathcal{V}^2}{kT}\\
&-\frac{3mV^2\mathcal{V}^4}{4kT}-\frac{m(V_i\mathcal{V}_i)^3}{kT}-\frac{5m\mathcal{V}^6}{16kT}+\frac{m\Pi_iV_i\mathcal{V}^2}{kT}\\
&-\frac{9m(V_i\mathcal{V}_i)^2\mathcal{V}^2}{4kT}+\frac{m\Pi_i\mathcal{V}_i(V_i\mathcal{V}_i)}{kT}-\frac{3m(V_i\mathcal{V}_i)\mathcal{V}^4}{2kT}\\
&+\frac{m\Pi_i\mathcal{V}_i\mathcal{V}^2}{kT}-\frac{2m\Phi\mathcal{V}^2}{kT}+\frac{m\mathcal{V}_i\mathcal{V}_j\Psi_{ij}}{2kT}+\frac{15U\mathcal{V}^2}{4}\\
&+\frac{15V^2\mathcal{V}^2}{16}+\frac{15(V_i\mathcal{V}_i)^2}{16}+\frac{15(V_i\mathcal{V}_i)\mathcal{V}^2}{8}+\frac{45\mathcal{V}^4}{64}\Bigg]\Bigg\}. \qquad (4.39)
\end{aligned}
$$

4.2.3 Post-Newtonian macroscopic fields

To determine the particle four-flow and the energy-momentum tensor components we need to know the integration element $\sqrt{-g}\,d^3u/u_0$ in the second post-Newtonian approximation. To obtain it we begin by determining the Jacobian matrix of the transformation $d^3u = |J|d^3v$ from (4.36) which leads to

$$
\begin{aligned}
d^3u = \Bigg[1+\frac{1}{c^2}\left(\frac{5v^2}{2}+3U\right)+\frac{1}{c^4}\Bigg(&\frac{35v^4}{8}+\frac{9U^2}{2}+6\Phi\\
&+\frac{35Uv^2}{2}-4\Pi_iv_i\Bigg)\Bigg]d^3v. \qquad (4.40)
\end{aligned}
$$

Moreover from the expressions for the components of the metric tensor (3.2) – (3.4) and of the particle four-velocity (4.36) we have that

$$u_0 = (g_{00}u^0 + g_{0i}u^i) = u^0\left[1 - 2\frac{U}{c^2} + \frac{1}{c^4}\left(2U^2 - 4\Phi + \Pi_i v_i\right)\right], \tag{4.41}$$

$$\sqrt{-g} = 1 + \frac{2U}{c^2} - \frac{1}{c^4}\left(U^2 + 2\Phi + \frac{\Psi_{kk}}{2}\right). \tag{4.42}$$

Hence up to the $1/c^4$ order we get the following relationship

$$\frac{\sqrt{-g}\,d^3u}{u_0} = \left[1 + \frac{1}{c^2}\left(\frac{5v^2}{2} + 7U\right) + \frac{1}{c^4}\left(\frac{35v^4}{8} + \frac{55Uv^2}{2} + \frac{43U^2}{2} + 8\Phi - \frac{\Psi_{kk}}{2} - 5\Pi_i v_i\right)\right]\frac{d^3v}{u^0}$$
$$= \left\{1 + \frac{1}{c^2}\left[2v^2 + 6U\right] + \frac{1}{c^4}\left[3v^4 + 20Uv^2 + 15U^2 + 6\Phi - 4\Pi_i v_i - \frac{\Psi_{kk}}{2}\right]\right\}\frac{d^3v}{c}. \tag{4.43}$$

From the knowledge of the equilibrium Maxwell-Jüttner distribution function (4.39) and of the integration element (4.43) it is possible to determine the components of the particle four-flow and energy-momentum tensor in the second post-Newtonian approximation. Indeed, the insertion of (4.39) and (4.43) into the definition of the particle four-flow $(4.14)_1$ and integration of the resulting equation leads to

$$N^0 = nc\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} + U\right) + \frac{1}{c^4}\left(\frac{3V^4}{8} + \frac{5V^2U}{2}\right.\right.$$

$$+\frac{U^2}{2}+2\Phi-\Pi_iV_i\Big)\Big],\qquad(4.44)$$

and $N^i = N^0V_i/c$.

The components of the energy-momentum tensor follow from the insertion of the Maxwell-Jüttner distribution function (4.39) and the integration element (4.43) into $(4.14)_2$ and integration of the resulting equations. The time component of the energy-momentum tensor reads

$$T^{00}=\rho c^2\Bigg[1+\frac{1}{c^2}\left(V^2+2U+\frac{3kT}{2m}\right)+\frac{1}{c^4}\Bigg(V^4+6UV^2 +2U^2+\frac{5kTV^2}{2m}+\frac{15(kT)^2}{8m^2}+\frac{3kTU}{m}-2\Pi_iV_i+4\Phi\Bigg)\Bigg].\qquad(4.45)$$

If we make use of the thermal equation of state and the expression of the specific internal energy

$$p=\frac{\rho kT}{m},\qquad \varepsilon=\frac{3kT}{2m}\left(1+\frac{5kT}{4mc^2}\right),\qquad(4.46)$$

the resulting expression for the time component of the energy-momentum tensor becomes

$$T^{00}=\rho c^2\Bigg\{1+\frac{1}{c^2}\left(V^2+2U+\varepsilon\right)+\frac{1}{c^4}\Bigg[V^4+6UV^2 +2U^2+\left(\varepsilon+\frac{p}{\rho}\right)V^2+2\varepsilon U-2\Pi_iV_i+4\Phi\Bigg]\Bigg\}.\qquad(4.47)$$

The final expression for the energy-momentum tensor space-time components is

$$T^{0i} = \rho c V_i \left[1 + \frac{1}{c^2}\left(V^2 + 2U + \frac{5kT}{2m}\right) + \frac{1}{c^4}\left(V^4 + 6UV^2 \right.\right.$$
$$\left.\left. + 2U^2 + \frac{5kTV^2}{2m} + \frac{15(kT)^2}{8m^2} + \frac{5kTU}{m} - 2\Pi_j V_j + 4\Phi\right)\right]$$
$$- \frac{\rho kT\Pi_i}{mc^4} = \rho c V_i \left\{1 + \frac{1}{c^2}\left(V^2 + 2U + \varepsilon + \frac{p}{\rho}\right)\right.$$
$$+ \frac{1}{c^4}\left[V^4 + 6UV^2 + 2U^2 + \left(\varepsilon + \frac{p}{\rho}\right)\left(V^2 + 2U\right)\right.$$
$$\left.\left. - 2\Pi_j V_j + 4\Phi\right]\right\} - \frac{p\Pi_i}{c^4}, \tag{4.48}$$

while the one for the spatial components reads

$$T^{ij} = \rho\left(V_iV_j + \frac{kT}{m}\delta_{ij}\right) + \frac{\rho}{c^2}\left[\left(V^2 + 2U + \frac{3kT_0}{2m}\right)V_iV_j\right.$$
$$\left. - \frac{2kTU}{m}\delta_{ij}\right] + \frac{\rho}{c^4}\left[\left(V^4 + 6UV^2 + 2U^2 + \frac{5kTV^2}{2m}\right.\right.$$
$$\left. + \frac{15(kT)^2}{8m^2} + \frac{5kTU}{m} - 2\Pi_k V_k + 4\Phi\right)V_iV_j + \frac{4kTU^2}{m}\delta_{ij}$$
$$\left. + \frac{kT\Psi_{ij}}{m}\right] = \rho\left(V_iV_j + \frac{p}{\rho}\delta_{ij}\right) + \frac{\rho}{c^2}\left[\left(V^2 + 2U + \varepsilon\right.\right.$$
$$\left.\left. + \frac{p}{\rho}\right)V_iV_j - 2\frac{p}{\rho}U\delta_{ij}\right] + \frac{\rho}{c^4}\left[\left(V^4 + 6UV^2 + 2U^2\right.\right.$$
$$\left. + \left(\varepsilon + \frac{p}{\rho}\right)(V^2 + 2U) - 2\Pi_k V_k + 4\Phi\right)V_iV_j$$

$$+\frac{p}{\rho}(4U^2\delta_{ij}+\Psi_{ij})\Big]. \tag{4.49}$$

The above expressions for the components of the particle four-flow and energy-momentum tensor match the ones given in Section 3.6.

4.2.4 Post-Newtonian hydrodynamic equations

The post-Newtonian hydrodynamic equation for the mass density is obtained from the multiplication of the Boltzmann equation (4.36) by $m^4\sqrt{-g}d^3u/u_0$, the use of Maxwell-Jüttner distribution function (4.39) and of the integration element (4.43). The integration of the resulting equation, yields

$$\frac{\partial}{\partial t}\Bigg\{\rho\Bigg[1+\frac{1}{c^2}\left(\frac{V^2}{2}+U\right)+\frac{1}{c^4}\Bigg(\frac{3V^4}{8}+\frac{5V^2U}{2}+\frac{U^2}{2}$$
$$+2\Phi-\Pi_jV_j\Bigg)\Bigg]\Bigg\}+\frac{\partial}{\partial x^i}\Bigg\{\rho V_i\Bigg[1+\frac{1}{c^2}\left(\frac{V^2}{2}+U\right)$$
$$+\frac{1}{c^4}\left(\frac{3V^4}{8}+\frac{5V^2U}{2}+\frac{U^2}{2}+2\Phi-\Pi_jV_j\right)\Bigg]\Bigg\}$$
$$+\underline{2\frac{\rho}{c^2}\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)}+\underline{\frac{\rho}{c^4}\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)\left(V^2-4U\right)}$$
$$\underline{-\frac{\rho}{c^4}\left[2\left(\frac{\partial \Phi}{\partial t}+V_i\frac{\partial \Phi}{\partial x^i}\right)+\frac{1}{2}\left(\frac{\partial \Psi_{kk}}{\partial t}+V_i\frac{\partial \Psi_{kk}}{\partial x^i}\right)\right]}=0. \tag{4.50}$$

In order to get the continuity equation (3.74) we have to transform the underlined terms as follows. The first underlined term can be rewritten as

$$2\frac{\rho}{c^2}\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)=\frac{2}{c^2}\left(\frac{\partial\rho U}{\partial t}+\frac{\partial\rho UV_i}{\partial x^i}\right)-\frac{2U}{c^2}\left(\frac{\partial\rho}{\partial t}\right.$$
$$\left.+\frac{\partial\rho V_i}{\partial x^i}\right)=\frac{2}{c^2}\left(\frac{\partial\rho U}{\partial t}+\frac{\partial\rho V_iU}{\partial x^i}\right)+\frac{U}{c^4}\left[\left(\frac{\partial\rho V^2}{\partial t}\right.\right.$$
$$\left.\left.+\frac{\partial\rho V^2V_i}{\partial x^i}\right)+6\left(\frac{\partial\rho U}{\partial t}+\frac{\partial\rho UV_i}{\partial x^i}\right)\right],\quad(4.51)$$

where the expression for the continuity equation in the first post-Newtonian approximation (2.122) was used. The second underlined term can be transformed according to

$$\frac{\rho}{c^4}\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)\left(V^2-4U\right)=\frac{1}{c^4}\left\{\frac{\partial\rho UV^2}{\partial t}\right.$$
$$+\frac{\partial\rho UV^2V_i}{\partial x^i}-U\left(\frac{\partial\rho V^2}{\partial t}+\frac{\partial\rho V^2V_i}{\partial x^i}\right)-4U\left[\frac{\partial\rho U}{\partial t}\right.$$
$$\left.\left.+\frac{\partial\rho UV_i}{\partial x^i}-U\underline{\left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho V_i}{\partial x^i}\right)}\right]\right\}.\qquad(4.52)$$

Here we note that for the above underlined term we can use the Newtonian continuity equation so that this term vanishes. Now by adding the two equations (4.51) and (4.52) we get

$$2\frac{\rho}{c^2}\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)+\frac{\rho}{c^4}\left(\frac{\partial U}{\partial t}+V_i\frac{\partial U}{\partial x^i}\right)\left(V^2-4U\right)$$

$$= \frac{2}{c^2}\left(\frac{\partial \rho U}{\partial t} + \frac{\partial \rho U V_i}{\partial x^i}\right) + \frac{1}{c^4}\left\{\frac{\partial \rho U V^2}{\partial t} + \frac{\partial \rho U V^2 V_i}{\partial x^i} + 2U\left(\frac{\partial \rho U}{\partial t} + \frac{\partial \rho U V_i}{\partial x^i}\right)\right\}. \quad (4.53)$$

Furthermore, the last term of the above equality can be expressed as

$$\frac{2U}{c^4}\left(\frac{\partial \rho U}{\partial t} + \frac{\partial \rho U V_i}{\partial x^i}\right) = \frac{1}{c^4}\left(\frac{\partial \rho U^2}{\partial t} + \frac{\partial \rho U^2 V_i}{\partial x^i}\right) + \mathcal{O}\left(c^{-6}\right). \quad (4.54)$$

The last underlined term in (4.50) can be written as

$$-\frac{\rho}{c^4}\left[2\left(\frac{\partial \Phi}{\partial t} + V_i\frac{\partial \Phi}{\partial x^i}\right) + \frac{1}{2}\left(\frac{\partial \Psi_{kk}}{\partial t} + V_i\frac{\partial \Psi_{kk}}{\partial x^i}\right)\right]$$
$$= -\frac{1}{c^4}\left[\frac{\partial \rho\left(2\Phi + \frac{\Psi_{kk}}{2}\right)}{\partial t} + \frac{\partial \rho V_i\left(2\Phi + \frac{\Psi_{kk}}{2}\right)}{\partial x^i}\right] + \frac{1}{c^4}\left(2\Phi + \frac{\Psi_{kk}}{2}\right)\underline{\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_i}{\partial x^i}\right)}, \quad (4.55)$$

where the underlined term above vanishes thanks to the Newtonian continuity equation.

The continuity equation in the second post-Newtonian approximation is obtained from (4.50) by using (4.53)– (4.55), yielding

$$\frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial \widetilde{\rho} V_i}{\partial x^i} = 0, \quad (4.56)$$

which together with the expression for the mass density in the second post-Newtonian approximation

$$\widetilde{\rho} = \rho\Bigg[1 + \frac{1}{c^2}\left(\frac{V^2}{2} + 3U\right) + \frac{1}{c^4}\left(\frac{3}{8}V^4 + \frac{7}{2}UV^2 + \frac{3}{2}U^2 - \frac{1}{2}\Psi_{kk} - \Pi_i V_i\right)\Bigg], \tag{4.57}$$

matches equation (3.74) from the phenomenological theory.

The mass-energy hydrodynamic equation is obtained by applying the same methodology, i.e. the Boltzmann equation (4.36) is multiplied by the term $m^4 u^0 \sqrt{-g} d^3 u / u_0$, the Maxwell-Jüttner distribution function (4.39) and the integration element (4.43) are used. From the integration of the resulting equation it follows that

$$\frac{\partial}{\partial t}\Bigg\{\rho\Bigg[1 + \frac{1}{c^2}\left(V^2 + 2U + \varepsilon\right) + \frac{1}{c^4}\Bigg(V^4 + 6V^2U + 2U^2 + V^2\left(\varepsilon + \frac{p}{\rho}\right) + 2U\varepsilon + 4\Phi - 2\Pi_j V_j\Bigg)\Bigg]\Bigg\} + \frac{\partial}{\partial x^i}\Bigg\{\rho V_i\Bigg[1 + \frac{1}{c^2}\left(V^2 + 2U + \varepsilon + \frac{p}{\rho}\right) + \frac{1}{c^4}\Bigg(V^4 + 6V^2U + 2U^2 + 4\Phi + (2U + V^2)\left(\varepsilon + \frac{p}{\rho}\right) - 2\Pi_j V_j\Bigg)\Bigg] - \frac{p\Pi_i}{c^4}\Bigg\} + \frac{\rho}{c^2}\frac{\partial U}{\partial t} + \frac{\rho}{c^4}\Bigg[\left(\frac{3kT}{2m} + 2U + 2V^2\right)\frac{\partial U}{\partial t} - 3\left(\frac{\partial U^2}{\partial t} + V_i\frac{\partial U^2}{\partial x^i}\right) + \Pi_i\frac{\partial U}{\partial x^i} - \frac{1}{2}\left(\frac{\partial \Psi_{kk}}{\partial t} + V_i\frac{\partial \Psi_{kk}}{\partial x^i}\right)$$

$$-\left(4\frac{\partial\Phi}{\partial t}+6V_i\frac{\partial\Phi}{\partial x^i}\right)+V_iV_j\frac{\partial\Pi_i}{\partial x^j}\Bigg]=0. \tag{4.58}$$

This equation can be put in the following form

$$\frac{\partial\widetilde{\sigma}}{\partial t}+\frac{\partial\widetilde{\sigma}V_i}{\partial x^i}+\frac{1}{c^2}\left(\rho\frac{\partial U}{\partial t}-\frac{\partial p}{\partial t}\right)+\frac{2\rho}{c^4}\Bigg[\varphi\frac{\partial U}{\partial t}-V_i\frac{\partial\Phi}{\partial x^i}$$
$$+\frac{1}{\rho}\frac{\partial pUV_i}{\partial x^i}-\frac{V_i}{2}\frac{\partial\Pi_i}{\partial t}\Bigg]=0. \tag{4.59}$$

by introducing the abbreviations $\varphi = V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}$ from (2.111) and

$$\widetilde{\sigma}=\rho\Bigg\{1+\frac{1}{c^2}\left(V^2+2U+\varepsilon+\frac{p}{\rho}\right)+\frac{1}{c^4}\Bigg[V^4+6V^2U-U^2$$
$$+2U\varepsilon+V^2\left(\varepsilon+\frac{p}{\rho}\right)-\Pi_iV_i-\frac{1}{2}\Psi_{kk}\Bigg]\Bigg\}. \tag{4.60}$$

Note that in (4.59) the term

$$\frac{1}{c^4}\left(3U^2+4\Phi+\frac{\Psi_{kk}}{2}\right)\left[\frac{\partial\rho}{\partial t}+\frac{\partial\rho V_i}{\partial x^i}\right]$$
$$-\frac{\Pi_i}{c^4}\left[\frac{\partial\rho V_i}{\partial t}+\frac{\partial\rho V_iV_j}{\partial x^j}+\frac{\partial p}{\partial x^i}-\rho\frac{\partial U}{\partial x^i}\right], \tag{4.61}$$

was neglected, since the Newtonian continuity equation and the momentum hydrodynamic equation for the first and the second terms within the brackets can be used, respectively. Equation (4.59) corresponds to (3.81) of the phenomenological theory.

4.3 Post-Newtonian Jeans equations

In astrophysics the so-called Jeans equation refers to the momentum density hydrodynamic equation for stationary symmetrical self-gravitating systems which is derived from the collisionless Boltzmann equation (see e.g. the book [6] and the references therein). In this section we shall derive the Jeans equation from the collisionless first post-Newtonian Boltzmann equation (4.8) for stationary spherically symmetrical and axisymmetrical self-gravitating systems. The mean velocity of stationary systems, represented by the hydrodynamic velocity vanishes, i.e., $\mathbf{V} = \mathbf{0}$ and the post-Newtonian Maxwell-Jüttner distribution function (4.13) reduces to

$$f = f_0 \left\{1 - \frac{15kT}{8mc^2} - \frac{m}{kTc^2}\left[\frac{3v^4}{8} - 2\phi v^2\right]\right\}$$
$$= \frac{ne^{-\frac{mv^2}{2kT}}}{(2\pi mkT)^{\frac{3}{2}}}\left\{1 - \frac{15kT}{8mc^2} - \frac{m}{kTc^2}\left[\frac{3v^4}{8} - 2\phi v^2\right]\right\}. \quad (4.62)$$

Above f_0 is the Maxwellian distribution function.

Furthermore, for stationary systems the component of the energy-momentum tensor T^{0i} given by (4.23) vanishes and the Poisson equation (2.66) for the gravitational potential ξ_i reduces to $\nabla^2\xi_i = 0$. Hence we can consider $\vec{\xi}$ as a Laplacian vector field such that $\nabla \times \vec{\xi} = 0$ and $\nabla \cdot \vec{\xi} = 0$.

The expressions for the post-Newtonian collisionless Boltzmann equation in spherical and cylindrical coordinates by considering $\vec{\xi}$ a Laplacian vector field are given in the Appendices B and C.

4.3.1 Stationary and spherically symmetrical self-gravitating systems

For stationary spherically symmetrical self-gravitating systems the Jeans equation is a differential equation for the determination of the radial velocity dispersion, which is represented by the square root of the mean value of the radial velocity square $\sqrt{\langle v_r^2\rangle}$. The equation for the radial velocity dispersion is obtained from the multiplication of the post-Newtonian Boltzmann equation in spherical coordinates (4.89) by $m^4 v_r u^0/c$ and integration of the resulting equation by taking into account the invariant integration element (4.20), yielding

$$\frac{d}{dr}\left\{\rho\left[\langle v_r^2\rangle+\frac{3}{c^2}\langle v_r^2v^2\rangle-\frac{8}{c^2}\phi\langle v_r^2\rangle-\frac{m}{kTc^2}\left(\frac{3}{8}\langle v_r^2v^4\rangle\right.\right.\right.$$
$$\left.\left.\left.-2\phi\langle v_r^2v^2\rangle\right)-\frac{15}{8c^2}\frac{kT}{m}\langle v_r^2\rangle\right]\right\}-\frac{\rho}{r}\left\{\left(1-\frac{8}{c^2}\phi\right.\right.$$
$$\left.-\frac{15}{8c^2}\frac{kT}{m}\right)\big(\langle v_\theta^2\rangle+\langle v_\varphi^2\rangle-2\langle v_r^2\rangle\big)+\frac{1}{c^2}\left(3+\frac{2m}{kTc^2}\right)$$
$$\times\big(\langle v_\theta^2v^2\rangle+\langle v_\varphi^2v^2\rangle-2\langle v_r^2v^2\rangle\big)-\frac{3m}{8kTc^2}\big(\langle v_\theta^2v^4\rangle$$
$$\left.+\langle v_\varphi^2v^4\rangle-2\langle v_r^2v^4\rangle\big)\right\}-\frac{\rho}{r}\mathrm{cotan}\,\theta\left\{\langle v_rv_\theta\rangle+\frac{3}{c^2}\langle v_rv_\theta v^2\rangle\right.$$
$$+\frac{6}{c^2}\langle v_rv_\theta v_\varphi^2\rangle-\frac{8}{c^2}\phi\langle v_rv_\theta\rangle-\frac{15}{8c^2}\frac{kT}{m}\langle v_rv_\theta\rangle$$
$$\left.-\frac{m}{kTc^2}\left[\frac{3}{8}\langle v_rv_\theta v^4\rangle-2\phi\langle v_rv_\theta v^2\rangle\right]\right\}$$

$$+\rho\frac{d\phi}{dr}\left[1+\frac{4}{c^2}\langle v^2\rangle-\frac{4}{c^2}\phi-\frac{4}{c^2}\langle v_r^2\rangle-\frac{15}{8c^2}\frac{kT}{m}\right.$$
$$\left.-\frac{m}{kTc^2}\left(\frac{3}{8}\langle v^4\rangle-2\phi\langle v^2\rangle\right)\right]+\frac{\rho}{c^2}\frac{d\psi}{dr}=0. \qquad (4.63)$$

Note that the assumption of a spherically symmetrical system implies that the gravitational potentials and the distribution function do not depend on the angles θ and φ but only on r. In (4.63) the mean values are defined by

$$\rho=\int m^4 f_0 d^3v, \qquad \rho\langle v^n v_r^2\rangle=\int m^4 v^n v_r^2 f_0 d^3 v, \qquad (4.64)$$

and so on. In the derivation of (4.63) we have consider that the integrals with the derivatives of the distribution function with respect to the components of the velocity can be reduced according to

$$\int v^2\frac{\partial f}{\partial v_r}dv_r dv_\theta dv_\varphi=\int\frac{\partial v^2 f}{\partial v_r}dv_r dv_\theta dv_\varphi$$
$$-2\int v_r f dv_r dv_\theta dv_\varphi=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}v^2 f\Big|_{-\infty}^{+\infty}dv_\theta dv_\varphi$$
$$-2\int v_r f dv_r dv_\theta dv_\varphi=-2\langle v_r\rangle, \qquad (4.65)$$

since the distribution function vanishes for great values of the particle velocity.

The mean values in (4.63) can be expressed in terms of $\langle v_r^2\rangle, \langle v_\theta^2\rangle$ and $\langle v_\varphi^2\rangle$. Indeed, by introducing the Maxwellian distribution function $(4.62)_2$ in $(4.64)_2$ and integrating the resulting equations follow the relationships

$$\langle v^2\rangle = 3\frac{kT}{m}, \quad \langle v^4\rangle = 15\left(\frac{kT}{m}\right)^2, \quad \langle v^2 v_r^2\rangle = 5\frac{kT}{m}\langle v_r^2\rangle, \tag{4.66}$$

$$\langle v^4 v_r^2\rangle = 35\left(\frac{kT}{m}\right)^2\langle v_r^2\rangle, \quad \langle v^2 v_\theta^2\rangle = 5\frac{kT}{m}\langle v_\theta^2\rangle, \tag{4.67}$$

$$\langle v^4 v_\theta^2\rangle = 35\left(\frac{kT}{m}\right)^2\langle v_\theta^2\rangle, \quad \langle v^2 v_\varphi^2\rangle = 5\frac{kT}{m}\langle v_\varphi^2\rangle, \tag{4.68}$$

$$\langle v^4 v_\varphi^2\rangle = 35\left(\frac{kT}{m}\right)^2\langle v_\varphi^2\rangle, \tag{4.69}$$

$$\langle v_r v_\theta\rangle = \langle v_r v_\theta v^2\rangle = \langle v_r v_\theta v^4\rangle = \langle v_r v_\theta v_\varphi^2\rangle = 0. \tag{4.70}$$

If we multiply the post-Newtonian Boltzmann equation in spherical coordinates (4.89) by $m^4 v_\theta u^0/c$ or $m^4 v_\varphi u^0/c$ and integrate the resulting equations by taking into account the invariant integration element (4.20), we get – by considering that the odd moments vanish – that $\langle v_\theta^2\rangle = \langle v_\varphi^2\rangle$. Now from the above results we can express (4.63) as a function of $\langle v_r^2\rangle, \langle v_\theta^2\rangle, \langle v_\varphi^2\rangle$ and obtain the post-Newtonian Jeans equation for stationary spherically symmetrical systems:

$$\frac{d}{dr}\left[\rho\langle v_r^2\rangle\left(1+\frac{2\phi}{c^2}\right)\right] + 2\rho\frac{\langle v_r^2\rangle\beta}{r}\left(1+\frac{2\phi}{c^2}\right) + \frac{\rho}{c^2}\frac{d\psi}{dr}$$
$$+\rho\frac{d\phi}{dr}\left[1+\frac{2\phi}{c^2} - \frac{4}{c^2}\langle v_r^2\rangle + \frac{9}{2}\frac{kT}{mc^2}\right] = 0. \tag{4.71}$$

Here we have introduced the velocity anisotropy parameter $\beta = 1 - \langle v_\theta^2\rangle/\langle v_r^2\rangle$, by assuming that $\langle v_\theta^2\rangle = \langle v_\varphi^2\rangle$.

The radial velocity dispersion $\sqrt{\langle v_r^2\rangle}$ can be found as a solution of equation (4.71) together with the Poisson equations for the gravitational potentials ϕ and ψ once we know the velocity anisotropy parameter β and the dependence of the mass density ρ on the radial distance r.

4.3.2 Stationary and axisymmetrical self-gravitating systems

Another interesting problem in astrophysics is the analysis of stationary and axisymmetrical self-gravitating systems. The equations that rule the behavior of such systems are obtained from the post-Newtonian Boltzmann equation in cylindrical coordinates (4.93) as follows. First we multiply (4.93) by $m^4 v_r u^0/c$ and the integration element (4.20). The integration of the resulting equation, yields

$$\frac{\partial}{\partial r}\left\{\rho\left[\langle v_r^2\rangle+\frac{3}{c^2}\langle v_r^2v^2\rangle-\frac{m}{kTc^2}\left(\frac{3}{8}\langle v_r^2v^4\rangle-2\phi\langle v_r^2v^2\rangle\right)\right.\right.$$
$$\left.\left.-\frac{15}{8c^2}\frac{kT}{m}\langle v_r^2\rangle-\frac{8}{c^2}\phi\langle v_r^2\rangle\right]\right\}+\frac{\partial}{\partial z}\left\{\rho\left[\langle v_rv_z\rangle+\frac{3}{c^2}\langle v_rv_zv^2\rangle\right.\right.$$
$$-\frac{8}{c^2}\phi\langle v_rv_z\rangle-\frac{15}{8c^2}\frac{kT}{m}\langle v_rv_z\rangle-\frac{m}{kTc^2}\left(\frac{3}{8}\langle v_rv_zv^4\rangle\right.$$
$$\left.\left.\left.-2\phi\langle v_rv_zv^2\rangle\right)\right]\right\}+\frac{\rho}{r}\left\{\frac{3}{c^2}\left(\langle v_r^2v^2\rangle-\langle v_\varphi^2v^2\rangle\right)\right.$$
$$+\left(\langle v_r^2\rangle-\langle v_\varphi^2\rangle\right)\left(1-8\frac{\phi}{c^2}-\frac{15}{8c^2}\frac{kT}{m}\right)$$

$$
-\frac{m}{kTc^2}\left[\frac{3}{8}\big(\langle v_r^2 v^4\rangle - \langle v_\varphi^2 v^4\rangle\big) - 2\phi\big(\langle v_r^2 v^2\rangle - \langle v_\varphi^2 v^2\rangle\big)\right]\Bigg\}
-\frac{4}{c^2}\rho\frac{\partial\phi}{\partial z}\langle v_r v_z\rangle + \rho\frac{\partial\phi}{\partial r}\bigg[1 + \frac{4}{c^2}\langle v^2\rangle - \frac{4}{c^2}\langle v_r^2\rangle - \frac{4}{c^2}\phi
-\frac{15}{8c^2}\frac{kT}{m} - \frac{m}{kTc^2}\left(\frac{3}{8}\langle v^4\rangle - 2\phi\langle v^2\rangle\right)\bigg] + \frac{\rho}{c^2}\frac{\partial\psi}{\partial r} = 0. \tag{4.72}
$$

Next we follow the same methodology but multiply the post-Newtonian Boltzmann equation in cylindrical coordinates (4.93) by $m^4 v_z u^0/c$, resulting

$$
\frac{\partial}{\partial z}\Bigg\{\rho\bigg[\langle v_z^2\rangle + \frac{3}{c^2}\langle v_z^2 v^2\rangle - \frac{8}{c^2}\phi\langle v_z^2\rangle - \frac{m}{kTc^2}\bigg(\frac{3}{8}\langle v_z^2 v^4\rangle
-2\phi\langle v_z^2 v^2\rangle\bigg) - \frac{15}{8c^2}\frac{kT}{m}\langle v_z^2\rangle\bigg]\Bigg\} + \frac{\partial}{\partial r}\Bigg\{\rho\bigg[\langle v_r v_z\rangle
+\frac{3}{c^2}\langle v_r v_z v^2\rangle - \frac{8}{c^2}\phi\langle v_r v_z\rangle - \frac{15}{8c^2}\frac{kT}{m}\langle v_r v_z\rangle
-\frac{m}{kTc^2}\left(\frac{3}{8}\langle v_r v_z v^4\rangle - 2\phi\langle v_r v_z v^2\rangle\right)\bigg]\Bigg\}
+\frac{\rho}{r}\Bigg\{\frac{3}{c^2}\langle v_r v_z v^2\rangle + \langle v_r v_z\rangle\left(1 - 8\frac{\phi}{c^2} - \frac{15}{8c^2}\frac{kT}{m}\right)
-\frac{m}{kTc^2}\left[\frac{3}{8}\langle v_r v_z v^4\rangle - 2\phi\langle v_r v_z v^2\rangle\right]\Bigg\}
+\rho\frac{\partial\phi}{\partial z}\bigg[1 + \frac{4}{c^2}\langle v^2\rangle - \frac{4}{c^2}\langle v_z^2\rangle - \frac{m}{kTc^2}\left(\frac{3}{8}\langle v^4\rangle - 2\phi\langle v^2\rangle\right)
-\frac{4}{c^2}\phi - \frac{15}{8c^2}\frac{kT}{m}\bigg] - \frac{4}{c^2}\rho\frac{\partial\phi}{\partial r}\langle v_r v_z\rangle + \frac{\rho}{c^2}\frac{\partial\psi}{\partial z} = 0. \tag{4.73}
$$

Finally the multiplication of the post-Newtonian Boltzmann equation in cylindrical coordinates (4.93) by $m^4 v_\varphi u^0/c$ and following the same methodology leads to

$$\frac{\partial}{\partial r}\left\{\rho\left[\langle v_r v_\varphi\rangle+\frac{3}{c^2}\langle v_r v_\varphi v^2\rangle-\frac{m}{kTc^2}\left(\frac{3}{8}\langle v_r v_\varphi v^4\rangle\right.\right.\right.$$
$$\left.\left.\left.-2\phi\langle v_r v_\varphi v^2\rangle\right)-\frac{8}{c^2}\phi\langle v_r v_\varphi\rangle-\frac{15}{8c^2}\frac{kT}{m}\langle v_r v_\varphi\rangle\right]\right\}$$
$$+\frac{\partial}{\partial z}\left\{\rho\left[\langle v_\varphi v_z\rangle+\frac{3}{c^2}\langle v_\varphi v_z v^2\rangle-\frac{8}{c^2}\phi\langle v_\varphi v_z\rangle-\frac{15}{8c^2}\frac{kT}{m}\langle v_\varphi v_z\rangle\right.\right.$$
$$\left.\left.-\frac{m}{kTc^2}\left(\frac{3}{8}\langle v_\varphi v_z v^4\rangle-2\phi\langle v_\varphi v_z v^2\rangle\right)\right]\right\}+2\frac{\rho}{r}\left\{\left(1-8\frac{\phi}{c^2}\right.\right.$$
$$\left.-\frac{15}{8c^2}\frac{kT}{m}\right)\langle v_r v_\varphi\rangle-\frac{m}{kTc^2}\left[\frac{3}{8}\langle v_r v_\varphi v^4\rangle-2\phi\langle v_r v_\varphi v^2\rangle\right]$$
$$\left.+\frac{3}{c^2}\langle v_r v_\varphi v^2\rangle\right\}-\frac{4}{c^2}\rho\frac{\partial\phi}{\partial r}\langle v_r v_\varphi\rangle-\frac{4}{c^2}\rho\frac{\partial\phi}{\partial z}\langle v_z v_\varphi\rangle=0. \qquad (4.74)$$

The mean values can be expressed in terms of $\langle v_r^2\rangle$ and $\langle v_\varphi^2\rangle$ and $\langle v_z^2\rangle$ from the integrations by using the Maxwellian distribution function $(4.62)_2$. Here we have that

$$\langle v^2 v_z^2\rangle=5\frac{kT}{m}\langle v_z^2\rangle, \qquad \langle v^4 v_z^2\rangle=35\left(\frac{kT}{m}\right)^2\langle v_z^2\rangle, \qquad (4.75)$$

while the mean values with odd velocities vanish.

Since the odd moments vanish, (4.74) becomes trivial and (4.72) and (4.73) reduce to the post-Newtonian Jeans equations for stationary and axisymmetrical self-gravitating systems, namely

$$\frac{\partial}{\partial r}\left[\rho\langle v_r^2\rangle\left(1+\frac{2\phi}{c^2}\right)\right]+\frac{\rho\left(\langle v_r^2\rangle-\langle v_\varphi^2\rangle\right)}{r}\left(1+\frac{2\phi}{c^2}\right)$$
$$+\rho\frac{\partial\phi}{\partial r}\left[1+\frac{2\phi}{c^2}-\frac{4}{c^2}\langle v_r^2\rangle+\frac{9}{2}\frac{kT}{mc^2}\right]+\frac{\rho}{c^2}\frac{\partial\psi}{\partial r}=0, \quad (4.76)$$
$$\frac{\partial}{\partial z}\left[\rho\langle v_z^2\rangle\left(1+\frac{2\phi}{c^2}\right)\right]+\frac{\rho}{c^2}\frac{\partial\psi}{\partial z}+\rho\frac{\partial\phi}{\partial z}\left[1+\frac{2\phi}{c^2}\right.$$
$$\left.-\frac{4}{c^2}\langle v_z^2\rangle+\frac{9}{2}\frac{kT}{mc^2}\right]=0. \quad (4.77)$$

Appendix A

For the integration of the equations in this chapter we have used the following Gaussian integrals from the kinetic theory of gases (see e.g.[7])

$$I_n=\int\mathcal{V}^n e^{-\frac{m\mathcal{V}^2}{kT}}d\mathcal{V}=\frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\left(\frac{kT}{m}\right)^{\frac{n+1}{2}} \quad (4.78)$$

$$\Gamma(n+1)=n\Gamma(n),\qquad \Gamma(1)=1,\qquad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \quad (4.79)$$

$$\int e^{-\frac{m\mathcal{V}^2}{kT_0}}\mathcal{V}_i\mathcal{V}_j d^3\mathcal{V}=\frac{I_2}{3}\delta_{ij}, \quad (4.80)$$

$$\int e^{-\frac{m\mathcal{V}^2}{kT_0}}\mathcal{V}_i\mathcal{V}_j\mathcal{V}_k\mathcal{V}_l d^3\mathcal{V}=\frac{I_4\left[\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right]}{15}, \quad (4.81)$$

$$\int e^{-\frac{m\mathcal{V}^2}{kT_0}}\mathcal{V}_i\mathcal{V}_j\mathcal{V}_k\mathcal{V}_l\mathcal{V}_n\mathcal{V}_n d^3\mathcal{V}=\frac{I_6}{105}\left[\delta_{ij}\left(\delta_{kl}\delta_{mn}+\delta_{km}\delta_{ln}\right.\right.$$

$$+\delta_{kn}\delta_{lm}\big) + \delta_{ik}\big(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln} + \delta_{jn}\delta_{lm}\big)$$
$$+\delta_{il}\big(\delta_{jk}\delta_{mn} + \delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}\big) + \delta_{im}\big(\delta_{jk}\delta_{ln}$$
$$+\delta_{jl}\delta_{kn} + \delta_{jn}\delta_{kl}\big) + \delta_{in}\big(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}\big)\big]. \quad (4.82)$$

Appendix B: Boltzmann equation in spherical coordinates

In order to write the post-Newtonian Boltzmann equation (4.8) in spherical coordinates we make use of the relationships between Cartesian coordinates (x^1, x^2, x^3) and spherical coordinates (r, θ, φ)

$$x^1 = r\sin\theta\cos\varphi, \quad x^2 = r\sin\theta\sin\varphi, \quad x^3 = r\cos\theta, \quad (4.83)$$

and the relationships between the velocities (v^1, v^2, v^3) in Cartesian coordinates and the ones in spherical coordinates $(v_r = \dot{r}, v_\theta = r\dot{\theta}, v_\varphi = r\sin\theta\dot{\varphi})$

$$v^1 = v_r\sin\theta\cos\varphi + v_\theta\cos\theta\cos\varphi - v_\varphi\sin\varphi, \quad (4.84)$$
$$v^2 = v_r\sin\theta\sin\varphi + v_\theta\cos\theta\sin\varphi + v_\varphi\cos\varphi, \quad (4.85)$$
$$v^3 = v_r\cos\theta - v_\theta\sin\theta. \quad (4.86)$$

The relationship between the material time derivative of the distribution function $f = f(t, r, \theta, \varphi, v_r, v_\theta, v_\varphi)$ in spherical coordinates is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\dot{r} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial \varphi}\dot{\varphi}$$

$$+\frac{\partial f}{\partial v_r}\dot{v}_r + \frac{\partial f}{\partial v_\theta}\dot{v}_\theta + \frac{\partial f}{\partial v_\varphi}\dot{v}_\varphi, \tag{4.87}$$

which implies

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial f}{\partial \varphi}$$
$$+\left(\frac{v_\theta^2 + v_\varphi^2}{r}\right)\frac{\partial f}{\partial v_r} + \left(\frac{v_\varphi^2 \operatorname{cotan}\theta}{r} - \frac{v_r v_\theta}{r}\right)\frac{\partial f}{\partial v_\theta}$$
$$-\left(\frac{v_\theta v_\varphi \operatorname{cotan}\theta}{r} + \frac{v_r v_\varphi}{r}\right)\frac{\partial f}{\partial v_\varphi}. \tag{4.88}$$

Hence the Boltzmann equation (4.8) in spherical coordinates becomes

$$\left(1 + \frac{v^2}{2c^2} - \frac{\phi}{c^2}\right)\left[\frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial f}{\partial \varphi}\right.$$
$$+\left(\frac{v_\theta^2 + v_\varphi^2}{r}\right)\frac{\partial f}{\partial v_r} + \left(\frac{v_\varphi^2 \operatorname{cotan}\theta}{r} - \frac{v_r v_\theta}{r}\right)\frac{\partial f}{\partial v_\theta}$$
$$\left.-\left(\frac{v_\theta v_\varphi \operatorname{cotan}\theta}{r} + \frac{v_r v_\varphi}{r}\right)\frac{\partial f}{\partial v_\varphi}\right] - \left(1 + \frac{3v^2}{2c^2} + \frac{3\phi}{c^2}\right)$$
$$\times\left(\frac{\partial f}{\partial v_r}\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial v_\theta}\frac{\partial \phi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial v_\varphi}\frac{\partial \phi}{\partial \varphi}\right)$$
$$+\frac{1}{c^2}\left\{3\left(v_r\frac{\partial f}{\partial v_r} + v_\theta\frac{\partial f}{\partial v_\theta} + v_\varphi\frac{\partial f}{\partial v_\varphi}\right)\frac{\partial \phi}{\partial t} + 4\left(v_r\frac{\partial f}{\partial v_r}\right.\right.$$
$$\left.+v_\theta\frac{\partial f}{\partial v_\theta} + v_\varphi\frac{\partial f}{\partial v_\varphi}\right)\left(v_r\frac{\partial \phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial \phi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}\right)$$

$$-\left(\frac{\partial f}{\partial v_r}\frac{\partial \psi}{\partial r}+\frac{1}{r}\frac{\partial f}{\partial v_\theta}\frac{\partial \psi}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial v_\varphi}\frac{\partial \psi}{\partial \varphi}\right)$$
$$\left.-\frac{\partial f}{\partial v_r}\frac{\partial \xi_r}{\partial t}-\frac{\partial f}{\partial v_\theta}\frac{\partial \xi_\theta}{\partial t}-\frac{\partial f}{\partial v_\varphi}\frac{\partial \xi_\varphi}{\partial t}\right\}=0. \tag{4.89}$$

Appendix C: Boltzmann equation in cylindrical coordinates

The relationships between Cartesian coordinates (x^1, x^2, x^3) and cylindrical coordinates (r, φ, z) are

$$x^1 = r\cos\varphi, \qquad x^2 = r\sin\varphi, \qquad x^3 = z, \tag{4.90}$$

while the relations between the velocities (v^1, v^2, v^3) in Cartesian coordinates and the ones in cylindrical coordinates read

$$v^1 = v_r\cos\varphi - v_\varphi\sin\varphi,\ v^2 = v_r\sin\varphi + v_\varphi\cos\varphi,\ v^3 = \dot{z}, \tag{4.91}$$

where $v_r = \dot{r}$ and $v_\varphi = r\dot{\varphi}$.

The material time derivative of the distribution function in cylindrical coordinates $f = f(t, r, \varphi, z, v_r, v_\varphi, v_z)$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\varphi}{r}\frac{\partial f}{\partial \varphi} + v_z\frac{\partial f}{\partial z} + \frac{v_\varphi^2}{r}\frac{\partial f}{\partial v_r} - \frac{v_r v_\varphi}{r}\frac{\partial f}{\partial v_\varphi}, \tag{4.92}$$

so that the Boltzmann equation (4.8) in cylindrical coordinates reads

$$\left(1+\frac{v^2}{2c^2}-\frac{\phi}{c^2}\right)\left[\frac{\partial f}{\partial t}+v_r\frac{\partial f}{\partial r}+\frac{v_\varphi}{r}\frac{\partial f}{\partial \varphi}+v_z\frac{\partial f}{\partial z}+\frac{v_\varphi^2}{r}\frac{\partial f}{\partial v_r}\right.$$

$$
\begin{aligned}
&-\frac{v_r v_\varphi}{r}\frac{\partial f}{\partial v_\varphi}\Bigg] - \left(1+\frac{3v^2}{2c^2}+\frac{3\phi}{c^2}\right)\left(\frac{\partial f}{\partial v_r}\frac{\partial \phi}{\partial r}+\frac{1}{r}\frac{\partial f}{\partial v_\varphi}\frac{\partial \phi}{\partial \varphi}\right.\\
&\left.+\frac{\partial f}{\partial v_z}\frac{\partial \phi}{\partial z}\right)+\frac{1}{c^2}\Bigg\{3\left(v_r\frac{\partial f}{\partial v_r}+v_\theta\frac{\partial f}{\partial v_\theta}+v_\varphi\frac{\partial f}{\partial v_\varphi}\right)\frac{\partial \phi}{\partial t}\\
&-\left(\frac{\partial f}{\partial v_r}\frac{\partial \psi}{\partial r}+\frac{1}{r}\frac{\partial f}{\partial v_\varphi}\frac{\partial \psi}{\partial \varphi}+\frac{\partial f}{\partial v_z}\frac{\partial \psi}{\partial z}\right)+4\left(v_r\frac{\partial f}{\partial v_r}\right.\\
&\left.+v_\varphi\frac{\partial f}{\partial v_\varphi}+v_z\frac{\partial f}{\partial v_z}\right)\left(v_r\frac{\partial \phi}{\partial r}+\frac{v_\varphi}{r}\frac{\partial \phi}{\partial \varphi}+v_z\frac{\partial \phi}{\partial z}\right)\\
&-\frac{\partial f}{\partial v_r}\frac{\partial \xi_r}{\partial t}-\frac{\partial f}{\partial v_\varphi}\frac{\partial \xi_\varphi}{\partial t}-\frac{\partial f}{\partial v_z}\frac{\partial \xi_z}{\partial t}\Bigg\}=0. \qquad (4.93)
\end{aligned}
$$

References

[1] V. Rezania and Y. Sobouti, Liouville's equation in post Newtonian approximation I. Static solutions, *Astron. Astrophys.* **354**, 1110 (2000).

[2] C. A. Agón, J. F. Pedraza and J. Ramos-Caro, Kinetic theory of collisionless self-gravitating gases: Post-Newtonian polytropes, *Phys. Rev. D* **83**, 123007 (2011).

[3] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann equation: theory and applications* (Birkhäuser, Basel, 2002).

[4] G. M. Kremer, M. G. Richarte and K. Weber, Self-

gravitating systems of ideal gases in the 1PN approximation, Phys. Rev. D **93**, 064073 (2016).

[5] S. Weinberg, *Gravitation and cosmology. Principles and applications of the theory of relativity* (Wiley, New York, 1972).

[6] J. Binney and S. Tremaine *Galactic Dynamics*, 2nd. ed. (Princeton University Press, Princeton, 2008).

[7] G. M. Kremer, *An introduction to the Boltzmann equation and transport processes in gases* (Springer, Berlin, 2010).

[8] G.M. Kremer, Post-Newtonian kinetic theory, *Annals of Physics* **426**, 168400 (2021).

CHAPTER 5

STELLAR STRUCTURE MODELS

In astrophysics, the Lane-Emden equation is used to model self-gravitating spherically symmetrical stellar interiors characterized by a polytropic equation of state. The solution of the Lane-Emden equation allows to determine some physical quantities for these systems, such as pressure, density, and temperature. A full description of the Newtonian version of the Lane-Emden equation with applications to stellar structures can be found in the books by Eddington [1] and Chandrasekhar [2]. In this chapter the post-Newtonian Lane-Emden equation is derived and the physical properties of some stars are analysed.

5.1 The polytropic equation of state

The first law of thermodynamics connects the internal energy U of the system with the heat Q supplied to the system and the work W done by the system. For infinitesimal quasi-static changes this law is represented by $dU = d'Q - d'W$, where $d'Q$ and $d'W$ refer to the fact that both quantities are not differentials. Furthermore, for quasi-static changes the work done by the system is given by the product of the pressure p and the infinitesimal volume dV, i.e., $d'W = pdV$.

For a perfect gas the equation of state is given by $pV = \mathfrak{n}RT$, where $\mathfrak{n}$ is the number of moles of the gas and $R = 8.314$ J/(K mol) is the universal gas constant. The internal energy of a perfect gas is only a function of the absolute temperature $U = U(T)$, so that the first law of thermodynamics becomes

$$d'Q = \frac{dU}{dT}dT + \mathfrak{n}RT\frac{dV}{V}. \tag{5.1}$$

If we divide the quantities by the amount of the substance in moles and introduce the heat per mole $q = Q/\mathfrak{n}$, the internal energy per mole $u = U/\mathfrak{n}$ and the volume per mole $v = V/\mathfrak{n}$ the above equation can be rewritten as

$$d'q = \frac{du}{dT}dT + RT\frac{dv}{v}. \tag{5.2}$$

The molar heat capacities are defined in terms of the ratio $d'q/dT$ and are commonly referred as specific heat capacities.

For an isochoric process the volume remains constant and we have from (5.2) the specific heat capacity at constant volume

$$c_v = \left.\frac{d'q}{dT}\right|_v = \frac{du}{dT}. \tag{5.3}$$

If we use the equation of state $pV = \mathfrak{n}RT$ we can write

$$\frac{dv}{v} = \frac{dV}{V} = \frac{dT}{T} - \frac{dp}{p}, \tag{5.4}$$

and (5.2) reduces to

$$d'q = (c_v + R)dT - \frac{RT}{p}dp. \tag{5.5}$$

For an isobaric process the pressure remains constant and we have the specific heat capacity at constant pressure

$$c_p = c_v + R = \left.\frac{d'q}{dT}\right|_p. \tag{5.6}$$

In a polytropic process the specific heat capacity $c = d'q/dT$ remains constant in a quasi-static process so that we can write from (5.2) that

$$(c_v - c)\frac{dT}{T} + (c_p - c_v)\frac{dv}{v} = 0, \tag{5.7}$$

which can be integrated, yielding

$$T^{c_v - c} v^{c_p - c_v} = \text{constant}, \quad \text{or} \quad Tv^{\gamma - 1} = \text{constant}, \tag{5.8}$$

where $\gamma = (c_p - c)(c_v - c)$.

Now by using the equation of state to eliminate the absolute temperature from (5.8) we get the equation of state for polytropic quasi-static processes

$$pV^{\gamma} = \text{constant}, \qquad \text{or} \qquad p = \kappa\rho^{\gamma} = \kappa\rho^{\frac{n+1}{n}}, \tag{5.9}$$

where κ is a constant and we have introduced the so-called polytropic index $n = 1/(\gamma - 1)$.

The specific heat capacity c vanishes for adiabatic processes where $d'q = 0$ and $\gamma = c_p/c_v$ reduces to the ratio of the specific heat capacities at constant pressure and constant volume. For isothermal processes $dT = 0$ and $c \to \infty$.

The relationship between the specific internal energy ε and the polytropic equation of state $p = \kappa\rho^{\gamma}$ can be established from the integrability condition of the Gibbs equation

$$ds = \frac{1}{T}\left(d\varepsilon - \frac{p}{\rho^2}d\rho\right) = \frac{1}{T}\left[\frac{\partial\varepsilon}{\partial T}dT + \left(\frac{\partial\varepsilon}{\partial\rho} - \frac{p}{\rho^2}\right)d\rho\right], \tag{5.10}$$

where s denotes the specific entropy and $\varepsilon = \varepsilon(\rho, T)$. The integrability condition that follows from the above equation reads

$$\rho^2\frac{\partial\varepsilon}{\partial\rho} = p - T\frac{\partial p}{\partial T} = \kappa\rho^{\gamma}. \tag{5.11}$$

The integration of the above equation leads to

$$\varepsilon = \frac{1}{\gamma - 1}\frac{p}{\rho} = n\frac{p}{\rho} = \frac{\kappa\rho^{\gamma-1}}{\gamma - 1}, \tag{5.12}$$

showing that the specific internal energy of a polytropic fluid is only a function of the mass density.

5.2 Stellar mean molecular weight

We consider a star as a self-gravitating spherically symmetrical mass of a highly ionized gas at equilibrium. The mass of the gas is held together by its own gravity and has three kinds of species: hydrogen, helium and heavy elements, which for the purpose of the calculations are not specified.

The pressure of the star is a sum of the partial pressures due to each species

$$p = \sum_a p_a = \sum_a n_a kT, \tag{5.13}$$

where n_a is the particle number density of the species a.

For a highly ionized gas each atom contributes with $Z_a + 1$ particles where Z_a is the atomic number of species a. Hence the particle number density n_a of species a, which is the number of atoms per volume is given by

$$n_a = \frac{\rho x_a (Z_a + 1)}{M_a m_\mu}, \tag{5.14}$$

where ρ is the mass density of the star, $m_\mu = 1.66 \times 10^{-27}$ kg the unified atomic mass, x_a the mass fraction of species a and M_a the corresponding atomic mass.

From (5.13), (5.14) and the equation of state of the star we have

$$\sum_a \frac{\rho x_a (Z_a + 1)}{M_a m_\mu} kT = \frac{\rho}{m_\mu \mu} kT, \tag{5.15}$$

and the mean molecular weight μ of the stellar material becomes

$$\mu^{-1} = \sum_a \frac{x_a(Z_a + 1)}{M_a}. \tag{5.16}$$

Let X, Y and Z be the mass fraction of hydrogen, helium and heavy elements, respectively. For a mixture with these three species we must have that $X + Y + Z = 1$. The atomic number and the corresponding atomic mass for hydrogen are $Z_a = 1$ and $M_a = 2$, for helium $Z_a = 2$ and $M_a = 4$, while for the heavy materials with atomic mass greater then $M_a > 4$ we can approximate $(Z_a + 1)/M_a \approx 1/2$. Hence, the mean molecular weight (5.16) becomes

$$\mu = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z} = \frac{4}{2 + 6X + Y}, \tag{5.17}$$

by eliminating the mass fraction of the heavy elements $Z = 1 - X - Y$.

The stellar structures we are interested in are the neutron stars, white and brown dwarfs, red giants and the Sun.

White dwarfs are dense stars with low luminosity whose masses are of order of the Sun and radii comparable to the one of the Earth. In the structure of white dwarfs there exists almost heavy metals $Z \approx 1$, they are devoid of hydrogen and helium so that $X = Y \approx 0$ and the mean molecular weight is $\mu = 2$. Brown dwarfs are small stars with sizes approximately of a planet like Jupiter and unlike a regular star the fusion of hydrogen does not occur. Brown dwarfs are composed by hydrogen, helium and heavy metals in the approximate proportion

$X = 0.70$, $Y = 0.28$ and $Z = 0.02$ and the mean molecular weight is $\mu = 0.62$. Red giants represent the final evolution phase of stars of intermediate or low masses after the hydrogen fusion. For red giants the mean molecular weight is $\mu = 1.34$ since they are devoid of hydrogen and helium predominates in the proportion $Y = 0.98$, $X = 0$ and $Z = 0.02$. Neutron stars are formed from a gravitational collapse of massive stars at the end of their life. The neutron stars have only neutrons so that $\mu = 1$. The mass fractions for the Sun are $X = 0.73$, $Y = 0.25$ and $Z = 0.02$ and its mean molecular weight is $\mu = 0.6$.

5.3 Newtonian Lane-Emden equation

We start by considering the Newtonian momentum density hydrodynamic equation (2.129) for a stationary self-gravitating system where the hydrodynamic velocity vanishes $\mathbf{V} = \mathbf{0}$. In spherical coordinates where the only dependence of ρ, p, and ϕ is on the radial variable r this equation reduces to

$$\frac{dp}{dr} - \rho\frac{d\Phi}{dr} = 0. \tag{5.18}$$

Here we have adopted the convention for $\phi = -\Phi$, so that the Poisson equation $\nabla^2\phi = 4\pi G\rho = -\nabla^2\Phi$ in spherical coordinates reads

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = -4\pi G\rho. \tag{5.19}$$

By considering the polytropic equation of state $p = \kappa\rho^{\frac{1+n}{n}}$ one can obtain from (5.18) a differential equation which connects the mass density ρ and the gravitational potential Φ, namely

$$\kappa \frac{1+n}{n} \rho^{\frac{1-n}{n}} \frac{d\rho}{dr} = \frac{d\Phi}{dr}. \tag{5.20}$$

From the integration of the above equation it follows that

$$\kappa(1+n)\rho^{\frac{1}{n}} = \Phi + \text{constant}. \tag{5.21}$$

As was pointed by Eddington [1] the usual convention is to consider that the gravitational potential has a zero value at infinity, but this choice is arbitrary. Hence for convenience, here it is assumed that the gravitational potential vanishes at the boundary of the star where the mass density also vanishes. In this case the integration constant is zero and we get from (5.21) the following relationship between the mass density and gravitational potential

$$\rho = \left(\frac{\Phi}{(n+1)\kappa}\right)^n. \tag{5.22}$$

The polytropic equation of state $p = \kappa\rho^{\frac{1+n}{n}}$ in terms of the gravitational potential reads

$$p = \kappa\left(\frac{\Phi}{(n+1)\kappa}\right)^{n+1} = \frac{\rho\Phi}{n+1}. \tag{5.23}$$

Now we can obtain the differential equation for the gravitational potential Φ from (5.19) and (5.22)

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} + \frac{4\pi G}{[(n+1)\kappa]^n}\Phi^n = 0. \tag{5.24}$$

The solution of (5.24) for the gravitational potential $\Phi(r)$ follows from the knowledge of mass density $\rho(r)$ and the pressure $p(r)$ given by (5.22) and (5.23), respectively.

In order to solve (5.24) we introduce the dimensionless quantities

$$u = \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}, \quad z = \frac{r}{a}, \quad a = \sqrt{\frac{\kappa(n+1)}{4\pi G}\rho_c^{\frac{1-n}{n}}}, \quad (5.25)$$

where the quantities Φ_c and ρ_c refer to their values at the center of the star. Hence (5.24) can be rewritten as

$$\frac{d^2u(z)}{dz^2} + \frac{2}{z}\frac{du(z)}{dz} + u(z)^n = 0, \quad (5.26)$$

which is known in the literature as the Lane-Emden equation of index n.

The Lane-Emden equation (5.26) can be solved numerically for different values of the polytropic index n provided we specify two boundary conditions at the center of the star. The boundary conditions are

$$u(0) = 1, \qquad \text{and} \qquad \frac{du}{dz}(0) = 0. \quad (5.27)$$

The first boundary condition is a direct consequence of $(5.25)_1$. The second one follows from (5.18) by writing it as

$$\lim_{r\to 0}\frac{dp}{dr} = \lim_{r\to 0}\rho(r)\frac{d}{dr}\left(\frac{GM(r)}{r}\right) \approx \lim_{r\to 0}\rho(r)\frac{d}{dr}\left(\frac{4\pi}{3}Gr^2\rho_c\right) = 0, \quad (5.28)$$

where the Newtonian gravitational potential $\Phi = GM(r)/r$ was introduced and the mass near the center was approximated by $M(r) \approx 4\pi r^3 \rho_c/3$. Hence from the polytropic equation of state $p = \kappa\rho^{\frac{1+n}{n}}$ we have that $\frac{d\rho}{dr}(0) = 0$ and $(5.27)_2$ follows from the definition $(5.25)_1$.

5.4 Post-Newtonian Lane-Emden equation

For the Lane-Emden equation in the post-Newtonian approximation, we consider the momentum density hydrodynamic equation (2.131) for stationary self-gravitating systems where the hydrodynamic velocity vanishes, i.e, $\mathbf{V} = \mathbf{0}$. Here we write the potentials ϕ and ψ as $\Phi = -\phi$ and $\Psi = -\psi$. Since in spherical coordinates the fields ρ, p, Φ and Ψ depend only on the radial variable r, equation (2.131) becomes

$$\frac{d}{dr}\left[p\left(1+\frac{2\Phi}{c^2}\right)\right]-\rho\left[1+\frac{2}{c^2}\left(\Phi+\frac{\varepsilon}{2}+\frac{3p}{2\rho}\right)\right]\frac{d\Phi}{dr}-\frac{\rho}{c^2}\frac{d\Psi}{dr}=0. \tag{5.29}$$

The above equation can be rewritten as

$$\left(1+\frac{2\Phi}{c^2}\right)\frac{dp}{dr}-\rho\left[1+\frac{1}{c^2}\left(2\Phi+(n+1)\frac{p}{\rho}\right)\right]\frac{d\Phi}{dr}-\frac{\rho}{c^2}\frac{d\Psi}{dr}=0, \tag{5.30}$$

by taking into account the relationship $\varepsilon = np/\rho$, which according to (5.12) is valid for a polytropic fluid. By neglecting the terms of order $1/c^2$ the above equation reduces to the Newtonian equation (5.18).

If we consider terms up to $1/c^2$ we can obtain from (5.30) the equation

$$\frac{1}{\rho}\frac{dp}{dr}\left(1-\frac{n+1}{c^2}\frac{p}{\rho}\right)-\frac{d}{dr}\left(\Phi+\frac{\Psi}{c^2}\right)=0, \tag{5.31}$$

which can be solved for the mass density ρ as function of the potentials Φ, Ψ once we consider the polytropic equation of state. Indeed, from the insertion of the polytropic equation of state $p=\kappa\rho^{\frac{n+1}{n}}$ into (5.31) and integration of the resulting equation yields

$$\Phi+\frac{\Psi}{c^2}=(n+1)\kappa\rho^{\frac{1}{n}}\left(1-\frac{\kappa(1+n)}{2c^2}\rho^{\frac{1}{n}}\right). \tag{5.32}$$

Here it is assumed that the gravitational potentials Φ and Ψ vanish at the boundary of the star where the mass density also vanishes.

Equation (5.32) can be solved for ρ up to order $1/c^2$, yielding

$$\begin{aligned}\rho &= \left[\frac{\Phi+\frac{\Psi}{c^2}}{(n+1)\kappa\left(1-\frac{\kappa(1+n)}{2c^2}\rho^{\frac{1}{n}}\right)}\right]^n \\ &\approx \left(\frac{\Phi}{(n+1)\kappa}\right)^n\left\{1+\frac{n}{c^2}\left[\left(\frac{\Psi}{\Phi}\right)+\frac{\kappa(1+n)\rho^{\frac{1}{n}}}{2}\right]\right. \\ &\approx \left(\frac{\Phi}{(n+1)\kappa}\right)^n\left\{1+\frac{n}{c^2}\left[\left(\frac{\Psi}{\Phi}\right)+\frac{\Phi}{2}\right]\right\}. \end{aligned} \tag{5.33}$$

In spherical coordinates the Poisson equation for the gravitational potential Φ is given by (5.19) while the one for the gravitational Ψ which follow from (2.100) for a stationary process reads

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -8\pi G\rho\left(\Phi + \frac{3+n}{2}\frac{p}{\rho}\right). \tag{5.34}$$

Now we combine the Poisson equations (5.19) and (5.34) and write

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\Phi + \frac{\Psi}{c^2}\right)\right] = -4\pi G\rho\left[1 + \frac{2}{c^2}\left(\Phi + \frac{(3+n)p}{2\rho}\right)\right]. \tag{5.35}$$

If we eliminate the potentials Φ, Ψ from the above equation by using (5.32) and use the polytropic equation of state $p = \kappa\rho^{\frac{n+1}{n}}$ we get the following differential equation for the mass density

$$\kappa(n+1)\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\rho^{\frac{1}{n}} - \frac{(1+n)\kappa}{2c^2}\rho^{\frac{2}{n}}\right)\right]$$
$$= -4\pi G\rho\left[1 + \frac{(5+3n)\kappa}{c^2}\rho^{\frac{1}{n}}\right]. \tag{5.36}$$

In order to get the Lane-Emden equation we introduce the dimensionless variables (5.25) and in these new variables (5.36) becomes

$$\left(1 - \frac{(1+n)p_c}{c^2\rho_c}u(z)\right)\left[\frac{d^2u(z)}{dz^2} + \frac{2}{z}\frac{du(z)}{dz}\right]$$
$$-\frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2 = -u(z)^n\left(1 + \frac{(5+3n)p_c}{c^2\rho_c}u(z)\right). \tag{5.37}$$

If we consider terms up to the order $1/c^2$ we get the following equivalent version of the Lane-Emden equation in the first post-Newtonian approximation:

$$\left(1-\frac{(6+4n)p_c}{c^2\rho_c}u(z)\right)\left[\frac{d^2u(z)}{dz^2}+\frac{2}{z}\frac{du(z)}{dz}\right]$$
$$-\frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2+u(z)^n=0. \tag{5.38}$$

The Newtonian limit of the Lane-Emden equation (5.26) is recovered when the terms with $1/c^2$ are not taken into account.

If we assume that the equation of state for pressure at the center of the star is the one of a perfect fluid $p_c = \rho_c kT_c/m = \rho_c kT_c/\mu m_\mu$, where T_c is the temperature at the star center, we have that

$$\frac{p_c}{\rho_c c^2}=\frac{kT_c}{mc^2}=\frac{kT_c}{\mu m_\mu c^2}, \tag{5.39}$$

which represents the ratio of the thermal energy of the fluid at the star center kT_c and the rest energy of its particles mc^2.

5.5 Lane-Emden equation in the post-Newtonian Brans-Dicke theory

In Section 2.5 the Brans-Dicke theory was analyzed and the corresponding post-Newtonian hydrodynamic equations were determined. Two gravitational constants were introduced, $\mathbb{G}$ from the Poisson equation (2.189) and $\mathcal{G}$ which is of order of the gravitational constant G. Here the difference between these two gravitational constants will be determined by a parameter $\beta = \mathbb{G}/\mathcal{G}$ so that when $\beta = 1$ both gravitational constants coincide. From (2.188) it is possible to relate the dimensionless coupling constant ω with β, yielding

$$\omega = \frac{4 - 3\beta}{2(\beta - 1)} \tag{5.40}$$

The stationary momentum density hydrodynamic equation (2.219) in the post-Newtonian Brans-Dicke theory reads

$$\frac{\partial}{\partial x^i}\left[p\left(1 + \frac{2(3-2\beta)}{\beta}\frac{\Phi}{c^2}\right)\right] - \rho\frac{\partial \Phi}{\partial x^i}\left\{1 + \frac{1}{c^2}\left[\frac{2(3-2\beta)}{\beta}\Phi \right.\right.$$
$$\left.\left. + \left(n + \frac{3(2-\beta)}{\beta}\right)\frac{p}{\rho}\right]\right\} - \frac{1}{\beta}\frac{\rho}{c^2}\frac{\partial \Psi}{\partial x^i} = 0, \tag{5.41}$$

while the the Poisson equations (2.189) and (2.213) become

$$\nabla^2\Phi = -4\pi\beta\mathcal{G}\rho, \quad \nabla^2\Psi = -8\pi\beta\mathcal{G}\rho\left(\Phi + \frac{\beta}{2}\varepsilon + \frac{3(2-\beta)}{2}\frac{p}{\rho}\right), \tag{5.42}$$

respectively.

Following the same methodology of the last section we obtain the same equation (5.31) for the stationary momentum density hydrodynamic equation (5.41) and through integration the following relationship which connects the two gravitational potential with the mass density, namely

$$\Phi + \frac{\Psi}{\beta c^2} = (n+1)\kappa\rho^{\frac{1}{n}}\left(1 - \frac{\kappa(1+n)}{2c^2}\rho^{\frac{1}{n}}\right). \tag{5.43}$$

Moreover, the combination of the Poisson equations (5.42) leads to the following post-Newtonian Lane-Emden equation in the Brans-Dicke theory

$$\left(1-\frac{[2n(\beta+1)+(8-2\beta)]p_c}{\beta c^2\rho_c}u(z)\right)\left[\frac{d^2u(z)}{dz^2}+\frac{2}{z}\frac{du(z)}{dz}\right]$$
$$-\frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2+\beta u(z)^n=0. \qquad (5.44)$$

Note that when $\beta = 1$ the results of the last section are recovered.

5.6 The physical quantities of stars

We rely on the books by Eddington [1] and Chandrasekhar [2] and give here the expressions for the mass, radius, pressure, mass density and temperature of the stars which follow from the Lane-Emden equations.

5.6.1 Newtonian theory

The numerical solution of the Lane-Emden equation (5.26) with the boundary conditions (5.27) represents a monotonically decreasing behavior of $u(z)$ and its first zero, which will be denoted by $z|_{u=0} = R_N$, corresponds to the surface of the star. The radius of the star is given by

$$R = aR_N = \sqrt{\frac{(n+1)\kappa}{4\pi G}\rho_c^{\frac{1-n}{n}}}\,R_N, \qquad (5.45)$$

thanks to (5.25).

The mass of the star is calculated from

$$\begin{aligned} M(R) &= \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c \int_0^{R_N} z^2 u^n dz \\ &= -4\pi a^3 \rho_c \int_0^{R_N} d\left(z^2 \frac{du}{dz}\right) = -4\pi \rho_c a^3 R_N^2 \left.\frac{du}{dz}\right|_{R_N} \\ &= 4\pi \rho_c a^3 M_N, \end{aligned} \tag{5.46}$$

where we used the relationship (5.25), the Lane-Emden equation (5.26) and – by following Eddington [1] – introduced the quantity

$$M_N = -R_N^2 \left.\frac{du}{dz}\right|_{R_N}. \tag{5.47}$$

If we eliminate a and ρ_c from the above equation by using (5.25) and (5.45) we get that the mass of the star can be written as

$$M(R) = 4\pi \left[\frac{(n+1)\kappa}{4\pi G}\right]^{\frac{n}{n-1}} \left(\frac{R}{R_N}\right)^{\frac{n-3}{n-1}} M_N. \tag{5.48}$$

so that we can build the following mass-radius relationships by taking into account (5.22), (5.45) and (5.46)

$$\frac{GM(R)}{M_N}\frac{R_N}{R} = (n+1)\kappa\rho_c^{\frac{1}{n}} = \Phi_c, \tag{5.49}$$

$$\left(\frac{GM(R)}{M_N}\right)^{n-1}\left(\frac{R_N}{R}\right)^{n-3} = \frac{[(n+1)\kappa]^n}{4\pi G}. \tag{5.50}$$

From the knowledge of the mass $M(R)$ and radius R of a star the quantities R_N and M_N can be determined from the Lane-Emden equation (5.26) and the values of κ, Φ_c and ρ_c can be obtained from the above equations for fixed values of the polytropic index n.

Another way to determine the central mass density of the star is to express it as function of the mean mass density of the star

$$\overline{\rho} = \frac{M(R)}{4\pi R^3/3} = \left[-\frac{3}{R_N}\frac{du}{dz}\bigg|_{R_N}\right]\rho_c, \text{ and } \frac{\rho_c}{\overline{\rho}} = \left[-\frac{3}{R_N}\frac{du}{dz}\bigg|_{R_N}\right]^{-1}, \tag{5.51}$$

thanks to (5.46).

The central pressure of the star follows from the polytropic equation of state $p_c = \kappa\rho_c^{\frac{1+n}{n}}$ together with (5.49) and (5.51), yielding

$$p_c = \frac{GM(R)}{M_N}\frac{R_N}{R}\frac{\rho_c}{n+1} = \frac{GM(R)}{M_N}\frac{R_N}{R}\frac{\overline{\rho}}{n+1}\left[-\frac{3}{R_N}\frac{du}{dz}\bigg|_{R_N}\right]^{-1}. \tag{5.52}$$

From the equation of state of a perfect fluid one can obtain the temperature at the center of the star

$$T_c = \frac{\mu m_\mu}{k}\frac{p_c}{\rho_c} = \frac{\mu m_\mu}{k(n+1)}\frac{GM(R)}{M_N}\frac{R_N}{R}. \tag{5.53}$$

The mass density and pressure and temperature as functions of the dimensionless radial distance z follows from (5.25) and the polytropic equation of state, namely

$$\rho(z) = \rho_c\, u(z)^n, \quad p(z) = p_c\, u(z)^{n+1}, \quad T(z) = T_c\, u(z). \tag{5.54}$$

5.6.2 Post-Newtonian theory

The first zero of the numerical solution of the post-Newtonian Lane-Emden equation (5.38) for $u(z)$ will be denoted here by $z|_{u=0} = R_{PN} = R/a$.

The inner mass $M(R)$ of a sphere with radius R is given by

$$M(R) = \int_0^R 4\pi\sqrt{\gamma_*}\rho r^2 dr, \tag{5.55}$$

where γ_* is the determinant of the spatial metric tensor. Here up to $1/c^2$ order we have

$$\begin{aligned} \sqrt{\gamma_*} = \sqrt{\frac{-g}{g_{00}}} &= \left(1 + \frac{3\Phi}{c^2}\right) = \left(1 + \frac{3(n+1)\kappa\rho^{\frac{1}{n}}}{c^2}\right) \\ &= \left(1 + \frac{3(n+1)p_c}{c^2\rho_c} u(z)\right), \end{aligned} \tag{5.56}$$

thanks to (5.25) and (5.32). Hence the mass of the star which follows from the Lane-Emden equation (5.38) is

$$\begin{aligned} M(R) &= 4\pi a^3 \rho_c \int_0^{R_{PN}} \left(1 + \frac{3(n+1)p_c}{c^2\rho_c} u(z)\right) z^2 u^n dz \\ &= -4\pi a^3 \rho_c \int_0^{R_{PN}} \left\{ \left(1 - \frac{(3+n)p_c}{c^2\rho_c} u(z)\right) \left[\frac{d^2u(z)}{dz^2}\right.\right. \end{aligned}$$

$$+\frac{2}{z}\frac{du(z)}{dz}\bigg] - \frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2\bigg\}z^2dz = 4\pi\rho_c a^3 M_{PN}. \tag{5.57}$$

In the above equation we have introduced the abbreviation

$$M_{PN} = -\int_0^{R_{PN}} \left\{\left(1 - \frac{(3+n)p_c}{c^2\rho_c}u(z)\right)\left[\frac{d^2u(z)}{dz^2} + \frac{2}{z}\frac{du(z)}{dz}\right] - \frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2\right\}z^2dz. \tag{5.58}$$

The mass-radius relationships are given by

$$\frac{GM(R)}{M_{PN}}\frac{R_{PN}}{R} = (n+1)\kappa\rho_c^{\frac{1}{n}} = \Phi_c, \tag{5.59}$$

$$\left(\frac{GM(R)}{M_{PN}}\right)^{n-1}\left(\frac{R_{PN}}{R}\right)^{n-3} = \frac{[(n+1)\kappa]^n}{4\pi G}, \tag{5.60}$$

while the central mass density, pressure and temperature read

$$\frac{\rho_c}{\overline{\rho}} = \frac{R_{PN}^3}{3M_{PN}}, \tag{5.61}$$

$$p_c = \frac{GM(R)}{M_{PN}}\frac{R_{PN}}{R}\frac{\overline{\rho}}{n+1}\frac{R_{PN}^3}{3M_{PN}}, \tag{5.62}$$

$$T_c = \frac{\mu m_\mu}{k(n+1)}\frac{GM(R)}{M_{PN}}\frac{R_{PN}}{R}. \tag{5.63}$$

5.6.3 Brans-Dicke post-Newtonian theory

The first zero of the numerical solution of the post-Newtonian Brans-Dicke Lane-Emden equation (5.44) for $u(z)$ will be denoted here by $z|_{u=0} = R_{BD} = R/a$. The equations for the physical quantities can be obtained from the results of the post-Newtonian theory above by replacing M_{PN} by

$$M_{BD} = -\int_0^{R_b} \left\{ \left(1 - \frac{[n(2\beta-1)+5-2\beta]p_c}{c^2\rho_c} u(z)\right) \times \left[\frac{d^2u(z)}{dz^2} + \frac{2}{z}\frac{du(z)}{dz}\right] - \frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2 \right\} \frac{dz}{\beta}. \quad (5.64)$$

5.7 Polytropic solutions of the Lane-Emden equations

In this section we shall search for polytropic solutions of the Lane-Emden equations for the *Sun* and some other stars.

The *Sun* has a mass $M_\odot = 1.989 \times 10^{30}$kg, radius $R_\odot = 6.96 \times 10^8$m and the polytropic index usually adopted for it is $n = 3$ so that $p = \kappa\rho^{\frac{4}{3}}$. This equation of state corresponds to a completely degenerate ultra-relativistic Fermi gas (see e. g. [6]).

The polytropic index $n = 3$ is also considered for white dwarf stars with higher masses. Here we are interested in the white dwarf *Sirius B* which is the companion that orbits around the star *Sirius*. Its mass and radius are $M = 1.5M_\odot$ and $R = 8.4 \times 10^{-3}R_\odot$, respectively.

The equation of state of a non-relativistic completely degenerate Fermi gas is $p = \kappa\rho^{\frac{5}{3}}$ which corresponds to the polytropic index $n = 3/2$ (see e. g. [6]). Convective core stars of red giants and brown dwarfs are represented by this polytropic index. We shall analyze the red giant star *Aldebaran* with mass $M = 1.5M_\odot$ and radius $R = 44.2R_\odot$ and the brown dwarf star *Teide 1* with mass $M = 5.3\times10^{-2}M_\odot$ and radius $R = 10^{-1}R_\odot$.

Neutron stars can be represented by an equation of state with a polytropic index $n \simeq 1$. Here we will focus our attention to the neutron stars *PSR J0348+0432* with mass $M = 2.01M_\odot$ and radius $R = 1.87\times10^{-5}R_\odot$ and *PSR J1614–2230* with mass $M = 1.91M_\odot$ and radius $R = 1.87 \times 10^{-5}R_\odot$.

Let us analyze first the results that follow from the Newtonian Lane-Emden equation for the *Sun*, *Teide 1*, *Aldebaran* and *Sirius B*, which are represented in Table 5.1. The first zeros were found as a solution of the Newtonian Lane-Emden equation (5.26) and the central mass density, central pressure and central temperature were calculated from (5.51), (5.52) and (5.53), respectively. Note that first zeros for the *Sun* and for *Sirius B* are equal as well as the ones for *Teide 1* and for *Aldebaran*, which is a consequence that the *Sun* and *Sirius B* have the same polytropic index $n = 3$, while *Teide 1* and *Aldebaran* have the polytropic index $n = 3/2$. We infer from this table that the central pressure of the *Sun* and *Teide 1* are of the same order but the central mass density and temperature of *Teide 1* are one order of magnitude lower than the *Sun*, since the former has mass and radius smaller than the latter. The red giant *Aldebaran* has mass and radius larger than those of the *Sun* and *Teide 1* but its central density, pressure and temperature are smaller. The central quantities of the white dwarf *Sirius B* are several orders of magnitude greater than those of the *Sun*, since its has a smaller radius and a greater mass than the latter.

	first zero	ρ_c (kg/m^3)	p_c (Pa)	T_c (K)
Sun	6.90	7.64×10^4	1.25×10^{16}	1.18×10^7
Teide 1	3.65	4.46×10^5	2.43×10^{16}	4.05×10^6
Aldebaran	3.65	1.45×10^{-1}	5.04×10^8	5.60×10^5
Sirius B	6.90	1.56×10^{11}	3.34×10^{24}	5.14×10^9

Table 5.1: First zeros, central mass densities, pressures and temperatures from the Newtonian Lane-Emden equation.

The Lane-Emden in the Brans-Dicke post-Newtonian theory (5.44) differs from the post-Newtonian one (5.38) by the parameter β, which is a very small quantity. Indeed, according to the Cassini probe [7] the value of the dimensionless coupling constant ω in the Brans-Dicke theory should be $\omega > 40,000$ and the constrains due to Planck's data [8] imply that $\omega > 181.65$. These restriction on dimensionless coupling constant imply that the parameter is approximate $\beta \approx 1$ and the corresponding Lane-Emden equation in the Brans-Dicke post-Newtonian theory (5.44) reduces to the one of the post-Newtonian theory (5.38). Hence, the values that follow from these two theories are quite the same.

The difference between the Lane-Emden equations for the Newtonian (5.26) and post-Newtonian (5.38) theories lies on the terms that are multiplied by $p_c/\rho c^2 = kT_c/mc^2$ corresponding to the ratio of the thermal energy of the fluid at the star center kT_c and the rest energy of its particles mc^2. Here this parameter was determined from the central temperature T_c obtained from the Newtonian theory and the values found are: $kT_c/mc^2 =$

1.19×10^{-6} for the *Sun*, $kT_c/mc^2 = 2.37 \times 10^{-4}$ for *Sirius B*, $kT_c/mc^2 = 6.04 \times 10^{-7}$ for *Teide 1* and $kT_c/mc^2 = 3.86 \times 10^{-8}$for *Aldebaran*. Hence, the values given in Table 5.1 remain practically unchanged for these stars if we take into account the post-Newtonian Lane-Emden equation. Post-Newtonian corrections are important for more massive stars like the neutron stars *PSR J0348+0432* and *PSR J1614–2230* whose values are given in Table 5.2. In neutron stars the central temperature is at least three orders of magnitude greater than those of the other stars analysed here and the ratio of the thermal energy at the star center and the rest energy of the particle is $kT_c/mc^2 = 1.14 \times 10^{-1}$ for *PSR J0348+0432* and $kT_c/mc^2 = 1.09 \times 10^{-1}$ for *PSR J1614–2230*. From this table we can infer that the post-Newtonian corrections for the central pressure and temperature are about fifty percent larger than those from the Newtonian theory. The results for the two neutron stars are quite the same, since the only difference between both is in their masses which are of the same magnitude.

PSR J0348+0432	first zero	ρ_c (kg/m^3)	p_c (Pa)	T_c (K)
Newtonian	3.14	1.42×10^{18}	1.46×10^{34}	1.23×10^{12}
Post-Newtonian	2.53	1.38×10^{18}	2.12×10^{34}	1.84×10^{12}
PSR J1614–2230	first zero	ρ_c (kg/m^3)	p_c (Pa)	T_c (K)
Newtonian	3.14	1.37×10^{18}	1.34×10^{34}	1.18×10^{12}
Post-Newtonian	2.55	1.34×10^{18}	1.95×10^{34}	1.75×10^{12}

Table 5.2: Neutron Stars *PSR J0348+0432* and *PSR J1614–2230*. First zeros, central mass densities, pressures and temperatures from Newtonian and post-Newtonian Lane-Emden equations.

References

[1] A. S. Eddington, *The internal constitution of the stars* (Cambridge Univ. Press, Cambridge, 1926).

[2] S. Chandrasekhar, *An introduction to the study of stellar structure* (Dover, New York, 1957).

[3] S. Capozziello and V. Faraoni, *Beyond Einstein gravity* (Springer, Dordrecht, 2011).

[4] R. André and G. M. Kremer, Stellar structure model in hydrostatic equilibrium in the context of $f(R)$–gravity, *Res. Astron. Astrophys.* **17**, 122 (2017).

[5] G. M. Kremer, *An introduction to the Boltzmann equation and transport processes in gases* (Springer, Berlin, 2010).

[6] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann equation: theory and applications* (Birkhäuser, Basel, 2002).

[7] B. Bertotti, L. Iess and P. Tortora, A test of general relativity using radio links with the Cassini spacecraft, *Nature* **425**, 374 (2003).

[8] Y.-C. Li, F.-Q. Wu and X. Chen, Constraints on the Brans-Dicke gravity theory with the Planck data, *Phys. Rev. D* **88**, 084053 (2013).

CHAPTER 6

SPHERICALLY SYMMETRICAL ACCRETION

An important area of research in astrophysics is related to the spherically symmetrical steady state problem where a compact massive object (e.g. white dwarfs, neutron stars or black holes) captures gravitationally the particles of the surrounded matter of an interstellar plasma. This process is known as spherically symmetrical accretion. The pioneers works in this subject were published by Hoyle and Lyttleton [1, 2], Bondi and Hoyle [3], Bondi [4] and Michel [5]. In this chapter the spherically symmetrical accretion in the Newtonian and post-Newtonian approximation are analysed.

6.1 Newtonian spherically symmetrical accretion

6.1.1 Newtonian Bernoulli equation

In the analysis of the spherically symmetrical accretion a compact massive object of mass M at rest is surrounded by an infinite gas cloud of an interstellar plasma which is moving with a velocity V relative to it. At large distances from the compact massive object the gas cloud is at rest with uniform density and pressure denoted by ρ_∞ and p_∞, respectively. The flow of the gas cloud is steady-state and spherically symmetrical and the resulting mass increase of the compact massive object is not taken into account. The gas is characterized by a polytropic equation of state and by a sound speed a given by

$$p = \kappa\rho^\gamma = \kappa\rho^{\frac{n+1}{n}}, \qquad a = \sqrt{\frac{dp}{d\rho}} = \sqrt{\gamma\kappa\rho^{\gamma-1}} = \sqrt{\frac{\gamma p}{\rho}}, \quad (6.1)$$

where κ is a constant and n the polytropic index.

For steady states the Newtonian hydrodynamic equations for mass density (2.119) and momentum density (2.129) for an Eulerian fluid become

$$\frac{\partial \rho V_i}{\partial x_i} = 0, \qquad V_j \frac{\partial V_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0, \qquad (6.2)$$

respectively. Here V_i is the gas flow velocity and ϕ the Newtonian gravitational potential. By considering that the fields depend only on the radial direction r the above equations in spherical coordinates can be written as

$$\frac{1}{r^2}\frac{d(r^2\rho V)}{dr} = 0, \qquad V\frac{dV}{dr} + \frac{1}{\rho}\frac{dp}{dr} + \frac{d\phi}{dr} = 0, \tag{6.3}$$

with V denoting the component of the gas flow velocity in the radial direction.

The integration of the continuity equation $(6.3)_1$ implies the constant mass accretion rate

$$\dot{M} = 4\pi r^2 \rho V = \text{constant}. \tag{6.4}$$

If we take into account the polytropic equation of state (6.1) we can rewritten the momentum density hydrodynamic equation $(6.3)_2$ as

$$\frac{1}{2}\frac{dV^2}{dr} + \frac{\gamma\kappa}{\gamma-1}\frac{d\rho^{\gamma-1}}{dr} + \frac{d\phi}{dr} = 0. \tag{6.5}$$

The integration of the above equation results the Newtonian Bernoulli equation

$$\frac{V^2}{2} + \frac{\gamma\kappa\rho^{\gamma-1}}{\gamma-1} + \phi = \frac{V^2}{2} + \frac{a^2}{\gamma-1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma-1}. \tag{6.6}$$

In the above equation we have introduced the sound speed $a^2 = \gamma\kappa\rho^{\gamma-1}$ and the expression for the Newtonian gravitational potential $\phi = -GM/r$. Moreover, it was supposed that the radial gas flow velocity and the Newtonian gravitational potential vanish at large distances from the compact massive object where a_∞ denotes the sound speed there.

If we denote by a prime the derivative with respect to r, i.e., $' \equiv d/dr$ we can rewrite (6.3) as

$$\frac{\rho'}{\rho} + \frac{V'}{V} + \frac{2}{r} = 0, \qquad VV' + a^2 \frac{\rho'}{\rho} + \frac{GM}{r^2} = 0. \tag{6.7}$$

The system of equations (6.7) can be solved for V' and ρ', yielding

$$\frac{V'}{V} = \frac{2a^2/r - GM/r^2}{V^2 - a^2}, \qquad \frac{\rho'}{\rho} = -\frac{2V^2/r - GM/r^2}{V^2 - a^2}, \tag{6.8}$$

which implies the relationship

$$\left(2V^2 - \frac{GM}{r}\right)\frac{dV}{V} = -\left(2a^2 - \frac{GM}{r}\right)\frac{d\rho}{\rho}. \tag{6.9}$$

We infer from the above equation that a critical point is attained when both expressions within the parenthesis vanish, since both imply turning points for the functions ρ and V. The critical values of the gas flow velocity V_c and sound speed a_c are given by

$$V_c^2 = \frac{GM}{2r_c} = a_c^2, \tag{6.10}$$

where r_c denotes the critical radius. The existence of a critical point prevent singularities in the gas flow solution and guarantees a smooth monotonic increase of the gas flow velocity when

r decreases. Note from (6.10) that at the critical point the gas flow velocity is equal to the sound speed so that the critical radius represents the transonic radius.

If we insert (6.10) into the Bernoulli equation (6.6) and solve for the gas flow velocity (or the sound speed) at the critical point we get

$$a_c^2 = V_c^2 = \frac{2}{5-3\gamma} a_\infty^2, \qquad r_c = \frac{5-3\gamma}{4} \frac{GM}{a_\infty^2}, \tag{6.11}$$

which is valid for $\gamma \neq 5/3$.

From the expression for the sound speed $a^2 = \gamma \kappa \rho^{\gamma-1}$ we can obtain the following relationship for the mass density

$$\rho = \rho_\infty \left(\frac{a}{a_\infty} \right)^{\frac{2}{\gamma-1}}, \tag{6.12}$$

so that mass accretion rate (6.4) in terms of the variables at the critical point can be rewritten as

$$\dot{M} = 4\pi r_c^2 \rho_c V_c = 4\pi \lambda_c \left(GM\right)^2 \rho_\infty a_\infty^{-3}, \tag{6.13}$$

thanks to (6.11) and (6.12). Here λ_c is a dimensionless parameter that depends only on γ, namely

$$\lambda_c = 2^{\frac{9-7\gamma}{2(\gamma-1)}} \left(5 - 3\gamma\right)^{\frac{3\gamma-5}{2(\gamma-1)}}. \tag{6.14}$$

Some values of the dimensionless parameter λ_c as function of γ are given in Table 6.1, where $\gamma = 1$ refers to an isothermal equation of state, $\gamma = 7/5$ and $\gamma = 5/3$ to adiabatic equations of state for diatomic and monatomic gases, respectively, and $\gamma = 4/3$ to an equation of state for a completely degenerate ultra-relativistic Fermi gas. The value $\gamma = 5/3$ represents also the equation of state of a completely degenerate non-relativistic Fermi gas.

γ	1	4/3	7/5	5/3
λ_c	1.120	0.707	0.625	0.250

Table 6.1: Values of λ_c for some values of γ.

The temperature can be related with the sound speed and the density by using the equation of state for a perfect fluid $p = \rho T/\mu m_\mu$ together with the expression for the sound speed $a^2 = \gamma p/\rho$, yielding

$$\frac{T}{T_\infty} = \left(\frac{a}{a_\infty}\right)^2 = \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}. \tag{6.15}$$

Let us investigate the behavior of the gas flow velocity, mass density and absolute temperature for some special cases.

We begin with the case where $\gamma = 5/3$ so that the Bernoulli equation (6.6) can be rewritten as

$$\frac{V^2 + 3a^2}{2GM/r} = 1 + \frac{3a_\infty^2 r}{2GM}. \tag{6.16}$$

From (6.10) together with (6.16) and also from $(6.11)_2$ we infer that $r_c = 0$, the critical radius vanishes when $\gamma = 5/3$. We estimate the last term in (6.16) by considering that the mass of the compact object is of order of the Sun $M \simeq M_\odot$ and that the sound speed of the gas far from the compact massive object is of order $a_\infty \simeq 10^4$ m/s. In this case

$$\frac{3a_\infty^2 r}{2GM} \approx \frac{r}{6\text{au}}, \tag{6.17}$$

where au$=1.496\times10^{11}$ m is the astronomical unit. If we restrict ourselves to distances from the compact massive object where $a_\infty^2 r/GM \ll 1$ we get from (6.16) that

$$V^2 + 3a^2 \approx \frac{2GM}{r}. \tag{6.18}$$

At the critical point we can approximate the gas flow velocity with the speed of sound and it follows that

$$V \approx a \approx \sqrt{\frac{GM}{2r}}. \tag{6.19}$$

Hence, the mass density (6.12) and the absolute temperature (6.15) at the critical point become

$$\frac{\rho}{\rho_\infty} \approx \left(\frac{GM}{2a_\infty^2}\right)^{\frac{3}{2}} \frac{1}{r^{\frac{3}{2}}}, \qquad \frac{T}{T_\infty} \approx \left(\frac{GM}{2a_\infty^2}\right) \frac{1}{r}. \tag{6.20}$$

From (6.19) and (6.20) one infers that the gas flow velocity, mass density and the absolute temperature increase when the radial distance from the compact massive object decreases. Furthermore, the increase in the mass density is more accentuated than those of the temperature and gas flow velocity.

Two limiting cases can be analyzed for $\gamma \neq 5/3$. The first one is when $r \gg r_c$ where the gas flow velocity V and gravitational potential ϕ become very small and according to Bernoulli equation (6.6) we can approximate the sound speed with its value far from the compact massive object $a \approx a_\infty$. In this case we have from (6.12) and (6.15) that the mass density and the temperature are equal to their values far from the massive compact object, i.e., $\rho \approx \rho_\infty$ and $T \approx T_\infty$. The gas flow velocity can be obtained from the mass accretion rate (6.4) and (6.13), namely

$$\dot{M} = 4\pi r^2 \rho V = 4\pi \lambda_c \left(GM\right)^2 \rho_\infty a_\infty^{-3}, \implies V \approx \frac{\lambda_c \left(GM\right)^2 a_\infty^{-3}}{r^2}. \tag{6.21}$$

Hence the gas flow velocity increases when the distance from the compact massive object decreases.

In the other limiting case the condition $r \ll r_c$ holds and we can rewrite the Bernoulli equation (6.6) as

$$\frac{V^2}{2a_\infty^2} = \frac{4}{5-3\gamma}\frac{r_c}{r} - \frac{1}{\gamma-1}\left(\frac{a^2}{a_\infty^2} - 1\right) \approx \frac{4}{5-3\gamma}\frac{r_c}{r} = \frac{GM}{a_\infty^2 r}, \tag{6.22}$$

thanks to $(6.11)_2$ and the condition that $r \ll r_c$. Hence we have that

$$V \approx \sqrt{\frac{2GM}{r}}, \tag{6.23}$$

and the gas flow velocity is a function of the inverse of the square of the radial distance from the compact massive object and increases by decreasing this distance. From the mass accretion rate we have that the dependence of the mass density on r is

$$\dot{M} = 4\pi r^2 \rho V = 4\pi\lambda_c \left(GM\right)^2 \rho_\infty a_\infty^{-3}, \qquad \text{hence} \tag{6.24}$$

$$\frac{\rho}{\rho_\infty} \approx \frac{\lambda_c \left(GM\right)^{\frac{3}{2}} a_\infty^{-3}}{\sqrt{2}\, r^{\frac{3}{2}}}, \tag{6.25}$$

while the dependence of the absolute temperature on r is obtained from (6.15), yielding

$$\frac{T}{T_\infty} \approx \left[\frac{\lambda_c \left(GM\right)^{\frac{3}{2}} a_\infty^{-3}}{\sqrt{2}\, r^{\frac{3}{2}}}\right]^{\gamma-1}. \tag{6.26}$$

Both fields increase by decreasing the distance from the compact massive object.

6.1.2 Gas flow velocity as function of radial distance

For the determination of the dependence of the gas flow velocity as function of the radial distance we follow Bondi [4] and introduce the dimensionless quantities

$$r_* = \frac{r a_\infty^2}{GM}, \qquad s_* = \frac{V}{a_\infty}, \qquad t_* = \frac{\rho}{\rho_\infty} = \left(\frac{a}{a_\infty}\right)^{\frac{2}{\gamma-1}}, \tag{6.27}$$

which are related to the radial distance from the compact massive object, gas flow velocity and mass density, respectively.

The mass accretion rate (6.4) in terms of these dimensionless quantities reads

$$\dot{M} = 4\pi\rho r^2 V = 4\pi\lambda \left(GM\right)^2 \rho_\infty a_\infty^{-3}, \tag{6.28}$$

where $\lambda = r_*^2 s_* t_*$ is a constant.

We shall write the Bernoulli equation (6.6) in terms of the Mach number $u_* = V/a$ – which gives the ratio of the gas flow velocity V and the sound speed a – and the dimensionless radial distance r_*. The Mach number is usually denoted by Ma, but the expressions in the sequence become less cumbersome by denoting it as u_*. For this end we express the dimensionless parameters s_* and t_* as functions of (r_*, u_*) by using the relationships (6.27) and $\lambda = r_*^2 s_* t_*$, namely

$$s_* = u_*^{\frac{2}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{\gamma-1}{\gamma+1}}, \qquad t_* = \left(\frac{\lambda}{u_* \, r_*^2}\right)^{\frac{2}{\gamma+1}}. \tag{6.29}$$

The gas flow velocity V and sound speed a in terms of (r_*, u_*) follow from (6.27) and $a/a_\infty = t_*^{\frac{\gamma-1}{2}}$, yielding

$$V = a_\infty u_*^{\frac{2}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{\gamma-1}{\gamma+1}}, \qquad a = a_\infty \left(\frac{\lambda}{u_* \, r_*^2}\right)^{\frac{\gamma-1}{\gamma+1}}, \tag{6.30}$$

while the gravitational potential ϕ as function of r_* reads

$$\phi = -\frac{GM}{r} = -\frac{a_\infty^2}{r_*}. \tag{6.31}$$

The dependence of the Mach number u_* with the dimensionless radial distance r_* follows from the Newtonian Bernoulli equation (6.6) together with (6.29) – (6.31) resulting

$$\frac{u_*^{\frac{4}{\gamma+1}}}{2} \left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} + \frac{1}{\gamma-1} \left(\frac{\lambda}{u_* \, r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} = \frac{1}{r_*} + \frac{1}{\gamma-1}. \tag{6.32}$$

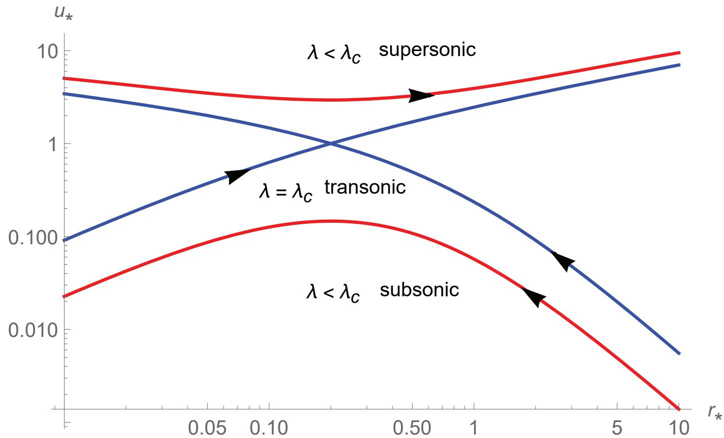

Figure 6.1: LogLogPlot of Mach number u_* as function of the dimensionless radial distance r_* for $\gamma = 7/5$ and two values of the dimensionless parameter: $\lambda = \lambda_c = 0.625$ and $\lambda = \lambda_c/4$.

In Figure 6.1 the Mach number u_* is plotted as function of the dimensionless radial distance r_* for $\gamma = 7/5$ where two different values of the dimensionless parameter λ were adopted: $\lambda = \lambda_c = 0.625$ and $\lambda = \lambda_c/4$. These values are the same as those adopted in the work of Bondi [4]. From this figure we infer that we have two physically accepted kinds of flows which correspond to an infall of matter to the compact massive object and an outflow of matter from the compact massive object. The matter infalls correspond to accretion flows and are represented

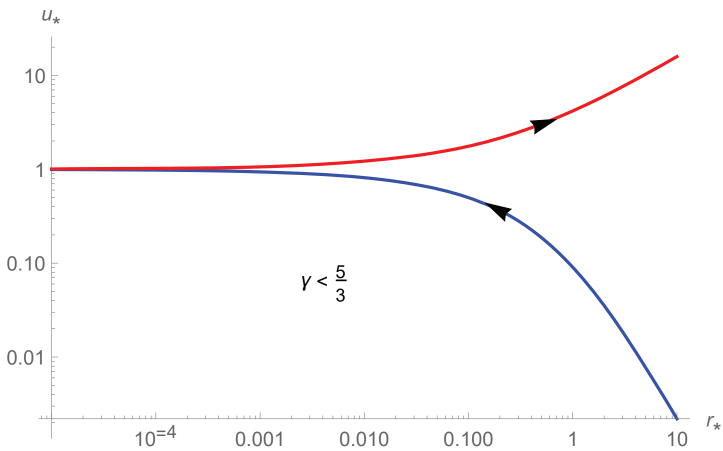

Figure 6.2: LogLogPlot of Mach number u_* as function of dimensionless radial distance r_* for $\lambda_c = 0.25$ corresponding to $\gamma = 5/3$.

in the figure by the left arrows. The matter outflows are wind flows and in the figure are represented by right arrows. For the critical value of the dimensionless parameter $\lambda_c = 0.625$ the solution goes through the critical radial distance $r_c = 0.20$ where the critical Mach number is one corresponding to the transonic point. These solutions are represented by the blue curves. For dimensionless parameters smaller than the critical dimensionless parameter $\lambda < \lambda_c$ there are two solutions, one refers to a subsonic accretion flow while the other to a supersonic

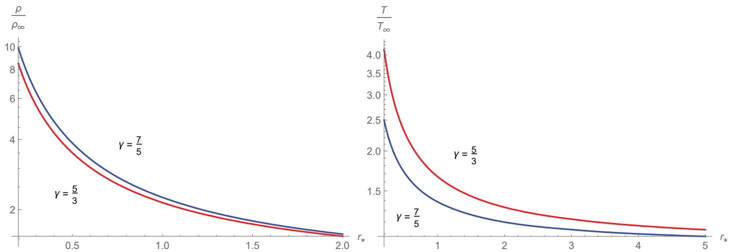

Figure 6.3: LogPlot of the ratios of the mass densities ρ/ρ_∞ (left frame) and absolute temperatures T/T_∞ (right frame) as functions of radial distance r_* for $\gamma = 7/5$ (blue curve) and $\gamma = 5/3$ (red curve).

wind flow. Both solutions for $\lambda < \lambda_c$ are represented in the figure by red curves.

As was pointed previously for the case where $\gamma = 5/3$ the critical dimensionless radial distance vanishes $r_c = 0$. The Mach number u_* as function of the dimensionless radial distance r_* for this case is plotted in Figure 6.2. The two physically accepted flows correspond to an accretion represented by the blue curve and a wind flow represented by the red one. Both solutions converge to the Mach number $u_* = 1$ at the transonic point.

The ratios of the mass densities

$$\frac{\rho}{\rho_\infty} = \left(\frac{\lambda}{u_* \, r_*^2} \right)^{\frac{2}{\gamma+1}}, \tag{6.33}$$

for the accretion solutions are plotted in the left frame of Figure 6.3 as functions of the dimensionless radial distance r_*. The blue curve represents the case $\gamma = 7/5$ while the red one refers to $\gamma = 5/3$. As expected both mass densities ratios increase by decreasing the radial distance from the compact massive object and the mass densities ρ tend to ρ_∞ far from the compact massive object. We infer from this figure that the increase in the mass density for $\gamma = 7/5$ is more accentuate than that for $\gamma = 5/3$.

In the right frame of Figure 6.3 the ratios of the temperatures

$$\frac{T}{T_\infty} = \left(\frac{\lambda}{u_* \, r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}, \tag{6.34}$$

are plotted as functions of the dimensionless radial distance where the blue curve represents $\gamma = 7/5$ while the red one $\gamma = 5/3$. From this curve we observe that the absolute temperatures T far from the compact massive object tend to T_∞ while the ratio T/T_∞ for $\gamma = 5/3$ is bigger than the one for $\gamma = 7/5$.

6.2 Post-Newtonian accretion

6.2.1 Post-Newtonian Bernoulli equation

Let us now analyse the same problem but within the framework of the post-Newtonian hydrodynamic equations. The steady state hydrodynamic equations for the mass density (2.122), the mass-energy density (2.127) and the momentum density (2.131) become

$$\frac{\partial \rho_* V_i}{\partial x^i} = 0, \qquad \frac{\partial \sigma V_i}{\partial x^i} = 0, \tag{6.35}$$

$$\sigma V_j \frac{\partial V_i}{\partial x^j} + \rho \frac{\partial \phi}{\partial x^i}\left[1 + \frac{2}{c^2}\left(V^2 - \phi + \frac{3\gamma - 2}{2(\gamma - 1)}\frac{p}{\rho}\right)\right]$$
$$+ \frac{\partial}{\partial x^i}\left[p\left(1 - \frac{2\phi}{c^2}\right)\right] - \frac{4\rho}{c^2}V_j\frac{\partial \phi V_i}{\partial x^j}$$
$$+ \frac{\rho}{c^2}\frac{\partial \psi}{\partial x^i} + \frac{\rho}{c^2}V_j\left(\frac{\partial \xi_i}{\partial x^j} - \frac{\partial \xi_j}{\partial x^i}\right) = 0. \tag{6.36}$$

In the steady state momentum density hydrodynamic equation (6.36) we have used the corresponding mass-energy density hydrodynamic equation $(6.35)_2$ and the relationship between the specific internal energy and pressure $\varepsilon = p/(\gamma - 1)\rho$ for a polytropic fluid.

The fields in spherical coordinates depend only on the radial coordinate r and for a spherically symmetrical flow the gas flow velocity has only the radial component $V_i = (V(r), 0, 0)$. Hence we get from (6.35) and (6.36)

$$\frac{d\left\{r^2\rho\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} - 3\phi\right)\right]V\right\}}{dr} = 0, \tag{6.37}$$

$$\frac{d\left\{r^2\rho\left[1 + \frac{1}{c^2}\left(V^2 - 2\phi + \frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)\right]V\right\}}{dr} = 0, \tag{6.38}$$

$$\rho\left[1 + \frac{1}{c^2}\left(V^2 - 6\phi + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right)\right]V\frac{dV}{dr}$$

$$+\rho\frac{d\phi}{dr}\left[1-\frac{2}{c^2}\left(V^2+\phi-\frac{3\gamma-2}{2(\gamma-1)}\frac{p}{\rho}\right)\right]$$
$$+\frac{d}{dr}\left[p\left(1-\frac{2\phi}{c^2}\right)\right]+\frac{\rho}{c^2}\frac{d\psi}{dr}=0, \tag{6.39}$$

by considering the definitions of ρ_* and σ from (2.123) and (2.126), namely

$$\rho_*=\rho\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}-3\phi\right)\right],\ \sigma=\rho\left[1+\frac{1}{c^2}\left(V^2+\varepsilon-2\phi+\frac{p}{\rho}\right)\right]. \tag{6.40}$$

At this point it is more appropriate to analyse the accretion flow by introducing the proper velocity of the flow v_r which is measured by a local stationary observer (see e.g [6, 7]). The definition of the proper velocity reads

$$v_r=\frac{U^r}{U_0/c}=\frac{U^r}{(U^0/c)\,(1+2\phi/c^2)}. \tag{6.41}$$

The relationship between the proper velocity and the radial component of the four-velocity follows from $V_i=U^i/(U^0/c)$ and (6.41), yielding

$$v_r=\frac{V}{(1+2\phi/c^2)},\qquad \text{or}\qquad V=v_r\left(1+2\frac{\phi}{c^2}\right). \tag{6.42}$$

The system of differential equations (6.37) – (6.39) can be rewritten in terms of the proper velocity v_r as

$$\frac{d\left\{r^2\rho\left[1+\frac{1}{c^2}\left(\frac{v_r^2}{2}-\phi\right)\right]v_r\right\}}{dr}=0, \tag{6.43}$$

$$\frac{d\left\{r^2\rho\left[1+\frac{1}{c^2}\left(v_r^2+\frac{a^2}{\gamma-1}\right)\right]v_r\right\}}{dr}=0, \tag{6.44}$$

$$\left[1+\frac{1}{c^2}\left(v_r^2-2\phi+\frac{a^2}{\gamma-1}\right)\right]v_r\frac{dv_r}{dr}+\frac{d\rho}{dr}\frac{a^2}{\rho}\left(1-\frac{2\phi}{c^2}\right)$$
$$+\frac{d\phi}{dr}\left[1-\frac{1}{c^2}\left(2\phi-\frac{a^2}{\gamma-1}\right)\right]+\frac{1}{c^2}\frac{d\psi}{dr}=0, \tag{6.45}$$

thanks to (6.42) and the expression for the sound speed of a gas with a polytropic equation of state $a^2=\gamma p/\rho$.

The mass-density and mass-energy accretion rates are obtained from the integration of (6.43) and (6.44) resulting

$$\dot{M}_{\rho_*}=4\pi\rho r^2\left[1+\frac{1}{c^2}\left(\frac{v_r^2}{2}-\phi\right)\right]v_r, \tag{6.46}$$

$$\dot{M}_{\sigma}=4\pi\rho r^2\left[1+\frac{1}{c^2}\left(v_r^2+\frac{a^2}{\gamma-1}\right)\right]v_r. \tag{6.47}$$

The relationship between both accretion rates follow from

$$\frac{\dot{M}_\sigma}{\dot{M}_{\rho_*}}=\frac{\left[1+\frac{1}{c^2}\left(v_r^2+\frac{a^2}{\gamma-1}\right)\right]v_r}{\left[1+\frac{1}{c^2}\left(\frac{v_r^2}{2}-\phi\right)\right]v_r}\approx\left[1+\frac{1}{c^2}\underline{\left(\frac{v_r^2}{2}+\phi+\frac{a^2}{\gamma-1}\right)}\right]. \tag{6.48}$$

Here the approximation $1/(1+x)\approx 1-x$ for the $1/c^2$ – term was used. The underlined term is of $1/c^2$ order and we can use the Newtonian Bernoulli equation

$$\frac{v_r^2}{2}+\frac{a^2}{\gamma-1}+\phi=\frac{a_\infty^2}{\gamma-1}, \tag{6.49}$$

to simplify (6.48) and get

$$\dot{M}_\sigma = \dot{M}_{\rho_*}\left[1 + \frac{1}{\gamma - 1}\frac{a_\infty^2}{c^2}\right]. \tag{6.50}$$

Note that the mass density and mass-energy density accretion rates are related to each other and differ by a term of order $1/c^2$ which refers to square of the ratio of the sound speed far from the compact massive object and the speed of light. Both accretion rates coincide in the Newtonian limiting case for small values of $a_\infty^2/c^2 \ll 1$, i.e., $\dot{M}_\sigma = \dot{M}_{\rho_*} = 4\pi\rho r^2 V$.

For the determination of the post-Newtonian Bernoulli equation we multiply the momentum density equation (6.45) by

$$\frac{1}{\rho}\left[1 + \frac{1}{c^2}\left(2\phi - \frac{a^2}{\gamma - 1}\right)\right],$$

which, by considering terms up to the $1/c^2$ order, leads to the following differential equation

$$v_r\frac{dv_r}{dr}\left[1 + \frac{v_r^2}{c^2}\right] + \frac{d\rho}{dr}\frac{a^2}{\rho}\left[1 - \frac{a^2}{c^2(\gamma - 1)}\right] + \frac{d\phi}{dr} + \frac{1}{c^2}\frac{d\psi}{dr} = 0. \tag{6.51}$$

If we take into account the equation of state $p = \kappa\rho^\gamma$ and the Newtonian Bernoulli equation (6.6) for the underlined $1/c^2$ – term, the above equation reduces to

$$v_r dv_r\left[1 + \frac{v_r^2}{c^2}\right] + \gamma\kappa\rho^{\gamma-2}d\rho\left[1 - \frac{\gamma}{c^2(\gamma - 1)}\kappa\rho^{\gamma-1}\right] + d\phi + \frac{1}{c^2}d\psi = 0. \tag{6.52}$$

The integration of (6.52) leads to the Bernoulli equation in the post-Newtonian approximation

$$\frac{v_r^2}{2}\left[1+\frac{v_r^2}{2c^2}\right]+\frac{a^2}{\gamma-1}\left[1-\frac{a^2}{2c^2(\gamma-1)}\right]+\phi+\frac{\psi}{c^2}$$
$$=\frac{a_\infty^2}{\gamma-1}\left[1-\frac{a_\infty^2}{2c^2(\gamma-1)}\right]. \tag{6.53}$$

Above it was assumed that the gravitational potentials ϕ, ψ and the proper velocity v_r vanish far from the compact massive object. We call attention to the fact that (6.53) reduces to the Newtonian Bernoulli equation (6.6) by neglecting the $1/c^2$ – terms.

As in the Newtonian analysis we denote the differentiation with respect to the radial coordinate r by a prime and expand the derivatives in the hydrodynamic equation for the mass density (6.43) and get

$$2r\rho v_r + r^2\rho' v_r + r^2\rho v_r' + \frac{\frac{r^2\rho v_r}{c^2}\left(v_r v_r' - \phi'\right)}{1+\frac{1}{c^2}\left(\frac{v_r^2}{2}-\phi\right)} = 0. \tag{6.54}$$

This equation reduces to

$$\frac{2}{r}+\frac{\rho'}{\rho}+\frac{v_r'}{v_r}+\frac{1}{c^2}\left(v_r v_r' - \phi'\right)=0, \tag{6.55}$$

by considering terms up to $1/c^2$. Likewise one can obtain from the hydrodynamic equation for the mass-energy density (6.44) the following expression

$$\frac{2}{r}+\frac{\rho'}{\rho}+\frac{v_r'}{v_r}+\frac{1}{c^2}\left(2v_r v_r' + a^2\frac{\rho'}{\rho}\right)=0. \tag{6.56}$$

In terms of the derivative with respect to the radial coordinate, equation (6.52) can be rewritten as

$$\left(1+\frac{v_r^2}{c^2}\right)v_r v_r' + \left[1-\frac{a^2}{c^2(\gamma-1)}\right]a^2\frac{\rho'}{\rho} + \phi' + \frac{\psi'}{c^2} = 0. \quad (6.57)$$

By neglecting the $1/c^2$ – terms the mass density (6.55) and the mass-energy density (6.56) equations coincide and these equations together with (6.57) reduce to the Newtonian ones (6.7).

The system of differential equations (6.55) – (6.57) can be solved algebraically for v_r', ρ' and ψ' resulting

$$\frac{v_r'}{v_r} = \frac{2}{r}\frac{a^2\left(1-\frac{r\phi'}{2c^2}\right)-\frac{r\phi'}{2}}{v_r^2\left(1-\frac{a^2}{c^2}\right)-a^2}, \qquad \frac{\rho'}{\rho} = -\frac{2}{r}\frac{v_r^2\left(1-\frac{r\phi'}{c^2}\right)-\frac{r\phi'}{2}}{v_r^2\left(1-\frac{a^2}{c^2}\right)-a^2}, \quad (6.58)$$

$$\psi' = \frac{N_1}{r(\gamma-1)\left[v_r^2\left(1-\frac{a^2}{c^2}\right)-a^2\right]}, \quad (6.59)$$

where N_1 denotes the following abbreviation

$$N_1 = a^4\left[r\phi' - 2v_r^2\left(1-\frac{r\phi'}{c^2}\right)\right] + r\phi' v_r^4(\gamma-1) - 2a^2(\gamma-1)v_r^4\left(1-\frac{r\phi'}{2c^2}\right). \quad (6.60)$$

From (6.58) we can build the relationship

$$\left[v_r^2\left(1-\frac{r\phi'}{c^2}\right)-\frac{r\phi'}{2}\right]\frac{dv_r}{v_r} = -\left[a^2\left(1-\frac{r\phi'}{2c^2}\right)-\frac{r\phi'}{2}\right]\frac{d\rho}{\rho}. \quad (6.61)$$

The same analysis as in the Newtonian case follows from the above equation, i.e., the turning points for the functions ρ and v_r are attained when both expressions within the parenthesis vanish, yielding

$$V_c^2 = \frac{r_c\phi_c'}{2\left(1 - \frac{r_c\phi_c'}{c^2}\right)} \approx \frac{r_c\phi_c'}{2}\left(1 + \frac{r_c\phi_c'}{c^2}\right), \qquad (6.62)$$

$$a_c^2 = \frac{r_c\phi_c'}{2\left(1 - \frac{r_c\phi_c'}{2c^2}\right)} \approx \frac{r_c\phi_c'}{2}\left(1 + \frac{r_c\phi_c'}{2c^2}\right). \qquad (6.63)$$

Hence, the solution must pass through a critical point, which is defined by a critical radius r_c, a critical proper velocity V_c and a critical sound velocity a_c. As was commented in the Newtonian case, the existence of a critical point prevent singularities in the flow solution and guarantees a smooth monotonic increase of the flow velocity when r decreases. Note that the approximation above is valid since we are working with a first post-Newtonian theory. The relationship between the critical gas flow velocity and speed of sound follows from (6.62) and (6.63) resulting

$$\frac{a_c^2}{V_c^2} = \frac{\left(1 - \frac{r_c\phi_c'}{c^2}\right)}{\left(1 - \frac{r_c\phi_c'}{2c^2}\right)} \approx \left(1 - \frac{r_c\phi_c'}{2c^2}\right) \approx \left(1 - \frac{a_c^2}{c^2}\right) \approx \left(1 - \frac{V_c^2}{c^2}\right). \qquad (6.64)$$

Hence, unlike the Newtonian case, the critical gas flow is not equal to the sound speed.

For the determination of ψ_c' at the critical point we substitute (6.64) into (6.59) and take into account the expression for the Newtonian gravitational potential $\phi = -GM/r = -r\phi'$, resulting

$$\frac{\psi_c'}{c^2} = \frac{\phi_c^2}{2c^2 r_c} = \frac{G^2M^2}{2c^2 r_c^3}. \tag{6.65}$$

We can also determine ψ' by considering that the scalar gravitational potential ψ is of order $1/c^2$ and we can approximate (6.59) by

$$\frac{\psi'}{c^2} = \frac{-a^4\left[\phi + 2v_r^2\right] - 2a^2(\gamma-1)v_r^4 - \phi v_r^4(\gamma-1)}{c^2 r(\gamma-1)\left[v_r^2 - a^2\right]}. \tag{6.66}$$

Here we have neglected the terms proportional to $1/c^2$ and taken into account the relationship $\phi' r = -\phi$. If we rely on the virial theorem $2K + W = 0$ – where K and W represent the kinetic and potential energies – we can assume that $2v_r^2 + \phi = 0$ and (6.66) reduces to

$$\frac{\psi'}{c^2} = \frac{\phi^2}{2c^2 r}. \tag{6.67}$$

Now from the integration of (6.67) and the use of $\phi = -GM/r = -r\phi'$ the scalar gravitational potential ψ can be determined, yielding

$$\frac{\psi}{c^2} = -\frac{G^2M^2}{4c^2r^2} = -\frac{\phi^2}{4c^2}, \qquad \text{so that} \qquad \frac{\psi_c}{c^2} = -\frac{\phi_c^2}{4c^2}. \tag{6.68}$$

Here it was considered that the scalar gravitational potential ψ vanishes at distances far from the massive object $r \to \infty$. Equation $(6.68)_2$ can be seen as the integral of (6.65) with respect to r_c.

The determination of the critical values follows from the Bernoulli equation (6.53) when the expressions for the sound speed (6.63), proper velocity (6.64) and gravitational potential $(6.68)_2$ are taken into account. One obtains an algebraic equation for the determination of ϕ_c at the critical point which can be solved and its value up to order $1/c^2$ is

$$\phi_c = -\frac{4a_\infty^2}{(5-3\gamma)}\left[1 + \frac{15-11\gamma}{8(\gamma-1)(5-3\gamma)}\frac{a_\infty^2}{c^2}\right]. \tag{6.69}$$

The critical radius is obtained from $\phi = -GM/r$ and reads

$$r_c = \frac{(5-3\gamma)}{4}\frac{GM}{a_\infty^2}\left[1 - \frac{15-11\gamma}{8(\gamma-1)(5-3\gamma)}\frac{a_\infty^2}{c^2}\right]. \tag{6.70}$$

From (6.63) and (6.64) together with (6.69) and $\phi_c' = -\phi_c/r_c$ follow the critical values of the sound speed a_c and proper velocity V_c, namely

$$a_c^2 = \frac{2a_\infty^2}{(5-3\gamma)}\left[1 - \frac{1-5\gamma}{8(\gamma-1)(5-3\gamma)}\frac{a_\infty^2}{c^2}\right], \tag{6.71}$$

$$V_c^2 = \frac{2a_\infty^2}{(5-3\gamma)}\left[1 - \frac{17-21\gamma}{8(\gamma-1)(5-3\gamma)}\frac{a_\infty^2}{c^2}\right]. \tag{6.72}$$

The critical value of the mass density is obtained from the expression for the sound speed $a^2 = \gamma p/\rho$ together with its critical value (6.71) and the equation of state $p = \kappa\rho^\gamma$, yielding

$$\frac{\rho_c}{\rho_\infty} = \left(\frac{a_c}{a_\infty}\right)^{\frac{2}{\gamma-1}} = \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}}\left[1 - \frac{1-5\gamma}{8(\gamma-1)^2(5-3\gamma)}\frac{a_\infty^2}{c^2}\right], \tag{6.73}$$

while the critical value of the absolute temperature follows from the equation of state $p = kT\rho/\mu m_\mu$ and reads

$$\frac{T_c}{T_\infty} = \frac{2}{5-3\gamma}\left[1 - \frac{1-5\gamma}{8(\gamma-1)(5-3\gamma)}\frac{a_\infty^2}{c^2}\right]. \tag{6.74}$$

At the critical point the mass-density accretion rate (6.46) becomes

$$\dot{M}_{\rho_*} = 4\pi\rho_c r_c^2\left[1 + \frac{1}{c^2}\left(\frac{V_c^2}{2} - \phi_c\right)\right]V_c = 4\pi\lambda_c\left(\frac{GM}{a_\infty^2}\right)^2\rho_\infty a_\infty, \tag{6.75}$$

where the critical value for the dimensionless parameter is given by

$$\lambda_c = (5-3\gamma)^{\frac{3\gamma-5}{2\gamma-2}}\,2^{\frac{9-7\gamma}{2\gamma-2}}\left[1 + \frac{121-216\gamma+103\gamma^2}{16(5-3\gamma)(\gamma-1)^2}\frac{a_\infty^2}{c^2}\right]. \tag{6.76}$$

When the ratio of the sound speed far from the compact massive object and the speed of light is very small $a_\infty/c \ll 1$ the Newtonian critical values are recovered, namely

$$\phi_c = -\frac{4a_\infty^2}{(5-3\gamma)}, \qquad r_c = \frac{(5-3\gamma)}{4}\frac{GM}{a_\infty^2}, \tag{6.77}$$

$$a_c^2 = V_c^2 = \frac{2a_\infty^2}{(5-3\gamma)}, \qquad \frac{\rho_c}{\rho_\infty} = \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}}, \tag{6.78}$$

$$\lambda_c = (5-3\gamma)^{\frac{3\gamma-5}{2\gamma-2}}\,2^{\frac{9-7\gamma}{2\gamma-2}}. \tag{6.79}$$

The critical values above are valid for $\gamma \neq 5/3$ and in the case when $\gamma = 5/3$ the post-Newtonian Bernoulli equation (6.53) reduces to

$$\left[\frac{v_r^2}{2}\left(1+\frac{v_r^2}{2c^2}\right)+\frac{3a^2}{2}\left(1-\frac{3a^2}{4c^2}\right)\right]\frac{r}{GM}-\left(1+\frac{GM}{4rc^2}\right) = \frac{3a_\infty^2 r}{2GM}\left(1-\frac{3a_\infty^2}{4c^2}\right). \tag{6.80}$$

If as in the Newtonian case we restrict ourselves to distances from the compact massive object where $a_\infty^2 r/GM \ll 1$ and use the relationship (6.64) – which connects the gas flow velocity with the sound speed at the critical point – we get from (6.80):

$$v_r^2\left[1-\frac{19v_r^2}{16c^2}\right]\approx\frac{GM}{2r}\left[1+\frac{GM}{4rc^2}\right],\ \text{or}\ v_r^2\approx\frac{GM}{2r}\left[1+\frac{27GM}{32rc^2}\right]. \tag{6.81}$$

The expression for the sound speed, mass density and absolute temperature for $\gamma = 5/3$ in the post-Newtonian approximation read

$$a^2\approx v_r^2\left[1-\frac{v_r^2}{c^2}\right]\approx\frac{GM}{2r}\left[1-\frac{5GM}{32rc^2}\right], \tag{6.82}$$

$$\frac{\rho}{\rho_\infty}=\left(\frac{a}{a_\infty}\right)^3\approx\left(\frac{GM}{2a_\infty^2 r}\right)^{\frac{3}{2}}\left[1-\frac{15GM}{64rc^2}\right], \tag{6.83}$$

$$\frac{T}{T_\infty}=\left(\frac{a}{a_\infty}\right)^2\approx\frac{GM}{2a_\infty^2 r}\left[1-\frac{5GM}{32rc^2}\right]. \tag{6.84}$$

The post-Newtonian contributions to the fields given in the equations (6.81) – (6.84) are small, since for a compact massive object with a mass $M \simeq M_\odot$, $GM/c^2 \approx 10^2$m.

As in Section 6.1.1 we shall analyse the post-Newtonian approximation limiting cases for $\gamma \neq 5/3$ when the radial distance from the compact massive object is smaller or bigger than the critical radius.

When $r \gg r_c$ the gas proper velocity v_r and Newtonian gravitational potential ϕ become very small so that the sound speed can be approximated by its value far from the compact massive object $a \approx a_\infty$ as well as the mass density and the absolute temperature, i.e., $\rho \approx \rho_\infty$ and $T \approx T_\infty$. As a consequence the gas flow velocity for $r \gg r_c$ is the same as that for the Newtonian case (6.21).

The post-Newtonian Bernoulli equation (6.53) for $r \ll r_c$ reduces to

$$\frac{v_r^2}{2}\left[1+\frac{v_r^2}{2c^2}\right] \approx \frac{GM}{r}\left[1+\frac{GM}{4rc^2}\right], \tag{6.85}$$

if we use the same arguments that were applied to derive (6.22). From the above equation we obtain that

$$v_r \approx \sqrt{\frac{2GM}{r}}\left[1-\frac{3GM}{8rc^2}\right]. \tag{6.86}$$

Furthermore, from the mass accretion rate (6.75) the dependence of the mass density on r is

$$\frac{\rho}{\rho_\infty} \approx \frac{\lambda_c\,(GM)^{\frac{3}{2}}\,a_\infty^{-3}}{\sqrt{2}\,r^{\frac{3}{2}}}\left[1-\frac{13GM}{8rc^2}\right]. \tag{6.87}$$

We infer from (6.86) that the gas proper velocity in the post-Newtonian approximation for $r \ll r_c$ is smaller than the Newtonian one as well the ratio of the mass density (6.87).

6.2.2 Mach number as function of the radial distance

In the post-Newtonian approximation we write Bondi's dimensionless quantities (6.27) as

$$r_* = \frac{ra_\infty^2}{GM}\left(1 + \frac{GM}{2rc^2}\right), \; s_* = \frac{v_r}{a_\infty}\left(1 + \frac{v_r^2}{2c^2}\right), \; t_* = \frac{\rho}{\rho_\infty}, \tag{6.88}$$

which are related to the radial distance, proper velocity and mass density, respectively. Another dimensionless quantity is the Mach number, which is the ratio of the proper velocity and the sound speed $u_* = v_r/a$.

If we solve (6.88) for r and v_r by considering terms up to $1/c^2$ we obtain

$$r = \frac{GM}{a_\infty^2} r_* \left(1 - \frac{\beta^2}{2r_*}\right), \qquad v_r = a_\infty s_* \left(1 - \frac{\beta^2}{2} s_*^2\right). \tag{6.89}$$

Here we have introduced the relativistic dimensionless parameter $\beta = a_\infty/c$ which refers to the ratio of the sound speed far from the compact massive object and the speed of light.

The mass-density accretion rate (6.46) can be rewritten in terms of the new variables (6.88) and by taking into account the Newtonian gravitational potential $\phi = -GM/r$ as

$$\dot{M}_{\rho_*} = 4\pi\lambda \left(\frac{GM}{a_\infty^2}\right)^2 \rho_\infty a_\infty, \tag{6.90}$$

where $\lambda = r_*^2 s_* t_*$.

The dependence of the proper velocity as a function of the radial velocity follows from the post-Newtonian Bernoulli equation (6.53) by expressing it as function of the dimensionless quantities (r_*, u_*).

Let us write first the dimensionless parameters s_* and t_* as functions of (r_*, u_*), namely

$$s_* = u_*^{\frac{2}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{\gamma-1}{\gamma+1}} \left[1 + \frac{\beta^2}{\gamma+1} u_*^{\frac{4}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right], \quad (6.91)$$

$$t_* = \left(\frac{\lambda}{u_* r_*^2}\right)^{\frac{2}{\gamma+1}} \left[1 - \frac{\beta^2}{\gamma+1} u_*^{\frac{4}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]. \quad (6.92)$$

Above we have taken into account $(6.88)_2$ and the following relationships $u_* = v_r/a$, $\lambda = r_*^2 s_* t_*$ and $t_* = \rho/\rho_\infty = (a/a_\infty)^{\frac{2}{\gamma-1}}$.

Now we can rewrite v_r and a in terms of (r_*, u_*) from $(6.89)_2$ and $a/a_\infty = t_*^{\frac{\gamma-1}{2}}$, yielding

$$v_r = a_\infty u_*^{\frac{2}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{\gamma-1}{\gamma+1}} \left[1 - \frac{\beta^2(\gamma-1)}{2(\gamma+1)} u_*^{\frac{4}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right], (6.93)$$

$$a = a_\infty \left(\frac{\lambda}{u_* r_*^2}\right)^{\frac{\gamma-1}{\gamma+1}} \left[1 - \frac{\beta^2(\gamma-1)}{2(\gamma+1)} u_*^{\frac{4}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]. (6.94)$$

The gravitational potentials ϕ and ψ as functions of r_* are given by

$$\phi = -\frac{GM}{r} = -\frac{a_\infty^2}{r_*}\left(1+\frac{\beta^2}{2r_*}\right), \qquad \frac{\psi}{c^2} = -\frac{\phi^2}{4c^2} = -\frac{a_\infty^2}{4r_*^2}\beta^2. \tag{6.95}$$

We call attention to the fact that in (6.91) – (6.95) we have considered only terms up to the order $1/c^2$.

Now from the post-Newtonian Bernoulli equation (6.53) together with (6.93) – (6.95) we get the final equation which gives the dependence of the Mach number u_* with the dimensionless radial distance r_*

$$\frac{u_*^{\frac{4}{\gamma+1}}}{2}\left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\left[1+\frac{\beta^2}{2(\gamma+1)}u_*^{\frac{4}{\gamma+1}}\left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]$$
$$-\frac{1}{r_*}\left(1+\frac{3\beta^2}{4r_*}\right)+\frac{1}{\gamma-1}\left(\frac{\lambda}{u_* r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}$$
$$\times\left[1-\left(1+\frac{(\gamma+1)}{2(\gamma-1)^2u_*^2}\right)\frac{\beta^2(\gamma-1)}{\gamma+1}u_*^{\frac{4}{\gamma+1}}\left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]$$
$$=\frac{1}{\gamma-1}\left(1-\frac{\beta^2}{2(\gamma-1)}\right). \tag{6.96}$$

The above equation reduces to the Newtonian one (6.32) if we neglect the β^2 terms.

6.3 Relativistic Bondi accretion

In this section we shall analyze the relativistic Bondi accretion and its weak field limiting case. The determination of the relativistic Bernoulli equation and the analysis of the critical point follow the work by Michel [5].

6.3.1 Relativistic Bernoulli equation

The Schwarzschild metric is the solution of Einstein's field equations that describes the gravitational field outside a spherical mass. The line element in spherical coordinates (r, θ, φ) reads

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)\left(dx^0\right)^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)}\left(dr\right)^2 - r^2\left[\left(d\theta\right)^2 + \sin^2\theta\left(d\varphi\right)^2\right]. \qquad (6.97)$$

The Schwarzschild radius corresponds to the radius which defines the event horizon of a Schwarzschild black hole and is given by $r_S = 2GM/rc^2$.

The gas cloud of the interstellar plasma is characterized by the particle four-flow N^μ and energy-momentum tensor $T^{\mu\nu}$ of a perfect fluid

$$N^\mu = nU^\mu, \qquad T^{\mu\nu} = (p + \epsilon)\frac{U^\mu U^\nu}{c^2} - pg^{\mu\nu}. \qquad (6.98)$$

The balance equations for the particle four-flow and energy-momentum tensor can be written as

$$N^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}N^\mu}{\partial x^\mu} = 0, \qquad (6.99)$$

$$T_\mu{}^\nu{}_{;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}\, T_\mu{}^\nu}{\partial x^\nu} - \frac{1}{2} T^{\nu\sigma} \frac{\partial g_{\nu\sigma}}{\partial x^\mu} = 0. \qquad (6.100)$$

In the analysis of the spherically symmetrical accretion the non-vanishing components of the four-velocity are

$$(U^\mu) = \left(U^0 = \frac{dx^0}{d\tau}, U^r = \frac{dr}{d\tau}, 0, 0 \right), \qquad (6.101)$$

and due to the constraint $g_{\mu\nu} U^\mu U^\nu = c^2$ the component U^0 is connected with U^r by

$$\frac{U^0}{c} = \frac{\sqrt{1 - \frac{2GM}{rc^2} + \left(\frac{U^r}{c}\right)^2}}{1 - \frac{2GM}{rc^2}}. \qquad (6.102)$$

The balance equation for the particle four-flow (6.99) can be integrated furnishing the relationship

$$\sqrt{-g} n U^r = \text{constant}, \qquad (6.103)$$

which together with $\sqrt{-g} = r^2 \sin^2\theta$ implies the spherically symmetrical mass accretion rate

$$\dot{M} = 4\pi \rho r^2 U^r = \text{constant}. \qquad (6.104)$$

Here we have introduced the mass density $\rho = mn$ where m denotes the rest mass of the gas particles.

The time component of the balance of the energy-momentum tensor (6.100) reduces to

$$\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}\, T_0{}^\nu}{\partial x^\nu} = 0, \qquad (6.105)$$

since the components of the metric tensor do not depend on the time. The integration of the above equation yields

$$\sqrt{-g}{T_0}^1 = \sqrt{-g}\,(p+\epsilon)\,\frac{U^r}{c}\frac{U_0}{c} = \text{constant}. \tag{6.106}$$

The elimination of U^r from (6.106) by using (6.103) leads to

$$\frac{p+\epsilon}{\rho}\frac{U_0}{c} = \text{constant}, \tag{6.107}$$

which can be rewritten as

$$\left(\frac{p+\epsilon}{\rho}\right)^2\left[1-\frac{2GM}{rc^2}+\left(\frac{U^r}{c}\right)^2\right] = \text{constant}, \tag{6.108}$$

thanks to the relationship (6.102).

At this point it is interesting to introduce the sound speed a whose expression in relativity is given by (see e.g. [8])

$$\frac{a^2}{c^2} = \frac{\rho}{p+\epsilon}\left(\frac{\partial p}{\partial \rho}\right)_s, \tag{6.109}$$

where the differentiation is taken at constant entropy s.

From now on we consider a gas with a polytropic equation of state and energy density equation given by

$$p = \kappa\rho^\gamma, \qquad \epsilon = \rho c^2 + \frac{\kappa\rho^\gamma}{\gamma-1}. \tag{6.110}$$

From (6.109) and (6.110) we can obtain the expressions

$$\frac{p+\epsilon}{\rho} = \frac{\kappa\gamma\rho^{\gamma-1}}{\gamma-1} + c^2 = \frac{\kappa\gamma\rho^{\gamma-1}}{a^2/c^2}, \tag{6.111}$$

which implies that we can write

$$\kappa\gamma\rho^{\gamma-1} = \frac{(\gamma-1)a^2}{\gamma-1-a^2/c^2}, \quad \frac{p+\epsilon}{\rho} = \frac{c^2}{1-a^2/(\gamma-1)c^2}. \tag{6.112}$$

Furthermore, based on the above expressions the sound speed can be expressed as

$$\frac{a^2}{c^2} = \frac{\rho^2}{p+\epsilon}\frac{d}{d\rho}\left(\frac{p+\epsilon}{\rho}\right). \tag{6.113}$$

The relativistic Bernoulli equation is obtained from the substitution of (6.112) into (6.108), yielding

$$\left(1-\frac{a^2}{c^2(\gamma-1)}\right)^2 = \left(1-\frac{a_\infty^2}{c^2(\gamma-1)}\right)^2\left[1-\frac{2GM}{rc^2}+\left(\frac{U^r}{c}\right)^2\right], \tag{6.114}$$

where it was supposed that far from the Schwarzschild black hole $2GM/rc^2$ and U^r vanish while the sound speed becomes a_∞.

To determine the critical point of the gas flow we differentiate (6.104) and (6.108) and get respectively

$$\frac{d\rho}{\rho} + \frac{dU^r}{U^r} + 2\frac{dr}{r} = 0, \tag{6.115}$$

$$2\left(\frac{p+\epsilon}{\rho}\right)\frac{d}{d\rho}\left(\frac{p+\epsilon}{\rho}\right)d\rho\left[1-\frac{2GM}{rc^2}+\left(\frac{U^r}{c}\right)^2\right] + \left(\frac{p+\epsilon}{\rho}\right)^2\left[\frac{2GM}{r^2c^2}dr + \frac{2}{c^2}U^r dU^r\right] = 0. \tag{6.116}$$

Another expression for (6.116) is obtained by eliminating $d\rho$ from (6.115) and considering the relationships (6.112) and (6.113). Hence it follows that

$$\frac{dU^r}{U^r}\left[\frac{a^2}{c^2}-\frac{(U^r/c)^2}{1-2GM/rc^2+(U^r/c)^2}\right] + \frac{dr}{r}\left[2\frac{a^2}{c^2}-\frac{GM/rc^2}{1-2GM/rc^2+(U^r/c)^2}\right]=0. \quad (6.117)$$

The critical point is determined when both expressions in the parenthesis in (6.117) vanish resulting the following expressions for the critical gas flow velocity and sound speed

$$(U_c^r)^2=\frac{GM}{2r_c}, \quad a_c^2=\frac{(U_c^r)^2}{1-3(U_c^r/c)^2}, \quad (U_c^r)^2=\frac{a_c^2}{1+3(a_c/c)^2}. \quad (6.118)$$

Note that as in the post-Newtonian case the critical sound speed does not coincide with the critical gas flow velocity.

It is interesting to compare the post-Newtonian approximation with the weak field limit of the relativistic case. For that end we begin by writing the Bernoulli equation (6.114) at the critical point and taking into account (6.118) as

$$\left(1+3\frac{a_c^2}{c^2}\right)\left(1-\frac{a_c^2}{c^2(\gamma-1)}\right)^2=\left(1-\frac{a_\infty^2}{c^2(\gamma-1)}\right)^2. \quad (6.119)$$

This is a third order algebraic equation for the determination of the critical sound speed a_c^2 which was solved in [9]. Here we are interested in its weak field approximation and solve (6.119) by considering terms up to $1/c^2$ – order

$$a_c^2 = \frac{2a_\infty^2}{5-3\gamma}\left[1 - \frac{3(3\gamma+1)}{2(5-3\gamma)(\gamma-1)}\frac{a_\infty^2}{c^2}\right]. \tag{6.120}$$

The approximate expressions for the critical gas flow velocity U_c and critical radial distance follows from (6.118) and (6.120), yielding

$$U_c{}^2 = \frac{2a_\infty^2}{5-3\gamma}\left[1 + \frac{3(7-11\gamma)}{4(5-3\gamma)(\gamma-1)}\frac{a_\infty^2}{c^2}\right], \tag{6.121}$$

$$r_c = \frac{(5-3\gamma)}{4}\frac{GM}{a_\infty^2}\left[1 - \frac{3(7-11\gamma)}{4(\gamma-1)(5-3\gamma)}\frac{a_\infty^2}{c^2}\right]. \tag{6.122}$$

The critical mass density is obtained from the knowledge of the critical sound speed and reads

$$\frac{\rho_c}{\rho_\infty} = \left(\frac{a_c}{a_\infty}\right)^{\frac{2}{\gamma-1}} = \left(\frac{2}{5-3\gamma}\right)^{\frac{1}{\gamma-1}}\left[1 - \frac{3(3\gamma+1)}{2(5-3\gamma)(\gamma-1)^2}\frac{a_\infty^2}{c^2}\right]. \tag{6.123}$$

The critical absolute temperature follows from the equation of state of a perfect gas $p = kT\rho/\mu m_\mu$ namely

$$\frac{T_c}{T_\infty} = \left(\frac{a_c}{a_\infty}\right)^2 = \frac{2}{5-3\gamma}\left[1 - \frac{3(3\gamma+1)}{2(5-3\gamma)(\gamma-1)}\frac{a_\infty^2}{c^2}\right]. \tag{6.124}$$

We can rewrite the mass accretion rate (6.104) as

$$\dot{M} = 4\pi\rho_c r_c^2 V_c = 4\pi\lambda_c (GM)^2 \rho_\infty a_\infty^{-3} \tag{6.125}$$

thanks to (6.121) – (6.123). Here λ_c is given by

$$\lambda_c = (5-3\gamma)^{\frac{3\gamma-5}{2\gamma-2}}\, 2^{\frac{9-7\gamma}{2\gamma-2}} \left[1 + \frac{3(17-66\gamma+33\gamma^2)}{8(\gamma-1)^2(5-3\gamma)}\frac{a_\infty^2}{c^2}\right]. \quad (6.126)$$

Note that all expressions above for the critical quantities differ from those of the post-Newtonian approximation given in Section 6.2.1.

The weak field limit of the relativistic Bernoulli equation is obtained by considering terms up to the $1/c^2$ order in (6.114), yielding

$$\frac{(U^r)^2}{2}\left[1 - \left(\frac{U^r}{c}\right)^2 - \frac{4\phi}{c^2}\right] + \frac{a^2}{(\gamma-1)}\left[1 - \frac{a^2}{2c^2(\gamma-1)} - \frac{2\phi}{c^2} - \left(\frac{U^r}{c}\right)^2\right] + \phi\left(1 - \frac{2\phi}{c^2}\right) = \frac{a_\infty^2}{(\gamma-1)}\left(1 - \frac{a_\infty^2}{2c^2(\gamma-1)}\right). \quad (6.127)$$

Above we have introduced the Newtonian gravitational potential $\phi = -\frac{GM}{r}$. Without the $1/c^2$ – terms (6.127) reduces to the non-relativistic Bernoulli equation (6.6), but this expression differs from the post-Newtonian Bernoulli equation (6.53).

The weak field approximation of the Bernoulli equation given by (6.127) can also be expressed in terms of the proper velocity of the flow v_r defined by (6.41). For that end we use the relationship between the components U^r and U^0 which follows from $U_\mu U^\mu = c^2$

$$\left(\frac{U^0}{c}\right)^2 = \frac{1 + \left(1 - \frac{2\phi}{c^2}\right)\left(\frac{U^r}{c}\right)^2}{1 + \frac{2\phi}{c^2}}, \tag{6.128}$$

so that the proper velocity (6.41) becomes

$$v_r = \frac{U^r}{\sqrt{1 + 2\frac{\phi}{c^2}}\sqrt{1 + \left(1 - \frac{2\phi}{c^2}\right)\left(\frac{U^r}{c}\right)^2}}. \tag{6.129}$$

The expression of the radial four-velocity component in terms of the proper velocity by retaining terms up to the $1/c^2$ order is

$$U^r = v_r\left[1 + \frac{\phi}{c^2} + \frac{v_r^2}{2c^2}\right]. \tag{6.130}$$

The weak field limit of Bernoulli equation written in terms of the proper velocity is obtained from the insertion of (6.130) into (6.127) and considering terms up to $1/c^2$ order, yielding

$$\begin{aligned}
&\frac{v_r^2}{2}\left[1 - \frac{2\phi}{c^2}\right] + \frac{a^2}{(\gamma-1)}\left[1 - \frac{a^2}{2c^2(\gamma-1)} - \frac{2\phi}{c^2} - \frac{v_r^2}{c^2}\right] \\
&+\phi\left(1 - \frac{2\phi}{c^2}\right) - \frac{a_\infty^2}{(\gamma-1)}\left(1 - \frac{a_\infty^2}{2c^2(\gamma-1)}\right) \\
&= \frac{v_r^2}{2}\left[1 - \frac{2\phi}{c^2}\right] + \frac{a^2}{(\gamma-1)}\left[1 + \frac{3a^2 - 4a_\infty^2}{2c^2(\gamma-1)}\right] \\
&+\phi\left(1 - \frac{2\phi}{c^2}\right) = \frac{a_\infty^2}{(\gamma-1)}\left(1 - \frac{a_\infty^2}{2c^2(\gamma-1)}\right).
\end{aligned} \tag{6.131}$$

Note that for the underlined term of the above equation we have used the Newtonian Bernoulli equation (6.49), since it is of $1/c^2$ order.

6.3.2 Mach number as function of the radial distance

For the weak field limit the mass density accretion rate follows from (6.103) which in terms of the proper velocity reads

$$\dot{M} = 4\pi\rho r^2 U^r = 4\pi\rho r^2 v_r \left[1 - \frac{GM}{rc^2} + \frac{v_r^2}{2c^2}\right]. \qquad (6.132)$$

If we introduce the dimensionless quantities

$$r_* = \frac{ra_\infty^2}{GM}\left(1 - \frac{GM}{2rc^2}\right), \quad s_* = \frac{v_r}{a_\infty}\left(1 + \frac{v_r^2}{2c^2}\right), \quad t_* = \frac{\rho}{\rho_\infty}, \qquad (6.133)$$

the mass density accretion rate becomes

$$\dot{M} = 4\pi\lambda\left(\frac{GM}{a_\infty^2}\right)^2 \rho_\infty a_\infty, \qquad \text{where} \qquad \lambda = r_*^2 s_* t_*. \qquad (6.134)$$

By considering terms up to the $1/c^2$ order we can obtain from (6.133) that

$$r = \frac{GMr_*}{a_\infty^2}\left(1 + \frac{\beta^2}{2r_*}\right), \qquad v_r = a_\infty s_*\left(1 - \frac{\beta^2 s_*}{2}\right). \qquad (6.135)$$

The expression for s_* given above is the same as the one in the post-Newtonian approximation (6.88) so that we can use

(6.93) and (6.94) for the proper velocity and sound speed as a function of the Mach number u_* and dimensionless radial distance r_*, respectively. The Newtonian gravitational potential $\phi = -GM/r$ in terms of the dimensionless radial distance is

$$\phi = -\frac{a_\infty^2}{r_*}\left(1 - \frac{\beta^2}{2r_*}\right). \tag{6.136}$$

Now from the weak field Bernoulli equation (6.131) together with (6.93), (6.94) and (6.136) follow the expression which relate the Mach number $u_* = v^r/a$ as function of the dimensionless radial distance r_*, namely

$$\begin{aligned}
&\frac{u_*^{\frac{4}{\gamma+1}}}{2}\left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\left[1 - \frac{\beta^2(\gamma-1)}{\gamma+1}u_*^{\frac{4}{\gamma+1}}\left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}} + \frac{2\beta^2}{r_*}\right] \\
&\quad -\frac{1}{r_*}\left(1 + \frac{3\beta^2}{2r_*}\right) + \frac{1}{\gamma-1}\left(\frac{\lambda}{u_*\, r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\left[1 - \frac{2\beta^2}{\gamma-1}\right. \\
&\quad \left. -\left(1 - \frac{3(\gamma+1)}{2(\gamma-1)^2 u_*^2}\right)\frac{\beta^2(\gamma-1)}{(\gamma+1)}u_*^{\frac{4}{\gamma+1}}\left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right] \\
&\quad = \frac{1}{\gamma-1}\left(1 - \frac{\beta^2}{2(\gamma-1)}\right).
\end{aligned} \tag{6.137}$$

The relativistic Bernoulli equation (6.114) can also be written in terms of a Mach number defined by $U_* = U^r/a$ which refers to the ratio of the radial component of the four-velocity and the sound speed. Here we can use the Bondi dimensionless quantities (6.27) and write

$$\left[1-\frac{\beta^2}{\gamma-1}\left(\frac{\lambda}{U_* r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]^2=\left(1-\frac{\beta^2}{\gamma-1}\right)^2\left[1-\frac{2\beta^2}{r_*}\right.$$
$$\left.+\beta^2 U_*^2\left(\frac{\lambda}{U_* r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]. \tag{6.138}$$

In terms of the Mach number related with the proper velocity $u_*=v_r/a$ the Mach number U_* is given by

$$u_*=\frac{v_r}{a}=\frac{U_*}{\sqrt{1-2\frac{\beta^2}{r}}\sqrt{1+\left(1+2\frac{\beta^2}{r}\right)\beta^2 U_*^2\left(\frac{\lambda}{U_* r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}}}, \tag{6.139}$$

thanks to (6.129).

6.4 Numerical results

Let us analyse the behavior of the solutions for the Mach number u_* as function of the dimensionless radial distance r_* which follow from the different approximations of the Bernoulli equation, namely, the Newtonian, the post-Newtonian approximation, the relativistic and its weak field approximation.

The Newtonian solution of (6.32) will be denoted by (N), the post-Newtonian solution of (6.96) by (PN) and the weak field approximation solution of (6.137) by (WF). For the relativistic accretion, denoted by (R), the Bernoulli equation (6.138) was first solved for the Mach number with respect to the radial four-velocity U_* and then from (6.139) the Mach number for the proper velocity u_* was obtained.

In the tables and figures below – which show the Mach number u_* as a function of the dimensionless radial distance r_* – it was considered that the ratio of the sound velocity far from the massive body and the light speed was equal to $\beta = a_\infty/c = 10^{-2}$, which is of relativistic order.

Table 6.2 shows the values for the Mach number u_* as function of the dimensionless radial distance r_* in the range $5 \times 10^{-4} \leq r_* \leq 2.5 \times 10^{-2}$ for a ultra-relativistic Fermi gas where $\gamma = 4/3$. The critical radius in the Newtonian approximation is $r_* = 0.25$ where the critical Mach number assumes the value $u_* = 1$. From this table we infer that by increasing the dimensionless radial distances r_* from the massive body the Mach number decreases. The values of the Mach number for the relativistic case are bigger than the Newtonian ones while the values of the Mach number for the post-Newtonian and weak field approximations are practically the same and are smaller than those for the Newtonian case. By increasing the dimensionless radial distance the difference between the Newtonian, post-Newtonian and weak field solutions becomes very small and the solutions practically coincide at $r_* = 10^{-3}$. The contour plots for the Newtonian (6.32), post-Newtonian (6.96) and weak field (6.137) Bernoulli equations are shown in the left frame Figure 6.4. In this figure the Newtonian solution is represented by a straight line, the post-Newtonian by a dashed line and the weak field approximations by a dot-dashed line. We note that the difference between the Newtonian, post-Newtonian and weak field are very small and coincide by increasing the dimensionless radial distance.

r_*	u_* (N)	u_* (PN)	u_* (WF)	u_* (R)
2.5×10^{-2}	1.00	1.07	1.02	2.09
5×10^{-2}	2.40	2.41	2.41	3.31
10^{-2}	4.35	4.34	4.34	4.97
5×10^{-3}	5.40	5.37	5.37	6.07
10^{-3}	8.53	8.24	8.26	11.26
5×10^{-4}	10.29	9.66	9.66	20.26

Table 6.2: Mach number $u_* = v_r/a$ as function of the dimensionless radial distance r_* for a ultra-relativistic Fermi gas $\gamma = 4/3$.

For a diatomic gas where $\gamma = 7/5$ the Mach number u_* as function of the dimensionless radial distance r_* in the range $5 \times 10^{-4} \leq r_* \leq 2 \times 10^{-2}$ is shown in Table 6.3. One may infer the same conclusions as those in the former case, i.e., in comparison with the Newtonian solutions the dependence of Mach number with respect to the dimensionless radial distance for the relativistic case is bigger, the post-Newtonian and the weak field solutions are smaller and both have practically the same values. The critical radius for the Newtonian case is $r_* = 0.2$ where the Mach number attains the value $u_* = 1$. The contour plots of the Newtonian (straight line), the post-Newtonian (dashed line) and the weak field (dot-dashed line) solutions are displayed in the right frame of Figure 6.4 showing that the values of the Mach number for the Newtonian solution is bigger than those

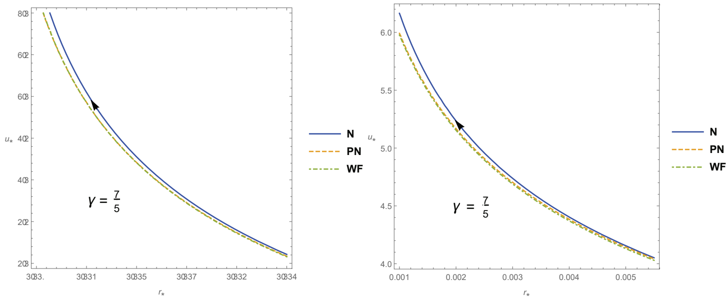

Figure 6.4: Contour plots of Mach number u_* as function of dimensionless radial distance r_* for ultra-relativistic Fermi gas $\gamma = 4/3$ (left frame) and for diatomic gas $\gamma = 7/5$ (right frame). Newtonian – straight line, post-Newtonian – dashed line and weak field – dot-dashed line.

for the post-Newtonian and weak field solutions and that the difference between them becomes very small by increasing the dimensionless radial distance.

The contour plots for a monatomic gas or a non-relativistic Fermi gas with $\gamma = 5/3$ are shown in Figure 6.5 where of the Newtonian solution is represented by a straight line, the post-Newtonian solution by a dashed line and weak field solution by a dot-dashed line. For the Newtonian case the critical dimensionless radial distance is $r_* = 0$ where the the Mach number becomes equal to $u_* = 1$. We call attention to the fact that $r_* = 0$ is a turning point for the Newtonian solution where a transition occurs from an accretion flow to a wind flow. We in-

r_*	u_* (N)	u_* (PN)	u_* (WF)	u_* (R)
2×10^{-2}	1.00	1.02	1.06	2.21
5×10^{-2}	2.00	2.00	2.00	2.86
10^{-2}	3.43	3.43	3.43	4.02
5×10^{-3}	4.16	4.14	4.13	4.77
10^{-3}	6.16	5.99	5.98	8.22
5×10^{-4}	7.21	6.83	6.81	14.22

Table 6.3: Mach number $u_* = V/a$ as function of the dimensionless radial distance r_* for a diatomic gas $\gamma = 7/5$.

fer from this figure that the values of the weak field solution are smaller than the Newtonian ones and the turning point is about $r_* \approx 4 \times 10^{-3}$. Furthermore, for the post-Newtonian solution there is no turning point which corresponds to a transition from an accretion flow to a wind flow.

Once the Mach numbers as functions of the dimensionless radial distances are determined one can obtain the ratios of the mass density and absolute temperature with respect to their values far from the compact massive object as function of the dimensionless radial distance from (6.33) and (6.34), respectively. The ratios ρ/ρ_∞ and T/T_∞ for a ultra-relativistic Fermi gas ($\gamma = 4/3$) and diatomic gas ($\gamma = 7/5$) which follow from the Newtonian and relativistic solutions are compared in table 6.4. The values of the density and temperature ratios correspond to a dimensionless radial distance $r_* = 10^{-2}$. One observes from this table that the values of the ratios ρ/ρ_∞ and T/T_∞ for the

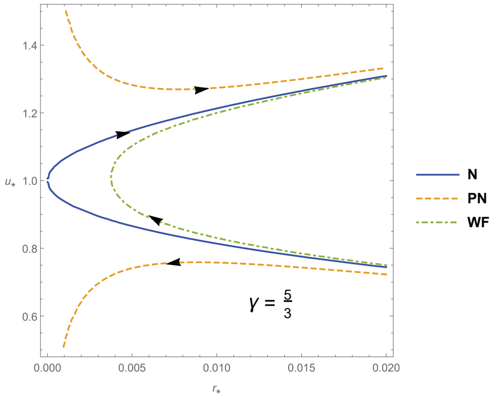

Figure 6.5: Contour plots of Mach number u_* as function of dimensionless radial distance r_* for non-relativistic Fermi gas $\gamma = 5/3$. Dashed line Newtonian, straight line post-Newtonian and weak field.Newtonian – straight line, post-Newtonian – dashed line and weak field – dot-dashed line.

relativistic solution are smaller than those for the Newtonian solution. This can be easily understood by noting that the mass accretion rate is a constant and proportional to $\rho r^2 v_r$, hence by increasing the velocity for a given radial distance the density must diminish. Note that the velocity of the relativistic solution is bigger than the one of the Newtonian case. There is no sensible difference in these ratios from the post-Newtonian and weak field approximations with respect to those from the Newtonian theory.

γ	ρ/ρ_∞ (N)	ρ/ρ_∞ (R)	T/T_∞ (N)	T/T_∞ (R)
4/3	5.65×10^2	4.87×10^2	2.64×10^6	1.87×10^6
7/5	5.21×10^2	4.00×10^2	3.31×10^6	1.76×10^6

Table 6.4: Comparison of ratios ρ/ρ_∞ and T/T_∞ at $r_* = 10^{-2}$ for $\gamma = 4/3$ and $\gamma = 7/5$. (N) Newtonian solution and (R) relativistic solution.

Here it is necessary to comment the behaviors of the post-Newtonian and weak field solutions found in the above analysis when both are compared with the Newtonian and relativistic solutions. One expects that the post-Newtonian and weak field solutions should be more close to the relativistic one and not smaller than the Newtonian solution. By inspecting the Newtonian (6.32), the post-Newtonian (6.96) and the weak field (6.137) equations we note that the two latter equations have corrections from the Newtonian one and their solutions should furnish different results for the dependence of the Mach number as function of the dimensionless radial distance. We can ask why the values of the Mach number for the post-Newtonian and weak field are smaller than in the Newtonian case. The only clue is to look at the expression for the proper velocity (6.93) for the post-Newtonian and weak field which can be written as

$$v_r = a_\infty u_*^{\frac{2}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{\gamma-1}{\gamma+1}} \left[1 - \frac{\beta^2(\gamma-1)}{2(\gamma+1)} u_*^{\frac{4}{\gamma+1}} \left(\frac{\lambda}{r_*^2}\right)^{\frac{2(\gamma-1)}{\gamma+1}}\right]$$

$$= v_r^N \left[1 - \frac{\beta^2(\gamma-1)}{2(\gamma+1)} u_*^{\frac{4}{\gamma+1}} \left(\frac{\lambda}{r_*^2} \right)^{\frac{2(\gamma-1)}{\gamma+1}} \right], \tag{6.140}$$

where v_r^N is the Newtonian expression for the proper velocity. One infers from the above equation that the proper velocities for the post-Newtonian and weak field should be smaller than the one for the Newtonian case, which could explain the difference in the behavior of the solutions.

To sum up the above results: (i) the Mach number for the Newtonian, post-Newtonian and weak field accretions have practically the same values for radial distances of order of the critical radial distance; (ii) by decreasing the radial distance the Mach number for the Newtonian accretion is bigger than the one for the post-Newtonian and weak field accretions; (iii) the effect of the correction terms in post-Newtonian and weak field Bernoulli equations are more perceptive for the lowest values of the radial distance; (iv) practically there is no difference between the Newtonian, post-Newtonian and weak field Mach numbers when the ratio $a_\infty/c \ll 10^{-2}$; (v) the solutions for $a_\infty/c > 10^{-2}$ does not lead to a continuous inflow and outflow velocities at the critical point; (vi) from the comparison of the solutions with those that follow from the relativistic Bernoulli equation shows that the Mach number of the former is bigger than the Newtonian, post-Newtonian and weak field Mach numbers.

References

[1] R. A. Lyttleton and F. Hoyle, The evolution of the stars, *The Observatory* **63**, 39 (1940).

[2] F. Hoyle and R. A. Lyttleton, On the accretion theory of stellar evolution, *Mon. Not. R. Astron. Soc.* **227**, 101 (1941).

[3] H. Bondi and F. Hoyle, On the mechanism of accretion by stars *Mon. Not. R. Astron. Soc.* **104**, 273 (1944).

[4] H. Bondi, On spherically symmetrical accretion, *Mon. Not. R. Astron. Soc.* **112**, 195 (1952).

[5] F. C. Michel, Accretion of matter by condensed objects, *Astrophys. Space Sci.* **15**, 153 (1972).

[6] S. L. Shapiro and S. A. Teukolsky, *Black holes, white dwarfs, and neutron stars* (Wiley-VCH, Weinheim, 2004).

[7] F. Banyuls, J. A. Font, J. M. Ibáñez, J. M. Martí and J. A. Miralles, Numerical $\{3+1\}$ general relativistic hydrodynamics: a local characteristic approach, *Ap. J.* **476**, 221 (1997).

[8] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann equation: theory and applications* (Birkhäuser, Basel, 2002).

[9] C. B. Richards, T. W. Baumgarte and S. L. Shapiro, Relativistic Bondi accretion for stiff equations of state, *MNRAS* **502**, 3003 (2021).

[10] G. M.Kremer and L. C. Mehret, Post-Newtonian Spherically Symmetrical Accretion, to be published in *Phys. Rev. D* (2021).

CHAPTER 7

JEANS INSTABILITY: HYDRODYNAMIC EQUATIONS

The standard cosmological model is based on the fact that at scales larger than 100 Mpc (3.08×10^{18} m) the Universe is basically uniform and homogeneous. The inhomogeneities at small scales account for the existence of galaxies and clusters of galaxies. A question that arises is how to explain the growth and the basic mechanism of matter aggregation which generate these structures. The first determination of the fluid instabilities for a static Universe – where small perturbations of a gas cloud could grow exponentially leading to its collapse – was due to Sir James Jeans in 1912 [1]. By using the fluid hydrodynamic equations he determined a dispersion relation which implied, apart from the harmonic perturbations, a growing and a decaying propagating modes. In other words, he found a physical cutoff – nowadays called Jeans' wavelength – such that the perturbations with wavelength shorter than the Jeans wavelength will not grow in time and evolve as harmonic oscillations, whereas the perturbations with larger wavelength may grow or decay exponentially in time. A very simple argument can be used to understand the gravitational instability associated with the Jeans mechanism: let us consider a spherical volume of radius λ which encloses a given mass M and where there exists a mass density inhomogeneity. In this spherical volume two forces are present, namely, the gravity force per unit of mass F_g and the pressure force per unit of mass F_p. The inhomogeneity will grow if $F_g > F_p$, i.e, if the gravity force per unit mass is greater than the opposed pressure force per unit of mass, namely,

$$F_g = \frac{GM}{\lambda^2} \propto \frac{G\rho\lambda^3}{\lambda^2} = G\rho\lambda > F_p \propto \frac{p\lambda^2}{\rho\lambda^3} \propto \frac{c_s^2}{\lambda}, \implies \lambda^2 > \frac{c_s^2}{G\rho}.$$

Here we have taken the sound speed as $c_s^2 \propto (p/\rho)$ and $M = \rho\lambda^3$. The Jeans wavelength is given in terms of the Jeans wave number by $\lambda_J = 2\pi/\kappa_J = 2\pi c_s/\sqrt{4\pi G\rho}$. By introducing the wave number $\kappa = 2\pi/\lambda$ the instability comes out if the condition $\lambda > \lambda_J$ or $\kappa < \kappa_J$ holds. Furthermore, if t_p represents the timescale associated with the pressure exerted over a region with matter and t_g the timescale needed to start the gravitational collapse of the matter due to its own weight, the Jeans instability occurs whenever the pressure timescale is greater than the gravitational one, i.e., $t_p = (\lambda/c_s) > t_g = 1/\sqrt{G\rho}$.

In this chapter we shall investigate the Jeans instability on the basis of the hydrodynamic equations for the Newtonian, first and second post-Newtonian approximations.

7.1 Newtonian Jeans instability

The Newtonian analysis of the Jeans instability is based on the mass density (1.8) and momentum density (1.9) balance equations for an Eulerian fluid where $p_{ij} = p\delta_{ij}$, which we reproduce here

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_i}{\partial x^i} = 0, \tag{7.1}$$

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x^j} + \frac{\partial p}{\partial x^i} - \rho \frac{\partial U}{\partial x^i} = 0, \tag{7.2}$$

where the Newtonian gravitational potential $U = -\phi$ obeys the Poisson equation

$$\nabla^2 U = -4\pi G \rho. \tag{7.3}$$

Here we shall investigate the Jeans instability for an isentropic flow with a polytropic equation of state $p = \kappa\rho^\gamma$ and sound speed $c_s^2 = dp/d\rho$.

Equations (7.1) – (7.3) represent a system of partial differential equations for the determination of the mass density $\rho(\mathbf{x}, t)$, hydrodynamic velocity $V_i(\mathbf{x},t)$ and Newtonian gravitational potential $U(\mathbf{x},t)$. We shall assume that the fluid is initially at rest with constant mass density ρ_0, vanishing hydrodynamic velocity $V_i^0 = 0$ and constant Newtonian gravitational potential U_0. The homogeneity condition implies that the equilibrium fields do not depend on the spatial coordinates.

The equilibrium fields are superposed by the field perturbations $\rho_1(\mathbf{x},t)$, $V_i^1(\mathbf{x},t)$ and $U_1(\mathbf{x},t)$, namely

$$\rho(\mathbf{x},t)=\rho_0+\rho_1(\mathbf{x},t),\ V_i(\mathbf{x},t)=V_i^1(\mathbf{x},t),\ U(\mathbf{x},t)=U_0+U_1(\mathbf{x},t), \tag{7.4}$$

where the field perturbations are supposed to be small quantities.

While the equilibrium fields satisfy the balance equations of mass density (7.1) and momentum density (7.2), the Poisson equation (7.3) leads to an inconsistency, namely, $0 = -4\pi G\rho_0$. To remover this inconsistency one relies on the "Jeans swindle", which asserts that the Poisson equation is valid only for the perturbed values of the Newtonian gravitational potential and mass density.

From the insertion of the representations (7.4) into (7.1) – (7.3) and subsequent linearization of the resulting equations it follows that

$$\frac{\partial\rho_1}{\partial t} + \rho_0\frac{\partial V_i^1}{\partial x^i} = 0, \tag{7.5}$$

$$\rho_0\frac{\partial V_i^1}{\partial t} + c_s^2\frac{\partial\rho_1}{\partial x^i} - \rho_0\frac{\partial U_1}{\partial x^i} = 0, \tag{7.6}$$

$$\nabla^2 U_1 = -4\pi G\rho_1. \tag{7.7}$$

In (7.6) we have considered $\partial p_1/\partial x^i = c_s^2 \partial \rho_1/\partial x^i$.

By taking the time derivative of (7.5) and the divergence of (7.6) we get

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \frac{\partial^2 V_i^1}{\partial t \partial x^i} = 0, \quad \rho_0 \frac{\partial^2 V_i^1}{\partial x^i \partial t} + c_s^2 \nabla^2 \rho_1 - \rho_0 \nabla^2 U_1 = 0. \tag{7.8}$$

If we eliminate the velocity derivatives from the mass density equation $(7.8)_1$ by using the momentum density equation $(7.8)_2$ and the Laplacian of the Newtonian gravitational potential by considering the Poisson equation (7.7) we get

$$\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0, \tag{7.9}$$

which is an equation which involves only the mass density perturbation.

Now we characterize the mass density perturbation by a plane wave of small amplitude $\overline{\rho}$, wave number vector $\mathbf{k}$ and frequency ω:

$$\rho_1(\mathbf{x}, t) = \overline{\rho} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]. \tag{7.10}$$

If we insert (7.10) into (7.9) we get the following dispersion relation which relates the frequency ω with the modulus of the wave number vector $\kappa = \sqrt{\mathbf{k} \cdot \mathbf{k}}$:

$$\omega^2 = c_s^2 \kappa_J^2 \left(\frac{\kappa^2}{\kappa_J^2} - 1 \right) = c_s^2 k_J^2 \left(\frac{\lambda_J^2}{\lambda^2} - 1 \right) \text{ where } \kappa_J = \frac{\sqrt{4\pi G \rho_0}}{c_s}, \tag{7.11}$$

denotes Jeans' wavelength.

The dispersion relation (7.11) represents the solution given by Jeans. For small wavelengths $\lambda_J > \lambda$ the frequency ω is a real quantity and the mass density perturbation propagates as harmonic waves in time. For big wavelengths $\lambda_J < \lambda$ the frequency ω becomes a pure imaginary quantity so that the mass density perturbation will grow or decay depending on the sign $\pm$. The one which grows in time refers to Jeans' instability.

The Jeans mass is defined as the minimum mass for an over-density to begin the gravitational collapse and it is defined as the mass contained within a sphere of radius λ_J, namely

$$M_J = \frac{4\pi}{3}\lambda_J^3\rho_0. \tag{7.12}$$

7.2 Jeans instability in expanding Universe

Another problem which is interesting to examine is the Jeans instability in an expanding Universe. This problem was first analyzed by Bonnor [2] in 1957 by using the Newtonian balance equations coupled with the Poisson equation. The description of this problem can also be found in the books by Weinberg [3] and Coles and Lucchin [4]. We note that the Newtonian gravity is valid in regions whose radius are small compared with the Hubble radius and the velocities are non-relativistic. The Hubble radius is of order of c/H_0 where the Hubble constant has an approximate value of $H_0 = 73$ km/s/Mpc.

We consider that the expanding Universe is ruled by the spatially flat Friedmann-Lamaître-Robertson-Walker (FLRW) metric $ds^2 = (cdt)^2 - a(t)(dx^2 + dy^2 + dz^2)$, where $a(t)$ is the cosmic scale factor.[1]

Here we shall follow the work [5] and write the Newtonian balance equations and the Poisson equation in terms of the comoving coordinates $\mathbf{x}_0$ which are related with the physical coordinates $\mathbf{x}$ by $\mathbf{x}(t) = a(t)\mathbf{x}_0$. For that end we have to transform the time and spatial derivatives as follows

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}_0} + \left.\frac{\partial x_0^i}{\partial t}\right|_{\mathbf{x}} \left.\frac{\partial}{\partial x_0^i}\right|_t = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}_0} - \frac{\dot{a}}{a} x_0^i \left.\frac{\partial}{\partial x_0^i}\right|_t, \quad (7.13)$$

$$\left.\frac{\partial}{\partial x^i}\right|_t = \frac{1}{a} \left.\frac{\partial}{\partial x_0^i}\right|_t. \quad (7.14)$$

For the analysis of Jeans instability we shall consider the background solutions

$$\rho_B = \rho_0 \left(\frac{a_0}{a}\right)^3, \quad \mathbf{V}_B = \dot{a}\mathbf{x}_0, \quad U_B = -\frac{2\pi}{3} G\rho_B\, \mathbf{x}\cdot\mathbf{x}, \quad (7.15)$$

superposed by the field perturbations ρ_1, V_i^1 and U_1 i.e., $\rho = \rho_B + \rho_1$, $V_i = V_i^B + V_i^1$ and $U = U_B + U_1$. The mass-energy density background results from Einstein's field equations for a dust dominated Universe where $p \approx 0$ (see (7.93)), while the velocity background follows from Hubble-Lamaître's law $\mathbf{V} = (\dot{a}/a)\mathbf{x}$. The Newtonian gravitational potential background satisfy the Poisson equation (7.3) without the necessity to invoke the "Jeans swindle".

[1] For an overview of the field equations which follow from Einstein's field equations the reader is referred to the Appendix of this chapter.

By taking into account (7.13) and considering only the linear terms the mass-energy density balance equation (7.1) can be written as

$$\frac{\partial \rho_B}{\partial t} + \frac{\partial \rho_1}{\partial t} - \frac{\dot{a}}{a} x_0^i \frac{\partial \rho_1}{\partial x_0^i} + \frac{\rho_B}{a}\left(\frac{\partial V_i^B}{\partial x_0^i} + \frac{\partial V_i^1}{\partial x_0^i}\right) + \frac{\rho_1}{a}\frac{\partial V_i^B}{\partial x_0^i} + \frac{V_i^B}{a}\frac{\partial \rho_1}{\partial x_0^i} = \frac{\partial \rho_1}{\partial t} + \frac{\rho_B}{a}\frac{\partial V_i^1}{\partial x_0^i} + \frac{\rho_1}{a}\frac{\partial V_i^B}{\partial x_0^i} = 0, \tag{7.16}$$

where in the last equality above we have used the relationships given in (7.15). Now we introduce the mass-energy density and velocity contrasts

$$\widetilde{\rho} = \frac{\rho_1}{\rho_B}, \qquad \widetilde{V}_i = \frac{V_i^1}{a}, \tag{7.17}$$

so that (7.16) thanks to (7.15) becomes

$$\frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial \widetilde{V}_i}{\partial x_0^i} = 0. \tag{7.18}$$

The momentum density balance equation (7.2) by considering (7.13) and taking in account only linear terms reads

$$(\rho_B + \rho_1)\left[\frac{\partial V_i^B}{\partial t} - \frac{\dot{a}}{a} x_0^j \frac{\partial V_i^B}{\partial x_0^j}\right] + \rho_B\left[\frac{\partial V_i^1}{\partial t} - \frac{\dot{a}}{a} x_0^j \frac{\partial V_i^1}{\partial x_0^j}\right] + \frac{\rho_B V_j^B}{a}\frac{\partial V_i^1}{\partial x_0^j} + \frac{(\rho_B V_j^B + \rho_B V_j^1 + \rho_1 V_j^B)}{a}\frac{\partial V_i^B}{\partial x_0^j} + \frac{c_s^2}{a}\frac{\partial \rho_1}{\partial x_0^1} - \frac{(\rho_B + \rho_1)}{a}\frac{\partial U_0}{\partial x_0^j} - \frac{\rho_B}{a}\frac{\partial U_1}{\partial x_0^i}$$

$$= (\rho_B + \rho_1)\left[\ddot{a} - \frac{\dot{a}^2}{a}\right]x_0^i + \rho_B\left[\frac{\partial V_i^1}{\partial t} - \frac{\dot{a}}{a}x_0^j\frac{\partial V_i^1}{\partial x_0^j}\right]$$

$$+(\rho_B V_i^B + \rho_B V_i^1 + \rho_1 V_i^B)\frac{\dot{a}}{a} + \frac{c_s^2}{a}\frac{\partial \rho_1}{\partial x_0^i} + \frac{\rho_B V_j^B}{a}\frac{\partial V_i^1}{\partial x_0^j}$$

$$+\frac{4\pi}{3}G\rho_B x_0^i(\rho_B + \rho_1) - \frac{\rho_B}{a}\frac{\partial U_1}{\partial x_0^i} = 0. \qquad (7.19)$$

Here we have introduced the sound speed by considering the relation $\partial p_1/\partial x_0^i = c_s^2\partial\rho_1/\partial x_0^i$. Furthermore, in the last equality the relationships (7.15) were used. Now by considering the Hubble-Lamaître's law $V_i^B = \dot{a}x_0^i$, the acceleration equation for a dust dominated Universe $(7.92)_1$ and introducing the mass-energy density $\widetilde{\rho}$ and velocity $\widetilde{V}_i$ contrasts the above equation reduces to

$$\frac{\partial \widetilde{V}_i}{\partial t} + 2\frac{\dot{a}}{a}\widetilde{V}_i + \frac{c_s^2}{a^2}\frac{\partial\widetilde{\rho}}{\partial x_0^i} - \frac{1}{a^2}\frac{\partial U_1}{\partial x_0^i} = 0. \qquad (7.20)$$

The Poisson equation (7.3) for the perturbations is given by

$$\nabla^2_{\mathbf{x}_0}U_1 = -4\pi G\rho_B a^2\widetilde{\rho}. \qquad (7.21)$$

Following the same methodology of the previous section we derive the mass-energy density contrast equation (7.18) with respect to time and the velocity contrast equation (7.20) with respect to the comoving coordinates. By eliminating the derivatives of the velocity contrast from the former equation by using the latter equation, yields

$$\frac{\partial^2\widetilde{\rho}}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\widetilde{\rho}}{\partial t} - \frac{c_s^2}{a^2}\nabla^2_{\mathbf{x}_0}\widetilde{\rho} - 4\pi G\rho_B\widetilde{\rho} = 0. \qquad (7.22)$$

Here we have used the Poisson equation (7.21) to eliminate the Laplacian of the perturbed Newtonian gravitational potential U_1.

The mass-energy density contrast $\widetilde{\rho}$ is now expanded in a plane wave base where the comoving wave number is $\mathbf{q}$, while the physical one is $\mathbf{q}/a(t)$. The factor $1/a(t)$ in the physical wave number takes into account that the wave number is stretched out in an expanding Universe. Hence we write the mass-energy density contrast as

$$\widetilde{\rho} = \delta\rho(t) \exp\left[i\mathbf{q} \cdot \mathbf{x}_0\right]. \tag{7.23}$$

Insertion of the plane wave representation (7.23) into (7.22) leads to the following differential equation for the amplitude of the mass-energy density contrast $\delta\rho(t)$:

$$\frac{d^2\delta\rho}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta\rho}{dt} + c_s^2\frac{q^2}{a^2}\delta\rho - 4\pi G\rho_B\delta\rho = 0. \tag{7.24}$$

Now we shall write (7.24) in term of dimensionless quantities. For that end we shall use the time dependence of the scale factor $a = a_0\left(6\pi G\rho_B t^2\right)^{\frac{1}{3}}$ which follows from the Friedmann equation for a pressureless fluid (see (7.94) in the Appendix) and the dimensionless quantities

$$\lambda_0 = \frac{2\pi a_0}{q}, \qquad \lambda_J = \frac{2\pi c_s}{\sqrt{4\pi G\rho_B}}, \qquad \tau = t\sqrt{6\pi G\rho_B}, \tag{7.25}$$

which represent the mass-energy density contrast wavelength, Jeans wavelength and dimensionless time, respectively. The

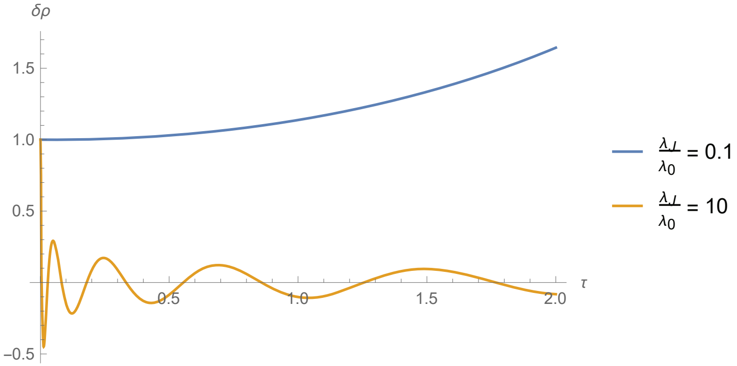

Figure 7.1: Amplitude of the mass-energy density contrast $\delta\rho$ as function of the dimensionless time τ for big $\lambda_J/\lambda_0 = 0.1$ and small $\lambda_J/\lambda_0 = 10$ wavelengths in comparison with Jeans wavelength.

resulting differential equation for the amplitude of the mass-energy density contrast becomes

$$\tau^2\delta\rho'' + \frac{4}{3}\tau\delta\rho' + \frac{2}{3}\left(\frac{\lambda_J^2}{\lambda_0^2}\tau^{\frac{2}{3}} - \tau^2\right)\delta\rho = 0. \tag{7.26}$$

In the above equation the prime denotes differentiation with respect to the dimensionless time τ and it was used the relationships $a = a_0\left(6\pi G\rho_B t^2\right)^{\frac{1}{3}}$ and $a'/a = 2/3\tau$.

The differential equation (7.26) was solved numerically for the following initial conditions: $\delta\rho(0.001) = 1$ and $\delta\rho'(0.001) =$

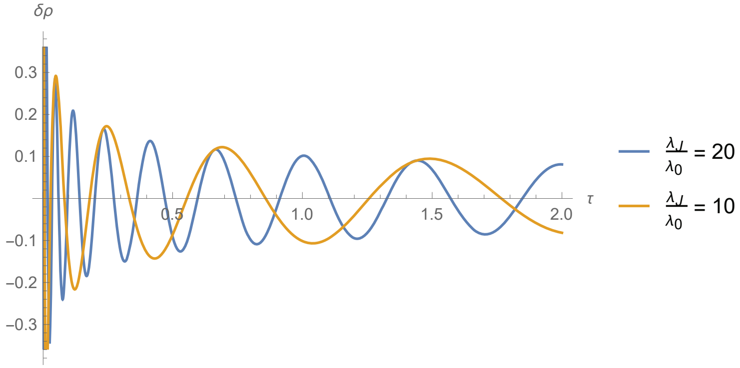

Figure 7.2: Amplitude of the mass-energy density contrast $\delta\rho$ as function of the dimensionless time τ in the oscillatory regime for two ratios $\lambda_J/\lambda_0 = 10$ and $\lambda_J/\lambda_0 = 20$.

0. In Figure 7.1 it is shown the behavior of the mass-energy density contrast as function of the dimensionless time τ for two values of the ratio between the Jeans and mass-energy density contrast wavelengths λ_J/λ_0. The case $\lambda_J/\lambda_0 = 0.1$ represents the Jeans instability where the mass-energy density contrast grows with time (big wavelengths), while the one where $\lambda_J/\lambda_0 = 10$ shows an oscillatory behavior of the mass-energy density contrast (small wavelengths). In Figure 7.2 a comparison of the oscillatory behavior for the ratios $\lambda_J/\lambda_0 = 10$ and $\lambda_J/\lambda_0 = 20$ shows that by decreasing the wavelength of the mass-energy density contrast the period of the oscillation decreases.

7.3 Post-Newtonian Jeans instability I

Let us investigate the Jeans instability within the framework of the first post-Newtonian Eulerian hydrodynamic equations. We begin by reproducing here the balance equations for the mass density (2.122)

$$\frac{\partial \rho_*}{\partial t} + \frac{\partial \rho_* V_i}{\partial x^i} = 0, \qquad \text{where} \qquad \rho_* = \rho\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} + 3U\right)\right], \tag{7.27}$$

and for the momentum density (2.133)

$$\rho\frac{dV_i}{dt} + \frac{\partial p}{\partial x^i}\left[1 - \frac{1}{c^2}\left(V^2 + 4U + \varepsilon + \frac{p}{\rho}\right)\right]$$
$$-\rho\frac{\partial U}{\partial x^i}\left[1 + \frac{1}{c^2}\left(V^2 - 4U\right)\right] - \frac{\rho}{c^2}\left[2\frac{\partial \Phi}{\partial x^i} + \frac{d\Pi_i}{dt}\right.$$
$$\left. -V_j\frac{\partial \Pi_j}{\partial x^i} + V_i\left(\frac{\partial U}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t} - 4\frac{dU}{dt}\right)\right] = 0. \tag{7.28}$$

The above equations were written in terms of the Chandrasekhar potentials $U = -\phi$, $\Phi = -\psi/2$ and $\Pi_i = -\xi_i$, which obey the Poisson equations (2.115) and (2.116)

$$\nabla^2 U = -4\pi G\rho, \qquad \nabla^2\Phi = -4\pi G\rho\left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right), \tag{7.29}$$

$$\nabla^2\Pi_i = -16\pi G\rho V_i + \frac{\partial^2 U}{\partial t\partial x^i}, \tag{7.30}$$

and the gauge condition (2.117)

$$3\frac{\partial U}{\partial t}+\frac{\partial \Pi_i}{\partial x^i}=0. \tag{7.31}$$

The system of partial differential equations (7.27) – (7.30) is closed by assuming the polytropic equation of state $p=\kappa\rho^\gamma$ and the specific internal energy equation $\varepsilon=\frac{1}{\gamma-1}\frac{p}{\rho}$, which follows from the integrability condition of the Gibbs equation (see Section 5.1).

We shall consider that the fluid is initially at rest where the fields of mass density and scalar gravitational potentials assume constant values $\rho=\rho_0$, $U=U_0$ and $\Phi=\Phi_0$, while the hydrodynamic velocity V_i and the gravitational potential vector Π_i vanish. These values for the fields satisfy the mass density (7.27) and the momentum density (7.28) balance equations but not the Poisson equations (7.29) and (7.30). Hence we assume again the "Jeans swindle" by considering that the Poisson equations are valid only for the perturbed fields. Note that we have assumed non-vanishing values for the unperturbed gravitational potentials $U=U_0$ and $\Phi=\Phi_0$. This assumption is different from the work [7] on the post-Newtonian Jeans analysis, since there it was considered vanishing values for the unperturbed potentials. It will be shown here that the unperturbed Newtonian gravitational potential U_0 plays an important role in the determination of the Jeans mass.

The equilibrium fields are superposed by small perturbations denoted by the subscript 1 such that

$$\rho(\mathbf{x},t)=\rho_0+\rho_1(\mathbf{x},t),\ V_i(\mathbf{x},t)=V_i^1(\mathbf{x},t),\ \Pi_i(\mathbf{x},t)=\Pi_i^1(\mathbf{x},t), \tag{7.32}$$

$$U(\mathbf{x},t)=U_0+U_1(\mathbf{x},t),\qquad \Phi(\mathbf{x},t)=\Phi_0+\Phi_1(\mathbf{x},t). \tag{7.33}$$

Insertion of the representations (7.32) and (7.33) into the balance equations of mass density (7.27) and momentum density (7.28) and linearization of the resulting equations, lead to

$$\left[\frac{\partial\rho_1}{\partial t}+\rho_0\frac{\partial V_i^1}{\partial x^i}\right]\left[1+\frac{3U_0}{c^2}\right]+\frac{3\rho_0}{c^2}\frac{\partial U_1}{\partial t}=0, \tag{7.34}$$

$$\rho_0\frac{\partial V_i^1}{\partial t}+c_s^2\frac{\partial\rho_1}{\partial x^i}\left[1-\frac{1}{c^2}\left(4U_0+\varepsilon_0+\frac{p_0}{\rho_0}\right)\right]$$
$$-\rho_0\frac{\partial U_1}{\partial x^i}\left(1-\frac{4U_0}{c^2}\right)-\frac{\rho_0}{c^2}\left(2\frac{\partial\Phi_1}{\partial x^i}+\frac{\partial\Pi_i^1}{\partial t}\right)=0. \tag{7.35}$$

If we multiply (7.34) by $(1-3U_0/c^2)$ and retain only terms up to $1/c^2$ order the resulting equation reduces to

$$\frac{\partial\rho_1}{\partial t}+\rho_0\frac{\partial V_i^1}{\partial x^i}+\frac{3\rho_0}{c^2}\frac{\partial U_1}{\partial t}=0. \tag{7.36}$$

By taking the time derivative of (7.36) and the divergence of (7.35) it follows respectively that

$$\frac{\partial^2\rho_1}{\partial t^2}+\rho_0\frac{\partial^2 V_i^1}{\partial x^i\partial t}+\frac{3\rho_0}{c^2}\frac{\partial^2 U_1}{\partial t^2}=0, \tag{7.37}$$

$$\rho_0\frac{\partial^2 V_i^1}{\partial x^i\partial t}+c_s^2\nabla^2\rho_1\left[1-\frac{1}{c^2}\left(4U_0+\varepsilon_0+\frac{p_0}{\rho_0}\right)\right]$$

$$-\rho_0\nabla^2 U_1\left(1-\frac{4U_0}{c^2}\right)-\frac{\rho_0}{c^2}\left(2\nabla^2\Phi_1+\frac{\partial^2\Pi_i^1}{\partial x^i\partial t}\right)=0. \quad (7.38)$$

Now the elimination of the velocity derivatives from (7.37) by using (7.38), yields

$$\frac{\partial^2\rho_1}{\partial t^2}-c_s^2\nabla^2\rho_1\left[1-\frac{1}{c^2}\left(4U_0+\varepsilon_0+\frac{p_0}{\rho_0}\right)\right]+2\frac{\rho_0}{c^2}\nabla^2\Phi_1$$
$$+\rho_0\nabla^2 U_1\left(1-\frac{4U_0}{c^2}\right)+\frac{\rho_0}{C^2}\frac{\partial}{\partial t}\underline{\left[3\frac{\partial U_1}{\partial t}+\frac{\partial\Pi_i^1}{\partial x^i}\right]}=0, \quad (7.39)$$

where the underlined term above vanishes thanks to the gauge condition (7.31).

The perturbed Poisson equations are obtained from the insertion of the representations (7.32) and (7.33) into (7.29), yielding

$$\nabla^2 U_1=-4\pi G\rho_1, \quad (7.40)$$

$$\nabla^2\Phi_1=-4\pi G\rho_1\left(U_0+\frac{\varepsilon_0}{2}\right)-4\pi G\rho_0\left(U_1+\frac{\varepsilon_1}{2}+\frac{3p_1}{2\rho_0}\right), \quad (7.41)$$

since the linearization of the ratio p/ρ is given by

$$\frac{p}{\rho}=\frac{p_0}{\rho_0}\left(\frac{1+p_1/p_0}{1+\rho_1/\rho_0}\right)\approx\frac{p_0}{\rho_0}\left(1+\frac{p_1}{p_0}-\frac{\rho_1}{\rho_0}\right). \quad (7.42)$$

Note that the perturbed gravitational potential vector Π_i^1 does not appear in (7.39), so that its Poisson equation (7.30) will not be used in the following analysis.

The elimination of the Laplacians of the scalar gravitational potentials U_1 and Φ_1 from (7.39) by the use of (7.40) and (7.41) results in the following differential equation for the perturbed mass density:

$$\frac{\partial^2\rho_1}{\partial t^2} - c_s^2\nabla^2\rho_1\left[1-\frac{1}{c^2}\left(4U_0+\varepsilon_0+\frac{p_0}{\rho_0}\right)\right] - \frac{4\pi G\rho_0^2}{c^2}\left(2U_1 +\varepsilon_1+\frac{3p_1}{\rho_0}\right) - 4\pi G\rho_0\rho_1\left[1-\frac{1}{c^2}\left(2U_0-\varepsilon_0\right)\right]=0. \qquad (7.43)$$

The above equation will turn into a differential equation for the determination of the mass density perturbation if we know the perturbations of the pressure p_1, specific internal energy ε_1 and gravitational potential U_1 as functions of ρ_1. The relationship between U_1 and ρ_1 is given by the Poisson equation (7.40). From the expressions of the sound speed $c_s^2 = dp/d\rho$ and of the specific internal energy $\varepsilon = p/\rho(\gamma-1)$ it is easy to obtain for a polytropic fluid that

$$p_0 = \frac{c_s^2}{\gamma}\rho_0, \qquad p_1 = c_s^2\rho_1, \qquad \varepsilon_0 = \frac{c_s^2}{\gamma(\gamma-1)}, \qquad \varepsilon_1 = \frac{c_s^2}{\gamma}\frac{\rho_1}{\rho_0}. \qquad (7.44)$$

Hence from (7.43) by using (7.44) we obtain the following expression

$$\frac{\partial^2\rho_1}{\partial t^2} - c_s^2\nabla^2\rho_1\left[1-\frac{c_s^2}{c^2}\left(4\frac{U_0}{c_s^2}+\frac{1}{\gamma-1}\right)\right] - 8\pi G\rho_0^2\frac{U_1}{c^2} -4\pi G\rho_0\rho_1\left[1-\frac{c_s^2}{c^2}\left(2\frac{U_0}{c_s^2}-\frac{3\gamma-2}{\gamma-1}\right)\right]=0. \qquad (7.45)$$

We consider now the mass density and the Newtonian gravitational potential as plane waves of frequency ω and wave number vector $\mathbf{k}$ according to

$$\rho_1(\mathbf{x},t) = \overline{\rho} e^{[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]}, \qquad U_1(\mathbf{x},t) = \overline{U} e^{[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]}, \quad (7.46)$$

where $\overline{\rho}$ and $\overline{U}$ represent small amplitudes of the mass density and Newtonian gravitational potential, respectively.

Insertion of the representations (7.46) into the Poisson equation (7.40) leads to a relationship between the mass density and the Newtonian gravitational potential amplitudes

$$\kappa^2 \overline{U} = 4\pi G \overline{\rho}. \quad (7.47)$$

The dispersion relation – which relates the frequency ω to the modulus of the wave number vector $\kappa = \sqrt{\mathbf{k}\cdot\mathbf{k}}$ – is obtained from the insertion of the plane wave representations (7.46) into the differential equation for the mass density perturbation (7.45) and taking into account (7.47), yielding

$$\omega^2 = c_s^2\kappa^2\left[1 - \frac{c_s^2}{c^2}\left(4\frac{U_0}{c_s^2} + \frac{1}{\gamma-1}\right)\right] - 2\left(\frac{4\pi G\rho_0}{\kappa c}\right)^2$$
$$-4\pi G\rho_0\left[1 - \frac{c_s^2}{c^2}\left(2\frac{U_0}{c_s^2} - \frac{3\gamma-2}{\gamma-1}\right)\right]. \quad (7.48)$$

By introducing the dimensionless frequency ω_* and wave number κ_* defined in terms of the Jeans wave number $\kappa_J = \sqrt{4\pi G\rho_0}/c_s$ as

$$\omega_* = \frac{\omega}{\sqrt{4\pi G\rho_0}}, \qquad \kappa_* = \frac{\kappa}{\kappa_J}, \quad (7.49)$$

the dispersion relation (7.48) becomes

$$\omega_*^2 = \kappa_*^2 - 1 - \frac{c_s^2}{c^2}\left[\left(\frac{4U_0}{c_s^2} + \frac{1}{\gamma-1}\right)\kappa_*^2 + \frac{2}{\kappa_*^2} - \frac{2U_0}{c_s^2} + \frac{3\gamma-2}{\gamma-1}\right]. \tag{7.50}$$

We note from the above equation the post-Newtonian contribution to the dispersion relation which is the factor of the ratio c_s^2/c^2. As it should be, without this term (7.50) reduces to the Newtonian dispersion relation (7.11).

The harmonic wave solutions in time are obtained from the real roots of the dispersion relation (7.50), while the instabilities which will grow or decay in time come from the pure imaginary roots of this equation. The Jeans instability refers to the one which grows in time. The value of κ_* where ω_* changes from the pure imaginary value to the real value is obtained by taking $\omega_* = 0$ in (7.50) and if we solve the resulting equation for κ_* by considering only terms up to the $1/c^2$ order, we get

$$\kappa_* = \frac{\kappa}{\kappa_J} = 1 + \frac{c_s^2}{c^2}\left[\frac{5\gamma-3}{2(\gamma-1)} + \frac{U_0}{c_s^2}\right] = \frac{\lambda_J}{\lambda}. \tag{7.51}$$

This equation gives the ratio of the Jeans wavelength and the wavelength of the perturbation.

From (7.51) we can determine the amount of mass which is necessary for an overdensity to initiate the gravitational collapse in the post-Newtonian theory. We recall that in the case of the Newtonian theory it is given by (7.12). Here we build the ratio of the Jeans masses corresponding to the post-Newtonian M_J^{PN} and Newtonian M_J^N wavelengths. The two masses are related to the masses contained in a sphere of radius equal to their corresponding wavelengths. Hence, we have

$$\frac{M_J^{PN}}{M_J^N} = \left(\frac{\lambda}{\lambda_J}\right)^3 = 1 - \frac{c_s^2}{c^2}\left[\frac{3(5\gamma-3)}{2(\gamma-1)} + \frac{3U_0}{c_s^2}\right], \qquad (7.52)$$

by considering terms up to the $1/c^2$ order.

We can infer from (7.52) that the mass necessary to begin the gravitational collapse in the post-Newtonian theory is smaller than the one in the Newtonian theory. The difference between the post-Newtonian and Newtonian Jean masses is small, since it depends on the square of the ratio between the sound and light speeds. Furthermore, the difference between the two masses depend on the polytropic index $n = 1/(\gamma - 1)$ and on the unperturbed Newtonian gravitational potential U_0. From the virial theorem the unperturbed Newtonian gravitational potential can be taken as the square of the velocity dispersion, which is a mean velocity of a group of astronomical objects. Furthermore, a non-vanishing background Newtonian gravitational potential implies in a smaller post-Newtonian Jeans mass in comparison with the one where the background Newtonian gravitational potential is not taken into account.

For a monatomic or Fermi non-relativistic gas $\gamma = 5/3$ and (7.52) becomes

$$\frac{M_J^{PN}}{M_J^N} = 1 - 3\frac{c_s^2}{c^2}\left[4 + \frac{U_0}{c_s^2}\right]. \qquad (7.53)$$

7.4 Post-Newtonian Jeans instability II

In this section we shall investigate the Jeans instability within the framework of the second post-Newtonian Eulerian equations. The balance equations for the mass density and momentum density are given by (3.74) and (3.93), respectively. Furthermore, the Poisson equations for the gravitational potentials are given by (3.47) – (3.51) and the gauge conditions by (3.46) and (3.52).

Here we shall also consider that the fluid is initially at rest with a constant mass density and vanishing hydrodynamic velocity, but for simplicity we take into account that the background values of the gravitational potentials are zero. The representation of these fields are

$$\rho(\mathbf{x},t) = \rho_0 + \rho_1(\mathbf{x},t) \qquad V_i(\mathbf{x},t) = V_i^1(\mathbf{x},t), \tag{7.54}$$

$$\Pi_i(\mathbf{x},t) = \Pi_i^1(\mathbf{x},t), \qquad U(\mathbf{x},t) = U_1(\mathbf{x},t), \tag{7.55}$$

$$\Phi(\mathbf{x},t) = \Phi_1(\mathbf{x},t), \qquad \Psi_{0i}(\mathbf{x},t) = \Psi_{0i}^1(\mathbf{x},t), \tag{7.56}$$

$$\Psi_{ij}(\mathbf{x},t) = \Psi_{ij}^1(\mathbf{x},t), \quad \Psi_{00}(\mathbf{x},t) = \Psi_{00}^1(\mathbf{x},t), \tag{7.57}$$

where the subscripts 1 denote the perturbed values of the fields.

By using the above representations into (3.75), the linearized mass density becomes

$$\widetilde{\rho} = \rho_1 + \rho_0\left[1 + \frac{3U_1}{c^2} - \frac{\Psi_{kk}^1}{2c^4}\right], \tag{7.58}$$

while from from (3.94) the linearized momentum density reads

$$\rho\mathfrak{V}_i = \rho_0 \left[V_i^1 - \frac{\Pi_i^1}{c^2}\right]\left[1 + \frac{1}{c^2}\left(\varepsilon_0 + \frac{p_0}{\rho_0}\right)\right] - \frac{\rho_0}{c^4}\Psi_{0i}^1. \quad (7.59)$$

Now the linearized mass density balance equation (3.74) reduces to

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial V_i^1}{\partial x^i} + \frac{3\rho_0}{c^2}\frac{\partial U_1}{\partial t} - \frac{\rho_0}{2c^4}\frac{\partial \Psi_{kk}^1}{\partial t} = 0, \quad (7.60)$$

and the linearized momentum density balance equation (3.93) becomes

$$\rho_0 \left[1 + \frac{1}{c^2}\left(\varepsilon_0 + \frac{p_0}{\rho_0}\right)\right]\left[\frac{\partial V_i^1}{\partial t} - \frac{\partial U_1}{\partial x^i} - \frac{2}{c^2}\frac{\partial \Phi_1}{\partial x^i} - \frac{1}{c^2}\frac{\partial \Pi_i^1}{\partial t}\right]$$
$$+ c_s^2 \frac{\partial \rho_1}{\partial x^i} + \frac{\rho_0}{c^4}\left(\frac{1}{2}\frac{\partial \Psi_{00}^1}{\partial x^i} - \frac{\partial \Psi_{0i}^1}{\partial t}\right) = 0. \quad (7.61)$$

For the gravitational potentials we shall invoke the "Jeans swindle" and consider that their Poisson equations are valid only for the perturbed fields. Hence, it follows from (3.47), (3.48), (3.50) and (3.51) that

$$\nabla^2 U_1 = -4\pi G\rho_1, \; \nabla^2 \Pi_i^1 = -16\pi G\rho_0 V_i^1 + \frac{\partial^2 U_1}{\partial t \partial x^i}, \quad (7.62)$$

$$\nabla^2 \Phi_1 = -2\pi G\varepsilon_0\rho_1 - 4\pi G\rho_0\left(U_1 + \frac{\varepsilon_1}{2} + \frac{3p_1}{2\rho_0}\right), \quad (7.63)$$

$$\nabla^2 \Psi_{00}^1 = -32\pi G\rho_0\Phi_1, \quad (7.64)$$

$$\nabla^2 \Psi_{0i}^1 = -16\pi G\rho_0\left[V_i^1\left(\varepsilon_0 + \frac{p_0}{\rho_0}\right) - \frac{\Pi_i^1}{2}\right]. \quad (7.65)$$

In (7.63) we have used the relationship (7.42).

The linearized gauge conditions (3.46) and (3.52) read

$$3\frac{\partial U_1}{\partial t}+\frac{\partial \Pi_i^1}{\partial x^i}+\frac{1}{c^2}\left[\frac{\partial \Psi_{0j}^1}{\partial x^j}-\frac{1}{2}\frac{\partial \Psi_{kk}^1}{\partial t}\right]=0, \quad (7.66)$$

$$\nabla^2\Psi_{kk}^1=32\pi G\varepsilon_0\rho_1+32\pi G\rho_0\left(4U_1+\varepsilon_1\right). \quad (7.67)$$

As in the previous section we have to evaluate the perturbed specific internal energy. For that end we shall rely on the result which comes out from the kinetic theory of relativistic monatomic gases for the specific internal energy, namely (see e.g. [8])

$$\varepsilon=\frac{3kT}{2m}\left(1+\frac{5kT}{4mc^2}\right), \quad (7.68)$$

which by using the relationship $\varepsilon=\frac{1}{\gamma-1}\frac{p}{\rho}$ can be rewritten as

$$\varepsilon=\frac{1}{\gamma-1}\frac{p}{\rho}\left(1+\frac{5}{6(\gamma-1)}\frac{p}{c^2\rho}\right). \quad (7.69)$$

Hence, the following relationships hold

$$\varepsilon_0=\frac{c_s^2}{\gamma(\gamma-1)}\left(1+\frac{5}{6\gamma(\gamma-1)}\frac{c_s^2}{c^2}\right), \quad (7.70)$$

$$\varepsilon_1=\frac{c_s^2}{\gamma}\left(1+\frac{5}{3\gamma(\gamma-1)}\frac{c_s^2}{c^2}\right)\frac{\rho_1}{\rho_0}, \quad (7.71)$$

thanks to (7.42) and (7.44).

By following the same methodology developed in the previous sections we take the time derivative of the mass density balance equation (7.60) resulting

$$\frac{\partial^2\rho_1}{\partial t^2}+\rho_0\frac{\partial^2 V_i^1}{\partial t\partial x^i}+\frac{3\rho_0}{c^2}\frac{\partial^2 U_1}{\partial t^2}-\frac{\rho_0}{2c^4}\frac{\partial^2\Psi_{kk}^1}{\partial t^2}=0. \qquad (7.72)$$

Next the divergence of the momentum density balance equation (7.61), yields

$$\rho_0\left[1+\frac{1}{c^2}\left(\varepsilon_0+\frac{p_0}{\rho_0}\right)\right]\left[\frac{\partial^2 V_i^1}{\partial t\partial x^i}-\nabla^2 U_1-\frac{2}{c^2}\nabla^2\Phi_1\right.$$
$$\left.-\frac{1}{c^2}\frac{\partial^2\Pi_i^1}{\partial t\partial x^i}\right]+c_s^2\nabla^2\rho_1+\frac{\rho_0}{c^4}\left(\frac{1}{2}\nabla^2\Psi_{00}^1-\frac{\partial^2\Psi_{0i}^1}{\partial t\partial x^i}\right)=0, \quad (7.73)$$

whose division by $\left[1+\frac{1}{c^2}\left(\varepsilon_0+\frac{p_0}{\rho_0}\right)\right]$ leads to an equation, which by considering terms up to the $1/c^4$ order, reads

$$\rho_0\left[\frac{\partial^2 V_i^1}{\partial t\partial x^i}-\nabla^2 U_1-\frac{2}{c^2}\nabla^2\Phi_1-\frac{1}{c^2}\frac{\partial^2\Pi_i^1}{\partial t\partial x^i}\right]$$
$$+c_s^2\nabla^2\rho_1\left[1-\frac{1}{c^2}\left(\varepsilon_0+\frac{p_0}{\rho_0}\right)+\frac{1}{c^4}\left(\varepsilon_0+\frac{p_0}{\rho_0}\right)^2\right]$$
$$+\frac{\rho_0}{c^4}\left(\frac{1}{2}\nabla^2\Psi_{00}^1-\frac{\partial^2\Psi_{0i}^1}{\partial t\partial x^i}\right)=0. \qquad (7.74)$$

Equation (7.74) is used to eliminate the velocity derivatives from (7.72), yielding

$$\frac{\partial^2\rho_1}{\partial t^2}-c_s^2\left[1-\frac{1}{c^2}\left(\varepsilon_0+\frac{p_0}{\rho_0}\right)+\frac{1}{c^4}\left(\varepsilon_0+\frac{p_0}{\rho_0}\right)^2\right]\nabla^2\rho_1$$

$$+\underline{\rho_0 \frac{\partial}{\partial t}\left[3\frac{\partial U_1}{\partial t} + \frac{\partial \Pi_i^1}{\partial x^i} + \frac{1}{c^2}\left(\frac{\partial \Psi_{0j}^1}{\partial x^j} - \frac{1}{2}\frac{\partial \Psi_{kk}^1}{\partial t}\right)\right]}$$

$$+\rho_0\left(\nabla^2 U_1 + \frac{2}{c^2}\nabla^2\Phi_1\right) - \frac{\rho_0}{2c^4}\nabla^2\Psi_{00}^1 = 0. \qquad (7.75)$$

Note that the underlined term vanishes, thanks to the gauge condition (7.66).

In order to obtain the dispersion relation we assume the plane wave representation of the perturbed fields

$$\rho_1(\mathbf{x}, t) = \overline{\rho} e^{[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]}, \qquad U_1(\mathbf{x}, t) = \overline{U} e^{[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]}, \quad (7.76)$$

$$\Phi_1(\mathbf{x}, t) = \overline{\Phi} e^{[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]}, \quad \Psi_{00}^1(\mathbf{x}, t) = \overline{\Psi}_{00} e^{[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]}, \quad (7.77)$$

where $\overline{\rho}$, $\overline{U}$, $\overline{\Phi}$ and $\overline{\Psi}_{00}$ are small amplitudes.

Insertion of the plane wave representations (7.76) and (7.77) into (7.75) leads to

$$\left\{\omega^2 - c_s^2\kappa^2\left[1 - \frac{1}{c^2}\left(\varepsilon_0 + \frac{p_0}{\rho_0}\right) + \frac{1}{c^4}\left(\varepsilon_0 + \frac{p_0}{\rho_0}\right)^2\right]\right\}\overline{\rho}$$

$$+\rho_0\kappa^2\left(\overline{U} + \frac{2\overline{\Phi}}{c^2} - \frac{\overline{\Psi}_{00}}{2c^4}\right) = 0. \qquad (7.78)$$

We have to eliminate from (7.78) the amplitudes $\overline{U}$, $\overline{\Phi}$ and $\overline{\Psi}_{00}$ and for that end we use the Poisson equations (7.62) – (7.64) together with (7.76) and (7.77), resulting

$$\kappa^2\overline{U} = 4\pi G\overline{\rho}, \qquad \kappa^2\overline{\Psi}_{00} = 32\pi G\rho_0\overline{\Phi}, \qquad (7.79)$$

$$\kappa^2\overline{\Phi} = 4\pi G\rho_0\overline{U} + 2\pi G\frac{c_s^2}{\gamma - 1}\left[3\gamma - 2 + \frac{5(2\gamma - 1)}{6\gamma^2(\gamma - 1)}\frac{c_s^2}{c^2}\right]\overline{\rho}, (7.80)$$

where the relationships (7.70) and (7.71) were taken into account.

The dispersion relation follows from (7.78) together with (7.49), (7.70), (7.79) and (7.80), yielding

$$\omega_*^2 = \kappa_*^2 - 1 - \frac{c_s^2}{c^2}\left[\frac{\kappa_*^2}{\gamma-1} + \frac{2}{\kappa_*^2} + \frac{3\gamma-2}{\gamma-1}\right]$$
$$-\frac{c_s^4}{c^4}\left[\frac{(5-6\gamma)\kappa_*^2}{6\gamma^2(\gamma-1)^2} - \frac{2(3\gamma-2)}{(\gamma-1)\kappa_*^2} + \frac{5(2\gamma-1)}{6(\gamma-1)^2\gamma^2} - \frac{4}{\kappa_*^4}\right]. \quad (7.81)$$

In the above equation we have considered only the terms up to the $1/c^4$ order.

By taking $\omega_* = 0$ in (7.81) we get the value of κ_* where ω_* changes from the real value – corresponding to harmonic waves – to the pure imaginary value – corresponding to growing (Jeans instability) or decaying waves. By solving the resulting equation for κ_* when $\omega_* = 0$ and consider only terms up to the $1/c^4$ order, we get

$$\kappa_* = 1 + \frac{5\gamma-3}{2(\gamma-1)}\frac{c_s^2}{c^2} + \frac{20-9\gamma(\gamma-1)(35\gamma-27)}{24\gamma(\gamma-1)^2}\frac{c_s^4}{c^4}, \quad (7.82)$$

which shows the contribution of the second post-Newtonian approximation to the dimensionless modulus of the wave number.

The contribution of the term c_s^4/c^2 is negative, since if we choose $\gamma = 5/3$ which corresponds to a monatomic or a Fermi non-relativistic gas we get

$$\kappa_* = 1 + 4\frac{c_s^2}{c^2} - \frac{33}{2}\frac{c_s^4}{c^4}. \quad (7.83)$$

In this case the Jeans mass becomes

$$\frac{M_J^{PN}}{M_J^N} = \left(\frac{\lambda}{\lambda_J}\right)^3 = 1 - 12\frac{c_s^2}{c^2} + \frac{291}{2}\frac{c_s^4}{c^4}. \tag{7.84}$$

Although the above equation shows that the Jeans mass in the second post-Newtonian approximation is larger than the one in the first post-Newtonian approximation, the difference between the two is very small. Indeed, for the ratio $c_s/c \simeq 10^{-1}$ the difference is only 1.6%

Appendix

The cosmological models are based on the Cosmological Principle that asserts that at large scales the Universe is spatially homogeneous and isotropic. The homogeneity refers to the property that the Universe is identically uniform at any place while in the isotropy property the uniformity is identical in all directions. The solution of Einstein's field equations for a homogeneous and isotropic Universe is given by the metric derived by Friedmann–Lemaître–Robertson–Walker (FLRW metric). Here we shall consider the FLRW metric for a spatially flat Universe where the line element is given by $ds^2 = (cdt)^2 - a(t)^2\left(dx^2 + dy^2 + dz^2\right)$, with $a(t)$ denoting the cosmic scale factor. We recall that Einstein's field equations are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}. \tag{7.85}$$

If we consider that the source of the gravitational field is a perfect fluid the energy-momentum tensor reads

$$T^{\mu\nu} = \left(\rho c^2 + p\right)\frac{U^\mu U^\nu}{c^2} - pg^{\mu\nu}. \tag{7.86}$$

For the FLRW spatially flat metric the non-vanishing components of the Christoffel symbols read

$$\Gamma^0{}_{11} = \Gamma^0{}_{22} = \Gamma^0{}_{33} = a'a, \quad \Gamma^1{}_{01} = \Gamma^2{}_{02} = \Gamma^3{}_{03} = \frac{a'}{a}, \tag{7.87}$$

where the prime refers to a differentiation with respect to $x^0 = ct$. The curvature scalar and the components of the Ricci tensor for the FLRW spatially flat metric are given by

$$R = 6\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right), \; R_{00} = 3\frac{a''}{a}, \tag{7.88}$$

$$R_{11} = R_{22} = R_{33} = -2a'^2 - a''a. \tag{7.89}$$

From the time and spatial components of Einstein's field equations it follows a coupled system of differential equations

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right), \tag{7.90}$$

where the dot represents the differentiation with respect to the proper time t. The above equations are known as the Friedmann and acceleration equations, respectively.

By differentiating the Friedmann equation $(7.90)_1$ with respect to time and eliminating the acceleration term by using

$(7.90)_2$ it follows the equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{7.91}$$

In the analysis of the matter dominated phase of the Universe the pressure is negligible in comparison of the mass-energy density and we have a "dust Universe" where $p \approx 0$. In this case the equations $(7.90)_2$ and (7.91) reduces to

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho, \qquad \dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0. \tag{7.92}$$

The integration of the last above equation furnishes a relationship between the mass-energy density and the scale factor which reads

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3. \tag{7.93}$$

The insertion of (7.93) into the Friedman equation (7.90) and subsequent integration of the resulting equation implies the knowledge of the dependence of the scale factor with time, namely

$$a = a_0 \left(6\pi G\rho_0 t^2\right)^{\frac{1}{3}}. \tag{7.94}$$

Hence it follows from (7.93) and (7.94) the dependence of the mass-energy density with respect to time $\rho = 1/(6\pi G t^2)$.

If we relate the observable measured coordinates $\mathbf{x}$ – known as the physical or proper coordinates – with the comoving coordinates $\mathbf{x}_0$ by $\mathbf{x}(t) = a(t)\mathbf{x}_0$, we obtain by differentiating it with respect to time – and considering that the peculiar velocities are absent – the Hubble-Lamaître's law $\mathbf{V} = (\dot{a}/a)\mathbf{x}$.

References

[1] J. H. Jeans, The Stability of a Spherical Nebula, *Philos. Trans. R. Soc. A* **199**, 1 (1902).

[2] W. B. Bonnor, Jeans' formula for gravitational instability, *Mon. Not. R. Astr. Soc.* **117**, 104 (1957).

[3] S. Weinberg, *Gravitation and Cosmology. Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[4] P. Coles and F. Lucchin, *Cosmology. The Origin and Evolution of Cosmic Structures* (Wiley, Chichester,2002).

[5] G. M. Kremer, M. G. Richarte and F. Teston, Jeans Instability in a Universe with Dissipation, *Phys. Rev. D* **97**, 023515 (2018).

[6] G. M. Kremer, *An introduction to the Boltzmann equation and transport processes in gases* (Springer, Berlin, 2010).

[7] E. Nazari, A. Kazemi, M. Roshan and S. Abbassi, Post-Newtonian Jeans analysis, *Ap. J.* **839**, 839 (2017).

[8] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann equation: theory and applications* (Birkhäuser, Basel, 2002).

CHAPTER 8

JEANS INSTABILITY: BOLTZMANN EQUATION

The aim of this chapter is to investigate the Jeans instability for self-gravitating gases within the framework of a kinetic theory based on the Boltzmann equation which is coupled with the Poisson equation. By considering perturbations of the one-particle distribution function and the gravitational potential from an equilibrium state, it is possible to derive a dispersion relation where the Jeans instability can be determined. Two analysis are developed here, in one the perturbed one-particle distribution function is left unspecified [1, 2] while in the other it is supposed as a function of the summational invariants [3].

8.1 Jeans instability for a single gas

We start by analyzing the Jeans instability of a self-gravitating single gas described by the collisionless Boltzmann and Poisson equations

$$\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial x^i}+\frac{\partial U}{\partial x^i}\frac{\partial f}{\partial v^i}=0, \qquad \nabla^2 U=-4\pi G\int mfd^3v, \tag{8.1}$$

respectively, which can be seen as a coupled system of differential equations for the determination of the one-particle distribution function $f(\mathbf{x},\mathbf{v},t)$ and Newtonian gravitational potential $U(\mathbf{x},t)$.

To analyze the gas instabilities we shall write the distribution function and the Newtonian gravitational potential as a sum of background terms denoted by the subscript zero and perturbed terms with the subscript 1, i.e.,

$$f(\mathbf{x},\mathbf{v},t)=f_0(\mathbf{v})+\epsilon f_1(\mathbf{x},\mathbf{v},t), \quad U(\mathbf{x},t)=U_0(\mathbf{x})+\epsilon U_1(\mathbf{x},t), \tag{8.2}$$

where we have introduced a small parameter ϵ multiplying the perturbed quantities to control that only linear terms in this parameter should be taken into account. The background distribution function is the Maxwellian one

$$f_0(\mathbf{v}) = \frac{\rho_0}{m}\frac{e^{-v^2/2\sigma^2}}{(2\pi\sigma^2)^{3/2}}, \tag{8.3}$$

which depends only on the gas particle velocity $\mathbf{v}$. The mass density ρ_0 and the temperature T_0 of the background are constants and $\sigma = \sqrt{kT_0/m}$ denotes the velocity (thermal) dispersion of the self-gravitating gas.

If we insert the representations (8.2) into the collisionless Boltzmann equation $(8.1)_1$ and equate the terms of the same ϵ–order we get two hierarchy equations:

$$\frac{\partial U_0}{\partial x^i}\frac{\partial f_0}{\partial v^i} = 0, \qquad \frac{\partial f_1}{\partial t} + v_i\frac{\partial f_1}{\partial x^i} + \frac{\partial U_1}{\partial x^i}\frac{\partial f_0}{\partial v^i} + \frac{\partial U_0}{\partial x^i}\frac{\partial f_1}{\partial v^i} = 0. \tag{8.4}$$

The first equation $(8.4)_1$ is satisfied if the background Newtonian gravitational potential of the self-gravitating gas does not depend on the spatial coordinates. This condition follows also from symmetry considerations, because in a homogeneous system there is no preference in the direction of the gravitational potential gradient. However, the condition $\nabla U_0 = 0$ does not satisfy the Poisson equation $(8.1)_2$ due to the fact that its right-hand side refers to the mass density of the self-gravitating system. In order to remove this inconsistency we again make use of the "Jeans swindle", which considers that the Poisson equation is valid only for the perturbed one-particle distribution function and perturbed gravitational potential, i.e.

$$\nabla^2 U_1 = -4\pi G \int m f_1 d^3v. \tag{8.5}$$

We represent the perturbations of the distribution function and of the Newtonian gravitational potential in terms of plane waves of frequency ω and wave number vector $\mathbf{k}$ as follows:

$$f_1(\mathbf{x},\mathbf{v},t)=\overline{f}_1 f_0(\mathbf{v})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \quad U_1(\mathbf{x},t)=\overline{U}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{8.6}$$

where $\overline{f}_1$ and $\overline{U}_1$ are the corresponding amplitudes which are considered to be small.

Insertion of the representations (8.6) into $(8.4)_2$ leads to a relationship between the amplitudes

$$(\mathbf{v}\cdot\mathbf{k}-\omega)\,\overline{f}_1-\frac{\mathbf{v}\cdot\mathbf{k}}{\sigma^2}\overline{U}_1=0. \tag{8.7}$$

Furthermore, the Poisson equation (8.5) with (8.6) becomes

$$\kappa^2\overline{U}_1=4\pi G\int m f_0\overline{f}_1 d^3v, \tag{8.8}$$

where $\kappa=\sqrt{\mathbf{k}\cdot\mathbf{k}}$ is the modulus of the wave number vector.

If we eliminate $\overline{f}_1$ from the Poisson equation (8.8) by using (8.7) we get

$$\kappa^2=\frac{4\pi G\rho_0}{(2\pi\sigma^2)^{\frac{3}{2}}\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{v_x^2+v_y^2+v_z^2}{2\sigma^2}}\mathbf{v}\cdot\mathbf{k}}{\mathbf{v}\cdot\mathbf{k}-\omega}dv_x dv_y dv_z, \tag{8.9}$$

Here we introduced Maxwellian distribution function (8.3) and the integrals in the velocity components $-\infty<(v_x,v_y,v_z)<\infty$.

Without loss of generality we can suppose the wave number vector in the x direction, i.e., $\mathbf{k}=(\kappa,0,0)$. The integration of

(8.9) with respect to the velocity components $-\infty < (v_y, v_z) < \infty$ leads to

$$\kappa^2 = \frac{4\pi G\rho_0}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty}\frac{e^{-\frac{v_x^2}{2\sigma^2}}v_x\kappa\left(v_x\kappa+\omega\right)}{v_x^2\kappa^2-\omega^2}dv_x. \tag{8.10}$$

Here we have multiplied the nominator and denominator of the integrand by $(v_x\kappa + \omega)$. Note that the integral of the term which is linear (odd) in v_x vanishes and the remaining integral in v_x can be written as

$$\kappa^2 = \frac{4\pi G\rho_0}{\sigma^2}I_2 \tag{8.11}$$

where I_2 is defined in terms of the integrals I_n given by

$$I_n(\kappa,\omega) = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{x^n e^{-x^2}}{x^2-(\omega/\sqrt{2}\sigma\kappa)^2}dx, \quad \text{where} \quad x = \frac{v_x}{\sqrt{2}\sigma}. \tag{8.12}$$

Equation (8.11) is a dispersion relation, since it relates the modulus of the wave number vector with the frequency $\kappa = \kappa(\omega)$.

The unstable solutions are those where $\Re(\omega) = 0$ and $\Im(\omega) > 0$, since in this case the solutions grow exponentially with time. By considering $\omega = i\omega_I$ with $\omega_I > 0$ the integrals I_n can be evaluated and (8.11) becomes

$$\kappa^2 = \frac{4\pi G\rho_0}{\sigma^2}\left[1-\sqrt{\frac{\pi}{2}}\frac{\omega_I}{\kappa\sigma}\exp\left(\frac{\omega_I^2}{2\kappa^2\sigma^2}\right)\operatorname{erfc}\left(\frac{\omega_I}{\sqrt{2}\kappa\sigma}\right)\right], \tag{8.13}$$

where erfc(x) denotes the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-x^2} dx. \tag{8.14}$$

If we introduce the Jeans wavelength $\kappa_J = \sqrt{4\pi G\rho_0}/\sigma$ with respect to the velocity dispersion σ and the following dimensionless quantities related with the frequency and modulus of the wave number vector

$$\omega_* = \frac{\omega_I}{\kappa_J \sigma}, \qquad \kappa_* = \frac{\kappa}{\kappa_J}, \tag{8.15}$$

we can express the dispersion relation (8.13) as

$$\kappa_*^2 = 1 - \sqrt{\frac{\pi}{2}} \frac{\omega_*}{\kappa_*} \exp\left(\frac{\omega_*^2}{2\kappa_*^2}\right) \operatorname{erfc}\left(\frac{\omega_*}{\sqrt{2}\kappa_*}\right). \tag{8.16}$$

In Figure 8.1 it is shown the contour plot of the dimensionless frequency as function of the dimensionless modulus of the wave vector. Two limiting cases can be inferred from this figure: (i) when the dimensionless frequency vanishes $\omega_* = 0$ the modulus of the wave number vector becomes equal to the Jeans wave number $\kappa = \kappa_J$ which is related to the limiting value of the frequency where the instability occurs and the corresponding minimum mass – the Jeans mass – for an overdensity to begin the gravitational collapse; (ii) the dimensionless frequency tends to one when the modulus of the wave number vector tends to zero. Both cases correspond to the Newtonian analysis described in Section 7.1 (see equation (7.11)).

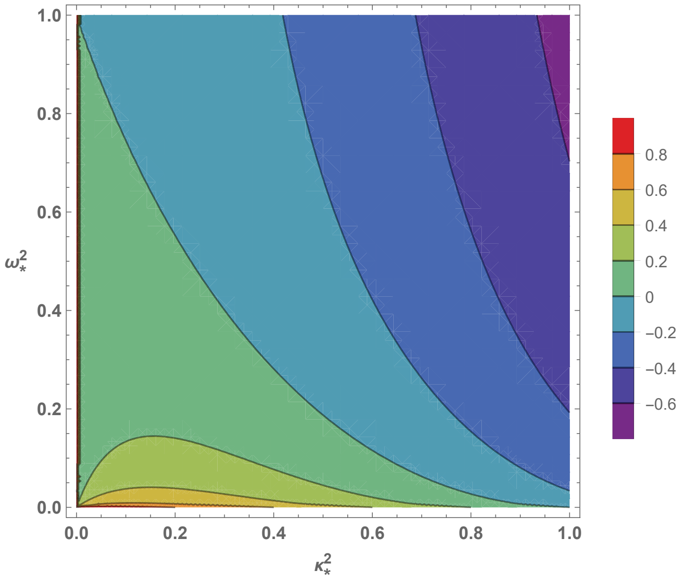

Figure 8.1: Contour plot of the dimensionless frequency as function of the dimensionless modulus of the wave number vector.

Another methodology to solve Jeans instability from Boltzmann equation was proposed in [3] and makes use of the summational invariants which are related with the rest mass m, the momentum $m\mathbf{v}$ and the energy $m\mathbf{v}^2/2$ of a particle. In this analysis the amplitude $\overline{f}_1$ is written as a linear combination of the summational invariants, namely

$$\overline{f}_1 = A + \mathbf{B} \cdot \mathbf{v} + Dv^2, \tag{8.17}$$

where A, $\mathbf{B}$ and D are unknowns quantities that do not depend on $\mathbf{v}$.

If we insert the representations (8.6) and (8.17) together with (8.3) into the perturbed Boltzmann $(8.4)_2$ and Poisson (8.3) equations we get

$$f_0 \left[\left(A + \mathbf{B} \cdot \mathbf{v} + D\mathbf{v}^2\right)(\mathbf{k} \cdot \mathbf{v} - \omega) - \mathbf{k} \cdot \mathbf{v} \frac{\overline{U}_1}{\sigma^2} \right] = 0, \tag{8.18}$$

$$\kappa^2 \overline{U}_1 = 4\pi G \int m f_0 \big(A + \mathbf{B} \cdot \mathbf{v} + Dv^2\big) d^3 v$$
$$= 4\pi G \rho_0 \big(A + 3\sigma^2 D\big). \tag{8.19}$$

We have performed the integration of the last equation above by using the formulas for the Gaussian integrals given in the Appendix A of Chapter 4 and introduced $\kappa = |\mathbf{k}|$.

Now we can build a system of algebraic equations for A, $\mathbf{B}\cdot\mathbf{k}$, D and $\overline{U}_1$. Indeed, if we multiply (8.18) by each of the summational invariants $\big(1, \mathbf{v}, \mathbf{v}^2\big)$ and integrate the resulting equations by using the Gaussian integrals we get respectively

$$\omega \left(A + 3\sigma^2 D\right) - \sigma^2 \mathbf{B} \cdot \mathbf{k} = 0, \tag{8.20}$$

$$\omega\, \mathbf{B} \cdot \mathbf{k} - \left[A + 5\sigma^2 D - \frac{\overline{U}_1}{\sigma^2} \right] \kappa^2 = 0, \tag{8.21}$$

$$\omega \left(3A + 15\sigma^2 D\right) - 5\sigma^2 \mathbf{B} \cdot \mathbf{k} = 0. \tag{8.22}$$

Note that (8.21) results from the scalar multiplication of the vector equation for $\mathbf{B}$ by $\mathbf{k}$ and by using $\kappa^2 = \mathbf{k} \cdot \mathbf{k}$.

Equations (8.19) – (8.22) represent a system of algebraic equations for A, $\mathbf{B} \cdot \mathbf{k}$, D and $\overline{U}_1$. It has a non-trivial solution if the determinant of the coefficients A, B, D and $\overline{U}_1$ vanishes, which implies the dispersion relation:

$$\omega^2 = c_s^2 \kappa_J^2 \left(\frac{\kappa^2}{\kappa_J^2} - 1 \right), \tag{8.23}$$

which is the same dispersion relation which comes from the Newtonian analysis of the Jeans instability of Section 7.1 given by (7.11).

The previous analyses were based on a collisionless Boltzmann equation where the evolution of the one-particle distribution function in the phase space does not consider the particle collisions. In the theory of the Boltzmann equation the collisional term is responsible for the irreversible processes characterized by the viscous and heat conduction effects. A question which is important to answer refers to the modifications introduced by the irreversible processes in the analysis of the Jeans instability. This problem was analyzed in [4] within the framework of a collisional Boltzmann equation and recently in [5] from a phenomenological theory based on the balance equations. Here we shall investigate the same problem by using the collisional Boltzmann equation with the same methodology described above, which makes use of the summational invariants.

We start by writing the BGK model of the Boltzmann equation where the structure of the collision term is simplified but preserves the basic properties of the full Boltzmann equation. The collision term in the BGK model is given in terms of the difference between the one-particle distribution function and the Maxwellian one multiplied by a frequency ν which is of order of the collision frequency. In this case the Boltzmann equation reads (see e.g. [6])

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + \frac{\partial U}{\partial x^i}\frac{\partial f}{\partial v^i} = -\nu(f - f_0). \tag{8.24}$$

As in the previous analysis the Boltzmann equation (8.24) is linked with the Poisson equation

$$\nabla^2 U = -4\pi G \int m f d^3 v. \tag{8.25}$$

The Boltzmann and the Poisson equations for the perturbed one-particle distribution function f_1 and perturbed Newtonian gravitational potential U_1 are

$$\frac{\partial f_1}{\partial t} + v_i \frac{\partial f_1}{\partial x^i} + \frac{\partial U_1}{\partial x^i}\frac{\partial f_0}{\partial v^i} = -\nu f_1, \quad \nabla^2 U_1 = -4\pi G \int m f_1 d^3 v. \tag{8.26}$$

Now we represent the perturbations in terms of plane waves with wave number vector $\mathbf{k}$ and time dependent amplitudes, namely

$$f_1(\mathbf{x}, \mathbf{v}, t) = \overline{f}_1(t) f_0(\mathbf{v}) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad U_1(\mathbf{x}, t) = \overline{U}_1(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{8.27}$$

where the amplitude $\overline{f}_1(t)$ is a linear combination of the summational invariants

$$\overline{f}_1(t) = A(t) + \mathbf{B}(t)\cdot\mathbf{v} + D(t)v^2. \tag{8.28}$$

Insertion of the representations (8.27) and (8.28) into the perturbed equations (8.26) imply

$$\frac{dA}{dt} + \frac{d\mathbf{B}}{dt}\cdot\mathbf{v} + \frac{dD}{dt}v^2 + i(\mathbf{v}\cdot\mathbf{k})\left[A + \mathbf{B}\cdot\mathbf{v} + Dv^2 - \frac{\overline{U}_1}{\sigma^2}\right]$$
$$= -\nu\left(A + \mathbf{B}\cdot\mathbf{v} + Dv^2\right), \tag{8.29}$$

$$\kappa^2\overline{U}_1 = 4\pi G\int m\left(A + \mathbf{B}\cdot\mathbf{v} + Dv^2\right)d^3v$$
$$= 4\pi G\rho_0\left(A + 3\sigma^2 D\right). \tag{8.30}$$

Following the same methodology above we multiply (8.29) by the summational invariants $(1, \mathbf{v}, v^2)$ and integrate the resulting equations by using the Gaussian integrals given in the Appendix A of Chapter 4, yielding

$$\frac{dA}{dt} + 3\sigma^2\frac{dD}{dt} + i\sigma^2\mathbf{B}\cdot\mathbf{k} = -\nu\left(A + 3\sigma^2 D\right), \tag{8.31}$$

$$\frac{d\mathbf{B}}{dt} + i\mathbf{k}\left(A + 5\sigma^2 D - \frac{\overline{U}_1}{\sigma^2}\right) = -\nu\mathbf{B}, \tag{8.32}$$

$$3\frac{dA}{dt} + 15\sigma^2\frac{dD}{dt} + i5\sigma^2\mathbf{B}\cdot\mathbf{k} = -\nu\left(3A + 15\sigma^2 D\right). \tag{8.33}$$

First, we subtract (8.33) from (8.31) multiplied by 5 and get

$$\frac{dA}{dt} = -\nu A, \qquad \text{hence} \qquad A = e^{-\nu t}. \tag{8.34}$$

Here we introduce the density contrast which is a parameter that indicates where a local increase in the matter density occurs. It is defined by the ratio of perturbed and unperturbed mass densities and given by

$$\delta\rho = \frac{\int m f_0 \overline{f}_1 d^3v}{\rho_0} = A + 3\sigma^2 D. \tag{8.35}$$

In terms of the density contrast (8.31) becomes

$$\frac{d\delta\rho}{dt} + i\sigma^2 \mathbf{B}\cdot\mathbf{k} = -\nu\delta\rho, \quad \text{and} \quad \frac{d^2\delta\rho}{dt^2} + i\sigma^2\frac{d\mathbf{B}}{dt}\cdot\mathbf{k} = -\nu\frac{d\delta\rho}{dt}, \tag{8.36}$$

where the last equation is the differentiation with respect to time of the first one.

We eliminate $\mathbf{B}$ and its time derivative from $(8.36)_2$ by using (8.32) and $(8.36)_1$, resulting

$$\frac{d^2\delta\rho}{dt^2} + \kappa^2\sigma^2\left(A + 5\sigma^2 D - \frac{\overline{U}_1}{\sigma^2}\right) + \nu\left(\frac{d\delta\rho}{dt} + \nu\delta\rho\right) = -\nu\frac{d\delta\rho}{dt}. \tag{8.37}$$

Finally the elimination from (8.37) of the amplitudes $\overline{U}_1$, A and D by using (8.30), (8.34) and (8.35), respectively, yields

$$\frac{d^2\delta\rho}{dt^2} + 2\nu\frac{d\delta\rho}{dt} - \frac{2}{3}\kappa^2\sigma^2 e^{-\nu t} + \left(\nu^2 - 4\pi G\rho_0 + \frac{5}{3}\kappa^2\sigma^2\right)\delta\rho = 0. \tag{8.38}$$

If we introduce the dimensionless quantities

$$\tau = t\sqrt{4\pi G\rho_0}, \qquad \nu_* = \frac{\nu}{\sqrt{4\pi G\rho_0}}, \qquad \kappa_J = \sqrt{\frac{4\pi G\rho_0}{\frac{5}{3}\sigma^2}}, \tag{8.39}$$

which are related to a dimensionless time, a dimensionless collision frequency, and Jeans wave number, respectively, the differential equation (8.38) becomes

$$\frac{d^2\delta\rho}{d\tau^2}+2\nu_*\frac{d\delta\rho}{d\tau}-\frac{2}{5}\frac{\kappa^2}{\kappa_J^2}e^{-\nu_*\tau}+\left(\nu_*^2-1+\frac{\kappa^2}{\kappa_J^2}\right)\delta\rho=0. \quad (8.40)$$

The solution of the differential equation (8.40) is

$$\delta\rho=\left[C_1\exp\left(\tau\sqrt{1-\frac{\lambda_J^2}{\lambda^2}}\right)+C_2\exp\left(-\tau\sqrt{1-\frac{\lambda_J^2}{\lambda^2}}\right)\right.$$
$$\left.+\frac{2\frac{\lambda_J^2}{\lambda^2}}{5\left(\frac{\lambda_J^2}{\lambda^2}-1\right)}\right]e^{-\nu_*\tau}, \quad (8.41)$$

where we have introduced the wavelength λ and Jeans' wavelength λ_J by considering the relationship $\kappa/\kappa_J=\lambda_J/\lambda$.

Let us analyse the solution (8.41) of the differential equation. We note that for wavelength values bigger than the Jeans wavelength $\lambda/\lambda_J>1$ the terms in the exponential are real. The first exponential will grow with time implying a growth of the density contrast and the Jeans instability shows up. When the wavelength values are smaller than the Jeans wavelength $\lambda/\lambda_J<1$ the term in the exponential is imaginary and the density contrast has an oscillatory character. These behaviors are modulated by the factor $e^{-\nu_*\tau}$ which implies that the growth and oscillatory behaviors become smaller when the collision frequency is taken into account, i.e. we may associate the collision frequency with a damped effect on the solution of the density contrast.

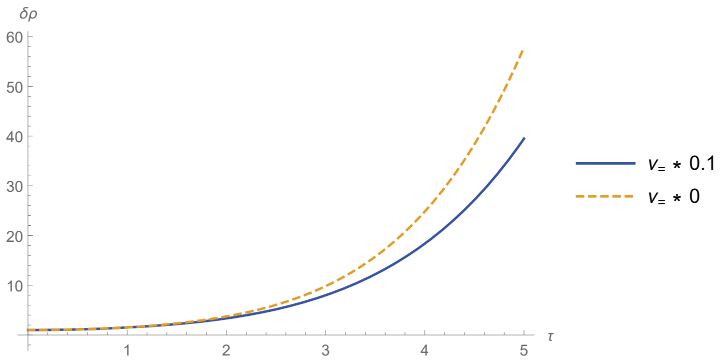

Figure 8.2: Density contrast $\delta\rho$ as function of the dimensionless time τ for $\lambda/\lambda_J = 10$, for the cases with ($\nu_* = 0.1$) and without ($\nu_* = 0$) collision frequency.

The differential equation (8.40) was solved numerically for the initial conditions $\delta\rho(0) = 1$ and $\delta\rho'(0) = 0$ for two values of the dimensionless collision frequency $\nu_* = 0$ and $\nu_* = 0.1$ that show the Jeans instability. Here the ratio of the wavelengths was $\lambda/\lambda_J = 10$. In Figure 8.2 the density contrast $\delta\rho$ is plotted as function of the dimensionless time τ. The straight line corresponds to the case $\nu_* = 0.1$ where the collisions are taken into account while the dashed line represents a collisionless Boltzmann equation where $\nu_* = 0$. Note that in the case where the collisions are considered the growth of the density contrast is less accentuated than the one that corresponds to the collisionless Boltzmann equation, since an energy dissipation comes out due to the particle collisions.

8.2 Jeans instability for systems of two fluids

At the present time it is known that the matter content of the Universe is composed by baryonic matter and dark matter. The baryonic matter consists of atoms of all categories while dark matter refers to a still unknown component which does not emit or interact with electromagnetic radiation. For the process of structure formation it is consider that cold dark matter is consisted of weakly interacting massive particles with velocities much smaller than the speed of light. Cold dark matter has a prominent role in the structure formation since it interacts only with gravity and is not opposed by any force such as the pressure of radiation. Hence, dark matter collapses first forming seeds into which the baryons fall later. The dark matter has an important role, since the epoch of structure formation would occur later than it is observed if dark matter was not present.

We shall analyze Jeans instability by considering two collisionless Boltzmann equations – one for the baryonic matter and another for the dark matter – which are connected with the Poisson equation [3, 7, 8]. Here we shall use the indices b and d to denote the baryonic and dark matter, respectively. The collisionless Boltzmann equations for the one-particle distribution functions of baryonic matter $f_b \equiv f(\mathbf{x}, \mathbf{v}_b, t)$ and dark matter

$f_d \equiv f(\mathbf{x}, \mathbf{v}_d, t)$ are given by

$$\frac{\partial f_b}{\partial t} + v_i^b \frac{\partial f_b}{\partial x^i} + \frac{\partial U}{\partial x^i}\frac{\partial f_b}{\partial v_b^i} = 0, \quad \frac{\partial f_d}{\partial t} + v_i^d \frac{\partial f_d}{\partial x^i} + \frac{\partial U}{\partial x^i}\frac{\partial f_d}{\partial v_d^i} = 0, \tag{8.42}$$

which are connected with the Poisson equation

$$\nabla^2 U = -4\pi G\left(\int m_b f_b d^3 v_b + \int m_d f_d d^3 v_d\right)$$
$$= -4\pi G(\rho_b + \rho_d). \tag{8.43}$$

Here (ρ_b, ρ_d) and (m_b, m_d) are the mass densities and the particle rest masses of the baryonic and dark matter, respectively.

As previously we suppose that the one-particle distribution functions $f(\mathbf{x}, \mathbf{v}_b, t)$, $f(\mathbf{x}, \mathbf{v}_d, t)$ and the Newtonian gravitational potential U are subjected to small perturbations from their equilibrium values $f_b^0(\mathbf{v}_b)$, $f_d^0(\mathbf{v}_d)$ and U_0 which reads

$$f(\mathbf{x}, \mathbf{v}_b, t) = f_0^b(\mathbf{v}_b) + \epsilon f_1^b(\mathbf{x}, \mathbf{v}_b, t), \tag{8.44}$$
$$f(\mathbf{x}, \mathbf{v}_d, t) = f_0^d(\mathbf{v}_d) + \epsilon f_1^d(\mathbf{x}, \mathbf{v}_d, t), \tag{8.45}$$
$$U(\mathbf{x}, t) = U_0 + \epsilon U_1(\mathbf{x}, t). \tag{8.46}$$

Here the equilibrium distribution functions are the Maxwellians

$$f_0^b(\mathbf{v}_b) = \frac{\rho_b}{m_b}\frac{e^{-\mathbf{v}_b^2/2\sigma_b^2}}{(2\pi\sigma_b^2)^{3/2}}, \qquad f_0^d(\mathbf{v}_d) = \frac{\rho_d}{m_d}\frac{e^{-\mathbf{v}_d^2/2\sigma_d^2}}{(2\pi\sigma_d^2)^{3/2}}. \tag{8.47}$$

In the above equations $\sigma_b = \sqrt{kT_b/m_b}$ and $\sigma_d = \sqrt{kT_d/m_d}$ are the dispersion velocities of the baryonic and dark matter which are connected with their absolute temperatures T_b and T_d, respectively.

Following the same methodology of the last section we insert the representations (8.44) – (8.46) into the Boltzmann (8.42) and Poisson (8.43) equations and get

$$\frac{\partial f_1^b}{\partial t} + v_i^b \frac{\partial f_1^b}{\partial x^i} + \frac{\partial U_1}{\partial x^i} \frac{\partial f_0^b}{\partial v_b^i} = 0, \tag{8.48}$$

$$\frac{\partial f_1^d}{\partial t} + v_i^d \frac{\partial f_1^d}{\partial x^i} + \frac{\partial U_1}{\partial x^i} \frac{\partial f_0^d}{\partial v_d^i} = 0, \tag{8.49}$$

$$\nabla^2 U_1 = -4\pi G \left(\int m_b f_1^b d^3 v_b + \int m_d f_1^d d^3 v_d \right). \tag{8.50}$$

Note that we have also considered that the Poisson equation is valid only for the perturbations, i.e. we have used the "Jeans swindle".

The next step is to consider the perturbations as plane waves of frequency ω and wave number vector $\mathbf{k}$ as follows

$$f_1^b(\mathbf{x}, \mathbf{v}_b, t) = f_0^b(\mathbf{v}_b) \left(A_b + \mathbf{B}_b \cdot \mathbf{v}_b + D_b \mathbf{v}_b^2 \right) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{8.51}$$

$$f_1^d(\mathbf{x}, \mathbf{v}_d, t) = f_0^d(\mathbf{v}_d) \left(A_d + \mathbf{B}_d \cdot \mathbf{v}_d + D_d \mathbf{v}_d^2 \right) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{8.52}$$

$$U_1(\mathbf{x}, t) = \overline{U}_1 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{8.53}$$

where the amplitude of the Newtonian gravitational potential is constant and the amplitudes of the perturbed distribution functions are given in terms of linear combinations of the summational invariants $\left(1, \mathbf{v}_b, \mathbf{v}_b^2\right)$ and $\left(1, \mathbf{v}_d, \mathbf{v}_d^2\right)$. Furthermore, $A_b, A_d, \mathbf{B}_b, \mathbf{B}_d, D_b, D_d$ are unknowns that do not depend on the particle velocities $\mathbf{v}_b$ and $\mathbf{v}_d$.

We insert the representations of the perturbations (8.51) – (8.53) into the Boltzmann and Poisson equations (8.48) – (8.50) and get the following system of algebraic equations

$$f_0^b \left[\left(A_b + \mathbf{B}_b \cdot \mathbf{v}_b + D_b \mathbf{v}_b^2 \right) (\omega - \mathbf{k} \cdot \mathbf{v}_b) + \mathbf{k} \cdot \mathbf{v}_b \frac{\overline{U}_1}{\sigma_b^2} \right] = 0, \quad (8.54)$$

$$f_0^d \left[\left(A_d + \mathbf{B}_d \cdot \mathbf{v}_d + D_d \mathbf{v}_d^2 \right) (\omega - \mathbf{k} \cdot \mathbf{v}_d) + \mathbf{k} \cdot \mathbf{v}_d \frac{\overline{U}_1}{\sigma_d^2} \right] = 0, \quad (8.55)$$

$$\kappa^2 \overline{U}_1 - 4\pi G \left[\left(A_b + 3\sigma_b^2 D_b \right) \rho_b + \left(A_d + 3\sigma_d^2 D_d \right) \rho_d \right] = 0. \quad (8.56)$$

From (8.54) – (8.56) we can build a system of algebraic equations for $A_b, A_d, B_b = \mathbf{B}_b \cdot \mathbf{k}, B_d = \mathbf{B}_d \cdot \mathbf{k}, D_b, D_d$ and $\overline{U}_1$. Indeed, from the multiplication of (8.54) by the summational invariants $\left(1, \mathbf{v}_b, \mathbf{v}_b^2\right)$ and (8.55) by the summational invariants $\left(1, \mathbf{v}_d, \mathbf{v}_d^2\right)$ and the integration of the resulting equations we get respectively the system of algebraic equations for $A_b, A_d, B_b, B_d, D_b, D_d$:

$$\omega \left(A_\alpha + 3\sigma_\alpha^2 D_\alpha \right) - B_\alpha \sigma_\alpha^2 = 0, \quad (8.57)$$

$$\omega B_\alpha - \left[A_\alpha + 5\sigma_\alpha^2 D_\alpha + \frac{\overline{U}_1}{\sigma_\alpha^2} \right] \kappa^2 = 0, \quad (8.58)$$

$$\omega \left(A_\alpha + \frac{5}{3} \sigma_\alpha^2 D_\alpha \right) - \frac{5}{3} B_\alpha \sigma_\alpha^2 = 0. \quad (8.59)$$

Here we have six equations corresponding to $\alpha = b, d$. To obtain the above expressions the Gaussian integrals of the Appendix A of Chapter 4 were used. Furthermore, as in the previous section a scalar multiplication with $\mathbf{k}$ was performed for the vector equations of $\mathbf{B}_b$ and $\mathbf{B}_d$.

The algebraic system of equations (8.56) – (8.59) has a non-trivial solution if the determinant of the coefficients $A_b, A_d, B_b, B_d, D_b, D_d, \overline{U}_1$ vanishes, which implies the dispersion relation:

$$\omega_*^4 + \left[1 + \frac{\rho_b}{\rho_d} - \left(1 + \frac{\sigma_b^2}{\sigma_d^2}\right)\kappa_*^2\right]\omega_*^2 + \frac{\sigma_b^2}{\sigma_d^2}\left[\kappa_*^2 - 1 - \frac{\rho_b\sigma_d^2}{\rho_d\sigma_b^2}\right]\kappa_*^2 = 0. \tag{8.60}$$

In the above equation we have introduced the dimensionless wave number κ_* and the dimensionless frequency ω_* defined by

$$\kappa_* = \frac{\kappa}{\kappa_J^d} = \frac{\kappa\, c_s^d}{\sqrt{4\pi G\rho_d}}, \qquad \omega_* = \frac{\omega}{\sqrt{4\pi G\rho_d}}, \tag{8.61}$$

which are given in terms of the dark matter Jeans wave number $\kappa_J^d = \sqrt{4\pi G\rho_d}/c_s^d$ where $c_s^d = \sqrt{5/3}\,\sigma_d$ denotes the dark matter sound speed. We have taken the dark matter to build the dimensionless quantities, since as it was explained before the dark matter begins to collapse into a complex network of dark matter halos well before the ordinary matter.

If we consider only one fluid corresponding to the dark matter we may neglect the baryonic matter by taking $\rho_b = \sigma_b = 0$ and the dispersion relation (8.60) reduces to:

$$\omega_*^4 + \left[1 - \kappa_*^2\right]\omega_*^2 = 0. \tag{8.62}$$

The above equation has the solutions $\omega_* = \pm\sqrt{\kappa^2/\kappa_J^2 - 1}$ and $\omega_* = 0$, so that the Jeans solution for one component (8.23) is recovered.

The roots of the dispersion relation (8.60) furnishes four solutions for the dimensionless frequency:

$$\omega_* = \pm\frac{\sigma_b}{\sqrt{2}\sigma_d}\sqrt{\kappa_*^2\left(1+\frac{\sigma_d^2}{\sigma_b^2}\right)-\frac{\sigma_d^2}{\sigma_b^2}\left(1+\frac{\rho_b}{\rho_d}\right)\pm\Delta\omega}, \quad (8.63)$$

where $\Delta\omega$ is given by

$$\Delta\omega^2 = \left[\kappa_*^2\left(1+\frac{\sigma_d^2}{\sigma_b^2}\right)-\frac{\sigma_d^2}{\sigma_b^2}\left(1+\frac{\rho_b}{\rho_d}\right)\right]^2 -4\frac{\sigma_d^2}{\sigma_b^2}\left[\kappa_*^4-\left(1+\frac{\rho_b\sigma_d^2}{\rho_d\sigma_b^2}\right)\kappa_*^2\right]. \quad (8.64)$$

We can infer that the dispersion relation (8.60) is a function of two ratios ρ_d/ρ_b and σ_d/σ_b. The mass density ratio has not changed considerably during the evolution of the Universe so that we can associate it with the present value of the density parameter ratio $\rho_d/\rho_b = \Omega_d/\Omega_b \approx 5.4$ [9]. There is no fixed value for the dispersion velocities ratio σ_d/σ_b and here we shall rely on the simulations of Milky Way-like galaxies which included baryonic and dark matter [10]. We have inferred from one of the simulations of this work – where Maxwellian distributions are considered – that this ratio can be taken as $\sigma_d/\sigma_b = 170/93 \approx 1.83$.

The real roots of the dispersion relation (8.63) will imply harmonic wave solutions in time, while the pure imaginary roots will provide instabilities which will grow or decay in time, and the one which grows refers to the Jeans instability. It is interesting to investigate the value of κ_* where ω_* changes from the pure imaginary value to the real value. If we take $\omega_* = 0$ in

(8.63) we get

$$\kappa_* = \frac{\kappa_J^{\rm db}}{\kappa_J^{\rm d}} = \sqrt{1 + \frac{\rho_b \sigma_d^2}{\rho_d \sigma_b^2}} = \frac{\lambda_J^{\rm d}}{\lambda_J^{\rm db}}, \tag{8.65}$$

which can be interpreted as the ratio of two Jeans wave numbers, the one denoted by $\kappa_J^{\rm db}$ refers to the system dark-baryonic matter while the other $\kappa_J^{\rm d}$ refers to the dark matter.

Let us analyze the amount of mass which is necessary for an overdensity of a dark-baryonic matter system to initiate the gravitational collapse. This is related to the Jeans mass contained in a sphere of radius equal to the wavelength $\lambda = 2\pi/\kappa$. If $M_J^{\rm db}$ denotes the Jeans mass of the dark-baryonic matter system and $M_J^{\rm d}$ the one for the dark matter system we can build the ratio of Jeans masses:

$$\frac{M_J^{\rm db}}{M_J^{\rm d}} = \frac{\rho_b + \rho_d}{\rho_d} \left(\frac{\lambda_J^{\rm db}}{\lambda_J^{\rm d}}\right)^3 = \left(1 + \frac{\rho_b}{\rho_d}\right) \left(\sqrt{1 + \frac{\rho_b \sigma_d^2}{\rho_d \sigma_b^2}}\right)^{-3} . \tag{8.66}$$

σ_d/σ_b	1.00	1.20	1.40	1.60	1.83	2.00	2.20
$M_J^{\rm db}/M_J^{\rm d}$	0.92	0.83	0.74	0.66	0.57	0.52	0.45

Table 8.1: Ratio of Jeans masses $M_J^{\rm db}/M_J^{\rm d}$ as functions of the ratio of the dispersion velocities σ_d/σ_b for $\rho_d/\rho_b = 5.4$.

The ratio of the Jeans masses of the systems dark-baryonic matter and dark matter for fixed values of the mass densities ratio $\rho_d/\rho_b = 5.4$, are given in Table 8.2 as functions of the dispersion velocities ratio. We infer from this table that the increase in the dispersion velocities ratio implies that the mass needed to begin the gravitational collapse becomes smaller than the mass where only one constituent is present.We can understand this behavior by noting that for large values of σ_d/σ_b the dispersion velocity of the baryonic matter is smaller than the one of the dark matter so that the baryonic matter hardly overcome the escape velocity of a given gravitational field.

8.3 Jeans instability in an expanding Universe

The aim of this section is to analyze Jeans instability by taking into account the collisionless Boltzmann and Poisson equations (8.1) in an expanding Universe where the source of the gravitational field is a pressureless fluid [3].

Here the equilibrium one-particle distribution function must be written in a comoving frame

$$f_0(\mathbf{v},t) = \frac{\rho(t)}{m}\frac{1}{[2\pi\sigma(t)^2]^{3/2}}\exp\left[-\frac{(\mathbf{v}-\dot{a}\mathbf{x}_0)^2}{2\sigma(t)^2}\right], \qquad (8.67)$$

thanks to Hubble-Lamaître's law $\dot{\mathbf{x}} = \dot{a}\mathbf{x}_0$. We note that the dispersion velocity $\sigma(t)$ and the mass density $\rho(t)$ are functions of time (see the Appendix of Chapter 7).

For the background Newtonian gravitational potential we adopt the same expression as that given in Section 7.2, namely

$$U_0(\mathbf{x},t) = -\frac{2\pi}{3}G\rho\mathbf{x}\cdot\mathbf{x} = -\frac{2\pi}{3}G\rho a^2\mathbf{x}_0\cdot\mathbf{x}_0. \tag{8.68}$$

In terms of the comoving coordinates $\mathbf{x}_0$ the Boltzmann equation $(8.1)_1$ can be rewritten, by taking into account the relationships (7.13), as

$$\left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}_0} + \frac{(v_i - \dot{a}x_i^0)}{a}\left.\frac{\partial f}{\partial x_0^i}\right|_t + \frac{1}{a}\left.\frac{\partial U}{\partial x_0^i}\right|_t\left.\frac{\partial f}{\partial v^i}\right|_{t,\mathbf{x}_0} = 0. \tag{8.69}$$

If we insert the background distribution function (8.67) into the Boltzmann equation (8.68) we get

$$\begin{aligned}
&\left.\frac{\partial f_0}{\partial t}\right|_{\mathbf{x}_0} + \frac{(v_i - \dot{a}x_i^0)}{a}\left.\frac{\partial f_0}{\partial x_0^i}\right|_t + \frac{1}{a}\left.\frac{\partial U}{\partial x_0^i}\right|_t\left.\frac{\partial f_0}{\partial v^i}\right|_{t,\mathbf{x}_0} \\
&= f_0\left\{\frac{\dot{\rho}}{\rho} + \frac{(\mathbf{v}-\dot{a}\mathbf{x}_0)^2}{\sigma^2}\frac{\dot{a}}{a} + \left[\frac{(\mathbf{v}-\dot{a}\mathbf{x}_0)^2}{\sigma^2} - 3\right]\frac{\dot{\sigma}}{\sigma}\right. \\
&\left. + \frac{(\mathbf{v}-\dot{a}\mathbf{x}_0)\cdot\mathbf{x}_0}{\sigma^2}\left(\ddot{a} + \frac{4\pi}{3}G\rho a\right)\right\} = 0.
\end{aligned} \tag{8.70}$$

By taking into account that for a pressureless fluid (7.92) holds and considering that the dispersion velocity is proportional to the inverse of the cosmic scale factor $\sigma(t)/\sigma_0 = a_0/a(t)$, the Boltzmann equation for the background distribution (8.70) is identically verified.

Furthermore, the Poisson equation is also identically verified for the background value of the gravitational potential (8.68), since

$$\nabla^2 U_0 = -4\pi G\rho = -4\pi G \int m f_0 d^3 v. \tag{8.71}$$

Now we require that the background distribution function (8.67) and Newtonian gravitational potential (8.68) are subjected to small perturbations characterized by $f_1(\mathbf{x}, \mathbf{v}, t)$ and $U_1(\mathbf{x}, t)$ such that

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{v}, t) + f_1(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{v}, t)\left[1 + h_1(\mathbf{x}, \mathbf{v}, t)\right], \tag{8.72}$$

$$U(\mathbf{x}, t) = U_0(\mathbf{x}, t) + U_1(\mathbf{x}, t). \tag{8.73}$$

Furthermore, we represent the perturbations h_1 and U_1 by plane waves where the physical wave number vector is $\mathbf{q}/a(t)$ while the comoving one is simply $\mathbf{q}$. The factor $1/a(t)$ takes into account that the wavelength is stretched out in an expanding Universe. Hence

$$h_1(\mathbf{x}, \mathbf{v}, t) = \overline{h}(\mathbf{x}, \mathbf{v}, t) e^{i\frac{\mathbf{q}\cdot\mathbf{x}}{a(t)}} = \overline{h}(\mathbf{x}, \mathbf{v}, t) e^{i\mathbf{q}\cdot\mathbf{x}_0}, \tag{8.74}$$

$$U_1(\mathbf{x}, t) = \overline{U}(t) e^{i\frac{\mathbf{q}\cdot\mathbf{x}}{a(t)}} = \overline{U}(t) e^{i\mathbf{q}\cdot\mathbf{x}_0}. \tag{8.75}$$

The amplitudes $\overline{h}(\mathbf{x}, \mathbf{v}, t)$ and $\overline{U}(t)$ are considered to be small and we assume that $\overline{h}$ is given as a linear combination of the comoving summational invariants 1, $(\mathbf{v} - \dot{a}\mathbf{x}_0)$ and $(\mathbf{v} - \dot{a}\mathbf{x}_0)^2$, namely

$$\overline{h}(\mathbf{x}, \mathbf{v}, t) = A(t) + \mathbf{B}(t) \cdot (\mathbf{v} - \dot{a}\mathbf{x}_0) + D(t)\left(\mathbf{v} - \dot{a}\mathbf{x}_0\right)^2, \tag{8.76}$$

where $A(t)$, $\mathbf{B}(t)$ and $D(t)$ are unknown functions of time that do not depend on the comoving summational invariants.

The insertion of (8.72) – (8.76) into the collisionless Boltzmann (8.69) and Poisson equations (8.5) implies the following system of equations:

$$
\begin{aligned}
&f_0\left[\left.\frac{\partial\overline{h}}{\partial t}\right|_{\mathbf{x}_0}+\frac{\left(v_i-\dot{a}x_0^i\right)}{a}\left.\frac{\partial\overline{h}}{\partial x_0^i}\right|_t+\frac{1}{a}\left.\frac{\partial U_0}{\partial x_0^i}\right|_t\left.\frac{\partial\overline{h}}{\partial v^i}\right|_{t,\mathbf{x}_0}\right]\\
&\quad+\frac{1}{a}\left.\frac{\partial U_1}{\partial x_0^i}\right|_t\left.\frac{\partial f_0}{\partial v^i}\right|_{t,\mathbf{x}_0}=f_0\left\{\frac{dA}{dt}+(\mathbf{v}-\dot{a}\mathbf{x}_0)\cdot\frac{d\mathbf{B}}{dt}\right.\\
&+(\mathbf{v}-\dot{a}\mathbf{x}_0)^2\frac{dD}{dt}-\frac{\dot{a}\big(\mathbf{v}-\dot{a}\mathbf{x}_0\big)\cdot\big[\mathbf{B}+2D\big(\mathbf{v}-\dot{a}\mathbf{x}_0\big)\big]}{a}\\
&\quad+\frac{i\mathbf{q}\cdot(\mathbf{v}-\dot{a}\mathbf{x}_0)}{a}\Big[A+\mathbf{B}\cdot(\mathbf{v}-\dot{a}\mathbf{x}_0)+D\,(\mathbf{v}-\dot{a}\mathbf{x}_0)^2\\
&\left.-\frac{\overline{U}}{\sigma^2}\right]-\underline{\left(\ddot{a}+\frac{4\pi}{3}\rho a\right)}\mathbf{x}_0\cdot[\mathbf{B}+2D\,(\mathbf{v}-\dot{a}\mathbf{x}_0)]\Big\}=0,
\end{aligned}
\tag{8.77}
$$

$$
\begin{aligned}
\frac{q^2}{a^2}\overline{U}&=4\pi G\int mf_0\Big[A+\mathbf{B}\cdot(\mathbf{v}-\dot{a}\mathbf{x}_0)\\
&+D\cdot(\mathbf{v}-\dot{a}\mathbf{x}_0)^2\Big]d^3v=4\pi G\rho\left(A+3\sigma^2D\right).
\end{aligned}
\tag{8.78}
$$

Thanks to the acceleration equation $(7.92)_1$ the underlined in (8.77) vanishes. Furthermore, in the last equation we have used the Gaussian integrals in the Appendix A of Chapter 4.

If we multiply (8.77) by the comovig summational invariants 1, $(\mathbf{v}-\dot{a}\mathbf{x}_0)$ and $(\mathbf{v}-\dot{a}\mathbf{x}_0)^2$ and integrate the resulting equations by using the Gaussian integrals in the Appendix A of Chapter 4 we get the following system of differential equations

$$\frac{dA}{dt} + 3\sigma^2\frac{dD}{dt} + i\frac{\sigma^2}{a}B - 6\frac{\dot{a}}{a}\sigma^2 D = 0, \tag{8.79}$$

$$\frac{dB}{dt} + i\frac{q^2}{a}\left[A + 5\sigma^2 D - \frac{\overline{U}}{\sigma^2}\right] - \frac{\dot{a}}{a}B = 0, \tag{8.80}$$

$$3\frac{dA}{dt} + 15\sigma^2\frac{dD}{dt} + i5\frac{\sigma^2}{a}B - 30\frac{\dot{a}}{a}\sigma^2 D = 0. \tag{8.81}$$

Equation (8.80) results from the scalar multiplication of the integrated equation by $\mathbf{q}$ and the introduction of $B(t) = \mathbf{B}(t)\cdot\mathbf{q}$.

If we subtract (8.81) from (8.79) multiplied by five, we get that $dA/dt = 0$ and for simplicity we choose $A = 1$.

Here the density contrast is also given by

$$\delta\rho = \frac{\int m f_0 \overline{h} d^3 v}{\rho} = A + 3\sigma^2 D. \tag{8.82}$$

In terms of the density contrast (8.79) or (8.81) becomes

$$\frac{d\delta\rho}{dt} + i\frac{\sigma^2}{a}B = 0, \tag{8.83}$$

by considering that $\sigma/\sigma_0 = a_0/a$. From the differentiation of the above equation with respect to time and elimination of B, dB/dt and $\overline{U}$ by taking into account (8.83), (8.80) and (8.78), respectively, we obtain the following differential equation for the density contrast

$$\frac{d^2\delta\rho}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta\rho}{dt} + \left(\frac{5q^2\sigma^2}{3a^2} - 4\pi G\rho\right)\delta\rho - \frac{2q^2\sigma^2}{3a^2} = 0. \tag{8.84}$$

If we introduce the dimensionless quantities

$$\lambda_0 = \frac{2\pi a_0}{q}, \qquad \lambda_J = \frac{2\pi\sqrt{5/3}\,\sigma}{\sqrt{4\pi G\rho}}, \qquad \tau = t\sqrt{6\pi G\rho}, \quad (8.85)$$

the differential equation for the density contrast (8.84) becomes

$$\tau^2\delta\rho'' + \frac{4}{3}\tau\delta\rho' + \frac{2}{3}\left(\frac{\lambda_J^2}{\lambda_0^2}\tau^{\frac{2}{3}} - \tau^2\right)\delta\rho - \underline{\frac{4\lambda_J^2}{15\lambda_0}\tau^{\frac{2}{3}}} = 0. \quad (8.86)$$

In the above equation we have taken into account that $a'/a = 2/3\tau$ and $a = a_0\tau^{\frac{2}{3}}$, moreover the primes refer to the differentiation with respect to τ. The difference of this equation from the one of the phenomenological theory (7.26) is due to the underlined term.

For the case of Jeans instability – where the density contrast grows with time – there is no difference between the numerical solutions of (8.86) and (7.26) for big wavelengths in comparison with Jeans' wavelength ($\lambda_0 > \lambda_J$), since the underlined term becomes small. For small wavelengths ($\lambda_0 < \lambda_J$) – which corresponds to the oscillatory behavior of the density contrast – there are differences between the two solutions but they are not very significant to comment here.

It is also interesting to analyse the influence of irreversible processes in the Jeans instability for an expanding Universe. As in Section 8.1 we shall consider the BGK model of the Boltzmann equation given by (8.24).

For the BGK model of the Boltzmann equation we get that

(8.79) – (8.81) become

$$\frac{dA}{dt} + 3\sigma^2 \frac{dD}{dt} + i\frac{\sigma^2}{a}B - 6\frac{\dot{a}}{a}\sigma^2 D = -\nu\left(A + 3\sigma^2 D\right), \quad (8.87)$$

$$\frac{dB}{dt} + i\frac{q^2}{a}\left[A + 5\sigma^2 D - \frac{\overline{U}}{\sigma^2}\right] - \frac{\dot{a}}{a}B = -\nu B, \quad (8.88)$$

$$3\frac{dA}{dt} + 15\sigma^2 \frac{dD}{dt} + i5\frac{\sigma^2}{a}B - 30\frac{\dot{a}}{a}\sigma^2 D = -\nu\left(3A + 15\sigma^2 D\right). \quad (8.89)$$

From the combination of (8.87) and (8.89) it follows that $dA/dt = -\nu A$ which implies that $A = e^{-\nu t}$. If we introduce the density contrast (8.82) we can rewrite (8.87) as

$$\frac{d\delta\rho}{dt} + i\frac{\sigma^2}{a}B = -\nu\delta\rho. \quad (8.90)$$

From the differentiation of (8.90) with respect to time and following the same steps above we arrive at

$$\frac{d^2\delta\rho}{dt^2} + \left(2\frac{\dot{a}}{a} + \nu\right)\left(\frac{d\delta\rho}{dt} + \nu\delta\rho\right) + \left(\frac{5q^2\sigma^2}{3a^2} - 4\pi G\rho\right)\delta\rho - \frac{2e^{-\nu t}q^2\sigma^2}{3a^2} = -\nu\frac{d\delta\rho}{dt}. \quad (8.91)$$

In terms of the dimensionless quantities (8.85) the above equation can be rewritten as

$$\delta\rho'' + \left(\frac{4}{3\tau} + 2\nu_*\right)\delta\rho' + \delta\rho\left[\nu_*\left(\frac{4}{3\tau} + \nu_*\right) + \frac{2}{3}\left(\frac{\lambda_J^2}{\lambda_0^2\tau^{\frac{4}{3}}} - 1\right)\right] - \frac{4\lambda_J^2}{15\lambda_0}\frac{e^{-\nu_*\tau}}{\tau^{\frac{4}{3}}} = 0. \quad (8.92)$$

Here the dimensionless collision frequency is $\nu_* = \nu/\sqrt{6\pi G\rho}$.

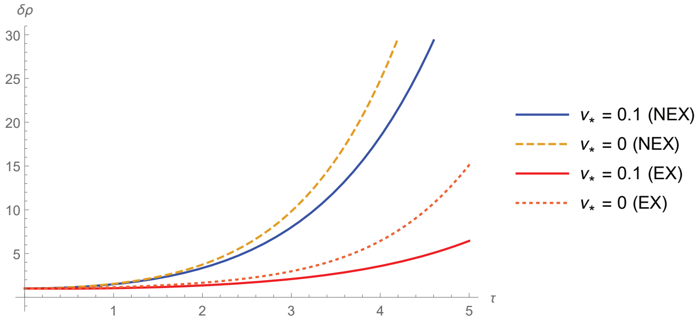

Figure 8.3: Density contrast $\delta\rho$ as function of the dimensionless time τ when $\lambda/\lambda_J = 10$, for the cases of non-expanding (NEX) and expanding (EX) Universe with dimensionless collision frequencies $\nu_* = 0.1$ and $\nu_* = 0$.

We have solved numerically the differential equation (8.92) with the same values adopted before for the initial conditions $\delta\rho(0) = 1$ and $\delta\rho'(0) = 0$, for the two values of the dimensionless collision frequency $\nu_* = 0$ and $\nu_* = 0.1$ and for the ratio of the wavelengths $\lambda/\lambda_J = 10$ that correspond to the Jeans instability. In Figure 8.3 we plotted the solutions of the density contrast $\delta\rho$ as function of the dimensionless time τ for the expanding (8.92) and non-expanding (8.40) Universe. The straight lines represent the case $\nu_* = 0.1$ where the collisions are taken into account while the dashed lines correspond to a collisionless Boltzmann equation where $\nu_* = 0$. As in the previous analysis of Figure 8.2

due to the presence of the particle collisions an energy dissipation comes out implying a less accentuate growth of the density contrast in comparison to the one for a collisionless Boltzmann equation. Furthermore, as a consequence that in an expanding Universe the solution refers to a comoving frame, the density contrast growth is smaller than the one for a non-expanding Universe.

8.4 Post-Newtonian Jeans instability

In this section we shall analyze Jeans instability from the collisionless post-Newtonian Boltzmann equation (4.8) which we rewrite by introducing the gravitational potentials in the Chandrasekhar description $U = -\phi$, $\Pi_i = -\xi_i$ and $\Phi = -\psi/2$:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + \frac{\partial U}{\partial x^i}\frac{\partial f}{\partial v^i} + \frac{1}{c^2}\Bigg[\left(v^2 - 4U\right)\frac{\partial U}{\partial x^i} - 4v_i v_j \frac{\partial U}{\partial x^j}$$
$$-3v_i \frac{\partial U}{\partial t} + 2\frac{\partial \Phi}{\partial x^i} + \frac{\partial \Pi_i}{\partial t} + v_j\left(\frac{\partial \Pi_i}{\partial x^j} - \frac{\partial \Pi_j}{\partial x^i}\right)\Bigg]\frac{\partial f}{\partial v^i} = 0. \quad (8.93)$$

The above equation results when (4.8) is multiplied by the factor $\left[1 - \frac{1}{c^2}\left(\frac{v^2}{2} + U\right)\right]$ and terms up to the $1/c^2$ order are considered.

The gravitational potentials are given in terms of the Poisson equations by (see Section 2.3.3)

$$\nabla^2 U = -\frac{4\pi G}{c^2}\overset{0}{T}{}^{00}, \quad \nabla^2 \Phi = -2\pi G\left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}\right), \quad (8.94)$$

$$\nabla^2 \Pi^i = -\frac{16\pi G}{c}\overset{1}{T}{}^{0i} + \frac{\partial^2 U}{\partial t \partial x^i}, \quad (8.95)$$

where the energy-momentum tensor is defined in terms of the one-particle distribution function by

$$T^{\mu\nu} = m^4 c \int u^\mu u^\nu f \frac{\sqrt{-g}\, d^3u}{u_0}. \tag{8.96}$$

We recall that the first post-Newtonian approximation for the components of the four-velocity (4.10) read

$$u^0 = c\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + U\right)\right], \qquad u^i = \frac{u^0 v^i}{c}. \tag{8.97}$$

Furthermore, the Maxwell-Jüttner distribution function (4.13) – denoted here by f_{MJ} – in a stationary equilibrium background where the hydrodynamic velocity vanishes $V_i = 0$ reduces to

$$f_{MJ} = f_0\left\{1 - \frac{\sigma^2}{c^2}\left[\frac{15}{8} + \frac{3v^4}{8\sigma^4} + \frac{2Uv^2}{\sigma^4}\right]\right\}, \tag{8.98}$$

$$f_0 = \frac{\rho_0}{m^4(2\pi\sigma^2)^{\frac{3}{2}}} e^{-\frac{v^2}{2\sigma^2}}. \tag{8.99}$$

Here f_0 is the Maxwellian distribution function which is a function of the gas particle velocities $\mathbf{v}$, while $\sigma = \sqrt{kT_0/m}$ represents the dispersion velocity of the gas. The mass density ρ_0 and the dispersion velocity σ are considered to be constants. Note that the factor $1/m^4$ is due to the fact that the Maxwell-Jüttner distribution function is written in terms of the four-momentum p^μ.

The invariant integration element which appears in the definition of the energy-momentum tensor (8.96) is given by (see

(4.20))

$$\frac{\sqrt{-g}\,d^3u}{u_0} = \left\{1 + \frac{1}{c^2}\left[2v^2 + 6U\right]\right\}\frac{d^3v}{c}. \tag{8.100}$$

For the analysis of Jeans instability we will make use of the Boltzmann equation (8.93) together with the Poisson equations (8.94) and (8.95). For that end we write the one particle distribution function and the gravitational potentials as a sum of background terms denoted by the subscript zero and perturbed terms with the subscript 1, namely

$$f(\mathbf{x}, \mathbf{v}, t) = f_{MJ}(\mathbf{x}, \mathbf{v}, t) + \epsilon f_1(\mathbf{x}, \mathbf{v}, t), \tag{8.101}$$

$$U(\mathbf{x}, \mathbf{v}, t) = U_0(\mathbf{x}) + \epsilon U_1(\mathbf{x}, \mathbf{v}, t), \tag{8.102}$$

$$\Phi(\mathbf{x}, \mathbf{v}, t) = \Phi_0(\mathbf{x}) + \epsilon \Phi_1(\mathbf{x}, \mathbf{v}, t), \tag{8.103}$$

$$\Pi_i(\mathbf{x}, \mathbf{v}, t) = \Pi_i^0(\mathbf{x}) + \epsilon \Pi_i^1(\mathbf{x}, \mathbf{v}, t), \tag{8.104}$$

where ϵ is a small parameter which is introduce in order to control that only linear terms in this parameter should be taken into account. Later we shall take it equal to one.

Now we insert the representations (8.101) – (8.104) into the Boltzmann equation (8.93), equate the terms of the same ϵ-order and get two hierarchy of equations that read

$$\frac{\partial U_0}{\partial x^i}\frac{\partial f_{MJ}^0}{\partial v^i} - \frac{2v^2 f_0}{\sigma^2 c^2} v_i \frac{\partial U_0}{\partial x^i} + \frac{1}{c^2}\Bigg[\left(v^2 - 4U_0\right)\frac{\partial U_0}{\partial x^i}$$
$$-4v_i v_j \frac{\partial U_0}{\partial x^j} + 2\frac{\partial \Phi_0}{\partial x^i} + v_j\left(\frac{\partial \Pi_i^0}{\partial x^j} - \frac{\partial \Pi_j^0}{\partial x^i}\right)\Bigg]\frac{\partial f_0}{\partial v^i} = 0, \tag{8.105}$$

$$\frac{\partial f_1}{\partial t}+v_i\frac{\partial f_1}{\partial x^i}+\frac{\partial U_1}{\partial x^i}\frac{\partial f^0_{MJ}}{\partial v^i}-\frac{2v^2f_0}{\sigma^2c^2}\left(\frac{\partial U_1}{\partial t}+v_i\frac{\partial U_1}{\partial x^i}\right)$$
$$+\frac{\partial U_0}{\partial x^i}\frac{\partial f_1}{\partial v_i}-\frac{4v_iU_1}{c^2\sigma^2}\frac{\partial U_0}{\partial x^i}+\frac{1}{c^2}\Bigg[\left(v^2-4U_0\right)\frac{\partial U_0}{\partial x^i}$$
$$+2\frac{\partial \Phi_0}{\partial x^i}-4v_iv_j\frac{\partial U_0}{\partial x^j}+v_j\left(\frac{\partial \Pi^0_i}{\partial x^j}-\frac{\partial \Pi^0_j}{\partial x^i}\right)\Bigg]\frac{\partial f_1}{\partial v^i}$$
$$+\frac{1}{c^2}\Bigg[\left(v^2-4U_0\right)\frac{\partial U_1}{\partial x^i}+2\frac{\partial \Phi_1}{\partial x^i}+\frac{\partial \Pi^1_i}{\partial t}-4v_iv_j\frac{\partial U_1}{\partial x^j}$$
$$-4U_1\frac{\partial U_0}{\partial x^i}-3v_i\frac{\partial U_1}{\partial t}+v_j\left(\frac{\partial \Pi^1_i}{\partial x^j}-\frac{\partial \Pi^1_j}{\partial x^i}\right)\Bigg]\frac{\partial f_0}{\partial v^i}=0. \quad (8.106)$$

Here the Maxwell-Jüttner distribution function was written as

$$f_{MJ}=f_0\left\{1-\frac{\sigma^2}{c^2}\left[\frac{15}{8}+\frac{3v^4}{8\sigma^4}+2\frac{U_0v^2}{\sigma^4}\right]\right\}-2f_0\epsilon\frac{U_1v^2}{c^2\sigma^2}$$
$$=f^0_{MJ}-2f_0\epsilon\frac{U_1v^2}{c^2\sigma^2}, \quad (8.107)$$

where f^0_{MJ} denotes the background Maxwell-Jüttner distribution function.

The background terms refer to a stationary equilibrium state and the background equation (8.105) is identically satisfied when the gradients of the potential gravitational backgrounds vanish, i.e., $\nabla U_0=\nabla\Phi_0=0$ and $\nabla\Pi^0_i=0$. In this case the Boltzmann equation for the perturbations (8.106) reduces to

$$\frac{\partial f_1}{\partial t}+v_i\frac{\partial f_1}{\partial x^i}+\frac{\partial U_1}{\partial x^i}\frac{\partial f^0_{MJ}}{\partial v^i}-\frac{2v^2f_0}{\sigma^2c^2}\left(\frac{\partial U_1}{\partial t}+v_i\frac{\partial U_1}{\partial x^i}\right)$$

$$+\frac{1}{c^2}\Bigg[\left(v^2-4U_0\right)\frac{\partial U_1}{\partial x^i}+2\frac{\partial \Phi_1}{\partial x^i}+\frac{\partial \Pi_i^1}{\partial t}-3v_i\frac{\partial U_1}{\partial t}$$
$$-4v_iv_j\frac{\partial U_1}{\partial x^j}+v_j\left(\frac{\partial \Pi_i^1}{\partial x^j}-\frac{\partial \Pi_j^1}{\partial x^i}\right)\Bigg]\frac{\partial f_0}{\partial v^i}=0. \qquad (8.108)$$

However, the Poisson equations (8.94) and (8.95) are not satisfied by the conditions of vanishing potential gradients, because the right-hand side of these equations are functions of the energy-momentum tensor which is non-zero at equilibrium. This inconsistency is removed by assuming "Jeans swindle" which considers that the Poisson equations are valid only for the perturbed distribution function and gravitational potentials.

In order to determine the components of energy-momentum tensor we note that we can write the components of the four-velocity (8.97) and the invariant integration element (8.100) as

$$u^0=c\left[1+\frac{1}{c^2}\left(\frac{v^2}{2}+U_0+\epsilon U_1\right)\right], \qquad u^i=\frac{u^0v^i}{c}, \quad (8.109)$$
$$\frac{\sqrt{-g}\,d^3u}{u_0}=\left\{1+\frac{1}{c^2}\left[2v^2+6U_0+6\epsilon U_1\right]\right\}\frac{d^3v}{c}, \quad (8.110)$$

by considering the representation of the Newtonian gravitational potential (8.102).

Now we have from (8.107), (8.109) and (8.110) by considering $\epsilon=1$ that

$$f\frac{\sqrt{-g}\,d^3u}{u_0}=\Bigg\{1-\frac{\sigma^2}{c^2}\Bigg[\frac{15}{8}+\frac{3v^4}{8\sigma^4}+2\frac{U_0v^2}{\sigma^4}$$

$$-2\frac{v^2}{\sigma^2}-6\frac{U_0}{\sigma^2}\bigg]\bigg\}f_0\frac{d^3v}{c}-\epsilon\frac{U_1}{c^2}\left(\frac{2v^2}{\sigma^2}-6\right)f_0\frac{d^3v}{c}$$
$$+\epsilon\left\{1+\frac{1}{c^2}\left[2v^2+6U_0\right]\right\}f_1\frac{d^3v}{c}. \tag{8.111}$$

The expressions for the energy-momentum tensor components that appear in the right-hand side of the Poisson equations (8.94) and (8.95) can be written thanks to the relationships (8.109) and (8.111) as

$$\overset{0}{T}{}^{00}+\overset{2}{T}{}^{00}=m^4c\int u^0u^0f\frac{\sqrt{-g}\,d^3u}{u_0}$$
$$=m^4c^2\int f_0\Bigg[1-\frac{\sigma^2}{c^2}\Bigg(\frac{15}{8}+\frac{3v^4}{8\sigma^4}+\frac{2U_0v^2}{\sigma^4}-\frac{3v^2}{\sigma^2}$$
$$-\frac{8U_0}{\sigma^2}\Bigg)\Bigg]d^3v+\epsilon\,m^4c^2\int\Bigg\{f_1\left[1+\frac{3v^2}{c^2}+\frac{8U_0}{c^2}\right]$$
$$-\left(\frac{2v^2}{\sigma^2}-8\right)\frac{f_0U_1}{c^2}\Bigg\}d^3v, \tag{8.112}$$

$$\overset{2}{T}{}^{ij}=m^4c\int u^iu^jf\frac{\sqrt{-g}\,d^3u}{u_0}=m^4\int v_iv_j(f_0+\epsilon f_1)d^3v, \tag{8.113}$$

$$\overset{1}{T}{}^{0i}=m^4c\int u^0u^if\frac{\sqrt{-g}\,d^3u}{u_0}=m^4c\int v_i(f_0+\epsilon f_1)d^3v. \tag{8.114}$$

Hence, the perturbed Poisson equations (8.94) and (8.95) become

$$\nabla^2 U_1 = -\frac{4\pi G}{c^2}[\overset{0}{T}{}^{00}]_1 = -4\pi G m^4 \int f_1 d^3 v, \tag{8.115}$$

$$\nabla^2 \Pi_1^i = -\frac{16\pi G}{c}[\overset{1}{T}{}^{0i}]_1 + \frac{\partial^2 U_1}{\partial t \partial x^i}$$

$$= -16\pi G m^4 \int v_i f_1 d^3 v + \frac{\partial^2 U_1}{\partial t \partial x^i}, \tag{8.116}$$

$$\nabla^2 \Phi_1 = -2\pi G \left([\overset{2}{T}{}^{00}]_1 + [\overset{2}{T}{}^{ii}]_1\right) = -2\pi G m^4 \int \Big[\big(4v^2$$

$$+8U_0\big) f_1 - \left(\frac{2v^2}{\sigma^2} - 8\right) U_1 f_0 \Big] d^3 v. \tag{8.117}$$

Above $[\overset{0}{T}{}^{00}]_1$, $[\overset{1}{T}{}^{0i}]_1$ and so one denote the energy-momentum tensor components calculated with the perturbed distribution function f_1.

Now we represent the perturbations in terms of plane waves of frequency ω and wave number vector $\mathbf{k}$, namely

$$f_1(\mathbf{x}, \mathbf{v}, t) = \overline{f}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \; U_1(\mathbf{x}, \mathbf{v}, t) = \overline{U}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{8.118}$$

$$\Phi_1(\mathbf{x}, \mathbf{v}, t) = \overline{\Phi}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \; \Pi_1^i(\mathbf{x}, \mathbf{v}, t) = \overline{\Pi_i^1} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{8.119}$$

where $\overline{f}_1, \overline{U}_1, \overline{\Phi}_1$ and $\overline{\Pi_1^i}$ are small amplitudes.

Insertion of the representations (8.118) and (8.119) into the perturbed Boltzmann (8.108) and Poisson equations (8.115) – (8.117), yields

$$(\mathbf{v}\cdot\mathbf{k} - \omega)\overline{f}_1 - \frac{f_0}{\sigma^2}(\mathbf{v}\cdot\mathbf{k})\overline{U}_1 \Bigg\{ \Bigg[1 - \frac{\sigma^2}{c^2}\Bigg(\frac{15}{8} + \frac{3v^4}{8\sigma^4} - \frac{v^2}{2\sigma^2}$$

$$+\frac{2v^2U_0}{\sigma^4}\Bigg)\Bigg] + \frac{1}{c^2}\Big[v^2\omega\overline{U}_1 + 2(\mathbf{v}\cdot\mathbf{k})\overline{\Phi}_1 - \omega v_i\overline{\Pi_i^1}\Big]\Bigg\} = 0, \quad (8.120)$$

$$\kappa^2\overline{U}_1 = 4\pi G m^4 \int \overline{f}_1 d^3v, \quad (8.121)$$

$$\kappa^2\overline{\Pi_i^1} = 16\pi G m^4 \int v_i\overline{f}_1 d^3v - k_i\omega\overline{U}_1, \quad (8.122)$$

$$\kappa^2\overline{\Phi}_1 = 8\pi G m^4 \int (v^2 + 2U_0)\overline{f}_1 d^3v + 4\pi G\rho_0\overline{U}_1. \quad (8.123)$$

In (8.120) we have used the expression (8.107) to determine $\partial f_{MJ}/\partial v^i$.

For the calculation of the integrals in (8.121) – (8.123) we choose, without loss of generality, the wave number vector in the x-direction, i.e., $\mathbf{k} = (\kappa, 0, 0)$ and start with the substitution of f_1 from (8.120) into (8.122), yielding

$$\kappa^2\overline{\Pi_i^1} = \frac{16\pi G\rho_0}{(2\pi\sigma^2)^{\frac{3}{2}}}\int \frac{v_i(v_x\kappa+\omega)e^{-\frac{v^2}{2\sigma^2}}\,d^3v}{\sigma^2[(v_x\kappa)^2-\omega^2]}\Bigg\{\Bigg[1 - \frac{\sigma^2}{c^2}\Bigg(\frac{15}{8} + \frac{3v^4}{8\sigma^4} - \frac{v^2}{2\sigma^2} + \frac{2U_0v^2}{\sigma^4}\Bigg)\Bigg]v_x\kappa\overline{U}_1 + \frac{1}{c^2}\Big[v^2\omega\overline{U}_1 + 2v_x\kappa\overline{\Phi}_1 - \omega v_j\overline{\Pi_j^1}\Big]\Bigg\} - k_i\omega\overline{U}_1. \quad (8.124)$$

Above we have multiplied the numerator and denominator of the integrand by $(v_x\kappa+\omega)$. Now we have to perform the integration of (8.124) in the ranges $-\infty < (v_x, v_y, v_z) < \infty$.

For the components $i = y, z$ the integration of (8.124) leads to

$$\kappa^2 \overline{\Pi_i^1} = -8\pi G\rho_0 \frac{\omega^2}{\kappa^2\sigma^2 c^2} I_0 \overline{\Pi_i^1}, \qquad i = y, z, \tag{8.125}$$

and infer that $\overline{\Pi_y^1} = \overline{\Pi_z^1} = 0$. In the above equation I_0 refers to the integral I_n defined by (8.12).

The integration of (8.124) by considering the component $i = x$ yields

$$\kappa^2 \overline{\Pi_x^1} = \frac{16\pi G\rho_0 \omega}{\kappa\sigma^2} \left\{ \left[I_2 - \frac{3\sigma^2}{2c^2} \left(I_6 + \frac{5}{4} I_2 \right) - \frac{4U_0}{c^2} \left(I_2 + I_4 \right) \right] \overline{U}_1 + \frac{I_2}{c^2} \left[2\overline{\Phi}_1 - \frac{\omega}{\kappa} \overline{\Pi_x^1} \right] \right\} - \kappa\omega \overline{U}_1. \tag{8.126}$$

Following the same methodology the substitution of (8.120) into (8.121) and (8.123) and subsequent integration of the resulting equations lead to

$$\kappa^2 \overline{U}_1 = \frac{4\pi G\rho_0}{\sigma^2} \left\{ \left[I_2 + (I_0 + I_2) \frac{\omega^2}{c^2\kappa^2} - \frac{3\sigma^2}{2c^2} \left(I_6 + \frac{4}{3} I_4 + \frac{31}{12} I_2 \right) - \frac{4U_0}{c^2} \left(I_2 + I_4 \right) \right] \overline{U}_1 + \frac{I_2}{c^2} \left[2\overline{\Phi}_1 - \frac{\omega}{\kappa} \overline{\Pi_x^1} \right] \right\}, \tag{8.127}$$

$$\kappa^2 \overline{\Phi}_1 = 4\pi G\rho_0 \overline{U}_1 + 16\pi G\rho_0 \left\{ I_2 + I_4 + 2 \left(I_0 + I_2 + \frac{I_4}{2} \right) \frac{\omega^2}{\kappa^2 c^2} - \frac{3\sigma^2}{2c^2} \left(I_8 + \frac{7}{3} I_6 + \frac{71}{12} I_4 + \frac{71}{12} I_2 \right) \right.$$

$$+\frac{U_0}{\sigma^2}\left[I_2-\frac{\sigma^2}{c^2}\left(\frac{11}{2}I_6+10I_4+\frac{95}{8}I_2\right)\right.$$
$$\left.\left.-\frac{4U_0}{c^2}(I_2+I_4)+\frac{\omega^2}{c^2\kappa^2}(I_0+I_2)\right]\right\}\overline{U}_1$$
$$+\frac{16\pi G\rho_0}{c^2}\left(I_2+I_4+I_2\frac{U_0}{\sigma^2}\right)\left[2\overline{\Phi}_1-\frac{\omega}{\kappa}\overline{\Pi^1_x}\right]. \quad (8.128)$$

Equations (8.126) – (8.128) represent an algebraic system of equations for the amplitudes $\overline{\Pi^1_x}$, $\overline{U}_1$ and $\overline{\Phi}_1$. This system of equations has a solution if the determinant of the coefficients vanishes, which implies the following dispersion relation

$$\varkappa_*^4-\varkappa_*^2\left[I_2+\frac{\sigma^2}{c^2}\left(\frac{33}{8}I_2+6I_4-\frac{3}{2}I_6+4(I_2-I_4)\frac{U_0}{\sigma^2}\right)\right]$$
$$-\frac{\sigma^2}{c^2}\left[2I_2+(I_0-2I_2)\,\omega_*^2\right]=0. \quad (8.129)$$

Here we have not considered terms of order higher than $\mathcal{O}\left(c^{-2}\right)$, since we are dealing with the first post-Newtonian approximation. The dispersion relation (8.129) relates the dimensionless wave number $\varkappa_*$ with the dimensionless frequency ω_*, which are defined by

$$\varkappa_*=\frac{\kappa}{\varkappa_J},\quad \omega_*=\frac{\omega}{\sqrt{4\pi G\rho_0}},\quad \text{where}\quad \varkappa_J=\frac{\sqrt{4\pi G\rho_0}}{\sigma}, \quad (8.130)$$

denotes the Jeans wave number. Note that here the Jeans wave number and the dimensionless wave number $\varkappa_*$ are defined in terms of the dispersion velocity σ instead of the sound speed c_s.

We are searching for unstable solutions which correspond to Jeans instability and for these solutions we have that $\omega_* = i\omega_I$, i.e., $\Re(\omega_*) = 0$ and $\omega_I = \Im(\omega_*) > 0$. In this case the integrals (8.12) become

$$I_0 = \frac{\varkappa_*}{\omega_I}\sqrt{2\pi}\exp\left(\frac{\omega_I^2}{2\varkappa_*^2}\right)\mathrm{erfc}\left(\frac{\omega_I}{\sqrt{2}\varkappa_*}\right), \quad (8.131)$$

$$I_2 = 1 - \frac{\omega_I^2}{2\varkappa_*^2}I_0, \; I_4 = \frac{1}{2} - \frac{\omega_I^2}{2\varkappa_*^2}I_2, \; I_6 = \frac{3}{4} - \frac{\omega_I^2}{2\varkappa_*^2}I_4, \quad (8.132)$$

where erfc(x) denotes the complementary error function

$$\mathrm{erfc(x)} = \frac{2}{\sqrt{\pi}}\int_{\mathrm{x}}^{\infty} \mathrm{e}^{-\mathrm{x}^2}\mathrm{dx}. \quad (8.133)$$

In Figure 8.4 the contour plots obtained from the dispersion relation (8.129) are shown for two different values of the ratio between the dispersion velocity and the light speed. One of them refers to the Newtonian theory where $\sigma/c = 0$ and the other to the post-Newtonian approximation with $\sigma/c = 0.05$. In the evaluation of (8.129) for the post-Newtonian approximation it was consider that $U_0 \approx \sigma^2$, i.e., by considering the virial theorem where the Newtonian gravitational potential can be approximated with the square of the dispersion velocity. We infer from this figure that the post-Newtonian approximation has an influence in the limit of instability, since for a given frequency the corresponding modulus of the wave number vector in the post-Newtonian theory is greater than that of the Newtonian theory, which implies a decrease in the mass limit of interstellar gas clouds necessary to start the gravitational collapse.

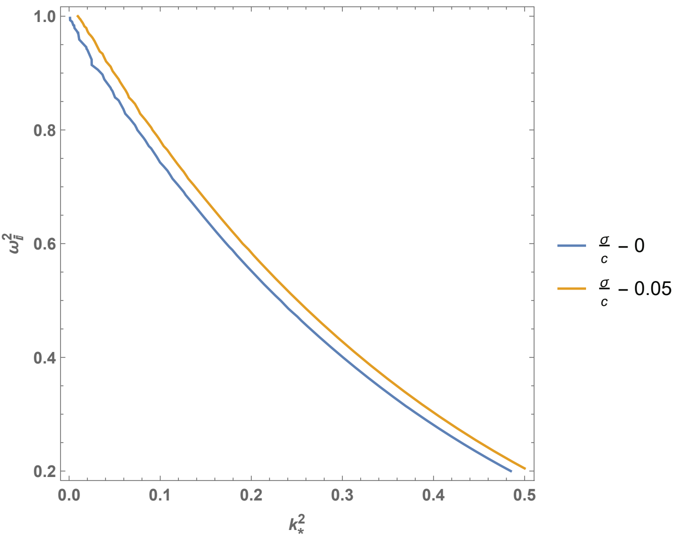

Figure 8.4: Contour plots of the dimensionless frequency as function of the dimensionless modulus of the wave number vector for the Newtonian ($\sigma/c = 0$) and post-Newtonian ($\sigma/c = 0.05$) theories.

In order to determine the amount of mass necessary to occur the gravitational collapse we set $\omega_I = 0$ in (8.129). This is the limiting value of the frequency where the instability occurs and corresponds to a minimum mass for an overdensity to begin the gravitational collapse. Hence, it follows

$$\varkappa_*^4 - \left[1 + \frac{\sigma^2}{c^2}\left(6 + \frac{2U_0}{\sigma^2} + \frac{2}{\varkappa_*^2}\right)\right]\varkappa_*^2 = 0. \tag{8.134}$$

The real positive solution of (8.134) is

$$\varkappa_* = \sqrt{\frac{1}{2} + \frac{\sigma^2}{c^2}\left[3 + \frac{U_0}{\sigma^2} + \Delta\varkappa\right]}, \tag{8.135}$$

where $\Delta\varkappa$ is the following abbreviation

$$\Delta\varkappa = \sqrt{\frac{1}{4} + \frac{\sigma^2}{c^2}\left[5 + \frac{U_0}{\sigma^2} + \frac{\sigma^2}{c^2}\left(9 + \frac{6U_0}{\sigma^2} + \frac{U_0^2}{\sigma^4}\right)\right]}. \tag{8.136}$$

By considering terms up to the $1/c^2$ order (8.135) reduces to

$$\varkappa_* = 1 + \frac{\sigma^2}{c^2}\left[4 + \frac{U_0}{\sigma^2}\right]. \tag{8.137}$$

As previously stated the amount of mass for an overdensity to initiate the gravitational collapse is related to the Jeans mass contained in a sphere of radius equal to the wavelength of the perturbation. The ratio of the Jean masses corresponding to the post-Newtonian M_J^{PN} and Newtonian M_J^N wavelengths is given here by

$$\frac{M_J^{PN}}{M_J^N} = \frac{\lambda^3}{\lambda_J^3} = \frac{\varkappa_J^3}{\varkappa_*^3} \approx 1 - 3\frac{\sigma^2}{c^2}\left[4 + \frac{U_0}{\sigma^2}\right], \tag{8.138}$$

which shows that in the post-Newtonian framework the mass needed to begin the gravitational collapse is smaller than in the Newtonian case. Note that (8.138) is similar to the one found in the hydrodynamic theory for a monatomic or Fermi non-relativistic gas (7.53), the only difference is that (8.138) is a function of the dispersion velocity σ while (7.53) is a function of the sound speed c_s.

As in Section 8.1 we shall analyse the Jeans instability which follows from the post-Newtonian Boltzmann equation but by considering the summational invariants.

We begin by recalling that the relativistic summational invariants are the rest mass m and the momentum four-vector p^μ. Here the perturbed distribution function will be written as a function of a linear combination of the summational invariants $\widetilde{A}+\widetilde{B}_\mu p^\mu$, where $\widetilde{A}$ and $\widetilde{B}_\mu$ are unknowns which do not depend on the momentum four-vector p^μ.

Let us determine the post-Newtonian approximation of $\widetilde{A}+\widetilde{B}_\mu p^\mu$ and for that end we make use of the expressions for the metric tensor components (3.2), (3.3), (3.4) and four-velocity components (3.10), (3.11). Hence, we write

$$\begin{aligned}\widetilde{A}+g_{\mu\nu}\widetilde{B}^\mu p^\nu &= \widetilde{A}+g_{00}\widetilde{B}^0p^0+g_{0i}\widetilde{B}^0p^i+g_{0i}\widetilde{B}^ip^0\\ &+g_{ij}\widetilde{B}^ip^j=\widetilde{A}+mc\bigg[1+\frac{1}{c^2}\left(\frac{v^2}{2}-U_0\right)+\frac{1}{c^4}\bigg(\frac{3v^4}{8}\\ &+\frac{3v^2U_0}{2}+\frac{U_0^2}{2}-6\Phi_0-\Pi_i^0v_i\bigg)\bigg]\widetilde{B}^0+m\frac{\Pi_i^0v_i}{c^3}\widetilde{B}^0\\ &+m\frac{\Pi_i^0}{c^2}\widetilde{B}^i-mv_i\left[1+\frac{1}{c^2}\left(\frac{v^2}{2}+3U_0\right)\right]\widetilde{B}_i. \qquad (8.139)\end{aligned}$$

In the above equations we have consider terms up to $1/c^4$ order. Now we introduce new unknowns

$$A = \widetilde{A} + mc\widetilde{B}^0 \left[1 - \frac{U_0}{c^2} + \frac{1}{c^4}\left(\frac{U_0^2}{2} - 6\Phi_0\right)\right] + m\frac{\Pi_i^0}{c^2}\widetilde{B}^i, \quad (8.140)$$

$$D = \frac{m\widetilde{B}^0}{2c}, \qquad B_i = -m\widetilde{B}^i, \quad (8.141)$$

which implies that (8.139) reduces to

$$\widetilde{A} + g_{\mu\nu}\widetilde{B}^\mu p^\nu = A + v^2\left[1 + \frac{1}{c^2}\left(\frac{3v^2}{4} + 3U_0\right)\right]D + v_i\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + 3U_0\right)\right]B_i. \quad (8.142)$$

Here we may identify the post-Newtonian summational invariants

$$1,\ v^2\left[1 + \frac{1}{c^2}\left(\frac{3v^2}{4} + 3U_0\right)\right],\ v_i\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + 3U_0\right)\right]. (8.143)$$

The perturbed distribution function is obtained from the product of the Maxwell-Jüttner distribution function (8.98) and (8.142), yielding

$$\overline{f}_1 = f^0_{MJ}\left(\widetilde{A} + \widetilde{B}_\mu p^\mu\right) = f_0\Bigg\{\left[1 - \frac{\sigma^2}{c^2}\left(\frac{15}{8} + \frac{3v^4}{8\sigma^4} + \frac{2U_0 v^2}{\sigma^4}\right)\right]A + v^2\left[1 - \frac{\sigma^2}{c^2}\left(\frac{15}{8} + \frac{3v^4}{8\sigma^4}\right.\right.$$

$$-\frac{3v^2}{4\sigma^2}+\frac{2U_0v^2}{\sigma^4}-\frac{3U_0}{\sigma^4}\Big)\Big]D+v_i\Big[1-\frac{\sigma^2}{c^2}\Big(\frac{15}{8}$$
$$+\frac{3v^4}{8\sigma^4}-\frac{v^2}{2\sigma^2}+\frac{2U_0v^2}{\sigma^4}-\frac{3U_0}{\sigma^4}\Big)\Big]B_i\Big\}. \qquad (8.144)$$

The perturbed potentials $\overline{U}_1$, $\overline{\Pi}_1^1$ and $\overline{\Phi}_1$ as functions of A, D and B_i are obtained from the insertion of the perturbed distribution function (8.144) into (8.121), (8.122) (8.123) and by integrating the resulting equations. It follows respectively

$$\kappa^2\overline{U}_1=4\pi G\rho_0\Big\{A\left[1-\frac{\sigma^2}{c^2}\left(\frac{15}{2}+\frac{6U_0}{\sigma^2}\right)\right]$$
$$+3\sigma^2D\left[1-\frac{\sigma^2}{c^2}\left(\frac{45}{4}+\frac{7U_0}{\sigma^2}\right)\right]\Big\}, \qquad (8.145)$$

$$\kappa^2\overline{\Pi}_i^1=16\pi G\rho_0\sigma^2B_i\left[1-\frac{\sigma^2}{c^2}\left(\frac{25}{2}+\frac{7U_0}{\sigma^2}\right)\right]+k_i\omega\overline{U}_1, \quad (8.146)$$

$$\kappa^2\overline{\Phi}_1=4\pi G\rho_0\Big\{\overline{U}_1+6\sigma^2A\Big[1+\frac{2U_0}{3\sigma^2}-\frac{\sigma^2}{c^2}\Big(15$$
$$+\frac{15U_0}{\sigma^2}+\frac{4U_0^2}{\sigma^4}\Big)\Big]+30\sigma^4D\Big[1+\frac{2U_0}{5\sigma^2}$$
$$-\frac{\sigma^2}{c^2}\Big(\frac{81}{4}+\frac{31U_0}{2\sigma^2}+\frac{14U_0^2}{5\sigma^4}\Big)\Big]\Big\}. \qquad (8.147)$$

We insert the perturbed distribution function (8.144) into the perturbed Boltzmann equation (8.120) and multiply the resulting equation by each of the summational invariants given in

(8.143). The integration of the resulting equations by considering the invariant element of integration (8.100) leads to the following system of algebraic equations

$$\left[1-\frac{3\sigma^2}{2c^2}\right]A+3\sigma^2\left[1-\frac{\sigma^2}{c^2}\left(\frac{5}{4}+\frac{U_0}{c^2}\right)\right]D+\frac{3\overline{U}_1}{c^2}$$
$$-\frac{\sigma^2}{\omega}\left[1-\frac{\sigma^2}{c^2}\left(\frac{5}{2}+\frac{U_0}{c^2}\right)\right]\mathbf{B}\cdot\mathbf{k}=0, \qquad (8.148)$$

$$\left[1-\frac{\sigma^2}{c^2}\left(\frac{5}{4}+\frac{U_0}{c^2}\right)\right]A+5\sigma^2\left[1-\frac{\sigma^2}{c^2}\left(1+\frac{2U_0}{c^2}\right)\right]D$$
$$+\frac{5\overline{U}_1}{c^2}-\frac{5}{3}\frac{\sigma^2}{\omega}\left[1-\frac{\sigma^2}{c^2}\left(\frac{11}{4}+\frac{2U_0}{c^2}\right)\right]\mathbf{B}\cdot\mathbf{k}=0, \quad (8.149)$$

$$\left\{\left[1-\frac{\sigma^2}{c^2}\left(\frac{5}{2}+\frac{U_0}{c^2}\right)\right]A+5\sigma^2\left[1-\frac{\sigma^2}{c^2}\left(\frac{11}{4}\right.\right.\right.$$
$$\left.\left.\left.+\frac{2U_0}{c^2}\right)\right]D-\frac{2\overline{\Phi}_1}{\sigma^2c^2}-\frac{\overline{U}_1}{\sigma^2}\left(1-\frac{U_0}{c^2}\right)\right\}k_i$$
$$-\omega\left[B_i\left(1+\frac{2U_0}{c^2}\right)-\frac{\overline{\Pi}_i^1}{\sigma^2c^2}\right]=0. \qquad (8.150)$$

In order to get a system algebraic equations for the perturbations $\overline{U}_1$, $\overline{\Pi}_i^1 k_i$, $\overline{\Phi}^1$, A, D and B_ik_i we build first the scalar product of the vector equations (8.146) and (8.150) with k_i. The resulting equations together with (8.145), (8.147), (8.148) and (8.149) becomes a system of algebraic equations for these perturbations. This system of equations has a non-vanishing solution if the determinant of the coefficients vanishes. Hence, it follows the dispersion relation

$$\omega_*^2 = \kappa_*^2 - 1 - \left[\frac{5}{2} + \frac{4}{\kappa_*^2} + \frac{27\kappa^2}{10} + \frac{2U_0}{c_s^2}\left(2\kappa_*^2 - 1\right)\right]\frac{c_s^2}{c^2}, \quad (8.151)$$

for the dimensionless frequency $\omega_* = \omega/\sqrt{4\pi G\rho_0}$ and dimensionless wave number $\kappa_* = \kappa/\kappa_J = \kappa c_s^2/\sqrt{4\pi G\rho_0}$. In (8.151) we have taken into account only terms up to the $1/c^2$ order.

The real root of the κ_* when $\omega_* = 0$ is

$$\kappa_* = 1 + \frac{c_s^2}{c^2}\left[\frac{23}{5} + \frac{U_0}{c_s^2}\right], \quad (8.152)$$

by considering terms up to the $1/c^2$ order. Note that the above equation is not equal to (8.137), since the former is given in terms of the sound speed and the latter in terms of the dispersion velocity. Furthermore, the c_s^2/c^2 factor in (8.152) is $23/5 = 4.6$, while the one in the phenomenological theory (7.51) when $\gamma = 5/3$ is 4.

References

[1] J. Binney and S. Tremaine, *Galactic Dynamics*, 2nd. ed. (Princeton University Press, Princeton, 2008).

[2] P. Coles and F. Lucchin, *Cosmology. The Origin and Evolution of Cosmic structures*, 2nd, edn. (John Wiley, Chichester, 2002).

[3] G. M. Kremer, Analysis of Jeans instability from the Boltzmann equation, *AIP Conference Proceedings* **1786**, 160002 (2016).

[4] S. A. Trigger, A. I. Ershkovich, G. J. F. van Heijst and P.P J. M. Schram, Kinetic theory of Jeans instability, *Phys. Rev. E* **69** 066403 (2004).

[5] G. M. Kremer, M. G. Richarte and F. Teston, Jeans Instability in a Universe with Dissipation, *Phys. Rev. D* **97**, 023515 (2018).

[6] G. M. Kremer, *An introduction to the Boltzmann equation and transport processes in gases* (Springer, Berlin, 2010).

[7] G. M. Kremer and R. André, Analisis of instability of systems composed by dark and baryonic matter, *Int. J. Mod. Phys. D* **25**, 1650012 (2016).

[8] G. M. Kremer, M. G. Richarte and E. M. Schiefer, Using kinetic theory to examine a self-gravitating system composed of baryons and cold dark matter, *Eur. Phys. J. C* **79**, 492 (2019).

[9] L. Baudis and S. Profumo, Dark matter, in *Review of Particle Physics* https://pdg.lbl.gov/2020/reviews/rpp2020-rev-dark-matter.pdf

[10] F. -S. Ling, E. Nezri, E. Athanassoula and R. Theyssier, Dark matter direct detection signals inferred from a cosmological N-body simulation with baryons *J. Cosmol. Astropart. Phys.* 012 (2010).

[11] G. M. Kremer, Jeans instability from post-Newtonian Boltzmann equation, *Eur. Phys. J. C* **81**, 927 (2021).

CHAPTER 9

GALAXY ROTATION CURVES

The galaxies are astronomical objects composed by stars, stellar remnants, interstellar gas, dust, and dark matter which are gravitationally bound due to gravitational interaction amongst their constituents.

The rotation curves obtained from the Newtonian theory increase linearly near the origin up to a maximum and for large radii vanish. However the measured circular velocity curves for the galaxies show a small value near the center that increases linearly up to a small cusp and for large radii tends to a finite nonzero value. Since the Newtonian gravity do not succeed to predict the mass distribution of astronomical objects one introduces a dark matter component which till nowadays cannot be observed or measured directly, but can be detected through its gravitational effects with the astronomical objects, although it does not interact directly with the standard matter.

The inclusion of post-Newtonian corrections make it possible to reduce the amount of dark matter which is needed to explain the rotation curves which flatten at large radii. However these corrections cannot solve the whole problem of generating flat rotation curves, but may help to reduce the dark matter amount in relation with the Newtonian models.

The post-Newtonian corrections to the problem of galaxy rotation curves was first investigated in [1, 2, 3] by using a polytropic equation of state. Here we shall follow the work [4] where the components of the energy-momentum tensor were obtained from a post-Newtonian Maxwell-Jüttner distribution function. In this chapter we also analyse an application of the spherically symmetrical Jeans equation which is related with the effect of a central massive black hole on the velocity dispersion profile of the host galaxy.

9.1 Post-Newtonian particle dynamics

In this section we shall follow Weinberg [5] and determine the Lagrangian of a single particle in the first post-Newtonian approximation. The relation between the proper time τ and the time coordinate t for a free falling particle of velocity $\mathbf{V}$ is given by (2.85). In terms of the Chandrasekhar potentials $U = -\phi$, $\Phi = \psi/2$ and $\Pi_i = -\xi_i$ we have

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(2U + V^2\right) - \frac{2}{c^4}\left(2\Phi - U^2 + UV^2 - \Pi_i V_i\right). \tag{9.1}$$

From the above equation we can obtain the relationship

$$\frac{d\tau}{dt} = 1 - \frac{1}{c^2}\left(\frac{V^2}{2} + U\right) - \frac{1}{c^4}\left(\frac{V^4}{8} - \frac{3UV^2}{2} - \frac{U^2}{2} + 2\Phi - \Pi_i V_i\right), \tag{9.2}$$

by using the approximation $\sqrt{1+x} \approx 1 + x/2 - x^2/8$.

As was pointed out by Weinberg $\int (d\tau/dt)dt$ is stationary so that we can define the Lagrangian of a single particle per rest mass m by

$$\frac{L}{m} = 1 - \frac{d\tau}{dt} = \frac{1}{c^2}\left(\frac{V^2}{2} + U\right) + \frac{1}{c^4}\left(\frac{V^4}{8} - \frac{3UV^2}{2} - \frac{U^2}{2} + 2\Phi - \Pi_i V_i\right). \tag{9.3}$$

From the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x^i} \tag{9.4}$$

we can obtain the equation of motion

$$\left\{\left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} + 3U\right)\right]\delta_{ij} + \frac{V_i V_j}{c^2}\right\}\frac{dV_j}{dt}$$

$$+\frac{1}{c^2}\left(3V_i\frac{dU}{dt}-\frac{d\Pi_i}{dt}\right)=\frac{\partial U}{\partial x^i}$$
$$+\frac{1}{c^2}\left(\frac{3V^2}{2}\frac{\partial U}{\partial x^i}-U\frac{\partial U}{\partial x^i}+2\frac{\partial \Phi}{\partial x^i}-V_j\frac{\partial \Pi_j}{\partial x^i}\right). \quad (9.5)$$

In the Appendix of this Chapter it is shown that the inverse of the second order tensor

$$\mathbf{S}_{ij}=\left[1+\frac{1}{c^2}\left(\frac{V^2}{2}+3U\right)\right]\delta_{ij}+\frac{V_iV_j}{c^2}, \quad (9.6)$$

up to the $1/c^2$ order is

$$(\mathbf{S}^{-1})_{ij}=\left[1-\frac{1}{c^2}\left(\frac{V^2}{2}+3U\right)\right]\delta_{ij}-\frac{V_iV_j}{c^2}. \quad (9.7)$$

Hence the multiplication of (9.5) by (9.7) and considering terms up to the $1/c^2$ order results the acceleration equation of a single particle (see (4.34))

$$\frac{dV_i}{dt}=\frac{\partial U}{\partial x^i}+\frac{1}{c^2}\left[V^2\frac{\partial U}{\partial x^i}-4V_iV_j\frac{\partial U}{\partial x^j}-4U\frac{\partial U}{\partial x^i}+2\frac{\partial \Phi}{\partial x^i}\right.$$
$$\left.-3V_i\frac{\partial U}{\partial t}+\frac{\partial \Pi_i}{\partial t}+V_j\left(\frac{\partial \Pi_i}{\partial x^j}-\frac{\partial \Pi_j}{\partial x^i}\right)\right], \quad (9.8)$$

To derive the above equation we have considered the material time derivative $d/dt=\partial/\partial t+V_i\partial/\partial x^i$.

The energy of the single particle follows from

$$E=V_i\frac{\partial L}{\partial V^i}-L=\frac{m}{c^2}\left[\frac{V^2}{2}-U+\frac{1}{c^2}\left(\frac{3V^4}{8}\right.\right.$$
$$\left.\left.+\frac{3UV^2}{2}+\frac{U^2}{2}-2\Phi\right)\right]. \quad (9.9)$$

9.2 Maxwell-Jüttner distribution function

Here we shall derive the post-Newtonian Maxwell-Jüttner distribution function for a system characterized by a reference state where the mass density ρ_0 and absolute temperature T_0 assume constants values.

From the two representations of the Maxwell-Jüttner distribution function (1.19) and (1.40) we can write

$$f = \frac{n}{4\pi m^2 ckTK_2(\zeta)} \exp\left(-\frac{p^\mu U_\mu}{kT}\right) = \exp\left[\frac{\mu}{kT} - 1 - \frac{p_\mu U^\mu}{kT}\right]. \tag{9.10}$$

Hence we can build the following relationship

$$\begin{aligned} \frac{n}{4\pi m^2 ckTK_2(\zeta)} &= \exp\left[\frac{\mu}{kT} - 1\right] \\ &= \exp\left[\frac{\mu_0}{kT_0} - 1\right] = \frac{n_0}{4\pi m^2 ckT_0K_2(\zeta_0)}, \end{aligned} \tag{9.11}$$

since the ratio $\mu/T = \mu_0/T_0$ is a consequence of the Tolman (1.74) and Klein (1.75) laws.

Tolman's law (1.74) implies that the temperature T can be expressed in terms of the reference temperature by

$$T = \frac{T_0}{\sqrt{g_{00}}} = T_0\left[1 + \frac{U}{c^2} + \frac{1}{c^4}\left(\frac{U^2}{2} + 2\Phi\right)\right], \tag{9.12}$$

where only terms up to the $1/c^4$ order were considered.

As in the Section 4.1.2 we obtain that the exponential factor in the Maxwell-Jüttner distribution function (9.10) can be written as

$$\frac{g_{\mu\nu}p^\mu U^\nu}{kT} = \frac{mc^2}{kT_0}\left\{1 + \frac{\mathcal{V}^2}{2c^2} - \frac{U}{c^2} + \frac{1}{c^4}\left[\frac{3\mathcal{V}^4}{8} + \frac{3\mathcal{V}^2 U}{2}\right.\right.$$
$$\left.\left. + \frac{(V_i\mathcal{V}_i)^2}{2} + \frac{\mathcal{V}^2 V^2}{2} + (V_i\mathcal{V}_i)\mathcal{V}^2 + \frac{U^2}{2} - 2\Phi\right]\right\}, \quad (9.13)$$

by making use of (9.12) and introducing the peculiar velocity $\mathcal{V}_i = v_i - V_i$.

The modified Bessel function of second kind up to the $1/c^2$ order reads

$$\frac{1}{K_2(\zeta_0)} = \sqrt{\frac{2mc^2}{\pi kT_0}}\, e^{\frac{mc^2}{kT_0}} \left(1 - \frac{15kT_0}{8mc^2} + \dots\right). \qquad (9.14)$$

Now we can get the post-Newtonian Maxwell-Jüttner distribution function from (9.10) together with (9.13), (9.14), yielding

$$f = \frac{n_0}{(2\pi mkT_0)^{\frac{3}{2}}} e^{-\frac{m\mathcal{V}^2}{2kT_0} + \frac{mU}{kT_0}} \left\{1 - \frac{15kT_0}{8mc^2} - \frac{m}{kT_0c^2}\left[\frac{3\mathcal{V}^4}{8}\right.\right.$$
$$\left.\left. + \frac{3\mathcal{V}^2 U}{2} + \frac{(V_i\mathcal{V}_i)^2}{2} + \frac{V^2\mathcal{V}^2}{2} + (V_i\mathcal{V}_i)\mathcal{V}^2 + \frac{U^2}{2} - 2\Phi\right]\right\}. \quad (9.15)$$

Above we have considered the terms with the factor $1/c^2$ of small order and use the approximation $e^{-x} \approx 1 - x$.

In the following sections we shall search for static solutions of a self-gravitating system where the hydrodynamic velocity $\mathbf{V}$

vanishes and $\mathcal{V}_i = v_i$. In this case the post-Newtonian Maxwell-Jüttner distribution function (9.15) reduces to

$$f = \frac{n_0}{(2\pi mkT_0)^{\frac{3}{2}}} e^{-\frac{mv^2}{2kT_0}-\frac{mU}{kT_0}} \left\{1 - \frac{15kT_0}{8mc^2} - \frac{m}{kT_0c^2}\left[\frac{3v^4}{8} + \frac{3v^2U}{2} + \frac{U^2}{2} - 2\Phi\right]\right\}. \quad (9.16)$$

9.3 Search for static solutions

The search for static solutions of a self-gravitating system in the post-Newtonian approximation is based on the equations for the scalar gravitational potentials U and Φ, given respectively by (2.101) and (2.110). For the static case these equations read

$$\nabla^2 U = -\frac{4\pi G}{c^2} \overset{0}{T}{}^{00}, \qquad \nabla^2 \Phi = -2\pi G \left(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}\right). \quad (9.17)$$

The equation (2.116) for the vector gravitational potential Π_i in the static case becomes $\nabla^2 \Pi_i = 0$ and this equation does not couple with the equations for the scalar gravitational potential U and Φ.

The energy-momentum tensor and the invariant element of integration are given respectively by (4.14) and (4.20) which are reproduce here

$$T^{\mu\nu} = m^4 c \int u^\mu u^\nu \sqrt{-g} f \frac{d^3u}{u_0}, \quad (9.18)$$

$$\frac{\sqrt{-g}\, d^3u}{u_0} = \left\{1 + \frac{1}{c^2}\left[2v^2 + 6U\right]\right\} \frac{d^3v}{c}. \quad (9.19)$$

To evaluate the components of the energy-momentum tensor we have to consider the maximum limit of the velocity where the gas particle is unable to leave the matter distribution. This is the escape velocity which is obtained from the expression for the energy (9.9) by considering that its maximum value is attained when the energy vanishes. In the first post-Newtonian approximation the maximum value of the velocity is $v_e = \sqrt{2U}$.

The components of the energy-momentum tensor in the different orders which follow from the integration in the interval $[0, v_e]$ are given by

$$\overset{0}{T}{}^{00} = -\rho_0 c^2 \left[2\sqrt{\frac{U_*}{\pi}} - e^{U_*}\text{erf}\left(\sqrt{U_*}\right)\right], \tag{9.20}$$

$$\overset{2}{T}{}^{00} = -\frac{\rho_0 kT_0}{m}\left[\left(3+\frac{23}{2}U_* - 10U_*^2 + 4\Phi_*\right)\sqrt{\frac{U_*}{\pi}} - \left(\frac{3}{2}+\frac{7}{2}U_* - \frac{1}{2}U_*^2 + 2\Phi_*\right)e^{U_*}\text{erf}\left(\sqrt{U_*}\right)\right], \tag{9.21}$$

$$\overset{2}{T}{}^{ii} = -\frac{\rho_0 kT_0}{m}\left[(6+4U_*)\sqrt{\frac{U_*}{\pi}} - 3e^{U_*}\text{erf}\left(\sqrt{U_*}\right)\right]. \tag{9.22}$$

Here we have introduced the dimensionless quantities

$$U_* = \frac{m}{kT_0}U, \qquad \Phi_* = \left(\frac{m}{kT_0}\right)^2 \Phi. \tag{9.23}$$

Furthermore, $\text{erf}\left(\sqrt{U_*}\right)$ is the error function

$$\text{erf}\left(\sqrt{U_*}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{U_*}} e^{-x^2}dx. \tag{9.24}$$

From the insertion of (9.20) – (9.22) into the Poisson equations (9.17) we get a coupled system of differential equations for the gravitational potentials U_* and Φ_*, namely

$$\nabla^2 U_* = k_J^2\left[2\sqrt{\frac{U_*}{\pi}} - e^{U_*}\mathrm{erf}\left(\sqrt{U_*}\right)\right], \tag{9.25}$$

$$\nabla^2 \Phi_* = \frac{k_J^2}{2}\Bigg[\left(9+\frac{31}{2}U_* - 10U_*^2 + 4\Phi_*\right)\sqrt{\frac{U_*}{\pi}} - \left(\frac{9}{2}+\frac{7}{2}U_* - \frac{1}{2}U_*^2 + 2\Phi_*\right)e^{U_*}\mathrm{erf}\left(\sqrt{U_*}\right)\Bigg]. \tag{9.26}$$

Here $k_J = \sqrt{4\pi G\rho_0}/\sigma$ can be identified as the Jeans wave number with $\sigma = \sqrt{kT_0/m}$ denoting the dispersion (thermal) velocity of the self-gravitating fluid.

We shall consider a spherical coordinate system where the gravitational potentials are only functions of the radial coordinate r. By introducing the dimensionless radial coordinate $r_* = rk_J$ the system of differential equations (9.25) and (9.26) becomes

$$\frac{1}{r_*^2}\frac{d}{dr_*}\left(r_*^2\frac{dU_*}{dr_*}\right) = \left[2\sqrt{\frac{U_*}{\pi}} - e^{U_*}\mathrm{erf}\left(\sqrt{U_*}\right)\right], \tag{9.27}$$

$$\frac{2}{r_*^2}\frac{d}{dr_*}\left(r_*^2\frac{d\Phi_*}{dr_*}\right) = \Bigg[\left(9+\frac{31}{2}U_* - 10U_*^2 + 4\Phi_*\right)\sqrt{\frac{U_*}{\pi}} - \left(\frac{9}{2}+\frac{7}{2}U_* - \frac{1}{2}U_*^2 + 2\Phi_*\right)e^{U_*}\mathrm{erf}\left(\sqrt{U_*}\right)\Bigg]. \tag{9.28}$$

Now we specify appropriate boundary conditions for the system of differential equations (9.27) and (9.28) in order to solve it numerically. Here we assume that the boundary conditions at the center of the configuration for the gravitational potentials are:

$$U_*(0) = \Phi_*(0) = 1, \qquad \left.\frac{dU_*}{dr_*}\right|_{r_*=0} = \left.\frac{d\Phi_*}{dr_*}\right|_{r_*=0} = 0. \tag{9.29}$$

9.4 Numerical analysis of some fields

In this section we shall analyze the profiles of the mass density, pressure and gravitational potential energy as functions of the radial distance, which follow when the system of differential equations (9.27) and (9.28) are solved for the boundary conditions (9.29).

The mass density corresponds to the time component of the energy-momentum tensor T^{00} and follows from (9.20) and (9.21), namely

$$\rho_* = \frac{\overset{0}{T}{}^{00} + \overset{2}{T}{}^{00}}{\rho_0 c^2} = \rho_*^{\mathrm{N}} + \rho_*^{\mathrm{PN}}. \tag{9.30}$$

Here the mass density is written as a sum of a Newtonian ρ_*^{N} and a post-Newtonian ρ_*^{PN} contribution given by

$$\rho_*^{\mathrm{N}} = e^{U_*}\mathrm{erf}\left(\sqrt{U_*}\right) - 2\sqrt{\frac{U_*}{\pi}} \tag{9.31}$$

$$\rho_*^{\mathrm{PN}} = \frac{1}{\zeta_0}\left[\left(\frac{3}{2} + \frac{7}{2}U_* - \frac{1}{2}U_*^2 + 2\Phi_*\right)e^{U_*}\mathrm{erf}\left(\sqrt{U_*}\right)\right.$$

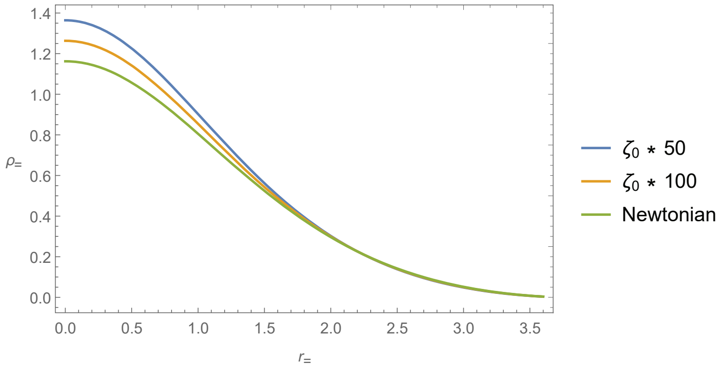

Figure 9.1: Dimensionless mass density ρ_* as function of the dimensionless radial distance r_* for the Newtonian theory and post-Newtonian theory with $\zeta_0 = 50$ and $\zeta_0 = 100$.

$$-\left(3+\frac{23}{2}U_*-10U_*^2+4\Phi_*\right)\sqrt{\frac{U_*}{\pi}}\Bigg]. \tag{9.32}$$

The dimensionless mass density ρ_* as function of the dimensionless radial distance r_* for the Newtonian and the post-Newtonian theories are plotted in Figure 9.1. We note that the post-Newtonian curves are functions of the parameter $\zeta_0 = mc^2/kT_0$ which is the ratio of the rest energy of the gas particles and the thermal energy of the gas. From this figure one can infer that the contributions to the mass density becomes larger at the configuration center when the values of ζ_0 decrease, i.e., when the values of the temperature of the gas T_0 increase . The mass densities tend to zero for large values of the radial distance r_* and are always positive. Here it is noteworthy to call attention to the fact that the solutions for the gravitational potentials become complex for values larger than $r_* \approx 3.6$.

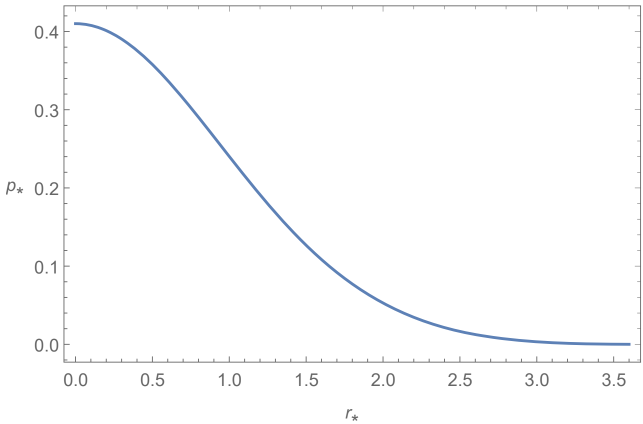

Figure 9.2: Dimensionless pressure p_* as function of the dimensionless radial distance r_*.

The pressure of the gas is given by $p = \overset{2}{T}{}^{ii}/3$ and its expression in terms of dimensionless quantities which follows from

(9.22) is given by

$$p_* = \frac{mp}{k\rho_0 T_0} = \frac{m\overset{2}{T}{}^{ii}}{3k\rho_0 T_0} = e^{U_*}\mathrm{erf}\left(\sqrt{U_*}\right) - \left(2 + \frac{4}{3}U_*\right)\sqrt{\frac{U_*}{\pi}}. \tag{9.33}$$

In Figure 9.2 the dimensionless pressure p_* is plotted as a function of the dimensionless radial coordinate r_*. The pressure behavior matches the one for the mass density, i.e. its maximum value occurs at the configuration center and it tends to zero for large values of the dimensionless radial distance. We call attention to the fact that in the first post-Newtonian approximation there is no contribution for the pressure which depends on the factor $1/\zeta_0$, because these contributions will appear only in the order of $\overset{4}{T}{}^{ii}$.

The behavior of the pressure-density ratio p_*/ρ_* as function of the dimensionless radial coordinate is shown in Figure 9.3. The behavior is the same as the one for the dimensionless mass density, namely, for small radii, the ratio p_*/ρ_* goes to a constant value whereas tends to zero at large radii.

We can be analyzed also the gravitational potential energy of a gas particle, which can be obtained from the expression for the energy of a single particle (9.9) by taking $\mathbf{V} = \mathbf{0}$. The dimensionless Newtonian E_*^{N} and post-Newtonian E_*^{PN} gravitational potential energy read

$$E_* = \frac{Ec^2}{kT_0} = E_*^{\mathrm{N}} + E_*^{\mathrm{PN}} \tag{9.34}$$

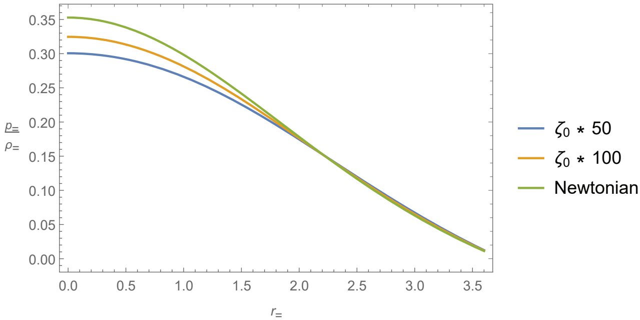

Figure 9.3: Pressure-density ratio as function of the dimensionless radial distance r_* for the Newtonian theory and post-Newtonian theory with $\zeta_0 = 50$ and $\zeta_0 = 100$.

where we can identify

$$E_*^{\rm N} = -U_*, \qquad E_*^{\rm PN} = \frac{1}{\zeta_0}\left(\frac{U_*^2}{2} - 2\Phi_*\right). \tag{9.35}$$

The plot of the dimensionless gravitational potential energy E_* as function of the dimensionless radial distance r_* is shown in Figure 9.4. From this figure we note that the Newtonian gravitational potential energy is always negative, but the post-Newtonian gravitational potential energies may change their sign for large values of the radial distance from the configuration center. Indeed, the temperature of the gas in the post-Newtonian term $U_*^2/2\zeta_0$ determines the sign change of the gravitational potential energy.

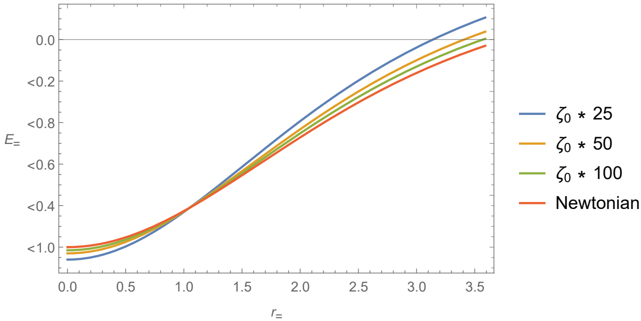

Figure 9.4: Dimensionless gravitational potential energy E_* as function of the dimensionless radial distance r_* for the Newtonian theory and post-Newtonian theory with $\zeta_0 = 25$, $\zeta_0 = 50$ and $\zeta_0 = 100$.

9.5 Circular rotation curves

In order to determine the post-Newtonian corrections to the rotation curves we begin by writing the equation for the acceleration of a free falling particle (9.8) for the case of stationary gravitational fields, namely[1]

$$\frac{dV_i}{dt} = \frac{\partial U}{\partial x^i} + \frac{1}{c^2}\left[V^2\frac{\partial U}{\partial x^i} - 4V_iV_j\frac{\partial U}{\partial x^j} - 4U\frac{\partial U}{\partial x^i} + 2\frac{\partial \Phi}{\partial x^i}\right]. \quad (9.36)$$

[1]Here we have considered that the vector gravitational potential is a Laplacian vector field, where $\nabla \times \vec{\Pi} = 0$.

Next we consider spherical coordinates (r, θ, φ) and restrict to circular orbits of particles in the equatorial plane where $\dot{r} = 0$, $\dot{\theta} = 0$ and $\theta = \pi/2$. In this case the velocity reads $\mathbf{V} = (0, 0, r\dot{\varphi})$ and the radial component of the acceleration is $r\dot{\varphi}^2$. Hence the radial component of (9.36) reduces to

$$r\dot{\varphi}^2 \left(1 + \frac{r}{c^2}\frac{\partial U}{\partial r}\right) = -\frac{\partial U}{\partial r}\left[1 - \frac{4U}{c^2}\frac{\partial U}{\partial r}\right] - \frac{2}{c^2}\frac{\partial \Phi}{\partial r}. \tag{9.37}$$

By considering terms up to the $1/c^2$ order we obtain from (9.37) the circular velocity $V_\varphi = r\dot{\varphi}$ in terms of the gravitational potentials

$$V_\varphi = \sqrt{r\frac{\partial U}{\partial r}\left(\frac{4U}{c^2} + \frac{r}{c^2}\frac{\partial U}{\partial r} - 1\right) - \frac{r}{c^2}\frac{\partial \Phi}{\partial r}}. \tag{9.38}$$

If in the above equation we neglect the $1/c^2$ terms we get the Newtonian circular velocity $V_\varphi = \sqrt{-r\partial U/\partial r}$.

We introduce now the dimensionless circular velocity $V_\varphi^* = V_\varphi\sqrt{m/kT_0}$ so that (9.38) can be written in terms of the dimensionless gravitational potentials U_*, Φ_*, dimensionless radial coordinate r_* and dimensionless parameter $\zeta_0 = mc^2/kT_0$ as

$$V_\varphi^* = \sqrt{r_*\frac{\partial U_*}{\partial r_*}\left(\frac{4U_*}{\zeta_0} + \frac{r_*}{\zeta_0}\frac{\partial U_*}{\partial r_*} - 1\right) - \frac{r}{\zeta_0}\frac{\partial \Phi_*}{\partial r_*}}. \tag{9.39}$$

Using the solutions of the system of differential equations (9.27) and (9.28) with the boundary conditions (9.29) the dimensionless circular velocities V_φ^* are plotted in Figure 9.5 as

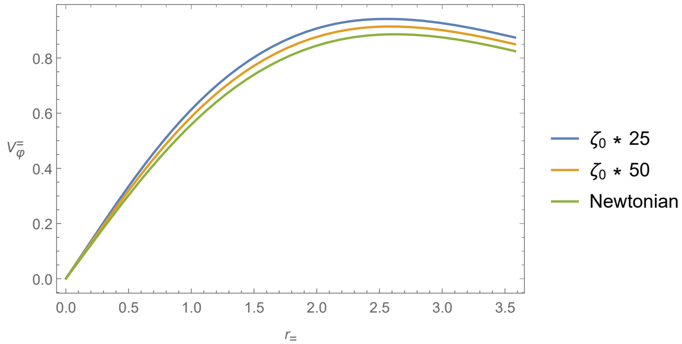

Figure 9.5: Dimensionless circular velocity V_φ^* as function of the dimensionless radial distance r_* for the Newtonian theory and post-Newtonian theory with $\zeta_0 = 25$ and $\zeta_0 = 50$.

functions of the dimensionless radial coordinate r_*. We infer from this figure that the circular velocity profiles for the Newtonian and post-Newtonian approximations have the same behaviors, but the circular velocities for the post-Newtonian approximations have large values. Note that by increasing the reference temperature T_0 the dimensionless parameter ζ_0 decreases and large values for the circular velocity are attained. This behavior is related with the increase of the thermal velocity of the gas particles $\sqrt{kT_0/m}$.

One can infer from the observational data of the galaxies rotation curves that there exist three distinct regimes for the circular velocity as a function of the radial coordinate. Beginning with a linear regime for small radii the circular velocity passes through a cusp and ends at large radii with a flatten shape (see e.g. [6]). Here we shall show how to determine this shape from the model described above. First we note from Figure 9.5 that the model has a good description for the inner zone which corresponds to $r_* \leq r_*^c \approx 3.6$. Indeed, the circular rotation curve grows linear with the radial distance, then it passes through small cusp but becomes ill-defined for $r_* > r_*^c$ due to the fact that the gravitational potentials become complex and imply that the solutions are unphysical.

This issue can be solved by matching the solutions in the inner zone – denoted here by $U_*^{(1)}(r_*)$ and $\Phi_*^{(1)}(r_*)$ – with two other gravitational potentials, namely $U_*^{(2)}(r_*)$ and $\Phi_*^{(2)}(r_*)$. In order to have a well-defined boundary problem these two gravitational potentials must fulfill the Laplace equations, namely $\nabla^2 U_*^{(2)}(r_*) = 0$ and $\nabla^2 \Phi_*^{(2)}(r_*) = 0$, and theirs values and corresponding first derivatives must be glued at r_c^*.

The most simple proposal which fulfill the Laplace equations is to consider

$$U_*^{(2)}(r_*) = \frac{\alpha}{r_*} + \beta \qquad \Phi_*^{(2)}(r_*) = \frac{\gamma}{r_*} + \delta, \tag{9.40}$$

where α, β, γ and δ are constants. Let us analyse these potentials, first by imposing the continuity of the potentials and their derivatives at r_*^c, which leads to

$$\alpha = -r_c^{*2} U_*^{(1)\prime}(r_c^*), \qquad \beta = U_*^{(1)}(r_c^*) + r_c^* U_*^{(1)\prime}(r_c^*), \tag{9.41}$$

$$\gamma = -r_c^{*2} \Phi_*^{(1)\prime}(r_c^*), \qquad \delta = \Phi_*^{(1)}(r_c^*) + r_c^* \Phi_*^{(1)\prime}(r_c^*). \tag{9.42}$$

Next we insert the potentials (9.40) into the expression for the dimensionless circular velocity (9.39), yielding

$$V_\varphi^* = \sqrt{\frac{\alpha}{r}\left[1 - \frac{1}{\zeta}\left(\frac{3\alpha}{r} + 4\beta\right)\right] + \frac{2\gamma}{r\zeta}} \tag{9.43}$$

However with the gravitational potentials (9.40) we cannot extend physically the first solution beyond the critical radius r_*^c, since the circular velocity (9.43) vanishes in the limit of large radii and cannot reproduce a flatten circular rotation curve in the outer zone.

From the above analysis we conclude that it is necessary to introduce other potentials in order to get a flatten circular rotation curve for large radii. Another proposal is to consider the previous gravitational potential $U_*(2)(r_*) = \alpha/r + \beta$ which fulfills the Laplace equation while the gravitational potential

$$\Phi_*^{(2)}(r_*) = \gamma\frac{e^{-kr_*}}{r_*} + \delta\ln r_*, \tag{9.44}$$

satisfies a Poisson equation. In (9.44) γ and δ are integration constants and k a parameter of the Yukawa term. This gravitational potentials must also satisfy at r_c^* the same boundary conditions given above. Hence, we can get the following system of equations

$$\Phi_*^{(2)}(r_c^*) = \gamma\frac{e^{-kr_c^*}}{r_c^*} + \delta\ln r_c^*, \tag{9.45}$$

$$\Phi_*^{(2)\prime}(r_*) = -\gamma\frac{e^{-kr_*}}{r_*}\left(k + \frac{1}{r_c^*}\right) + \frac{\delta}{r_c^*}. \tag{9.46}$$

Now from the conditions

$$U_*^{(1)}(r_c^*) = U_*^{(2)}(r_c^*), \qquad U_*^{(1)\prime}(r_c^*) = U_*^{(2)\prime}(r_c^*), \tag{9.47}$$

$$\Phi_*^{(1)}(r_c^*) = \Phi_*^{(2)}(r_c^*), \qquad \Phi_*^{(1)\prime}(r_c^*) = \Phi_*^{(2)\prime}(r_c^*) \tag{9.48}$$

we obtain numerically the values of the integration constants at $r_c^* \simeq 3.4001$: $\alpha \simeq 2.4175$, $\beta \simeq -0.6431$, $\gamma \simeq -29.7253$ and $\delta \simeq -2.6906$. Note that these values do not depend on the parameter ζ_0. The dimensionless circular velocity (9.39) in this case becomes

$$V_\varphi^* = \sqrt{\frac{\alpha}{r}\left[1 - \frac{1}{\zeta}\left(\frac{3\alpha}{r} + 4\beta\right)\right] + \frac{1}{\zeta}\left[\gamma e^{-kr_*}\left(k + \frac{1}{r_c^*}\right) - \delta\right]}. \tag{9.49}$$

From this equation we note that the logarithm contribution introduces a constant in the dimensionless circular velocity which dominates for large radii since the Coulomb and Yukawa terms fade away. Another option is to include a Coulomb term in $\Phi_*^{(2)}(r_c^*)$ instead of the Yukawa term. This option will also lead to a flatten curve, but the Yukawa term is better due to its smoother behavior for large radii.

In Figure 9.6 the dimensionless circular velocities V_φ^* are plotted as functions of the dimensionless radial distance r_* for the cases of Newtonian and post-Newtonian theories. The gravitational potentials in the inner zone are given by the solutions of the system of differential equations (9.27) and (9.28) with the boundary conditions (9.29) and in the outer zone by $(9.40)_1$ and (9.44). We note that the curves flatten at very large radii. Moreover, the values of the dimensionless circular velocity are bigger

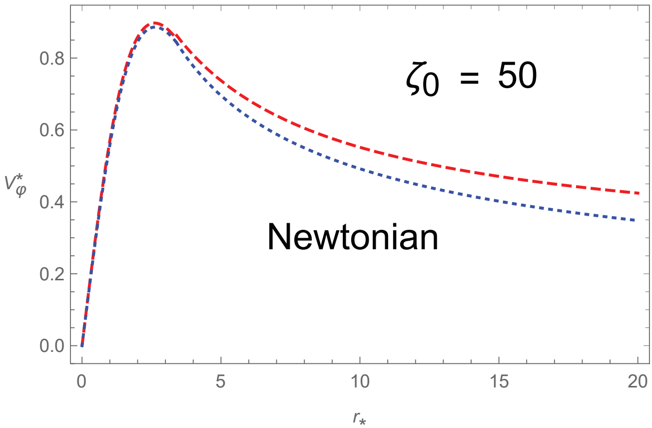

Figure 9.6: Dimensionless circular velocities as functions of the dimensionless radial coordinate r_* for the Newtonian and post-Newtonian approximation.

in the post-Newtonian theory than the ones in the Newtonian theory. Hence we can assert that the post-Newtonian theory furnishes corrections for the circular velocities of the Newtonian theory which can help to reduce the amount of dark matter needed to explain the rotation curves which flatten at large radii. These corrections by their own cannot overcome the whole problem of generating flat rotation curves, however, can reduce the amount of dark matter in relation with the Newtonian models.

A final remark refers to the influence of the boundary conditions in the behavior of the analyzed fields. The boundary condition that has more influence on the solutions refers to the Newtonian gravitational potential $U_*(r_*)$ at $r_* = 0$, but by restricting the values of $U_*(0)$ to the range $[0.5, 3]$ there is no change in the behavior of the curves, the only difference refers to the absolute values of the fields which become larger or smaller than the ones obtained for $U_*(0) = 1$.

9.6 Stationary spherically symmetrical systems

In this section we shall investigate the Jeans equation for stationary and spherically symmetrical self-gravitating fluid. The Jeans equation (4.71) was obtained in Section 4.3.1, which can be written as

$$\frac{d\rho\langle v_r^2\rangle}{dr}\left(1+\frac{2\phi}{c^2}\right)+\frac{2\rho\langle v_r^2\rangle}{c^2}\frac{d\phi}{dr}+2\rho\frac{\langle v_r^2\rangle\beta}{r}\left(1+\frac{2\phi}{c^2}\right)$$
$$+\frac{\rho}{c^2}\frac{d\psi}{dr}+\rho\frac{d\phi}{dr}\left[1+\frac{2\phi}{c^2}-\frac{4}{c^2}\langle v_r^2\rangle+\frac{9}{2}\frac{kT}{mc^2}\right]=0. \quad (9.50)$$

Here $\beta = 1-\langle v_\theta^2\rangle/\langle v_r^2\rangle$ is the velocity anisotropy parameter and it was assumed that $\langle v_\theta^2\rangle = \langle v_\varphi^2\rangle$. From the multiplication of the above equation by $\left(1-\frac{2\phi}{c^2}\right)$ and retaining terms up to the order $1/c^2$ we get that

$$\frac{d\rho\langle v_r^2\rangle}{dr}+2\rho\frac{\beta\langle v_r^2\rangle}{r}+\rho\frac{d\phi}{dr}\left[1-\frac{2}{c^2}\langle v_r^2\rangle+\frac{9}{2}\frac{kT}{mc^2}\right]+\frac{\rho}{c^2}\frac{d\psi}{dr}=0. \tag{9.51}$$

As was pointed out in Section 4.3.1 the radial velocity dispersion $\sqrt{\langle v_r^2\rangle}$ can be found as a solution of the above equation together with the Poisson equations for the gravitational potentials ϕ and ψ once we know the velocity anisotropy parameter β and the dependence of the mass density ρ on the radial distance r.

The two gravitational potentials in the Jeans equation (9.51) are determined from the Poisson equations

$$\nabla^2\phi=\frac{4\pi G}{c^2}\overset{0}{T}{}^{00},\qquad \nabla^2\psi=4\pi G\left(\overset{2}{T}{}^{00}+\overset{2}{T}{}^{ii}\right), \tag{9.52}$$

once the energy-momentum tensor components are known. The energy-momentum tensor components follow from the insertion of the distribution function (4.13) – for a stationary system where the hydrodynamic velocity vanishes $\mathbf{V}=0$ – and of the invariant integration element (4.20) into the definition of the energy-momentum tensor (9.18) and integration of the resulting equation, yielding

$$\overset{0}{T}{}^{00}+\overset{2}{T}{}^{00}=\rho c^2\left[1+\frac{1}{c^2}\left(\frac{3kT}{2m}-2\phi\right)\right], \tag{9.53}$$

$$\overset{2}{T}{}^{rr}=\rho\langle v_r^2\rangle,\qquad \overset{2}{T}{}^{\theta\theta}=\rho\langle v_\theta^2\rangle,\qquad \overset{2}{T}{}^{\varphi\varphi}=\rho\langle v_\varphi^2\rangle. \tag{9.54}$$

By taking into account (9.53) and (9.54) the Poisson equations (9.52) – in spherical coordinates where the gravitational potentials depend only on the radial coordinate $\phi = \phi(r)$ and $\psi = \psi(r)$ – become

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho, \tag{9.55}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = 4\pi G\rho\left[\langle v_r^2\rangle(3-2\beta) - 2\phi + \frac{3kT}{2m}\right]. \tag{9.56}$$

In order to solve the system of differential equations composed by the Jeans (9.51) and Poisson (9.55) and (9.56) equations we introduce the dimensionless variables

$$\sigma_r^2 = \frac{m}{kT}\langle v_r^2\rangle, \qquad \phi_* = \frac{m}{kT}\phi, \qquad \psi_* = \frac{m^2}{k^2T^2}\psi, \tag{9.57}$$

$$\zeta = \frac{mc^2}{kT}, \qquad \rho_* = \frac{\rho}{\rho_0}, \qquad r_* = \frac{\sqrt{4\pi G\rho_0}}{kT/m}\,r = k_J r. \tag{9.58}$$

Here, ρ_0 is a reference mass density, ζ a relativistic parameter – which depends on the temperature of the system and is related with the ratio of the rest energy of a particle and the thermal energy of the system – and k_J the Jeans wave number.

In terms of the dimensionless quantities (9.57) and (9.58) the dimensionless Jeans (9.51) and Poisson (9.55) and (9.56) equations become

$$\frac{d\rho_*\sigma_r^2}{dr_*} + 2\rho_*\frac{\beta\sigma_r^2}{r_*} + \rho_*\frac{d\phi_*}{dr_*}\left[1 - \frac{2}{\zeta}\sigma_r^2 + \frac{9}{2\zeta}\right] + \frac{\rho_*}{\zeta}\frac{d\psi_*}{dr_*} = 0, \tag{9.59}$$

$$\frac{1}{r_*^2}\frac{d}{dr_*}\left(r_*^2\frac{d\phi_*}{dr_*}\right) = \rho_*, \tag{9.60}$$

$$\frac{1}{r_*^2}\frac{d}{dr_*}\left(r_*^2\frac{d\psi_*}{dr_*}\right) = \rho_*\left[\sigma_r^2(3-2\beta) - 2\phi_* + \frac{3}{2}\right]. \tag{9.61}$$

For a given mass density profile and relativistic parameter ζ one can solve the system of differential equations (9.59) – (9.61).

As an application we shall investigate the effect of a central massive black hole on the velocity dispersion profile of the host galaxy following the method of the book of Binney and Tremaine [6]. The galaxy is assumed to have a constant mass-to-light ratio and the mass density and the Newtonian gravitational potential are given by the Hernquist model of scale-length a. Here the dimensionless mass density and Newtonian gravitational potential for the Hernquist model are written as

$$\rho_* = \frac{2}{r_*(r_*+1)^3}, \qquad \phi_* = -\frac{1}{r_*+1} - \frac{\mu}{r_*}. \tag{9.62}$$

The scale-length a is identified with the inverse of Jeans wave number $a = 1/k_J$ and ρ_0 is associated with the galaxy mass M_g. Furthermore $\mu = M_\bullet/M_g$ is the ratio of the black hole mass $M_\bullet$ and the galaxy mass M_g.

The Newtonian Poisson equation (9.60) is identically satisfied with the mass density and Newtonian gravitational potential representations of the Hernquist model (9.62).

The system of coupled differential equations given by (9.59) and (9.61) can be solved for the dimensionless radial velocity dispersion σ_r and gravitational potential ψ_* by assuming values for the velocity anisotropy parameter $\beta = 1 - \langle v_\theta^2\rangle/\langle v_r^2\rangle$, the ratio of the black hole and galaxy masses $\mu = M_\bullet/M_g$ and the relativistic parameter $\zeta = mc^2/kT$. This system was solved numerically with the boundary conditions $\sigma_r(3) = 0.1$, $\psi_*(3) = 0$ and $d\psi_*(3)/dr_* = 0$ for different values of the relativistic parameter ζ. Furthermore, the values of the two other parameters adopted here are similar to the ones given in the book by Binney and Tremaine [6]. Here the value for the ratio of the black hole and galaxy masses is $\mu = 0.004$, while the values for the velocity anisotropy parameter are $\beta = 0.1$ and $\beta = -0.1$ which correspond to a radial and a tangential bias, respectively.

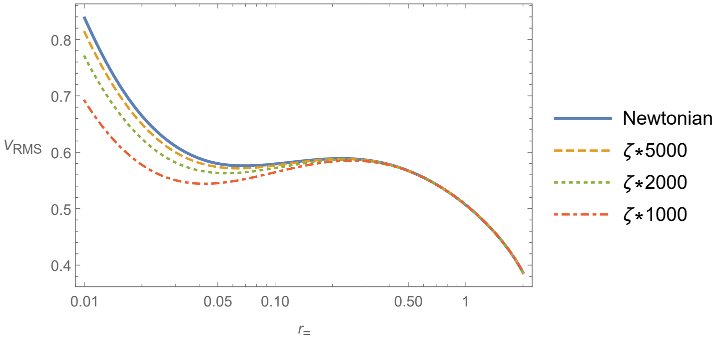

Figure 9.7: Dimensionless root mean square of the velocity V_{RMS} as function of the dimensionless radial distance r_* for $\beta = 0.1$ and $\mu = 0.004$. Newtonian solution (straight line) and post-Newtonian solutions (dashed lines) for different values of the relativistic parameter ζ.

In Figure 9.7 it is plotted the dimensionless root mean square velocity dispersion

$$V_{RMS} = \sqrt{\frac{\langle v_r^2\rangle + \langle v_\theta^2\rangle + \langle v_\varphi^2\rangle}{kT/m}} = \sqrt{3-2\beta}\,\sigma_r, \qquad (9.63)$$

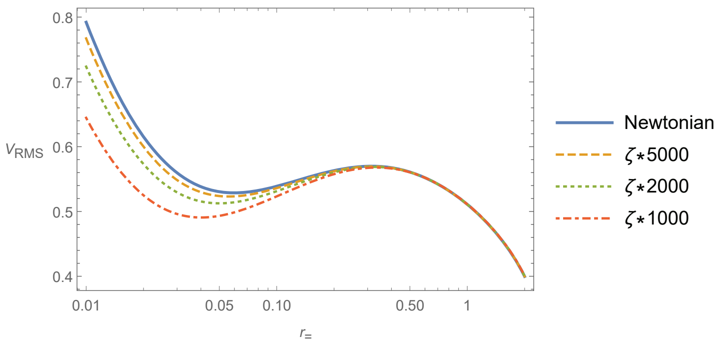

Figure 9.8: Dimensionless root mean square of the velocity V_{RMS} as function of the dimensionless radial distance r_* for $\beta = -0.1$ and $\mu = 0.004$. Newtonian solution (straight line) and post-Newtonian solutions (dashed lines) for different values of the relativistic parameter ζ.

as function of the dimensionless radial distance r_* for the case of a radial bias $\beta = 0.1$, ratio of the black hole and galaxy masses $\mu = 0.004$ and different values of the relativistic parameter $\zeta = mc^2/kT$. The Newtonian solution is represented by a straight line while the post-Newtonian solutions by dashed lines when the relativistic parameter assumes the values $\zeta = 5000, 2000, 1000$. We note from this figure that the black hole has influence on the dimensionless root mean square velocity dispersion which increases at small radii, because the deep potential well of the black hole increases the velocity of the stars.

Furthermore, by increasing the absolute temperature of the system the relativistic parameter ζ decreases as well as the dimensionless root mean square velocity dispersion.

The root mean square velocity dispersion V_{RMS} as function of the dimensionless radial distance r_* is plotted in Figure 9.8 for the case where the ratio of the black hole and galaxy masses is $\mu = 0.04$ and the velocity anisotropy parameter is $\beta = -0.1$ corresponding to a tangential bias. The behavior of the curves are the same as those in the preceding case, the difference lies in the values of the dimensionless root mean square velocity dispersion which are smaller in comparison with the previous case.

Appendix

Let S_{ij} be a Cartesian second order tensor in a three dimensional Euclidean space. According to Cayley-Hamilton theorem S_{ij} satisfies the characteristic polynomial equation

$$\left(\mathbf{S}^3\right)_{ij} - I_1\left(\mathbf{S}^2\right)_{ij} + I_2\mathbf{S}_{ij} - I_3\delta_{ij} = 0, \tag{9.64}$$

where the invariants are given by

$$I_1 = \mathbf{S}_{ii}, \qquad I_2 = \frac{1}{2}\left[(\mathbf{S}_{ii})^2 - \left(\mathbf{S}^2\right)_{ii}\right], \tag{9.65}$$

$$I_3 = \frac{1}{3}\left(\mathbf{S}^3\right)_{ii} - \frac{1}{2}\mathbf{S}_{ii}\left(\mathbf{S}^2\right)_{jj} + \frac{1}{6}\left(\mathbf{S}_{ii}\right)^3. \tag{9.66}$$

The inverse of $\mathbf{S}_{ij}$ follows from (9.64) and reads

$$\left(\mathbf{S}^{-1}\right)_{ij} = \frac{1}{I_3}\left[\left(\mathbf{S}^2\right)_{ij} - I_1\mathbf{S}_{ij} + I_2\delta_{ij}\right]. \tag{9.67}$$

In Section 9.2 the second order tensor $\mathbf{S}_{ij}$ is given by

$$\mathbf{S}_{ij} = \left[1 + \frac{1}{c^2}\left(\frac{V^2}{2} + 3U\right)\right]\delta_{ij} + \frac{V_iV_j}{c^2}, \tag{9.68}$$

so that the invariants up to the $1/c^2$ order are

$$I_1 = 3\left[1 + \frac{1}{c^2}\left(\frac{5V^2}{6} + 3U\right)\right], \tag{9.69}$$

$$I_2 = 3\left[1 + \frac{1}{c^2}\left(\frac{5V^2}{3} + 6U\right)\right], \tag{9.70}$$

$$I_3 = 1 + \frac{1}{c^2}\left(\frac{5V^2}{2} + 9U\right). \tag{9.71}$$

Hence the inverse up to the $1/c^2$ order calculated from (9.67) becomes

$$(\mathbf{S}^{-1})_{ij} = \left[1 - \frac{1}{c^2}\left(\frac{V^2}{2} + 3U\right)\right]\delta_{ij} - \frac{V_iV_j}{c^2}. \tag{9.72}$$

References

[1] C. A. Agón, J. F. Pedraza and J. Ramos-Caro, Kinetic theory of collisionless self-gravitating gases: Post-Newtonian polytropes, *Phys. Rev. D* **83**, 123007 (2011).

[2] J. Ramos-Caro, C. A. Agón and J. F. Pedraza, Kinetic theory of collisionless self-gravitating gases. II. Relativistic corrections in galatic dynamics, Phys.. Rev. D **86**, 043008 (2012).

[3] P.H. Nguyen and J. F. Pedraza, Anisotropic models for globular cluster, Galatic bulges, and dark halos, Phys. Rev. D **88**, 064020 (2013).

[4] G. M. Kremer, M. G. Richarte and K. Weber, Self-gravitating systems of ideal gases in the 1PN approximation, Phys. Rev. D **93**, 064073 (2016).

[5] S. Weinberg, *Gravitation and Cosmology. Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[6] J. Binney and S. Tremaine, *Galactic Dynamics*, 2nd. ed. (Princeton University Press, Princeton, 2008).

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