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Shai Dekel

POINTWISE VARIABLE ANISOTROPIC FUNCTION SPACES ON $\mathbb{R}^n$

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To my loving wife Adi and our amazing children Or, Chen, Ran, and Paz.
To my wonderful parents Sam and Dina... you are my inspiration!
My brother Asaf, my sister Vered, and the entire Dekel family... love!

Dedicated to my dear teachers Nira Dyn and Dany Leviatan on the occasion of their 80th birthday

Preface

> "One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis."
>
> Yves Meyer, preface to [33]

This book is in many ways the brainchild of Pencho Petrushev. During the author's visit to University of South Carolina in 2004, Pencho began drawing ellipses on the board whose shape changed from point to point and scale to scale and said: "Shai, I have a dream…". Pencho was looking for the right geometric setup that would bridge the gap between the classical isotropic setting of $\mathbb{R}^n$ equipped with the Euclidean metric and the more abstract setup of spaces of homogeneous type. The "dream" was to extend work that began as early as the 1960s and to generalize, in highly anisotropic setting, the entire scope of classical approximation, modern harmonic analysis, and function space theories. Meanwhile, Wolfgang Dahmen added his vision to the project. He was interested in establishing solid theoretical background for "meshless methods", which serve as a platform for the numerical solutions of partial differential equations. Indeed, solutions of many classes of differential equations exhibit anisotropic phenomena.

The author was fortunate enough to be invited by these two incredible mathematicians for a two week visit at the University of Aachen in 2005. In Aachen, the author's two main contributions were: being a good listener during the days' working sessions and being a reasonably good beer drinking companion during the evenings. The main outcomes of the visit were the first joint paper [22] and the basic foundations of [23]. The fundamental insights that lay the basis for the construction of the ellipsoid covers (see Section 2.2) were:

(i) The anisotropic construction should take place in $\mathbb{R}^n$ and use multilevel convex elements, so as to have the machinery of local algebraic polynomial approximation available. Since ellipsoids are the prototype of convex domains (see Proposition 1.6), they are the natural selection as building blocks.

(ii) The setup should support a generalized form of *pointwise variable anisotropy* and thus include as a very particular case the theory of classic anisotropic spaces, where the "directionality" is fixed over all points $x \in \mathbb{R}^n$. Therefore the setup should allow the ellipsoids' shape to change rapidly from point to point and from scale to scale.

(iii) The collection of ellipsoids should satisfy the notions of the abstract "balls" as in Stein's book [61, Section 1.1], since this implies a corresponding induced quasi-distance. As we will see, this necessitates that locally, in space and scale, intersecting ellipsoids need to have "equivalent" shapes.

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As will become apparent in Section 2.5, any space of homogeneous type over $\mathbb{R}^n$, equipped with the Lebesgue measure, whose anisotropic balls are “quasi-convex”, naturally fits into this framework.

In January 2020 the author visited Marcin Bownik at the University of Oregon to work on topics relating to Chapter 7 in this book. This visit served as an inception point for the book and Marcin, who is an amazing mathematician and wonderful person, provided tremendous help during the writing process.

Anisotropic phenomena naturally appear in nature and in various contexts in mathematical analysis and its applications. One example is the formation of shocks, which results in jump discontinuities of solutions of hyperbolic conservation laws across lower-dimensional manifolds. Another example arises in signal processing, where input functions have sharp edge or surface discontinuities separating between smooth areas. The central objective of this book is a very flexible framework, where the geometry of the anisotropic phenomena may change rapidly across space and scale.

Obviously, there is an incredible body of work that addresses the generalization of the isotropic theory to more general setting. Already in 1967, in proving the hypoellipticity of certain operators, Hörmander [47] studied differentiability and L_2 Lipschitz continuity along noncommuting vector fields. In the early 1970s, the development of modern “real-variable” harmonic analysis enabled Coifman and Weiss to begin developing parts of the theory such as singular operators and Hardy spaces in the setting of spaces of homogeneous type [19, 20]. Calderón and Torchinsky began studying in 1975 maximal operators based on an anisotropic dilation matrix subgroups [16, 17]. This line of research was generalized by Folland and Stein [39] in 1982, where they investigated the Hardy spaces over homogeneous groups. Nagel, Stein, and Wainger [56] established results in 1985, relating to basic properties of certain balls and metrics that can be naturally defined in terms of a given family of vector fields. As an application, they used these properties to obtain estimates for the kernels of approximate inverses of some nonelliptic partial differential operators, such as Hörmander’s sum of squares. In their book from 1987, Schmeisser and Triebel [59] devoted a full chapter to anisotropic function spaces, equipped with a fixed directional anisotropy. In 2003, Bownik [7] further developed and expanded anisotropic spaces based on powers of an anisotropic dilation matrix. In fact, his book is the main precursor to this book and in many ways inspired its writing. Marcin Bownik and Baode Li also helped with useful comments during the writing of the book.

Shai Dekel 2022

Contents

1 Local polynomial approximation over convex domains in $\mathbb{R}^n$

In this chapter, we review the theory of local approximation using multivariate algebraic polynomials of fixed total degree over "regular" domains in $\mathbb{R}^n$. By "regular" domains we mean domains that have nice geometric properties precisely defined in Section 1.1. The local smoothness analysis and approximation by algebraic polynomials are the critical components that allow us to construct anisotropic spaces that are a "true" generalization of the classical isotropic function spaces over $\mathbb{R}^n$. This is in contrast to general spaces of homogeneous type (see Definition 2.2) that do not have enough "structure", and thus function spaces defined over them are limited in various ways. In Section 1.2, we review the analysis tools we use to quantify local function smoothness. In Section 1.3, we provide some properties of algebraic polynomials over convex domains. We then provide estimates for the degree of polynomial approximation over domains, where Section 1.4 is focused on approximation in the p-norm, with $1 \le p \le \infty$, of the Sobolev class, and Section 1.5 is mostly dedicated to approximation in the p quasi-norm, with $0 < p < 1$.

1.1 Geometric properties of regular bounded domains

Definition 1.1. We denote by $B(x_0, r)$ the Euclidean ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. The image of the Euclidean unit ball $B^* := B(0,1)$ via an affine transformation is called an *ellipsoid*. For a given ellipsoid θ, we let A_θ be an affine transformation such that $\theta = A_\theta(B^*)$. Denoting by $v_\theta := A_\theta(0)$ the center of θ, we have

$$A_\theta(x) = M_\theta x + v_\theta, \quad \forall x \in \mathbb{R}^n, \tag{1.1}$$

where we may assume M_θ is a positive definite $n \times n$ matrix.

Any positive definite $n \times n$ real-valued matrix M may be represented in the form $M = UDU^{-1}$, where the matrix U is an $n \times n$ orthogonal matrix, and the matrix $D = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ is diagonal with $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n > 0$. It is easy to see that $\sigma_1 \ge \cdots \ge \sigma_n$ are the eigenvalues of M and $\sigma_1^{-1} \le \cdots \le \sigma_n^{-1}$ are the eigenvalues of M^{-1}. Hence

$$\|M\|_{\ell_2\to\ell_2} = \sigma_1 \quad \text{and} \quad \left\|M^{-1}\right\|_{\ell_2\to\ell_2} = 1/\sigma_n. \tag{1.2}$$

These norms have a clear geometric meaning. Thus if M_θ is as in (1.1), then $\operatorname{diam}(\theta) = 2\|M_\theta\|_{\ell_2\to\ell_2} = 2\sigma_1$. We may also say that the width of θ is $2\sigma_n$, since σ_n is the length of the smallest axis of θ.

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Lemma 1.2. *If two ellipsoids $\theta = M_\theta(B^*) + v_\theta$ and $\eta = M_\eta(B^*) + v_\eta$ satisfy $\eta \subseteq \theta$, then $M_\eta(B^*) \subseteq M_\theta(B^*)$.*

Proof. Without loss of generality, we can assume that $v_\eta = 0$. This implies that $B^* \subseteq M(B^*) + v$, where $M := M_\eta^{-1} M_\theta$, and $v := M_\eta^{-1} v_\theta$, and therefore it suffices to prove that $B^* \subseteq M(B^*)$. We first show that if $B^* \subseteq D(B^*) + v$, where $D := \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ is a diagonal matrix and $v \in \mathbb{R}^n$, then $B^* \subseteq D(B^*)$. Indeed, if $B^* - v \subseteq D(B^*)$, then $|\sigma_i| \geq \max(|1 - v_i|, |-1 - v_i|) \geq 1$, Therefore $B^* \subseteq D(B^*)$.

Next, since MM^T is a positive symmetric matrix, there exist a diagonal matrix D and an orthogonal matrix U such that $UMM^T U^T = D^2$. Then

$$\begin{aligned} D(B^*) &= \{Dx \in \mathbb{R}^n : xx^T \leq 1\} \\ &= \{y \in \mathbb{R}^n : y^T D^{-2} y \leq 1\} \\ &= \{y \in \mathbb{R}^n : y^T (UM(UM)^T)^{-1} y \leq 1\} \\ &= \{UMz \in \mathbb{R}^n : zz^T \leq 1\} \\ &= UM(B^*). \end{aligned}$$

Since $B^* \subseteq M(B^*) + v$, we obtain

$$B^* = U(B^*) \subseteq UM(B^*) + Uv = D(B^*) + Uv.$$

From the first part of the proof this implies $B^* \subseteq D(B^*) = UM(B^*) \Rightarrow B^* = U^T(B^*) \subseteq M(B^*)$. □

Definition 1.3. Let $\theta \subset \mathbb{R}^n$ be an ellipsoid such that $\theta = v_\theta + M_\theta(B^*)$, and let $Q > 0$. We denote by

$$Q \cdot \theta := v_\theta + QM_\theta(B^*)$$

the Q-dilation of θ.

Theorem 1.4 ([13]). *For two ellipsoids θ, η in $\mathbb{R}^n$ that satisfy $\eta \subseteq \theta$, the following converse is true:*

$$\theta \subseteq 2\frac{|\theta|}{|\eta|} \cdot \eta,$$

where $|\Omega|$ denotes the volume (Lebesgue measure) of a measurable set $\Omega \subset \mathbb{R}^n$. Furthermore, if the two ellipsoids have the same center, then this holds without the factor of 2.

Proof. Let $\theta = M_\theta(B^*) + v_\theta$ and $\eta = M_\eta + v_\eta$. Without loss of generality, we may assume that $v_\eta = 0$. Let $M := M_\eta^{-1} M_\theta$. By Lemma 1.2

$$\eta \subseteq \theta \Rightarrow M_\eta(B^*) \subseteq M_\theta(B^*) \Rightarrow B^* \subseteq M(B^*).$$

Also, as in the proof of Lemma 1.2, let $UMM^TU^T = D^2$, where U is orthogonal and $D = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$, $|\sigma_i| \geq 1$, $1 \leq i \leq n$, and $U^T D(B^*) = M(B^*)$. We have with $\sigma_{\max} := \max_{1\leq i\leq n} |\sigma_i|$

$$\frac{|\theta|}{|\eta|} = \frac{|M_\eta^{-1} M_\theta(B^*)|}{|B^*|} = \frac{|U^T D(B^*)|}{|B^*|} = \prod_{i=1}^n |\sigma_i| \geq \sigma_{\max}.$$

Therefore

$$M_\eta^{-1} M_\theta(B^*) = M(B^*) = U^T D(B^*) \subseteq \sigma_{\max} B^* \subseteq \frac{|\theta|}{|\eta|} B^*.$$

This gives

$$M_\theta(B^*) \subseteq \frac{|\theta|}{|\eta|} M_\eta(B^*), \tag{1.3}$$

which also proves the theorem for the case where $v_\theta = v_\eta$.

Next, since $\eta \subseteq \theta$,

$$\begin{aligned} B^* &= U(B^*) = UM_\eta^{-1}(\eta) \\ &\subseteq UM_\eta^{-1}(M_\theta(B^*) + v_\theta) \\ &= UM(B^*) + UM_\eta^{-1} v_\theta \\ &= D(B^*) + UM_\eta^{-1} v_\theta. \end{aligned}$$

In particular, since $0 \in D(B^*) + UM_\eta^{-1} v_\theta$, this gives that

$$-UM_\eta^{-1} v_\theta \in D(B^*) \subseteq \sigma_{\max} B^*,$$

and so

$$M_\eta^{-1} v_\theta \in \sigma_{\max} B^* \subseteq \frac{|\theta|}{|\eta|} B^*. \tag{1.4}$$

We conclude using (1.3) and (1.4) that

$$\theta = M_\eta M_\eta^{-1}(M_\theta(B^*) + v_\theta) \subseteq 2\frac{|\theta|}{|\eta|} M_\eta(B^*) = 2\frac{|\theta|}{|\eta|} \cdot \eta. \qquad \square$$

The ellipsoids are in fact the prototypical example of bounded convex domains.

Definition 1.5. A set $\Omega \subseteq \mathbb{R}^n$ is *convex* if for any two points $x, y \in \Omega$, the line segment $[x, y]$ is contained in Ω. The *convex hull* of a set $A \subset \mathbb{R}^n$ is the "minimal" convex set containing A, which is given by the intersection of all convex sets containing A.

Proposition 1.6 (John's lemma [48]). *For any bounded convex domain $\Omega \subset \mathbb{R}^n$, there exists an ellipsoid $\theta \subseteq \Omega$ such that*

$$\theta \subseteq \Omega \subseteq n \cdot \theta.$$

As depicted in Figure 1.1, this implies that the affine transformation $A_\theta^{-1}(x) := M_\theta^{-1}(x - v_\theta)$ gives

$$B(0,1) \subseteq A_\theta^{-1}(\Omega) \subseteq B(0,n). \tag{1.5}$$

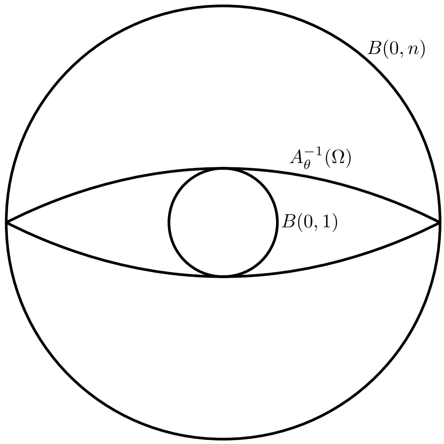

Figure 1.1: $A_\theta^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded convex domain.

It is interesting to note that John's ellipsoid θ is the ellipsoid with maximal volume such that $\theta \subseteq \Omega$. In some sense, this means that θ "covers" Ω sufficiently well. Our approximation theoretical applications of John's lemma use the fact that bounded convex domains are essentially equivalent to the Euclidean ball B^* up to an affine transformation and scale n.

Definition 1.7. A domain $\Omega \subset \mathbb{R}^n$ is *star-shaped* with respect to a Euclidean ball $B \subseteq \Omega$ (or a point $x_0 \in \Omega$), if for any point $x \in \Omega$, the convex hull of $\{x\} \cup B$ (or the line segment $[x, x_0]$) is contained in Ω.

Definition 1.8. We call the set

$$V := \{x \in \mathbb{R}^n : \ 0 \le |x| \le \rho, \angle(x, v) \le \kappa/2\},$$

a finite cone of axis direction v, height ρ, and aperture angle κ, where $\angle(x, v)$ is the angle between x and v. For $z \in \mathbb{R}^n$, the set $z + V := \{z + y, y \in V\}$ is a translate of V, which is a finite cone with head vertex at z. A cone V' is congruent to V if it can be obtained from V through a rigid motion.

We now define notions of "minimally smooth" domains (see [1, pp. 81–83], [60, p. 189]). Although we will be mostly dealing with bounded convex domains and, in particular, the particular case of ellipsoids, some of the results we use or prove hold for more general types of domains.

Definition 1.9. A domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the uniform cone property if there exist numbers $\delta > 0, L > 0$, a finite cover of open sets $\{U_j\}_{j=1}^J$ of $\partial\Omega$, and a corresponding collection $\{V_j\}_{j=1}^J$ of finite cones, each congruent to some fixed cone V, such that

(i) $\operatorname{diam}(U_j) \leq L, 1 \leq j \leq J$.

(ii) For any $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) < \delta$, we have $x \in \bigcup_{j=1}^J U_j$.

(iii) If $x \in \Omega \cap U_j$, then $x + V_j \subseteq \Omega, 1 \leq j \leq J$.

We will say the domain satisfies the overlapping uniform cone property if in addition the following condition is satisfied:

(iv) For every pair of points $x_1, x_2 \in \Omega$ such that $|x_1 - x_2| < \delta$ and $\operatorname{dist}(x_i, \partial\Omega) < \delta$, $i = 1, 2$, there exists an index j such that $x_i \in U_j$, $i = 1, 2$.

Theorem 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain such that $B(0, R_1) \subseteq \Omega \subseteq B(0, R_2)$ for some fixed $0 < R_1 < R_2$. Then Ω satisfies the overlapping uniform cone property with parameters that depend only on n, R_1, and R_2. Moreover, there exist $\delta > 0$ and a fixed cover $\{U_j\}_{j=1}^J$ with cones $\{V_j\}_{j=1}^J$, all congruent to a fixed cone V, that may be uniformly applied to all such convex domains.*

Proof. Our construction is based on the fact that if $B(0, R_1) \subsetneq \Omega$, then for any $x \in \Omega \setminus B(0, R_1)$, the convex closure of $\{x\} \cup B(0, R_1)$ is, by convexity, contained in Ω and also contained in a cone with head at x, axis direction of $-x$, and an aperture angle $\geq 2\arcsin(R_1/R_2)$.

Let $\{v_j\}_{j=1}^J$ be a finite set of normalized vector directions from the origin to be selected later. Let $V_{j,1}$ be the cone with head at the origin, axis v_j, height $9R_1/10$, and aperture angle $\kappa < \min(\pi/4, \arcsin(R_1/R_2))$. Let $V_{j,2}$, be the cone with head at the origin, axis direction v_j, height $R_2 + 1$, and the same aperture angle κ. Our covering of $\partial\Omega$ consists of $\{U_j\}_{j=1}^J$, $U_j := V_{j,2} \setminus V_{j,1}$. Thus $\operatorname{diam}(U_j) \leq \operatorname{diam}(B(0, R_2 + 1)) \leq 2(R_2 + 1) =: L$, and property (i) of Definition 1.9 is satisfied. With sufficient distribution of axis directions $\{v_j\}$, the cones $\{V_{j,2}\}$ overlap, cover $B(0, R_2)$, and thus also cover any $\Omega \subseteq B(0, R_2)$. Observe that this requires

$$J > \frac{S_{n-1}}{\kappa} = \frac{2\pi^{n/2}}{\Gamma(n/2)\kappa}.$$

Therefore $\{U_j\}_{j=1}^J$ cover $\overline{B(0, R_2)} \setminus B(0, R_1)$ and, in particular, $\partial\Omega$, since $\partial\Omega \subset \overline{B(0, R_2)} \setminus B(0, R_1)$. Thus property (ii) is satisfied for any $0 < \delta < R_1/10$.

We now construct for each U_j the corresponding cone V_j. It is in fact the cone with axis direction $-v_j$, aperture angle κ, and height of $R_1/10$. Thus all cones V_j are congruent to a single cone V.

Now for any convex Ω, $B(0,R_1) \subseteq \Omega \subseteq B(0,R_2)$, let $x \in \Omega \cap U_j$. There are two cases. If $x \in B(0,R_1)$, then $|x| > 9R_1/10$, and so $x + V_j \subset B(0,R_1) \subseteq \Omega$. The second case is $x \in \Omega \setminus B(0,R_1)$. Let Ω_x be the convex closure of $\{x\} \cup B(0,R_1)$. By convexity, $\Omega_x \subset \Omega$. We have that the angle between $-x$ and $-v_j$ is smaller than $\kappa/2$, the aperture angle of V_j is κ, whereas the aperture angle of Ω_x is $\geq 2\kappa$. Also, the height of V_j is $R_1/10$, which implies that $x + V_j \subset \Omega_x \subset \Omega$, which ensures property (iii). It remains to select $\{v_j\}_{j=1}^J$ to be sufficiently dense, so that $\{U_j\}_{j=1}^J$ have sufficient overlap, to ensure property (iv). It is sufficient to ensure that if $x, y \in B(0,R_2) \setminus B(0, 9R_1/10)$ and $|x-y| < \delta < R_1/10$, then there exists $1 \leq j \leq J$ such that $x, y \in V_{2,j}$. □

1.2 Moduli of smoothness

From this point, we assume that domains $\Omega \subset \mathbb{R}^n$ are measurable with a nonempty interior and that all functions are measurable as well.

1.2.1 Definitions and basic properties

Definition 1.11. Let $W_p^r(\Omega)$, $1 \leq p < \infty$, $r \in \mathbb{N}$, denote the *Sobolev spaces*, namely, the spaces of functions $g : \Omega \to \mathbb{C}$, $g \in L_p(\Omega)$, that have all their distributional derivatives of order up to r as functions in $L_p(\Omega)$. For $p = \infty$, we take $W_\infty^r(\Omega) = C^r(\Omega)$, that is, the functions with continuous bounded derivatives of order up to r. The norm of the Sobolev space is given by

$$\|g\|_{W_p^r(\Omega)} := \|g\|_{r,p} = \sum_{|\alpha| \leq r} \left\|\partial^\alpha g\right\|_{L_p(\Omega)}, \tag{1.6}$$

where for $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := \sum_{i=1}^n \alpha_i$, whereas the seminorm is given by

$$|g|_{W_p^r(\Omega)} := |g|_{r,p} = \sum_{|\alpha| = r} \left\|\partial^\alpha g\right\|_{L_p(\Omega)}. \tag{1.7}$$

It is known [1] that

$$\|g\|_{W_p^r(\Omega)} \sim \|g\|_{L_p(\Omega)} + |g|_{W_p^r(\Omega)}. \tag{1.8}$$

Definition 1.12. The K-functional of order r of $f \in L_p(\Omega)$, $1 \leq p \leq \infty$ (see, e. g., [35]) is defined by

$$K_r(f,t)_p := K(f,t,L_p(\Omega), W_p^r(\Omega)) := \inf_{g \in W_p^r(\Omega)} \{\|f-g\|_p + t|g|_{r,p}\}, \quad t > 0. \tag{1.9}$$

For a bounded domain Ω, we denote

$$K_r(f,\Omega)_p := K(f, \operatorname{diam}(\Omega)^r)_p. \tag{1.10}$$

It is important to note that the K-functional is unsuitable as a measure of smoothness if $0 < p < 1$. In fact, it is shown in [36] that for any finite interval $[a,b] \subset \mathbb{R}$, $0 < p < 1$, $0 < q \le \infty$, $r \ge 1$, and $t > 0$, $K_r(f, t^r, L_q([a,b]), W_p^r([a,b])) = 0$ for any $f \in L_q([a,b])$. This necessitates using other forms of smoothness in the range $0 < p < 1$.

For $f : \Omega \to \mathbb{C}$, $f \in L_p(\Omega)$, $0 < p \le \infty$, $h \in \mathbb{R}^n$, and $r \in \mathbb{N}$, we define the rth order difference operator $\Delta_h^r : L_p(\Omega) \to L_p(\Omega)$ by

$$\Delta_h^r(f,x) := \Delta_h^r(f,\Omega,x) := \begin{cases} \sum_{k=0}^{r} (-1)^{r+k}\binom{r}{k} f(x+kh), & [x, x+rh] \subset \Omega, \\ 0, & \text{otherwise}, \end{cases} \tag{1.11}$$

where $[x,y]$ denotes the line segment connecting any two points $x, y \in \mathbb{R}^n$.

Definition 1.13. The *modulus of smoothness of order* $\boldsymbol{r}$ is defined by

$$\omega_r(f,t)_p = \omega_r(f,\Omega,t)_p := \sup_{|h|\le t} \left\|\Delta_h^r(f,\Omega,\cdot)\right\|_{L_p(\Omega)}, \quad t > 0, \tag{1.12}$$

where $|h|$ denotes the l_2-norm of a vector $h \in \mathbb{R}^n$. For a bounded domain Ω, we also denote

$$\omega_r(f,\Omega)_p := \omega_r(f, \operatorname{diam}(\Omega))_p. \tag{1.13}$$

We list some of the properties of the modulus of smoothness that we will use throughout the book (see [35]) for more detail),

Proposition 1.14. *Let $\Omega \subseteq \mathbb{R}^n$ and $f, g \in L_p(\Omega)$, $0 < p \le \infty$. Then, for any $t > 0$:*
- (i) $\omega_r(f,t)_p \le c(r,p)\|f\|_p$. *In a more general form, for any $0 \le k < r$, $\omega_r(f,t)_p \le c(r,k,p)\omega_k(f,t)_p$ (where $\omega_0(f,\cdot)_p = \|f\|_p$).*
- (ii) $\omega_r(f+g,t)_p \le c(p)(\omega_r(f,t)_p + \omega_r(g,t)_p)$.
- (iii) *For any $\lambda \ge 1$, $\omega_r(f,\lambda t)_p \le (\lambda+1)^r \omega_r(f,t)_p$ for $1 \le p \le \infty$, and $\omega_r(f,\lambda t)_p^p \le (\lambda+1)^r \omega_r(f,t)_p^p$ for $0 < p < 1$.*
- (iv) *If $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$, then*

$$\omega_r(f,\Omega_1,t)_p \le \omega_r(f,\Omega_2,t)_p.$$

Also, for any vector $h \in \mathbb{R}^n$ and domain $\Omega \subseteq \mathbb{R}^n$,

$$\left\|\Delta_h^r(f,\Omega_1,\cdot)\right\|_{L_p(\Omega)} \le \left\|\Delta_h^r(f,\Omega_2,\cdot)\right\|_{L_p(\Omega)}. \tag{1.14}$$

1.2.2 K-functionals and moduli of smoothness

We now present the relationship of the difference and derivative operators using B-splines. We recall the univariate B-spline of order 1 (degree 0), $N_1(u) := \mathbf{1}_{[0,1]}(u)$. Then the B-spline of order r (degree $r-1$) is defined by $N_r := N_{r-1} * N_1$. The B-spline of order r is supported on $[0,r]$, is in C^{r-1}, and is a piecewise polynomial of degree $r-1$ over the integer intervals. For $h_1 > 0$, we define $N_r(u,h_1) := h_1^{-1}N_r(h_1^{-1}u)$. Let $g \in C^r(\Omega)$, and let $h \in \mathbb{R}^n$ with $|h| = h_1 > 0$. If the segment $[x, x+h]$ is contained in Ω, then for $\xi := h_1^{-1}h$ and $G(u) := g(x+u\xi)$, $u \in \mathbb{R}$, we have

$$\begin{aligned} h_1^{-1}\Delta_h(g,x) &= h_1^{-1}\int_0^{h_1} G'(u)du \\ &= \int_{\mathbb{R}} G'(u)N_1(u,h_1)du \\ &= \int_{\mathbb{R}} D_\xi g(x+u\xi)N_1(u,h_1)du, \end{aligned}$$

where

$$D_\xi g(y) := \lim_{u\to 0}\frac{g(y+u\xi)-g(y)}{u}.$$

By induction, for $r \ge 1$, we get

$$h_1^{-r}\Delta_h^r(g,x) = \int_{\mathbb{R}} G^{(r)}(u)N_r(u,h_1)du = \int_{\mathbb{R}} D_\xi^r g(x+u\xi)N_r(u,h_1)du. \tag{1.15}$$

Based on relation (1.15), we can bound the modulus of smoothness of the Sobolev class.

Theorem 1.15. *For $g \in W_p^r(\Omega)$, $r \ge 1$, $1 \le p \le \infty$,*

$$\omega_r(g,t)_p \le c(n,r)t^r|g|_{r,p}, \quad t > 0. \tag{1.16}$$

Proof. Let $g \in C^r(\Omega) \cap W_p^r(\Omega)$. Since $D_\xi g = \sum_{i=1}^n \xi_i \frac{\partial g}{\partial x_i}$ and $|\xi| = 1$, we have that $\|D_\xi g\|_p \le |g|_{1,p}$. We can see by induction that $D_\xi^r g = \sum_{|\alpha|=r} c_\alpha \partial^\alpha g$ with $|c_\alpha| \le c(n,r)$. This implies that $\|D_\xi^r g\|_p \le c(n,r)|g|_{r,p}$. Let $h \in \mathbb{R}^n$ with $0 < |h| = h_1 \le t$, let $\xi := h_1^{-1}h$, and denote $\Omega_{r,h} := \{x \in \Omega : [x, x+rh] \subset \Omega\}$. Applying (1.11), (1.15), and then Minkowski's inequality for $1 \le p \le \infty$ yields

$$\begin{aligned} \left\|\Delta_h^r(g,\cdot)\right\|_{L_p(\Omega)} &= \left\|\Delta_h^r(g,\cdot)\right\|_{L_p(\Omega_{h,r})} \\ &\le t^r\left\|\int_{\mathbb{R}} D_\xi^r g(\cdot+u\xi)N_r(u,h_1)du\right\|_{L_p(\Omega_{h,r})} \end{aligned}$$

$$\leq t^r \|D_\xi^r g\|_{L_p(\Omega)}$$
$$\leq c(n,r)t^r |g|_{r,p}.$$

Taking the supremum over all $h \in \mathbb{R}^n$, $|h| \leq t$, gives (1.16) for functions in $C^r(\Omega)$. For $1 \leq p < \infty$, we apply a standard density argument to obtain (1.16) for the Sobolev class. □

Proposition 1.16 ([49]). *Let $\Omega \subset \mathbb{R}^n$ satisfy the uniform cone property (Definition 1.9), and let $1 \leq p \leq \infty$ and $r \geq 1$. Then there exist constants $c_1(\Omega, p, n, r) > 0$ and $c_2(n, r) > 0$ such that for any $f \in L_p(\Omega)$,*

$$c_1 K_r(f, t^r)_p \leq \omega_r(f, t)_p \leq c_2 K_r(f, t^r)_p, \quad 0 < t \leq \operatorname{diam}(\Omega). \tag{1.17}$$

Proof. To see the right-hand side of (1.17), let g be any function in $W_p^r(\Omega)$. We apply (1.16) to obtain

$$\begin{aligned} \omega_r(f,t)_p &\leq \omega_r(f-g,t)_p + \omega_r(g,t)_p \\ &\leq 2^r \|f-g\|_p + C(n,r)t^r |g|_{r,p} \\ &\leq C(n,r)\big(\|f-g\|_p + t^r |g|_{r,p}\big). \end{aligned}$$

Therefore by taking the infimum over all such $g \in W_p^r(\Omega)$ we obtain the right-hand side of (1.17). The left-hand side is the main result of [49]. We note that the uniform cone property is a slightly stronger assumption than that used in [49]. □

Note that although c_2 in (1.17) depends only on n and r, the constant c_1 may further depend on the geometry of Ω (e. g., the parameters of the uniform cone property). We can obtain a more specific left-hand side inequality for convex domains. A first result for convex domains is the following:

Corollary 1.17. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain such that $B(0, R_1) \subseteq \Omega \subseteq B(0, R_2)$ for some fixed $0 < R_1 < R_2$. Then for $f \in L_p(\Omega)$, $1 \leq p \leq \infty$, $r \geq 1$, and $0 < t \leq 2R_2$,*

$$c_1(r,p,n,R_1,R_2) K_r(f,t^r)_{L_p(\Omega)} \leq \omega_r(f,t)_{L_p(\Omega)} \leq c_2(n,r) K_r(f,t^r)_{L_p(\Omega)}. \tag{1.18}$$

Proof. The right-hand side of (1.18) holds by (1.17) for more general domains. To prove the left-hand side inequality, we observe that by Theorem 1.10 Ω satisfies the uniform cone property with parameters that depend only on n, R_1, and R_2. Therefore by the method of proof of [49] the left-hand side of (1.18) holds with constant $c_1(r, p, n, R_1, R_2)$. □

The proof of the second result on the relationship between K-functional and moduli of smoothness over convex domains actually requires using the "local" polynomial approximation results of the next chapter. We state it here.

Proposition 1.18 ([26]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then, for any $f \in L_p(\Omega)$, $1 \le p \le \infty$, and $r \ge 1$,*

$$K_r(f, t^r) \le c(n, r, p)\left(\left(1 - \frac{t^r}{\operatorname{diam}(\Omega)^r}\right)\mu(\Omega, t)^{-(r-1+1/p)} + 1\right)\omega_r(f, t)_p,$$

where

$$\mu(\Omega, t) := \min_{x\in\Omega} \frac{|B(x,t) \cap \Omega|}{|B(x,t)|}, \quad 0 < t \le \operatorname{diam}(\Omega).$$

1.2.3 Marchaud inequalities

We saw that the modulus of smoothness has the property that for any $1 \le k < r$, $\omega_r(f,t)_p \le c(r,k,p)\omega_k(f,t)_p$, $t > 0$. Marchaud-type inequalities serve as the inverse. They are easier to prove for the simple cases of $\Omega = \mathbb{R}^n$ or where Ω is a univariate segment. We will require the following results over regular domains.

Proposition 1.19 ([49]). *Let Ω be a domain with the uniform cone property, and let $f \in L_p(\Omega)$, $1 \le p \le \infty$. Then for any $1 \le k < r$ and $0 < t < 1$,*

$$\omega_k(f,t)_p \le ct^k\left(\int_t^1 \frac{\omega_r(f,s)_p}{s^{k+1}} ds + \|f\|_p\right),$$

where the constant c depends on n, k, r and the uniform cone properties of Ω.

The proof of Proposition 1.19 for the case $1 \le p \le \infty$ is facilitated by the equivalence (1.17). In the case $0 < p < 1$, we are not equipped with the K-functional and need construct a "direct" proof [30]. We begin with a technical lemma.

Lemma 1.20. *Let Ω be a bounded open domain, $\tilde{t} > 0$, and $H \in \mathbb{R}^n$ a unit vector. Let $U \subset \Omega$ be an open subdomain such that for any $x \in U$, $[x, x + \tilde{t}H] \subset \Omega$. Let $S(x, \Omega)$, be the connected segment of the line passing through $x \in U$ with direction H that is contained in Ω. We denote*

$$\Omega_U := \bigcup_{x\in U} S(x, \Omega).$$

For $1 \le k \le r$ and $f \in L_p(\Omega)$, $0 < p \le 1$, we denote

$$\omega_k^H(f,t)_p^p := \sup_{|s|\le t} \int_{\Omega_U} \left|\Delta_{sH}^k(f,\Omega,x)\right|^p dx, \quad 0 < t \le \frac{\tilde{t}}{2r}. \tag{1.19}$$

Then

$$\omega_k^H(f,t)_p^p \le c(r,p,\tilde{t})t^{kp}\left(\int_t^{\tilde{t}} \frac{\omega_r(f,\Omega,s)_p^p}{s^{kp+1}}ds + \|f\|_{L_p(\Omega)}^p\right).$$

Proof. The setup of the lemma enables us to apply an induction process similar to the proof in the univariate case (see, e. g., Theorems 2.8.1 and 2.8.2 in [35]). First, assume that $r = k+1$. We partition $\Omega_U = \Omega_1 \cup \Omega_2$, where $\Omega_1 := \bigcup_{x\in U} S_1(x,\Omega)$ and $\Omega_2 := \bigcup_{x\in U} S_2(x,\Omega)$ with each segment partitioned $S(x,\Omega) = S_1(x,\Omega) \cup S_2(x,\Omega)$ at its midpoint. Observe that we are ensured that the length of each $S(x,\Omega)$ is at least $\tilde{t}$.

Now let $h = sH$, where $0 < s \le t \le \tilde{t}/4r$. For any $x \in \Omega_1$, we have that $[x, x+2kh] \subset \Omega_U$. This implies that

$$(T_h - I)^k = 2^{-k}(T_{2h} - I)^k + Q(T_h)(T_h - I)^{k+1} \tag{1.20}$$

is well defined on $L_p(\Omega_1)$ with $T_h f := f(\cdot + h)$ and

$$Q(z) := \frac{1 - 2^{-k}(z+1)^k}{z-1} \in \Pi_{k-1}(\mathbb{R}).$$

Next, observe that if $Q(z) = \sum_0^{k-1} a_i z^i$ and $g \in L_p(\Omega)$, then

$$\|Q(T_h)g\|_{L_p(\Omega_1)}^p \le \sum_0^{k-1} a_i^p \|T_h^i g\|_{L_p(\Omega_1)}^p \le C(k,p)\|g\|_{L_p(\Omega_U)}^p.$$

Applying (1.20) with definition (1.19) gives

$$\begin{aligned}\|\Delta_h^k f\|_{L_p(\Omega_1)}^p &\le 2^{-kp}\|\Delta_{2h}^k f\|_{L_p(\Omega_1)}^p + C\|\Delta_h^{k+1} f\|_{L_p(\Omega_U)}^p \\ &\le 2^{-kp}\|\Delta_{2h}^k f\|_{L_p(\Omega_1)}^p + C\omega_{k+1}^H(f,s)_p^p.\end{aligned}$$

By repeated application we get, for $2^m s \le \tilde{t}/4r$,

$$\|\Delta_h^k f\|_{L_p(\Omega_1)}^p \le C\left(2^{-mkp}\|f\|_{L_p(\Omega_U)}^p + \sum_{j=0}^m 2^{-jkp}\omega_{k+1}^H(f,2^j s)_p^p\right).$$

Our next step is bounding the kth difference operator on $L_p(\Omega_2)$. If $x+kh \in \Omega_2$, then there exists $x_0 \in U$ such that $x+kh \in S_2(x_0,\Omega)$. This implies that $[x-kh, x+kh] \subset \Omega_U$. Using the equality $|\Delta_h^k(f,x)| = |\Delta_{-h}^k(f(\cdot + kh),x)|$, we can apply the same machinery as above on Ω_2 for the function $f(\cdot + kh)$ and the difference vector $-h$ to obtain

$$\begin{aligned}\|\Delta_h^k f\|_{L_p(\Omega_2)}^p &= \|\Delta_{-h}^k f(\cdot + kh)\|_{L_p(\Omega_2)}^p \\ &\le C\left(2^{-mkp}\|f\|_{L_p(\Omega_U)}^p + \sum_{j=0}^m 2^{-jkp}\omega_{k+1}^H(f,2^j s)_p^p\right).\end{aligned}$$

Combining the above two estimates on Ω_1 and Ω_2 gives

$$\omega_k^H(f,t)_p^p \le C\left(2^{-mkp}\|f\|_{L_p(\Omega_U)}^p + \sum_{j=0}^{m} 2^{-jkp}\omega_{k+1}^H(f,2^jt)_p^p\right).$$

By induction we may conclude that, for $2^m t \le \tilde{t}/4r$,

$$\omega_k^H(f,t)_p^p \le C\left(2^{-mkp}\|f\|_{L_p(\Omega_U)}^p + \sum_{j=0}^{m} 2^{-jkp}\omega_r^H(f,2^jt)_p^p\right).$$

Now choose m such that

$$2^m t \le \frac{\tilde{t}}{4r} \le 2^{m+1}t.$$

Then

$$2^{-m} \le \frac{8rt}{\tilde{t}} \Rightarrow 2^{-mkp} \le C(r,p,\tilde{t})t^{kp}.$$

This allows us to obtain the desired result:

$$\begin{aligned}
\omega_k^H(f,\Omega,t)_p^p &\le Ct^{kp}\left(\|f\|_{L_p(\Omega_U)}^p + \sum_{j=0}^{m}\omega_r^H(f,2^jt)_p^p\int_{2^jt}^{2^{(j+1)}t}\frac{1}{s^{kp+1}}ds\right)\\
&\le Ct^{kp}\left(\|f\|_{L_p(\Omega_U)}^p + \sum_{j=0}^{m}\int_{2^jt}^{2^{(j+1)}t}\frac{\omega_r^H(f,s)_p^p}{s^{kp+1}}ds\right)\\
&\le Ct^{kp}\left(\|f\|_{L_p(\Omega_U)}^p + \int_{t}^{\tilde{t}/4r}\frac{\omega_r^H(f,s)_p^p}{s^{kp+1}}ds\right)\\
&\le Ct^{kp}\left(\|f\|_{L_p(\Omega)}^p + \int_{t}^{\tilde{t}}\frac{\omega_r(f,\Omega,s)_p^p}{s^{kp+1}}ds\right).
\end{aligned}$$

□

Theorem 1.21. *Let Ω satisfy the overlapping uniform cone property, and let $f \in L_p(\Omega)$, $0 < p \le 1$. Then for any $r \ge 2$, there exists $\tilde{t} > 0$ such that for $0 < t \le \tilde{t}$,*

$$\omega_1(f,t)_p^p \le ct^p\left(\int_t^{\tilde{t}}\frac{\omega_r(f,s)^p}{s^{p+1}}ds + \|f\|_p^p\right), \tag{1.21}$$

where the constant c depends on n, p, r and overlapping uniform cone properties of Ω.

Proof. Using Definition 1.9, it is easy to see that we may "normalize" the collection of finite cones $\{V_j\}_{j=1}^J$ to all be congruent to a single fixed cone V by taking the minimum over the cones' heights and aperture angles. Obviously, after this process, we still have that $x+V_j \subset \Omega$ for any $x \in U_j$. We also ensure that the height of the fixed cone is smaller than δ. We denote this height by $\rho := \rho(\Omega)$. Then we add $U_{J+1} := \{x \in \Omega : \quad \operatorname{dist}(x, \partial\Omega) > \delta\}$ to the cover. If U_{J+1} is not empty, then we can apply Lemma 1.20 with $U = U_{J+1}$ and arbitrary unit vector H to obtain

$$\omega_1(f, \Omega, t)^p_{L_p(U_{J+1})} \le Ct^p \left(\int_t^\delta \frac{\omega_r(f,s)^p}{s^{p+1}} ds + \|f\|_p^p \right), \quad t \le \delta/r.$$

Later we will need that for any constant $\tilde{t} \le \delta$,

$$\int_t^\delta \frac{\omega_r(f,s)^p}{s^{p+1}} ds + \|f\|_p^p \le \int_t^{\tilde{t}} \frac{\omega_r(f,s)^p}{s^{p+1}} ds + C(\tilde{t}, r, p)\|f\|_p^p, \quad t \le \tilde{t}/r,$$

which gives

$$\omega_1(f, \Omega, t)^p_{L_p(U_{J+1})} \le Ct^p \left(\int_t^{\tilde{t}} \frac{\omega_r(f,s)^p}{s^{p+1}} ds + \|f\|_p^p \right), \quad t \le \tilde{t}/r. \tag{1.22}$$

We now proceed to estimate on the regions "near" $\partial\Omega$. Let $h \in \mathbb{R}^n$, $|h| \le \tilde{t}/r$, where $\tilde{t}$ satisfies $0 < \tilde{t}(\rho, \kappa) \le \rho \le \delta$ and will be determined later. We argue that for this difference vector, it only remains to estimate $\|\Delta_h(f, \cdot)\|_{L_p(U_{\partial\Omega,h})}$, with

$$U_{\partial\Omega,h} := \{x \in \Omega : \ [x, x+h] \subset \Omega, \operatorname{dist}(x, \partial\Omega) < \delta, \operatorname{dist}(x+h, \partial\Omega) < \delta\}.$$

Indeed, if $[x, x+h]$ is not a subset of Ω, then by definition $\Delta_h(f, x) = 0$. If either x or $x+h$ are away from the boundary, then $|f(x+h) - f(x)|$ was already part of the integration over U_{J+1}. The technical difficulty we are facing when dealing with $U_{\partial\Omega,h}$ is that there might not be "sufficient intersection" of the infinite line going through $[x, x+h]$ with Ω. This requires to use the overlapping uniform cone properties of Ω. Since $|h| \le \delta$, by property (iv) in Definition 1.9 there exists $1 \le j \le J$ such that $x, x+h \in U_j$ (note that $[x, x+h]$ may not be a subset of U_j), $x + V_j$, and $x + h + V_j \subset \Omega$.

By geometric consideration, as depicted in Figure 1.2, there exist $\tilde{c} > 0$ and $0 < \tilde{t} \le \rho$ such that if $|h| \le \tilde{c}$, then the cones $x + V_j$ and $x + h + V_j$ intersect, and there is a point $z \in (x + V_j) \cap (x + h + V_j)$ such that $|x - z|, |x + h - z| \le \tilde{t}$, where the constants depend on the hight ρ and the head-angle κ of the reference cone V. For example, if $x + h \in x + V_j$, then we may choose $z = x + h$. In any case, now the lines going through the segments $[x, z]$ and $[x + h, z]$ have "sufficient intersection" with Ω of the height of the reference cone at least ρ.

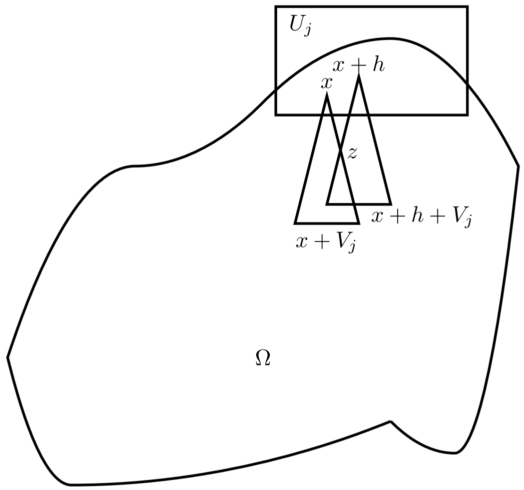

Figure 1.2: The points x and $x+h$ are contained in some U_j, and $z \in (x+V_j) \cap (x+h+V_j)$.

This leads to the partition

$$U_{\partial\Omega,h} = \bigcup_{j=1}^{J} U_{h,j}, \quad U_{h,j} := \{x \in U_{\partial\Omega,h} : x, x+h \in U_j\}.$$

It follows from the discussion above that there exist two unit vectors $H_{j,1}$ and $H_{j,2}$ such that if $x \in U_{h,j}$, then

(i) $h = a_1 H_{j,1} - a_2 H_{j,2}$ with $0 \le a_1, a_2 \le C|h|$,
(ii) $[x, x + a_1 H_{j,1}] \subset x + V_j$ and $[x+h, x+h+a_2 H_{j,2}] \subset x + h + V_j$,
(iii) the connected components of the intersection of Ω with the infinite lines containing the segments $[x, x+a_1 H_{j,1}]$ and $[x+h, x+h+a_2 H_{j,2}]$ are at least of length $\tilde{t}$.

The above properties allow us to apply Lemma 1.20 twice with $U = U_{J+1}$ and $H = H_{j,1}, H_{j,2}$ which gives

$$\begin{aligned}\|\Delta_h f\|_{L_p(U_{h,j})} &\le \|\Delta_{a_1 H_{j,1}} f\|_{L_p(U_{h,j})} + \|\Delta_{a_2 H_{j,2}} f\|_{L_p(U_{h,j})} \\ &\le C t^p \left(\int_t^{\tilde{t}} \frac{\omega_r(f,\Omega,s)_p^p}{s^{p+1}} ds + \|f\|_{L_p(\Omega)}^p \right).\end{aligned}$$

We now sum this estimate over all $U_{h,j}$ and then take the supremum on $h \le \tilde{t}/r$. Finally, the proof of the theorem is completed by adding estimate (1.22) over U_{J+1} to the estimate over $\bigcup_{j=1}^{J} U_j$. □

Corollary 1.22. *Let Ω be a convex domain with $B(0,R_1) \subseteq \Omega \subseteq B(0,R_2)$ for $0 < R_1 < R_2 < \infty$. Then for any $r \ge 2$, there exists $\tilde{t} > 0$ such that for any $0 < t \le \tilde{t}/r$, $0 < p \le 1$, and*

$f \in L_p(\Omega)$,

$$\omega_1(f,t)_p^p \le ct^p \left(\int_t^{\tilde{t}} \frac{\omega_r(f,s)^p}{s^{p+1}} ds + \|f\|_p^p \right), \tag{1.23}$$

where the constant c depends on R_1, R_2, n, p, r.

1.3 Algebraic polynomials over domains

Let $\Pi_{r-1} := \Pi_{r-1} = \Pi_{r-1}(\mathbb{R}^n)$ denote the multivariate polynomials of total degree $r-1$ (order r) in n variables. This is the collection of functions of the type $P(x) = \sum_{|\alpha|<r} c_\alpha x^\alpha$, where for $\alpha \in \mathbb{Z}_+^n$, $c_\alpha \in \mathbb{C}$, $|\alpha| := \sum_{i=1}^n \alpha_i$, and $x \in \mathbb{R}^n$, $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$. By $|\Omega|$ we denote the Lebesgue measure of a set Ω.

Lemma 1.23 ([30]). *Let* $P \in \Pi_{r-1}$, *and let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *be bounded convex domains such that* $\Omega_1 \subseteq \Omega_2$ *and* $|\Omega_2| \le \rho|\Omega_1|$ *for some* $\rho > 1$. *Then for* $0 < p \le \infty$,

$$\|P\|_{L_p(\Omega_2)} \le c(n,r,p,\rho)\|P\|_{L_p(\Omega_1)}.$$

Proof. Let $Ax = Mx + b$ be the affine transformation for which (1.5) holds for Ω_1. Since $A^{-1}(\Omega_1) \subseteq B(0,n)$, we have

$$\begin{aligned} \left|A^{-1}(\Omega_2)\right| &= \left|A^{-1}(\Omega_1)\right| \frac{|A^{-1}(\Omega_2)|}{|A^{-1}(\Omega_1)|} \\ &\le \left|B(0,n)\right|\rho := C(n,\rho). \end{aligned} \tag{1.24}$$

Observe that $A^{-1}(\Omega_2)$ is a convex domain that contains $A^{-1}(\Omega_1)$ and therefore also contains $B(0,1)$. Together with (1.24), this implies that the diameter of $A^{-1}(\Omega_2)$ must be bounded by a constant that depends on n and ρ, i. e., $A^{-1}(\Omega_2) \subseteq B(0,R)$, $R := R(n,\rho)$. Hence applying the equivalence of finite-dimensional (quasi-)normed spaces, we obtain

$$\begin{aligned} \|P\|_{L_p(\Omega_2)} &= \left|\det(M)\right|^{1/p} \|P\|_{L_p(A^{-1}(\Omega_2))} \\ &\le \left|\det(M)\right|^{1/p} \|P\|_{L_p(B(0,R))} \\ &\le C\left|\det(M)\right|^{1/p} \|P\|_{L_p(B(0,1))} \\ &\le C\left|\det(M)\right|^{1/p} \|P\|_{L_p(A^{-1}(\Omega_1))} \\ &= C\|P\|_{L_p(\Omega_1)}. \end{aligned}$$

□

Lemma 1.24 ([30]). *For any bounded convex domain* $\Omega \subset \mathbb{R}^n$, $P \in \Pi_{r-1}$, *and* $0 < p, q \le \infty$, *we have*

$$\|P\|_{L_q(\Omega)} \sim |\Omega|^{1/q-1/p} \|P\|_{L_p(\Omega)} \tag{1.25}$$

with constants of equivalency depending only on n, r, p, and q.

Proof. Let $Ax = Mx + b$ be the affine transformation for which (1.5) holds. Since $A(B(0,1)) = \theta$, from the properties of John's ellipsoid we get $|\det(M)| \sim |\Omega|$ with constants of equivalency depending only on n. Also, by the equivalence of finite-dimensional (quasi-)normed spaces, for any polynomial $\tilde{P} \in \Pi_{r-1}$, we have that $\|\tilde{P}\|_{L_p(B(0,1))} \sim \|\tilde{P}\|_{L_q(B(0,n))}$ with constants of equivalency that depend only on n, r, p, and q. Let $P \in \Pi_{r-1}$, and denote $\tilde{P} := P(A\cdot)$. Then

$$\begin{aligned}
\|P\|_{L_q(\Omega)} &= \left|\det(M)\right|^{1/q} \|\tilde{P}\|_{L_q(A^{-1}(\Omega))} \\
&\le \left|\det(M)\right|^{1/q} \|\tilde{P}\|_{L_q(B(0,n))} \\
&\le C\left|\det(M)\right|^{1/q} \|\tilde{P}\|_{L_p(B(0,1))} \\
&\le C\left|\det(M)\right|^{1/q} \|\tilde{P}\|_{L_p(A^{-1}(\Omega))} \\
&\le C\left|\det(M)\right|^{1/q-1/p} \|P\|_{L_p(\Omega)} \\
&\le C|\Omega|^{1/q-1/p} \|P\|_{L_p(\Omega)}.
\end{aligned}$$

□

We will need the following Bernstein–Markov-type inequality, which provides an estimate for the norms of derivatives of algebraic polynomials (see also [51]):

Proposition 1.25 ([57]). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded convex domain. Then, for* $1 \le p \le \infty$, *any polynomial* $P \in \Pi_{r-1}$, *and* $\alpha \in \mathbb{Z}_+^n$ *such that* $|\alpha| := \sum_{i=1}^n \alpha_i \le r-1$,

$$\left\|\partial^\alpha P\right\|_{L_p(\Omega)} \le C(n, |\alpha|)\, \mathrm{width}(\Omega)^{-|\alpha|} \|P\|_{L_p(\Omega)}, \tag{1.26}$$

where width(Ω) *is the diameter of the largest n-dimensional Euclidean ball contained in* Ω.

Theorem 1.26. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain, and let* $0 < p < \infty$. *Then, for any* $P \in \Pi_{r-1}$, *we have that* $\omega_r(P,t)_p = 0$, $0 < t \le \mathrm{diam}(\Omega)$. *In the other direction, if* Ω *is also open and connected and* $f \in L_p(\Omega)$ *is such that* $\omega_r(f,\Omega)_p = 0$ *for some* $r \ge 1$, *then there exists a polynomial* $P \in \Pi_{r-1}$ *such that* $f = P$ *a. e. on* Ω.

Proof. The first part is a direct application of identity (1.15), since it implies that $\Delta_h^r(P,x) = 0$ for any $x \in \Omega$ and $h \in \mathbb{R}^n$. To prove the second part, we apply the Whitney decomposition of Ω into interior disjoint cubes (see, e. g., the appendix in [41]). Namely, there exists a family of closed cubes $\{Q_k\}_{k=1}^\infty$ such that:

(i) $\bigcup_k Q_k = \Omega$, and the cubes Q_k, have disjoint interiors,
(ii) $\sqrt{n}l(Q_k) \le \operatorname{dist}(Q_k, \Omega^c) \le 4\sqrt{n}l(Q_k)$, where $l(Q_k)$ is the side length of Q_k,
(iii) if the boundaries of Q_k and Q_j touch, then

$$\frac{1}{4} \le \frac{l(Q_j)}{l(Q_k)} \le 4,$$

(iv) for any Q_k, there are at most 12^n cubes Q_j that touch it.

Now from the Whitney decomposition we construct a cover of "substantially" overlapping cubes $\{\tilde{Q}_k\}_{k=1}^{\infty}$, simply by symmetrically extending the lengths of the cubes, such that $l(\tilde{Q}_k) = 2l(Q_k)$, $1 \le k \le \infty$. By property (ii) of the Whitney decomposition we know that each $\tilde{Q}_k$ is contained in Ω, and thus $\bigcup_k \tilde{Q}_k = \Omega$. Also, for touching cubes Q_k and Q_j, the extensions have a "substantial" intersection, i. e.,

$$|\tilde{Q}_k \cap \tilde{Q}_j| \ge \min\{l(Q_k)/2, l(Q_j)/2\}^n.$$

As we will see, in the subsequent sections, we work hard to prove the anisotropic theory of "local" polynomial approximation. In particular, we produce uniform bounds for polynomial approximation on bounded convex domains in the p-norms, $0 < p \le \infty$. However, here, on the cubes $\{\tilde{Q}_k\}$, we may apply the isotropic theory. Namely, we may use the Whitney-type inequality on the unit cube [62], which by the invariance under dilations implies that there exists a constant $c(p, n, r) > 0$ such that $E_{r-1}(f, \tilde{Q}_k)_p := \inf_{P\in\Pi_{r-1}} \|f - P\|_{L_p(\tilde{Q}_k)} \le c\omega_r(f, \tilde{Q}_k)_p$. This means that $f = P_k$ a. e. on $\tilde{Q}_k$ for some $P_k \in \Pi_{r-1}$, $1 \le k \le \infty$. Since Ω is a connected domain, using the "substantial" intersections of the extended cubes of touching cubes yields that for touching cubes Q_k, Q_j, we have that $P_k = P_j$. From this we may conclude by induction (on a sequence of cubes touching at least one cube from the set of previous cubes) that there exists a unique $P \in \Pi_{r-1}$ such that $P = P_k$ for all k. This concludes the proof. □

Remark 1.27. Note that we should take care not to use the anisotropic Whitney theorem (Theorem 1.34) in the proof of the second part of Theorem 1.26 for the case $0 < p < 1$, since we would end up with a circular argument.

1.4 The Bramble–Hilbert lemma for convex domains

Given a bounded regular domain $\Omega \subset \mathbb{R}^n$, our goal is estimating the degree of approximation of a function $f \in L_p(\Omega)$, $0 < p \le \infty$, by algebraic polynomials of total degree $r - 1$,

$$E_{r-1}(f, \Omega)_p := \inf_{P\in\Pi_{r-1}} \|f - P\|_{L_p(\Omega)}.$$

For a star-shaped domain Ω (see Definition 1.7), we denote

$$\rho_{\max} := \max\{\rho \mid \Omega \text{ is star-shaped with respect to a ball } B \subseteq \Omega \text{ of radius } \rho\}.$$

The *chunkiness parameter* of Ω [15] is defined as

$$\gamma := \frac{\operatorname{diam}(\Omega)}{\rho_{\max}}. \tag{1.27}$$

Note that the chunkiness parameter γ becomes larger in cases where the domain is longer and thinner. This leads to the following Bramble–Hilbert formulation (see, e. g., [15]).

Theorem 1.28 (Bramble–Hilbert lemma for star-shaped domains). *Let Ω be a bounded domain that is star-shaped with respect to some ball B with chunkiness parameter γ, and let $g \in W_p^r(\Omega)$, $1 \le p \le \infty$, $r \ge 1$. Then there exists a polynomial $P \in \Pi_{r-1}$ such that*

$$|g - P|_{k,p} \le C(n,r)(1+\gamma)^n \operatorname{diam}(\Omega)^{r-k} |g|_{r,p}, \quad k = 0,1,\dots,r-1. \tag{1.28}$$

Before we proceed with the proof of Theorem 1.28, we need some preparation. Let $g \in C^r(\Omega)$ and recall that the classical *Taylor polynomial* of order r (degree $r-1$) at $x \in \Omega$ about a point $y \in B$ is given by

$$T_y^r g(x) := \sum_{|\alpha|<r} \frac{\partial^\alpha g(y)}{\alpha!}(x-y)^\alpha, \tag{1.29}$$

where $\alpha! := \prod_{i=1}^n \alpha_i!$. Then the *Taylor remainder* of order r is given by

$$R_y^r g(x) := g(x) - T_y^r g(x) = r \sum_{|\alpha|=r} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 s^{r-1} \partial^\alpha g\big(x + s(y-x)\big) ds, \tag{1.30}$$

which is meaningful, since the segment $[x,y]$ is contained in Ω. Then we have

$$g(x) = T_y^r g(x) + R_y^r g(x), \quad x \in \Omega.$$

Our construction of an approximating polynomial relies on averaging the Taylor polynomials over the ball B. It can be shown that there exists a cut-off function $\phi \in C^\infty$ for $B(0,1)$ with the following properties:

(i) $\int_{\mathbb{R}^n} \phi(x)dx = 1$,
(ii) $\operatorname{supp}(\phi) = B(0,1)$,
(iii) $\|\phi\|_\infty \le 1$.

For any ball $B(x_0,\rho)$, the cut-off function $\phi_B := \rho^{-n}\phi(\rho^{-1}(\cdot - x_0))$ satisfies the following properties:

(i) $\int_{\mathbb{R}^n} \phi_B(x)dx = 1$,
(ii) $\operatorname{supp}(\phi_B) = B(x_0, \rho)$,
(iii) $\|\phi_B\|_\infty \le \rho^{-n}$.

The *averaged Taylor polynomial* of $g \in C^r(\Omega)$ over $B \subseteq \Omega$ of order r (degree $r-1$) is given by

$$T_B^r g(x) := \int_B T_y^r g(x)\phi_B(y)dy, \quad x \in \Omega. \tag{1.31}$$

We also denote the *averaged Taylor remainder* by

$$R_B^r g(x) := g(x) - T_B^r g(x).$$

Lemma 1.29. *For $x \in \Omega$, where Ω is star-shaped with respect to $B(x_0, \rho) \subset \Omega$, and $g \in C^r(\Omega)$,*

$$R_B^r g(x) = r \sum_{|\alpha|=r} \int_{V(x)} K_\alpha(x,z)\partial^\alpha g(z)dz, \tag{1.32}$$

where $V(x)$ is the convex closure of $\{x\} \cup B$, and $K_\alpha = \frac{1}{\alpha!}(x-z)^\alpha K(x,z)$ with

$$|K(x,z)| \le C(\gamma+1)^n |x-z|^{-n}, \quad \gamma = \frac{\operatorname{diam}(\Omega)}{\rho}. \tag{1.33}$$

Proof. We fix $x \in \Omega$ and observe that, by properties (i) and(ii) of ϕ_B,

$$\begin{aligned} R_B^r g(x) &= g(x) - T_B^r g(x) \\ &= \int_B (g(x) - T_y^r g(x))\phi_B(y)dy \\ &= \int_B R_y^r g(x)\phi_B(y)dy \\ &= r \sum_{|\alpha|=r} \int_B \frac{(x-y)^\alpha}{\alpha!}\phi_B(y) \int_0^1 s^{r-1}\partial^\alpha g(x+s(y-x))ds dy. \end{aligned}$$

We now make the change of variables (y,s) to (z,s) with $z = x + s(y-x)$ and define the integration domain

$$A := \{(z,s) : s \in [0,1], |s^{-1}(z-x) + x - x_0| \le \rho\}$$

to obtain

$$\begin{aligned} R_B^r g(x) &= r \sum_{|\alpha|=r} \frac{1}{\alpha!} \int_A (x-z)^\alpha \phi_B(s^{-1}(z-x)+x) \partial^\alpha g(z) s^{-n-1} dz ds \\ &= r \sum_{|\alpha|=r} \int_{V(x)} \partial^\alpha g(z) \frac{1}{\alpha!}(x-z)^\alpha \int_0^1 \mathbf{1}_A(z,s)\phi_B(s^{-1}(z-x)+x) s^{-n-1} ds dz \\ &= r \sum_{|\alpha|=r} \int_{V(x)} \partial^\alpha g(z) K_\alpha(x,z) dz, \end{aligned}$$

where

$$K_\alpha(x,z) := \frac{1}{\alpha!}(x-z)^\alpha K(x,z), \quad K(x,z) := \int_0^1 \mathbf{1}_A(z,s)\phi_B(s^{-1}(z-x)+x)s^{-n-1} ds.$$

We now prove estimate (1.33). Observe that

$$(z,s) \in A \Rightarrow \frac{|z-x|}{|x-x_0|+\rho} < s.$$

So with $t := |z-x|/(|x-x_0|+\rho)$ and property (iii) of ϕ_B, we get

$$\begin{aligned} |K(x,z)| &= \left| \int_0^1 \mathbf{1}_A(z,s)\phi_B(s^{-1}(z-x)+x)s^{-n-1} ds \right| \\ &\le \|\phi_B\|_\infty \int_t^1 s^{-n-1} ds \\ &\le C(n)\rho^{-n} t^{-n} \\ &= C(n)\rho^{-n}|x-z|^{-n}(|x-x_0|+\rho)^n \\ &= C(n)\left(1+\frac{1}{\rho}|x-x_0|\right)^n |x-z|^{-n} \\ &\le C(n)(1+\gamma)^n |x-z|^{-n}. \end{aligned}$$

□

Next, we provide the following commutativity of Taylor polynomials and differentiation under affine transformations.

Lemma 1.30 ([29]). *Let $A(x) = Mx + b$ be a nonsingular affine transformation, and let $g \in C^r(\Omega)$. Then, for any $x \in \Omega$ and $\alpha \in \mathbb{Z}_+^n$ with $1 \le |\alpha| \le r$, we have*

$$\partial_x^\alpha [T_y^r(g(A\cdot))(A^{-1}x)] = T_y^{r-|\alpha|}(\partial^\alpha g(A\cdot))(A^{-1}x), \tag{1.34}$$

which implies that for a star-shaped domain (with respect to B),

$$\partial_x^\alpha [T_B^r(g(A\cdot))(A^{-1}x)] = T_B^{r-|\alpha|}(\partial^\alpha g(A\cdot))(A^{-1}x). \tag{1.35}$$

Proof. Observe that it is sufficient to prove that for any $1 \le k \le r-1$ and $1 \le s \le n$,

$$\partial_x^{e_s}\left[\sum_{|\beta|=k} \frac{\partial_y^\beta \tilde{g}(y)}{\beta!}(A^{-1}x-y)^\beta\right] = \sum_{|\gamma|=k-1} \frac{\partial_y^\gamma \widetilde{g_{x_s}}(y)}{\gamma!}(A^{-1}x-y)^\gamma, \tag{1.36}$$

where $\tilde{g} := g(A\cdot)$, $\widetilde{g_{x_s}} := g_{x_s}(A\cdot)$, $g_{x_s} := \frac{\partial g}{\partial x_s}$, and $\{e_s\}_{s=1,\dots,n}$ is the standard basis of $\mathbb{R}^n$. The case of a general multivariate derivative ∂_x^α follows by repeated applications of (1.36), and the Taylor series formulation (1.34) is obtained by adding all the degrees $1 \le k \le r-1$. To prove the above, let $M =: (a_{i,j})_{1\le i,j\le n}$ and $M^{-1} =: (b_{i,j})_{1\le i,j\le n}$. In the calculations below, if $\beta_i = 0$, then differentiating $(A^{-1}x-y)^\beta$ with respect to x_s does not produce the term $\beta_i b_{i,s}(A^{-1}x-y)^{\beta-e_i}$, we rather have 0, and it does not appear in the summation. Hence in this case, we regard $\beta_i b_{i,s}(A^{-1}x-y)^{\beta-e_i} := 0$ and $(\beta-e_i)! = \infty$, and again the term is not there. This takes care of itself automatically when we switch below the summation from β to $\gamma = \beta - e_i$:

$$\begin{aligned}
\partial_x^{e_s}\left[\sum_{|\beta|=k} \frac{\partial_y^\beta \tilde{g}(y)}{\beta!}(A^{-1}x-y)^\beta\right] &= \sum_{|\beta|=k} \frac{\partial_y^\beta \tilde{g}(y)}{\beta!}\partial_x^{e_s}\big((A^{-1}x-y)^\beta\big) \\
&= \sum_{|\beta|=k} \frac{\partial_y^\beta \tilde{g}(y)}{\beta!}\sum_{i=1}^n \beta_i b_{i,s}(A^{-1}x-y)^{\beta-e_i} \\
&= \sum_{|\beta|=k}\sum_{i=1}^n \frac{\partial_y^\beta \tilde{g}(y)}{(\beta-e_i)!} b_{i,s}(A^{-1}x-y)^{\beta-e_i} \\
&= \sum_{|\gamma|=k-1} \frac{(A^{-1}x-y)^\gamma}{\gamma!}\sum_{i=1}^n b_{i,s}\partial_y^{\gamma+e_i}\tilde{g}(y) \\
&= \sum_{\gamma|=k-1} \frac{(A^{-1}x-y)^\gamma}{\gamma!}\sum_{i=1}^n b_{i,s}\partial_y^{\gamma}\left(\sum_{j=1}^n a_{j,i} g_{x_j}(Ay)\right) \\
&= \sum_{|\gamma|=k-1} \frac{(A^{-1}x-y)^\gamma}{\gamma!}\sum_{j=1}^n \partial_y^\gamma(g_{x_j}(Ay))\sum_{i=1}^n a_{j,i}b_{i,s} \\
&= \sum_{|\gamma|=k-1} \frac{(A^{-1}x-y)^\gamma}{\gamma!}\sum_{j=1}^n \partial_y^\gamma(g_{x_j}(Ay))\delta_{j,s} \\
&= \sum_{|\gamma|=k-1} \frac{\partial_y^\gamma(\widetilde{g_{x_s}}(y))}{\gamma!}(A^{-1}x-y)^\gamma.
\end{aligned}$$

□

Proof of Theorem 1.28. We first assume that $g \in C^r(\Omega)$ and $\operatorname{diam}(\Omega) = 1$. We need the following Riesz potential inequality [15, Lemma 4.3.6]: for a given

$$h(x) = \int_\Omega |x-z|^{r-n} |f(z)| dz,$$

where $f \in L_p(\Omega)$, $1 \le p \le \infty$, we have

$$\|h\|_{L_p(\Omega)} \le C(n,r) \operatorname{diam}(\Omega)^r \|f\|_{L_p(\Omega)}. \tag{1.37}$$

For $k = 0$, we use (1.32), (1.33), and (1.37) with $\operatorname{diam}(\Omega) = 1$ to proceed with

$$\begin{aligned} \|g - T_B^r g\|_{L_p(\Omega)} &= \|R_B^r g\|_{L_p(\Omega)} \\ &\le r \sum_{|\alpha|=r} \left\| \int_\Omega |K_\alpha(\cdot,z)| |\partial^\alpha g(z)| dz \right\|_{L_p(\Omega)} \\ &\le C(n,r)(\gamma+1)^n \sum_{|\alpha|=r} \left\| \int_\Omega |x-z|^{r-n} |\partial^\alpha g(z)| dz \right\|_{L_p(\Omega)} \\ &\le C(n,r)(\gamma+1)^n |g|_{W_p^r(\Omega)}. \end{aligned}$$

For $0 < k < r$, let $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k$, and let $h := \partial^\alpha g$. Applying (1.35) with $A(x) = x$ and the estimate above for h give

$$\begin{aligned} \|\partial^\alpha (g - T_B^r g)\|_{L_p(\Omega)} &= \|h - T_B^{r-k} h\|_{L_p(\Omega)} \\ &\le C(n,r)(\gamma+1)^n |h|_{W_p^{r-k}(\Omega)} \\ &\le C(n,r)(\gamma+1)^n |g|_{W_p^r(\Omega)}. \end{aligned}$$

Summing up over all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k$, we conclude

$$|g - T_B^r g|_{W_p^k(\Omega)} \le C(n,r)(\gamma+1)^n |g|_{W_p^r(\Omega)}, \quad k = 0, \dots, r-1.$$

This finishes the proof for the case $g \in C^r(\Omega)$ and $\operatorname{diam}(\Omega) = 1$. For an arbitrary bounded domain Ω that is star-shaped with respect to B, let $\tilde{\Omega} = A^{-1}(\Omega)$, where A is an affine transform defined through its inverse $A^{-1}(x) := \operatorname{diam}(\Omega)^{-1}(x - x_0)$, where x_0 is the center of B. Observe that $\tilde{\Omega}$ satisfies $\operatorname{diam}(\tilde{\Omega}) = 1$ and is star-shaped with respect to the ball $A^{-1}(B)$, having the same chunkiness parameter γ as Ω. For $\tilde{g} := g(A\cdot)$, by the previous part in the proof

$$|\tilde{g} - T_{A^{-1}(B)}^r \tilde{g}|_{W_p^k(\tilde{\Omega})} \le C(n,r)(\gamma+1)^n |\tilde{g}|_{W_p^r(\tilde{\Omega})}, \quad k = 0, \dots, r-1.$$

Thus, with $P := T^r_{A^{-1}(B)}\tilde{g}(A^{-1}\cdot) \in \Pi_{r-1}$, for $1 \le p < \infty$ (the proof for $p = \infty$ is exactly the same with no need for the change of variables), we obtain

$$\begin{aligned}
\|g - P\|_{L_p(\Omega)} &= \operatorname{diam}(\Omega)^{-1/p}\|\tilde{g} - T^r_{A^{-1}(B)}\tilde{g}\|_{L_p(\tilde{\Omega})} \\
&\le C(n,r)\operatorname{diam}(\Omega)^{-1/p}(\gamma+1)^n|\tilde{g}|_{W^r_p(\tilde{\Omega})} \\
&\le C(n,r)\operatorname{diam}(\Omega)^{-1/p}(\gamma+1)^n\operatorname{diam}(\Omega)^r\sum_{|\alpha|=r}\|\partial^\alpha g(A\cdot)\|_{L_p(\tilde{\Omega})} \\
&= C(n,r)(\gamma+1)^n\operatorname{diam}(\Omega)^r\sum_{|\alpha|=r}\|\partial^\alpha g\|_{L_p(\Omega)} \\
&= C(n,r)(\gamma+1)^n\operatorname{diam}(\Omega)^r|g|_{W^r_p(\Omega)}.
\end{aligned}$$

For $0 < k < r$, let $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| = k$, and let $h := \partial^\alpha g$. Applying (1.35) with the affine transformation A defined above and the above estimate for h gives

$$\begin{aligned}
\|\partial^\alpha(g - P)\|_{L_p(\Omega)} &= \|h - \partial^\alpha[T^r_{A^{-1}(B)}\tilde{g}(A^{-1}\cdot)]\|_{L_p(\Omega)} \\
&= \|h - T^{r-k}_{A^{-1}(B)}h\|_{L_p(\Omega)} \\
&\le C(n,r)(\gamma+1)^n\operatorname{diam}(\Omega)^{r-k}|h|_{W^{r-k}_p(\Omega)} \\
&\le C(n,r)(\gamma+1)^n\operatorname{diam}(\Omega)^{r-k}|g|_{W^r_p(\Omega)}.
\end{aligned}$$

Summing up over all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| = k$, we conclude

$$|g - P|_{W^k_p(\Omega)} \le C(n,r)(\gamma+1)^n\operatorname{diam}(\Omega)^{r-k}|g|_{W^r_p(\Omega)}, \quad k = 0,\dots,r-1.$$

This concludes the proof for $g \in C^r(\Omega)$. Since $C^\infty(\Omega)$ is dense in $W^r_p(\Omega)$, $1 \le p < \infty$, we may apply a standard density argument to obtain (1.28) for $g \in W^r_p(\Omega)$, that is, there exist sequences $\{g_k\}$, $g_k \in C^r(\Omega)$, and $\{P_k\}$, $P_k \in \Pi_{r-1}$, $k \ge 1$, for which (1.28) is satisfied and also $\|g - g_k\|_{W^r_p(\Omega)} \to 0$. Then from $\{P_k\}$ we may extract a subsequence converging to $P \in \Pi_{r-1}$ (e. g., in the L_∞ norm), such that (1.28) is satisfied for g with P. □

The Bramble–Hilbert lemma for star-shaped domains implies that for Ω, a star-shaped domain with respect to some ball B, with chunkiness parameter γ and $f \in L_p(\Omega)$, $1 \le p \le \infty$, we have

$$K_r(f,\Omega)_p \le E_{r-1}(f,\Omega)_p \le C(n,r)(\gamma+1)^n K_r(f,\Omega)_p. \tag{1.38}$$

If we further assume that the domain satisfies the uniform cone property, then applying (1.17), for $t = \operatorname{diam}(\Omega)$, we obtain the equivalence

$$E_{r-1}(f,\Omega)_p \sim K_r(f,\Omega)_p \sim \omega_r(f,\Omega)_p \tag{1.39}$$

for $1 \le p \le \infty$ with constants that also depend on the shape of the domain Ω. An application of Theorem 1.28 is the following:

Theorem 1.31 ([29]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let A be a nonsingular affine map such that $B(0,1) \subseteq A^{-1}(\Omega) \subseteq B(0,n)$ and $A^{-1}(\Omega)$ is star-shaped with respect to $B(0,1)$. Then, for $g \in W_p^r(\Omega)$, $r \ge 1$, $1 \le p \le \infty$, there exists a polynomial $P \in \Pi_{r-1}$ such that*

$$|g - P|_{W_p^k(\Omega)} \le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{W_p^k(\Omega)}, \quad k = 0,1,\dots,r. \tag{1.40}$$

For the case of $g \in C^r(\Omega)$, $P(x) = T^r_{B(0,1)}(g(A\cdot))(A^{-1}x)$ satisfies (1.40).

Proof. Note that we can bound the chunkiness parameter (1.27) as follows:

$$\gamma(A^{-1}(\Omega)) \le 2n. \tag{1.41}$$

Since $A(x) = Mx + b$ maps $B(0,1)$ into Ω, we get that $\|M\|_2 \le \operatorname{diam}(\Omega)$. This gives that $\max_{1\le i,j\le n}|a_{i,j}| \le \operatorname{diam}\Omega$, where $M = (a_{i,j})_{1\le i,j\le n}$. With $\tilde{g} := g(A\cdot)$ and $\tilde{\Omega} := A^{-1}(\Omega)$, for $y \in \tilde{\Omega}$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k$, $k = 1,\dots,r$, we get

$$\left|\partial^\alpha \tilde{g}(y)\right| \le \operatorname{diam}(\Omega)^k \sum_{|\beta|=k} \left|(\partial^\beta g)(Ay)\right|.$$

In particular,

$$\sum_{|\alpha|=r} \|\partial^\alpha \tilde{g}\|_{L_p(\tilde{\Omega})} \le c(n,r)\operatorname{diam}(\Omega)^r \sum_{|\alpha|=r} \|(\partial^\alpha g)(A\cdot)\|_{L_p(\tilde{\Omega})}. \tag{1.42}$$

We can now prove (1.40) for $k = 0$. Let $\tilde{P} := T^r_{B(0,1)}\tilde{g} \in \Pi_{r-1}$ and $P := \tilde{P}(A^{-1}\cdot)$. Then since the chunkiness parameter of $\tilde{\Omega}$ satisfies (1.41), using (1.28) and (1.42), for $1 \le p < \infty$ (the proof for $p = \infty$ is exactly the same with no need for the change of variables), we obtain

$$\begin{aligned}
\|g - P\|_{L_p(\Omega)} &= \left|\det(M)\right|^{1/p}\|\tilde{g} - \tilde{P}\|_{L_p(\tilde{\Omega})} \\
&\le c(n,r)\left|\det(M)\right|^{1/p}|\tilde{g}|_{W_p^r(\tilde{\Omega})} \\
&\le c(n,r)\left|\det(M)\right|^{1/p}\operatorname{diam}(\Omega)^r \sum_{|\alpha|=r} \|\partial^\alpha g(A\cdot)\|_{L_p(\tilde{\Omega})} \\
&= c(n,r)\operatorname{diam}(\Omega)^r \sum_{|\alpha|=r} \|\partial^\alpha g\|_{L_p(\Omega)} \\
&= c(n,r)\operatorname{diam}(\Omega)^r|g|_{W_p^r(\Omega)}.
\end{aligned}$$

For $0 < k < r$, we proceed as in the proof of Theorem 1.28. Let $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k$, and let $h := \partial^\alpha g$. Applying (1.35) with the affine transformation A defined above, in the case $k = 0$ for h, we get

$$\begin{aligned}\|\partial^\alpha(g-P)\|_{L_p(\Omega)} &= \|h - \partial^\alpha[T^r_{B(0,1)}\tilde{g}(A^{-1}\cdot)]\|_{L_p(\Omega)}\\ &= \|h - T^{r-k}_{B(0,1)}h\|_{L_p(\Omega)}\\ &\le C(n,r)\operatorname{diam}(\Omega)^{r-k}|h|_{W_p^{r-k}(\Omega)}\\ &\le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{W_p^r(\Omega)}.\end{aligned}$$

Summing up over all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| = k$, we conclude

$$|g-P|_{W_p^k(\Omega)} \le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{W_p^r(\Omega)}, \quad k = 0,\dots,r-1.$$

This concludes the proof for $g \in C^r(\Omega)$. Since $C^\infty(\Omega)$ is dense in $W_p^r(\Omega)$, $1 \le p < \infty$, we may apply a standard density argument as in the proof of Theorem 1.28 to obtain (1.40) for $g \in W_p^r(\Omega)$. □

An immediate application of John's lemma (Proposition 1.6) and Theorem 1.31 gives the following:

Corollary 1.32 (Bramble–Hilbert lemma for convex domains [29]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $g \in W_p^r(\Omega)$, $r \in \mathbb{N}$, $1 \le p \le \infty$. Then there exists a polynomial $P \in \Pi_{r-1}$ such that*

$$|g-P|_{k,p} \le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{r,p}, \quad k = 0,1,\dots,r-1. \tag{1.43}$$

For the case of $g \in C^r(\Omega)$, $P(x) = T^r_{B(0,1)}(g(A\cdot))(A^{-1}x)$ satisfies (1.43), *where $T^r_B h$ is the averaged Taylor polynomial of h with respect to the ball B, given by* (1.31). *In particular, for the case $k = 0$, we obtain*

$$E_{r-1}(g,\Omega)_p \le C(n,r)\operatorname{diam}(\Omega)^r|g|_{r,p}. \tag{1.44}$$

For the general case of functions in $L_p(\Omega)$, we also get the following:

Corollary 1.33. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $f \in L_p(\Omega)$, $1 \le p \le \infty$. Then, for any $r \ge 1$,*

$$E_{r-1}(f,\Omega)_p \sim K_r(f,\Omega)_p, \tag{1.45}$$

where the constants of equivalency depend only on n and r and not on f or Ω.

Proof. Let $g_i \in W_p^r(\Omega)$, $i \ge 1$, be a sequence such that

$$K_r(f,\operatorname{diam}(\Omega)^r)_p = \inf_i\{\|f-g_i\|_p + \operatorname{diam}(\Omega)^r|g_i|_{r,p}\}.$$

By (1.43) there exist polynomials $P_i \in \Pi_{r-1}$, $i \geq 1$, such that

$$\|g_i - P_i\|_p \leq C(n,r)\,\mathrm{diam}(\Omega)^r |g_i|_{r,p}.$$

Therefore

$$\begin{aligned}
E_{r-1}(f,\Omega)_p &\leq \inf_i \|f - P_i\|_p \\
&\leq \inf_i \{\|f - g_i\|_p + \|g_i - P_i\|_p\} \\
&\leq \inf_i \{\|f - g_i\|_p + C(n,r)\,\mathrm{diam}(\Omega)^r |g_i|_{r,p}\} \\
&\leq C(n,r) K_r(f, \mathrm{diam}(\Omega)^r)_p \\
&= C(n,r) K_r(f,\Omega)_p.
\end{aligned}$$

To prove $K_r(f,\Omega)_p \leq E_{r-1}(f,\Omega)_p$, let P be an arbitrary polynomial in Π_{r-1}. Then using (1.9), it is easy to see that

$$K_r(f, \mathrm{diam}(\Omega)^r)_p \leq \|f - P\|_p + \mathrm{diam}(\Omega)^r |P|_{r,p} = \|f - P\|_p.$$

Since P was chosen arbitrarily, we get that

$$K_r(f,\Omega)_p = K_r(f, \mathrm{diam}(\Omega)^r)_p \leq \inf_{P \in \Pi_{r-1}} \|f - P\|_p = E_{r-1}(f,\Omega)_p. \qquad \square$$

1.5 The Whitney theorem for convex domains

In the previous section, where the polynomial approximation was taking place in the L_p space with $1 \leq p \leq \infty$, we were able to apply the tools of Sobolev spaces and the K-functional. However, for the case of $0 < p < 1$, we need to directly estimate "local" low-order polynomial approximation over convex domains explicitly using moduli of smoothness. The critical emphasis is on estimates where the leading constant does not further depend on the geometry of the domain. The main result of this section is the following:

Theorem 1.34 ([30]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $f \in L_p(\Omega)$, $0 < p \leq \infty$. Then for any $r \geq 1$,*

$$E_{r-1}(f,\Omega)_p \leq C(n,r,p)\,\omega_r(f,\Omega)_p, \tag{1.46}$$

where $\omega_r(f,\Omega)_p$ is defined in (1.13).

Before we proceed with the proof of Theorem 1.34, we review two corollaries that can be derived from it. By the first part of Theorem 1.26 we have that $\omega_r(P,\Omega)_p = 0$ for any polynomial $P \in \Pi_{r-1}$. Thus

$$\omega_r(f,\Omega)_p \le \omega_r(f-P,\Omega)_p \le C\|f-P\|_p,$$

which gives

$$\omega_r(f,\Omega)_p \le CE_{r-1}(f,\Omega)_p.$$

Combining this with (1.45) and (1.46) yields the following:

Corollary 1.35. *For all bounded convex domains $\Omega \subset \mathbb{R}^n$, functions $f \in L_p(\Omega)$, and $r \ge 1$, for $1 \le p \le \infty$, we have the equivalence*

$$E_{r-1}(f,\Omega)_p \sim K_r(f,\Omega)_p \sim \omega_r(f,\Omega)_p, \tag{1.47}$$

and for $0 < p < 1$, we have the equivalence

$$E_{r-1}(f,\Omega)_p \sim \omega_r(f,\Omega)_p, \tag{1.48}$$

where the constants depend on n, r, and p but not on Ω or f.

Corollary 1.36. *For any bounded convex domain $\Omega \subset \mathbb{R}^n$, $r \ge 1$, and $1 \le p < \infty$, there exists a linear projector $P_{\Omega,p} : L_p(\Omega) \to \Pi_{r-1}$ that realizes the Whitney inequality*

$$\|f - P_{\Omega,p}f\|_{L_p(\Omega)} \le C(n,r,p)\omega_r(f,\Omega)_p.$$

This also implies that the projectors $\{P_{\Omega,p}\}_\Omega$ are uniformly bounded over all bounded convex domains.

Proof. Recall that by (1.43), for any $g \in C^r(\Omega)$, the linear projector

$$P_\Omega g(x) := T^r_{B(0,1)}\big(g(A\cdot)\big)(A^{-1}x)$$

realizes the Bramble–Hilbert lemma

$$\|g - P_\Omega g\|_{L_p(\Omega)} \le C(n,r)\,\mathrm{diam}(\Omega)^r |g|_{W_p^r(\Omega)}.$$

By (1.47) this further implies that

$$\|g - P_\Omega g\|_{L_p(\Omega)} \le C(n,r,p)\omega_r(g,\Omega)_p.$$

Observe that this also gives that P_Ω is bounded on $C^r(\Omega) \cap L_p(\Omega)$:

$$\begin{aligned}\|P_\Omega g\|_p &\le \|P_\Omega g - g\|_p + \|g\|_p\\ &\le C\omega_r(g,\Omega)_p + \|g\|_p\\ &\le C\|g\|_p.\end{aligned}$$

Since $C^r(\Omega)$ is dense in $L_p(\Omega)$, $1 \le p < \infty$, we may extend P_Ω to a bounded projector $P_{\Omega,p}$ that realizes the Whitney estimate for functions in $L_p(\Omega)$. □

We prove Theorem 1.34 separately for $1 \le p \le \infty$ and $0 < p < 1$. As we will see, in the former case, we can use the equivalence of the modulus of smoothness and the K-functional and then apply the machinery of K-functionals. In the latter case, we have to work significantly harder as the classical K-functional in L_p, $0 < p < 1$, is trivial.

Proof of Theorem 1.34 *for the case* $1 \le p \le \infty$. Let $A(x) = Mx + b$ be the affine transformation for which (1.5) holds. Corollary 1.33 implies that for $\widetilde{\Omega} := A^{-1}(\Omega)$ and $\widetilde{f} := f(A\cdot)$, there exists a polynomial $\widetilde{P} \in \Pi_{r-1}$ such that

$$\|\widetilde{f} - \widetilde{P}\|_{L_p(\widetilde{\Omega})} \le C(n,r)K_r(\widetilde{f},\widetilde{\Omega})_p.$$

Since $B(0,1) \subseteq \widetilde{\Omega} \subseteq B(0,n)$, $\widetilde{\Omega}$ fulfills the conditions of Corollary 1.17 with $R_1 = 1$ and $R_2 = n$, we may apply (1.18) with $t = \operatorname{diam}(\widetilde{\Omega})$ to obtain

$$\begin{aligned}\|\tilde{f} - \tilde{P}\|_{L_p(\tilde{\Omega})} &\le C(n,r)K_r(\tilde{f},\tilde{\Omega})_p \\ &\le C(n,r,p)\omega_r(\tilde{f},\tilde{\Omega})_p.\end{aligned}$$

Denoting $P := \tilde{P}(A^{-1}\cdot)$ yields

$$\begin{aligned}\|f - P\|_{L_p(\Omega)} &= \left|\det(M)\right|^{1/p}\|\tilde{f} - \tilde{P}\|_{L_p(\tilde{\Omega})} \\ &\le C\left|\det(M)\right|^{1/p}\omega_r(\tilde{f},\tilde{\Omega})_p \\ &= C\omega_r(f,\Omega)_p.\end{aligned}$$

This proves Theorem 1.34 for the case $1 \le p \le \infty$. □

We now turn to the proof of the Whitney theorem for $0 < p < 1$ [30]. We first consider the case $r = 1$.

Lemma 1.37. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $f \in L_p(\Omega)$, $0 < p < \infty$. Then there exists a constant c such that*

$$\int_\Omega \left|f(x) - c\right|^p dx \le \frac{1}{|\Omega|}\int_{|h|\le\operatorname{diam}(\Omega)}\int_\Omega \left|\Delta_h(f,\Omega,x)\right|^p dxdh, \tag{1.49}$$

where $|\Omega|$ denotes the volume of Ω.

Proof. By a standard density argument we may assume that f is continuous. Consider the function $\phi(y) := \int_\Omega |f(x) - f(y)|^p\,dx$, $y \in \Omega$. Clearly, there exists $y_0 \in \Omega$ such that

$$\phi(y_0) \le \frac{1}{|\Omega|}\int_\Omega \phi(y)\,dy.$$

Therefore with $c := f(y_0)$ we get

$$\begin{aligned}\int_\Omega |f(x) - c|^p\,dx &= \phi(y_0)\\ &\le \frac{1}{|\Omega|}\int_\Omega \phi(y)\,dy\\ &= \frac{1}{|\Omega|}\int_\Omega\int_\Omega |f(x) - f(y)|^p\,dx\,dy.\end{aligned}$$

By definition, for any domain Ω and every $x \in \Omega$, if $x + h \notin \Omega$, then $\Delta_h(f, \Omega, x) = 0$. Therefore the substitution $h = y - x$ yields (1.49). □

Corollary 1.38. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $f \in L_p(\Omega)$, $0 < p < \infty$. Then there exists a constant c such that*

$$\|f - c\|_{L_p(\Omega)} \le (2n)^{n/p}\omega_1(f, \Omega)_p. \tag{1.50}$$

Proof. Let $\widetilde{\Omega} := A^{-1}(\Omega)$, where A is the affine transformation for which (1.5) holds. Denote $\tilde{f} := f(A\cdot)$. By Lemma 1.37 there exists a constant c such that

$$\int_{\widetilde{\Omega}} |\tilde{f}(x) - c|^p\,dx \le \frac{1}{|\widetilde{\Omega}|}\int_{|h|\le 2n}\int_{\widetilde{\Omega}} |\Delta_h(\tilde{f}, \widetilde{\Omega}, x)|^p\,dx\,dh.$$

Hence

$$\begin{aligned}\int_{\widetilde{\Omega}} |\tilde{f}(x) - c|^p\,dx &\le \frac{|B(0, 2n)|}{|B(0, 1)|}\omega_1(\tilde{f}, \widetilde{\Omega})_p^p\\ &= (2n)^n\omega_1(\tilde{f}, \widetilde{\Omega})_p^p.\end{aligned}$$

As we have seen in the proof of Theorem 1.34 for the case $1 \le p \le \infty$, the Whitney inequality is invariant under affine maps, and therefore the above inequality implies (1.50). □

Lemma 1.39. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain such that $B(0, R_1) \subseteq \Omega \subseteq B(0, R_2)$ for some $0 < R_1 < R_2$, and let $f \in L_p(\Omega)$, $0 < p < \infty$. Then, for each $m \in \mathbb{N}$, there exists a step function*

$$\phi = \sum_{k=1}^{K} \mathbf{1}_{Q_k} c_k$$

with the following properties:

(1) *Q_k, $1 \le k \le K \le C_1(n, R_2)m^n$, are cubes taken from the uniform grid of side length m^{-1} and thus have disjoint interiors;*

(2) $\Omega \subseteq \bigcup_{k=1}^{K} Q_k$;
(3) $\|f - \phi\|_{L_p(\Omega)} \le C(n, R_1, R_2)\omega_1(f, 1/m)_{L_p(\Omega)}$;
(4) $\|\phi\|_{L_p(\mathbb{R}^n)} \le C(n, R_1, R_2, p)\|f\|_{L_p(\Omega)}$.

Proof. For $m \in \mathbb{N}$, we select from the uniform grid of length m^{-1} all the cubes Q_k, $1 \le k \le \tilde{K} \le (2R_2)^n m^n$, for which $\mathrm{int}(Q_k \cap \Omega) \ne \emptyset$. For each $1 \le k \le \tilde{K}$, we construct from Q_k, by a symmetric extension, the cube $\tilde{Q}_k$ with side length $3m^{-1}$. For example, the cube $[0, m^{-1}]^n$ is extended to $[-m^{-1}, 2m^{-1}]^n$. We claim that there exists a constant $C_2(n, R_1, R_2)$ such that

$$|\tilde{Q}_k \cap \Omega| \ge C_2(n, R_1, R_2)m^{-n}, \quad 1 \le k \le \tilde{K}. \tag{1.51}$$

Indeed, given $1 \le k \le \tilde{K}$, take a point $x_0 \in Q_k \cap \Omega$. If $x_0 \in B(0, R_1)$, then it is easy to see that there exists a constant $C_3(n, R_1)$ for which

$$|\Omega \cap \tilde{Q}_k| \ge |B(0, R_1) \cap \tilde{Q}_k| \ge C_3(n, R_1)m^{-n}.$$

Otherwise, $x_0 \notin B(0, R_1)$, and we denote by $V(x_0)$ the cone defined by the convex closure of the set $\{x_0\} \cup B(0, R_1) \subseteq \Omega$. Since $B(0, R_1) \subset V(x_0) \subset B(0, R_2)$, it follows that the head angle α of the cone $V(x_0)$ satisfies $\sin(\alpha/2) \ge R_1/R_2$. Therefore the volume of $V(x_0) \cap \tilde{Q}_k$ is bounded from below by the volume of a cone in $\mathbb{R}^n$ with head angle $2\arcsin(R_1/R_2)$ and height m^{-1}. This implies that there exists a constant $C_4(n, R_1, R_2)$ such that

$$|\Omega \cap \tilde{Q}_k| \ge |V(x_0) \cap \tilde{Q}_k| \ge C_4(n, R_1, R_2)m^{-n}.$$

We conclude that (1.51) holds with $C_2 := \min(C_3, C_4)$.

Next, we augment cubes Q_k, $\tilde{K} < k \le K$, with $K \le C_1(n, R_2)m^n$, taken from the uniform grid of length m^{-1}, to ensure that $\bigcup_{k=1}^{K} Q_k = \bigcup_{k=1}^{\tilde{K}} \tilde{Q}_k$.

We first assume that $f \ge 0$. This will allow us to show that ϕ constructed below satisfies property (4). We also focus on the case $0 < p \le 1$. Lemma 1.37 implies that for each $1 \le j \le \tilde{K}$, there exists a constant $\tilde{c}_j$ that satisfies

$$\int_{\tilde{Q}_j \cap \Omega} |f(x) - \tilde{c}_j|^p \, dx \le \frac{1}{|\tilde{Q}_j \cap \Omega|} \int_{|h| \le 3\sqrt{n}m^{-1}} \int_{\Omega} |\Delta_h(f, \tilde{Q}_j \cap \Omega, x)|^p \, dx \, dh.$$

We denote by $\{\tilde{Q}_{k,j} : 1 \le j \le J(k) \le 3^n\}$ the collection of larger cubes that contain the cube Q_k, $1 \le k \le K$, and set

$$c_k := \frac{1}{J(k)} \sum_{j=1}^{J(k)} \tilde{c}_{k,j}.$$

We claim that

$$\phi := \sum_{k=1}^{K} \mathbf{1}_{Q_k} c_k$$

satisfies properties (3) and (4). We proceed to prove property (3). Recalling that only the cubes Q_k, $1 \le k \le \tilde{K}$, intersect with the interior of Ω and applying the properties of the modulus of smoothness from Proposition 1.14 and (1.51), we have

$$\begin{aligned}
\|f-\phi\|_{L_p(\Omega)}^p &= \sum_{k=1}^{\tilde{K}} \int_{Q_k\cap\Omega} |f(x)-c_k|^p\,dx \\
&= \sum_{k=1}^{\tilde{K}} \int_{Q_k\cap\Omega} \left|\frac{1}{J(k)}\sum_{j=1}^{J(k)}(f(x)-\tilde{c}_{k,j})\right|^p dx \\
&\le \sum_{k=1}^{\tilde{K}}\sum_{j=1}^{J(k)} \int_{Q_k\cap\Omega} |f(x)-\tilde{c}_{k,j}|^p\,dx \\
&= \sum_{j=1}^{\tilde{K}} \int_{\tilde{Q}_j\cap\Omega} |f(x)-\tilde{c}_j|^p\,dx \\
&\le \sum_{j=1}^{\tilde{K}} \frac{1}{|\tilde{Q}_j\cap\Omega|} \int_{|h|\le 3\sqrt{n}m^{-1}} \int_{\tilde{Q}_j\cap\Omega} |\Delta_h(f,\tilde{Q}_j\cap\Omega,x)|^p\,dx\,dh \\
&\le C(n,R_1,R_2)m^n \sum_{k=1}^{\tilde{K}} \int_{|h|\le 3\sqrt{n}m^{-1}} \int_{Q_k\cap\Omega} |\Delta_h(f,\Omega,x)|^p\,dx\,dh \\
&= C(n,R_1,R_2)m^n \int_{|h|\le 3\sqrt{n}m^{-1}} \int_{\Omega} |\Delta_h(f,\Omega,x)|^p\,dx\,dh \\
&\le C(n,R_1,R_2)\omega_1(f,3\sqrt{n}/m)_{L_p(\Omega)}^p \\
&\le C(n,R_1,R_2)\omega_1(f,1/m)_{L_p(\Omega)}^p.
\end{aligned}$$

This proves (3). To prove property (4), we note that since we assumed that $f \ge 0$, it follows from the proof of Lemma 1.37 that we may take $\tilde{c}_j \ge 0$, $1 \le j \le \tilde{K}$, and hence that $c_k \ge 0$, $1 \le k \le K$. Applying (1.51) yields

$$\begin{aligned}
\|\phi\|_{L_p(\mathbb{R}^n)}^p &= m^{-n}\sum_{k=1}^{K} c_k^p \\
&= m^{-n}\sum_{j=1}^{K}\left(\frac{1}{J(k)}\sum_{j=1}^{j(k)} \tilde{c}_{k,j}\right)^p
\end{aligned}$$

$$\leq C(n,R_1,R_2)\sum_{j=1}^{K}\sum_{j=1}^{J(k)}\tilde{c}_{k,j}^p|\tilde{Q}_{k,j}\cap\Omega|$$
$$\leq C(n,R_1,R_2)\sum_{j=1}^{\tilde{K}}\tilde{c}_j^p|\tilde{Q}_j\cap\Omega|.$$

Using the norm equivalence of finite-dimensional spaces, we may proceed with

$$\begin{aligned}\sum_{j=1}^{\tilde{K}}\tilde{c}_j^p|\tilde{Q}_j\cap\Omega| &= \sum_{k=1}^{\tilde{K}}\left(\sum_{j=1}^{J(k)}\tilde{c}_{k,j}^p\right)|Q_k\cap\Omega|\\ &\leq C(n,p)\sum_{k=1}^{\tilde{K}}\int_{Q_k\cap\Omega}c_k^p\,dx\\ &= C(n,p)\|\phi\|_{L_p(\Omega)}^p\\ &\leq C(n,p)\big(\|f\|_{L_p(\Omega)}^p+\|f-\phi\|_{L_p(\Omega)}^p\big)\\ &\leq C(n,p)\big(\|f\|_{L_p(\Omega)}^p+\omega_1(f,1/n)_{L_p(\Omega)}^p\big)\\ &\leq C(n,p)\|f\|_{L_p(\Omega)}^p.\end{aligned}$$

The proof of the case $1 \leq p < \infty$ is similar, and this completes the proof of (4) for nonnegative functions.

For an arbitrary function $f \in L_p(\Omega)$, $0 < p < \infty$, we use the representation $f = f_+ - f_-$ where $f_+(x) := \max(0, f(x))$ and $f_-(x) := \max(0, -f(x)) \geq 0$. Using the above method, we construct approximating step functions ϕ_1, ϕ_2 such that

$$\|f_+-\phi_1\|_{L_p(\Omega)}\leq C\omega_1(f_+,1/n)_p,\quad \|f_--\phi_2\|_{L_p(\Omega)}\leq C\omega_1(f_-,1/n)_p,$$

and

$$\|\phi_1\|_{L_p(\mathbb{R}^n)}\leq C\|f_+\|_{L_p(\Omega)},\quad \|\phi_2\|_{L_p(\mathbb{R}^n)}\leq C\|f_-\|_{L_p(\Omega)}.$$

It is easy to see that for any $x, h \in \mathbb{R}^n$, $|\Delta_h(f_\pm, x)| \leq |\Delta_h(f, x)|$. Therefore $\omega_1(f_\pm \cdot)_p \leq \omega_1(f, \cdot)_p$. Also, it is clear that $\|f_\pm\|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega)}$. We conclude that the step function $\phi := \phi_1 - \phi_2$ fulfills properties (1)–(4). □

Definition 1.40. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain containing the origin. We denote by $\phi_\Omega \in C(\mathbb{S}^{n-1})$ the unique continuous function that describes $\partial\Omega$, where $\mathbb{S}^{n-1}$ is the unit sphere. Namely, for $\theta \in \mathbb{S}^{n-1}$, $\phi_\Omega(\theta) = r$ if and only if (r, θ) is the unique point in $\mathbb{R}^n$ in polar representation for which $(r, \theta) \in \partial\Omega$. Observe that the norm of $C(\mathbb{S}^{n-1})$ induces a metric on the collection of such domains.

Lemma 1.41. *Let* $\{\Omega_m\}_{m\geq 1}$ *be convex domains in* $\mathbb{R}^n$ *such that* $B(0,R_1) \subseteq \Omega_m \subseteq B(0,R_2)$ *for some* $0 < R_1 < R_2$. *Then there exists a subsequence* $\{\Omega_{m_i}\}_{i\geq 1}$ *that converges in the sense of Definition* 1.40 *to a convex domain* Ω *such that* $B(0,R_1) \subseteq \Omega \subseteq B(0,R_2)$.

Proof. Let $\phi_{\Omega_m}(\theta)$, $m \geq 1$, $\theta \in \mathbb{S}^{n-1}$, be the corresponding continuous function that describes the boundary of Ω_m. A similar argument to that used in Theorem 1.10 shows that all the functions $\{\phi_{\Omega_m}\}$ are uniformly bounded in the Lip-1 norm with a uniform constant $M := M(n,R_1,R_2)$. By the Arzelà–Ascoli theorem there exists a convergent subsequence to some function ϕ. It is easy to verify that the function ϕ describes the boundary of a convex domain Ω with $B(0,R_1) \subseteq \Omega \subseteq B(0,R_2)$. □

Proof of Theorem 1.34 *for the case* $0 \leq p \leq 1$. Estimate (1.50) is (1.46) for $r = 1$. Assume on the contrary that for fixed parameters n, $r > 1$, and $0 < p \leq 1$, there does not exist a constant $C(n,r,p)$ for which (1.46) holds for all bounded convex domains $\Omega \subset \mathbb{R}^n$ and functions $f \in L_p(\Omega)$. In view of the invariance of the Whitney estimate under affine maps, by John's lemma (Proposition 1.6) this implies the existence of a sequence of convex domains $\{\tilde{\Omega}_m\}_{m\geq 1}$, $B(0,1) \subseteq \tilde{\Omega}_m \subseteq B(0,n)$, and functions $\tilde{f}_m \in L_p(\tilde{\Omega}_m)$ for which

$$E_{r-1}(\tilde{f}_m,\tilde{\Omega}_m)_p^p > m\omega_r(\tilde{f}_m,\tilde{\Omega}_m)_p^p, \quad m \geq 1.$$

By Lemma 1.41 we may assume that $\{\tilde{\Omega}_m\}_{m\geq 1}$ converges to a convex domain Ω such that $B(0,1) \subseteq \Omega \subseteq B(0,n)$ in the sense of Definition 1.40. For any sequence $\epsilon_k \downarrow 0$, there exist $m_k \uparrow \infty$ such that

$$B(0,1/2) \subseteq \Omega_{m_k} := (1-\epsilon_k)\tilde{\Omega}_{m_k} \subseteq \Omega \subseteq B(0,n).$$

Hence, for the functions $f_{m_k} := (1-\epsilon_k)^{-n/p}\tilde{f}_{m_k}((1-\epsilon_k)^{-1}\cdot)$, we have

$$\begin{aligned} E_{r-1}(f_{m_k},\Omega_{m_k})_p^p &= E_{r-1}(\tilde{f}_{m_k},\tilde{\Omega}_{m_k})^p \\ &> m_k\omega_r(\tilde{f}_{m_k},\tilde{\Omega}_{m_k})_p^p \\ &= m_k\omega_r(f_{m_k},\Omega_{m_k})_p^p. \end{aligned}$$

Clearly, $\{\Omega_{m_k}\}_{k\geq 1}$, $B(0,1/2) \subseteq \Omega_{m_k} \subseteq \Omega \subseteq B(0,n)$, also converges to Ω in the sense of Definition 1.40. We simplify the notation by setting $f_k := f_{m_k}$ and $\Omega_k := \Omega_{m_k}$, $\Omega_k \subseteq \Omega$, and we let $P_k \in \Pi_{r-1}$ be the best approximation to f_k on Ω_k, i. e.,

$$\|f_k - P_k\|_{L_p(\Omega_k)}^p = E_{r-1}(f_k,\Omega_k)_p^p > k\omega_r(f_k,\Omega_k)_p^p.$$

Setting $g_k := \lambda_k(f_k - P_k)$ with λ_k defined by $\|g_k\|_{L_p(\Omega_k)} = 1$, we have a sequence of domains $\{\Omega_k\}_{k\geq 1}$ and functions $\{g_k\}_{k\geq 1}$ with the following properties:

(i) $\|g_k\|_{L_p(\Omega_k)} = E_{r-1}(g_k,\Omega_k)_p = 1$,
(ii) $\omega_r(g_k,\Omega_k)_p^p \leq 1/k$,
(iii) $B(0,1/2) \subseteq \Omega_k \subseteq \Omega$, and $\{\Omega_k\}$ converges to Ω in the sense of Definition 1.40.

By Corollary 1.22 the Marchaud inequality holds with a uniform constant for all the above domains $\{\Omega_k\}$. Thus, for sufficiently small $0 < \delta < \tilde{t}$, where $\tilde{t}(n, r, p)$ is determined in Corollary 1.22, from property (ii) we get

$$\omega_1(g_k, \delta)^p_{L_p(\Omega_k)} \le C(n, r, p)\delta^p \left(\int_\delta^{\tilde{t}} u^{-(p+1)} \frac{1}{k} \, du + 1 \right) \le C(n, r, p)\left(\frac{1}{k} + \delta^p \right).$$

It follows that for each $\epsilon > 0$, there exist δ_0 and k_0 such that

$$\omega_1(g_k, \delta)^p_{L_p(\Omega_k)} \le \epsilon \quad \text{for } \delta < \delta_0 \text{ and } k \ge k_0.$$

Applying Lemma 1.39 with $R_1 = 1/2$ and $R_2 = n$, we get that for any $\epsilon > 0$, there exist functions $\phi_{k,m}$, $k \ge k_0$, $m := m(\epsilon)$, that are piecewise constant over the grid of length m^{-1} and for which

$$\|g_k - \phi_{k,m}\|^p_{L_p(\Omega_k)} \le C\omega_1(g_k, m^{-1})^p_{L_p(\Omega_k)} \le \epsilon, \quad k \ge k_0(\epsilon). \tag{1.52}$$

Lemma 1.39(4) and property (i) also yield

$$\|\phi_{k,m}\|^p_{L_p(\mathbb{R}^n)} \le C(n, p). \tag{1.53}$$

Since $\phi_{k,m}$ is constant over the cubes of side length m^{-1}, we may apply (1.53) to obtain

$$\|\phi_{k,m}\|_{L_\infty(\Omega)} \le C\left(m^n \int_\Omega |\phi_{k,m}(x)|^p dx \right)^{1/p} \le Cm^{n/p} =: M.$$

Consider the set $\Phi := \Phi(\epsilon)$ of all step functions over the uniform grid of side length m^{-1} that take the values

$$j\epsilon^{1/p}|B(0, n)|^{-1/p}, \quad j = 0, \pm 1, \dots, \pm\lceil \epsilon^{-1/p}|B(0, n)|^{1/p} M \rceil.$$

Clearly,

$$\inf_{\varphi \in \Phi} \|\phi_{k,m} - \varphi\|^p_{L_p(\Omega)} \le \int_\Omega (\epsilon^{1/p}|B(0, n)|^{-1/p})^p dx \le \epsilon.$$

Hence the set Φ is a finite ϵ-net for $\{\phi_{k,m}\}_{k=k_0(\epsilon)}^\infty$ in $L_p(\Omega)$. Thus there exist $\varphi_\epsilon \in \Phi$ and infinite subsequences $\{\phi^\epsilon_{k,m}\}_{k \ge 1}$ and $\{g^\epsilon_k\}_{k \ge 1}$ such that $\|\phi^\epsilon_{k,m} - \varphi_\epsilon\|^p_{L_p(\Omega)} \le \epsilon$, and, in turn,

$\|g_k^\epsilon - \varphi_\epsilon\|^p_{L_p(\Omega_k)} \le 2\epsilon$. Applying the above process for $\epsilon_i := 1/(2i)$, $i \ge 2$, we can construct a sequence $\{\varphi_i\}_{i\ge 2}$ with the following properties:

(i) $0 < C_1 \le \|\varphi_i\|_{L_p(\Omega)} \le C_2 < \infty$.

(ii) For each $i \ge 2$, $\|\varphi_i - g_{i,j}\|_{L_p(\Omega_{i,j})} \le 1/i$ for all $j \ge 1$, where $\{g_{i,j}\}_{j\ge 1}$ is an infinite subsequence of $\{g_k\}$.

(iii) $E_{r-1}(\varphi_i, \Omega)_p^p \ge 1/2$.

(iv) $\omega_r(\varphi_i, \Omega)_p^p \le C/i$, where $C = C(r)$.

Let us prove property (iii). Since $\Omega_{i,j} \subseteq \Omega$, $j \ge 1$, it follows that

$$\begin{aligned} E_{r-1}(\varphi_i, \Omega)_p^p &\ge \inf_{Q\in\Pi_{r-1}} \|\varphi_i - Q\|^p_{L_p(\Omega_{i,j})} \\ &\ge \inf_{Q\in\Pi_{r-1}} \|g_{i,j} - Q\|^p_{L_p(\Omega_{i,j})} - \|\varphi_i - g_{i,j}\|^p_{L_p(\Omega_{i,j})} \\ &\ge 1 - 1/i \ge 1/2. \end{aligned}$$

We now prove property (iv). For a fixed $i \ge 2$, let $h \in \mathbb{R}^n$, $|h| \le \operatorname{diam}(\Omega)$, be such that

$$\omega_r(\varphi_i, \Omega)_p^p \le 2 \int_\Omega \left|\Delta_h^r(\varphi_i, \Omega, x)\right|^p dx.$$

Now let

$$\Omega_{i,j,h} := \{x \in \Omega : [x, x + rh] \subset \Omega,\ [x, x + rh] \not\subset \Omega_{i,j}\}$$

and

$$\tilde{\Omega}_{i,j,h} := \bigcup_{x\in\Omega_{i,j,h}} [x, x + rh].$$

As the domains $\Omega_{i,j}$ converge to Ω as $j \to \infty$ in the sense of Definition 1.40, it follows that the measure of the sets $\tilde{\Omega}_{i,j,h}$ tends to zero as $j \to \infty$. Consequently,

$$\int_{\tilde{\Omega}_{i,j,h}} \left|\varphi_i(x)\right|^p dx \to 0, \quad j \to \infty. \tag{1.54}$$

This gives

$$\begin{aligned} \omega_r(\varphi_i, \Omega)_p^p &\le 2 \int_\Omega \left|\Delta_h^r(\varphi_i, \Omega, x)\right|^p dx \\ &\le 2\Big(\int_{\Omega\setminus\Omega_{i,j,h}} \left|\Delta_h^r(\varphi_i, \Omega, x)\right|^p dx + \int_{\Omega_{i,j,h}} \left|\Delta_h^r(\varphi_i, \Omega, x)\right|^p dx \Big) \end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\int_{\Omega_{i,j}} \big|\Delta_h^r(\varphi_i,\Omega_{i,j},x)\big|^p\,dx + \int_{\tilde\Omega_{i,j,h}} \big|\varphi_i(x)\big|^p\,dx\Big) \\
&\le C\Big(\int_{\Omega_{i,j}} \big|\Delta_h^r(g_{i,j},\Omega_{i,j},x)\big|^p\,dx + \|\varphi_i - g_{i,j}\|_{L_p(\Omega_{i,j})}^p + \int_{\tilde\Omega_{i,j,h}} \big|\varphi_i(x)\big|^p\,dx\Big) \\
&\le C\Big(\omega_r(g_{i,j},\Omega_{i,j})_p^p + \|\varphi_i - g_{i,j}\|_{L_p(\Omega_{i,j})}^p + \int_{\tilde\Omega_{i,j,h}} \big|\varphi_i(x)\big|^p\,dx\Big) \\
&=: C(I_1 + I_2 + I_3).
\end{aligned}
$$

Finally, $I_1 = \omega_r(g_{i,j},\Omega_{i,j})_p^p \to 0$ as $j \to \infty$, and by (1.54) $I_3 \to 0$ as $j \to \infty$, whereas by (ii) $I_2 \le 1/i$ for all $j \ge 1$. This completes the proof of (iv).

We now repeat the proof with the sequence $\{\varphi_i\}_{i\ge2}$ on the fixed domain Ω in place of sequences $\{g_k\}_{k\ge1}$ and $\{\Omega_k\}_{k\ge1}$. This can be done because properties (i), (iii), and (iv) of $\{\varphi_i\}$ are almost the same as properties (i) and (ii) of $\{g_k\}$, and, in addition, we have the major advantage of a fixed domain Ω. Thus we obtain sequences $\{\Psi_{i,m}\}$ of piecewise constants on the grid of length m^{-1} for which

$$\|\varphi_i - \Psi_{i,m}\|_{L_p(\Omega)}^p \le \epsilon$$

and which possess the finite ϵ-net property, that is, for each $\epsilon > 0$, we have Ψ^ϵ such that $\|\varphi_i^\epsilon - \Psi^\epsilon\|_{L_p(\Omega)}^p \le 2\epsilon$ for an infinite subsequence of $\{\varphi_i\}$. Taking $\epsilon_l = 1/(2l)$ and repeating the argument for $l = 2, 3, \ldots$, each time taking a subsequence of the previous one, in summary, we obtain a sequence $\{\Psi_l\}_{l\ge2}$ and a sequence $\{\varphi_j\}_{j\ge2}$ such that

$$\|\Psi_l - \varphi_j\|_{L_p(\Omega)}^p \le \frac{1}{l}, \quad \forall j \ge l.$$

Hence $\{\Psi_l\}_{l\ge2}$ is a Cauchy sequence in $L_p(\Omega)$ and therefore converges to some $\Psi \in L_p(\Omega)$. This implies that $\varphi_j \to \Psi$ in $L_p(\Omega)$ and, in turn, that, on the one hand, $\omega_r(\Psi,\Omega)_p = 0$, whereas, on the other hand,

$$
\begin{aligned}
E_{r-1}(\Psi,\Omega)_p^p &\ge \inf_{Q\in\Pi_{r-1}} \|\varphi_j - Q\|_{L_p(\Omega)}^p - \|\Psi - \varphi_j\|_{L_p(\Omega)}^p \\
&\ge \frac{1}{2} - \|\Psi - \varphi_j\|_{L_p(\Omega)}^p \to \frac{1}{2} \quad \text{as } j \to \infty,
\end{aligned}
$$

contradicting Theorem 1.26.

We conclude that Theorem 1.34 holds, that is, there exists a constant $C(n,r,p)$ such that for all bounded convex domains Ω and all functions $f \in L_p(\Omega)$, $0 < p < 1$,

$$E_{r-1}(f)_p \le C(n,r,p)\omega_r(f,\Omega)_p. \qquad \square$$

2 Anisotropic multilevel ellipsoid covers of $\mathbb{R}^n$

Spaces of homogeneous type serve as a platform for significant generalization of the Euclidean space equipped with the Lebesgue measure [33]. However, function spaces defined over general spaces of homogeneous type are limited in many ways. Kernels can only have limited regularity, Hardy spaces are only defined for values of $p \leq 1$ "close" to 1, Besov spaces can only be defined for limited smoothness $\alpha > 0$, etc. In fact, these limitations are determined by the parameter α of Proposition 2.4. Thus the goal of the construction presented in Section 2.2 is providing a platform for spaces of homogeneous type that are sufficiently general on one hand but, at the same time, do not have these limitations and allow almost complete generalization of their Euclidean function space counterparts. Our setup is over the space $\mathbb{R}^n$ and uses the Lebesgue measure. However, the Euclidean distance is replaced by quasi-distances derived from replacing the Euclidean balls by (possibly) anisotropic ellipsoids that may change rapidly from point to point and from scale to scale. This pointwise variable control over the local geometry of the homogeneous space over $\mathbb{R}^n$ allows us to apply local smoothness analysis using machinery such as moduli of smoothness and representations/approximations by algebraic polynomials. In Section 2.5, we precisely characterize the spaces of homogeneous type that induce ellipsoid covers, which allows us to provide examples showing that our setting is quite comprehensive.

2.1 Spaces of homogeneous type

Definition 2.1. A *quasi-distance* on a set X is a mapping $\rho : X \times X \to [0, \infty)$ that satisfies the following conditions for all $x, y, z \in X$:

(i) $\rho(x,y) = 0 \Leftrightarrow x = y$,

(ii) $\rho(x,y) = \rho(y,x)$,

(iii) there exists $\kappa \geq 1$ such that

$$\rho(x,y) \leq \kappa\big(\rho(x,z) + \rho(z,y)\big). \tag{2.1}$$

Any quasi-distance ρ defines a topology for which the balls $B_\rho(x,r) := \{y \in X : \rho(x,y) < r\}$ form a base.

Definition 2.2 ([19]). A space of *homogeneous type* (X, ρ, μ) is a set X together with a quasi-distance ρ and a nonnegative measure μ such that $0 < \mu(B_\rho(x,r)) < \infty$ for all $x \in X$ and $r > 0$ and such that the following *doubling condition* holds for some fixed $c_0 > 0$:

$$\mu\big(B_\rho(x,2r)\big) \leq c_0 \mu\big(B_\rho(x,r)\big), \quad \forall x \in X,\ \forall r > 0. \tag{2.2}$$

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Obviously, we assume that μ is defined on a σ-algebra that contains all Borel sets and balls $B(x,r)$. Throughout the book, we will frequently use the notation $|\Omega| := \mu(\Omega)$ for measurable $\Omega \subseteq X$. The doubling condition (2.2) implies, with the *"upper dimension"* $d := \log_2 c_0$, the following growth condition on the volume of balls:

$$|B_\rho(x,\lambda r)| \le c_0\lambda^d |B_\rho(x,r)|, \quad \forall x \in \mathbb{R}^n,\ r>0,\ \lambda \ge 1. \tag{2.3}$$

A *Normal Space of Homogeneous Type* is a homogeneous space for which (2.2) is replaced by the stronger condition $\mu(B_\rho(x,r)) \sim r$ with constants that do not depend on x and r.

Remark 2.3. Given a metric space X equipped with distance ρ and measure μ, the condition $\mu(B_\rho(x,r)) \sim r$, $x \in X$, $r > 0$, is a particular case of *Ahlfors–David q-regularity* with $q = 1$. In fact, this condition, also known as *Ahlfors*-1 *regularity*, already appeared in Ahlfors' paper from 1935 [2].

Proposition 2.4 ([54]). *Let ρ be a quasi-distance on a set X satisfying* (2.1) *with $\kappa \ge 1$. Then there exist a quasi-distance ρ' on X and constants $c > 0$ and $0 < \alpha < 1$ such that any $x,y,z \in X$ and $r > 0$,*
(i) $\rho'(x,y) \sim \rho(x,y)$,
(ii) $|\rho'(x,z) - \rho'(y,z)| \le cr^{1-\alpha}\rho'(x,y)^\alpha$ *whenever* $\rho'(x,z), \rho'(y,z) \le r$.

Moreover, we may choose

$$\alpha := \frac{\log(2)}{\log(3\kappa^2)}, \tag{2.4}$$

where κ is given by (2.1).

Proposition 2.5 ([54]). *Let (X,ρ,μ) be a space of homogeneous type such that all the balls are open sets. Then the function*

$$\rho'(x,y) := \inf\{\mu(B_\rho) : B_\rho \text{ is a ball}, x,y \in B_\rho\}, \quad x,y \in X,\ x \ne y,$$

and $\rho'(x,x) := 0$, $x \in X$, is a quasi-distance on X inducing the same topology as ρ, and (X,ρ',μ) is a normal space of homogeneous type.

The above results (e. g., [33]) are typically applied to "correct" a given quasi-distance ρ of a space of homogeneous type (X,ρ,μ) and derive from it a quasi-distance ρ' such that (X,ρ',μ) is a normal space of homogeneous type, where also property (ii) of Proposition 2.4 holds.

In the Euclidean setting, when $n = 1$, the notions of distance and volume are equivalent, and therefore $\rho(x,y) = \rho'(x,y) = |x-y|$. However, it is interesting to note that "normalizing" the Euclidean distance in dimensions $n \ge 2$ as above by using the volume of minimal balls simplifies computations where the dimension n comes into play. As we will see later, the spaces constructed and analyzed in this book are normal spaces of homogeneous type. For normal spaces, the condition $\mu(B_\rho(x,r)) \sim r$ allows us to show that integration of the quasi-distance "behaves" similarly to integration of the Euclidean distance.

Theorem 2.6. *Let (X,ρ,μ) be a normal space of homogeneous type, Then, for any $\delta > 0$, there exist constants of equivalency such that for all $x \in X$, $r > 0$, and $\beta > 0$,*

$$\int_{B_\rho(x,r)} \rho(x,y)^{\delta-1} d\mu(y) \sim r^\delta, \tag{2.5}$$

$$\int_{B_\rho(x,r)^c} \rho(x,y)^{-(\delta+1)} d\mu(y) \sim r^{-\delta}, \tag{2.6}$$

$$\int_X \frac{1}{(\beta+\rho(x,y))^{(1+\delta)}} d\mu(y) \sim \beta^{-\delta}. \tag{2.7}$$

Proof. We will prove (2.5). The other two equivalences are proved in similar manner. For the upper bound, it is sufficient to use dyadic rings:

$$\begin{aligned} \int_{B_\rho(x,r)} \rho(x,y)^{\delta-1} d\mu(y) &= \sum_{k=0}^{\infty} \int_{2^{-(k+1)}r\le\rho(x,y)<2^{-k}r} \rho(x,y)^{\delta-1} d\mu(y) \\ &\le C\sum_{k=0}^{\infty} (2^{-k}r)^{\delta-1} |B_\rho(x,2^{-k}r)| \\ &\le Cr^\delta \sum_{k=0}^{\infty} 2^{-k\delta} \\ &\le Cr^\delta. \end{aligned}$$

For the lower bound, we need to make sure that the rings have "substantial" volume. Let $0 < c_1 < c_2 < \infty$ be constants such that $c_1 r \le \mu(B_\rho(x,r)) \le c_2 r$ for all $x \in X$ and $r > 0$. Then for $M \in \mathbb{N}$, satisfying $Mc_1 > c_2$, and $\tilde{c}_1 := Mc_1 - c_2$, we have that for all $x \in X$ and $r > 0$

$$|B_\rho(x,Mr) \setminus B_\rho(x,r)| \ge \tilde{c}_1 r.$$

We use these constants to estimate

$$\begin{aligned} \int_{B_\rho(x,r)} \rho(x,y)^{\delta-1} d\mu(y) &= \sum_{k=0}^{\infty} \int_{M^{-(k+1)}r\le\rho(x,y)<M^{-k}r} \rho(x,y)^{\delta-1} d\mu(y) \\ &\ge C\sum_{k=0}^{\infty} (M^{-k}r)^{\delta-1} |B_\rho(x,MM^{-(k+1)}r) \setminus B_\rho(x,M^{-(k+1)}r)| \end{aligned}$$

$$\geq C \sum_{k=0}^{\infty} \left(M^{-k}r\right)^{\delta-1} M^{-(k+1)}r$$
$$\geq Cr^{\delta} \sum_{k=0}^{\infty} M^{-k\delta}$$
$$\geq Cr^{\delta}. \qquad \square$$

Definition 2.7. Let (X, ρ, μ) be a space of homogeneous type. For $f \in L_1^{\mathrm{loc}}(X)$, we define the *maximal function*

$$Mf(x) := \sup_{x \in B_\rho} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y)| dy \tag{2.8}$$

and the *central maximal function*

$$M_B f(x) := \sup_{r>0} \frac{1}{|B_\rho(x,r)|} \int_{B_\rho(x,r)} |f(y)| dy. \tag{2.9}$$

It is well known and easy to see that $Mf(x) \sim M_B f(x)$ for all $x \in X$. Thus from this point we will use the central maximal function. It is a classic result [33, 61] that the maximal theorem holds in the general setup of spaces of homogeneous type.

Proposition 2.8 ([19]). *Let (X, ρ, μ) be a space of homogeneous type. Then there exists a constant $c > 0$ such that for all $f \in L^1(X)$ and $\alpha > 0$,*

$$\left|\{x : M_B f(x) > \alpha\}\right| \leq c\alpha^{-1} \|f\|_1. \tag{2.10}$$

For $1 < p < \infty$, there exists a constant $A_p > 0$ such that for all $f \in L^p(X)$,

$$\|M_B f\|_p \leq A_p \|f\|_p. \tag{2.11}$$

We will also need the Fefferman–Stein vector-valued maximal function inequality in the setting of spaces of homogeneous type.

Proposition 2.9 ([42]). *For $1 < p, q < \infty$, there exists a constant $c = c(p, q)$ such that for all measurable functions $\{f_j\}$ on X,*

$$\left\| \left(\sum_j |M_B f_j|^q \right)^{1/q} \right\|_{L_p(X)} \leq c \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L_p(X)}. \tag{2.12}$$

2.2 Construction and properties of ellipsoid covers

Definition 2.10. We say that

$$\Theta := \bigcup_{t\in\mathbb{R}} \Theta_t,$$

is a *continuous multilevel ellipsoid cover* of $\mathbb{R}^n$ if it satisfies the following, where $\mathbf{p}(\Theta) := \{a_1, \dots, a_6\}$ are positive constants:

For all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, there exist an ellipsoid $\theta(x,t) \in \Theta_t$ and an affine transformation $A_{x,t}(y) = M_{x,t}y + x$ such that $M_{x,t}$ is positive definite and $\theta(x,t) = A_{x,t}(B^*)$ (see Definition 1.1). We require the following two conditions:

(a) The "volume condition"

$$a_1 2^{-t} \le \left|\theta(x,t)\right| \le a_2 2^{-t}. \tag{2.13}$$

(b) The "shape condition": For any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $v \ge 0$, if $\theta(x,t) \cap \theta(y, t+v) \ne \emptyset$, then

$$a_3 2^{-a_4 v} \le 1/\left\|M_{y,t+v}^{-1} M_{x,t}\right\| \le \left\|M_{x,t}^{-1} M_{y,t+v}\right\| \le a_5 2^{-a_6 v}. \tag{2.14}$$

Here $\|\cdot\|$ is the matrix norm given by $\|M\| := \max_{v\in\mathbb{R}^n, |v|=1} |Mv|$. As depicted in Figure 2.1 and we explain below, the "shape condition" (b) ensures that, locally in scale and space, ellipsoids have similar shape. However, in some cases, for technical reasons, we will require an additional stronger assumption.

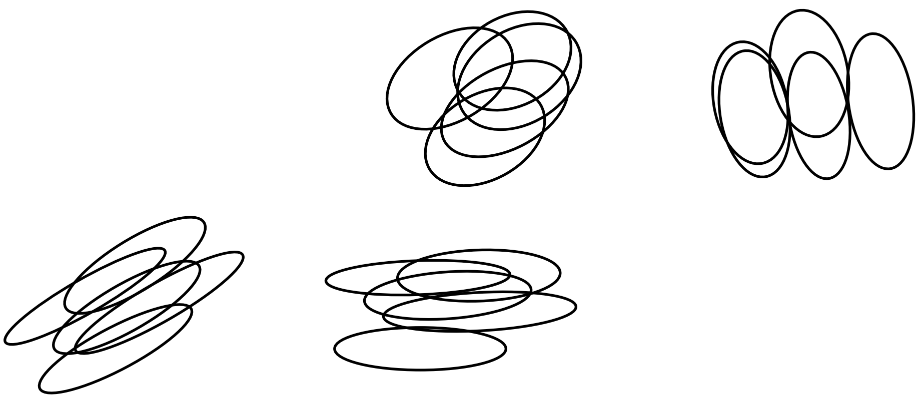

Figure 2.1: Ellipsoids at a fixed scale have equivalent volume, but their shape may change across space.

Definition 2.11. We say that a continuous cover Θ is a *pointwise continuous cover* if for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\|M_{x',t} - M_{x,t}\| \to 0 \quad \text{as } x' \to x. \tag{2.15}$$

As we will see (Theorem 2.28 below), the requirement of pointwise continuity does not impose a real restriction. However, we also define a rigid form of continuous covers, where the anisotropy does not changes across space.

Definition 2.12. A continuous cover Θ is said to be *t-continuous* if for each $t \in \mathbb{R}$, $M_{x,t}(B^*) = M_{x',t}(B^*)$ for all $x, x' \in \mathbb{R}^n$. This implies that for each $t \in \mathbb{R}$, we can select a fixed matrix M_t such that $\theta(x,t) = M_t(B^*) + x$ for all $x \in \mathbb{R}^n$. Obviously, a t-continuous cover is pointwise continuous.

Next, we proceed with a discretization of the scale parameter.

Definition 2.13. We call

$$\Theta = \bigcup_{m \in \mathbb{Z}} \Theta_m$$

a *multilevel semi-continuous ellipsoid cover* of $\mathbb{R}^n$ if the following conditions are obeyed, where $\mathbf{p}(\Theta) := \{a_1, \dots, a_6\}$ are positive constants:

(a) For all $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}$, there exist an ellipsoid $\theta(x,m) \in \Theta_m$ and an affine transform $A_{x,m}$ such that

$$a_1 2^{-m} \le |\theta(x,m)| \le a_2 2^{-m},$$

$\theta(x,m) = A_{x,m}(B^*)$, and $A_{x,m}$ is of the form $A_{x,m}(y) = M_{x,m} y + x$, where $M_{x,m}$ is positive definite.

(b) For any $v, y \in \mathbb{R}^n$, $m \in \mathbb{Z}$, and $\nu \ge 0$, if $\theta(v,m) \cap \theta(y, m+\nu) \ne \emptyset$, then

$$a_3 2^{-a_4 \nu} \le 1 / \|M_{y,m+\nu}^{-1} M_{v,m}\| \le \|M_{v,m}^{-1} M_{y,m+\nu}\| \le a_5 2^{-a_6 \nu}. \tag{2.16}$$

We readily see that any continuous ellipsoid cover Θ of $\mathbb{R}^n$ induces a semicontinuous ellipsoid cover by sampling at $t = m$, $m \in \mathbb{Z}$. As we will see in the next chapter, further discretization of the space variable at each level m facilitates the construction of multilevel function representations. This leads to the following:

Definition 2.14. We call

$$\Theta = \bigcup_{m \in \mathbb{Z}} \Theta_m$$

a *discrete multilevel ellipsoid cover* of $\mathbb{R}^n$ if the following conditions are obeyed, where $\mathbf{p}(\Theta) := \{a_1, \dots, a_8, N_1\}$ are positive constants:

(a) Every level Θ_m, $m \in \mathbb{Z}$, consists of a countable number of ellipsoids $\theta \in \Theta_m$ such that

$$a_1 2^{-m} \leq |\theta| \leq a_2 2^{-m} \tag{2.17}$$

and Θ_m is a cover of $\mathbb{R}^n$, i. e., $\mathbb{R}^n = \bigcup_{\theta \in \Theta_m} \theta$.

(b) For any $\theta \in \Theta_m$ and $\theta' \in \Theta_{m+\nu}$, $\nu \geq 0$, with $\theta \cap \theta' \neq \emptyset$, we have

$$a_3 2^{-a_4 \nu} \leq 1/\|M_{\theta'}^{-1} M_\theta\| \leq \|M_\theta^{-1} M_{\theta'}\| \leq a_5 2^{-a_6 \nu}. \tag{2.18}$$

(c) Each $\theta \in \Theta_m$ can intersect with at most $N_1 - 1$ other ellipsoids from Θ_m.

(d) For any $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}$, there exists $\theta \in \Theta_m$ such that $x \in \theta^\diamond$, where $\theta^\diamond := a_7 \cdot \theta$ is the dilated version of θ by a factor of $a_7 < 1$.

(e) If $\theta \cap \eta \neq \emptyset$ with $\theta \in \Theta_m$ and $\eta \in \Theta_m \cup \Theta_{m+1}$, then $|\theta \cap \eta| \geq a_8 |\eta|$.

Examples

(i) The regular cover of $\mathbb{R}^n$ consisting of all Euclidean balls is the simplest example of a t-continuous ellipsoid (ball) cover of $\mathbb{R}^n$. Observe that the induced quasi-distance ρ defined in (2.35) uses the volume of the Euclidean balls and not their radii, which provides a "normalized" quasi-distance where $|B_\rho(x,r)| \sim r$. In this case, we have $a_4 = a_6 = 1/n$.

(ii) Let $w : \mathbb{R}^n \to \mathbb{R}_+$ be a positive weight function such that $0 < c_1 \leq w(x) \leq c_2 < \infty$ for all $x \in \mathbb{R}^n$. Define the following distance: For any two points $x, y \in \mathbb{R}^n$, if $x = y$, then $\rho(x,y) = 0$, else

$$\rho(x,y) := \inf_\gamma \left\{ \int_0^{l(\gamma)} w(\gamma(t))dt, \gamma : [0, l(\gamma)] \to \mathbb{R}^n, \gamma \in C^1, \gamma(0) = x, \gamma(l(\gamma)) = y \right\},$$

where $l(\gamma)$ is the length of a curve γ in natural parameterization. It is easy to see that

$$B(x, c_2^{-1} r) \subseteq B_\rho(x,r) \subseteq B(x, c_1^{-1} r), \quad \forall x \in \mathbb{R}^n,\ r > 0.$$

This implies that ρ satisfies the doubling condition (2.2) and is a quasi-convex distance (see Definition 2.34). By Theorem 2.36 ρ induces a continuous ellipsoid cover.

(iii) The one parameter family of diagonal dilation matrices

$$D_t = \operatorname{diag}(2^{-tb_1}, 2^{-tb_2}, \ldots, 2^{-tb_n})$$

with $\sum_{j=1}^n b_j = 1$, $b_j > 0$, $j = 1, \ldots, n$, induces a t-continuous ellipsoid cover of $\mathbb{R}^n$, with $M_{x,t} = D_t$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Here $a_4 = \max_j b_j$ and $a_6 = \min_j b_j$.

(iv) Calderón and Torchinsky [16, 17] developed the so-called parabolic Hardy spaces generated by continuous dilation matrices associated with a continuous semigroup M_t, $t > 0$, $M_{st} := M_t M_s$, satisfying

$$t^\alpha \leq \|M_t\|_{\ell_2 \to \ell_2} \leq t^\beta, \quad t \geq 1,$$

where $1 \leq \alpha \leq \beta < \infty$. We easily see that any such semigroup of matrices gives rise to a t-continuous ellipsoid cover of $\mathbb{R}^n$.

(v) Consider an arbitrary $n \times n$ real matrix M with eigenvalues λ satisfying $|\lambda| > 1$. Then we easily see that the affine transformations $A_{x,m}(y) := M^{-m}y + x$, $x \in \mathbb{R}^n$, $m \in \mathbb{Z}$, define a semicontinuous ellipsoid cover (dilations) in the sense of Definition 2.13. The dilations of this particular kind are used in [7, 8, 11] for the development of anisotropic Hardy, Besov, and Triebel–Lizorkin spaces.

(vi) In Section 5.3.2, we present constructions of bivariate anisotropic continuous covers that are pointwise variable and adapted to the edge singularities of the indicator functions of a circle and a square [23]. This allows us to demonstrate that prototypical piecewise constant functions have higher anisotropic smoothness when compared with their classic isotropic Besov smoothness.

(vii) Consider the example of vector fields from [61, I.2.6], which is a relatively simple example from a general class of balls and metrics studied by Nagel, Stein, and Wainger [56]. Here, for some $k \in \mathbb{N}$, we define in $\mathbb{R}^2$ two vector fields $X_1 = \partial/\partial x_1$ and $X_2 = x_1^k \partial/\partial x_2$, which are associated with a quasi-distance ρ and the anisotropic balls

$$B_\rho(x, r) := \{y \in \mathbb{R}^2 : |x_1 - y_1| < r, |x_2 - y_2| < \max(r^{k+1}, |x_1|^k r)\}.$$

Observe that the pointwise variable anisotropic balls are in fact rectangles and hence convex. It is easy to see (this is a particular case of John's theorem 1.6) that there exist ellipses $\{\theta_{x,r}\}$ such that

$$\theta_{x,r} \subset B_\rho(x, r) \subset 2 \cdot \theta_{x,r}, \quad \forall x \in \mathbb{R}^2,\ r > 0.$$

As we will see in Section 2.5, since ρ is a quasi-convex quasi-distance satisfying the doubling condition, the ellipses $\{\theta_{x,r}\}$ induce a pointwise variable continuous ellipsoid cover associated with the vector fields.

We now list some properties of ellipsoid covers that are used throughout the book:

1. It is important to note that the set of all ellipsoid covers of $\mathbb{R}^n$ is invariant under affine transformations. More precisely, the images $A(\theta)$ of all ellipsoids $\theta \in \Theta$ of a given cover Θ of $\mathbb{R}^n$ via an affine transformation A of the form $A(x) = Mx + v$ with $|\det(M)| = 1$ form an ellipsoid cover of $\mathbb{R}^n$ with the same parameters as the parameters of Θ. If $|\det(M)| \neq 1$, then only the constants a_1 and a_2 in (2.13) or (2.17) change accordingly.

2. Conditions (2.14) or (2.18) of the covers indicate that if $\theta \cap \theta' \neq \emptyset$, then locally the ellipsoids θ and θ' cannot change uncontrollably in shape and orientation when they are from close levels. More precisely, denote $M := M_\theta^{-1} M_{\theta'}$ and let $M = UDV$ be the singular value decomposition of M, where U and V are orthogonal matrices, and $D = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ is diagonal with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. As in (1.2),
$$\|M\|_{\ell_2 \to \ell_2} = \sigma_1 \quad \text{and} \quad \|M_{\theta'}^{-1} M_\theta\|_{\ell_2 \to \ell_2} = \|M^{-1}\|_{\ell_2 \to \ell_2} = 1/\sigma_n.$$
Therefore conditions (2.14), (2.18) are equivalently expressed as
$$a_3 2^{-a_4 \nu} \leq \sigma_n \leq \cdots \leq \sigma_1 \leq a_5 2^{-a_6 \nu}. \tag{2.19}$$
This condition also has a clear geometric interpretation: The affine transformation A_θ^{-1}, which maps the ellipsoid θ onto the unit ball B^*, maps the ellipsoid θ' onto an ellipsoid with semiaxes $\sigma_1, \sigma_2, \dots, \sigma_n$ satisfying (2.19).
3. Evidently, the sign of ν in condition (b) can be reversed. Namely, condition (b) for discrete covers is equivalent to the following condition:
(b′) If $\theta \in \Theta_m$ and $\theta' \in \Theta_{m-\nu}$ $(\nu \geq 0)$ with $\theta \cap \theta' \neq \emptyset$, then
$$(1/a_5) 2^{a_6 \nu} \leq 1/\|M_{\theta'}^{-1} M_\theta\|_{\ell_2 \to \ell_2} \leq \|M_\theta^{-1} M_{\theta'}\|_{\ell_2 \to \ell_2} \leq (1/a_3) 2^{a_4 \nu}. \tag{2.20}$$
Therefore, if $M := M_\theta^{-1} M_{\theta'} = UDV$ with $D = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ as above, then (2.20) is equivalent to
$$(1/a_5) 2^{a_6 \nu} \leq \sigma_n \leq \cdots \leq \sigma_1 \leq (1/a_3) 2^{a_4 \nu}. \tag{2.21}$$
4. We need to interrelate the semiaxes of intersecting ellipsoids from Θ. For $\theta \in \Theta$, denote
$$\sigma_{\max}(\theta) := \|M_\theta\|_{\ell_2 \to \ell_2} \quad \text{and} \quad \sigma_{\min}(\theta) := \|M_\theta^{-1}\|_{\ell_2 \to \ell_2}^{-1}. \tag{2.22}$$
These are the maximum and minimum semiaxes of the ellipsoid θ.

Lemma 2.15. *If $\theta \in \Theta_t$ (or $\theta \in \Theta_m$ in the discrete case), $\theta' \in \Theta_{t+\nu}$ (or $\theta' \in \Theta_{m+\nu}$ in the discrete case), $\nu \geq 0$, and $\theta \cap \theta' \neq \emptyset$, then*
$$a_3 2^{-a_4 \nu} \sigma_{\max}(\theta) \leq \sigma_{\max}(\theta') \leq a_5 2^{-a_6 \nu} \sigma_{\max}(\theta) \tag{2.23}$$
and
$$a_3 2^{-a_4 \nu} \sigma_{\min}(\theta) \leq \sigma_{\min}(\theta') \leq a_5 2^{-a_6 \nu} \sigma_{\min}(\theta). \tag{2.24}$$

Proof. By the shape condition (2.14) we have
$$\|M_{\theta'}\| \leq \|M_\theta\| \|M_\theta^{-1} M_{\theta'}\| \leq a_5 2^{-a_6 \nu} \|M_\theta\|$$

and

$$\|M_\theta\| \le \|M'_\theta\| \|M_{\theta'}^{-1} M_\theta\| \le (1/a_3) 2^{a_4 \nu} \|M_{\theta'}\|,$$

which yield (2.23). We similarly prove (2.24). □

5. We can generalize the volume conditions (2.13) and (2.17) by adding an additional parameter a_0 and allowing the volume of $\theta \in \Theta_t$ ($\theta \in \Theta_m$ in the discrete case) to satisfy $a_1 2^{-a_0 t} \le |\theta| \le a_2 2^{-a_0 t}$ ($a_1 2^{-a_0 m} \le |\theta| \le a_2 2^{-a_0 m}$ in the discrete case). The methods introduced in this book still apply under this generalization without too much change.
6. The shape conditions (2.14) and (2.18) imply that the parameters a_3 and a_5 satisfy $0 < a_3 \le 1 \le a_5$. This is an immediate consequence of the trivial choice of the same ellipsoid (e. g., $x = y$ and $\nu = 0$ in the continuous case, so that $M_{x,t} = M_{y,t+\nu}$). Also, as we will see, we should assume that $a_6 \le a_4$.
7. By (2.23), if $(\theta_t)_{t \le 0}$ ($(\theta_m)_{m \le 0}$ in the discrete case) is a set of ellipsoids $\theta_t \in \Theta_t$ ($\theta_m \in \Theta_m$ in the discrete case) that contain a fixed point $x \in \mathbb{R}^n$, then $\bigcup_{t \le 0} \theta_t = \mathbb{R}^n$ ($\bigcup_{m \le 0} \theta_m = \mathbb{R}^n$). Also, we have the following:

Lemma 2.16. *For any bounded set* $\Omega \subset \mathbb{R}^n$,

$$\min\{\operatorname{diam}(\theta) : \theta \in \Theta_t, \theta \cap \Omega \ne \emptyset\} \to \infty, \quad t \to -\infty, \tag{2.25}$$

and in the other direction, there exists a constant $c(\Omega, \mathbf{p}(\Theta)) > 0$ *such that*

$$\|M_{x,t}\| \le c 2^{-a_6 t}, \quad \forall x \in \Omega,\ t \ge 0. \tag{2.26}$$

This gives

$$\max\{\operatorname{diam}(\theta) : \theta \in \Theta_t, \theta \cap \Omega \ne \emptyset\} \to 0, \quad t \to \infty. \tag{2.27}$$

Furthermore, there exist constants $0 < c_1 < c_2 < \infty$, *depending on* $\mathbf{p}(\Theta)$ *and* Ω, *such that for any ellipsoid* $\theta \in \Theta_0$ *with* $\theta \cap \Omega \ne \emptyset$, *we have that* $c_1 \le \operatorname{diam}(\theta) \le c_2$.

Proof. We may choose any point $x \in \Omega$ and a set of ellipsoids $\theta_t \in \Theta_t$, $t \le 0$, all containing x, and apply (2.23) to obtain that there exists an ellipsoid $\theta_\Omega \in \Theta_{t_0}$ for some $t_0 \le 0$ ($t_0 \in \mathbb{Z}$ in the discrete case) such that $\Omega \subset \theta_\Omega$. Let $\theta \in \Theta_t$ with $t \le t_0$ be such that $\theta \cap \Omega \ne \emptyset$. Therefore $\theta \cap \theta_\Omega \ne \emptyset$, and we may apply again (2.23) to obtain

$$\operatorname{diam}(\theta) \ge a_5^{-1} 2^{a_6 (t_0 - t)} \operatorname{diam}(\theta_\Omega),$$

which proves (2.25). In the other direction, any $\theta \in \Theta_t$, $t \ge 0$, that satisfies $\theta \cap \Omega \ne \emptyset$, also satisfies $\theta \cap \theta_\Omega \ne \emptyset$, and therefore (2.23) can be used to derive (2.27) by

$$\operatorname{diam}(\theta) \le a_5 2^{a_6 (t - t_0)} \operatorname{diam}(\theta_\Omega).$$

Finally, applying (2.23) implies that for any $\theta \in \Theta_0$,

$$c_1 := a_3 2^{-a_4 t_0} \operatorname{diam}(\theta_\Omega) \le \operatorname{diam}(\theta) \le a_5 2^{-a_6 t_0} \operatorname{diam}(\theta_\Omega) =: c_2. \qquad \square$$

8. Property (c) of discrete covers allows us to "color" ellipsoids at a fixed level using N_1 colors in a way that intersecting ellipsoids do not have the same color (see Section 3.2.1).
9. Property (d) of the discrete covers indicates that every point $x \in \mathbb{R}^n$ is contained in the "core" and thus is "well covered" by at least one ellipsoid from every level Θ_m.
10. The properties of ellipsoid covers imply the following multilevel relations.

Theorem 2.17. *Let Θ be a continuous cover. Then there exists $J_1(\mathbf{p}(\Theta)) > 0$ such that for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\lambda \ge 1$,*

$$\lambda \cdot \theta(x,t) = x + \lambda M_{x,t}(B^*) \subseteq \theta(x, t - J_1\lambda). \tag{2.28}$$

Choosing $\lambda = 1$ gives

$$\theta(x,t) \subset \theta(x, t - J_1), \tag{2.29}$$

whereas choosing $\lambda = 2$ and denoting $J := 2J_1$ give

$$M_{x,t}(B^*) \subseteq \frac{1}{2} M_{x,t-J}(B^*), \quad \theta(x,t) \subset \theta(x, t-J). \tag{2.30}$$

Proof. Fix $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Note that (2.28) holds if and only if

$$M^{-1}_{x,t-J_1\lambda} M_{x,t}(B^*) \subseteq \frac{1}{\lambda} B^*, \quad \lambda \ge 1.$$

From (2.14) we have $M^{-1}_{x,t-J_1\lambda} M_{x,t}(B^*) \subseteq a_5 2^{-a_6 J_1 \lambda} B^*$. Therefore we should choose large enough J_1 such that $a_5 2^{-a_6 J_1 \lambda} \le \lambda^{-1}$ for all $\lambda \ge 1$. Indeed, choosing

$$J_1 := \frac{\log(a_5) + 1}{\log(2) a_6}$$

gives, for all $\lambda \ge 1$,

$$J_1 \ge \frac{\log(a_5) + \lambda}{\lambda \log(2) a_6} \ge \frac{\log(a_5 \lambda)}{\lambda \log(2) a_6} \Rightarrow \log(2^{a_6 J_1 \lambda}) \ge \log(a_5 \lambda) \Rightarrow a_5 2^{-a_6 J_1 \lambda} \le \frac{1}{\lambda}. \qquad \square$$

Lemma 2.18. *For a cover Θ, there is a parameter $\gamma \in \mathbb{N}$ depending only on $\mathbf{p}(\Theta)$ such that for any ellipsoid $\theta \in \Theta_t$, $t \in \mathbb{R}$ ($\theta \in \Theta_m$, $m \in \mathbb{Z}$ in the discrete case), and any $\tilde{\gamma} \ge \gamma$ ($\tilde{\gamma} \in \mathbb{N}$ in the discrete case), there exists an ellipsoid $\eta \in \Theta_{m-\tilde{\gamma}}$ that satisfies the following: For any $\theta' \in \Theta_{t+\nu}$, $\nu \ge 0$ ($\theta' \in \Theta_{m+\nu}$, $\nu \in \mathbb{N}$, in the discrete case) with $\theta \cap \theta' \ne \emptyset$, we have* *that $\theta \cup \theta' \subset \eta$. Moreover, if Θ is a continuous cover, then we can choose $\eta = \theta(v, t - \tilde{\gamma})$ if $\theta = \theta(v, t)$.*

Proof. We prove the result for a discrete cover. The case of continuous cover is similar and easier. Let $\omega' := A_\theta^{-1}(\theta')$ and recall that $A_\theta^{-1}(\theta) = B^*$. By property (2.18) it follows that

$$\operatorname{diam}(\omega') = 2\|M_\theta^{-1} M_{\theta'}\| \leq 2a_5 2^{-\nu a_6} \leq 2a_5,$$

and hence, since $B^* \cap \omega' \neq \emptyset$

$$A_\theta^{-1}(\theta \cup \theta') = B^* \cup \omega' \subset B(0, 1 + 2a_5). \tag{2.31}$$

By property (d) of the discrete covers, for any $j \geq 1$, there exists $\theta_j \in \Theta_{m-j}$ such that $v_\theta \in \theta_j^\diamond$, where $\theta_j^\diamond := a_7 \cdot \theta_j$ is the dilated θ_j by a factor of $a_7 < 1$ (we note in passing that for a continuous cover, we simply choose $\theta_j = \theta(v_\theta, t - j)$ for any scalar $j > 0$ if $\theta = \theta(v_\theta, t)$). Denote $\omega_j := A_\theta^{-1}(\theta_j)$ and let $\omega_j^\diamond = a_7 \cdot \omega_j$. Also, denote by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ the semiaxes of the ellipsoid ω_j. By (2.21) it follows that

$$\sigma_n \geq \frac{1}{a_5} 2^{a_6 j}. \tag{2.32}$$

On the other hand, by a simple geometric property of ellipsoids

$$\operatorname{dist}(\omega_j^\diamond, \partial\omega_j) = (1 - a_7)\sigma_n,$$

where $\operatorname{dist}(E_1, E_2)$ denotes the (minimal) Euclidean distance between the sets $E_1, E_2 \subset \mathbb{R}^n$. From this and (2.32) it follows that

$$\operatorname{dist}(\omega_j^\diamond, \partial\omega_j) \geq \frac{1 - a_7}{a_5} 2^{a_6 j}. \tag{2.33}$$

Now choose $\gamma \geq 1$ so that

$$\frac{1 - a_7}{a_5} 2^{a_6 \gamma} \geq 1 + 2a_5. \tag{2.34}$$

Let $j \geq \gamma$. Since $v_\theta \in \theta_j^\diamond$, we have $0 \in \omega_j^\diamond$, and using (2.31), (2.33), and (2.34) we infer $A_\theta^{-1}(\theta \cup \theta') \subset \omega_j$, which implies $\theta \cup \theta' \subset A_\theta(\omega_j) = \theta_j =: \eta$. This completes the proof. □

Lemma 2.19. *Let Θ be a discrete cover. Then there is a positive integer $N_2(\mathbf{p}(\Theta))$ such that for any $\theta \in \Theta_m$, $m \in \mathbb{Z}$, the number of ellipsoids from Θ_{m+j}, $j \geq 1$, that intersect θ is bounded by $N_2 2^j$.*

Proof. Let $\theta \in \Theta_m$, $m \in \mathbb{Z}$. By Lemma 2.18 there exists $\eta \in \Theta_{m-\gamma}$ such that $\theta \cup \theta' \subset \eta$ for any $\theta' \in \Theta_{m+j}$, $\theta \cap \theta' \neq \emptyset$. Now pick any such $\theta' \in \Theta_{m+j}$, and let $\mathcal{X}_{\theta'} := \bigcup_{\theta^* \in \Theta_{m+j}, \theta^* \cap \theta' \neq \emptyset} \theta^*$. Consider $\mathcal{X}_{\theta'}$ as a first cluster. We then pick $\theta'' \in \Theta_{m+j}$ such that $\theta'' \cap \theta \neq \emptyset$, $\theta'' \cap \theta' = \emptyset$ and create a second cluster $\mathcal{X}_{\theta''}$ (that possibly intersects with the first). By property (c) of discrete covers each such cluster contains at most N_1 ellipsoids, and the number of them can be bounded by

$$\frac{|\eta|}{|\theta'|} \leq \frac{a_2 2^{-(m-\gamma)}}{a_1 2^{-(m+j)}} \leq \frac{a_2}{a_1} 2^{\gamma+j}, \quad \forall \theta' \in \Theta_{m+j},\ \theta \cap \theta' \neq \emptyset.$$

This implies that we can choose

$$N_2 := N_1 \lceil a_1^{-1} a_2 \rceil 2^{\gamma}. \qquad \square$$

The following two covering lemmas for ellipsoid covers are versions of classic results on ball coverings in arbitrary spaces of homogeneous type (see, e. g., [61]). They are essential for the Calderón–Zygmund decomposition, which is used for the analysis of Hardy spaces $H^p(\Theta)$, $0 < p \leq 1$ (see Chapter 6). The first is a Wiener-type lemma:

Lemma 2.20. *Let Θ be a continuous cover of $\mathbb{R}^n$. There exists a constant $\gamma(\mathbf{p}(\Theta)) > 0$ such that for any open set $\Omega \subset \mathbb{R}^n$ and a bounded from below function $t : \Omega \to \mathbb{Z}$ such that $\theta(x, t(x)) \subset \Omega$ for all $x \in \Omega$, the following holds: there exists a sequence of points $\{x_j\} \subset \Omega$ (finite or infinite) such that the ellipsoids $\theta(x_j, t(x_j))$ are mutually disjoint and $\Omega \subset \bigcup_j \theta(x_j, t(x_j) - \gamma)$.*

Proof. By Lemma 2.18 there exists $\gamma > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$, if $\theta(x, t) \cap \theta(y, s) \neq \emptyset$ with $t \leq s$, then $\theta(y, s) \subset \theta(x, t - \gamma)$.

Since $t : \Omega \to \mathbb{Z}$ is bounded from below, we may pick $x_1 \in \Omega$, with $t(x_1) = \min_{x \in \Omega} t(x)$. Next, if $\Omega \subseteq \theta(x_1, t(x_1) - \gamma)$, we are done. Otherwise, we proceed inductively. Assume that we have picked $x_1, \ldots, x_j$, and set $\Omega' = \Omega \setminus \bigcup_{i=1}^{j} \theta(x_i, t(x_i) - \gamma)$. If $\Omega' = \emptyset$, we are done. Else, pick x_{j+1}, with $t(x_{j+1}) = \min_{x \in \Omega'} t(x)$. We claim that during our construction process, it is not possible that $\theta(x_i, t(x_i)) \cap \theta(x_j, t(x_{j+1})) \neq \emptyset$ for $i < j + 1$. Indeed, if this holds, then there are two possible cases: If $t(x_i) \leq t(x_{j+1})$, then $\theta(x_{j+1}, t(x_{j+1})) \subset \theta(x_i, t(x_i) - \gamma)$, and so $x_{j+1} \notin \Omega'$, a contradiction. If $t(x_i) > t(x_{j+1})$, then $x_{j+1} \in \theta(x_k, t(x_k) - \gamma)$ for some $k < i$, since otherwise, x_{j+1} would have been picked before x_i. But this is a contradiction because it implies again that $x_{j+1} \notin \Omega'$. Thus we have proved that the ellipsoids $\{\theta(x_j, t(x_j))\}$ are mutually disjoint. Since the process terminates only when $\Omega' = \emptyset$, we obtain that any point $x \in \Omega$ is covered. $\square$

The next result is an anisotropic variant of the Whitney lemma.

Lemma 2.21. *Let Θ be a continuous cover. There exists a constant $\gamma(\mathbf{p}(\Theta)) > 0$ such that for any open $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$ and any $m \geq 0$, there exist a sequence of points $\{x_j\}_{j \in \mathbb{N}} \subset \Omega$ and a sequence $\{t_j\}_{j \in \mathbb{N}}$ such that*

(i) $\Omega = \bigcup_j \theta(x_j, t_j)$,
(ii) $\theta(x_j, t_j + \gamma)$ *are pairwise disjoint,*
(iii) *for every* $j \in \mathbb{N}$, $\theta(x_j, t_j - m - \gamma) \cap \Omega^c = \emptyset$, *but* $\theta(x_j, t_j - m - \gamma - 1) \cap \Omega^c \neq \emptyset$,
(iv) $\theta(x_j, t_j - m) \cap \theta(x_i, t_i - m) \neq \emptyset \Rightarrow |t_i - t_j| < \gamma + 1$,
(v) *for every* $j \in \mathbb{N}$,

$$\#\{i \in \mathbb{N} : \theta(x_i, t_i - m) \cap \theta(x_j, t_j - m) \neq \emptyset\} \leq L,$$

where L depends only on the parameters of the cover and m.

Proof. As in the Wiener lemma, we choose the constant $\gamma \in \mathbb{N}$ from Lemma 2.18. For every $x \in \Omega$, define

$$t(x) := \inf\{s \in \mathbb{Z} : \theta(x, s - m - \gamma) \subset \Omega\} + \gamma.$$

Since Ω is open and since by (2.27), for each point $x \in \mathbb{R}^n$, the diameters of the ellipsoids $\theta(x, t)$ decrease as $t \to \infty$, we get that $t(x)$ is well defined. Also, since Ω has finite volume, $t(x)$ is bounded from below on Ω. By Lemma 2.20 we can find for the function $t(x)$ a sequence $\{x_j\} \subset \Omega$ such that $\{\theta(x_j, t_j + \gamma)\}$ are disjoint and $\Omega = \bigcup_j \theta(x_j, t_j)$. This gives properties (i) and (ii). By construction, $\theta(x_j, t_j - m - \gamma) \cap \Omega^c = \emptyset$, but $\theta(x_j, t_j - m - \gamma - 1) \cap \Omega^c \neq \emptyset$, which implies property (iii). To prove property (iv), assume by contradiction that there exist indices i, j such that $\theta(x_i, t_i - m) \cap \theta(x_j, t_j - m) \neq \emptyset$ with $t_j \leq t_i - \gamma - 1$. This gives that $\theta(x_i, t_i - m - \gamma - 1) \cap \theta(x_j, t_j - m) \neq \emptyset$ with $t_j - m \leq t_i - m - \gamma - 1$. The choice of γ guarantees that $\theta(x_i, t_i - m - \gamma - 1) \subset \theta(x_j, t_j - m - \gamma)$. However, we arrive at a contradiction with the established property (iii), since

$$\emptyset \neq \theta(x_i, t_i - m - \gamma - 1) \cap \Omega^c \subset \theta(x_j, t_j - m - \gamma) \cap \Omega^c = \emptyset.$$

We now prove property (v). For $j \geq 1$, let $I(j) := \{i : \theta(x_i, t_i - m) \cap \theta(x_j, t_j - m) \neq \emptyset\}$. From property (iv) we derive that $t_j < t_i + \gamma + 1$ for all $i \in I(j)$. Therefore

$$\bigcup_{i \in I(j)} \theta(x_i, t_i - m) \subset \theta(x_j, t_j - m - 2\gamma - 1).$$

On the other hand, since $t_j > t_i - \gamma - 1$, we have by (2.13) that for all $i \in I(j)$,

$$\begin{aligned} |\theta(x_j, t_j - m - 2\gamma - 1)| &\leq a_2 2^{-(t_j - m - 2\gamma - 1)} \\ &\leq a_2 2^{-(t_i - m - 3\gamma - 2)} \\ &= L a_1 2^{-(t_i + \gamma)} \\ &\leq L|\theta(x_i, t_i + \gamma)|, \end{aligned}$$

where $L := a_1^{-1}a_2 2^{m+4\gamma+2}$. This, coupled with property (ii), gives

$$\#I(j) \le \frac{1}{\min_{i\in I(j)} |\theta(x_i, t_i+\gamma)|} \sum_{i\in I(j)} \big|\theta(x_i, t_i+\gamma)\big| \le \frac{|\theta(x_j, t_j - m - 2\gamma - 1)|}{\min_{i\in I(j)} |\theta(x_i, t_i+\gamma)|} \le L. \qquad \square$$

2.3 Quasi-distances induced by covers

The continuous and discrete ellipsoid covers induce a natural quasi-distance on $\mathbb{R}^n$. Let Θ be a cover. We define $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ by

$$\rho(x,y) := \inf_{\theta\in\Theta}\{|\theta| : \ x, y \in \theta\}. \tag{2.35}$$

Theorem 2.22. *The function* ρ *in* (2.35)*, induced by a discrete or a continuous ellipsoid cover, is a quasi-distance on* $\mathbb{R}^n$*. For a continuous cover,*

$$\rho(x,y) \sim \inf_{y\in\theta(x,t)} \big|\theta(x,t)\big| \sim \inf_{x\in\theta(y,t)} \big|\theta(y,t)\big|, \quad \forall x,y \in \mathbb{R}^n. \tag{2.36}$$

Proof. We need to ensure that ρ satisfies the three conditions of Definition 2.1. Property (i) of the quasi-distance is derived from (2.27). Property (ii) is obvious by the definition of $\rho(\cdot,\cdot)$ in (2.35). Let us show property (iii) of the quasi-triangle inequality in the case of a discrete cover. Let $x, y, z \in \mathbb{R}^n$ and assume that $\rho(x,z) = |\theta|$, $x, z \in \theta$, and $\rho(z,y) = |\theta'|$, $z, y \in \theta'$, where $\theta \in \Theta_m$ and $\theta' \in \Theta_{m+\nu}$. Without loss of generality, we may assume that $\nu \ge 0$. We now apply Lemma 2.18 to conclude that there exists an ellipsoid $\eta \in \Theta_{m-\gamma}$ such that $\theta \cup \theta' \subset \eta$, and hence

$$\begin{aligned} \rho(x,y) &\le |\eta| \le a_2 2^{-(m-\gamma)} \\ &\le a_2 a_1^{-1} 2^{\gamma} \big(|\theta| + \big|\theta'\big|\big) \\ &= \kappa\big(\rho(x,z) + \rho(z,x)\big), \end{aligned}$$

where $\kappa := a_2 a_1^{-1} 2^{\gamma}$.

We now prove (2.36) for the case of continuous covers. From the definition it is obvious that $\rho(x,y) \le \inf_{y\in\theta(x,t)} |\theta(x,t)|$ for all $x, y \in \mathbb{R}^n$. Now let $x \ne y$, $x, y \in \theta'$, $\theta' \in \Theta_s$, with $\rho(x,y) = |\theta'|$. By Lemma 2.18, $\theta(x,s) \cup \theta' \subset \theta(x, s-\gamma)$, and so $y \in \theta(x, s-\gamma)$. Thus

$$\inf_{y\in\theta(x,t)} \big|\theta(x,t)\big| \le \big|\theta(x, s-\gamma)\big| \le a_1^{-1} a_2 2^{\gamma} \big|\theta'\big| = C\rho(x,y). \qquad \square$$

Let Θ be an ellipsoid cover inducing a quasi-distance ρ. We recall the notation

$$B_\rho(x,r) := \{y \in \mathbb{R}^n : \ \rho(x,y) < r\}. \tag{2.37}$$

Evidently, (2.35) implies that

$$B_\rho(x,r) = \bigcup_{\theta\in\Theta} \{\theta : |\theta| < r,\ x \in \theta\}.$$

Theorem 2.23. *Let Θ be an ellipsoid cover inducing a quasi-distance ρ. For each ball $B_\rho(x,r)$, $x \in \mathbb{R}^n$, $r > 0$, there exist ellipsoids $\theta', \theta'' \in \Theta$ such that $\theta' \subset B_\rho(x,r) \subset \theta''$ and $|\theta'| \sim |B_\rho(x,r)| \sim |\theta''| \sim r$, where the constants of equivalency depend on $\mathbf{p}(\Theta)$. In particular, this implies that $(\mathbb{R}^n, \rho, dx)$, where dx is the Lebesgue measure, is a normal space of homogeneous type (see Definition 2.2). In the case where Θ is a continuous cover, we may choose θ' and θ'' with centers at x.*

In the other direction, for any ellipsoid $\theta \in \Theta$ with center v_θ, there exist balls B'_ρ, B''_ρ with center at v_θ such that $B'_\rho \subset \theta \subset B''_\rho$ and $|B'_\rho| \sim |\theta| \sim |B''_\rho|$, where the constants of equivalency depend on $\mathbf{p}(\Theta)$.

Proof. We mostly prove the case where $\rho(\cdot,\cdot)$ is generated by a discrete ellipsoid cover of $\mathbb{R}^n$ and point out in passing the technique for the case of continuous covers. Let $B_\rho(x,r)$, $x \in \mathbb{R}^n$, $r > 0$, be an anisotropic ball. Choose m so that $a_2 2^{-m} < r \le a_2 2^{-(m-1)}$. There exists $\theta' \in \Theta_m$ such that $x \in \theta'$ and hence, by the "volume property" of Θ,

$$a_1 2^{-m} \le |\theta'| \le a_2 2^{-m} < r.$$

From this and the definition of $\rho(\cdot,\cdot)$ it follows that $\theta' \subset B_\rho(x,r)$, and hence

$$|B_\rho(x,r)| \ge |\theta'| \ge a_1 2^{-m} \ge \frac{a_1}{2a_2} r =: c_1 r.$$

We note that in the case of a continuous cover, we may choose $\theta' = \theta(x,m)$. Next, observe that

$$B_\rho(x,r) = \bigcup_{\theta\in\Theta:\, x\in\theta, |\theta|<r} \theta.$$

Suppose $\theta \in \Theta_m$ is at the minimum level such that $x \in \theta$ and $|\theta| < r$. An application of Lemma 2.18 gives that there exists $\theta'' \in \Theta_{m-\gamma}$ such that $B_\rho(x,r) \subseteq \theta''$. Also,

$$|\theta''| \le a_2 2^{-(m-\gamma)} \le a_1^{-1} a_2 2^{\gamma} |\theta| \le c_2 r.$$

For the case of a continuous cover, let $t' := \inf\{t \in \mathbb{R} : \theta(x,t) \subseteq B_\rho(x,r)\}$. For any "small" $\varepsilon > 0$, let $t := t' + \varepsilon$. Then $\theta(x,t) \subseteq B_\rho(x,r)$. Next, observe that any $\theta' \subseteq B_\rho(x,r)$, $x \in \theta'$, is of scale $\ge t'$ and by Lemma 2.18 is contained in $\theta(x, t'-\gamma)$. Therefore $B_\rho(x,r) \subseteq \theta(x, t'-\gamma)$. We obtain that $\theta(x,t) \subseteq B_\rho(x,r) \subseteq \theta(x, t-\varepsilon-\gamma)$ with equivalent volumes. This completes the proof of the first part of the theorem.

Now let $\theta \in \Theta_m$. Denote $x'' := v_\theta$ (the center of θ), $r'' := a_2 2^{-m}$, and $B''_\rho := B''_\rho(x'', r'')$. Then since $x'' \in \theta$ and $|\theta| \le r''$, by definition $\theta \subset B''_\rho$. By the first part of the theorem we also have that $|B''| \le c_2 r'' \le c_2 a_1^{-1} a_2 |\theta|$.

Next, let $\theta' \in \Theta_{m+\nu}$, $\nu \ge 0$, be such that $x' := v_\theta \in \theta'$. Applying (2.23) on the cover $A_\theta^{-1}(\Theta)$ gives

$$\sigma_{\max}(A_\theta^{-1}(\theta')) \le a_5 2^{-a_6 \nu} \sigma_{\max}(A_\theta^{-1}(\theta)) = a_5 2^{-a_6 \nu}.$$

Therefore if $\nu \ge a_6^{-1} \log_2(a_5)$, then $\sigma_{\max}(A_\theta^{-1}(\theta')) \le 1$. Since we also know that $0 = A_\theta^{-1}(v_\theta) \in A_\theta^{-1}(\theta')$, we get that $A_\theta^{-1}(\theta') \subset B^* = A_\theta^{-1}(\theta)$, which in turn implies $\theta' \subset \theta$. Thus setting $\nu := \lceil a_6^{-1} \log_2(a_5) \rceil$ and $r' := a_1 2^{-(m+\nu)}$ gives that any $\theta' \in \Theta$ satisfying $x' \in \theta'$ and $|\theta'| \le r'$ must be contained in θ. Therefore for $B'_\rho := B_\rho(x', r') \subset \theta$, we have $|B'_\rho| \sim r' \sim |\theta|$. □

Remark 2.24. As Theorem 2.23 shows, the framework of ellipsoid covers is a special case of spaces of homogeneous type. However, we point out that since the construction supports pointwise variable anisotropy, there is no assumption of an underlying group structure, translation invariance, etc. and so there is actually no 'homogeneity' property associated with the setup (see also the discussion in [20, example (13), p. 590]).

Definition 2.25. Let ρ be a quasi-distance on $\mathbb{R}^n$, and let $\tau = (\tau_0, \tau_1)$, $0 < \tau_0 \le \tau_1 \le 1$. For any $x, y \in \mathbb{R}^n$ and $d > 0$ we define,

$$\tau(x,y,d) := \begin{cases} \tau_0, & \rho(x,y) < d, \\ \tau_1, & \rho(x,y) \ge d, \end{cases} \qquad \tilde{\tau}(x,y,d) := \begin{cases} \tau_1, & \rho(x,y) < d, \\ \tau_0, & \rho(x,y) \ge d. \end{cases} \tag{2.38}$$

For $t \in \mathbb{R}$, we define

$$\tau(t) := \begin{cases} \tau_1, & t \le 0, \\ \tau_0, & t > 0, \end{cases} \qquad \tilde{\tau}(t) := \begin{cases} \tau_0, & t \le 0, \\ \tau_1, & t > 0. \end{cases} \tag{2.39}$$

Theorem 2.26. *Let Θ be an ellipsoid cover, and let ρ be the quasi-distance* (2.35). *Denote $\tau := (\tau_0, \tau_1) = (a_6, a_4)$, where $0 < a_6 \le a_4 \le 1$ are the parameters from either* (2.14) *or* (2.18). *Then, for each fixed $y \in \mathbb{R}^n$ (or all $y \in \Omega$, where Ω is a bounded set), there exist constants $0 < c_1 < c_2 < \infty$ that depend on y (or Ω) and $\mathbf{p}(\Theta)$ such that*

$$c_1 \rho(x,y)^{\tilde{\tau}(x,y,1)} \le |x - y| \le c_2 \rho(x,y)^{\tau(x,y,1)}, \quad \forall x \in \mathbb{R}^n, \tag{2.40}$$

where $|x - y|$ is the usual Euclidean distance between x and y.

Proof. We prove the theorem for discrete covers (the proof for continuous covers is similar). Take an ellipsoid $\theta_0 \in \Theta_0$ such that $y \in \theta_0 \in \Theta_0$. For any $x \in \mathbb{R}^n$, let $\theta \in \Theta_m$ be such that $\rho(x,y) = |\theta|$. Applying (2.23) yields

$$\begin{aligned}|x-y| &\le \operatorname{diam}(\theta)\\ &\le C\operatorname{diam}(\theta_0)2^{-\tau(m)m}\\ &\le C\operatorname{diam}(\theta_0)a_1^{-\tau(m)}|\theta|^{\tau(m)}\\ &\le C\rho(x,y)^{\tau(x,y,1)}.\end{aligned}$$

We now prove the left-hand side of (2.40). Since $\theta \in \Theta_m$ is the ellipsoid with minimal volume containing both x and y, we may apply property (a) and then (d) of discrete covers to conclude that for an integer $\nu := \lceil \log_2(a_1^{-1}a_2)\rceil$, there exists $\theta_1 \in \Theta_{m+\nu}$ such that $y \in \theta_1^{\diamond}$ (the dilated version of θ_1 by a factor a_7) and $x \notin \theta_1$. Denote by $\sigma_{\min}(\theta_1)$ the minimal semiaxis of θ_1. From (2.24) we get that $\sigma_{\min}(\theta_1) \ge c\sigma_{\min}(\theta_0)2^{-\tilde\tau(m+\nu)(m+\nu)}$. Thus

$$\begin{aligned}|x-y| &\ge (1-a_7)\sigma_{\min}(\theta_1)\\ &\ge C2^{-\tilde\tau(m+\nu)(m+\nu)}\\ &\ge C\rho(x,y)^{\tilde\tau(m+\nu)}\\ &\ge C\rho(x,y)^{\tilde\tau(x,y,1)}.\end{aligned}$$

For the case of $y \in \Omega$, where Ω is a bounded set, the proof is almost identical, since by Lemma 2.16 we have that all ellipsoids $\theta_0 \in \Theta_0$, $\theta_0 \cap \Omega \ne \emptyset$, have equivalent shape. □

Observe that in the case where all ellipsoids in Θ_0 are equivalent in shape (for example, to the Euclidean ball), we get that the constants c_1, c_2 in (2.40) depend only on $p(\Theta)$ and not on the points y. In the particular case where the ellipsoid cover is composed of Euclidean balls, we have that the parameters in (2.14) and (2.18) satisfy $a_4 = a_6 = n^{-1}$, and (2.40) is easily verified by

$$|x-y| \sim \left|B(x,|x-y|)\right|^{1/n} \sim \rho(x,y)^{1/n} \sim \rho(x,y)^{\tau(x,y,1)} = \rho(x,y)^{\tilde\tau(x,y,1)}.$$

Although the equivalence (2.40) of the anisotropic quasi-distance and the Euclidean distance is very “rough” in nature, it is nevertheless sufficient to produce the equivalence of isotropic and anisotropic test functions. Recall that the space of Schwartz functions $\mathcal{S}$ is the set of all functions $\varphi \in C^{\infty}(\mathbb{R}^n)$ such that for any $\alpha \in \mathbb{Z}_+^n$ and $N \ge 1$, there exists a constant $C_{\alpha,N}$ for which

$$\left|\partial^{\alpha}\varphi(x)\right| \le C_{\alpha,N}\left(1+|x|\right)^{-N}, \quad \forall x \in \mathbb{R}^n.$$

We then define by $\mathcal{S}'$ the space tempered distributions, which is the space of linear functionals on $\mathcal{S}$. Applying the equivalence (2.40) for $y = 0$ yields

$$\left|\partial^{\alpha}\varphi(x)\right| \le \tilde C_{\alpha,N}\left(1+\rho(x,0)\right)^{-Na_6}, \quad \forall x \in \mathbb{R}^n.$$

In the other direction, we will also frequently use the fact that a function $\varphi \in C^\infty(\mathbb{R}^n)$ supported on some ellipsoid $\theta \in \Theta$ is by (2.40) also compactly supported with respect to the Euclidean distance and hence in $\mathcal{S}$.

2.4 Equivalency of covers

Definition 2.27. We say that two covers (continuous or discrete) $\Theta, \tilde{\Theta}$ are equivalent if for any $\theta \in \Theta$, there exist $\tilde{\theta}_1, \tilde{\theta}_2 \in \tilde{\Theta}$ such that $\tilde{\theta}_1 \subseteq \theta \subseteq \tilde{\theta}_2$ and $|\tilde{\theta}_1| \sim |\theta| \sim |\tilde{\theta}_2|$, and visa versa, with constants of equivalency depending only on $\mathbf{p}(\Theta)$ and $\mathbf{p}(\tilde{\Theta})$.

We have the following equivalent conditions for cases where both covers are of the same type:

(i) For the case of continuous covers, an equivalent condition is that there exists a constant $c > 0$ such that for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\tilde{\theta}(x, t + c) \subseteq \theta(x, t) \subseteq \tilde{\theta}(x, t - c).$$

(ii) Another equivalent condition for the case of continuous covers is the existence of constants $0 < c_1 \le 1 \le c_2 < \infty$ such that

$$x + c_1 \tilde{M}_{x,t}(B^*) = c_1 \cdot \tilde{\theta}(x, t) \subseteq \theta(x, t) \subseteq c_2 \cdot \tilde{\theta}(x, t) = x + c_2 \tilde{M}_{x,t}(B^*).$$

(iii) Two discrete covers are equivalent if there exists a constant $K \in \mathbb{N}$ such that for any $\theta \in \Theta_m$, there exist $\tilde{\theta}' \in \tilde{\Theta}_{m+K}$ and $\tilde{\theta}'' \in \tilde{\Theta}_{m-K}$ such that $\tilde{\theta}' \subseteq \theta \subseteq \tilde{\theta}''$ and visa versa.

We now show that in essence, requiring a cover to be pointwise continuous (see Definition 2.11) is not a significant restriction.

Theorem 2.28 ([12]). *Given a continuous cover, there exists an equivalent pointwise continuous cover.*

To prove the theorem, we first need the following lemmas.

Lemma 2.29. *For any ellipsoid cover* Θ *and fixed* $t \in \mathbb{R}$, *there exists a bounded continuous function* $r : \mathbb{R}^n \to (0, \infty)$ *such that* $B(x, r(x)) \subset \theta(x, t)$ *for all* $x \in \mathbb{R}^n$.

Proof. Fix $t \in \mathbb{R}$. For $x \in \mathbb{R}^n$, let $r_x := \sigma_{\min}(M_{x,t}) = \|M_{x,t}^{-1}\|^{-1}$. Note that by (2.13), $r_x \le c2^{-t/n}$, $\forall x \in \mathcal{R}^n$. Obviously, we have that

$$B(x, r_x) \subset x + M_{x,t}(B^*) = \theta(x, t). \tag{2.41}$$

This implies that if $B(x', r_{x'}) \cap B(x, r_x) \ne \emptyset$ for $x' \in \mathbb{R}^n$, then $\theta(x', t) \cap \theta(x, t) \ne \emptyset$, and we may apply the shape condition to obtain

$$\|M_{x',t}^{-1}\| \le \|M_{x',t}^{-1} M_{x,t}\| \|M_{x,t}^{-1}\| \le a_5 \|M_{x,t}^{-1}\|.$$

Hence $a_5^{-1} r_x \le \|M_{x',t}^{-1}\|^{-1} = r_{x'}$. Similarly, we have $r_{x'} \le a_5 r_x$. Therefore

$$a_5^{-1} r_x \le r_{x'} \le a_5 r_x, \quad \forall x' \in \mathbb{R}^n, \quad B(x', r_{x'}) \cap B(x, r_x) \ne \emptyset. \tag{2.42}$$

By (2.41) we have $|r_x|^n \sim |B(x, r_x)| \le |\theta(x,t)| \le a_2 2^{-t}$ for all $x \in \mathbb{R}^n$. Applying the classical isotropic Vitali covering lemma for the cover $\{B(x, \frac{1}{10} r_x)\}_{x\in\mathbb{R}^n}$, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ such that the balls $B(x_i, \frac{1}{10} r_{x_i})$, $i \in \mathbb{N}$, are mutually disjoint, and $\mathbb{R}^n = \bigcup_{i=1}^{\infty} B(x_i, \frac{1}{2} r_{x_i})$. For simplicity, we denote $r_i := r_{x_i}$. For $j \in \mathbb{N}$, we let

$$I(j) := \{i : B(x_i, r_i) \cap B(x_j, r_j) \ne \emptyset\}.$$

By (2.42) we have that for any $i \in I(j)$,

$$B\left(x_i, \frac{1}{10} r_i\right) \subset B(x_i, r_i) \subset B(x_j, (2a_5 + 1) r_j),$$

and hence $\bigcup_{i\in I(j)} B(x_i, \frac{1}{10} r_i) \subset B(x_j, (2a_5+1) r_j)$. From this and (2.42) it follows that

$$\begin{aligned} \sharp I(j) &\le \frac{\sum_{i\in I(j)} |B(x_i, r_i)|}{\min_{i\in I(j)} |B(x_i, r_i)|} \le \frac{\sum_{i\in I(j)} 10^n |B(x_i, \frac{1}{10} r_i)|}{|B(x_j, \frac{1}{a_5} r_j)|} \\ &\le \frac{10^n |B(x_j, (2a_5+1) r_j)|}{|B(x_j, \frac{1}{a_5} r_j)|} = \left[10 a_5 (2a_5 + 1)\right]^n =: L. \end{aligned} \tag{2.43}$$

Choose a function $\phi \in C^\infty$ such that $\operatorname{supp}(\phi) = B^*$, $0 \le \phi \le 1$, and $\phi \equiv 1$ on $\frac{1}{2} B^*$. For every $i \in \mathbb{N}$, define

$$\phi_i(x) := \frac{r_i^\circ}{a_5 L} \phi\left(\frac{x - x_i}{r_i}\right),$$

where $r_i^\circ := \min\{r_j : B(x_i, r_i) \cap B(x_j, r_j) \ne \emptyset\}$, and L is as in (2.43). For $x \in \mathbb{R}^n$, we define

$$r(x) := \sum_{i=1}^{\infty} \phi_i(x).$$

This is a well-defined continuous function since on each ball $B(x_j, r_j)$ the above series has $\le L$ nonzero terms corresponding to $i \in I(j)$. More precisely, if $x \in B(x_j, r_j)$, then

$$r(x) \le \sum_{i\in I(j)} \phi_i(x) \le \sum_{i\in I(j)} \frac{r_i^\circ}{a_5 L} \le \sum_{i\in I(j)} \frac{r_j}{a_5 L} \le \frac{r_j}{a_5} \le r_x. \tag{2.44}$$

This, together with (2.41), implies that $B(x, r(x)) \subset B(x, r_x) \subset \theta(x,t)$.

Also note that for any $x \in \mathbb{R}^n$, there exists i, such that $x \in B(x_i, \frac{1}{2}r_i)$, this gives that

$$r(x) \geq \frac{r_i^\circ}{a_5 L} > 0. \qquad \square$$

Lemma 2.30. *Assume that* Θ *is a continuous cover and that there exist a constant* $c > 0$ *and positive definite matrices* $\{\tilde{M}_{x,t}\}$ *such that* $\|M_{x,t}^{-1}\tilde{M}_{x,t}\| \leq c$, $\|\tilde{M}_{x,t}^{-1}M_{x,t}\| \leq c$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. *Then* $\tilde{\Theta} = \{\tilde{\theta}(x,t)\}$, *and* $\tilde{\theta}(x,t) = x + \tilde{M}_{x,t}(B^*)$ *is a valid continuous cover (as per Definition* 2.10*) that is also equivalent to* Θ.

Proof. From the conditions of the lemma it is obvious that $c^{-1}\|M_{x,t}\| \leq \|\tilde{M}_{x,t}\| \leq c\|M_{x,t}\|$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Therefore $|\tilde{\theta}(x,t)| \sim |\theta(x,t)| \sim 2^{-t}$, and $\tilde{\Theta}$ satisfies condition (2.13). Next, we see that there exists a constant $\mu(c, \mathbf{p}(\Theta))$ such that for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, if $\tilde{\theta}(x,t) \cap \tilde{\theta}(y, t+\nu) \neq \emptyset$ for some $\nu > 0$, then $\theta(x, t-\mu) \cap \theta(y, t+\nu) \neq \emptyset$. Therefore we may apply the right-hand side of the shape condition (2.14) for Θ to prove that $\tilde{\Theta}$ also satisfies it by

$$\begin{aligned}
\|\tilde{M}_{x,t}^{-1}\tilde{M}_{y,t+\nu}\| &= \|\tilde{M}_{x,t}^{-1}M_{x,t}M_{x,t}^{-1}M_{y,t+\nu}M_{y,t+\nu}^{-1}\tilde{M}_{y,t+\nu}\| \\
&\leq c^2\|M_{x,t}^{-1}M_{y,t+\nu}\| \\
&= c^2\|M_{x,t}^{-1}M_{x,t-\mu}M_{x,t-\mu}^{-1}M_{y,t+\nu}\| \\
&\leq c^2 a_3 2^{a_4\mu}\|M_{x,t-\mu}^{-1}M_{y,t+\nu}\| \\
&\leq c^2 a_3 2^{(a_4+a_6)\mu}2^{-a_6\nu} =: \tilde{a}_5 2^{-a_6\nu}.
\end{aligned}$$

The left-hand side of (2.14) is proved in a similar manner. We derive that $\tilde{\Theta}$ is a valid continuous cover with parameters $\mathbf{p}(\tilde{\Theta})$.

To see that the two covers are equivalent, observe that for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\tilde{M}_{x,t}(B^*) \subseteq cM_{x,t}(B^*)$, which implies $\tilde{\theta}(x,t) \subseteq x + cM_{x,t}(B^*)$. Application of Theorem 2.17 gives that there exists $J_1(\mathbf{p}(\Theta))$ for which $\tilde{\theta}(x,t) \subseteq \theta(x, t - cJ_1)$. Applying the same theorem to $\tilde{\Theta}$ gives the inclusion $\theta(x,t) \subseteq \tilde{\theta}(x, t - c\tilde{J}_1)$ for some fixed $\tilde{J}_1$. $\square$

Proof of Theorem 2.28. Fix $t \in \mathbb{R}$. Let $r := r_t : \mathbb{R}^n \to (0, \infty)$ be the continuous function as in Lemma 2.29. Choose a (nonredundant) sequence $\{x_k\}$ of points in $\mathbb{R}^n$ such that $\bigcup_{k\in\mathbb{N}} B(x_k, r(x_k)) = \mathbb{R}^n$. Choose a partition $\{E_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}^n$ into measurable sets such that $E_k \subseteq B(x_k, r(x_k))$ for all $k \in \mathbb{N}$. For example, define

$$E_k = \begin{cases} B(x_1, r(x_1)), & k = 1, \\ B(x_k, r(x_k)) \setminus \bigcup_{i=1}^{k-1} B(x_i, r(x_i)), & k \geq 2. \end{cases}$$

Define $\tilde{M}_{x,t} = M_{x_k,t}$ if $x \in E_k$ for some $k \in \mathbb{N}$. We now define the set of ellipsoids (which, as we will prove, is a valid continuous cover)

$$\Xi := \{\xi(x,t) := x + N_{x,t}(B^*) :\ x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

using positive definite matrices

$$N_{x,t} := \left(\frac{1}{|B(x,r(x))|} \int\limits_{B(x,r(x))} (\tilde{M}_{y,t})^{-2}\, dy \right)^{-1/2}. \tag{2.45}$$

Since the function r is bounded, for any $y \in \mathbb{R}^n$, we choose $k \in \mathbb{N}$ such that $y \in E_k$ and apply (2.44) to obtain the estimate

$$\|(\tilde{M}_{y,t})^{-1}\| = \|(M_{x_k,t})^{-1}\| \le \frac{1}{r(x_k)} \le \sup_{x \in B(y,\|r\|_\infty)} \frac{1}{r(x)}.$$

Therefore the vector-valued integral in (2.45) is well defined with values in positive definite matrices. By the continuity of the function r we can easily show that

$$x \mapsto \frac{1}{|B(x,r(x))|} \int\limits_{B(x,r(x))} (\tilde{M}_{y,t})^{-2}\, dy$$

is a continuous positive definite matrix-valued function. Then, using the facts that the square root mapping $M \mapsto M^{1/2}$ is continuous on the space of all positive definite $n \times n$ matrices M and that the inverse mapping $M \mapsto M^{-1}$ is continuous on the space of $n \times n$ invertible matrices, we deduce that $x \mapsto N_{x,t}$ is also continuous.

It remains to show that the positive definite matrices $\{N_{x,t}\}$ satisfy the conditions of Lemma 2.30 with respect to the reference cover Θ, since this will imply that Ξ is a valid continuous cover equivalent to Θ. Fix $x \in \mathbb{R}^n$. Take any $y \in B(x,r(x))$ and let $k \in \mathbb{N}$ be such that $y \in E_k$. Since $y \in \theta(x,t) \cap \theta(x_k,t) \neq \emptyset$ by the shape condition (2.14),

$$\|M_{x,t}^{-1} M_{x_k,t}\|, \|M_{x_k,t}^{-1} M_{x,t}\| \le a_5.$$

This implies that

$$a_5^{-2}(M_{x,t})^{-2} \le (M_{x_k,t})^{-2} \le a_5^2 (M_{x,t})^{-2},$$

where we recall that for two positive definite matrices M_1, M_2, the notation $M_1^{-2} \le M_2^{-2}$ means that

$$\langle M_1^{-2} v, v \rangle \le \langle M_2^{-2} v, v \rangle, \quad \forall v \in \mathbb{R}^n.$$

Hence

$$a_5^{-2}(M_{x,t})^{-2} \le (\tilde{M}_{y,t})^{-2} \le a_5^2 (M_{x,t})^{-2}.$$

Integrating the above inequality over $y \in B(x,r(x))$ as in (2.45) yields

$$a_5^{-2}(M_{x,t})^{-2} \le (N_{x,t})^{-2} \le a_5^2 (M_{x,t})^{-2}.$$

This in turn gives

$$\|M_{x,t}^{-1}N_{x,t}\|, \|N_{x,t}^{-1}M_{x,t}\| \le a_5.$$

Thus the conditions of Lemma 2.30 are satisfied, and we may conclude proof of the theorem. □

Condition (e) in Definition 2.14 of the discrete covers may also seem restrictive, but the next observation shows that this is not the case.

Theorem 2.31. *Suppose* Θ *is a discrete multilevel ellipsoid cover of* $\mathbb{R}^n$ *satisfying conditions* (a)–(d) *of Definition* 2.14. *Then there exists an equivalent discrete multilevel ellipsoid cover* $\widetilde{\Theta}$ *satisfying properties* (a)–(e) *(with possibly different constants) obtained by dilating every ellipsoid* $\theta \in \Theta$ *by a factor* r_θ *satisfying* $(a_7 + 1)/2 \le r_\theta \le 1$.

Proof. By Lemma 2.19 there exist constants N_0, N_1, and N_2, depending on the parameters of Θ, such that each ellipsoid $\theta \in \Theta_m$, $m \in \mathbb{Z}$, can be intersected by at most N_0 from Θ_{m-1}, $N_1 - 1$ ellipsoids from Θ_m, and N_2 ellipsoids from Θ_{m+1}. Now set $N := N_0 + N_1 + N_2$, $b := (a_7 + 1)/2$, and $\delta := (1 - b)/N$.

For a fixed $\theta \in \Theta_m$, denote $\theta_j := (b + j\delta) \cdot \theta$ and $\Gamma_j := \{\eta \in \Theta_{m-1} \cup \Theta_m \cup \Theta_{m+1} : \eta \neq \theta, \eta \cap \theta_j \neq \emptyset\}$, $j = 0, 1, \ldots, N$. We then start an inductive process in j where our initial candidate is $\theta = \theta_0$. If there exists some $\eta \in \Gamma_0 \backslash \Gamma_1$, then η potentially has no substantial intersection with θ_0, and so we proceed to inspect θ_1. Observe that the ellipsoids in $\Gamma_0 \setminus \Gamma_1$ from this point onward in the process will no longer intersect our candidate ellipsoid. If, on the other hand, $\Gamma_0 \setminus \Gamma_1 = \emptyset$, then this implies that all the ellipsoids in Γ_0 intersect with θ_1, which is a sufficiently substantial "core" of θ_0. Recalling that these ellipsoids have equivalent volume and shape with θ_0, we get that they have substantial intersection with θ_0, and we may terminate the process.

Since there are at most $N - 1$ intersecting ellipsoids and N possible steps, at some point in our process, we must arrive at an index $0 \le j_0 \le N$ such that $\Gamma_{j_0} \setminus \Gamma_{j_0+1} = \emptyset$. We then set θ_{j_0} as the new ellipsoid to replace θ.

We complete the proof of this proposition inductively by processing as above all ellipsoids from Θ ordered in a sequence. The rule is that once an ellipsoid from Θ has been processed, it will never be touched again. □

Theorem 2.32. *For every continuous or semicontinuous ellipsoid cover* Θ *of* $\mathbb{R}^n$*, through an adaptive sampling and dilation process, there is an equivalent discrete ellipsoid cover* $\widehat{\Theta}$.

Proof. We may assume that Θ is a semicontinuous ellipsoid cover of $\mathbb{R}^n$, since otherwise we would construct one from the given continuous cover.

We first construct for every $m \in \mathbb{Z}$ a countable set $\widehat{\Theta}_m \subset \Theta_m$ satisfying conditions (c)–(d) of Definition 2.14. This can be done, e. g., in two steps as follows: We first choose countably many ellipsoids from Θ_m so that condition (d) is fulfilled with $a_7 = 1/2$ and then inductively remove from this collection one-by-one all ellipsoids that do not destroy condition (d). After that, condition (c) will be automatically fulfilled with some constant N_1 because of condition (b) on Θ. Conditions (a)–(b) on $\widehat{\Theta}_m$ will be inherited from Θ_m.

Secondly, Theorem 2.31 enables us to correct $\{\widehat{\Theta}_m\}$ so that condition (e) is obeyed as well. □

2.5 Spaces of homogeneous type induced by covers

In this section, we strengthen the results of [13] and characterize the quasi-distances on $\mathbb{R}^n$ that may be induced by ellipsoid covers. We start with some definitions.

Definition 2.33. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ is *Q-quasi-convex* if there exists an ellipsoid θ such that

$$\theta \subseteq \Omega \subseteq Q \cdot \theta, \tag{2.46}$$

where $Q \cdot \theta$ is the Q-dilation of Definition 1.3. We say it is *Q-quasi-convex with respect to* $x \in \mathbb{R}^n$ if there exists an ellipsoid θ with center at x such that (2.46) holds.

Observe that by Theorem 1.6 any bounded convex domain $\Omega \subset \mathbb{R}^n$ is n-quasi-convex.

Definition 2.34. Let ρ be a quasi-distance on $\mathbb{R}^n$. We say that ρ is *quasi-convex* if there exists $Q \geq 1$ such that any ball $B_\rho(x,r)$, $x \in \mathbb{R}^n$, $r > 0$, is Q-quasi-convex with respect to x, that is, for all $x \in \mathbb{R}^n$ and $r > 0$, there exists an ellipsoid $\theta_{x,r}$ with center at x such that

$$\theta_{x,r} \subseteq B_\rho(x,r) \subseteq Q \cdot \theta_{x,r}.$$

This obviously implies

$$|\theta_{x,r}| \leq \left|B_\rho(x,r)\right| \leq Q^n |\theta_{x,r}|.$$

In this case, we define the corresponding (possibly not unique) family of ellipsoids

$$\Theta_\rho := \{\theta_{x,r} : x \in \mathbb{R}^n, r > 0\}. \tag{2.47}$$

Theorem 2.35. *Let Θ be a continuous cover, and let ρ be the quasi-distance* (2.35). *Then, ρ is Q-quasi-convex for any $Q > a_3^{-1} 2^{a_4 \gamma}$, where γ is given by Lemma* 2.18.

Proof. As in the proof of Theorem 2.23, let $t_r' := \inf\{t \in \mathbb{R} : \theta(x,t) \subseteq B_\rho(x,r)\}$. For any "small" $\varepsilon > 0$, let $t_r := t_r' + \varepsilon$. Then $\theta(x,t_r) \subseteq B_\rho(x,r)$. Next, since any $\theta' \in \Theta$,

$\theta' \subseteq B_\rho(x,r)$, $x \in \theta'$, is of scale $\geq t'_r$, by Lemma 2.18 it is contained in $\theta(x, t'_r - \gamma)$. Therefore $B_\rho(x,r) \subseteq \theta(x, t'_r - \gamma)$. We now conclude by (2.20) that

$$\|M^{-1}_{x,t_r} M_{x,t'_r-\gamma}\| \leq a_3^{-1} 2^{a_4(\gamma+\varepsilon)} \Rightarrow M_{x,t'_r-\gamma}(B^*) \subseteq a_3^{-1} 2^{a_4(\gamma+\varepsilon)} M_{x,t_r}(B^*)$$
$$\Rightarrow B_\rho(x,r) \subseteq \theta(x, t'_r - \gamma) \subseteq Q \cdot \theta(x, t_r)$$

with $Q := a_3^{-1} 2^{a_4(\gamma+\varepsilon)}$. □

We see that a continuous cover induces a quasi-distance that is quasi-convex. The main result of this section is the converse.

Theorem 2.36. *Let ρ be a quasi-distance on $\mathbb{R}^n$ that is quasi-convex and satisfies the doubling condition* (2.2)*. Then the corresponding family of ellipsoids Θ_ρ given by* (2.47) *induces a continuous cover of $\mathbb{R}^n$ satisfying all the conditions of Definition* 2.10.

Before we proceed with the proof of Theorem 2.36, we need some preparation.

Lemma 2.37. *Let ρ be a quasi-distance on $\mathbb{R}^n$ that is Q-quasi-convex and such that the doubling condition* (2.2) *holds. Let Θ_ρ be the corresponding family of ellipsoids given by* (2.47)*. Suppose that for $\tilde{c} > 0$, $x, y \in \mathbb{R}^n$, $r, s > 0$*

$$B_\rho(x,r) \cap B_\rho(y,s) \neq \emptyset, \quad s \leq \tilde{c} r.$$

Then there exists a constant $c > 0$, depending on c_0 of (2.2)*, Q, κ of* (2.1)*, and $\tilde{c}$, such that $\theta_{y,s} \subseteq c \cdot \theta_{x,r}$.*

Proof. Since $B_\rho(x,r) \cap B_\rho(y,s) \neq \emptyset$ and $s \leq \tilde{c} r$, the quasi-triangle inequality of ρ and the quasi-convexity yield

$$\theta_{y,s} \subseteq B_\rho(y,s) \subseteq B_\rho\big(x, (2\tilde{c}\kappa^2 + \kappa) r\big) = B_\rho(x, c_3 r) \subseteq Q \cdot \theta_{x,c_3 r}, \tag{2.48}$$

where $c_3 := 2\tilde{c}\kappa^2 + \kappa \geq 1$. Obviously, we also have

$$\theta_{x,r} \subseteq B_\rho(x,r) \subseteq B_\rho(x, c_3 r) \subseteq Q \cdot \theta_{x,c_3 r}. \tag{2.49}$$

By the quasi-convexity of ρ and the "upper dimension" inequality (2.3)

$$\begin{aligned}
|Q \cdot \theta_{x,c_3 r}| &= Q^n |\theta_{x,c_3 r}| \\
&\leq Q^n \big|B_\rho(x, c_3 r)\big| \\
&\leq Q^n c_0 c_3^d \big|B_\rho(x,r)\big| \\
&\leq Q^{2n} c_0 c_3^d |\theta_{x,r}| =: c|\theta_{x,r}|.
\end{aligned}$$

Combining this with (2.48), (2.49), and Theorem 1.4 allows us to conclude

$$\theta_{y,s} \subseteq Q \cdot \theta_{x,c_3 r} \subseteq \frac{|Q \cdot \theta_{x,c_3 r}|}{|\theta_{x,r}|} \cdot \theta_{x,r} \subseteq c \cdot \theta_{x,r}. \qquad \square$$

Definition 2.38. For any bounded set $\Omega \subset \mathbb{R}^n$, $v \in \mathbb{R}^n$, and a positive scalar $a > 0$, we denote

$$a(\Omega + v) := \{a(x + v) : x \in \Omega\}.$$

We then say that a quasi-distance ρ on $\mathbb{R}^n$ satisfies the *inner property* if there exist constants $0 < a, b \le 1$ such that for any $x \in \mathbb{R}^n$, $r > 0$, and $\lambda \ge 1$,

$$a\lambda^b (B_\rho(x, r) - x) \subseteq B_\rho(x, \lambda r) - x.$$

Observe that in the setting of spaces of homogeneous type, (2.3) holds with $d := \log_2 c_0$:

$$|B_\rho(x, \lambda r)| \le c_0 \lambda^d |B_\rho(x, r)|, \quad \forall x \in \mathbb{R}^n,\ r > 0,\ \lambda \ge 1,$$

whereas the inner property gives the inverse

$$a\lambda^b |B_\rho(x, r)| \le |B_\rho(x, \lambda r)|, \quad \forall x \in \mathbb{R}^n,\ r > 0,\ \lambda \ge 1. \tag{2.50}$$

Lemma 2.39. *Let ρ be a quasi-convex quasi-distance on $\mathbb{R}^n$, and let Θ_ρ be the corresponding family of ellipsoids as in* (2.47). *Then ρ satisfies the inner property iff there exist constants $\tilde{a}, \tilde{b} > 0$ such that for any $x \in \mathbb{R}^n$, $r > 0$, and $\lambda \ge 1$,*

$$\tilde{a}\lambda^{\tilde{b}} \cdot \theta_{x,r} \subseteq \theta_{x,\lambda r}. \tag{2.51}$$

Proof. Assume first that ρ satisfies the inner property. Since ρ is quasi-convex, for any $x \in \mathbb{R}^n$, $r > 0$, and $\lambda \ge 1$, we get

$$a\lambda^b (\theta_{x,r} - x) \subseteq a\lambda^b (B_\rho(x, r) - x) \subseteq B_\rho(x, \lambda r) - x \subseteq Q(\theta_{x,\lambda r} - x).$$

Therefore Θ_ρ satisfies (2.51) with $\tilde{a} = a/Q$, $\tilde{b} = b$. In the other direction, if Θ_ρ satisfies (2.51), then

$$\tilde{a}\lambda^{\tilde{b}} (B_\rho(x, r) - x) \subseteq \tilde{a}\lambda^{\tilde{b}} (Q \cdot \theta_{x,r} - x) \subseteq Q(\theta_{x,\lambda r} - x) \subseteq Q(B_\rho(x, \lambda r) - x).$$

Therefore ρ satisfies the inner property with $a = \tilde{a}/Q$, $b = \tilde{b}$. $\square$

Theorem 2.40. *Let ρ be a quasi-convex quasi-distance on $\mathbb{R}^n$ for which the doubling condition* (2.2) *holds. Then it satisfies the inner property.*

Proof. First, we will show that there exists $0 < a < 1$ such that

$$a^{-1}\big(B_\rho(x,r) - x\big) + x \subseteq B_\rho(x, 2\kappa r), \quad \forall x \in \mathbb{R}^n,\ r > 0. \tag{2.52}$$

For any $x \in \mathbb{R}^n$, $r > 0$, and $y \in B_\rho(x,r)$, it is obvious that $B_\rho(x,r) \cap B_\rho(y,r) \neq \emptyset$. Therefore the conditions of Lemma 2.37 are satisfied, and we have both inclusions

$$\theta_{y,r} \subseteq c \cdot \theta_{x,r}, \quad \theta_{x,r} \subseteq c \cdot \theta_{y,r}.$$

Applying further Lemma 1.2 gives

$$\theta_{y,r} - y \subseteq c(\theta_{x,r} - x), \quad \theta_{x,r} - x \subseteq c(\theta_{y,r} - y).$$

There exists $0 < a < 1$ such that $(a^{-1} - 1)Qc = 1$. With this choice,

$$(a^{-1} - 1)Q(\theta_{x,r} - x) \subseteq \theta_{y,r} - y. \tag{2.53}$$

For any $z \in a^{-1}(B_\rho(x,r) - x) + x$, let $y \in B_\rho(x,r)$ be such that

$$z = a^{-1}(y - x) + x = y + \big(a^{-1} - 1\big)(y - x).$$

Since $y - x \in B_\rho(x,r) - x \subseteq Q(\theta_{x,r} - x)$, using (2.53), we get that $z \in \theta_{y,r} \subseteq B_\rho(y,r)$. By the triangle inequality

$$\rho(z,x) \le \kappa\big(\rho(x,y) + \rho(y,z)\big) \le 2\kappa r,$$

which yields $z \in B_\rho(x, 2\kappa r)$ and proves (2.52).

We now define $b := \log(a^{-1})/\log(2\kappa)$. For any $\lambda \ge 1$, let $m \in \mathbb{N}_0$ be such that $(2\kappa)^m \le \lambda < (2\kappa)^{m+1}$. Then, using $a^{-1} = (2\kappa)^b$ and (2.52), we may conclude

$$\begin{aligned} a\lambda^b\big(B_\rho(x,r) - x\big) &\subseteq a(2\kappa)^{b(m+1)}\big(B_\rho(x,r) - x\big) \\ &= a^{-m}\big(B_\rho(x,r) - x\big) \\ &\subseteq B_\rho\big(x, (2\kappa)^m r\big) - x \\ &\subseteq B_\rho(x, \lambda r) - x. \end{aligned}$$

□

Proof of Theorem 2.36. For any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, let $\tilde{r}(x,t) := \sup\{r : |B_\rho(x,r)| \le 2^{-t}\}$ and then $r(x,t) := 0.75\tilde{r}(x,t)$. Observe that $\tilde{r}(x,t) < \infty$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, since (2.51) ensures sufficient growth of volume with increased radii. Define the cover Θ by $\theta(x,t) \in \Theta_t$, $\theta(x,t) := \theta_{x,r(x,t)}$, where $\{\theta_{x,r}\}$ are defined by (2.47).

First, we verify that Θ satisfies the volume condition (2.13). By definition of $r(x,t)$

$$\big|\theta(x,t)\big| = |\theta_{x,r(x,t)}| \le \big|B_\rho\big(x, r(x,t)\big)\big| \le 2^{-t}.$$

In the other direction, using the doubling condition we have

$$2^{-t} \le \left|B_\rho(x, 2r(x,t))\right| \le c_0\left|B_\rho(x, r(x,t))\right| \le c_0 Q|\theta_{x,r(x,t)}| = c_0 Q\left|\theta(x,t)\right|.$$

This implies that Θ satisfies the volume condition (2.13) with $a_1 = (c_0 Q)^{-1}$ and $a_2 = 1$.

Next, we show Θ satisfies the shape condition (2.14). Observe that it is sufficient to show there exist constants $a_3, a_4, a_5, a_6 > 0$ such that for any two ellipsoids $\theta(x,t), \theta(y, t+\nu) \in \Theta$, $\nu \ge 0$, such that $\theta(x,t) \cap \theta(y, t+\nu) \ne \emptyset$,

$$a_3 2^{-a_4\nu} M_{x,t}(B^*) \subseteq M_{y,t+\nu}(B^*) \subseteq a_5 2^{-a_6\nu} M_{x,t}(B^*). \tag{2.54}$$

To prove (2.54), it is sufficient to verify the following two sets of inclusions:

$$a_3' M_{x,t}(B^*) \subseteq M_{y,t}(B^*) \subseteq a_5' M_{x,t}(B^*), \tag{2.55}$$

$$a_3'' 2^{-a_4\nu} M_{y,t}(B^*) \subseteq M_{y,t+\nu}(B^*) \subseteq a_5'' 2^{-a_6\nu} M_{y,t}(B^*). \tag{2.56}$$

We start with (2.55). Let $s := r(y,t)$ and $r := r(x,t)$. We claim that $B_\rho(y,s) \cap B_\rho(x,r) \ne \emptyset$. Indeed, $s = r(y,t) \ge r(y, t+\nu)$, which gives

$$\theta(y, t+\nu) = \theta_{y, r(y,t+\nu)} \subseteq B_\rho(y, r(y, t+\nu)) \subseteq B_\rho(y,s),$$

and so

$$\theta(x,t) \cap \theta(y, t+\nu) \subseteq B_\rho(x,r) \cap B_\rho(y,s) \Rightarrow B_\rho(x,r) \cap B_\rho(y,s) \ne \emptyset.$$

Without loss of generality, $s \le r$, since otherwise we can prove the inclusions $a_3''' M_{y,t}(B^*) \subseteq M_{x,t}(B^*) \subseteq a_5''' M_{y,t}(B^*)$. Under the assumption $s \le r$, we may apply Lemma 2.37 to obtain that $\theta(y,t) = \theta_{y,s} \subseteq c \cdot \theta_{x,r} = c \cdot \theta(x,t)$. Using Lemma 1.2, we get that

$$M_{y,t}(B^*) \subseteq c M_{x,t}(B^*),$$

which is the right-hand side of (2.55) with $a_5' := c$. Next, by Theorem 1.4 we obtain that

$$c M_{x,t}(B^*) \subseteq 2\frac{c^n|\theta(x,t)|}{|\theta(y,t)|} M_{y,t}(B^*) \Rightarrow \frac{c^{1-n} a_1}{2a_2} M_{x,t}(B^*) \subseteq M_{y,t}(B^*).$$

This gives the left-hand side of (2.55) with

$$a_3' := \frac{c^{1-n} a_1}{2a_2}.$$

We now turn to prove (2.56). From the definition it is obvious that $s := r(y, t+\nu) \le r(y,t) =: r$. Under this condition, we may apply Lemma 2.37 to obtain that $\theta(y, t+\nu) =$

$\theta_{y,s} \subseteq c \cdot \theta_{y,r} = c \cdot \theta(y,t)$. Using Lemma 1.2, we get that

$$M_{y,t+v}(B^*) \subseteq cM_{y,t}(B^*).$$

Next, by Theorem 1.4 we obtain that

$$cM_{y,t}(B^*) \subseteq \frac{c^n|\theta(y,t)|}{|\theta(y,t+v)|} M_{y,t+v}(B^*) \Rightarrow \frac{c^{1-n}a_1}{2a_2} 2^{-v} M_{y,t}(B^*) \subseteq M_{y,t+v}(B^*).$$

This gives the left-hand side of (2.56) with

$$a_3'' := \frac{c^{1-n}a_1}{2a_2}, \quad a_4 := 1.$$

Next, we show that the right-hand side of (2.56) is satisfied. By Theorem 2.40 ρ satisfies the inner property, and so by Lemma 2.39 there exist constants $\tilde{a}, \tilde{b} > 0$ such that for any $y \in \mathbb{R}^n$, $r > 0$, and $\lambda \geq 1$, $\tilde{a}\lambda^{\tilde{b}} \cdot \theta_{y,r} \subseteq \theta_{y,\lambda r}$. The ellipsoid inner property (2.51) for $\lambda := r/s$ implies

$$\tilde{a}\lambda^{\tilde{b}} \cdot \theta(y,t+v) = \tilde{a}\lambda^{\tilde{b}} \cdot \theta_{y,s} \subseteq \theta_{y,\lambda s} = \theta_{y,r} = \theta(y,t).$$

We may apply Lemma 1.2 to obtain

$$\tilde{a}\lambda^{\tilde{b}} M_{y,t+v}(B^*) \subseteq M_{y,t}(B^*). \tag{2.57}$$

We now use (2.3) and the Q-quasi-convexity of ρ to derive

$$\begin{aligned} 2^{-t} &\leq a_1^{-1}|\theta(y,t)| \\ &= a_1^{-1}|\theta_{y,r}| \\ &\leq a_1^{-1}|B_\rho(y,r)| \\ &\leq a_1^{-1}c_0\left(\frac{r}{s}\right)^d |B_\rho(y,s)| \\ &\leq a_1^{-1}c_0 Q^n\left(\frac{r}{s}\right)^d |\theta_{y,s}| \\ &= a_1^{-1}c_0 Q^n\left(\frac{r}{s}\right)^d |\theta(y,t+v)| \\ &\leq a_1^{-1}c_0 Q^n\left(\frac{r}{s}\right)^d 2^{-(t+v)}. \end{aligned}$$

This gives

$$2^v \leq \tilde{c}\lambda^d$$

with $\tilde{c} := a_1^{-1} c_0 Q^n$. Combining this with (2.57) yields

$$M_{y,t+v}(B^*) \subseteq \tilde{a}^{-1}\lambda^{-\tilde{b}} M_{y,t}(B^*) \subseteq \tilde{a}^{-1}\tilde{c}^{\tilde{b}/d}(2^{-v})^{\tilde{b}/d} M_{y,t}(B^*).$$

This is the right-hand side of (2.56) with $a_5'' = \tilde{a}^{-1}\tilde{c}^{\tilde{b}/d}$ and $a_6 = \tilde{b}/d$. We may conclude that the ellipsoid cover Θ satisfies (2.54). This in turn implies that the shape condition (2.14) holds, and so Θ satisfies all the conditions of a continuous cover as per Definition 2.10. □

3 Anisotropic multiresolution analysis

In this chapter, we focus on multiresolution analysis constructions that are subordinate to the anisotropic quasi-distance induced by the ellipsoid covers. In contrast to the general case of spaces of homogeneous type, our multiresolution analysis constructions over $\mathbb{R}^n$ provide polynomial reproduction of arbitrary (but fixed) order and have arbitrarily high (but fixed) regularization.

3.1 Multiresolution kernel operators

In the setting of a general space of homogeneous type (see Definition 2.2), there exists a very useful construction of multiresolution kernel operators [33].

Definition 3.1. Let (X, ρ, μ) be a space of homogeneous type with κ satisfying (2.1) and α the corresponding constant from Proposition 2.4. A sequence $\{S_m\}_{m\in\mathbb{Z}}$ of kernel operators, formally defined by $S_m f(x) := \int_X S_m(x,y) f(y) d\mu(y)$, is said to be an *approximation to the identity* if there exist $0 < \tau \le \alpha$, $\delta > 0$, and $c > 0$ such that for all $x, x', y, y' \in X$ and $m \in \mathbb{Z}$,

$$\left|S_m(x,y)\right| \le c \frac{2^{-m\delta}}{(2^{-m} + \rho(x,y))^{1+\delta}},$$
$$\left|S_m(x,y) - S_m(x',y)\right| \le c\left(\frac{\rho(x,x')}{2^{-m} + \rho(x,y)}\right)^{\tau} \frac{2^{-m\delta}}{(2^{-m} + \rho(x,y))^{1+\delta}}$$

for $\rho(x,x') \le \frac{1}{2\kappa}(2^{-m} + \rho(x,y))$,

$$\left|S_m(x,y) - S_m(x,y')\right| \le c\left(\frac{\rho(y,y')}{2^{-m} + \rho(x,y)}\right)^{\tau} \frac{2^{-m\delta}}{(2^{-m} + \rho(x,y))^{1+\delta}}$$

for $\rho(y,y') \le \frac{1}{2\kappa}(2^{-m} + \rho(x,y))$,

$$\int_X S_m(x,y) d\mu(y) = 1, \quad \forall x \in X,$$
$$\int_X S_m(x,y) d\mu(x) = 1, \quad \forall y \in X.$$

Our setting of normal spaces of homogeneous spaces induced by ellipsoid covers of $\mathbb{R}^n$ allows us to generalize the above approximation to the identity of order one to arbitrary (but fixed) higher orders and regularity with kernels that reproduce polynomials of arbitrary (but fixed) higher degrees.

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Let $K(x,y)$ be a smooth kernel. For $x,y \in \mathbb{R}^n$, we have the Taylor representation of the kernel about the point x with y fixed:

$$K(z,y) = T_x^r(K(\cdot,y))(z) + R_x^r(K(\cdot,y))(z), \tag{3.1}$$

where T_x^r is the Taylor polynomial of degree $r-1$ (order r) from (1.29), and R_x^r is the Taylor remainder of order r from (1.30).

Definition 3.2. Let $(\mathbb{R}^n,\rho,\mu)$ be a normal space of homogeneous type, where μ is the Lebesgue measure. A sequence of kernel operators $\{S_m\}$, formally defined by $S_m f(x) := \int_{\mathbb{R}^n} S_m(x,y)f(y)dy$, is a *multiresolution of order* (τ,δ,r), $\tau = (\tau_0,\tau_1)$, $0 < \tau_0 \le \tau_1 \le 1$, $\delta > 0$, $r \ge 1$, if there exists a constant $c > 0$ such that for all $x,x',y,y',z \in \mathbb{R}^n$, and $1 \le k \le r$, the following conditions are satisfied:

$$|S_m(x,y)| \le c\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta}}, \tag{3.2}$$

$$\begin{aligned}&|R_x^k(S_m(\cdot,y))(z)|\\ &\quad\le c\rho(x,z)^{\tau(x,z,2^{-m})k}\left(\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau(x,z,2^{-m})k}}+\frac{2^{-m\delta}}{(2^{-m}+\rho(y,z))^{1+\delta+\tau(x,z,2^{-m})k}}\right),\end{aligned} \tag{3.3}$$

$$\begin{aligned}&|R_y^k(S_m(x,\cdot))(z)|\\ &\quad\le c\rho(y,z)^{\tau(y,z,2^{-m})k}\left(\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau(y,z,2^{-m})k}}+\frac{2^{-m\delta}}{(2^{-m}+\rho(x,z))^{1+\delta+\tau(y,z,2^{-m})k}}\right),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&|R_y^k(R_x^k(S_m(\cdot,\cdot))(x'))(y')|, |R_x^k(R_y^k(S_m(\cdot,\cdot))(y'))(x')|\\ &\quad\le c\rho(x,x')^{\tau(x,x',2^{-m})k}\rho(y,y')^{\tau(y,y',2^{-m})k}\\ &\qquad\times\left(\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau(x,x',2^{-m})k+\tau(y,y',2^{-m})k}}\right.\\ &\qquad+\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y'))^{1+\delta+\tau(x,x',2^{-m})k+\tau(y,y',2^{-m})k}}\\ &\qquad+\frac{2^{-m\delta}}{(2^{-m}+\rho(x',y))^{1+\delta+\tau(x,x',2^{-m})k+\tau(y,y',2^{-m})k}}\\ &\qquad\left.+\frac{2^{-m\delta}}{(2^{-m}+\rho(x',y'))^{1+\delta+\tau(x,x',2^{-m})k+\tau(y,y',2^{-m})k}}\right),\end{aligned} \tag{3.5}$$

$$P(x) = \int_{\mathbb{R}^n} S_m(x,y)P(y)dy, \quad P(y) = \int_{\mathbb{R}^n} S_m(x,y)P(x)dx, \quad \forall P \in \Pi_{r-1}. \tag{3.6}$$

Here we used $\tau(\cdot,\cdot,\cdot)$ defined by (2.38). Also, to clarify our notation, denote $g_m(x,x',y) := R_x^k(S_m(\cdot,y))(x')$. Then for fixed $x,x' \in \mathbb{R}^n$, $R_y^k(R_x^k(S_m(\cdot,\cdot))(x'))(y') = R_y^k(g_m(x,x',\cdot))(y')$.

We will use the fact that condition (3.3) implies

$$\left|R_x^k(S_m(\cdot,y))(z)\right| \le c\rho(x,z)^{\tau_0 k}\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau_0 k}} \tag{3.7}$$

if $\rho(x,z) \le \frac{1}{2\kappa}(2^{-m}+\rho(x,y))$ and condition (3.4) implies

$$\left|R_y^k(S_m(x,\cdot))(z)\right| \le c\rho(y,z)^{\tau_0 k}\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau_0 k}} \tag{3.8}$$

if $\rho(y,z) \le \frac{1}{2\kappa}(2^{-m}+\rho(x,y))$. Furthermore, condition (3.5) implies

$$\left|R_y^k(R_x^k(S_m(\cdot,\cdot))(x'))(y')\right| \le c\rho(x,x')^{\tau_0 k}\rho(y,y')^{\tau_0 k}\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+2\tau_0 k}} \tag{3.9}$$

if $\rho(x,x') \le \frac{1}{2\kappa}(2^{-m}+\rho(x,y))$ and $\rho(y,y') \le \frac{1}{2\kappa}(2^{-m}+\rho(x,y))$.

Given an ellipsoid cover, our goal is to construct multiresolution kernels for any given decay parameter $\delta > 0$ and given order r, satisfying all the above properties. Recall that in the cases where the ellipsoid cover is continuous or semicontinuous, we can apply Theorem 2.32 and sample from it a discrete cover that produces an equivalent quasi-distance. Therefore in our constructions below, we focus on discrete covers.

3.2 A multilevel system of bases

We will provide constructions of several types of anisotropic locally stable bases, beginning with the multiresolution $\{\Phi_m\}_{m\in\mathbb{Z}}$ described in the next subsection. The basis Φ_m consists of bumps supported over the ellipsoids of Θ_m, and thus its span may be regarded as the functions at level m of the anisotropic multiresolution. We will later also construct a second type of basis, also composed of bumps, this time supported over interactions of ellipsoids from adjacent levels. This basis will be used to construct "two-level splits", which in turn are used to represent the difference between two projections on adjacent levels of the multiresolution. Summing up over all such differences provides a wavelet-type representation of a given function.

3.2.1 Coloring the ellipsoids in Θ

Our construction begins with a coloring scheme of ellipsoids required for the construction of stable bases. We split a discrete ellipsoid cover Θ into no more than $2N_1$ disjoint subsets (colors) $\{\Theta^\nu\}_{\nu=1}^{2N_1}$ so that for any $m \in \mathbb{Z}$, none of two ellipsoids $\theta', \theta'' \in \Theta_m \cup \Theta_{m+1}$ with $\theta' \cap \theta'' \neq \emptyset$ are of the same color. Indeed, using property (c) of Θ (see Definition 2.14), it is easy to color (inductively) any level Θ_m by using no more than N_1 colors.

So we use at most N_1 colors to color the ellipsoids in $\{\Theta_{2j}\}_{j\in\mathbb{Z}}$ and further at most N_1 colors to color the ellipsoids in $\{\Theta_{2j+1}\}_{j\in\mathbb{Z}}$.

Thus we may assume that we have the following disjoint splitting:

$$\Theta = \bigcup_{\nu=1}^{2N_1} \Theta^{\nu} \quad \text{and} \quad \Theta_{2j} = \bigcup_{\nu=1}^{N_1} \Theta_{2j}^{\nu}, \quad \Theta_{2j+1} = \bigcup_{\nu=N_1+1}^{2N_1} \Theta_{2j+1}^{\nu}, \quad j \in \mathbb{Z}, \tag{3.10}$$

where if $\theta' \in \Theta_{m_1}^{\nu_1}$ and $\theta'' \in \Theta_{m_2}^{\nu_2}$ with $|m_1 - m_2| \le 1$ and $\theta' \cap \theta'' \neq \emptyset$, then $\nu_1 \neq \nu_2$.

Remark 3.3. In this chapter we will only use different colors of intersecting ellipsoids on a single level for the construction of $\{\Phi_m\}_{m\in\mathbb{Z}}$ below. The two-level coloring scheme will come into play in the next chapter when we construct the two-level splits.

3.2.2 Definition of single-level bases

We first introduce $2N_1$ smooth piecewise polynomial bumps associated with the colors from above. For fixed positive integers L and r, $L \ge r$, we define

$$\phi_\nu(x) := \left(1 - |x|^2\right)_+^{L+\nu r}, \quad \nu = 1, 2, \dots, 2N_1, \quad x_+ := \max\{x, 0\}. \tag{3.11}$$

Notice that $\phi_\nu \in C^{L+\nu r-1} \subset C^L$.

Remark 3.4. The bumps ϕ_ν can be modified to be C^∞ functions. To this end, let $h \in C^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} h = \overline{B(0,1)}$, $h \ge 0$, and $\int_{\mathbb{R}^n} h = 1$. Denote $h_\delta(x) := \delta^{-n} h(\delta^{-1}x)$. Then for $0 < \delta < 1$, the bumps $\phi_\nu^* := \phi_\nu * h_\delta$ apparently have the following properties: $\phi_\nu^* \in C^\infty$, ϕ_ν^* is a polynomial of degree exactly $2(L + \nu r)$ on $B(0, 1 - \delta)$, and $\operatorname{supp} \phi_\nu^* = \overline{B(0, 1 + \delta)}$. Now the bumps $\{\phi_\nu^*\}$, dilated by a factor of $1 + \delta$ with δ sufficiently small (depending on the parameters of Θ) can be successfully used in place of $\{\phi_\nu\}$.

For any $\theta \in \Theta$, let A_θ denote the affine transform from Definition 1.1 such that $A_\theta(B^*) = \theta$ (recall $B^* := B(0,1)$) and set

$$\phi_\theta := \phi_\nu \circ A_\theta^{-1} \quad \text{if } \theta \in \Theta^\nu,\ 1 \le \nu \le 2N_1. \tag{3.12}$$

By the properties of discrete covers there exist constants $0 < c_1 < c_2 < \infty$ such that

$$0 < c_1 \le \sum_{\theta\in\Theta_m} \phi_\theta(x) \le c_2, \quad \forall x \in \mathbb{R}^n. \tag{3.13}$$

Indeed, the constant c_2 is derived from property (c) of discrete covers, which assumes that a point $x \in \mathbb{R}^n$ is contained in at most N_1 ellipsoids. The constant c_1 is derived by property (d), which states that any point is contained in the "core" $\theta^\diamond = a_7 \cdot \theta$ of at least one ellipsoid $\theta \in \Theta_m$. This allows us to introduce locally stable mth-level partitions of unity by defining for any $\theta \in \Theta_m$

$$\varphi_\theta := \frac{\phi_\theta}{\sum_{\theta' \in \Theta_m} \phi_{\theta'}}, \quad \sum_{\theta \in \Theta_m} \varphi_\theta(x) = 1, \quad \forall x \in \mathbb{R}^n. \tag{3.14}$$

Let

$$\{P_\beta : |\beta| \le r-1\}, \quad \text{where } \deg P_\beta = |\beta|, \tag{3.15}$$

be an orthonormal basis in $L_2(B^*)$ for the space Π_{r-1} of all polynomials in n variables of total degree $r-1$. Since $\|P_\beta\|_{L_2(B^*)} = 1$

$$\|P_\beta \phi_\nu\|_{L_2(B^*)} \sim \|P_\beta \phi_\nu\|_{L_\infty(B^*)} \sim 1, \quad \nu = 1, 2, \ldots, 2N_1. \tag{3.16}$$

For any $\theta \in \Theta$ and $|\beta| < r$, we define

$$P_{\theta,\beta} := |\theta|^{-1/2} P_\beta \circ A_\theta^{-1}. \tag{3.17}$$

Let us now introduce the more compact notation

$$\Lambda_m := \{\lambda := (\theta, \beta) : \theta \in \Theta_m, |\beta| < r\}, \tag{3.18}$$

and if $\lambda := (\theta, \beta)$, then we denote by θ_λ and β_λ the components of λ. With this notation, we define

$$\varphi_\lambda := P_\lambda \varphi_{\theta_\lambda} = P_{\theta_\lambda, \beta_\lambda} \varphi_{\theta_\lambda}. \tag{3.19}$$

Notice that $\|\varphi_\lambda\|_2 \sim 1$ and, in general, $\|\varphi_\lambda\|_p \sim |\theta|^{1/p-1/2}$, $0 < p \le \infty$. Also, $\varphi_\lambda \in C^L$.

Definition 3.5. We define the mth-level basis Φ_m by

$$\Phi_m := \{\varphi_\lambda : \lambda \in \Lambda_m\} \tag{3.20}$$

and set

$$\mathcal{S}_m := \operatorname{span}(\Phi_m), \tag{3.21}$$

that is, $\mathcal{S}_m$ is the set of all functions f on $\mathbb{R}^n$ of the form

$$f(x) = \sum_{\lambda \in \Lambda_m} c_\lambda \varphi_\lambda(x), \quad x \in \mathbb{R}^n, \tag{3.22}$$

where $\{c_\lambda\}$ is an arbitrary collection of complex numbers.

Remarks

(i) Since each $x \in \mathbb{R}^n$ is contained in at most N_1 ellipsoids from Θ_m, the sum in (3.22) is finite and hence well defined.

(ii) By the partition of unity (3.14) it readily follows that $\Pi_{r-1} \subset \mathcal{S}_m$.

(iii) Φ_m is linearly independent, i. e., if $\sum_{\lambda\in\Lambda_m} c_\lambda\varphi_\lambda = 0$ a. e., then $c_\lambda = 0$ for all $\lambda \in \Lambda_m$. More importantly, Φ_m is locally linearly independent and L_p stable, as we establish in the next theorem.

Theorem 3.6. *Any function $f \in \mathcal{S}_m$ has a unique representation*

$$f(x) = \sum_{\lambda\in\Lambda_m} \langle f, \tilde{g}_\lambda\rangle \varphi_\lambda(x), \tag{3.23}$$

where for every $x \in \mathbb{R}^n$, the sum is finite, and the functions $\tilde{g}_\lambda$ have the following properties: For every $\theta \in \Theta_m$, there exists an ellipsoid $\theta^* := A_\theta(B^*_\theta) \subset \theta$ for some ball $B^*_\theta \subset B^*$ with $|\theta^*| \sim |\theta|$ such that for $0 < p \le \infty$,

$$\langle \varphi_\lambda, \tilde{g}_{\lambda'}\rangle = \delta_{\lambda,\lambda'}, \quad \forall \lambda, \lambda' \in \Lambda_m, \tag{3.24}$$

$$\operatorname{supp}(\tilde{g}_\lambda) \subset \overline{\theta^*_\lambda}, \quad \|\tilde{g}_\lambda\|_p \sim |\theta_\lambda|^{1/p-1/2}. \tag{3.25}$$

Moreover, if $f \in \mathcal{S}_m \cap L_p$, $0 < p \le \infty$, and $f = \sum_{\lambda\in\Lambda_m} c_\lambda\varphi_\lambda$, then

$$\|f\|_p \sim \left(\sum_{\lambda\in\Lambda_m} \|c_\lambda\varphi_\lambda\|_p^p\right)^{1/p}, \tag{3.26}$$

with the obvious modification when $p = \infty$. Here all constants of equivalence depend only on $\mathbf{p}(\Theta)$, L, p, and r.

Proof. We first construct the balls $B^*_\theta \subset B^*$. Fix $\theta \in \Theta_m$ ($m \in \mathbb{Z}$) and let $\mathcal{X}_\theta$ be the set of all $\theta' \in \Theta_m$ such that $\theta' \cap \theta \ne \emptyset$. Denote

$$\mathcal{X}^*_\theta := \{A_\theta^{-1}(\theta') : \theta' \in \mathcal{X}_\theta\}.$$

We claim that there exists a ball $B^*_\theta \subset B^*$ such that $|B^*_\theta| \sim 1$ and for each $\eta \in \mathcal{X}^*_\theta$, either $B^*_\theta \subset \eta$ or $B^*_\theta \cap \eta = \emptyset$. Indeed, $\mathcal{X}^*_\theta$ partitions B^* into a bounded number of interior disjoint subdomains $c(N_1) > 0$, so there exists at least one such subdomain $\Omega^* \subset B^*$ with $|\Omega^*| \ge c(N_1)^{-1}|B^*|$. Obviously, any Euclidean ball $B \subset \Omega^*$ satisfies the property that for each $\eta \in \mathcal{X}^*_\theta$, either $B \subset \eta$ or $B \cap \eta = \emptyset$. So it remains to prove that there exists a ball $B^*_\theta \subset \Omega^*$ of "substantial" volume. Indeed, by property (e) of discrete covers, for any $\eta \in \mathcal{X}^*_\theta$, we have $|\eta \cap B^*| \ge a_8|B^*|$. Also, the shape similarity of the set $\mathcal{X}_\theta$ with θ implies that the set $\mathcal{X}^*_\theta$ is similar in shape to B^*. Thus Ω^* is created by set operations of "unit ball" like ellipsoids. Define $\theta^* := A_\theta(B^*_\theta)$.

Denote by $\mathcal{Y}_\theta$ the set of all $\theta' \in \mathcal{X}_\theta$ such that $\theta^* \subset \theta'$ and set

$$\mathcal{F}_\theta := \{g^\diamond_{\theta',\beta} := \varphi_{\theta',\beta}\mathbf{1}_{\theta^*} : \theta' \in \mathcal{Y}_\theta, |\beta| < r\}.$$

It is an important observation that the set of functions $\mathcal{F}_\theta$ is linearly independent. Indeed, every two ellipsoids in $\mathcal{Y}_\theta$ contain θ^* and thus intersect and have distinct colors. If $\theta' \in \mathcal{Y}_\theta$ and $\theta' \in \Theta^\nu_m$ for some $1 \le \nu \le 2N_1$, then $\phi_{\theta'}P_{\theta',\beta'}$ is a polynomial of degree exactly $L + \nu r + |\beta'|$ on θ^*, and $L + \nu r \le L + \nu r + |\beta'| < L + (\nu + 1)r$. Consequently, the functions $\{\phi_{\theta'}P_{\theta',\beta}\mathbf{1}_{B_\theta} : \theta' \in \mathcal{Y}_\theta, |\beta| < r\}$ are linearly independent on θ^*, and hence $\mathcal{F}_\theta$ is linearly independent.

Define $g^*_{\theta',\beta} := |\theta|^{1/2} g^\diamond_{\theta',\beta} \circ A_\theta$. Notice that $\operatorname{supp} g^*_{\theta',\beta} = \overline{B^*_\theta}$ and $\|g^*_{\theta',\beta}\|_2 \sim \|g^*_{\theta',\beta}\|_\infty \sim 1$. Let

$$\mathcal{F}^*_\theta := \{g^*_{\theta',\beta} : \theta' \in \mathcal{Y}_\theta, |\beta| < r\} \quad \text{and} \quad \Lambda_\theta := \{\lambda := (\theta', \beta) : \theta' \in \mathcal{Y}_\theta, |\beta| < r\}.$$

As $\mathcal{F}_\theta$ is linearly independent, $\mathcal{F}^*_\theta$ is linearly independent as well. Consequently, the Gram matrix

$$G_\theta := \big(\langle g^*_{\theta',\beta'}, g^*_{\theta'',\beta''}\rangle\big)_{(\theta',\beta'),(\theta'',\beta'')\in\Lambda_\theta}$$

is nonsingular, and hence its inverse

$$G^{-1}_\theta =: (R_{(\theta',\beta'),(\theta'',\beta'')})_{(\theta',\beta'),(\theta'',\beta'')\in\Lambda_\theta}$$

exists.

We next show that the functions

$$\tilde{g}_{\theta,\beta} := \sum_{(\theta',\beta')\in\Lambda_\theta} R_{(\theta,\beta),(\theta',\beta')} g^\diamond_{\theta',\beta'} \tag{3.27}$$

form a dual system to Φ_m. Indeed, for $\theta \in \Theta_m$, $\operatorname{supp}(\tilde{g}_{\theta,\beta}) = \theta^*$. if $\theta' \in \Theta_m$ and $\theta' \notin \mathcal{Y}_\theta$, then $\theta' \cap \theta^* = \emptyset$, and hence $\langle \varphi_{\theta',\beta'}, \tilde{g}_{\theta,\beta}\rangle = 0$. Otherwise, for $\theta' \in \mathcal{Y}_\theta$ and $|\beta'| < r$,

$$\begin{aligned}\langle \varphi_{\theta',\beta'}, \tilde{g}_{\theta,\beta}\rangle = |\theta|\langle \varphi_{\theta',\beta'} \circ A_\theta, \tilde{g}_{\theta,\beta} \circ A_\theta\rangle &= \sum_{(\theta'',\beta'')\in\Lambda_\theta} R_{(\theta,\beta),(\theta'',\beta'')}\langle g^*_{\theta',\beta'}, g^*_{\theta'',\beta''}\rangle \\ &= \big(G^{-1}_\theta G_\theta\big)_{(\theta,\beta),(\theta',\beta')} = \delta_{(\theta,\beta),(\theta',\beta')},\end{aligned}$$

as claimed.

Our next and most important step is showing that

$$|R_{(\theta',\beta'),(\theta'',\beta'')}| \le c, \quad \forall\,(\theta',\beta'),(\theta'',\beta'') \in \Lambda_\theta, \tag{3.28}$$

where $c > 0$ depends only on $\mathbf{p}(\Theta)$, L, and r. We will use a compactness argument.

We readily see that the set $\mathcal{F}_\theta^*$ is a particular case of the general case where we have a collection of linearly independent functions

$$\mathcal{F} = \left\{ f_{j,\beta} := \frac{(\phi_{\nu_j} P_\beta) \circ L_j}{\sum_{j=1}^J \phi_{\nu_j} \circ L_j} \cdot \mathbf{1}_{B_0} : j = 1, 2, \ldots, J, |\beta| < k \right\},$$

where $B_0 \subset B^*$ is a ball with $|B_0| \geq c_1 > 0$, the indices

$$1 \leq \nu_1 < \nu_2 < \cdots < \nu_J \leq N_1$$

are fixed, ϕ_ν are from (3.11), and P_β are as in (3.15) with the normalization from (3.16), i. e., $\|P_\beta\|_2 = 1$, which implies $\|\phi_\nu P_\beta\|_\infty \sim 1$. We also assume that L_j, $j = 1, 2, \ldots, J$, are affine transforms of the form $L_j(x) = M_j x + v_j$ satisfying the following conditions:

(i) $M_j = U_j D_j V_j$, where U_j and V_j are orthogonal $n \times n$ matrices,

$$D_j = \operatorname{diag}(\tau_1^j, \tau_2^j, \ldots, \tau_n^j) \quad \text{with } 0 < c_2 \leq \min_\ell \tau_\ell^j \leq \max_\ell \tau_\ell^j \leq c_3, \quad \text{and} \quad |v_j| \leq c_4;$$

(ii) $0 < c_5 \leq \sum_{j=1}^J (\phi_{\nu_j} \circ L_j)(x) \leq c_6$ for $x \in B^*$;

(iii) $L_j(B_0) \subset B^*$.

Let $\Lambda := \{\lambda := (j, \beta) : j = 1, 2, \ldots, J, |\beta| < k\}$. Since F is linearly independent, the Gram matrix $G := (\langle f_\lambda, f_{\lambda'} \rangle)_{\lambda, \lambda' \in \Lambda}$ is nonsingular, and hence $G^{-1} =: (R_{\lambda,\lambda'})_{\lambda,\lambda' \in \Lambda}$ exists.

Each of the affine transforms L_j depends on parameters from a subset, say, K of the set $\mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$. The set of all orthogonal $n \times n$ matrices is a compact subset of $\mathbb{R}^{n\times n}$. Hence the parameters of all affine transforms L_j satisfying condition (i) belong to a compact subset, say, K_1 of $\mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$. On the other hand, condition (iii) on L_j can be expressed in the form

$$\max_{|x - x_0| \leq a} |M_j x + v_j| \leq 1,$$

where x_0 and a ($a \sim 1$) are the center and radius of B_0. Therefore conditions (ii) and (iii) define K as a closed subset of the compact K_1, and hence K is compact.

The entries of G and $\det(G)$ apparently depend continuously on the parameters of the affine transforms L_j, $j = 1, 2, \ldots, J$, and since K is compact,

$$|\langle f_\lambda, f_{\lambda'} \rangle| \leq c_7, \quad \forall \lambda, \lambda' \in \Lambda, \quad \text{and} \quad 0 < c_8 \leq \det G \leq c_9.$$

From this it follows that

$$|R_{\lambda,\lambda'}| \leq c_{10} \quad \forall \lambda, \lambda' \in \Lambda, \tag{3.29}$$

where c_{10} as well as c_7, c_8, and c_9 depend only on $c_1, \ldots, c_6$, $\mathbf{p}(\Theta)$, L, and r. Finally, using that there are only finitely many possibilities for the indices $1 \leq \nu_1 < \nu_2 < \cdots < \nu_J \leq$

N_1, we conclude that the constant c_{10} in estimate (3.29) can be selected independently of these indices.

Applying the above claim to the specific case at hand, it follows that estimate (3.28) holds. Then (3.24) follows by (3.27) and (3.28).

The stability properties (3.26) follow by a standard argument from the properties (3.24) of the dual, as we now show for $0 < p < \infty$ (the case $p = \infty$ is similar and simpler). Since for each $\theta \in \Theta_m$, by property (c) of discrete covers we have that $\#\mathcal{X}_\theta \le N_1$, $\forall \theta \in \Theta_m$, we get

$$\begin{aligned}
\|f\|_p^p &\le \sum_{\theta \in \Theta_m} \|f\|_{L_p(\theta)}^p \\
&\le C \sum_{\theta \in \Theta_m} \sum_{\theta' \in \mathcal{X}_\theta, |\beta| < r} \|c_{\theta',\beta} \varphi_{\theta',\beta}\|_{L_p(\theta')}^p \\
&\le C \sum_{\theta' \in \Theta_m, |\beta| < r} \|c_{\theta',\beta} \varphi_{\theta',\beta}\|_{L_p(\theta')}^p \\
&= C \sum_{\lambda \in \Lambda_m} \|c_\lambda \varphi_\lambda\|_p^p.
\end{aligned}$$

In the other direction, for $1 \le p < \infty$, using (3.24), Hölder inequality, and then (3.25) gives

$$\begin{aligned}
\|c_\lambda \varphi_\lambda\|_p &= \big\| \langle f, \tilde{g}_\lambda \rangle \varphi_\lambda \big\|_p \\
&\le \|f\|_{L_p(\theta_\lambda)} \|\tilde{g}_\lambda\|_{p'} \|\varphi_\lambda\|_p \\
&\le C \|f\|_{L_p(\theta_\lambda)}.
\end{aligned}$$

Combining this with property (c) of discrete covers yields

$$\begin{aligned}
\sum_{\lambda \in \Lambda_m} \|c_\lambda \varphi_\lambda\|_p^p &\le C \sum_{\lambda \in \Lambda_m} \|f\|_{L_p(\theta_\lambda)}^p \\
&\le C \|f\|_p^p.
\end{aligned}$$

□

3.2.3 Local projectors onto polynomials

The anisotropic regularity notions we are aiming at rely on appropriate operators, which map L_p^{loc} into $\mathcal{S}_m$, locally preserve Π_{r-1} and hence provide good local approximation. The form of the operators will differ somewhat for $p \ge 1$ and $p < 1$.

(a) *Case* $1 \le p \le \infty$. There are in fact a number of ways to construct suitable operators. A first obvious idea is using the bases $\{\Phi_m\}$ and their duals $\tilde{G}_m := \{\tilde{g}_\lambda : \lambda \in \Lambda_m\}$ from Theorem 3.6 to introduce projectors mapping L_p^{loc} onto the spaces $\mathcal{S}_m$,

$$Q_m f := \sum_{\lambda \in \Lambda_m} \langle f, \tilde{g}_\lambda \rangle \varphi_\lambda. \tag{3.30}$$

Alternatively, simpler local projectors onto polynomials are obtained as follows. Recall that for $\theta \in \Theta$, $\{P_{\theta,\beta}\}_{|\beta|<r}$ defined by (3.17) is an orthonormal basis for Π_{r-1} in $L_2(\theta)$. Using again our compact notation from (3.19), we define

$$P_m f := \sum_{\lambda \in \Lambda_m} \langle f, P_\lambda \rangle \varphi_\lambda. \tag{3.31}$$

Evidently, P_m is a linear operator that maps L_p^{loc} into $\mathcal{S}_m$ and preserves locally all polynomials from Π_{r-1}. To be more specific, setting

$$\theta^* := \cup\{\theta' \in \Theta_m : \theta \cap \theta' \neq \emptyset\} \quad \text{for } \theta \in \Theta_m, \tag{3.32}$$

we easily to see that if $f|_{\theta^*} = P|_{\theta^*}$ with $P \in \Pi_{r-1}$, then $P_m f|_\theta = P|_\theta$.

In Section 3.3, we construct yet another dual system for $\{\varphi_\lambda\}_{\lambda \in \Lambda_m}$ that leads to different projectors and allows the construction of high-order “approximation to the identity” kernel operators.

(b) *Case* $0 < p < 1$. Apparently, the above operators are no longer usable when working in L_p with $p < 1$. Hence we need to modify them. In fact, the following construction covers the full range $0 < p \le \infty$. For $0 < p \le \infty$ and a given ellipsoid $\theta \in \Theta$, we let $P_{\theta,p} : L_p(\theta) \to \Pi_{r-1}$ be a projector such that

$$\|f - P_{\theta,p} f\|_{L_p(\theta)} \le C(n, r, p)\omega_r(f, \theta)_p, \quad f \in L_p(\theta), \tag{3.33}$$

where $\omega_r(f, \theta)_p$ is the modulus of smoothness of f over θ defined in (1.13). Note that (3.33) is a consequence of Whitney’s theorem 1.34 and $P_{\theta,p} f$ can simply be defined as the best (or near best) approximation to f from Π_{r-1} in $L_p(\theta)$. Furthermore, by Corollary 1.36, for $1 \le p < \infty$, there exist linear projectors that realize (3.33).

We now define the operator $P_{m,p} : L_p^{\text{loc}} \to \mathcal{S}_m$ by

$$P_{m,p} f := \sum_{\theta \in \Theta_m} P_{\theta,p} f \varphi_\theta. \tag{3.34}$$

Since $P_{m,p} f \in \mathcal{S}_m$, it can be represented in terms of the basis functions $\{\varphi_\lambda\}_{\lambda \in \Lambda_m}$ as

$$P_{m,p} f := \sum_{\lambda \in \Lambda_m} b_\lambda(f) \varphi_\lambda, \tag{3.35}$$

where $b_\lambda(f) := \langle P_{m,p} f, \tilde{g}_\lambda \rangle$ depends nonlinearly on f if $p < 1$.

In summary, any $T_m \in \{Q_m, P_m, P_{m,p}\}$ defined by (3.30), (3.31), or (3.35) has the representation

$$T_m f = \sum_{\lambda \in \Lambda_m} b_\lambda(f) \varphi_\lambda, \quad \text{where } b_\lambda(f) = \begin{cases} \langle f, \tilde{g}_\lambda \rangle & \text{if } T_m = Q_m, \\ \langle f, P_\lambda \rangle & \text{if } T_m = P_m, \\ \langle T_{m,p} f, \tilde{g}_\lambda \rangle & \text{if } T_m = P_{m,p}. \end{cases} \tag{3.36}$$

Theorem 3.7. *Let T_m be the operator Q_m from* (3.30) *or P_m from* (3.31) *or $P_{m,p}$ from* (3.34) *if* $1 \le p \le \infty$, *and let* $T_m := P_{m,p}$ *if* $0 < p < 1$. *Then for $f \in L_p^{\mathrm{loc}}$ and $\theta \in \Theta_m$ ($m \in \mathbb{Z}$),*

$$\|T_m f\|_{L_p(\theta)} \le c\|f\|_{L_p(\theta^*)}, \tag{3.37}$$

where θ^* is from (3.32), *and*

$$\|f - T_m f\|_{L_p(\theta)} \le c \sum_{\theta' \in \Theta_m:\, \theta' \cap \theta \ne \emptyset} \omega_r(f, \theta')_p. \tag{3.38}$$

Furthermore, if $f \in L_p^{\mathrm{loc}}$, then

$$\|f - T_m f\|_{L_p(K)} \to 0 \quad \text{as } m \to \infty \text{ for any bounded } K \subset \mathbb{R}^n, \tag{3.39}$$

and if $f \in L_p$ ($L_\infty := C_0$), then

$$\|f - T_m f\|_p \to 0 \quad \text{as } m \to \infty. \tag{3.40}$$

Proof. We first prove (3.37) in the case $T_m = Q_m$ and $1 \le p \le \infty$ (the proof in the other cases is similar). By (3.30), (3.25), and property (c) of discrete covers it follows that

$$\begin{aligned}
\|Q_m f\|_{L_p(\theta)} &\le \sum_{\lambda \in \Lambda_m: \theta_\lambda \cap \theta \ne \emptyset} \big|\langle f, \tilde{g}_\lambda\rangle\big| \|\varphi_\lambda\|_p \\
&\le \sum_{\lambda \in \Lambda_m: \theta_\lambda \cap \theta \ne \emptyset} \|f\|_{L_p(\theta_\lambda)} \|\tilde{g}_\lambda\|_{p'} \|\varphi_\lambda\|_p \\
&\le C \sum_{\lambda \in \Lambda_m: \theta_\lambda \cap \theta \ne \emptyset} \|f\|_{L_p(\theta_\lambda)} \\
&\le C\|f\|_{L_p(\theta^*)} \quad (1/p + 1/p' = 1),
\end{aligned}$$

as claimed.

To prove (3.38), we first show that for $0 < p \le \infty$ and any $\theta \in \Theta_m$, there exists $P_\theta \in \Pi_{r-1}$ such that

$$E_{r-1}(f, \theta^*)_p \le \|f - P_\theta\|_{L_p(\theta^*)} \le C \sum_{\theta' \in \mathcal{X}_\theta} \omega_r(f, \theta')_p, \tag{3.41}$$

where θ^* is defined in (3.32), and $\mathcal{X}_\theta := \{\theta' \in \Theta_m : \theta' \cap \theta \ne \emptyset\}$. Indeed, by Whitney's theorem 1.34 for convex sets, $E_{r-1}(f, \theta)_p \le C\omega_r(f, \theta)_p$ for any ellipsoid θ. For any $\theta' \in \mathcal{X}_\theta$, let $P_{\theta'} \in \Pi_{r-1}$ be such that $\|f - P_{\theta'}\|_{L_p(\theta')} \le 2E_{r-1}(f, \theta')_p$. By condition (e) of discrete covers, for any $\theta' \in \mathcal{X}_\theta$, $|\theta'| \le a_8|\theta \cap \theta'|$. We combine this with an application of Lemma 1.23 to get

$$\begin{aligned}
\|P_{\theta'} - P_\theta\|_{L_p(\theta')} &\le C\|P_{\theta'} - P_\theta\|_{L_p(\theta' \cap \theta)} \\
&\le C\|f - P_{\theta'}\|_{L_p(\theta' \cap \theta)} + C\|f - P_\theta\|_{L_p(\theta' \cap \theta)}
\end{aligned}$$

$$\leq C\|f - P_{\theta'}\|_{L_p(\theta')} + C\|f - P_\theta\|_{L_p(\theta)}$$
$$\leq C\omega_r(f,\theta')_p + C\omega_r(f,\theta)_p.$$

By condition (c) on discrete covers we know that $\#\mathcal{X}_\theta \leq N_1$. From this and the preceding estimate we conclude that

$$\begin{aligned}\|f - P_\theta\|_{L_p(\theta^*)} &\leq C \sum_{\theta' \in \mathcal{X}_\theta} \|f - P_\theta\|_{L_p(\theta')} \\ &\leq C \sum_{\theta' \in \mathcal{X}_\theta} \|P_{\theta'} - P_\theta\|_{L_p(\theta')} + C \sum_{\theta' \in \mathcal{X}_\theta} \|f - P_{\theta'}\|_{L_p(\theta')} \\ &\leq C \sum_{\theta' \in \mathcal{X}_\theta} \omega_r(f,\theta')_p,\end{aligned}$$

which yields (3.41). Then, using that $T_m(P_\theta) = P_\theta$ and (3.37), we get

$$\begin{aligned}\|f - T_m f\|_{L_p(\theta)} &\leq C\|f - P_\theta\|_{L_p(\theta)} + C\|T_m(P_\theta - f)\|_{L_p(\theta)} \\ &\leq C\|f - P_\theta\|_{L_p(\theta^*)} \\ &\leq C \sum_{\theta' \in \mathcal{X}_\theta} \omega_r(f,\theta')_p,\end{aligned}$$

and (3.38) follows.

By (2.27) for any bounded $K \subset \mathbb{R}^n$,

$$\max\{\operatorname{diam}\theta : \theta \in \Theta_m, \theta \cap K \neq \emptyset\} \to 0 \quad \text{as } m \to \infty.$$

This and (3.38) readily imply (3.39), which leads to (3.40). □

3.3 Construction of the anisotropic multiresolution kernels

To construct kernels $\{S_m\}_{m\in\mathbb{Z}}$ that satisfy properties (3.2)–(3.6), for any given $\delta > 0$ and $r \geq 1$, we need to construct yet another dual basis to Φ_m. Let G_m be the Gram matrix given by

$$G_m := [A_{\lambda,\lambda'}]_{\lambda,\lambda' \in \Lambda_m}, \quad A_{\lambda,\lambda'} := \langle \varphi_\lambda, \varphi_{\lambda'} \rangle.$$

By (3.26), since $\|\varphi_\lambda\|_2 \sim 1$ for all $\lambda \in \Lambda_m$, for any sequence $\alpha \in l_2(\Lambda_m)$, we have

$$c_1\|\alpha\|_{l_2}^2 \leq \langle G_m\alpha, \alpha\rangle = \Big\|\sum_{\lambda\in\Lambda_m} \alpha_\lambda\varphi_\lambda\Big\|_2^2 \leq c_2\|\alpha\|_{l_2}^2,$$

where the constants $c_1, c_2 > 0$ do not depend on α or m. Thus the operator $G_m : l_2 \to l_2$ with matrix G_m is symmetric and positive, and $c_1 I \le G_m \le c_2 I$. Therefore G_m^{-1} exists, and $c_2^{-1} I \le G_m \le c_1^{-1} I$. Denote by $G_m^{-1} := [B_{\lambda,\lambda'}]_{\lambda,\lambda' \in \Lambda_m}$ the matrix of the operator G_m^{-1}.

We now introduce a *graph-distance* $\tilde{d}_m(\cdot,\cdot)$ on Λ_m. To this end, we first define the graph-distance $d_m(\theta, \theta')$ between any $\theta, \theta' \in \Theta_m$ as the length of the shortest *chain* connecting θ and θ'. A chain is a list of ellipsoids in Θ_m where each consecutive ellipsoids have a nonempty intersection and its length is the number of elements − 1. Evidently, d_m is a distance on Θ_m. Let us order in a sequence, indexed by $0, 1, \ldots$, the multiindices $\beta \in \mathbb{N}^n$ in such a way that if $N(\beta)$ denotes the index of β, then $N(\beta) < N(\beta')$ for $|\beta| < |\beta'|$. Denote also $N_{\max} := \max_{|\beta|<r} N(\beta) + 1$. After this preparation, we define the graph distance $\tilde{d}_m(\lambda, \lambda')$ between any $\lambda, \lambda' \in \Lambda_m$ by

$$\tilde{d}_m(\lambda, \lambda') := N_{\max} d_m(\theta_\lambda, \theta_{\lambda'}) + |N(\beta_\lambda) - N(\beta_{\lambda'})|.$$

We readily see that $\tilde{d}_m(\cdot,\cdot)$ is a true distance on Λ_m, which is dominated by the graph distance between the ellipsoids. Applying a generalization, given in [53], of a well-known result of Demko on the inverses of band matrices, we arrive at the following result,

Lemma 3.8. *There exist constants* $0 < q < 1$ *and* $c > 0$, *depending only on* $p(\Theta)$, r, *and our choice of* $\{\phi_\nu\}_{\nu=1,\ldots,N_1}$, *such that the following estimates hold for the entries of* G_m^{-1}, $m \in \mathbb{Z}$:

$$|B_{\lambda,\lambda'}| \le c q^{\tilde{d}_m(\lambda,\lambda')} \le c q^{d_m(\theta_\lambda, \theta_{\lambda'})}, \quad \forall \lambda, \lambda' \in \Lambda_m. \tag{3.42}$$

Further, we need an estimate of the entries $B_{\lambda,\lambda'}$ using the quasi-distance.

Lemma 3.9. *There exist constants* $0 < q_*, \alpha < 1$ *and* $c > 0$, *depending only on* $p(\Theta)$ *and* r, *such that for all entries* $B_{\lambda,\lambda'}$, $\lambda, \lambda' \in \Lambda_m$, *and points* $x \in \theta_\lambda$ *and* $y \in \theta_{\lambda'}$,

$$|B_{\lambda,\lambda'}| \le c q_*^{(2^m \rho(x,y))^\alpha}. \tag{3.43}$$

Proof. Let $\lambda, \lambda' \in \Lambda_m$. There exists a connected chain of ellipsoids in Θ_m of length $d_m(\theta_\lambda, \theta_{\lambda'})$ that starts at θ_λ and ends in $\theta_{\lambda'}$. By Lemma 2.18 there exists a fixed constant $\gamma(\mathbf{p}(\Theta)) \ge 1$ such that there exists a connected chain of ellipsoids in $\Theta_{m-\gamma}$ of length $\lceil d_m(\theta_\lambda, \theta_{\lambda'})/2 \rceil$ whose first element contains θ_λ and the last $\theta_{\lambda'}$. After at most $L := 2\log_2(d_m(\theta_\lambda, \theta_{\lambda'}))$ such iterations, we obtain an ellipsoid $\eta \in \Theta_{m-L\gamma}$ such that $\theta_\lambda, \theta_{\lambda'} \subset \eta$, and therefore

$$\rho(x, y) \le |\eta| \le a_2 2^{-(m-L\gamma)} = a_2 2^{-m} d_m(\theta_\lambda, \theta_{\lambda'})^{2\gamma}. \tag{3.44}$$

Denoting $q_* := q^{a_2^{-1/2\gamma}}$, where q is defined by (3.42), and $\alpha := 1/2\gamma$, we conclude that (3.43) holds by combining (3.42) and (3.44):

$$|B_{\lambda,\lambda'}| \le cq^{d_m(\theta_\lambda,\theta_{\lambda'})} \le cq^{(a_2^{-1}2^m\rho(x,y))^{1/2\gamma}} = cq_*^{(2^m\rho(x,y))^\alpha}. \qquad \square$$

Definition 3.10. We define the *dual basis* $\tilde{\Phi}_m := \{\tilde{\varphi}_\lambda\}_{\lambda\in\Lambda_m}$ by

$$\tilde{\varphi}_\lambda := \sum_{\lambda'\in\Lambda_m} B_{\lambda,\lambda'}\varphi_{\lambda'}, \quad \lambda \in \Lambda_m, \tag{3.45}$$

and the multiresolution kernel operators $\{S_m\}_{m\in\mathbb{Z}}$ by

$$S_m(x,y) := \sum_{\lambda\in\Lambda_m} \varphi_\lambda(x)\tilde{\varphi}_\lambda(y). \tag{3.46}$$

For $\lambda \in \Lambda_m$, let x_0 be any point in θ_λ. Combining (3.43) and (3.45), we see that

$$\left|\tilde{\varphi}_\lambda(x)\right| \le C2^{-m/2} \sum_{x\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| \le C2^{-m/2} q_*^{(2^m\rho(x,x_0))^\alpha}. \tag{3.47}$$

Therefore each $\tilde{\varphi}_\lambda$ has fast decay with respect to the quasi-distance induced by Θ, and thus, by Theorem 2.26 it also has fast decay with respect to the Euclidean distance. In fact, if $\{\phi_\nu\}_{\nu=1,\dots,2N_1}$ are constructed as C^∞ bumps (see Remark 3.4), then $\tilde{\varphi}_\lambda$ is in the Schwartz class $\mathcal{S}$ (we omit the proof). Also,

$$\langle\varphi_\lambda, \tilde{\varphi}_{\lambda'}\rangle = \sum_{\lambda''\in\Lambda_m} B_{\lambda',\lambda''}\langle\varphi_\lambda, \varphi_{\lambda''}\rangle = (G_m^{-1}G_m)_{\lambda',\lambda} = \delta_{\lambda,\lambda'}.$$

Our next step is showing that $\{S_m\}_{m\in\mathbb{Z}}$ form a high-order multiresolution analysis (see Definition 3.2). As we will see, the parameters $\tau = (\tau_0, \tau_1)$ depend on the parameters of the cover. We begin with the following lemmas.

Lemma 3.11. *For any $f \in C^r(\mathbb{R}^n)$, we have the following commutativity of Taylor polynomials of degree $k-1$, $k \le r$, and affine transformations A:*

$$T_x^k(f\circ A, z) = T_{A(x)}^k(f, A(z)), \quad \forall x, z \in \mathbb{R}^n.$$

Therefore we have

$$R_x^k(f\circ A, z) = R_{A(x)}^k(f, A(z)). \tag{3.48}$$

Proof. Let $Ax = Mx + b$ with $M = \{a_{i,j}\}_{1\le i,j\le n}$. Since $Az - Ax = Mz - Mx$, we have

$$\begin{aligned} T_x^k(f\circ A, z) &= \sum_{|\alpha|<k} \frac{\partial^\alpha [f\circ A](x)}{\alpha!}(z-x)^\alpha \\ &= \sum_{|\alpha|<k} \frac{\partial^\alpha f(Ax)}{\alpha!} \prod_{j=1}^n \left(\sum_{i=1}^n a_{i,j}\right)^{\alpha_j} \prod_{j=1}^n (z_j - x_j)^{\alpha_j} \\ &= \sum_{|\alpha|<k} \frac{\partial^\alpha f(Ax)}{\alpha!} \prod_{j=1}^n \left(\sum_{i=1}^n a_{i,j}(z_j - x_j)\right)^{\alpha_j} \\ &= \sum_{|\alpha|<k} \frac{\partial^\alpha f(Ax)}{\alpha!}(Az - Ax)^\alpha. \end{aligned}$$

□

Lemma 3.12. *Let Θ be a discrete ellipsoid cover of $\mathbb{R}^n$, denote $\tau := (a_6, a_4)$, and let $1 \le k \le r$. For any $\lambda \in \Lambda_m$ and $x, z \in \mathbb{R}^n$, we have*

$$\left|R_x^k(\varphi_\lambda, z)\right| \le c2^{m/2}\left(2^m \rho(x,z)\right)^{\tau(x,z,2^{-m})k}, \tag{3.49}$$

where $\tau(\cdot,\cdot,\cdot)$ is defined in (2.38), and $R_x^k(f,z)$ is the Taylor remainder of order k about the point x and at the point z. The constant depends on the parameters of the cover, n, and the choice of r and $\{\phi_\nu\}_{\nu=1,\dots,N_1}$ (in the construction of the bases in 3.2.2).

Proof. Assume first that $\lambda = (\theta, \beta)$, where $\theta \in \Theta_0$ and $\theta = B^*$ (recall B^* denotes the Euclidean unit ball in $\mathbb{R}^n$). Evidently, in this particular case, $|\varphi_\lambda|_{W_\infty^k} \le c^*$ with c^* depending on the aforementioned parameters, where $|\cdot|_{W_\infty^k}$ is the Sobolev seminorm defined in (1.7). By definition there exists an ellipsoid $\tilde\theta \in \Theta_j$ for some $j \in \mathbb{Z}$ such that $\rho(x,z) = |\tilde\theta|$. Since we may assume that either x or z is in B^* (otherwise, $R_x^k(\varphi_\lambda, z) = 0$, and (3.49) is obvious), we get that $\tilde\theta \cap B^* \ne \emptyset$. We may consider two cases.

Case 1: $j \ge 0$. Since $\tilde\theta \cap B^* \ne \emptyset$, then by (2.23) we have

$$|x - z| \le \operatorname{diam}(\tilde\theta) \le a_5 2^{-a_6 j}.$$

Also, since $\tilde\theta \in \Theta_j$, by (2.17) we have that $|\tilde\theta| \ge a_1 2^{-j}$. Combining these last two estimates yields

$$\begin{aligned} \left|R_x^k(\varphi_{B^*,\beta}, z)\right| &\le C|\varphi_{B^*,\beta}|_{W_\infty^k}|x-z|^k \\ &\le C2^{-a_6 jk} \\ &\le C|\tilde\theta|^{a_6 k} \\ &\le C\rho(x,z)^{a_6 k}. \end{aligned}$$

Case 2: $j < 0$. Since $\tilde{\theta} \cap B^* \neq \emptyset$, by (2.23) we have $|x - z| \leq \operatorname{diam}(\tilde{\theta}) \leq C2^{-a_4 j}$. Similarly as above, we arrive at

$$\left|R_x^k(\varphi_{B^*,\beta}, z)\right| \leq C\rho(x,z)^{a_4 k}.$$

These last two estimates prove (3.49) for the case $\theta_\lambda \in \Theta_0$ and $\theta_\lambda = B^*$. We now consider the case where both the ellipsoid and the cover are arbitrary. Let $\theta \in \Theta_m$ and A_θ be the affine transformation such that $\theta = A_\theta(B^*)$. Evidently, $\Theta^* := \{A^{-1}(\eta)\}_{\eta\in\Theta}$ is an ellipsoid cover of $\mathbb{R}^n$ with the same parameters a_3, a_4, a_5, a_6 as Θ. Denote by $\rho^*(\cdot,\cdot)$ the quasi-distance induced by Θ^*. It is easy to see that

$$\rho^*(A^{-1}(x), A^{-1}(z)) = |\theta|^{-1}\rho(x,z). \tag{3.50}$$

Denote $\varphi_{B^*,\beta} := \varphi_{B^*} P_\beta$ (this is a particular case of (3.19) for the unit ball). Notice that $\varphi_{\theta,\beta} = |\theta|^{-1/2}\varphi_{B^*,\beta} \circ A_\theta^{-1}$. We use (3.48), then (3.49) for the particular case of $A^{-1}(\theta) = B^* \in \Theta^*$, and finally (3.50) to obtain

$$\begin{aligned}\left|R_x^k(\varphi_{\theta,\beta}, z)\right| &= |\theta|^{-1/2}\left|R^k_{A_\theta^{-1}(x)}(\varphi_{B^*,\beta}, A_\theta^{-1}(z))\right| \\ &\leq C|\theta|^{-1/2}\rho^*\left(A_\theta^{-1}(x), A_\theta^{-1}(z)\right)^{\tau(A_\theta^{-1}(x), A_\theta^{-1}(z),1)k} \\ &= C|\theta|^{-1/2}\left(|\theta|^{-1}\rho(x,z)\right)^{\tau(x,z,2^{-m})k}.\end{aligned}$$

The proof of the lemma is complete. □

Theorem 3.13. *Suppose Θ is a discrete ellipsoid cover of $\mathbb{R}^n$, denote $\tau := (a_6, a_4)$, and let S_m, $m \in \mathbb{Z}$, be defined as in* (3.46). *Then there exist $0 < q_*, \alpha < 1$ and $c > 0$ such that for any $k \leq r$, $x, x', y, y', z \in \mathbb{R}^n$,*

$$\left|S_m(x,y)\right| \leq c2^m q_*^{(2^m\rho(x,y))^\alpha}, \tag{3.51}$$

$$\left|R_x^k(S_m(\cdot,y),z)\right| \leq c2^m\left(2^m\rho(x,z)\right)^{\tau(x,z,2^{-m})k}\left(q_*^{(2^m\rho(x,y))^\alpha} + q_*^{(2^m\rho(y,z))^\alpha}\right), \tag{3.52}$$

$$\left|R_y^k(S_m(x,\cdot),z)\right| \leq c2^m\left(2^m\rho(y,z)\right)^{\tau(x,z,2^{-m})k}\left(q_*^{(2^m\rho(x,y))^\alpha} + q_*^{(2^m\rho(x,z))^\alpha}\right), \tag{3.53}$$

$$\begin{aligned}\left|R_y^k R_x^k[S_m(\cdot,\cdot)](x',y')\right| &= \left|R_x^k R_y^k[S_m(\cdot,\cdot)](x',y')\right| \\ &\leq c2^m\left(2^m\rho(x,x')\right)^{\tau(x,x',2^{-m})k}\left(2^m\rho(y,y')\right)^{\tau(y,y',2^{-m})k} \\ &\quad\times\left(q_*^{(2^m\rho(x,y))^\alpha} + q_*^{(2^m\rho(x,y'))^\alpha} + q_*^{(2^m\rho(x',y))^\alpha} + q_*^{(2^m\rho(x',y'))^\alpha}\right).\end{aligned} \tag{3.54}$$

Proof. By (3.45) and (3.46) the kernel $S_m(x,y)$ has a representation

$$S_m(x,y) = \sum_{\lambda,\lambda'\in\Lambda_m} B_{\lambda,\lambda'}\varphi_\lambda(x)\varphi_{\lambda'}(y). \tag{3.55}$$

We now use $\|\varphi_\lambda\|_\infty \sim 2^{m/2}$ for all $\lambda \in \Lambda_m$, the fact that the points x and y are contained in a bounded number of ellipsoids $\theta \in \Theta_m$, and (3.43) to obtain (3.51):

$$|S_m(x,y)| \le \sum_{x\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}||\varphi_\lambda(x)||\varphi_{\lambda'}(y)| \le C2^m q_*^{(2^m\rho(x,y))^\alpha}.$$

For the proof of (3.52), we use the same tools and further apply (3.49):

$$\begin{aligned}|R_x^k(S_m(\cdot,y))(z)| &\le \sum_{x\in\theta_\lambda \vee z\in\theta_\lambda} \sum_{y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}||R_x^k(\varphi_\lambda, z)||\varphi_{\lambda'}(y)| \\ &\le C2^m(2^m\rho(x,z))^{\tau(x,z,2^{-m})}\Big(\sum_{x\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{z\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}|\Big) \\ &\le C2^m(2^m\rho(x,z))^{\tau(x,z,2^{-m})}\big(q_*^{(2^m\rho(x,y))^\alpha} + q_*^{(2^m\rho(y,z))^\alpha}\big).\end{aligned}$$

The proof of (3.53) is similar. Finally, we prove (3.54) using the same technique:

$$\begin{aligned}&|R_y^k(R_x^k(S_m(\cdot,\cdot))(x'))(y')| \\ &\le \sum_{x\in\theta_\lambda \vee x'\in\theta_\lambda} \sum_{y\in\theta_{\lambda'} \vee y'\in\theta_{\lambda'}} |B_{\lambda,\lambda'}||R_x^k(\varphi_\lambda, x')||R_y^k(\varphi_{\lambda'}, y')| \\ &\le C2^m(2^m\rho(x,x'))^{\tau(x,x',2^{-m})}(2^m\rho(y,y'))^{\tau(y,y',2^{-m})} \\ &\quad\times\Big(\sum_{x\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{x\in\theta_\lambda, y'\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{x'\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{x'\in\theta_\lambda, y'\in\theta_{\lambda'}} |B_{\lambda,\lambda'}|\Big) \\ &\le C2^m(2^m\rho(x,x'))^{\tau(x,x',2^{-m})}(2^m\rho(y,y'))^{\tau(y,y',2^{-m})} \\ &\quad\times\big(q_*^{(2^m\rho(x,y))^\alpha} + q_*^{(2^m\rho(x,y'))^\alpha} + q_*^{(2^m\rho(x',y))^\alpha} + q_*^{(2^m\rho(x',y'))^\alpha}\big).\end{aligned}$$ □

We can now prove that our construction is indeed a high-order multiresolution.

Corollary 3.14. *For a discrete ellipsoid cover* Θ, *the kernels* $\{S_m\}_{m\in\mathbb{Z}}$ *defined by* (3.46) *are a multiresolution of order* (τ,δ,r) *with respect to the quasi-distance* (2.35) *induced by the cover. The vector* τ *can be taken as* $\tau = (a_6, a_4)$, *the parameter* δ *can be any positive scalar, and the parameter r is the total order of the polynomials used in the construction of the local ellipsoid "bumps" in Section* 3.2.2.

Proof. For any $\tilde\delta > 0$, denote $\tilde q := q_*^{1/\tilde\delta}$, where q_* is given by (3.43). Evidently, for any $0 < \tilde q, \alpha < 1$, there exists a constant $c_1(\tilde q,\alpha) > 0$ such that $\tilde q^{t^\alpha} \le c_1(1+t)^{-1}$ for all $t \ge 0$. Therefore, for all $m \in \mathbb{Z}$, $x, y \in \mathbb{R}^n$, we have

$$q_*^{(2^m\rho(x,y))^\alpha} = \tilde q^{(2^m\rho(x,y))^\alpha\tilde\delta} \le c_1^{\tilde\delta}\left(\frac{1}{1+2^m\rho(x,y)}\right)^{\tilde\delta} = c\frac{2^{-m\tilde\delta}}{(2^{-m}+\rho(x,y))^{\tilde\delta}}. \tag{3.56}$$

Thus, for any $\delta > 0$, setting $\tilde{\delta} = 1 + \delta$ in (3.56), from (3.51) we get

$$\begin{aligned} |S_m(x,y)| &\le C2^m q_*^{(2^m\rho(x,y))^\alpha} \\ &\le C2^m \frac{2^{-m(1+\delta)}}{(2^{-m}+\rho(x,y))^{1+\delta}} \\ &= C\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta}}, \end{aligned}$$

which is property (3.2) in Definition 3.2. Properties (3.3) and (3.4) are proved similarly by applying (3.52) and (3.53) for $1 \le k \le r$ and setting $\tilde{\delta} = 1 + \delta + \tau_1 k$:

$$\begin{aligned} &|R_x^k(S_m(\cdot,y),z)| \\ &\quad\le C2^m(2^m\rho(x,z))^{\tau(x,z,2^{-m})k}\big(q_*^{(2^m\rho(x,y))^\alpha} + q_*^{(2^m\rho(y,z))^\alpha}\big) \\ &\quad\le C2^m(2^m\rho(x,z))^{\tau(x,z,2^{-m})k} \\ &\qquad\times\left(\left(\frac{2^{-m}}{2^{-m}+\rho(x,y)}\right)^{1+\delta+\tau(x,z,2^{-m})k} + \left(\frac{2^{-m}}{2^{-m}+\rho(y,z)}\right)^{1+\delta+\tau(x,z,2^{-m})k}\right) \\ &\quad= C\rho(x,z)^{\tau(x,z,2^{-m})k}\left(\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau(x,z,2^{-m})k}} + \frac{2^{-m\delta}}{(2^{-m}+\rho(y,z))^{1+\delta+\tau(x,z,2^{-m})k}}\right). \end{aligned}$$

Property (3.5) is proved similarly. Finally, we prove the polynomial reproduction property (3.6). Since $\Pi_{r-1} \subset \mathcal{S}_m$ for all $m \in \mathbb{Z}$ by construction, for any $P \in \Pi_{r-1}$, there exist coefficients $\{c_\lambda\}_{\lambda\in\Lambda_m}$ such that $P = \sum_{\lambda\in\Lambda_m} c_\lambda\varphi_\lambda$. For fixed $y \in \mathbb{R}^n$, we use the fast decay of the kernel $S_m(\cdot,y)$ away from y to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} S_m(x,y)P(x)dx &= \int_{\mathbb{R}^n}\Big(\sum_{\lambda,\lambda'\in\Lambda_m} B_{\lambda,\lambda'}\varphi_\lambda(x)\varphi_{\lambda'}(y)\Big)\Big(\sum_{\lambda''\in\Lambda_m} c_{\lambda''}\varphi_{\lambda''}(x)\Big)dx \\ &= \sum_{\lambda,\lambda',\lambda''\in\Lambda_m} c_{\lambda''}B_{\lambda,\lambda'}\varphi_{\lambda'}(y)\int_{\mathbb{R}^n}\varphi_\lambda(x)\varphi_{\lambda''}(x)dx \\ &= \sum_{\lambda',\lambda''\in\Lambda_m} c_{\lambda''}\varphi_{\lambda'}(y)\sum_{\lambda\in\Lambda_m} B_{\lambda,\lambda'}A_{\lambda'',\lambda} \\ &= \sum_{\lambda',\lambda''\in\Lambda_m} c_{\lambda''}\varphi_{\lambda'}(y)\delta_{\lambda',\lambda''} \\ &= \sum_{\lambda''\in\Lambda_m} c_{\lambda''}\varphi_{\lambda''}(y) = P(y). \end{aligned}$$

The proof that $P(x) = \int_{\mathbb{R}^n} S_m(x,y)P(y)dy$ is similar. This concludes the proof of the corollary. □

4 Anisotropic wavelets and two-level splits

4.1 Wavelet decomposition of spaces of homogeneous type

In the isotropic setting, wavelets [24, 34] are bases of $L_2(\mathbb{R}^n)$ that are well localized with respect to the Euclidean metric in space and frequency. Wavelet constructions have many applications in harmonic analysis, approximation theory, function space theory, signal processing, and numerical methods for PDEs. The simplest example is the univariate Haar orthonormal basis, which is perfectly localized in space since it is compactly supported (and somewhat localized in frequency). It is defined through dilations and translations $\{\psi_{j,k}\}$, $\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k)$, $j, k \in \mathbb{Z}$, where $\psi : \mathbb{R} \to \mathbb{R}$ is the "mother" wavelet

$$\psi(x) := \begin{cases} 1, & 0 \le x < 1/2, \\ -1, & 1/2 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

However, most isotropic wavelet constructions, including the Haar, are in fact derived from an isotropic multiresolution analysis with properties of localization and polynomial reproduction as in the previous chapter. For example, the span of the Haar wavelets $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ can be regarded as "differences" between two consecutive scales V_j, V_{j+1} in the multiresolution $\{V_j\}_{j\in\mathbb{Z}}$, where V_j is the closure of the span of $\varphi_{j,k} := 2^{j/2}\varphi(2^j \cdot -k)$, $k \in \mathbb{Z}$, with $\varphi := \mathbf{1}_{[0,1]}$.

For anisotropic function spaces, where the anisotropy is constant over $\mathbb{R}^n$, it is also possible to construct wavelet-type bases that are generated from a single function and aligned with the anisotropy. As in the isotropic case, these bases allow us to characterize the corresponding anisotropic function spaces such as Besov or Triebel–Lizorkin [38, 63]. As we will see below, in the general case of pointwise variable anisotropy, it is possible to start the construction from a fixed set of smooth "bumps"; however, we need to adapt their dilation pointwise and scalewise, making the wavelet-type constructions more complex.

There is a remarkable construction by Auscher and Hytönen using the technique of randomized dyadic cubes [4] of a wavelet orthonormal basis of $L_2(X)$, where (X, ρ, μ) is a space of homogeneous type, which is well localized with respect to the quasi-metric ρ. Moreover, the wavelet basis $\{\psi_i\}_{i\in I}$ satisfies the following properties:

(i) Vanishing moment: $\int_X \psi_i d\mu = 0$ for all $i \in I$;
(ii) Lipschitz regularity $0 < \eta < 1$ with respect to ρ;
(iii) Exponential decay with respect to ρ.

The construction of compactly supported wavelets in this generality remains an open problem. We note that for the particular case $X = \mathbb{R}^n$, we can directly construct an

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orthonormal basis with similar properties, over an anisotropic nested multilevel triangulation mesh [52], which induces a quasi-distance in a similar way to the ellipsoid covers.

Since our ellipsoid covers induce a quasi-distance and in turn a space of homogeneous type, the construction of [4], combined with Theorem 2.23, implies that we can construct an orthonormal wavelet basis that has Lipschitz regularity and is well localized with respect to the ellipsoid cover. However, it is still an open problem if a higher-order well-localized anisotropic orthonormal basis $\{\psi_i\}_{i\in I}$ can be constructed. By higher-order we mean that for an arbitrarily high but fixed $r \geq 1$,

$$\int_{\mathbb{R}^n} P\psi_i = 0, \quad \forall P \in \Pi_{r-1},\ \forall i \in I.$$

Therefore, we follow [28] and focus our attention on frame constructions, in view of the fact that frames can be thought of as some kind of "generalized bases", since they satisfy the following "quasi-Parseval" type property.

Definition 4.1. A family of elements $\{f_i\}_{i\in I}$ contained in a Hilbert space $\mathcal{H}$ is a *frame* if there exist constants $0 < A \leq B < \infty$ such that for any $f \in \mathcal{H}$,

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in I} \left|\langle f, f_i\rangle_{\mathcal{H}}\right|^2 \leq B\|f\|_{\mathcal{H}}^2. \tag{4.1}$$

Littlewood and Paley initiated a fundamental branch of harmonic analysis in the 1930s, where the Fourier series is split into dyadic blocks $f = \sum_j \Delta_j(f)$, and then most functional spaces can be characterized by size estimates on $\Delta_j(f)$. David, Journe, and Semmes [25] used an idea of R. Coifman to generalize the Littlewood–Paley analysis to the general setting on spaces of homogeneous type. Useful wavelet representations of functions (e. g., $L_2(X)$) are constructed [33] based on the approximation of the identity of Definition 3.1.

Definition 4.2. For an approximation of the identity $\{S_m\}_{m\in\mathbb{Z}}$, we define the *wavelet* operators

$$D_m := S_{m+1} - S_m,$$

so that, formally, the identity operator $I = \lim_{m\to\infty} S_m$ is decomposed by $I = \sum_{m=-\infty}^{\infty} D_m$.

In the general setting of spaces of homogeneous type, the properties of $\{S_m\}_{m\in\mathbb{Z}}$ with constants $0 < \tau < \alpha$ and $\delta > 0$ imply that there exists a constant $c > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$\left|D_m(x,y)\right| \leq c\frac{2^{-m\delta}}{(2^{-m} + \rho(x,y))^{1+\delta}}, \tag{4.2}$$

$$|D_m(x,y) - D_m(x',y)| \le c\rho(x,x')^\tau \frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau}} \tag{4.3}$$

for $\rho(x,x') \le \frac{1}{2\kappa}(2^{-m}+\rho(x,y))$,

$$|D_m(x,y) - D_m(x',y)| \le c\rho(y,y')^\tau \frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\tau}} \tag{4.4}$$

for $\rho(y,y') \le \frac{1}{2\kappa}(2^{-m}+\rho(x,y))$,

$$\int_X D_m(x,y)d\mu(y) = 0, \quad \forall x \in X, \quad \int_X D_m(x,y)d\mu(x) = 0, \quad \forall y \in X. \tag{4.5}$$

These properties of the wavelet kernels mean that for fixed $x_0, y_0 \in \mathbb{R}^n$, the functions $D_m(x_0,\cdot)$, $D_m(\cdot,y_0)$ are anisotropic molecules in the following sense.

Definition 4.3. Fix a quasi-distance ρ on $\mathbb{R}^n$. A function $f \in C(\mathbb{R}^n)$ belongs to the *anisotropic test function space* $\mathcal{M}(\tau,\delta,x_0,t)$, $\tau,\delta > 0$, $x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}$, if there exists a constant $c > 0$ such that:

(i) For all $x \in \mathbb{R}^n$,

$$|f(x)| \le c\frac{2^{-t\delta}}{(2^{-t}+\rho(x,x_0))^{1+\delta}};$$

(ii) For all $x, y \in \mathbb{R}^n$ such that

$$\rho(x,y) \le \frac{1}{2\kappa}(2^{-t}+\rho(x,x_0))$$

with κ defined in (2.1),

$$|f(x)-f(y)| \le c\rho(x,y)^\tau \frac{2^{-t\delta}}{(2^{-t}+\rho(x,x_0))^{1+\delta+\tau}}.$$

The norm $\|f\|_{\mathcal{M}(\tau,\delta,x_0,t)}$ is the infimum over all such constants. An anisotropic test function f is said to be a *molecule* in $\mathcal{M}_0(\tau,\delta,x_0,t)$ if $\int_{\mathbb{R}^n} f = 0$.

In Sections 6.6 and 7.2, we show that the setting of ellipsoid covers allows us to consider the generalization to anisotropic test functions and molecules of higher order of regularity. Meanwhile, the minimal regularity considered in Definition 4.3 is sufficient to guarantee the following wavelet reproducing formula.

Proposition 4.4 (Continuous Calderón reproducing formula, [33]). *Let* $(\mathbb{R}^n,\rho,dx)$ *be a normal space of homogeneous type, let* $\{S_m\}$ *be an approximation of the identity as per Definition* 3.1*, and let* $D_m := S_{m+1} - S_m$ *be the wavelet operators satisfying* (4.2)–(4.5) *for*

$0 < \tau, \delta < \alpha$. Then there exist linear kernel operators $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ and $\{\hat{D}_m\}_{m\in\mathbb{Z}}$, acting on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, such that

$$f(x) = \sum_{m\in\mathbb{Z}} \tilde{D}_m D_m(f)(x) = \sum_{m\in\mathbb{Z}} D_m \hat{D}_m(f)(x), \tag{4.6}$$

where the series converges in $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Furthermore, the kernels of $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ and $\{\hat{D}_m\}_{m\in\mathbb{Z}}$ also satisfy conditions (4.2)–(4.5) *for any $\tau' < \tau$ and $\delta' < \delta$.*

Proof. Here we only sketch the proof and refer the reader to [33] for an in-depth treatment of analysis of spaces of homogeneous type. What is coined as "Coifman's idea" (attributed to Ronald Coifman) consists of writing

$$I = \sum_m D_m = \sum_k D_k \sum_l D_l = \sum_{k,l} D_k D_l.$$

Then for some integer $N > 0$, we define the operators $D_k^N := \sum_{|j|\le N} D_{k+j}$ and the operators T_N and R_N by

$$I = \sum_{k,l} D_k D_l = \sum_{k\in\mathbb{Z}} D_k^N D_k + \sum_{k\in\mathbb{Z}} \sum_{|j|>N} D_{k+j} D_k =: T_N + R_N.$$

One then shows that for any $\tau' < \tau$ and $\delta' < \delta$, the singular operator R_N is uniformly bounded on $\mathcal{M}_0(\tau', \delta', x_0, t)$ for all $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Moreover, there exist constants $\varepsilon > 0$ and $c > 0$ that do not depend on N, such that for $f \in \mathcal{M}_0(\tau', \delta', x_0, t)$,

$$\|R_N f\|_{\mathcal{M}_0(\tau',\delta',x_0,t)} \le c2^{-N\varepsilon} \|f\|_{\mathcal{M}_0(\tau',\delta',x_0,t)}. \tag{4.7}$$

This allows us to choose a sufficiently large N such that $c2^{-N\varepsilon} < 1$, which implies that $\|R_N\| < 1$ in the operator norm. Therefore the operator

$$T_N^{-1} := (I - R_N)^{-1} = \sum_{k=0}^{\infty} R_N^k$$

exists as a kernel operator and is bounded on $\mathcal{M}_0(\tau', \delta', x_0, t)$. Subsequently,

$$I = T_N^{-1} T_N = \sum_m (T_N^{-1} D_m^N) D_m = \sum_m \tilde{D}_m D_m, \quad \tilde{D}_m := T_N^{-1} D_m^N.$$

The regularity and vanishing moment conditions of the kernels $\{D_m\}$ imply that for any fixed N and $y_0 \in \mathbb{R}^n$, the function $D_m^N(\cdot, y_0)$ is in $\mathcal{M}_0(\tau, \delta, y_0, m)$. This gives that $\tilde{D}_m(\cdot, y_0) = T_N^{-1} D_m^N(\cdot, y_0) \in \mathcal{M}_0(\tau', \delta', y_0, m)$. Similarly, for fixed $x_0 \in \mathbb{R}^n$, $\tilde{D}_m(x_0, \cdot) \in \mathcal{M}_0(\tau', \delta', x_0, m)$. This implies that $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ satisfy conditions (4.2)–(4.5) for any $\tau' < \tau$, $\delta' < \delta$.

We may also write

$$I = T_N T_N^{-1} = \sum_m D_m (D_m^N T_N^{-1}) = \sum_m D_m \hat{D}_m, \quad \hat{D}_m := D_m^N T_N^{-1}.$$

Similar arguments imply that $\{\hat{D}_m\}_{m\in\mathbb{Z}}$ also satisfy conditions (4.2)–(4.5) for all $\tau' < \tau$ and $\delta' < \delta$. □

In the general setting of spaces of homogeneous type, we also have the Littlewood–Paley characterization of L_p spaces.

Proposition 4.5 ([25]). *Let* (X, ρ, μ), *be a space of homogeneous type, and let* $\{S_m\}_{m\in\mathbb{Z}}$ *be an approximation of the identity satisfying the conditions of Definition* 3.1. *Then for* $D_m := S_{m+1} - S_m$, $m \in \mathbb{Z}$, *and* $1 < p < \infty$, *there exist constants* $0 < c_1 < c_2 < \infty$ *such that*

$$c_1 \|f\|_{L_p(X)} \le \left\| \left(\sum_{m\in\mathbb{Z}} |D_m(f)|^2 \right)^{1/2} \right\|_{L_p(X)} \le c_2 \|f\|_{L_p(X)}. \tag{4.8}$$

4.2 Two-level splits

We have seen that we may define wavelet operators as the differences of two-level adjacent quasi-projection kernels $D_m = S_{m+1} - S_m$. Using the representation of the operators S_m with localized anisotropic "bumps", we can construct useful localized representations of the difference between two adjacent scales in the multiresolution ladder and in particular D_m using two-level splits [23].

Definition 4.6. Let Θ be a discrete cover and denote

$$\mathcal{M}_m := \{\nu = (\eta, \theta, \beta) : \eta \in \Theta_{m+1}, \theta \in \Theta_m, \eta \cap \theta \ne \emptyset, |\beta| < r\}, \quad m \in \mathbb{Z}.$$

We define using (3.14) and (3.17) the *two-level split basis*

$$F_\nu := P_{\eta,\beta}\varphi_\eta\varphi_\theta = \varphi_{\eta,\beta}\varphi_\theta, \quad \nu = (\eta, \theta, \beta) \in \mathcal{M}_m. \tag{4.9}$$

We denote $\mathcal{F}_m := \{F_\nu : \nu \in \mathcal{M}_m\}$ and set $W_m := \operatorname{span}(\mathcal{F}_m)$. Finally, we also denote $\mathcal{F} := \{F_\nu \in \mathcal{M}\}$, where $\mathcal{M} := \{\mathcal{M}_m\}$.

Note that $F_\nu \in C^L$, $\operatorname{supp}(F_\nu) = \theta \cap \eta$ if $\nu = (\eta, \theta, \beta)$, and, by property (e) in Definition 2.14, $\|F_\nu\|_p \sim |\eta|^{1/p-1/2}$, $0 < p \le \infty$.

Let the coefficients $\{A_{\alpha,\beta}^{\theta,\eta}\}$ be determined from

$$P_{\theta,\alpha} = \sum_{|\beta|<r} A_{\alpha,\beta}^{\theta,\eta} P_{\eta,\beta}, \quad \theta \in \Theta_m,\ \eta \in \Theta_{m+1}. \tag{4.10}$$

We will use the fact that there exists a constant $c(\mathbf{p}(\Theta), n, r) > 0$ such that

$$|A^{\theta,\eta}_{\alpha,\beta}| \le c, \quad \forall \theta \in \Theta_m,\ \eta \in \Theta_{m+1},\ \theta \cap \eta \ne \emptyset. \tag{4.11}$$

For any $\lambda = (\theta, \alpha) \in \Lambda_m$, we obtain, through application of the partition of unity (3.14) at the level $m+1$ and then (4.10), the following *meshless two-scale relationship*

$$\begin{aligned}\varphi_\lambda &= P_{\theta,\alpha}\varphi_\theta \\ &= \sum_{\eta\in\Theta_{m+1},\ \eta\cap\theta\ne\emptyset} P_{\theta,\alpha}\varphi_\theta\varphi_\eta \\ &= \sum_{\eta\in\Theta_{m+1},\ \eta\cap\theta\ne\emptyset,\ |\beta|<r} A^{\theta,\eta}_{\alpha,\beta} P_{\eta,\beta}\varphi_\theta\varphi_\eta \\ &= \sum_{\eta\in\Theta_{m+1},\ \eta\cap\theta\ne\emptyset,\ |\beta|<r} A^{\theta,\eta}_{\alpha,\beta} F_{\eta,\theta,\beta},\end{aligned}$$

and hence $\varphi_\lambda \in W_m$. Also, if $\lambda \in \Lambda_{m+1}$ and $\lambda = (\eta, \beta)$, then the partition of unity at the level m gives

$$\varphi_\lambda = P_{\eta,\beta}\varphi_\eta = \sum_{\theta\in\Theta_m:\theta\cap\eta\ne\emptyset} P_{\eta,\beta}\varphi_\eta\varphi_\theta = \sum_{\theta\in\Theta_m:\theta\cap\eta\ne\emptyset} F_{\eta,\theta,\beta}.$$

Combining the last two results, we find that $\overline{\operatorname{span}}(\Phi_m \cup \Phi_{m+1}) \subset W_m$.

These two representations of the local bumps using two-scale splits naturally lead to a representation of the difference operator $T_{m+1} - T_m$, where $T_m \in \{Q_m, P_m, P_{m,p}\}$ are defined by (3.30), (3.31), or (3.35). Using the representation $T_m f = \sum_{\lambda\in\Lambda_m} b_\lambda(f)\varphi_\lambda$, defined by (3.36) and the polynomial two-scale relation (4.10), we obtain

$$T_{m+1}f - T_m f = \sum_{\nu\in\mathcal{M}_m} d_\nu(f) F_\nu, \tag{4.12}$$

where

$$d_\nu(f) = d_{(\eta,\theta,\beta)}(f) := b_{\eta,\beta}(f) - \sum_{|\alpha|<r} A^{\theta,\eta}_{\alpha,\beta} b_{\theta,\alpha}(f).$$

The next result concerns the local anisotropic smoothness of a two-level split element and will be applied later on in the setting of Besov spaces.

Lemma 4.7. *For any* $1 \le k \le r$ *and* $0 < p < \infty$, *there exists a constant* $c(\mathbf{p}(\Theta), k, p) > 0$ *such that for any* $\sigma \in \Theta_m$ *and* $F_\nu \in \mathcal{F}_j$, $j \le m$, $\sigma \cap \eta_\nu \ne \emptyset$,

$$\omega_k(F_\nu, \sigma)_p^p \le c 2^{j-m-a_6 k(m-j)p} \|F_\nu\|_p^p, \tag{4.13}$$

where $\omega_k(\cdot,\cdot)_p$ *is a moduli of smoothness defined in* (1.12).

Proof. Denote briefly $\eta := \eta_\nu$, $\theta := \theta_\nu$, $\beta := \beta_\nu$, and $F := F_\nu$. Also, let $\sigma^* := A_\eta^{-1}\sigma$, $\theta^* := A_\eta^{-1}\theta$, and $F^* := F \circ A_\eta$. Recall that $F = P_{\eta,\beta}\varphi_\eta\varphi_\theta$ with $P_{\eta,\beta} := |\eta|^{-1/2}P_\beta \circ A_\eta^{-1}$. Hence $F^* = |\eta|^{-1/2}P_\beta\varphi_\eta(A_\eta\cdot)\varphi_\theta(A_\eta\cdot)$. Applying (2.18) for any $\eta \in \Theta_{m+1}$ and $\theta \in \Theta_m$ such that $\eta \cap \theta \neq \emptyset$ implies that for $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \le L$,

$$\left\|\partial^\alpha(\varphi_\theta \circ A_\eta)\right\|_\infty \le C,$$

where C depends only on L and $\mathbf{p}(\Theta)$. This gives that $|F^*|_{W_\infty^k} \le C(\mathbf{p}(\Theta), k)$. Now for any $h \in \mathbb{R}^n$, with $h^* := M_\eta^{-1}h$, we have

$$\begin{aligned}\|\Delta_h^k F\|_{L_p(\sigma)}^p &= \left|\det(M_\eta)\right|\|\Delta_{h^*}^k F^*\|_{L_p(\sigma^*)}^p \\ &\le C|\eta|\left|h^*\right|^{kp}|F^*|_{W_\infty^k}^p\left|\sigma^*\right| \\ &\le C|\eta|^{1-p/2}\left|\sigma^*\right| \operatorname{diam}(\sigma^*)^{kp}.\end{aligned}$$

Here we assumed that $k|h^*| \le \operatorname{diam}(\sigma^*)$, since otherwise $\Delta_{h^*}^k F^*(x) \equiv 0$. Next, observe that

$$\left|\sigma^*\right| = \left|\det(M_\eta^{-1}M_\sigma)\right| = \left|\det(M_\eta^{-1})\right|\left|\det(M_\sigma)\right| = |\eta|^{-1}|\sigma|$$

and by (2.18)

$$\operatorname{diam}(\sigma^*) = 2\|M_\eta^{-1}M_\sigma\|_{\ell_2 \to \ell_2} \le 2a_5 2^{-a_6(m-j)}.$$

We use these observations to obtain

$$\begin{aligned}\omega_k(F,\sigma)_p^p &\le C|\sigma||\eta|^{-p/2}2^{-a_6(m-j)kp} \\ &\le C|\sigma||\eta|^{-1}2^{-a_6(m-j)kp}|\eta|^{1-p/2} \\ &\le C2^{j-m-a_6(m-j)kp}\|F\|_p^p.\end{aligned}$$

□

Next, we claim that for each $m \in \mathbb{Z}$, $\mathcal{F}_m = \{F_\nu : \nu \in \mathcal{M}_m\}$ satisfies the crucial property of representation stability.

Theorem 4.8. *If $f \in W_m \cap L_p(\mathbb{R}^n)$, $0 < p \le \infty$, and $f = \sum_{\nu \in \mathcal{M}_m} a_\nu F_\nu$, then*

$$\|f\|_p \sim \begin{cases}(\sum_{\nu\in\mathcal{M}_m}\|a_\nu F_\nu\|_p^p)^{1/p} \sim 2^{m(\frac{1}{2}-\frac{1}{p})}(\sum_{\nu\in\mathcal{M}_m}|a_\nu|^p)^{1/p}, & 0 < p < \infty,\\ \sup_{\nu\in\mathcal{M}_m}\|a_\nu F_\nu\|_\infty \sim 2^{m/2}\sup_{\nu\in\mathcal{M}_m}|a_\nu|, & p = \infty.\end{cases} \tag{4.14}$$

The proof of Theorem 4.8 is a mere repetition of the proof of Theorem 3.6, where the stability of $\Phi_m = \{\varphi_\lambda\}_{\lambda\in\Lambda_m}$ was established. Here we specifically use the two-level coloring scheme (3.10) to ensure the linear independence of $\{F_\nu\}_{\nu\in\mathcal{M}_m}$.

4.3 Anisotropic wavelet operators

Let $\{S_m\}_{m\in\mathbb{Z}}$ be a multiresolution analysis of order (τ,δ,r) (see Section 3.1). Then it is clear that the kernels of the *wavelet operators*

$$D_m := S_{m+1} - S_m \tag{4.15}$$

satisfy conditions (3.2)–(3.5) of Definition 3.2 (with possibly different constants), whereas the polynomial reproduction condition (3.6) is replaced with the *vanishing moments* property

$$\int_{\mathbb{R}^n} D_m(x,y)P(y)dy = 0, \quad \int_{\mathbb{R}^n} D_m(x,y)P(x)dx = 0, \quad \forall P \in \Pi_{r-1}. \tag{4.16}$$

In fact, the kernels $\{D_m\}_{m\in\mathbb{Z}}$ are a particular case of *Approximation of the Identity with Exponential Decay* [45], since they inherit their regularity from $\{S_m\}_{m\in\mathbb{Z}}$, which in turn satisfy by Theorem 3.13 the exponential decay properties.

Proposition 4.9. *The dual operators $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ and $\{\hat{D}_m\}_{m\in\mathbb{Z}}$, constructed for the continuous Calderón reproducing formula* (4.6)*, also satisfy the higher vanishing moments properties* (4.16).

Proof. Recall from the proof of Proposition 4.4 that for sufficiently large N, $I = T_N + R_N$ with $\|R_N\| < 1$, where the norm is the operator norm acting on molecules. This gives that for any $m \in \mathbb{Z}$,

$$\begin{aligned}\tilde{D}_m &= T_N^{-1} D_m^N \\ &= \left(\sum_{j=0}^{\infty} R_N^j\right) D_m^N \\ &= \left(\sum_{j=0}^{\infty}\left(\sum_{|i|>N}\sum_{k\in\mathbb{Z}} D_{k+i}D_k\right)^j\right) D_m^N.\end{aligned}$$

Therefore $\tilde{D}_m$ satisfies the rth vanishing moments conditions (4.16), since it is a limit of finite compositions of wavelet operators, all satisfying (4.16). A similar argument shows that $\hat{D}_m$ also satisfies (4.16). □

Remark 4.10. Whereas the dual operators $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ and $\{\hat{D}_m\}_{m\in\mathbb{Z}}$ inherit the vanishing moments properties from the operators $\{D_m\}_{m\in\mathbb{Z}}$, there remains an *open question* on their regularity. In the setting of ellipsoid covers of $\mathbb{R}^n$, the wavelet operators $\{D_m\}_{m\in\mathbb{Z}}$ inherit their regularity from the anisotropic multiresolution analysis, whose kernels $\{S_m\}_{m\in\mathbb{Z}}$ may be constructed to have any prescribed higher regularity, faster decay and higher-order Lipschitz properties (Definition 3.2). Meanwhile, the dual wavelet operators are constructed using a more general framework of singular operators acting on low-order molecules in spaces of homogeneous type. This means, for example, that we only claim “modest” decay for the duals

$$|\tilde{D}_m(x,y)|, |\hat{D}_m(x,y)| \le C \frac{2^{-m\delta'}}{(2^{-m} + \rho(x,y))^{1+\delta'}},$$

where δ' is given in Proposition 4.4, and α is given by (2.4) with $\delta' < \alpha < 1$.

Observe that the operator R_N is in fact an anisotropic singular operator. So the question is in what sense we can correctly define higher-order molecule spaces and if R_N are bounded operators on these higher-order molecule spaces with $\|R_N\|_* \le C2^{-\varepsilon N}$ for some fixed $C > 0$ and $\varepsilon > 0$, with an appropriate operator norm $\|\cdot\|_*$. In Section 7.2, we demonstrate how an anisotropic singular operator indeed maps a smooth atom to a smooth molecule, yet with some quantifiable regularity lost. Such an estimate is not applicable in a scenario where we wish to apply the singular operators R_N^j as $j \to \infty$.

Recall that our construction in Section 3.3 of multiresolution kernels over a discrete cover Θ yields the multiresolution analysis kernels

$$S_m(x,y) = \sum_{\lambda\in\Lambda_m} \tilde{\varphi}_\lambda(y)\varphi_\lambda(x),$$

where $\{\varphi_\lambda\}_{\lambda\in\Lambda_m}$ are supported over ellipsoids at level Θ_m, whereas (3.47) implies that the duals $\{\tilde{\varphi}_\lambda\}_{\lambda\in\Lambda_m}$ have rapid decay. We use the partition of unity (3.14) and the polynomial two-scale relation (4.10) to compute the following two-level split representation of the wavelet kernel:

$$\begin{aligned} D_m(x,y) &= \sum_{\lambda\in\Lambda_{m+1}} \tilde{\varphi}_\lambda(y)\varphi_\lambda(x) - \sum_{\lambda\in\Lambda_m} \tilde{\varphi}_\lambda(y)\varphi_\lambda(x) \\ &= \sum_{\theta\in\Theta_m} \sum_{(\eta,\beta)\in\Lambda_{m+1}} \tilde{\varphi}_{\eta,\beta}(y)\varphi_\eta(x)P_{\eta,\beta}(x)\varphi_\theta(x) \\ &\quad - \sum_{\eta\in\Theta_{m+1}} \sum_{(\theta,\alpha)\in\Lambda_m} \tilde{\varphi}_{\theta,\alpha}(y)\varphi_\theta(x)P_{\theta,\alpha}(x)\varphi_\eta(x) \\ &= \sum_{\eta\in\Theta_{m+1}} \sum_{\theta\in\Theta_m,\theta\cap\eta\neq\emptyset} \sum_{|\beta|<r} \Big(\tilde{\varphi}_{\eta,\beta}(y) - \sum_{|\alpha|<r} A^{\theta,\eta}_{\alpha,\beta}\tilde{\varphi}_{\theta,\alpha}(y)\Big)P_{\eta,\beta}(x)\varphi_\eta(x)\varphi_\theta(x) \\ &= \sum_{\nu\in\mathcal{M}_m} G_\nu(y)F_\nu(x), \end{aligned}$$

where $\{F_\nu\}_{\nu\in\mathcal{M}_m}$ are given by (4.9), and

$$G_\nu := G_{(\eta,\theta,\beta)} := \tilde{\varphi}_{\eta,\beta} - \sum_{|\alpha|<r} A^{\theta,\eta}_{\alpha,\beta}\tilde{\varphi}_{\theta,\alpha}. \tag{4.17}$$

Observe that since $\theta\cap\eta \neq \emptyset$ for each $\nu = (\eta,\theta,\beta) \in \mathcal{M}_m$, (3.47) implies that the duals $\{G_\nu\}$ have fast decay with respect to the quasi-distance induced by the cover. Consequently, we obtain the two-level split representation for the wavelet operators

$$D_m(f) = \sum_{\nu \in \mathcal{M}_m} \langle f, G_\nu \rangle F_\nu, \quad m \in \mathbb{Z}. \tag{4.18}$$

Theorem 4.11. *The duals* $\{G_\nu\}_{\nu \in \mathcal{M}}$ *of the two-level splits* $\{F_\nu\}_{\nu \in \mathcal{M}}$ *are a frame.*

Proof. For any $f \in L_2(\mathbb{R}^n)$, by (4.18) we have

$$f = \sum_m D_m(f) = \sum_m \sum_{\nu \in \mathcal{M}_m} \langle f, G_\nu \rangle F_\nu.$$

We combine (4.8) with (4.14) to obtain

$$\begin{aligned} \|f\|_2^2 &\sim \sum_m \int_{\mathbb{R}^n} \left|D_m(f)(x)\right|^2 dx \\ &\sim \sum_m \sum_{\nu \in \mathcal{M}_m} \left\|\langle f, G_\nu \rangle F_\nu\right\|_2^2 \\ &\sim \sum_m \sum_{\nu \in \mathcal{M}_m} \left|\langle f, G_\nu \rangle\right|^2. \end{aligned}$$

□

We now show that two wavelet operators (kernels) from different scales are "almost orthogonal". This generalizes known results for the isotropic case and the case $r = 1$ in the setting of spaces of homogeneous type (see [33, 45]).

Theorem 4.12. *Assume that two kernels operators* $\{D_m^1\}$ *and* $\{D_m^2\}$, $m \in \mathbb{Z}$, *satisfy conditions* (3.2)–(3.4) *of a multiresolution with order* $(\tau, \delta + \tau_1 r, r)$, $\tau = (\tau_0, \tau_1)$, $r \geq 1$, $\delta > \tau_1 r$, *and the vanishing moments condition with r* (4.16). *Then, for all* $k, l \in \mathbb{Z}$,

$$\left|D_k^1 D_l^2(x,y)\right| = \left|\int_{\mathbb{R}^n} D_k^1(x,z) D_l^2(z,y)\,dz\right| \leq C 2^{-|k-l|\tau_0 r} \frac{2^{-\min(k,l)\delta}}{(2^{-\min(k,l)} + \rho(x,y))^{1+\delta}}.$$

Proof. For simplicity of notation, we assume that $\{D_m\} = \{D_m^1\} = \{D_m^2\}$ and prove the bound on the kernel $D_k D_l(x,y)$. The proof of the other cases are similar, where the technique of using the vanishing moments property on the Taylor polynomial and the bound on the Taylor remainder is applied on the integration coordinate z. We further assume that $l \leq k$. The proof for the case $k < l$ is similar. We apply the vanishing moments property (4.16) to obtain

$$\begin{aligned} \left|D_k D_l(x,y)\right| &= \left|\int_{\mathbb{R}^n} D_k(x,z) D_l(z,y)\,dz\right| \\ &\leq \int_{\mathbb{R}^n} \left|D_k(x,z)\right| \left|R_x^r(D_l(\cdot,y))(z)\right| dz \end{aligned}$$

$$
\begin{aligned}
&\leq \int\limits_{\rho(x,z)\leq \frac{1}{2\kappa}(2^{-l}+\rho(x,y))} |D_k(x,z)||R_x^r(D_l(\cdot,y))(z)|dz \\
&\quad + \int\limits_{\rho(x,y)\leq\rho(y,z)} |D_k(x,z)||R_x^r(D_l(\cdot,y))(z)|dz \\
&\quad + \int\limits_{\rho(x,y)>\rho(y,z)\wedge\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y))} |D_k(x,z)||R_x^r(D_l(\cdot,y))(z)|dz \\
&=: I + II + III.
\end{aligned}
$$

We separately estimate the three integrals. Applying the properties of the kernels, (3.7), and then (2.7), we derive

$$
\begin{aligned}
I &= \int\limits_{\rho(x,z)\leq \frac{1}{2\kappa}(2^{-l}+\rho(x,y))} |D_k(x,z)||R_x^r(D_l(\cdot,y))(z)|dz \\
&\leq C \int\limits_{\rho(x,z)\leq \frac{1}{2\kappa}(2^{-l}+\rho(x,y))} \frac{2^{-k\delta}}{(2^{-k}+\rho(x,z))^{1+\delta}}\rho(x,z)^{\tau_0 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_0 r}}dz \\
&\leq C2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_0 r}}\int\limits_{\mathbb{R}^n}\frac{\rho(x,z)^{\tau_0 r}}{(2^{-k}+\rho(x,z))^{1+\delta}}dz \\
&\leq C2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_0 r}}2^{k(\delta-\tau_0 r)} \\
&\leq C2^{(l-k)\tau_0 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta}}.
\end{aligned}
$$

The estimate of the second integral is similar, only here we use property (3.3), the fact that $\rho(x,y)\leq\rho(y,z)$, and (2.7):

$$
\begin{aligned}
II &= \int\limits_{\rho(x,y)\leq\rho(y,z)} |D_k(x,z)||R_x^r(D_l(\cdot,y))(z)|dz \\
&\leq C \int\limits_{\rho(x,y)\leq\rho(y,z)} \frac{2^{-k\delta}}{(2^{-k}+\rho(x,z))^{1+\delta}}\rho(x,z)^{\tau(x,z,2^{-l})r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau(x,z,2^{-l})r}}dz \\
&\leq C2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_0 r}}\int\limits_{\rho(x,z)\leq 2^{-l}}\frac{\rho(x,z)^{\tau_0 r}}{(2^{-k}+\rho(x,z))^{1+\delta}}dz \\
&\quad + C2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_1 r}}\int\limits_{\rho(x,z)>2^{-l}}\frac{\rho(x,z)^{\tau_1 r}}{(2^{-k}+\rho(x,z))^{1+\delta}}dz
\end{aligned}
$$

$$\le C2^{-k\delta}\left(\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_0 r}}2^{k(\delta-\tau_0 r)}+\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_1 r}}2^{k(\delta-\tau_1 r)}\right)$$
$$\le C2^{(l-k)\tau_0 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta}}.$$

We proceed with the estimate of III and further subdivide the integration domain:

$$III=\int_{\rho(x,y)>\rho(y,z)\wedge\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y)),\rho(x,z)\le 2^{-l}}\cdot+\int_{\rho(x,y)>\rho(y,z)\wedge\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y)),\rho(x,z)>2^{-l}}\cdot$$
$$=: III_1+III_2.$$

We show the bound of III_2, where on the integration domain, $\tau(x,z,2^{-l})=\tau_1$ (the bound of III_1 is similar with $\tau(x,z,2^{-l})=\tau_0$):

$$III_2\le\int_{\rho(x,y)>\rho(y,z)\wedge\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y))}|D_k(x,z)||R_x^r(D_l(\cdot,y))(z)|dz$$
$$\le C\int_{\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y))}|D_k(x,z)|\rho(x,z)^{\tau_1 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(z,y))^{1+\delta+\tau_1 r}}dz$$
$$\le C2^{-l\delta}\int_{\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y))}\frac{2^{-k\delta}}{(2^{-k}+\rho(x,z))^{1+\delta}}\frac{\rho(z,y)^{\tau_1 r}}{(2^{-l}+\rho(z,y))^{1+\delta+\tau_1 r}}dz$$
$$+C2^{-l\delta}\int_{\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y))}\frac{2^{-k(\delta+\tau_1 r)}}{(2^{-k}+\rho(x,z))^{1+\delta+\tau_1 r}}\frac{\rho(x,y)^{\tau_1 r}}{(2^{-l}+\rho(z,y))^{1+\delta+\tau_1 r}}dz$$
$$\le C2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta}}\int_{\mathbb{R}^n}\frac{\rho(z,y)^{\tau_1 r}}{(2^{-l}+\rho(z,y))^{1+\delta+\tau_1 r}}dz$$
$$+2^{-k(\delta+\tau_1 r)}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_1 r}}\int_{\mathbb{R}^n}\frac{\rho(x,y)^{\tau_1 r}}{(2^{-l}+\rho(z,y))^{1+\delta+\tau_1 r}}dz$$
$$\le C2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta}}2^{l\delta}+C2^{-k(\delta+\tau_1 r)}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\tau_1 r}}\rho(x,y)^{\tau_1 r}2^{l(\delta+\tau_1 r)}$$
$$\le C2^{(l-k)\tau_0 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta}}.$$
□

4.4 Anisotropic discrete wavelet frames

Our goal is to construct frames of $L_2(\mathbb{R}^n)$ (see (4.1)), that are well localized with respect to the anisotropic distance induced by an ellipsoid cover. This is achieved through a discrete Calderón reproducing formula, which is obtained by “sampling” the continuous Calderón reproducing formula (4.6). First, we introduce the following sampling process.

Definition 4.13. Let ρ be a quasi-distance on $\mathbb{R}^n$. We call a set of closed domains $\{\Omega_{m,k}\}$, $m \in \mathbb{Z}$, $k \in I_m$, and points $y_{m,k} \in \Omega_{m,k}$, a *sampling set* if it satisfies the following properties:

(a) For each $m \in \mathbb{Z}$, the sets $\Omega_{m,k}$, $k \in I_m$, are pairwise interior disjoint;

(b) For all $m \in \mathbb{Z}$, $\mathbb{R}^n = \bigcup_{k \in I_m} \Omega_{m,k}$;

(c) Each set $\Omega_{m,k}$ satisfies $\Omega_{m,k} \subset B_\rho(x_{m,k}, c2^{-m})$ for some point $x_{m,k} \in \mathbb{R}^n$ and fixed $c > 0$;

(d) There exists a constant $c' > 0$ such that for any $m \in \mathbb{Z}$ and $k \in I_m$, we have that $\rho(y_{m,k}, y_{m,j}) > c'2^{-m}$ for all $j \in I_m$, $j \neq k$, except perhaps for a bounded set.

Examples

(i) We can construct a sampling set from a discrete ellipsoid cover. We begin by picking a maximal set of disjoint ellipsoids as follows: For each level Θ_m, we enumerate the ellipsoids as $\theta_{m,j}$, $j \geq 1$. We define $\theta'_{m,1} := \theta_{m,1}$ and then inductively for $k, j > 1$, $\theta'_{m,k} := \theta_{m,j}$ if $\operatorname{int}((\bigcup_{i=1}^{k-1} \theta'_{m,i}) \cap \theta_{m,j}) = \emptyset$. We also select $x_{m,k}$ and $y_{m,k}$ as the center of $\theta'_{m,k}$. After this step, we denote $\Omega'_{m,k} := \theta'_{m,k}$ and observe that these domains and the sampling points $\{x_{m,k}\}$, $\{y_{m,k}\}$ satisfy properties (a), (c), and (d) but are still open sets and do not satisfy property (b). To see that property (d) is indeed satisfied, recall that by Theorem 2.23 there exists a ball B'_ρ with center at $y_{m,k}$ such that $B'_\rho \subseteq \theta'_{m,k}$ and $|B'_\rho| \sim |\theta'_{m,k}|$. This immediately implies that there exists a constant $c' > 0$ such that $\rho(y_{m,k}, y_{m,j}) > c'2^{-m}$ for all $j \neq k$.
Next, observe that each $\theta_{m,j}$ that was not selected at the previous step must intersect one of the selected ellipsoids $\theta'_{m,k}$. We iterate on these ellipsoids and update the domains $\Omega'_{m,k}$. For each such ellipsoid $\theta_{m,j}$, we add the domain $\theta_{m,j} \setminus (\bigcup_{i=1}^{\infty} \Omega'_{m,i})$ (if not empty at this stage) to one of the domains $\Omega'_{m,k}$ only if $\theta_{m,j}$ intersects $\theta'_{m,k}$. Observe that the domains $\Omega'_{m,k}$ are possibly enlarged during this process, but this is controlled by the fact that each ellipsoid $\theta'_{m,k}$ has no more than $N_1 - 1$ neighbors from the level m. This means that property (c) can still hold by enlarging the constant c, so that the anisotropic ball contains $\bigcup_{\theta_{j,m} \cap \theta'_{m,k} \neq \emptyset} \theta_{j,m}$. Evidently, we attain domains $\{\Omega_{m,k}\}$ as the closures of $\{\Omega'_{m,k}\}$ that satisfy all the conditions.

(ii) Christ's "dyadic cube" construction for spaces of homogeneous type [18] also satisfies the above conditions. As the name suggests, it has similar properties to those of the regular isotropic dyadic cube cover of $\mathbb{R}^n$. For example, each sampling "cube" $\Omega_{m+1,k}$ is contained in a unique sampling "cube" $\Omega_{m,k'}$ for some $k' \in I_m$. Also, each sampling domain at the level m is "substantial" in the sense that it contains a ball of radius $\geq c'2^{-m}$. Therefore property (d) is satisfied, provided that the sampling points $y_{m,k} \in \Omega_{m,k}$ are picked from these inner balls.

Theorem 4.14 (Discrete Calderón reproducing formula). *Let $\{S_m\}_{m\in\mathbb{Z}}$ be an anisotropic multiresolution of order (τ,δ,r), $\tau = (\tau_0,\tau_1)$, with respect to the quasi-distance induced by a discrete ellipsoid cover Θ. Denote $D_m := S_{m+1} - S_m$ and let $\{\Omega_{m,k}\}$ and $\{y_{m,k}\}$, $y_{m,k} \in \Omega_{m,k}$, be a sampling set for Θ. Then there exist $N > 0$ and linear kernel operators $\{\hat{E}_m\}$ such that for all $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$,*

$$f(x) = \sum_{m\in\mathbb{Z}} \sum_{k\in I_{m+N}} |\Omega_{m+N,k}|\hat{E}_m(f)(y_{m+N,k})D_m(x,y_{m+N,k}). \tag{4.19}$$

Furthermore, the kernels of $\{\hat{E}_m\}$ satisfy conditions (4.2)–(4.4) for $0 < \tau_0, \delta_0 < \alpha$, ($\alpha$ is defined in Proposition 2.4) and the vanishing moments property (4.16) for r.

Proof. The proof is similar to that in [43]. The discrete formula (4.19) is obtained from the continuous formula (4.6) as follows. We fix some $N > 0$ and apply (4.6) to obtain, for $f \in L_p(\mathbb{R}^n)$,

$$\begin{aligned}
f(x) &= \sum_m D_m \hat{D}_m(f)(x) \\
&= \sum_m \sum_{k\in I_{m+N}} \int_{\Omega_{m+N,k}} D_m(x,y)\hat{D}_m(f)(y)dy \\
&= \sum_m \sum_{k\in I_{m+N}} |\Omega_{m+N,k}|D_m(x,y_{m+N,k})\hat{D}_m(f)(y_{m+N,k}) \\
&\quad + \Bigg\{\sum_m \sum_{k\in I_{m+N}} \int_{\Omega_{m+N,k}} [D_m(x,y) - D_m(x,y_{m+N,k})]\hat{D}_m(f)(y)dy \\
&\quad + \sum_m \sum_{k\in I_{m+N}} \int_{\Omega_{m+N,k}} D_m(x,y_{m+N,k})[\hat{D}_m(f)(y) - \hat{D}_m(f)(y_{m+N,k})]dy\Bigg\} \\
&=: \tilde{T}_N f(x) + \tilde{R}_N f(x).
\end{aligned}$$

It is shown in [43] that for sufficiently large $N > 0$, the operator $\tilde{R}_N$ is bounded on $\mathcal{M}_0(\tau_0,\delta_0,x_0,t)$ for $0 < \tau_0, \delta_0 < \alpha$ and any $x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}$, and its norm is strictly smaller than 1. Similarly, for sufficiently large N, it is bounded on L_p, $1 < p < \infty$, with norm smaller than 1. Therefore, there exists the inverse operator $\tilde{T}_N^{-1}$, and with $\hat{E}_m := \hat{D}_m \tilde{T}_N^{-1}$, we get

$$\begin{aligned}
f(x) &= \tilde{T}_N \tilde{T}_N^{-1}(f)(x) \\
&= \sum_m \sum_{k\in I_{m+N}} |\Omega_{m+N,k}|D_m(x,y_{m+N,k})\hat{D}_m(\tilde{T}_N^{-1}(f))(y_{m+N,k}) \\
&= \sum_m \sum_{k\in I_{m+N}} |\Omega_{m+N,k}|D_m(x,y_{m+N,k})\hat{E}_m(f)(y_{m+N,k}).
\end{aligned}$$

□

Denoting the index set $K_m := I_{m+N}$, the discrete wavelets

$$\psi_{m,k}(x) := |\Omega_{m+N,k}|^{1/2} D_m(x, y_{m+N,k}), \quad m \in \mathbb{Z},\ k \in K_m, \tag{4.20}$$

and the dual wavelets

$$\tilde{\psi}_{m,k}(x) := |\Omega_{m+N,k}|^{1/2} \hat{E}_m(y_{m+N,k}, x), \quad m \in \mathbb{Z},\ k \in K_m, \tag{4.21}$$

we obtain the following discrete wavelet representation:

$$f(x) = \sum_m \sum_{k \in K_m} \langle f, \tilde{\psi}_{m,k} \rangle \psi_{m,k}(x). \tag{4.22}$$

Observe that the anisotropic wavelet representation (4.22) resembles a classical isotropic wavelet representation (see [24]). However, here the wavelets are specifically "tuned" to the geometry of the given ellipsoid cover and the induced quasi-distance. Compared with the orthonormal wavelet basis constructed in [4], the wavelets $\{\psi_{m,k}\}$ also have fast decay but can be constructed to be smoother and have more vanishing moments. However, the duals $\{\tilde{\psi}_{m,k}\}$ only enjoy the higher vanishing moments property but potentially may suffer from slow decay and lower regularity. We now proceed to show that the anisotropic wavelets constitute a frame (see Definition 4.1).

Theorem 4.15. *Let $\{S_m\}_{m \in \mathbb{Z}}$ be an anisotropic multiresolution of order $(\tau, \delta + \tau_1 r, r)$ with $\tau = (\tau_0, \tau_1)$, $\delta > \tau_1 r$, and $r > \tau_0^{-1}$. Denote $D_m := S_{m+1} - S_m$ and let $\{\Omega_{m,k}\}$ and $\{y_{m,k}\}$, $y_{m,k} \in \Omega_{m,k}$, be a sampling set for Θ. Then there exist constants $0 < A \le B < \infty$ such that for any $f \in L_2(\mathbb{R}^n)$,*

$$A\|f\|_2^2 \le \sum_m \sum_{k \in K_m} \left|\langle f, \tilde{\psi}_{m,k} \rangle\right|^2 \le B\|f\|_2^2, \tag{4.23}$$

where $\{\tilde{\psi}_{m,k}\}$ are defined by (4.21).

Proof. The proof is similar to that in [43]. We begin with a proof of the right-hand side of (4.23). From (4.20), Theorem 4.12, and property (c) in Definition 4.13 of the sampling set we obtain

$$\begin{aligned}
\left|\langle \psi_{m,k}, \psi_{m',k'} \rangle\right| &= |\Omega_{m+N,k}|^{1/2} |\Omega_{m'+N,k'}|^{1/2} \left| \int_{\mathbb{R}^n} D_m(x, y_{m+N,k}) D_{m'}(x, y_{m'+N,k'}) dx \right| \\
&\le C |\Omega_{m+N,k}|^{1/2} |\Omega_{m'+N,k'}|^{1/2} \\
&\quad \times 2^{-|m-m'|\tau_0 r} \frac{2^{-\min(m,m')\delta}}{(2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'}))^{1+\delta}} \\
&\le C 2^{-|m-m'|\tau_0 r} \left(\frac{2^{-\min(m,m')}}{2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'})} \right)^{1+\delta}.
\end{aligned}$$

We denote $\omega(m,k) := 2^{-m}$ and apply this estimate, property (d) of the sampling set (Definition 4.13), and the condition $r > \tau_0^{-1}$ to compute for fixed $m \in \mathbb{Z}$, $k \in K_m$,

$$\begin{aligned}
&\sum_{m',k'} |\langle \psi_{m,k}, \psi_{m',k'}\rangle| \omega(m',k') \\
&\le C \sum_{m',k'} 2^{-m'} 2^{-|m-m'|\tau_0 r} \left(\frac{2^{-\min(m,m')}}{2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'})} \right)^{1+\delta} \\
&\le C \sum_{m'} 2^{-m'} 2^{-|m-m'|\tau_0 r} 2^{m'} \sum_{k'} 2^{-m'} \left(\frac{2^{-\min(m,m')}}{2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'})} \right)^{1+\delta} \\
&\le C \sum_{m'} 2^{-m'} 2^{-|m-m'|\tau_0 r} 2^{m'} 2^{-\min(m,m')} \\
&\le C \left(\sum_{m' \le m} 2^{-m'} 2^{-(m-m')\tau_0 r} + \sum_{m' > m} 2^{-m'} 2^{-(m'-m)\tau_0 r} 2^{m'} 2^{-m} \right) \\
&\le C \left(2^{-m} \sum_{m' \le m} 2^{-(m-m')(\tau_0 r - 1)} + 2^{-m} \sum_{m' > m} 2^{-(m'-m)\tau_0 r} \right) \\
&\le C\omega(m,k).
\end{aligned}$$

The above estimate is exactly the condition of Schur's lemma (see [55, Section 8.4] for the case of isotropic dyadic cubes and wavelets), which we use here to show that the infinite matrix $M := \{\langle \psi_{m,k}, \psi_{m',k'}\rangle\}$ is bounded on l_2 sequences over the "sampling" index space. In particular, for the sequence $\alpha := \{\langle f, \tilde{\psi}_{m,k}\rangle\}_{m\in\mathbb{Z}, k\in K_m}$, we obtain

$$\|f\|_2^2 = \langle M\alpha, \alpha\rangle \le \|M\| \|\alpha\|^2 \le B \sum_{m,k} \left|\langle f, \tilde{\psi}_{m,k}\rangle\right|^2.$$

Next, we prove the right-hand side inequality of (4.23). By definition we have

$$\begin{aligned}
\sum_m \sum_{k\in K_m} \left|\langle f, \tilde{\psi}_{m,k}\rangle\right|^2 &= \sum_m \sum_{k\in K_m} |\Omega_{m+N,k}| \left|\hat{E}_m(f)(y_{m+N,k})\right|^2 \\
&= \sum_m \sum_{k\in K_m} \int_{\Omega_{m+N,k}} \left|\hat{E}_m(f)(y_{m+N,k})\right|^2 dy.
\end{aligned}$$

Proposition 4.4 shows that there exist operators $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ that satisfy the regularity conditions (4.2)–(4.5) with constants $0 < \tilde{\tau}, \tilde{\delta} < \alpha$ and have r vanishing moments and for which $f = \sum_m \tilde{D}_m D_m(f)$. We can show (using a similar, but simpler, approach to the proof of Theorem 4.12) that for $m, j \in \mathbb{Z}$,

$$\left|\hat{E}_m \tilde{D}_j(x,y)\right| \le c 2^{-|m-j|\varepsilon} \frac{2^{-\min(m,j)\varepsilon}}{(2^{-\min(m,j)} + \rho(x,y))^{1+\varepsilon}}, \tag{4.24}$$

where $\varepsilon := \min(\tilde{\tau}, \tilde{\delta})$, We use the continuous Calderón formula, (4.24), and the maximal function (2.8) to estimate each coefficient:

$$
\begin{aligned}
\left|\langle f, \tilde{\psi}_{m,k}\rangle\right|^2 &= \int_{\Omega_{m+N,k}} \left|\hat{E}_m(f)(y_{m+N,k})\right|^2 dy \\
&= \int_{\Omega_{m+N,k}} \left|\sum_j \hat{E}_m \tilde{D}_j D_j(f)(y_{m+N,k})\right|^2 dy \\
&\le C \int_{\Omega_{m+N,k}} \left(\sum_j \int_{\mathbb{R}^n} 2^{-|m-j|\varepsilon} \frac{2^{-\min(m,j)\varepsilon}}{(2^{-\min(m,j)} + \rho(y_{m+N,k}, z))^{1+\varepsilon}} \left|D_j(f)(z)\right| dz\right)^2 dy \\
&\le C \int_{\Omega_{m+N,k}} \left(\sum_j 2^{-|m-j|\varepsilon} M D_j(f)(y)\right)^2 dy.
\end{aligned}
$$

Applying the discrete Hölder inequality, the maximal inequality (2.11) and then (4.8), we get

$$
\begin{aligned}
\sum_m \sum_{k \in K_m} \left|\langle f, \tilde{\psi}_{m,k}\rangle\right|^2 &\le C \sum_m \int_{\mathbb{R}^n} \left(\sum_j 2^{-|m-j|\varepsilon} M_B D_j(f)(y)\right)^2 dy \\
&\le C \sum_m \int_{\mathbb{R}^n} \left(\sum_j 2^{-|m-j|\varepsilon}\right)\left(\sum_j 2^{-|m-j|\varepsilon} \left(M_B D_j(f)(y)\right)^2\right) dy \\
&\le C \sum_j \left\|M_B D_j(f)\right\|_2^2 \\
&\le C \sum_j \left\|D_j(f)\right\|_2^2 \\
&\le C \|f\|_2^2.
\end{aligned}
$$

□

5 Anisotropic smoothness spaces

The classical anisotropic Sobolev spaces over $\mathbb{R}^n$ [5, 21, 58], introduced by the Russian school in the 1970s, are associated with directional vectors $l = (l_1, \ldots, l_n)$, $l_i \in \mathbb{N}_+$, $1 \le i \le n$. The space $W_p^l(\mathbb{R}^n)$, $1 \le p \le \infty$, is defined as the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $f, \partial^l f \in L_p(\mathbb{R}^n)$ with

$$\|f\|_{W_p^l} := \|f\|_p + \left\|\partial^l f\right\|_p.$$

The mean smoothness s is defined by

$$\frac{1}{s} = \frac{1}{n}\left(\frac{1}{l_1} + \cdots + \frac{1}{l_n}\right),$$

from which we derive the anisotropy vector $a = (a_1, \ldots, a_n)$, $a_i := s/l_i$, $1 \le i \le n$. Obviously, $a_1 + \cdots + a_n = n$. An anisotropic distance to the origin is a continuous function $v : \mathbb{R}^n \to \mathbb{R}$, $v(0) = 0$, $v(x) > 0$ for $x \ne 0$, satisfying $v(t^{a_1}x_1, \ldots, t^{a_n}x_n) = tv(x)$ for all $x \in \mathbb{R}^n$ and $t > 0$. For example, we may define

$$v_\lambda(x) := \left(\sum_{i=1}^n |x_i|^{\lambda/a_i}\right)^{1/\lambda}, \quad 0 < \lambda < \infty.$$

Farkas [38] proved that there exists a smooth distance to the origin $|\cdot|_a \in C^\infty(\mathbb{R}^n \setminus \{0\})$ with the following property: for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}_+^n$, there exists $c(\alpha,\beta) > 0$ such that

$$\left|\partial^\alpha(|x|_a^\alpha)\right| \le c|x|_a^{\alpha - a\cdot\beta}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

This allows us to adapt the isotropic notation of decomposition of frequency windows to this anisotropic setting. We construct $\phi_0 \in C^\infty$ such that

$$\phi_0(x) = 1, \quad \forall |x|_a \le 1, \quad \operatorname{supp}(\phi_0) \subseteq \{x \in \mathbb{R}^n : |x|_a \le 2\}.$$

Denoting

$$\phi_m(x) := \phi_0(2^{-ma_1}x_1, \ldots, 2^{-ma_n}x_n) - \phi_0(2^{(-m+1)a_1}x_1, \ldots, 2^{(-m+1)a_n}x_n), \quad m \in \mathbb{N},$$

we obtain the partition of unity subordinate to a,

$$\sum_{m=0}^\infty \phi_m = 1.$$

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Finally, the anisotropic Besov space with smoothness index $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, subordinate to a is defined as [59]

$$B_{pq}^{\alpha,a} := \left\{ f \in \mathcal{S}' : \|f\|_{B_{pq}^{\alpha,a}} := \left(\sum_{m=0}^{\infty} \left[2^{m\alpha} \left\| (\phi_m \hat{f})^{\vee} \right\|_p \right]^q \right)^{1/q} < \infty \right\}.$$

In this chapter, we generalize the above classic anisotropic spaces, where the anisotropy a is fixed over $\mathbb{R}^n$, to the pointwise variable anisotropic setting, where the anisotropic phenomena can change rapidly from point to point and across scale. Therefore we use the "local" machinery of moduli of smoothness over the ellipsoids of a discrete ellipsoid cover, whereas the multiresolution Fourier multipliers $\{(\phi_m \hat{f})^{\vee}\}_m$ are replaced by the pointwise variable projections $\{T_m f\}_m$ defined by (3.36).

5.1 Anisotropic moduli of smoothness

The moduli of smoothness over $\mathbb{R}^n$ introduced in Section 1.2 were used in the context of "local" approximation estimates. Yet, they are isotropic, i. e., associated with the standard Euclidean distance. We now generalize them to moduli that are subordinate to anisotropic ellipsoid covers or, equivalently, to the induced quasi-distances [23, 27].

5.1.1 Definition and properties

Definition 5.1. Let Θ be a discrete cover. For any $r \geq 1$ and $m \in \mathbb{Z}$, we define the *anisotropic moduli of smoothness* of $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ at parameters $t = 2^{-m}$, $m \in \mathbb{Z}$, by

$$\omega_{\Theta,r}(f, 2^{-m})_p := \begin{cases} \left(\sum_{\theta \in \Theta_m} \omega_r(f,\theta)_p^p\right)^{1/p}, & 0 < p < \infty, \\ \sup_{\theta \in \Theta_m} \omega_r(f,\theta)_\infty, & p = \infty, \end{cases} \tag{5.1}$$

where $\omega_r(\cdot,\theta)_p$ is defined in (1.13).

Although the underlying geometry can possibly be highly anisotropic, the anisotropic moduli (5.1) corresponding to ellipsoid covers have similar properties to the classic isotropic moduli (1.12).

Theorem 5.2. *Let Θ be a discrete cover inducing the quasi-distance ρ of* (2.35). *The moduli $\omega_{\Theta,r}(\cdot,\cdot)_p$ have the following properties:*

(a) *There exists a constant $c(r, N_1)$ such that for any $f \in L_p(\mathbb{R}^n)$, $0 < p \leq \infty$, we have $\omega_{\Theta,r}(f, 2^{-m})_p \leq c\|f\|_p$ for all $m \in \mathbb{Z}$. More generally, for any $0 \leq k < r$, there exists a constant $c(r, k, N_1)$ such that $\omega_{\Theta,r}(f, 2^{-m})_p \leq c\omega_{\Theta,k}(f, 2^{-m})_p$ for all $m \in \mathbb{Z}$.*

(b) *For any $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, we have that $\omega_{\Theta,r}(f, 2^{-m})_p \to 0$ as $m \to \infty$.*

(c) *For $r \geq 1$, $0 < p \leq \infty$, there exists a constant $\lambda(\Theta, r, p) \geq 1$ such that for any $f \in L_p(\mathbb{R}^n)$, $m \in \mathbb{Z}$, and $k \geq 1$,*

$$\omega_{\Theta,r}(f, 2^{-m})_p \leq \lambda^k \omega_{\Theta,r}(f, 2^{-(m+k)})_p. \tag{5.2}$$

(d) *If another discrete cover $\tilde{\Theta}$ induces an equivalent quasi-distance $\tilde{\rho}$, i. e., $c_1\rho(x,y) \leq \tilde{\rho}(x,y) \leq c_2\rho(x,y)$ for all $x, y \in \mathbb{R}^n$, then for any $r \geq 1$, $0 < p \leq \infty$, and $m \in \mathbb{Z}$,*

$$\omega_{\Theta,r}(f, 2^{-m})_p \sim \omega_{\tilde{\Theta},r}(f, 2^{-m})_p, \tag{5.3}$$

where the constants of equivalency depend only on c_1, c_2 and the parameters of the covers.

Proof. (a) The boundedness of $\omega_{\Theta,r}(f, \cdot)_p$ by $c(r, N_1)\|f\|_p$ is obvious from the fact that each ellipsoid $\theta \in \Theta_m$ intersects with at most $N_1 - 1$ neighbors from Θ_m and the bound $\omega_r(f, \theta)_p \leq C\|f\|_{L_p(\theta)}$ from Proposition 1.14.

(b) For any $\varepsilon > 0$, let $Q_\varepsilon := [-M, M]^n$, $M > 0$, be such that $\int_{\mathbb{R}^n \setminus Q_\varepsilon} |f|^p dx \leq \varepsilon$. By Lemma 2.16 there exists $d_0(M, p(\Theta)) > 0$ such that for any $\theta \in \Theta_0$, $\theta \cap Q_\varepsilon \neq \emptyset$, we have that $\operatorname{diam}(\theta) \leq d_0$. From (2.23) we get for any $\theta \in \Theta_m$, $m \geq 0$, that if $\theta \cap Q_\varepsilon \neq \emptyset$, then $\operatorname{diam}(\theta) \leq a_5 d_0 2^{-a_6 m}$. This "quasi-uniform" property on the compact set Q_ε ensures that, as in the uniform (isotropic) case,

$$\sum_{\theta \in \Theta_m, \theta \cap Q_\varepsilon \neq \emptyset} \omega_r(f, \theta)_p^p \to 0 \quad \text{as } m \to \infty.$$

We also have

$$\begin{aligned} \sum_{\theta \in \Theta_m, \theta \cap Q_\varepsilon = \emptyset} \omega_r(f, \theta)_p^p &\leq C \sum_{\theta \in \Theta_m, \theta \cap Q_\varepsilon = \emptyset} \|f\|_{L_p(\theta)}^p \\ &\leq C\|f\|_{L_p(\mathbb{R}^n \setminus Q_\varepsilon)}^p \leq C\varepsilon. \end{aligned}$$

(c) It is sufficient to prove that $\omega_{\Theta,r}(f, 2^{-m})_p \leq \lambda\omega_{\Theta,r}(f, 2^{-(m+1)})_p$, since the general case (5.2) follows by repeated application. By Lemma 2.19 there exists a positive integer $N_2(p(\Theta))$ such that for any $\theta \in \Theta_m$,

$$\#\{\eta \in \Theta_{m+1} : \eta \cap \theta \neq \emptyset\} \leq N_2.$$

It is sufficient to show that there exists a constant $\tilde{\lambda} = \tilde{\lambda}(\Theta, r, p)$ such that for each $\theta \in \Theta_m$,

$$\omega_r(f, \theta)_p \leq \tilde{\lambda} \begin{cases} (\sum_{\eta \in \Theta_{m+1}: \theta \cap \eta \neq \emptyset} \omega_r(f, \eta)_p^p)^{1/p}, & 0 < p < \infty, \\ \max_{\eta \in \Theta_{m+1}: \theta \cap \eta \neq \emptyset} \omega_r(f, \eta)_\infty, & p = \infty. \end{cases} \tag{5.4}$$

Assume first that $m = 0$ and $\theta = B^*$ (the Euclidean unit ball). From (2.23) and (2.24) it follows that each $\eta \in \Theta_1$ such that $\eta \cap \theta \neq \emptyset$ is an ellipsoid "equivalent" to a Euclidean ball with $a_3 2^{-a_4} \le \sigma_{\min}(\eta) \le \sigma_{\max}(\eta) \le a_5 2^{-a_6}$. Property (d) in Definition 2.14 ensures that for each $x \in \theta$, there exists $\eta \in \Theta_1$ such that x is in the "core" $\eta^\diamond$. Combining these two observations implies that $\operatorname{dist}(x, \partial\eta) \ge (1 - a_7) a_3 2^{-a_4} =: \tilde{c}$, and so $B(x, \tilde{c}) \subset \eta$.

Recall from Definition (1.13) that

$$\omega_r(f, \theta)_p = \omega_r(f, B^*)_p = \sup_{|h| \le 2/r} \|\Delta_h^r(f, \cdot)\|_{L_p(B^*)}.$$

Observe that for any $h \in \mathbb{R}^n$ such that $|h| \le 2/r$ and for $\tilde{h} := K^{-1} h$ with $K := 2\lceil \tilde{c}^{-1} \rceil$, we have that $|\tilde{h}| \le \tilde{c}/r$. Using a well-known identity for the difference operator (see, e. g., [35, Chapter 2]), we have

$$\Delta_h^r(f, x) = \sum_{k_1=0}^{K-1} \cdots \sum_{k_r=0}^{K-1} \Delta_{\tilde{h}}^r(f, x + k_1 \tilde{h} + \cdots + k_r \tilde{h}).$$

For any domain $\Omega \subseteq \mathbb{R}^n$, denote $X(\Omega, h) := \{x \in \Omega : [x, x + rh] \subset \Omega\}$. Then if $x \in X(B^*, h)$, then also $y := x + k_1 \tilde{h} + \cdots + k_r \tilde{h} \in B^*$ for all $0 \le k_1, \ldots, k_r < K$. Furthermore, since $r|\tilde{h}| \le \tilde{c}$, for any $y \in B^*$, there exists $\eta \in \Theta_1$, $\eta \cap B^* \neq \emptyset$, such that $B(y, \tilde{c}) \subset \eta \Rightarrow [y, y + r\tilde{h}] \subset \eta$. From this we conclude that for $0 < p < \infty$ and any $h \in \mathbb{R}^n$, $|h| \le 2/r$, there exists a constant $\tilde{\lambda}(p, K) > 0$ such that

$$\begin{aligned} \int_{B^*} |\Delta_h^r(f, x, B^*)|^p dx &= \int_{X(B^*, h)} |\Delta_h^r(f, x)|^p dx \\ &\le \tilde{\lambda}^p \sum_{\eta \in \Theta_1 : \eta \cap B^* \neq \emptyset} \int_{X(\eta, \tilde{h})} |\Delta_{\tilde{h}}^r(f, y)|^p dy \\ &\le \tilde{\lambda}^p \sum_{\eta \in \Theta_1 : \eta \cap B^* \neq \emptyset} \omega_r(f, \eta)_p^p. \end{aligned}$$

This proves (5.4) for the case $m = 0$, $\theta = B^*$, and $0 < p < \infty$ (the case $p = \infty$ is similar). In the case where Θ is an arbitrary cover and $\theta \in \Theta_m$, let $\tilde{\Theta} := A_\theta^{-1}(\Theta)$, where $A_\theta(x) = Mx + v$ is an affine transform satisfying $A_\theta(B^*) = \theta$. Observe that $\tilde{\Theta}$ is a discrete cover with parameters equivalent to $p(\Theta)$. Denoting $\tilde{f} := f(A_\theta \cdot)$, we have

$$\begin{aligned} \omega_r(f, \theta)_p^p &= |\det(M)| \omega_r(\tilde{f}, B^*)_p^p \\ &\le \tilde{\lambda}^p |\det(M)| \sum_{\tilde{\eta} \in \tilde{\Theta}_1 : B^* \cap \tilde{\eta} \neq \emptyset} \omega_r(\tilde{f}, \tilde{\eta})_p^p \\ &\le \tilde{\lambda}^p \sum_{\eta \in \Theta_{m+1} : \theta \cap \eta \neq \emptyset} \omega_r(f, \eta)_p^p. \end{aligned}$$

This proves (5.4) and completes the proof of (5.2) for $0 < p < \infty$. The proof for $p = \infty$ is similar.

(d) Let Θ, $\tilde{\Theta}$ be two discrete covers with parameters $p(\Theta)$, $p(\tilde{\Theta})$ and equivalent induced quasi-distances $\rho \sim \tilde{\rho}$. Let $\theta \in \Theta_m$. By Theorem 2.23 there exists a ball $B_\rho(x, c2^{-m})$ such that $\theta \subset B_\rho(x, c2^{-m})$. By the equivalence of the quasi-distances there exists a constant $\tilde{c}$ such that $B_\rho(x, c2^{-m}) \subseteq B_{\tilde{\rho}}(x, c\tilde{c}2^{-m})$, and in turn there exists a positive integer $K(c, \tilde{c}, \mathbf{p}(\tilde{\Theta}))$ such that there exists $\tilde{\theta} \in \tilde{\Theta}_{m-K}$ satisfying $B_{\tilde{\rho}}(x, c\tilde{c}2^{-m}) \subseteq \tilde{\theta}$. This gives $\theta \subseteq \tilde{\theta}$ with θ and $\tilde{\theta}$ on "equivalent" levels where the parameters of the equivalence depend on $\mathbf{p}(\Theta)$ and $\mathbf{p}(\tilde{\Theta})$. Evidently, $\omega_r(f, \theta)_p \le \omega_r(f, \tilde{\theta})_p$ for all $f \in L_p^{\mathrm{loc}}$. Using this and (5.2) for the cover $\tilde{\Theta}$, we conclude that

$$\omega_{\Theta,r}(f, 2^{-m})_p \le \omega_{\tilde{\Theta},r}(f, 2^{-m+K})_p \le C\omega_{\tilde{\Theta},r}(f, 2^{-m})_p.$$

The proof of the inverse inequality is identical. □

We can now formulate an anisotropic Jackson-type theorem.

Theorem 5.3. *For a cover Θ, $1 \le k \le r$, $0 < p \le \infty$, and any $m \in \mathbb{Z}$,*

$$\|f - T_m f\|_p \le c\omega_{\Theta,k}(f, 2^{-m})_p, \tag{5.5}$$

where $\{T_m\}_{m\in\mathbb{Z}}$ are the "projection" operators defined in (3.36)*, with $T_m = P_{m,p}$ from* (3.34) *for the case* $0 < p < 1$.

Proof. We prove the theorem for $0 < p < \infty$ (the case $p = \infty$ is similar). From (3.38) we get that for any $\theta \in \Theta_m$,

$$\|f - T_m f\|_{L_p(\theta)}^p \le C \sum_{\theta' \in \Theta_m : \theta' \cap \theta \ne \emptyset} \omega_k(f, \theta')_p^p.$$

Thus

$$\begin{aligned}\|f - T_m f\|_{L_p(\mathbb{R}^n)}^p &\le \sum_{\theta \in \Theta_m} \|f - T_m f\|_{L_p(\theta)}^p \\ &\le C \sum_{\theta \in \Theta_m} \sum_{\theta' \in \Theta_m : \theta' \cap \theta \ne \emptyset} \omega_k(f, \theta')_p^p \\ &\le C \sum_{\theta \in \Theta_m} \omega_k(f, \theta)_p^p \\ &= C\omega_{\Theta,k}(f, 2^{-m})_p^p.\end{aligned}$$

□

5.1.2 The anisotropic Marchaud inequality

From Proposition 1.14 we know that for isotropic moduli of smoothness, we have the following: for any $1 \le k < r$, $\omega_r(f, t)_p \le C\omega_k(f, t)_p$ for all $f \in L_p(\mathbb{R}^n)$ and $t > 0$. The classical isotropic Marchaud inequality over the domain $\mathbb{R}^n$ with $0 < p \le \infty$ (see Section 1.2 for the case of regular domains) is the following inverse [35]:

$$\omega_k(f,t)_p \le ct^k \left(\int_t^\infty \frac{\omega_r(f,s)_p^\gamma}{s^{k\gamma+1}} ds \right)^{1/\gamma}, \quad t > 0,$$

where $\gamma := \min(1,p)$. We easily obtain a discrete form for $t = 2^{-m}$ by estimating the above integral over dyadic intervals:

$$\omega_k(f,2^{-m})_p \le c2^{-mk} \left(\sum_{j=-\infty}^{m} [2^{jk}\omega_r(f,2^{-j})_p]^\gamma \right)^{1/\gamma}. \tag{5.6}$$

In the anisotropic setting, Theorem 5.2(a) gives an equivalent form to the first inequality, Namely, for any $1 \le k < r$, there exists a constant $c > 0$ such that $\omega_{\Theta,r}(f,2^{-m})_p \le c\omega_{\Theta,k}(f,2^{-m})_p$ for all $f \in L_p(\mathbb{R}^n)$ and $m \in \mathbb{Z}$. Next, we present an anisotropic generalization of the isotropic discrete form (5.6).

Theorem 5.4. *For a discrete cover* Θ, $1 \le k < r$, *and* $0 < p \le \infty$, *there exists a constant* $c(\mathbf{p}(\Theta),k,r,p) > 0$ *such that for any* $f \in L_p(\mathbb{R}^n)$ *and* $m \in \mathbb{Z}$,

$$\omega_{\Theta,k}(f,2^{-m})_p \le c2^{-a_6 mk} \left(\sum_{j=-\infty}^{m} [2^{a_6 jk}\omega_{\Theta,r}(f,2^{-j})_p]^\gamma \right)^{1/\gamma}, \tag{5.7}$$

where $\gamma := \min(1,p)$, *and* a_6 *is defined in* (2.18).

Proof. Assume first that $0 < p < \infty$. We use a telescopic sum of the operators $\{T_j\}$ from (3.36), which provide "local" approximation order r, and apply Theorem 5.2(a) and then (5.5) to obtain

$$\begin{aligned}\omega_{\Theta,k}(f,2^{-m})_p^\gamma &\le \omega_{\Theta,k}(f - T_m f,2^{-m})_p^\gamma + \sum_{j=-\infty}^{m} \omega_{\Theta,k}(T_j f - T_{j-1} f,2^{-m})_p^\gamma \\ &\le C\left(\omega_{\Theta,r}(f,2^{-m})_p^\gamma + \sum_{j=-\infty}^{m} \omega_{\Theta,k}((T_j - T_{j-1})f,2^{-m})_p^\gamma \right).\end{aligned}$$

It remains to show that

$$\omega_{\Theta,k}((T_j - T_{j-1})f,2^{-m})_p \le C2^{a_6 k(j-m)}\omega_{\Theta,r}(f,2^{-j})_p, \quad j \le m. \tag{5.8}$$

Recall that W_{j-1} is the span of $\mathcal{F}_{j-1} = \{F_\nu : \nu \in \mathcal{M}_{j-1}\}$ defined in (4.9) and that $\overline{\operatorname{span}}(\Phi_j \cup \Phi_{j-1}) \subset W_{j-1}$. Therefore $(T_j - T_{j-1})f \in W_{j-1}$, and there exists a representation

$$(T_j - T_{j-1})f = \sum_{\nu \in \mathcal{M}_{j-1}} c_\nu F_\nu$$

for some coefficients $\{c_\nu\}_{\nu \in \mathcal{M}_{j-1}}$. By (4.13) for any $\theta \in \Theta_m$ and $F_\nu \in \mathcal{F}_{j-1}$, $j \le m$, such that $\theta \cap \eta_\nu \ne \emptyset$, we have

$$\omega_k(F_\nu, \theta)_p^p \le C2^{j-m-a_6 k(m-j)p} \|F_\nu\|_p^p.$$

Applying this estimate, Lemma 2.19, Theorem 4.8, and then (5.5), we conclude (5.8) for $1 \le p < \infty$:

$$\begin{aligned}
\omega_{\Theta,k}\big((T_j - T_{j-1})f, 2^{-m}\big)_p^p &= \sum_{\theta \in \Theta_m} \omega_k\big((T_j - T_{j-1})f, \theta\big)_p^p \\
&\le C \sum_{\theta \in \Theta_m} \left(\sum_{\nu \in \mathcal{M}_{j-1}: \theta \cap \eta_\nu \ne \emptyset} \omega_k(c_\nu F_\nu, \theta)_p \right)^p \\
&\le C2^{j-m-a_6 k(m-j)p} \sum_{\theta \in \Theta_m} \left(\sum_{\nu \in \mathcal{M}_{j-1}: \theta \cap \eta_\nu \ne \emptyset} \|c_\nu F_\nu\|_p \right)^p \\
&\le C2^{j-m-a_6 k(m-j)p} \Big(\max_{\eta \in \Theta_j} \#\{\theta \in \Theta_m :\ \theta \cap \eta \ne \emptyset\} \Big) \sum_{\nu \in \mathcal{M}_{j-1}} \|c_\nu F_\nu\|_p^p \\
&\le C2^{-a_6 k(m-j)p} \big\| (T_j - T_{j-1})f \big\|_p^p \\
&\le C2^{-a_6 k(m-j)p} \omega_{\Theta,r}(f, 2^{-j})_p^p.
\end{aligned}$$

The proofs of (5.8) for the cases $0 < p < 1$ and $p = \infty$, are similar. □

5.1.3 The anisotropic Ul'yanov inequality

The classic Ul'yanov inequality relates moduli of smoothness for different indices $p \le q$. The first result proved by Ul'yanov [64] for periodic functions $f \in L_p(\mathbb{T})$ and $1 \le p \le q < \infty$ is

$$\omega_1(f, t)_q \le c \left(\int_0^t \big(u^{-(\frac{1}{p} - \frac{1}{q})} \omega_1(f, u)_p\big)^q \frac{du}{u} \right)^{1/q}.$$

A higher-order (but slightly weaker) version [35] for $f \in L_p(\mathbb{R})$ and $1 \le p \le q < \infty$ is

$$\omega_r(f, t)_q \le c \int_0^t u^{-(\frac{1}{p} - \frac{1}{q})} \omega_r(f, u)_p \frac{du}{u}.$$

The following anisotropic version is a discrete generalization of the sharp isotropic result of [37] for the “full range” of indices,

Theorem 5.5. *For a discrete cover* Θ, $r \geq 1$, *and* $0 < p \leq q \leq \infty$, *there exists a constant* $c(\mathbf{p}(\Theta), r, p, q)$ *such that for any* $f \in L_p(\mathbb{R}^n)$,

$$\|f\|_q \leq c\left(\left(\sum_{j=0}^{\infty} 2^{j(\frac{1}{p}-\frac{1}{q})\gamma}\omega_{\Theta,r}(f,2^{-j})_p^{\gamma}\right)^{1/\gamma} + \|f\|_p\right), \tag{5.9}$$

and for any $m \in \mathbb{Z}$,

$$\omega_{\Theta,r}(f,2^{-m})_q \leq c\left(\sum_{j=m}^{\infty} 2^{j(\frac{1}{p}-\frac{1}{q})\gamma}\omega_{\Theta,r}(f,2^{-j})_p^{\gamma}\right)^{1/\gamma}, \tag{5.10}$$

where

$$\gamma := \begin{cases} q, & 0 < q < \infty, \\ 1, & q = \infty. \end{cases}$$

To prove Theorem 5.5, we need some results. In all of them, it will be convenient to use the operators $T_m = P_{m,p}$ defined by (3.34). The first result is a Nikolskii-type estimate.

Theorem 5.6. *For* $f \in L_p(\mathbb{R}^n)$, $0 < p \leq q \leq \infty$, *and* $m \in \mathbb{Z}$,

$$\|T_{m+1}f - T_m f\|_q \leq c2^{m(\frac{1}{p}-\frac{1}{q})}\omega_{\Theta,r}(f,2^{-m})_p. \tag{5.11}$$

Proof. Recall again that W_m is the span of $\mathcal{F}_m = \{F_\nu : \nu \in \mathcal{M}_m\}$ defined in (4.9) and that $\overline{\text{span}}(\Phi_m \cup \Phi_{m+1}) \subset W_m$. Therefore $(T_{m+1} - T_m)f \in W_m$, where $T_m = T_{m,p}$, and there exists a representation

$$(T_{m+1} - T_m)f = \sum_{\nu\in\mathcal{M}_m} c_\nu F_\nu$$

with some coefficients $\{c_\nu\}_{\nu\in\mathcal{M}_j}$. Applying (4.14) for the q-norm, $q < \infty$, then the assumption $p \leq q$, then (4.14) for the p-norm and finally the Jackson inequality (5.5) at the levels m and $m+1$ yields

$$\begin{aligned}\|T_{m+1}f - T_m f\|_q &\leq C2^{m(\frac{1}{2}-\frac{1}{q})}\left(\sum_{\nu\in\mathcal{M}_m}|c_\nu|^q\right)^{1/q} \\ &\leq C2^{m(\frac{1}{p}-\frac{1}{q})}2^{m(\frac{1}{2}-\frac{1}{p})}\left(\sum_{\nu\in\mathcal{M}_m}|c_\nu|^p\right)^{1/p}\end{aligned}$$

$$\leq C2^{m(\frac{1}{p}-\frac{1}{q})}\|T_{m+1}f - T_m f\|_p$$
$$\leq C2^{m(\frac{1}{p}-\frac{1}{q})}\omega_{\Theta,r}(f, 2^{-m})_p.$$

The proof for the case $q = \infty$ is similar. □

Lemma 5.7. *For $f \in L_p(\mathbb{R}^n)$ and $0 < p \leq q \leq \infty$,*

$$\omega_{\Theta,r}(f, 2^{-m})_q \leq c(\|f - T_m f\|_q + 2^{m(\frac{1}{p}-\frac{1}{q})}\omega_{\Theta,r}(f, 2^{-m})_p), \tag{5.12}$$

where $T_m = T_{m,p}$ are defined by (3.34).

Proof. First, observe that

$$\omega_{\Theta,r}(f, 2^{-m})_q \leq C(\omega_{\Theta,r}(f - T_m f, 2^{-m})_q + \omega_{\Theta,r}(T_m f, 2^{-m})_q).$$

Since Theorem 5.2(a) gives

$$\omega_{\Theta,r}(f - T_m f, 2^{-m})_q \leq C\|f - T_m f\|_q,$$

it suffices to show that

$$\omega_{\Theta,r}(T_m f, 2^{-m})_q \leq C2^{m(\frac{1}{p}-\frac{1}{q})}\omega_{\Theta,r}(f, 2^{-m})_p.$$

By definition, for $0 < q < \infty$,

$$\omega_{\Theta,r}(T_m f, 2^{-m})_q^q = \sum_{\theta\in\Theta_m} \omega_r(T_m f, \theta)_q^q.$$

By the partition of unity of $\{\varphi_\theta\}_{\theta\in\Theta_m}$ and property (c) of discrete covers we have

$$\omega_r(T_m f, \theta)_q^q = \omega_r\Big(\sum_{\theta'\in\Theta_m, \theta'\cap\theta\neq\emptyset} P_{\theta',p}(f)\varphi_{\theta'}, \theta\Big)_q^q$$
$$= \omega_r\Big(P_{\theta,p}(f) + \sum_{\theta'\in\Theta_m, \theta'\cap\theta\neq\emptyset} (P_{\theta',p}(f) - P_{\theta,p}(f))\varphi_{\theta'}, \theta\Big)_q^q$$
$$\leq C \sum_{\theta'\in\Theta_m, \theta'\cap\theta\neq\emptyset, \theta'\neq\theta} \|P_{\theta',p}(f) - P_{\theta,p}(f)\|_{L_q(\theta)}^q.$$

By property (e) of discrete covers, Lemma 1.23, Lemma 1.24, and (3.33) we have, for each $\theta' \in \Theta_m$, $\theta' \neq \theta$, $\theta' \cap \theta \neq \emptyset$,

$$\|P_{\theta',p}(f) - P_{\theta,p}(f)\|_{L_q(\theta)}^q \leq C\|P_{\theta',p}(f) - P_{\theta,p}(f)\|_{L_q(\theta\cap\theta')}^q$$
$$\leq C2^{mq(\frac{1}{p}-\frac{1}{q})}\|P_{\theta',p}(f) - P_{\theta,p}(f)\|_{L_p(\theta\cap\theta')}^q$$

$$\leq C2^{mq(\frac{1}{p}-\frac{1}{q})}\big(\|f-P_{\theta,p}(f)\|^q_{L_p(\theta)}+\|f-P_{\theta',p}(f)\|^q_{L_p(\theta')}\big)$$
$$\leq C2^{mq(\frac{1}{p}-\frac{1}{q})}\big(\omega_r(f,\theta)^q_p+\omega_r(f,\theta')^q_p\big).$$

We apply this and $q \geq p$ to obtain

$$\begin{aligned}\omega_{\Theta,r}(T_m f, 2^{-m})^q_q &\leq C2^{mq(\frac{1}{p}-\frac{1}{q})}\sum_{\theta\in\Theta_m}\omega_r(f,\theta)^q_p \\ &\leq C2^{mq(\frac{1}{p}-\frac{1}{q})}\Big(\sum_{\theta\in\Theta_m}\omega_r(f,\theta)^p_p\Big)^{q/p} \\ &= C2^{mq(\frac{1}{p}-\frac{1}{q})}\omega_{\Theta,r}(f,2^{-m})^q_p.\end{aligned}$$

This concludes the proof of the lemma for $0 < q < \infty$. The proof for $q = \infty$ is similar. □

Proof of Theorem 5.5. By (5.12) we have

$$\omega_{\Theta,r}(f,2^{-m})_q \leq C\big(\|f-T_m f\|_q + 2^{m(\frac{1}{p}-\frac{1}{q})}\omega_{\Theta,r}(f,2^{-m})_p\big).$$

Let us replace for a moment the first right-hand side term $\|f - T_m f\|_q$ by $\|T_M f - T_m f\|_q$ for a “large” $M > m$. Observe that for any $j \in \mathbb{Z}$, $T_j f = T_{j,p} f \in L_q(\mathbb{R}^n)$, since using (4.14) with $q \geq p$,

$$\|T_j f\|_q \leq C2^{j(\frac{1}{p}-\frac{1}{q})}\|T_j f\|_p \leq C2^{j(\frac{1}{p}-\frac{1}{q})}\|f\|_p.$$

For $0 < q \leq 1$, by (5.11) we have

$$\|T_M f - T_m f\|^q_q \leq \sum_{j=m}^{M-1}\|T_{j+1}f - T_j f\|^q_q \leq C\sum_{j=m}^{M-1}2^{j(\frac{1}{p}-\frac{1}{q})q}\omega_{\Theta,r}(f,2^{-j})^q_p.$$

For $1 \leq q \leq \infty$, we similarly get

$$\|T_M f - T_m f\|_q \leq \sum_{j=m}^{M-1}\|T_{j+1}f - T_j f\|_q \leq C\sum_{j=m}^{M-1}2^{j(\frac{1}{p}-\frac{1}{q})}\omega_{\Theta,r}(f,2^{-j})_p.$$

However, note that for $1 < q < \infty$, we claim a sharper estimate in (5.10) using the l_q-norm of $\{2^{j(1/p-1/q)}\omega_{\Theta,r}(f,2^{-j})_p\}$ instead of the l_1-norm. Indeed, this is achieved using exactly the proof of Lemma 3.1 in [37], which requires the Nikolskii-type estimate (5.11) and gives, for $1 < q < \infty$,

$$\|T_M f - T_m f\|_q \leq C\Big(\sum_{j=m}^{M-1}2^{j(\frac{1}{p}-\frac{1}{q})q}\omega_{\Theta,r}(f,2^{-j})^q_p\Big)^{1/q}.$$

Therefore, to prove (5.10), it remains to show that if the right-hand side of (5.10) is finite, then

$$\|T_M f - T_m f\|_q \to \|f - T_m f\|_q \quad \text{as } M \to \infty. \tag{5.13}$$

Indeed, it is easy to see that if the right hand side of (5.10) is finite then $\{T_M f\}$ is a Cauchy sequence in L_q. At the same time, we know $\{T_M f\}$ converges to f in L_p as $M \to \infty$. Therefore $\{T_M f\}$ converge in L_q to f and thus (5.13) is proved.

From the above we can easily obtain (5.9) by

$$\begin{aligned}\|f\|_q &\le C(\|f - T_0 f\|_q + \|T_0 f\|_q)\\ &\le C\left(\left(\sum_{j=0}^{\infty} 2^{j(\frac{1}{p}-\frac{1}{q})\gamma}\omega_{\Theta,r}(f,2^{-j})_p^{\gamma}\right)^{1/\gamma} + \|f\|_p\right).\end{aligned}$$

□

5.2 Comparing the moduli $\omega_r(\cdot,\cdot)_p$ and $\omega_{\Theta,r}(\cdot,\cdot)_p$

Here we wish to show that the anisotropic moduli of smoothness over ellipsoid covers are a true generalization of the isotropic moduli. To this end, we need the following “inverse”-type inequality that bounds a sum of local moduli over the elements of a cover by the moduli over the covered domain ($\mathbb{R}^n$ in our application)

Proposition 5.8 ([37]). *Suppose the following conditions hold for a convex domain $\Omega \subseteq \mathbb{R}^n$ and $t > 0$:*

(i) *There exist convex sets $\tilde{\Omega}_i$, $i \in I$, where I is some countable index set, such that $\Omega = \bigcup_{i\in I} \tilde{\Omega}_i$.*
(ii) *Each point $x \in \Omega$ is contained in at most N_1 sets $\tilde{\Omega}_i$.*
(iii) *There exist $0 < c_1 < c_2 < \infty$ such that each $\tilde{\Omega}_i$ contains an Euclidean ball of radius $\ge c_1 t$ and is contained in an Euclidean ball of radius $\le c_2 t$.*

Then, for any $f \in L_p(\Omega)$, $0 < p < \infty$,

$$\sum_{i\in I} \omega_r(f,\tilde{\Omega}_i)_p^p \le C(n,r,p,N_1,c_1,c_2)\omega_r(f,t)_{L_p(\Omega)}^p,$$

and for $p = \infty$,

$$\sup_{i\in I} \omega_r(f,\tilde{\Omega}_i)_\infty \le C(n,r,c_2)\omega_r(f,t)_{L_\infty(\Omega)}.$$

Theorem 5.9. *Let Θ be a discrete cover of ellipsoids in $\mathbb{R}^n$ that are equivalent to Euclidean balls with fixed parameters. Then $\omega_{\Theta,r}(\cdot,2^{-mn})_p \sim \omega_r(\cdot,2^{-m})_p$, where $\omega_r(\cdot,\cdot)_p$ is the classic isotropic modulus of smoothness over $\mathbb{R}^n$ defined in* (1.12).

Proof. By our assumption, in this special case, there exist two fixed constants $0 < R_1 < R_2 < \infty$ such that for every $\theta \in \Theta_{mn}$, there exist two Euclidean balls satisfying $B(x_1, R_1 2^{-m}) \subseteq \theta \subseteq B(x_2, R_2 2^{-m})$. Also, from the properties of discrete covers we obtain that there exists a positive integer $J(p(\Theta), R_1, R_2, r)$ such that for any $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}$, there exists an ellipsoid $\theta \in \Theta_{mn-J}$ such that $B(x, r2^{-m}) \subseteq \theta$, where r is the order of the moduli.

For each $\theta \in \Theta_{mn-J}$, denote by $X(\theta)$ the set of points $x \in \mathbb{R}^n$ for which $B(x, r2^{-m}) \subset \theta$. Since $\mathbb{R}^n = \bigcup_{\theta \in \Theta_{mn-J}} X(\theta)$ and each set $X(\theta)$ intersects with at most N_1 neighboring sets, we get, for $0 < p < \infty$,

$$\begin{aligned}
\omega_r(f, \mathbb{R}^n, 2^{-m})_p^p &= \sup_{|h| \le 2^{-m}} \int_{\mathbb{R}^n} \left|\Delta_h^r(f, \mathbb{R}^n, x)\right|^p dx \\
&\le C \sup_{|h| \le 2^{-m}} \sum_{\theta \in \Theta_{mn-J}} \int_{X(\theta)} \left|\Delta_h^r(f, \mathbb{R}^n, x)\right|^p dx \\
&\le C \sum_{\theta \in \Theta_{mn-J}} \sup_{|h| \le 2^{-m}} \int_{\theta} \left|\Delta_h^r(f, \theta, x)\right|^p dx \\
&\le C \sum_{\theta \in \Theta_{mn-J}} \omega_r(f, \theta)_p^p \\
&= C\omega_{\Theta,r}(f, 2^{-(mn-J)})_p^p \\
&\le C\omega_{\Theta,r}(f, 2^{-mn})_p^p,
\end{aligned}$$

where we applied (5.2) to obtain the last inequality. The case $p = \infty$ is similar and easier.

In the other direction, observe that our conditions ensure that the conditions of Proposition 5.8 are satisfied, from which the inverse inequality is immediate. □

5.3 Anisotropic Besov spaces

5.3.1 Definitions and properties

The classical isotropic homogeneous Besov space $B_{p,q}^{\alpha}(\mathbb{R}^n)$ with $0 < p, q \le \infty$ and smoothness index $\alpha > 0$ is defined as the space of functions $f \in L_p(\mathbb{R}^n)$ such that

$$|f|_{B_{p,q}^{\alpha}} := \begin{cases} \left(\int_{\mathbb{R}^n} (t^{-\alpha}\omega_r(f,t)_p)^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{-\alpha}\omega_r(f,t)_p), & q = \infty, \end{cases} \tag{5.14}$$

is finite, where $r \geq \lfloor\alpha\rfloor + 1$. By sampling at dyadic points $t = 2^{-m}$, $m \in \mathbb{Z}$, one can show that [35]

$$|f|_{B^{\alpha}_{p,q}} \sim \begin{cases} (\sum_{m\in\mathbb{Z}} (2^{\alpha m}\omega_r(f,2^{-m})_p)^q)^{1/q}, & 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}} 2^{\alpha m}\omega_r(f,2^{-m})_p, & q = \infty. \end{cases} \tag{5.15}$$

This leads to the following generalization in the anisotropic setting [23, 27].

Definition 5.10. We define the homogeneous B-space $B^{\alpha}_{p,q}(\Theta)$ induced by a discrete ellipsoid cover Θ with $0 < p, q \leq \infty$ and smoothness index $\alpha > 0$ as the space of functions $f \in L_p(\mathbb{R}^n)$ such that

$$|f|_{B^{\alpha}_{p,q}(\Theta)} := \begin{cases} (\sum_{m\in\mathbb{Z}} (2^{\alpha m}\omega_{\Theta,r}(f,2^{-m})_p)^q)^{1/q}, & 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}} 2^{\alpha m}\omega_{\Theta,r}(f,2^{-m})_p, & q = \infty, \end{cases} \tag{5.16}$$

is finite, where $\omega_{\Theta,r}(\cdot,\cdot)_p$ are the anisotropic moduli of smoothness defined in (5.1), and $r \geq 1$ satisfies

$$r > \frac{\alpha}{a_6}, \tag{5.17}$$

where a_6 is defined in (2.18). We also have the (quasi-)norm

$$\|f\|_{B^{\alpha}_{p,q}(\Theta)} := \|f\|_p + |f|_{B^{\alpha}_{p,q}(\Theta)}. \tag{5.18}$$

In contrast to the classical case, our definition of anisotropic B-space is in fact "normalized" in the sense that the dimension n does not come into play later in various embeddings or inequalities. Referring to (5.16), we note that for each $m \in \mathbb{Z}$, 2^{-m} is equivalent to the volume of ellipsoids on the level m, whereas in the classical isotropic case (5.14), it is the side length of a dyadic cube with volume 2^{-mn}. By Theorem 5.9 we have that in the particular case where all the ellipsoids of a cover Θ are equivalent to Euclidean balls,

$$B^{\alpha}_{p,q}(\Theta) \sim B^{n\alpha}_{p,q}(\mathbb{R}^n).$$

In a similar manner to the isotropic case, we have the following:

Theorem 5.11. *The seminorms* (5.16) *are equivalent for different values of r satisfying* (5.17).

Proof. Assume that r, r' satisfy (5.17) with $r' < r$. By Theorem 5.2(a) we have that $\omega_{\Theta,r}(f,2^{-m})_p \leq C(r',r,N_1)\omega_{\Theta,r'}(f,2^{-m})_p$ for all $m \in \mathbb{Z}$, which gives the first direction. To obtain the inverse direction, we apply the anisotropic Marchaud inequality (5.7), which for any $m \in \mathbb{Z}$ gives

$$\omega_{\Theta,r'}(f,2^{-m})_p \le C2^{-a_6 mr'}\left(\sum_{j=-\infty}^{m}[2^{a_6 jr'}\omega_{\Theta,r}(f,2^{-j})_p]^\gamma\right)^{1/\gamma}, \tag{5.19}$$

where $\gamma := \min(1,p)$.

We now recall a certain variant of the discrete Hardy inequality [35]. For a sequence of nonnegative numbers $a := \{a_m\}_{m\in\mathbb{Z}}$, we denote

$$\|a\|_{\alpha,q} := \begin{cases} (\sum_{m\in\mathbb{Z}}(2^{\alpha m}a_m)^q)^{1/q}, & 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}} 2^{\alpha m}a_m, & q = \infty. \end{cases}$$

Then, if $a = \{a_m\}$ and $b = \{b_m\}$ are two sequence of non-negative numbers and for some $C_0 > 0$, $\gamma > 0$, and $\mu > \alpha > 0$,

$$b_m \le C_0 2^{-m\mu}\left(\sum_{j=-\infty}^{m}[2^{j\mu}a_j]^\gamma\right)^{1/\gamma}, \quad \forall m \in \mathbb{Z},$$

then

$$\|b\|_{\alpha,q} \le C\|a\|_{\alpha,q}.$$

Therefore, equipped with (5.19), we can apply this variant of the discrete Hardy inequality with $a_m := \omega_{\Theta,r}(f,2^{-m})_p$, $b_m := \omega_{\Theta,r'}(f,2^{-m})_p$, and $\mu := a_6 r' > \alpha$ to conclude the theorem. □

As in the isotropic case, the Ul'yanov inequality can be applied to obtain embedding results for the anisotropic Besov spaces beyond the obvious embedding $B^{\alpha_2}_{p,q}(\Theta) \subset B^{\alpha_1}_{p,q}(\Theta)$ for $\alpha_1 \le \alpha_2$,

Theorem 5.12. *Let* Θ *be a cover of* $\mathbb{R}^n$, $0 < p < q \le \infty$, *and denote* $\lambda := 1/p - 1/q$. *Then, for* $\alpha > 0$, *the following (continuous) embeddings hold:*
(i) $B^{\alpha+\lambda}_{p,\infty}(\Theta) \subset B^{\alpha}_{q,\infty}(\Theta)$,
(ii) $B^{\alpha+\lambda}_{p,q}(\Theta) \subset B^{\alpha}_{q,q}(\Theta)$.

Proof. (i) Let $f \in B^{\alpha+\lambda}_{p,\infty}(\Theta)$. For $0 < p \le q < \infty$, by (5.9) we have

$$\begin{aligned}\|f\|_q &\le C\left(\left(\sum_{j=0}^{\infty}2^{j\lambda q}\omega_{\Theta,r}(f,2^{-j})_p^q\right)^{1/q} + \|f\|_p\right) \\ &\le C\left(|f|_{B^{\alpha+\lambda}_{p,\infty}(\Theta)}\left(\sum_{j=0}^{\infty}2^{j\lambda q}2^{-j(\alpha+\lambda)q}\right)^{1/q} + \|f\|_p\right) \\ &\le C(|f|_{B^{\alpha+\lambda}_{p,\infty}(\Theta)} + \|f\|_p) \\ &= C\|f\|_{B^{\alpha+\lambda}_{p,\infty}}.\end{aligned}$$

Then, for any $m \in \mathbb{Z}$, by (5.10) we have

$$\begin{aligned}\omega_{\Theta,r}(f,2^{-m})_q^q &\le C\sum_{j=m}^{\infty} 2^{j\lambda q}\omega_{\Theta,r}(f,2^{-j})_p^q \\ &\le C|f|^q_{B^{\alpha+\lambda}_{p,\infty}(\Theta)}\sum_{j=m}^{\infty}2^{j\lambda q}2^{-j(\alpha+\lambda)q} \\ &\le C|f|^q_{B^{\alpha+\lambda}_{p,\infty}(\Theta)}2^{-m\alpha q}.\end{aligned}$$

The proof for $q=\infty$ is similar.

(ii) For $0<p\le q<\infty$, an application of (5.9) yields

$$\begin{aligned}\|f\|_q &\le C\left(\left(\sum_{j=0}^{\infty}2^{j\lambda q}\omega_{\Theta,r}(f,2^{-j})_p^q\right)^{1/q}+\|f\|_p\right)\\ &\le C\left(\left(\sum_{j=0}^{\infty}2^{j(\alpha+\lambda)q}\omega_{\Theta,r}(f,2^{-j})_p^q\right)^{1/q}+\|f\|_p\right)\\ &\le C\|f\|_{B^{\alpha+\lambda}_{p,q}}.\end{aligned}$$

Inequality (5.10) gives

$$\begin{aligned}|f|^q_{B^{\alpha}_{q,q}(\Theta)} &= \sum_m \left(2^{m\alpha}\omega_{\Theta,r}(f,2^{-m})_q\right)^q\\ &\le C\sum_m\sum_{j=m}^{\infty}2^{m\alpha q}2^{j\lambda q}\omega_{\Theta,r}(f,2^{-j})_p^q\\ &= C\sum_j 2^{j\lambda q}\omega_{\Theta,r}(f,2^{-j})_p^q\sum_{m=-\infty}^{j}2^{m\alpha q}\\ &= C\sum_j 2^{jq(\alpha+\lambda)}\omega_{\Theta,r}(f,2^{-j})_p^q\sum_{m=-\infty}^{j}2^{(m-j)\alpha q}\\ &\le C\sum_j 2^{jq(\alpha+\lambda)}\omega_{\Theta,r}(f,2^{-j})_p^q\\ &\le C|f|^q_{B^{\alpha+\lambda}_{p,q}(\Theta)}.\end{aligned}$$

The proof for $q=\infty$ is easier. □

5.3.2 Examples of adaptive covers

We consider two simple examples of discontinuous functions on $\mathbb{R}^2$, the characteristic function $\mathbf{1}_{B^*}$ of the unit disk and the characteristic function $\mathbf{1}_{\square}$ of a square. We will show that using ellipse covers that are adaptive to the curve singularities of these indicator functions, each of them has higher anisotropic Besov smoothness compared with its (classical) isotropic Besov space smoothness [23]. For a cover Θ, $\alpha > 0$, and $\tau > 0$, we denote

$$B_\tau^\alpha(\Theta) := B_{\tau\tau}^\alpha(\Theta).$$

Observe that for $0 < \tau < p$ satisfying

$$\frac{1}{\tau} = \alpha + \frac{1}{p},$$

by (5.9) we have an embedding analogous to the isotropic case

$$B_\tau^\alpha(\Theta) \subset L_p(\mathbb{R}^n).$$

Example 5.13. There exists an anisotropic ellipse cover Θ of $\mathbb{R}^2$ such that $\mathbf{1}_{B^*} \in B_\tau^\alpha(\Theta)$ for any $\alpha < \frac{2}{3\tau}$. In comparison, if $\tilde{\Theta}$ is a cover of Euclidean balls related to classical isotropic Besov smoothness, then $\mathbf{1}_{B^*} \in B_\tau^\alpha(\tilde{\Theta}) = B_\tau^{2\alpha}(\mathbb{R}^n)$ only for $\alpha < \frac{1}{3\tau}$. Here the bounds on α are sharp.

Proof. We begin by constructing an appropriate continuous ellipse cover Θ_c of $\mathbb{R}^2$. For arbitrary $t \le 0$ and $v \in \mathbb{R}^2$, we define

$$\theta(v,t) := 2^{-t/2}B^* + v,$$

that is, the disk of radius $2^{-t/2}$ centered at v.

For the scales $t > 0$, the cover is adaptive to the "geometry" of the function, i. e., to the boundary of the disk. The idea of construction is that the ellipses intersecting with the edge singularity of the indicator function at S^1 essentially have a semiaxis of length $\sim 2^{-t/3}$ aligned with the gradient of the boundary and a semiaxis of length $\sim 2^{-2t/3}$ aligned with the normal to the boundary. This allows for a tighter ellipse cover of the singularity at each scale when comparing to nonadaptive Euclidean balls. Let $t > 0$. For any $v = (v_1, 0)$, $v_1 > 0$, which obeys the condition $|1 - v_1| \le 2^{-t/3}$, we define $\theta(v,t)$ as the set of all point $x \in \mathbb{R}^2$ such that

$$\frac{(x_1 - v_1)^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} \le 1,$$

where

$$\sigma_1 := \left(|1 - v_1| + 2^{-t/2}\right)2^{-t/6}, \quad \sigma_2 := \left(|1 - v_1| + 2^{-t/2}\right)^{-1}2^{-5t/6}.$$

If $v = (v_1, 0)$, $v_1 \ge 0$, satisfies $|1 - v_1| > 2^{-t/3}$, then we set $\theta(v,t) := 2^{-t/2}B^* + v$. Observe that in both cases, $|\theta(v,t)| \sim 2^{-t}$, and so the "volume condition" (2.13) is satisfied. For any point v that does not lie on the positive x_1-axis, we define $\theta(v,t)$ by a rotation of the ellipse $\theta((|v|,0),t)$ defined above about the origin that takes $(|v|,0)$ to v.

We now show that the collection of ellipses Θ_c defined above is a continuous ellipse cover of $\mathbb{R}^2$ in the sense of Definition 2.10. Fix $v=(v_1,0)$, $v_1>0$. Let $t,s>0$ and assume that $|1-v_1|\le 2^{-(t+s)/3}$ (other cases are similar or easier to prove). Denote by $\sigma_1(t)$ the x_1-semiaxis of $\theta(v,t)$. By the definition we have

$$\frac{\sigma_1(t+s)}{\sigma_1(t)}=\frac{|1-v_1|+2^{-(t+s)/2}}{|1-v_1|+2^{-t/2}}\cdot 2^{-s/6},$$

which leads to

$$2^{-2s/3}\le\frac{\sigma_1(t+s)}{\sigma_1(t)}\le 2^{-s/6}.$$

A similar computation gives

$$2^{-5s/6}\le\frac{\sigma_2(t+s)}{\sigma_2(t)}\le 2^{-s/3}.$$

Together, these estimates imply

$$2^{-5s/6}\le 1/\|M_{v,t+s}^{-1}M_{v,t}\|\le\|M_{v,t}^{-1}M_{v,t+s}\|\le 2^{-s/6}. \tag{5.20}$$

The case where v does not lie on the positive x_1-axis reduces to the above by rotation.

Now fix $t>0$ and let $\theta:=\theta(v,t)$ and $\theta':=\theta(v',t)$ be such that $\theta\cap\theta'\ne\emptyset$. Assume that $|1-|v||\le 2^{-t/3}$ and $|1-|v'||\le 2^{-t/3}$ (other cases are similar or easier to prove). Since Θ_c is rotation invariant, we may assume that $v=(v_1,0)$, $v_1>0$. Denote by σ_1, σ_2 the semiaxes of θ and by σ_1', σ_2' ($\sigma_1'<\sigma_2'$) the semiaxes of θ'. It is easy to see that $||v|-|v'||\le 22^{-t/3}$; however, the assumption $\theta\cap\theta'\ne\emptyset$ provides the stronger bound

$$\big||v'|-|v|\big|\le\sigma_1'+\sigma_1=\big(\big|1-|v'|\big|+\big|1-|v|\big|+2\cdot 2^{-t/2}\big)2^{-t/6}\le 4\cdot 2^{-t/2},$$

which implies

$$\sigma_1'\le\big(\big|1-|v|\big|+\big||v'|-|v|\big|+2^{-t/2}\big)2^{-t/6}\le\big(\big|1-|v|\big|+5\cdot 2^{-t/2}\big)2^{-t/6}\le 5\sigma_1.$$

Therefore

$$1/5\le\frac{\sigma_1'}{\sigma_1}\le 5,\quad 1/5\le\frac{\sigma_2'}{\sigma_2}\le 5. \tag{5.21}$$

We may assume that $t\ge 3$ (the case $t<3$ is trivial). Then $1/2\le|v|,\ |v'|\le 3/2$. Since $\theta\cap\theta'\ne\emptyset$, the ellipse θ' can be obtained by rotating $\theta(|v'|,t)$ about the origin about an angle γ such that

$$|\gamma| \le 2\sigma_2 + 2\sigma_2' \le 4 \cdot 2^{-t/3}. \tag{5.22}$$

Let A_θ be the affine transform that maps B^* onto θ. Then $A_\theta(x) = M_\theta x + v$, where $M_\theta = \operatorname{diag}(\sigma_1, \sigma_2)$. The affine transform $A_{\theta'}$ mapping B^* onto θ' is of the form $A_{\theta'}(x) = M_{\theta'} x + v'$, where $M_{\theta'}$ can be represented as the product of a diagonal and a rotation matrix, namely,

$$M_{\theta'} = M_\gamma \operatorname{diag}(\sigma_1', \sigma_2'), \quad M_\gamma := \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}.$$

It is straightforward to show that

$$M_\theta^{-1} M_{\theta'} = \begin{pmatrix} (\sigma_1'/\sigma_1)\cos\gamma & -(\sigma_2'/\sigma_1)\sin\gamma \\ (\sigma_1'/\sigma_2)\sin\gamma & (\sigma_2'/\sigma_2)\cos\gamma \end{pmatrix}.$$

From (5.22) it follows that

$$(\sigma_2'/\sigma_1)|\sin\gamma| \le 4 \quad \text{and} \quad (\sigma_1'/\sigma_2)|\sin\gamma| \le 4.$$

This and (5.21) imply that all entries of $M_\theta^{-1} M_{\theta'}$ are bounded in absolute value by 5. Hence

$$\|M_\theta^{-1} M_{\theta'}\|_{\ell_2 \to \ell_2} \le C, \quad \|M_{\theta'}^{-1} M_\theta\|_{\ell_2 \to \ell_2} \le C.$$

Combining this with (5.20) implies that Θ_c also satisfies the “shape condition” (2.14) with $a_4 = 5/6$ and $a_6 = 1/6$, and so it is a valid continuous ellipse cover of $\mathbb{R}^2$.

Applying Theorem 2.32 to Θ_c, through an adaptive sampling and dilation process, implies there exists an equivalent discrete ellipse cover Θ of $\mathbb{R}^2$ satisfying the conditions of Definition 2.14. It remains to show that $\mathbf{1}_{B^*} \in B_\tau^\alpha(\Theta)$ for all $\alpha < \frac{2}{3\tau}$.

Denote by Θ_m' the set of all ellipses from Θ_m that intersect the unit circle S^1 in $\mathbb{R}^2$. We need to estimate $\#\Theta_m'$. By condition (c) on discrete ellipsoid covers only N_1 ellipses from Θ_m may intersect at a time. This and the construction of Θ_c and Θ yield, for $m > 0$,

$$\begin{aligned} \#\Theta_m' &\le C2^m \Big| \bigcup_{\theta \in \Theta_m'} \theta \Big| \\ &\le C2^m 2^{-2m/3} = C2^{m/3}. \end{aligned}$$

Evidently, $\#\Theta_m' \le c$ if $m \le 0$. Next, observe that $\omega_r(\mathbf{1}_{B^*}, \theta)_\tau = 0$ if $\theta \in \Theta_m \setminus \Theta_m'$. For $\theta \in \Theta_m'$, if $m \le 0$, then $\omega_r(\mathbf{1}_{B^*}, \theta)_\tau \le c\|\mathbf{1}_{B^*}\|_{L_\tau(\mathbb{R}^2)} \le c$, and if $m > 0$, then $\omega_r(\mathbf{1}_{B^*}, \theta)_\tau \le$

$c|\theta|^{1/\tau} \leq c2^{-m/\tau}$. We get, for $\alpha < \frac{2}{3\tau}$,

$$\begin{aligned}|\mathbf{1}_{B^*}|^{\tau}_{B^{\alpha}_{\tau}(\Theta)} &= \sum_{m\in\mathbb{Z}} \sum_{\theta\in\Theta'_m} |\theta|^{-\alpha\tau}\omega_r(\mathbf{1}_{B^*},\theta)^{\tau}_{\tau} \\ &\leq C\sum_{m=-\infty}^{0} 2^{m\alpha\tau} + C\sum_{m=1}^{\infty}(\#\Theta'_m)2^{-m(1-\alpha\tau)} \\ &\leq C + C\sum_{m=1}^{\infty} 2^{-m(2/3-\alpha\tau)} \leq C.\end{aligned}$$

Consequently, $\mathbf{1}_{B^*} \in B^{\alpha}_{\tau}(\Theta)$ for $\alpha < \frac{2}{3\tau}$. Here the bound is sharp since S^1 cannot be covered by $\leq C2^{m/3}$ ellipsoids of area 2^{-m} whenever $C > 0$ is sufficiently small. □

Example 5.14. For any square □ in $\mathbb{R}^2$, there exists an anisotropic ellipsoid cover Θ of $\mathbb{R}^2$ such that $\mathbf{1}_{\square} \in B^{\alpha}_{\tau}(\Theta)$ for any $0 < \alpha < \frac{1}{\tau}$. In comparison, if $\tilde{\Theta}$ is a cover of Euclidean balls relating to classical isotropic Besov smoothness, then $\mathbf{1}_{\square} \in B^{\alpha}_{\tau}(\tilde{\Theta}) = B^{2\alpha}_{\tau}(\mathbb{R}^n)$ only for $\alpha < \frac{1}{3\tau}$. Here the bounds for α are sharp.

Proof. Without loss of generality, using dilation of the function and the constructed cover, we may assume that $\square = [-1,1] \times [0,2]$. As in the previous example, we first construct an appropriate continuous ellipse cover and then discretize it. We first construct ellipses $\theta(v,t)$ of our continuous cover Θ_c with centers v from the triangle $\triangle_0 := [(0,0),(1,0),(0,1)]$ and $t > 0$. Then we use symmetry about the x_2-axis to define $\theta(v,t)$ for v in the triangle $[(-1,0),(0,0),(0,1)]$. We next apply symmetry about the x_1-axis to define the ellipses $\theta(v,t)$ for v in the triangle $[(-1,0),(0,-1),(1,0)]$. Again by symmetry about the line $x_2 = -x_1 + 1$ we define $\theta(v,t)$ on the square $[(1,0),(2,1),(1,2),(0,1)]$. Symmetry about the line $x_2 = x_1 + 1$ enables us to define $\theta(v,t)$ for v in the rectangle $[(-1,0),(1,2),(0,3),(-2,1)]$. In this way the ellipses $\theta(v,t)$ would be defined with centers v from the square $S := [(0,-1),(2,1),(0,3),(-2,1)]$. Finally, we define the ellipses $\theta(v,t)$ with centers $v \in \mathbb{R}^2 \setminus S$ by

$$\theta(v,t) := 2^{-t/2}B^* + v.$$

In going further, for $t \leq 0$, we define $\theta(v,t)$ for all centers $v \in \mathbb{R}^2$ by $\theta(v,t) := 2^{-t/2}B^* + v$.

It remains to define the ellipses $\theta(v,t)$ with centers $v \in \triangle_0$ and $t > 0$. We begin by introducing a parameter $\delta > 0$ satisfying the condition

$$\frac{\delta}{2} < 1 - \alpha\tau. \tag{5.23}$$

The idea of the construction is to have near the edges of □ long and thin ellipses that are aligned with the edges and Euclidean balls away from the edges of □ and at the vertices. To this end, for every $v = (v_1,v_2) \in \triangle_0$, set $u := u_v := 1 - v_1 - v_2$ and define

$\theta(v,t)$ as the set of all $x \in \mathbb{R}^2$ such that

$$\frac{(x_1 - v_1)^2}{\sigma_1^2} + \frac{(x_2 - v_2)^2}{\sigma_2^2} \le 1,$$

where $\sigma_1 := (u_v 2^{t/2} + 1)^{1-\delta} 2^{-t/2}$ and $\sigma_2 := (u_v 2^{t/2} + 1)^{\delta-1} 2^{-t/2}$. Evidently, $|\theta(v,t)| \sim \sigma_1\sigma_2 = 2^{-t}$, which implies that the cover Θ_c satisfies the "volume condition" (2.13).

We next show that Θ_c satisfies the "shape condition" (2.14) with parameters depending only on δ. Fix $v \in \triangle_0$ and for any $t \in \mathbb{R}$, denote by $\sigma_1(t)$ and $\sigma_2(t)$ the semiaxes of $\theta(v,t)$. Then from the construction we have that for $t, s > 0$,

$$\frac{\sigma_1(t+s)}{\sigma_1(t)} = \left(\frac{u_v + 2^{-(t+s)/2}}{u_v + 2^{-t/2}}\right)^{1-\delta} 2^{-\delta s/2},$$

which readily implies

$$2^{-s/2} \le \frac{\sigma_1(t+s)}{\sigma_1(t)} \le 2^{-\delta s/2}.$$

A similar computation gives

$$2^{-s(1-\delta/2)} \le \frac{\sigma_2(t+s)}{\sigma_2(t)} \le 2^{-s/2}.$$

Together, these two estimates imply

$$2^{-s(1-\delta/2)} \le 1/\|M_{v,t+s}^{-1} M_{v,t}\| \le \|M_{v,t}^{-1} M_{v,t+s}\| \le 2^{-\delta s/2}. \tag{5.24}$$

Now fix $t > 0$ and let $\theta(v,t) \cap \theta(v',t) \ne \emptyset$, $v, v' \in \triangle_0$. Assume that $u_{v'} > u_v$. Since by construction $\sigma_1(x,s) \ge \sigma_2(x,s)$ for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$, we may estimate

$$\begin{aligned}
u_{v'} &\le u_v + \sigma_1(v,t) + \sigma_1(v',t) \\
&\le u_v + \left(u_v + 2^{-t/2}\right)^{1-\delta} 2^{-\delta t/2} + \left(u_{v'} + 2^{-t/2}\right)^{1-\delta} 2^{-\delta t/2} \\
&\le u_v + u_v^{1-\delta} 2^{-\delta t/2} + 2^{-t/2} + u_{v'}^{1-\delta} 2^{-\delta t/2} + 2^{-t/2}.
\end{aligned}$$

If $u_{v'} \ge 2^{-t/2}$, then this leads to

$$u_{v'} \le u_v + u_v^{1-\delta} 2^{-\delta t/2} + 2 \cdot 2^{-t/2} \le 2\left(u_v + 2^{-t/2}\right),$$

which yields

$$\sigma_1(v')/\sigma_1(v) \le c, \quad c = c(\delta).$$

If $u_{v'} < 2^{-t/2}$, then the same estimate immediately follows with a different constant $c = c(\delta)$. This yields

$$\|M_{v,t}^{-1}M_{v',t}\|, \|M_{v',t}^{-1}M_{v,t}\| \le C.$$

Combining this with (5.24), we get that the "shape condition" (2.14) is also satisfied with $a_4 = 1 - \delta/2$ and $a_6 = \delta/2$ and that Θ_c is a continuous ellipse cover of $\mathbb{R}^2$ in the sense of Definition 2.10.

By Theorem 2.32, through an adaptive sampling and dilation process, the above cover Θ_c induces a discrete ellipse cover Θ of $\mathbb{R}^2$. Our next task is showing that $\mathbf{1}_\square \in B_\tau^\alpha(\Theta)$. To this end, we need an upper bound for the number of all ellipses from Θ_m that intersect the boundary of $\square$. Denote this set by Θ_m'. By condition (c) on discrete covers and the construction of Θ_c and Θ it follows that for $m > 0$,

$$\begin{aligned}\#\Theta_m' &\le C2^m \Big|\bigcup_{\theta\in\Theta_m'} \theta\Big| \\ &\le C2^m \int_0^1 \sigma_2(\theta(v_1,0), 2^{-m})dv_1 \\ &\le C2^{m/2} \int_0^1 \big((1-v_1)2^{m/2}+1\big)^{\delta-1} dv_1 = C2^{\delta m/2}.\end{aligned}$$

Evidently, $\#\Theta_m \le c$ if $m \le 0$.

We are now prepared to estimate $|\mathbf{1}_\square|_{B_\tau^\alpha(\Theta)}$. Using the estimate of $\#\Theta_m'$ and (5.23), we get

$$\begin{aligned}|\mathbf{1}_\square|_{B_\tau^\alpha(\Theta)}^\tau &= \sum_{m\in\mathbb{Z}} \sum_{\theta\in\Theta_m'} |\theta|^{-\alpha\tau}\omega_r(\mathbf{1}_\square, \theta)_\tau^\tau \\ &\le C\sum_{m=-\infty}^{0} 2^{m\alpha\tau} + C\sum_{m=1}^{\infty} (\#\Theta_m')2^{-m(1-\alpha\tau)} \\ &\le C + C\sum_{m=1}^{\infty} 2^{-m(1-\alpha\tau-\delta/2)} \le C.\end{aligned}$$

□

5.3.3 Equivalent seminorms

Let T_m, $m \in \mathbb{Z}$, be the operators from (3.36) with order r satisfying (5.17). For $f \in L_p(\mathbb{R}^n)$, we define

$$|f|_{B_{p,q}^\alpha(\Theta)}^T := \begin{cases} \left(\sum_{m\in\mathbb{Z}} (2^{\alpha m}\|(T_{m+1}-T_m)f\|_p)^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}} 2^{\alpha m}\|(T_{m+1}-T_m)f\|_p, & q = \infty. \end{cases} \tag{5.25}$$

Recall from (4.12) that we have the two-scale split representation $(T_{m+1} - T_m)f = \sum_{\nu\in\mathcal{M}_m} d_\nu(f)F_\nu$ and that by Theorem 4.8 for $0 < p < \infty$, we have

$$\|(T_{m+1} - T_m)f\|_p \sim \Big(\sum_{\nu\in\mathcal{M}_m} \|d_\nu(f)F_\nu\|_p^p\Big)^{1/p},$$

with a similar equivalence for $p = \infty$. Thus, for $0 < p < \infty$,

$$|f|^T_{B^\alpha_{p,q}(\Theta)} \sim \Big(\sum_{m\in\mathbb{Z}} \Big(2^{\alpha m}\Big(\sum_{\nu\in\mathcal{M}_m} \|d_\nu(f)F_\nu\|_p^p\Big)^{1/p}\Big)^q\Big)^{1/q}. \tag{5.26}$$

Using the "two-level splits" from Definition 4.6, we also define the atomic (quasi-) norm

$$|f|^A_{B^\alpha_{p,q}(\Theta)} := \inf_{f=\sum_{\nu\in\mathcal{M}} a_\nu F_\nu} \Big(\sum_{m\in\mathbb{Z}} \Big(\sum_{\nu\in\mathcal{M}_m} (|\eta_\nu|^{-\alpha}\|a_\nu F_\nu\|_p)^p\Big)^{q/p}\Big)^{1/q}. \tag{5.27}$$

Theorem 5.15 ([23]). *For a discrete cover* Θ, $0 < p, q \le \infty$, *and* $\alpha > 0$, *if* (5.17) *is obeyed, then the (quasi-)seminorms* $|\cdot|_{B^\alpha_{p,q}(\Theta)}$, $|\cdot|^T_{B^\alpha_{p,q}(\Theta)}$, *and* $|\cdot|^A_{B^\alpha_{p,q}(\Theta)}$ *are equivalent.*

Proof. By (5.5) and (5.2) we have that for any $m \in \mathbb{Z}$,

$$\begin{aligned}\|(T_{m+1} - T_m)f\|_p &\le C(\|f - T_{m+1}f\|_p + \|f - T_mf\|_p)\\ &\le C(\omega_{\Theta,r}(f, 2^{-(m+1)})_p + \omega_{\Theta,r}(f, 2^{-m})_p)\\ &\le C\omega_{\Theta,r}(f, 2^{-(m+1)})_p.\end{aligned}$$

This gives

$$|f|^T_{B^\alpha_{p,q}(\Theta)} \le C|f|_{B^\alpha_{p,q}(\Theta)}.$$

Using representation (4.12), we have

$$T_{m+1}f - T_mf = \sum_{\nu\in\mathcal{M}_m} d_\nu(f)F_\nu,$$

and by the equivalence (4.14) we get that for $0 < p < \infty$,

$$\|(T_{m+1} - T_m)f\|_p^p \sim \sum_{\nu\in\mathcal{M}_m} \|d_\nu(f)F_\nu\|_p^p,$$

and for $p = \infty$,

$$\|(T_{m+1} - T_m)f\|_\infty \sim \sup_{\nu\in\mathcal{M}_m} \|d_\nu(f)F_\nu\|_\infty.$$

Since for $\nu = (\eta, \theta, \beta) \in \mathcal{M}_m$, $|\eta_\nu| \sim 2^{-m}$, this yields

$$|f|^A_{B^\alpha_{p,q}(\Theta)} \le C|f|^T_{B^\alpha_{p,q}(\Theta)}.$$

It remains to prove that

$$|f|_{B^\alpha_{p,q}(\Theta)} \le C|f|^A_{B^\alpha_{p,q}(\Theta)}.$$

We only consider the least favorable case where $1 < p < q < \infty$. Let $f = \sum_{\nu\in\mathcal{M}} a_\nu F_\nu$, be a "near-best" atomic decomposition in the following sense:

$$\begin{aligned}\Big(\sum_{m\in\mathbb{Z}} 2^{m\alpha q}\Big(\sum_{\nu\in\mathcal{M}_m} \|a_\nu F_\nu\|_p^p\Big)^{q/p}\Big)^{1/q} &\le 2^{-\alpha} a_2^\alpha \Big(\sum_{m\in\mathbb{Z}}\Big(\sum_{\nu\in\mathcal{M}_m} \big(|\eta_\nu|^{-\alpha}\|a_\nu F_\nu\|_p\big)^p\Big)^{q/p}\Big)^{1/q}\\ &\le 2^{-\alpha} a_2^\alpha 2|f|^A_{B^\alpha_{p,q}(\Theta)}.\end{aligned}$$

For any ellipsoid $\sigma \in \Theta_j$, using (4.13) and (4.14), we have

$$\begin{aligned}\omega_r(f,\sigma)_p &\le \omega_r\Big(\sum_{\nu\in\mathcal{M}_m:\, m<j,\eta_\nu\cap\sigma\neq\emptyset} a_\nu F_\nu, \sigma\Big)_p + C\Big\|\sum_{\nu\in\mathcal{M}_m:\, m\ge j,\eta_\nu\cap\sigma\neq\emptyset} a_\nu F_\nu\Big\|_p\\ &\le C\sum_{k=1}^\infty\Big(\sum_{\nu:\,\eta_\nu\in\Theta_{j-k},\eta_\nu\cap\sigma\neq\emptyset} |a_\nu|^p \omega_r(F_\nu,\sigma)_p^p\Big)^{1/p}\\ &\quad + C\sum_{k=0}^\infty\Big(\sum_{\nu:\,\eta_\nu\in\Theta_{j+k},\eta_\nu\cap\sigma\neq\emptyset} \|a_\nu F_\nu\|_p^p\Big)^{1/p}\\ &\le C\sum_{k=1}^\infty\Big(\sum_{\nu:\,\eta_\nu\in\Theta_{j-k},\eta_\nu\cap\sigma\neq\emptyset} 2^{-k-a_6 rkp}\|a_\nu F_\nu\|_p^p\Big)^{1/p}\\ &\quad + C\sum_{k=0}^\infty\Big(\sum_{\nu:\,\eta_\nu\in\Theta_{j+k},\eta_\nu\cap\sigma\neq\emptyset} \|a_\nu F_\nu\|_p^p\Big)^{1/p}.\end{aligned}$$

We use this in the definition of $|f|_{B^\alpha_{p,q}(\Theta)}$ to obtain

$$\begin{aligned}|f|^q_{B^\alpha_{p,q}} &\le C\sum_{m\in\mathbb{Z}}\Big(\sum_{\sigma\in\Theta_m}\Big[\sum_{k=1}^\infty\Big(\sum_{\nu:\,\eta_\nu\in\Theta_{m-k},\eta_\nu\cap\sigma\neq\emptyset} 2^{m\alpha p-k-a_6 rkp}\|a_\nu F_\nu\|_p^p\Big)^{1/p}\Big]^p\Big)^{q/p}\\ &\quad + C\sum_{m\in\mathbb{Z}}\Big(\sum_{\sigma\in\Theta_m}\Big[\sum_{k=0}^\infty\Big(\sum_{\nu:\,\eta_\nu\in\Theta_{m+k},\eta_\nu\cap\sigma\neq\emptyset} 2^{m\alpha p}\|a_\nu F_\nu\|_p^p\Big)^{1/p}\Big]^p\Big)^{q/p}\\ &=: C(\Sigma_1 + \Sigma_2).\end{aligned}$$

To estimate Σ_1, we apply Hölder's inequality and condition (5.17) to get

$$\Sigma_1 \le C \sum_{m\in\mathbb{Z}} \Bigg(\sum_{\sigma\in\Theta_m} \Bigg(\sum_{k=1}^{\infty} \sum_{\nu:\,\eta_\nu\in\Theta_{m-k},\eta_\nu\cap\sigma\ne\emptyset} 2^{-k(a_6 r-\alpha)p/2} 2^{-k} 2^{(m-k)\alpha p} \|a_\nu F_\nu\|_p^p \Bigg) \times \Bigg(\sum_{k=1}^{\infty} 2^{-k(a_6 r-\alpha)p'/2} \Bigg)^{p/p'} \Bigg)^{q/p}$$

$$\le C \sum_{m\in\mathbb{Z}} \Bigg(\sum_{\sigma\in\Theta_m} \sum_{k=1}^{\infty} \sum_{\nu:\,\eta_\nu\in\Theta_{m-k},\eta_\nu\cap\sigma\ne\emptyset} 2^{-k(a_6 r-\alpha)p/2} 2^{-k} 2^{(m-k)\alpha p} \|a_\nu F_\nu\|_p^p \Bigg)^{q/p}.$$

In going further, we switch the order of summation, use Lemma 2.19, apply Hölder's inequality with $s := q/p > 1$, and switch the order again to obtain

$$\Sigma_1 \le C \sum_{m\in\mathbb{Z}} \Bigg(\sum_{k=1}^{\infty} 2^{-k(a_6 r-\alpha)p/2} 2^{-k} \max_{\eta\in\Theta_{m-k}} \#\{\sigma\in\Theta_m : \sigma\cap\eta\ne\emptyset\} \sum_{\nu\in\mathcal{M}_{m-k}} 2^{(m-k)\alpha p} \|a_\nu F_\nu\|_p^p \Bigg)^{q/p}$$

$$\le C \sum_{m\in\mathbb{Z}} \Bigg(\sum_{k=1}^{\infty} 2^{-k(a_6 r-\alpha)p/2} \sum_{\nu\in\mathcal{M}_{m-k}} 2^{(m-k)\alpha p} \|a_\nu F_\nu\|_p^p \Bigg)^{q/p}$$

$$\le C \sum_{m\in\mathbb{Z}} \Bigg[\sum_{k=1}^{\infty} 2^{-k(a_6 r-\alpha)q/4} \Bigg(\sum_{\nu\in\mathcal{M}_{m-k}} 2^{(m-k)\alpha p} \|a_\nu F_\nu\|_p^p \Bigg)^{q/p} \Bigg] \Bigg[\sum_{k=1}^{\infty} 2^{-k(a_6 r-\alpha)ps'/4} \Bigg]^{q/ps'}$$

$$\le C \sum_{m\in\mathbb{Z}} \sum_{j=-\infty}^{m-1} 2^{-(m-j)(a_6 r-\alpha)q/4} \Bigg(\sum_{\nu\in\mathcal{M}_j} 2^{j\alpha p} \|a_\nu F_\nu\|_p^p \Bigg)^{q/p}$$

$$\le C \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \Bigg(\sum_{\nu\in\mathcal{M}_j} \|a_\nu F_\nu\|_p^p \Bigg)^{q/p} \sum_{m=j+1}^{\infty} 2^{-(m-j)(a_6 r-\alpha)q/4}$$

$$\le C\big(|f|^A_{B^\alpha_{p,q}(\Theta)}\big)^q.$$

We estimate Σ_2 in a similar fashion. Recall that an ellipsoid $\eta \in \Theta_{m+k}$, $k \ge 0$, only intersects with a bounded number of ellipsoids $\sigma \in \Theta_m$. Applying Hölder's inequality, switching the order of summation, and applying Hölder's inequality again with $s := q/p > 1$, we get

$$\Sigma_2 \le C \sum_{m\in\mathbb{Z}} \Bigg[\sum_{\sigma\in\Theta_m} \Bigg(\sum_{k=0}^{\infty} 2^{k\alpha p/2} \sum_{\nu:\,\eta_\nu\in\Theta_{m+k},\eta_\nu\cap\sigma\ne\emptyset} 2^{(m-k)\alpha p} \|a_\nu F_\nu\|_p^p \Bigg) \Bigg(\sum_{k=0}^{\infty} 2^{-k\alpha p'/2} \Bigg)^{p/p'} \Bigg]^{q/p}$$

$$\le C \sum_{m\in\mathbb{Z}} \Bigg[\sum_{\sigma\in\Theta_m} \sum_{k=0}^{\infty} 2^{k\alpha p/2} \sum_{\nu:\,\eta_\nu\in\Theta_{m+k},\eta_\nu\cap\sigma\ne\emptyset} 2^{(m-k)\alpha p} \|a_\nu F_\nu\|_p^p \Bigg]^{q/p}$$

$$\le C\sum_{m\in\mathbb{Z}}\left[\sum_{k=0}^{\infty}2^{k\alpha p/2}\sum_{\nu\in\mathcal{M}_{m+k}}2^{(m-k)\alpha p}\|a_\nu F_\nu\|_p^p\right]^{q/p}$$
$$\le C\sum_{m\in\mathbb{Z}}\left[\sum_{k=0}^{\infty}2^{k\alpha q/4}\left(\sum_{\nu\in\mathcal{M}_{m+k}}2^{(m-k)\alpha p}\|a_\nu F_\nu\|_p^p\right)^{q/p}\right]\left[\sum_{k=0}^{\infty}2^{-k\alpha ps'/4}\right]^{q/ps'}$$
$$\le C\sum_{m\in\mathbb{Z}}\sum_{j=m}^{\infty}2^{-(j-m)\alpha q/4}\left(\sum_{\nu\in\mathcal{M}_j}2^{j\alpha p}\|a_\nu F_\nu\|_p^p\right)^{q/p}$$
$$\le C\sum_{j\in\mathbb{Z}}\left(\sum_{\nu\in\mathcal{M}_j}2^{j\alpha p}\|a_\nu F_\nu\|_p^p\right)^{q/p}\sum_{m=-\infty}^{j}2^{-(j-m)\alpha q/4}$$
$$\le C\left(|f|^A_{B^\alpha_{p,q}(\Theta)}\right)^q.$$ □

Theorem 5.15 provides a pointwise variable anisotropic variant of wavelet characterization of Besov spaces (see [46] for the case of fixed anisotropy). Namely, since $\|F_\nu\|_p \sim |\eta_\nu|^{1/p-1/2}$ for $\nu = (\eta_\nu, \theta_\nu, \beta_\nu)$, the characterization $B^\alpha_{p,q}(\Theta) \sim B^\alpha_{p,q}(\Theta)^T$ gives

$$|f|_{B^\alpha_{p,q}(\Theta)} \sim \left(\sum_{m\in\mathbb{Z}}\left(2^{m(\alpha+1/2-1/p)}\left(\sum_{\nu\in\mathcal{M}_m}|d_\nu(f)|^p\right)^{1/p}\right)^q\right)^{1/q}$$

for $0 < p, q < \infty$. This form of characterization of Besov spaces is exactly the same as characterizations over spaces of homogeneous type [33, Theorem 4.21], except that here the smoothness index α is not bounded from above by a constant related to the geometry of the space (see (2.4)), and the indices p, q are similarly not bounded from below.

5.4 Adaptive approximation using two-level splits

Our goal is approximating functions in the p-norm using N-term adaptive two-level split elements. Let

$$B^\alpha_\tau(\Theta) := B^\alpha_{\tau\tau}(\Theta),$$

where

$$\frac{1}{\tau} = \alpha + \frac{1}{p}. \tag{5.28}$$

Recall that by (5.9) we have the embedding

$$B^\alpha_\tau(\Theta) \subset L_p(\mathbb{R}^n).$$

Thus, if $f \in B^\alpha_\tau(\Theta)$, then $f \in L_p(\mathbb{R}^n)$, and we have a representation $f = \sum_{\nu\in\mathcal{M}} d_\nu(f)F_\nu$ in $L_p(\mathbb{R}^n)$. We claim that

$$|f|_{B^\alpha_\tau(\Theta)} \sim \Big(\sum_{\nu\in\mathcal{M}} \|d_\nu(f)F_\nu\|_p^\tau\Big)^{1/\tau}. \tag{5.29}$$

Indeed, by (5.26), the equivalence $\|F_\nu\|_p \sim |\eta_\nu|^{1/p-1/2}$, and (5.28) we have

$$\begin{aligned}|f|_{B^\alpha_\tau(\Theta)} &\sim \Big(\sum_{m\in\mathbb{Z}} 2^{\alpha m\tau} \sum_{\nu\in\mathcal{M}_m} \|d_\nu(f)F_\nu\|_\tau^\tau\Big)^{1/\tau}\\ &\sim \Big(\sum_{m\in\mathbb{Z}} 2^{\alpha m\tau} \sum_{\nu\in\mathcal{M}_m} |d_\nu(f)|^\tau 2^{m(1/2-1/\tau)\tau}\Big)^{1/\tau}\\ &\sim \Big(\sum_{m\in\mathbb{Z}} 2^{\alpha m\tau} \sum_{\nu\in\mathcal{M}_m} \|d_\nu(f)F_\nu\|_p^\tau 2^{-m\alpha\tau}\Big)^{1/\tau}\\ &= \Big(\sum_{\nu\in\mathcal{M}} \|d_\nu(f)F_\nu\|_p^\tau\Big)^{1/\tau}.\end{aligned}$$

Let us define the nonlinear set of all N-term two-level split elements by

$$\Sigma_N := \Big\{\sum_{i=1}^N a_i F_{\nu_i} : F_{\nu_i} \in \mathcal{M}, 1 \le i \le N\Big\}.$$

We denote the degree of nonlinear approximation from Σ_N in the p-norm by

$$\sigma_N(f)_p := \inf_{g\in\Sigma_N} \|f-g\|_p.$$

We have the following Jackson theorem.

Theorem 5.16. *If $f \in B^\alpha_\tau(\Theta)$, where $\alpha > 0$, $0 < p < \infty$, and $\tau > 0$ satisfy* (5.28)*, then*

$$\sigma_N(f)_p \le cN^{-\alpha}|f|_{B^\alpha_\tau(\Theta)}.$$

Proof. The proof follows the method of [50] (which can be applied in a more general setting). Since $f \in B^\alpha_\tau(\Theta)$, by the embedding $f \in L_p(\mathbb{R}^n)$ and (5.29) we have

$$\mathcal{N}_\tau(f) := \Big(\sum_{\nu\in\mathcal{M}} \|d_\nu(f)F_\nu\|_p^\tau\Big)^{1/\tau} \sim |f|_{B^\alpha_\tau(\Theta)}.$$

Let us reorder the two-scale split elements by their significance:

$$\|d_{\nu_1}(f)F_{\nu_1}\|_p \ge \|d_{\nu_2}(f)F_{\nu_2}\|_p \ge \cdots.$$

We then denote the reordered elements $\Phi_i := d_{v_i}(f)F_{v_i}$, $i = 1, 2, \ldots$, and define

$$f_N := \sum_{i=1}^{N} \Phi_i \in \Sigma_N.$$

Since $\sigma_N(f)_p \leq \|f - f_N\|_p$, it is sufficient to prove that

$$\|f - f_N\|_p \leq cN^{-\alpha}\mathcal{N}_\tau(f). \tag{5.30}$$

Case I: $0 < p < 1$. To estimate $\|f - f_N\|_p$, we will use the following inequality for an ordered nonnegative scalar sequence $\{a_i\}_{i\in\mathbb{N}}$, $a_1 \geq a_2 \geq \cdots$ and $0 < \tau < p$ [50, Appendix B]:

$$\left(\sum_{i=N+1}^{\infty} a_i^p\right)^{1/p} \leq N^{1/p-1/\tau}\left(\sum_{i=1}^{\infty} a_i^\tau\right)^{1/\tau}.$$

Applying this with $a_i := \|\Phi_i\|_p$ gives (5.30):

$$\begin{aligned}
\|f - f_N\|_p &= \left\|\sum_{i=N+1}^{\infty} \Phi_i\right\|_p \\
&\leq \left(\sum_{i=N+1}^{\infty} \|\Phi_i\|_p^p\right)^{1/p} \\
&\leq N^{1/p-1/\tau}\left(\sum_{i=1}^{\infty} \|\Phi_i\|_p^\tau\right)^{1/\tau} \\
&= N^{-\alpha}\mathcal{N}_\tau(f).
\end{aligned}$$

Case II: $1 \leq p < \infty$. Since $\Phi_i = d_{v_i}(f)F_{v_i}$, we have that $E_i := \operatorname{supp}(\Phi_i) = \operatorname{supp}(F_{v_i}) = \eta_{v_i}$. We claim that there exists $c(\mathbf{p}(\Theta), p) > 0$ such that for $x \in E_m$,

$$\sum_{x\in E_i, |E_i|\geq|E_m|} \left(\frac{|E_m|}{|E_i|}\right)^{1/p} \leq c. \tag{5.31}$$

Indeed, depending on the parameters of the cover, any ellipsoid $\theta \in \Theta_j$ intersects with a only bounded number of ellipsoids $\eta \in \Theta$ such that $|\eta| \geq |\theta|$, and they only appear in levels lower than $j + c_1$ for some fixed constant c_1. Since $|\theta| \geq a_1 2^{-j}$ and $|\eta| \leq a_2 2^{-k}$, for $\eta \in \Theta_k$, we get

$$\sum_{x\in\theta, |\eta|\geq|\theta|} \left(\frac{|\theta|}{|\eta|}\right)^{1/p} \leq C \sum_{k=-\infty}^{j+c_1} 2^{(k-j)/p} \leq C.$$

We need the following lemma.

Lemma 5.17. *Let* $H := \sum_{i\in\Lambda} |\Phi_i|$, *where* $\#\Lambda \le M$, *and* $\|\Phi_i\|_p \le L$ *for* $i \in \Lambda$, $1 \le p < \infty$. *Then*

$$\|H\|_p \le CLM^{1/p}.$$

Proof. The claim is obvious for $p = 1$. Let $1 < p < \infty$. Recall that $\|F_\nu\|_q \sim |\eta_\nu|^{1/q-1/2}$, which implies

$$\begin{aligned}\|\Phi_i\|_\infty &\le C\big|d_{\nu_i}(f)\big||\eta_{\nu_i}|^{-1/2}\\ &\le C\big|d_{\nu_i}(f)\big|\|F_{\nu_i}\|_p|\eta_{\nu_i}|^{-1/p}\\ &= C\|\Phi_i\|_p|\eta_{\nu_i}|^{-1/p}\\ &\le CL|E_i|^{-1/p}.\end{aligned}$$

Therefore

$$\begin{aligned}\|H\|_p &\le \Big\|\sum_{i\in\Lambda}\|\Phi_i\|_\infty \mathbf{1}_{E_i}(\cdot)\Big\|_p\\ &\le CL\Big\|\sum_{i\in\Lambda}|E_i|^{-1/p}\mathbf{1}_{E_i}(\cdot)\Big\|_p.\end{aligned}$$

Denote $E := \bigcup_{i\in\Lambda} E_i$ and $\varepsilon(x) := \min_{i\in\Lambda}\{|E_i| : x \in E_i\}$ for $x \in E$. For $x \notin E$, set $\varepsilon(x) = 0$. By (5.31) we have

$$\sum_{i\in\Lambda}|E_i|^{-1/p}\mathbf{1}_{E_i}(x) \le C\varepsilon(x)^{-1/p}.$$

Therefore

$$\begin{aligned}\|H\|_p &\le CL\big\|\varepsilon(\cdot)^{-1/p}\big\|_p\\ &= CL\Big(\int_E \varepsilon(x)^{-1}dx\Big)^{1/p}\\ &\le CL\Big(\sum_{i\in\Lambda}|E_i|^{-1}\int_{\mathbb{R}^n}\mathbf{1}_{E_i}\Big)^{1/p}\\ &= CLM^{1/p}.\end{aligned}$$ □

We may now complete the proof of (5.30) for the case $1 \le p < \infty$. Denote

$$\Lambda_k := \{i : 2^{-k}\mathcal{N}_\tau(f) < \|\Phi_i\|_p \le 2^{-k+1}\mathcal{N}_\tau(f)\},\quad k \ge 1.$$

Recall that the sequence space l_τ embeds into the weak sequence space $l_{\tau,\infty}$, and so for any nonnegative sequence $a = \{a_j\}$, we have $\|a\|_{\tau,\infty} \le \|a\|_\tau$. Thus

$$\begin{aligned}
\#\Lambda_m &\le \sum_{k\le m} \#\Lambda_k \\
&= \# \bigcup_{k\le m} \Lambda_k \\
&= \#\{i : 2^{-m}\mathcal{N}_\tau(f) < \|\Phi_i\|_p\} \\
&\le \big\|\{\|\Phi_i\|_p\}\big\|_{\tau,\infty}^{\tau} 2^{m\tau}\mathcal{N}_\tau(f)^{-\tau} \\
&\le \big\|\{\|\Phi_i\|_p\}\big\|_{\tau}^{\tau} 2^{m\tau}\mathcal{N}_\tau(f)^{-\tau} \\
&= \mathcal{N}_\tau(f)^\tau 2^{m\tau}\mathcal{N}_\tau(f)^{-\tau} = 2^{m\tau}.
\end{aligned}$$

Let $N := \sum_{k\le m} \#\Lambda_k \le 2^{m\tau}$, $f_N := \sum_{i\in\Lambda_k, k\le m} \Phi_i$, and $H_k := \sum_{i\in\Lambda_k} \Phi_i$. We apply Lemma 5.17, the estimate $\#\Lambda_k \le C2^{k\tau}$, and (5.28) to obtain (5.30) for these particular cases of N:

$$\begin{aligned}
\|f - f_N\|_p &\le \Big\|\sum_{k=m+1}^{\infty} H_k\Big\|_p \\
&\le \sum_{k=m+1}^{\infty} \|H_k\|_p \\
&\le C \sum_{k=m+1}^{\infty} 2^{-k+1}\mathcal{N}_\tau(f)(\#\Lambda_k)^{1/p} \\
&\le C\mathcal{N}_\tau(f) \sum_{k=m+1}^{\infty} 2^{-k+1}2^{k\tau/p} \\
&\le C\mathcal{N}_\tau(f)2^{-m(1-\tau/p)} \\
&= C\mathcal{N}_\tau(f)2^{-m\tau\alpha} \\
&\le C\mathcal{N}_\tau(f)N^{-\alpha}.
\end{aligned}$$

The proof for the cases where N is not perfectly aligned with a sum of slice sizes $\#\Lambda_k$ is almost identical, and we omit it. □

5.5 Anisotropic Campanato spaces

The Campanato spaces are a family of smoothness spaces, where the smoothness is measured locally.

Definition 5.18. Let Θ be an ellipsoid cover (continuous or discrete) over $\mathbb{R}^n$, and let $\alpha \ge 0, 1 \le q \le \infty$. We define the *Campanato-type space* $\mathcal{C}^\alpha_{q,r}(\Theta)$ as the space of functions

$f \in L_q^{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{C}^\alpha_{q,r}(\Theta)} := \sup_{\theta\in\Theta} |\theta|^{-\alpha}\omega_r(f,\theta)_q < \infty, \tag{5.32}$$

where $\omega_r(f,\theta)_q$ is defined in (1.13), and $r \geq 1$ satisfies (5.17). We denote $\mathcal{C}^\alpha_q(\Theta) := \mathcal{C}^\alpha_{q,r}(\Theta)$, where r is the smallest integer that satisfies (5.17).

A few remarks are in order.

(i) Observe that $\mathcal{C}^\alpha_\infty(\Theta) = B^\alpha_{\infty,\infty}(\Theta)$, where $B^\alpha_{\infty,\infty}(\Theta)$ is the Besov space defined by (5.16).

(ii) By (1.47), for any bounded convex domain $\Omega \subset \mathbb{R}^n$ and $f \in L_q(\Omega)$, we have the equivalence

$$E_{r-1}(f,\Omega)_q := \inf_{P\in\Pi_{r-1}} \|f-P\|_{L_q(\Omega)} \sim \omega_r(f,\Omega)_q, \tag{5.33}$$

where the equivalence constants are independent of f and Ω. This leads to the following equivalent form of the norm:

$$\|f\|_{\mathcal{C}^\alpha_{q,r}(\Theta)} \sim \sup_{\theta\in\Theta} |\theta|^{-\alpha} E_{r-1}(f,\theta)_q. \tag{5.34}$$

(iii) By Theorem 2.23 we can replace the ellipsoids in (5.32) by anisotropic balls to get an equivalent norm.

(iv) It is easily seen that Campanato-type spaces constructed over equivalent covers (see Definition 2.27) are equivalent.

(v) It is readily seen that $\mathcal{C}^\alpha_{q,r}(\Theta)/\Pi_{r-1}$ is a Banach space.

(vi) In Section 6.8, we will identify the Campanato spaces as duals of Hardy spaces. As a result (see Corollary 6.65), we will see that $\mathcal{C}^\alpha_{q,r_1}(\Theta)/\Pi_{r_1-1} \sim \mathcal{C}^\alpha_{q,r_2}(\Theta)/\Pi_{r_2-1}$ for $\alpha > 1/q$ and sufficiently high orders r_1, r_2.

Theorem 5.19. *For a discrete cover* Θ, $1 \leq q < \infty$, $\alpha \geq 1/q$, *and* $r \geq 1$, *the smallest integer satisfying condition* (5.17), *there exists a constant* $c(\mathbf{p}(\Theta),\alpha,r,q) > 0$ *such that for all* $f \in \mathcal{C}^\alpha_q(\Theta) := \mathcal{C}^\alpha_{q,r}(\Theta)$,

$$\|T_m f\|_{\mathcal{C}^\alpha_q(\Theta)} \leq c\|f\|_{\mathcal{C}^\alpha_q(\Theta)}, \quad \forall m \in \mathbb{Z}, \tag{5.35}$$

where $T_m = T_{m,q}$, $m \in \mathbb{Z}$, *are the operators* (3.36) *of order r defined over the cover* Θ *if it is discrete or over a "discretization" of a continuous cover per Theorem* 2.31.

Proof. Without loss of generality, Θ is a discrete cover, and the operators T_m are well defined over it. We need to show that for any $j \in \mathbb{Z}$ and $\theta \in \Theta_m$

$$|\theta|^{-\alpha}\omega_r(T_j f,\theta)_q \leq C\|f\|_{\mathcal{C}^\alpha_q(\Theta)}.$$

There are two cases.

Case I: $m \le j$. Let

$$\Lambda(\theta, j) := \{\eta \in \Theta_j : \eta \cap \theta \neq \emptyset\}, \quad \Omega(\theta, j) := \bigcup_{\eta \in \Lambda(\theta, j)} \eta.$$

By Proposition 1.14, (3.38), and Lemma 2.19, we have

$$\begin{aligned} \omega_r(f - T_j f, \theta)_q^q &\le C \|f - T_j f\|_{L_q(\theta)}^q \\ &\le C \sum_{\eta \in \Lambda(\theta, j)} \|f - T_j f\|_{L_q(\eta)}^q \\ &\le C \sum_{\eta' \in \Theta_j, \eta' \cap \Omega(\theta, j) \neq \emptyset} \omega_r(f, \eta')^q \\ &\le C 2^{j-m} 2^{-j\alpha q} \|f\|_{\mathcal{C}_q^\alpha(\Theta)}^q. \end{aligned}$$

Using $\alpha \ge 1/q$ and $m \le j$, this gives

$$\begin{aligned} 2^{m\alpha} \omega_r(T_j f, \theta)_q &\le 2^{m\alpha} \omega_r(T_j f - f, \theta)_q + 2^{m\alpha} \omega_r(f, \theta)_q \\ &\le C\big(2^{(m-j)(\alpha - 1/q)} \|f\|_{\mathcal{C}_{q',l}^s(\Theta)} + \|f\|_{\mathcal{C}_q^\alpha(\Theta)}\big) \\ &\le C \|f\|_{\mathcal{C}_q^\alpha(\Theta)}. \end{aligned}$$

Case II: $m > j$. We apply a telescopic sum argument:

$$\omega_r(T_j f, \theta)_q \le \sum_{k=j}^{m-1} \omega_r\big((T_k - T_{k+1})f, \theta\big)_q + \omega_r(T_m f, \theta)_q.$$

Assume for a moment that for $\beta := a_6 r - \alpha > 0$ and $k < m$,

$$2^{m\alpha} \omega_r\big((T_k - T_{k+1})f, \theta\big)_q \le c 2^{(k-m)\beta} \|f\|_{\mathcal{C}_q^\alpha(\Theta)}. \tag{5.36}$$

Then

$$\begin{aligned} 2^{m\alpha} \omega_r(T_j f, \theta)_q &\le \sum_{k=j}^{m-1} 2^{m\alpha} \omega_r\big((T_k - T_{k+1})f, \theta\big)_q + 2^{m\alpha} \omega_r(T_m f, \theta)_q \\ &\le C\left(\sum_{k=j}^{m-1} 2^{(k-m)\beta}\right) \|f\|_{\mathcal{C}_q^\alpha(\Theta)} + C \|f\|_{\mathcal{C}_q^\alpha(\Theta)} \\ &\le C \|f\|_{\mathcal{C}_q^\alpha(\Theta)}. \end{aligned}$$

To prove (5.36), we use the “two-level split” representation (4.12) at the level k over θ:

$$\big((T_k - T_{k+1})f\big)(x) = \sum_{\nu \in \mathcal{M}_k,\, \eta_\nu \cap \theta \neq \emptyset} a_\nu F_\nu(x), \quad \forall x \in \theta.$$

By Lemma 4.13, for $\theta \in \Theta_m$, $F_v \in \mathcal{F}_k$, $k \le m$, such that $\eta_v \cap \theta \neq \emptyset$, we have

$$\begin{aligned}\omega_r(F_v, \theta)_q^q &\le C 2^{(k-m)(1/q+a_6 r)q} \|F_v\|_q^q \\ &= C 2^{(k-m)(1/q+\alpha+\beta)q} \|F_v\|_q^q.\end{aligned}$$

Let

$$\Lambda(\theta, k+1) := \{\eta_v \in \Theta_{k+1} : \eta_v \cap \theta \neq \emptyset\}, \quad \Omega(\theta, k+1) := \bigcup_{\eta_v \in \Lambda(\theta,k+1)} \eta_v,$$

and

$$\Lambda(\theta, k) := \{\theta_v \in \Theta_k : \theta_v \cap \Omega(\theta, k+1) \neq \emptyset\}, \quad \Omega(\theta, k+1) := \bigcup_{\theta_v \in \Lambda(\theta,k)} \theta_v.$$

Since $k < m$, $\#\Lambda(\theta, k+1)$ is bounded, which also implies that $\#\Lambda(\theta, k)$ is bounded. This, together with Theorem 4.8, yields

$$\begin{aligned}\omega_r\big((T_k - T_{k+1})f, \theta\big)_q^q &\le C \sum_{v \in \mathcal{M}_k, \eta_v \in \Lambda(\theta,k+1)} \omega_r(a_v F_v, \theta)_q^q \\ &\le C 2^{(k-m)(1/q+\alpha+\beta)q} \sum_{v \in \mathcal{M}_k, \eta_v \in \Lambda(\theta,k+1)} \|a_v F_v\|_q^q \\ &\le C 2^{(k-m)(1/q+\alpha+\beta)q} \big\|(T_k - T_{k+1})f\big\|_{L_q(\Omega(\theta,k+1))}^q.\end{aligned}$$

By (3.38) we also have

$$\begin{aligned}\big\|(T_k - T_{k+1})f\big\|_{L_q(\Omega(\theta,k+1))} &\le \|T_k f - f\|_{L_q(\Omega(\theta,k))} + \|f - T_{k+1} f\|_{L_q(\Omega(\theta,k+1))} \\ &\le C\Big(\sum_{\theta_v \in \Lambda(\theta,k)} \|T_k f - f\|_{L_q(\theta_v)} + \sum_{\eta_v \in \Lambda(\theta,k+1)} \|T_k f - f\|_{L_q(\eta_v)}\Big) \\ &\le C\Big(\sum_{\theta' \in \Theta_k, \theta' \cap \Omega(\theta,k) \neq \emptyset} \omega_r(f, \theta')_q + \sum_{\eta' \in \Theta_{k+1}, \eta' \cap \Omega(\theta,k+1) \neq \emptyset} \omega_r(f, \eta')_q\Big) \\ &\le C 2^{-k\alpha} \|f\|_{\mathcal{C}_q^\alpha}.\end{aligned}$$

We apply the last two estimates to conclude (5.36) by

$$\begin{aligned}2^{m\alpha} \omega_r\big((T_k - T_{k+1})f, \theta\big)_q &\le C 2^{m\alpha} 2^{(k-m)(1/q+\alpha+\beta)} 2^{-k\alpha} \|f\|_{\mathcal{C}_q^\alpha(\Theta)} \\ &\le C 2^{(k-m)\beta} \|f\|_{\mathcal{C}_q^\alpha(\Theta)}.\end{aligned}$$

□

6 Anisotropic Hardy spaces

The theory of real Hardy spaces in more "geometric" settings has received much attention. Coifman and Weiss [19, 20] pioneered this field in the 1970s. Then Folland and Stein [39] in the 1980s studied Hardy spaces over homogeneous groups. The complete real-variable theory of Hardy spaces on spaces of homogeneous type appears in [44]. In this general setting the Hardy spaces are limited to the range $d/(d+\eta) < p \le 1$, where d is the "upper dimension" defined in (2.3), and $0 < \eta < 1$ is the Lipschitz regularity of the wavelets constructed in [4]. As we will see, for p values "closer" to zero, we need the machinery of higher order local approximation by algebraic polynomials or, conversely, higher order of vanishing moments of the building blocks, atoms, molecules, etc. The Hardy spaces we construct over ellipsoid covers of $\mathbb{R}^n$ have the required structure that allows us to deal with the full range $0 < p \le 1$ and to generalize the Hardy spaces of Bownik [7] to the case of pointwise variable anisotropy.

6.1 Ellipsoid maximal functions

Definition 6.1. Let Θ be a continuous ellipsoid cover. We define the following ellipsoid maximal function for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$:

$$M_\Theta f(x) := \sup_{t\in\mathbb{R}} \frac{1}{|\theta(x,t)|} \int_{\theta(x,t)} |f|. \tag{6.1}$$

Lemma 6.2. *Let Θ be a continuous ellipsoid cover. Then for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$,*

$$M_B f(x) \sim M_\Theta f(x), \quad \forall x \in \mathbb{R}^n, \tag{6.2}$$

where $M_B f$ is the central maximal function (2.9) *corresponding to the quasi-distance* (2.35)*, and the constants of equivalence depend only on* $\mathbf{p}(\Theta)$.

Proof. Let us fix $x \in \mathbb{R}^n$. By Theorem 2.23, for any anisotropic ball $B_\rho(x,r)$, there exists an ellipsoid $\theta \in \Theta$ with center at x such that $B_\rho(x,r) \subseteq \theta$ and $|\theta| \sim r$. Therefore

$$\frac{1}{|B_\rho(x,r)|} \int_{B_\rho(x,r)} |f| \le C \frac{1}{|\theta|} \int_\theta |f| \le C M_\Theta f(x).$$

Taking the supremum over all balls $B_\rho(x,r)$, $r > 0$, yields the first inequality of (6.2). In the other direction, for $\theta := \theta(x,t)$, we have by definition $\theta \subseteq B_\rho(x,|\theta|)$. Theorem 2.23 yields $|B_\rho(x,|\theta|)| \sim |\theta|$, which implies

$$\frac{1}{|\theta|} \int_\theta |f| \le C \frac{1}{|B_\rho(x,|\theta|)|} \int_{B_\rho(x,|\theta|)} |f| \le C M_B f(x).$$

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Taking the supremum over all ellipsoids $\theta(x,t)$, $t \in \mathbb{R}$, provides the second inequality of (6.2) and concludes the proof. □

Combining Lemma 6.2 with Proposition 2.8 yields a maximal function theorem for the ellipsoid maximal function.

Theorem 6.3. *Let* Θ *be a continuous ellipsoid cover. Then there exists a constant* $C(\mathbf{p}(\Theta), n) > 0$ *such that for all* $f \in L_1(\mathbb{R}^n)$ *and* $\alpha > 0$,

$$\left|\{x : M_\Theta f(x) > \alpha\}\right| \le C\alpha^{-1}\|f\|_1. \tag{6.3}$$

For $1 < p < \infty$, *there exists a constant* $A_p(\mathbf{p}(\Theta), n, p) > 0$ *such that for all* $f \in L_p(\mathbb{R}^n)$,

$$\|M_\Theta f\|_p \le A_p\|f\|_p. \tag{6.4}$$

Let $\mathcal{S}$ denote the Schwartz class of rapidly decreasing C^∞ functions (with respect to the Euclidean metric), and let $\mathcal{S}'$ the dual space of tempered distributions. In this book, we simply call $f \in \mathcal{S}'$ a distribution.

Definition 6.4. For a function $\varphi \in C^N(\mathbb{R}^n)$ and $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le N \le \widetilde{N}$, let

$$\|\varphi\|_{\alpha,\widetilde{N}} := \sup_{y\in\mathbb{R}^n}(1+|y|)^{\widetilde{N}}\left|\partial^\alpha\varphi(y)\right|,$$
$$\|\varphi\|_{N,\widetilde{N}} := \max_{|\alpha|\le N}\|\varphi\|_{\alpha,\widetilde{N}}.$$

We then define the class of normalized Schwartz functions

$$\mathcal{S}_{N,\widetilde{N}} := \{\varphi \in \mathcal{S} : \|\varphi\|_{N,\widetilde{N}} \le 1\}. \tag{6.5}$$

Let Θ be a continuous cover where $\theta(x,t) = M_{x,t}(B^*) + x$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For $\varphi \in \mathcal{S}$, denote the pointwise variable anisotropic dilation

$$\varphi_{x,t}(y) := \left|\det(M_{x,t}^{-1})\right|\varphi(M_{x,t}^{-1}y), \quad y \in \mathbb{R}^n. \tag{6.6}$$

A pointwise variable anisotropic dilated version of a Schwartz function $\varphi \in \mathcal{S}$, corresponding to a point $x \in \mathbb{R}^n$ and scale $t \in \mathbb{R}$, acts on a distribution $f \in \mathcal{S}'$ through a convolution

$$f * \varphi_{x,t}(y) = \left|\det(M_{x,t}^{-1})\right|\langle f, \varphi(M_{x,t}^{-1}(y - \cdot))\rangle.$$

We now provide the pointwise variable anisotropic variants that generalize the classical isotropic maximal functions [61]: the nontangential, the grand nontangential, the radial, the grand radial, and the tangential maximal functions.

Definition 6.5. Let Θ be a continuous ellipsoid cover of $\mathbb{R}^n$. Let $\varphi \in \mathcal{S}$, $f \in \mathcal{S}'$, and $N, \widetilde{N} \in \mathbb{N}$, $N \leq \widetilde{N}$. The *nontangential maximal function* is defined as

$$M_\varphi f(x) := \sup_{t\in\mathbb{R}} \sup_{y\in\theta(x,t)} \left|f * \varphi_{x,t}(y)\right|, \quad x \in \mathbb{R}^n. \tag{6.7}$$

The *grand nontangential maximal function* is defined as

$$M_{N,\widetilde{N}} f(x) := \sup_{\varphi\in\mathcal{S}_{N,\widetilde{N}}} M_\varphi f(x), \quad x \in \mathbb{R}^n. \tag{6.8}$$

The *radial maximal function* is defined as

$$M_\varphi^\circ f(x) := \sup_{t\in\mathbb{R}} \left|f * \varphi_{x,t}(x)\right|, \quad x \in \mathbb{R}^n. \tag{6.9}$$

The *grand radial maximal function* is defined as

$$M_{N,\widetilde{N}}^\circ f(x) := \sup_{\varphi\in\mathcal{S}_{N,\widetilde{N}}} M_\varphi^\circ f(x), \quad x \in \mathbb{R}^n. \tag{6.10}$$

The *tangential maximal function* is defined as

$$T_\varphi^N f(x) := \sup_{t\in\mathbb{R}} \sup_{y\in\mathbb{R}^n} \left|f * \varphi_{x,t}(y)\right| \left(1 + \left|M_{x,t}^{-1}(x-y)\right|\right)^{-N}, \quad x \in \mathbb{R}^n. \tag{6.11}$$

It is easy to see that we have the following pointwise estimates for the radial, nontangential, and tangential maximal functions: for any $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$,

$$M_\varphi^0 f(x) \leq M_\varphi f(x) \leq 2^N T_\varphi^N f(x), \quad x \in \mathbb{R}^n. \tag{6.12}$$

Another relatively simple pointwise equivalence is the following:

Lemma 6.6. *For any* $0 < N \leq \widetilde{N}$ *and* $f \in \mathcal{S}'$,

$$M_{N,\widetilde{N}}^\circ f(x) \leq M_{N,\widetilde{N}} f(x) \leq 2^{\widetilde{N}} M_{N,\widetilde{N}}^\circ f(x), \quad x \in \mathbb{R}^n. \tag{6.13}$$

Proof. The first inequality is obvious. To show the second inequality, note that

$$\begin{aligned} M_{N,\widetilde{N}} f(x) &= \sup\{\left|f * \varphi_{x,t}(x + M_{x,t}y)\right| : y \in B^*, t \in \mathbb{R}, \varphi \in \mathcal{S}_{N,\widetilde{N}}\} \\ &= \sup\{\left|f * \phi_{x,t}(x)\right| : \phi(z) := \varphi(z+y), y \in B^*, t \in \mathbb{R}, \varphi \in \mathcal{S}_{N,\widetilde{N}}\} \\ &= \sup\{M_\phi^\circ f(x) : \phi(z) = \varphi(z+y), y \in B^*, t \in \mathbb{R}, \varphi \in \mathcal{S}_{N,\widetilde{N}}\}. \end{aligned}$$

For $\phi(z) = \varphi(z + y)$ with $y \in B^*$, we have

$$\begin{aligned}\|\phi\|_{N,\widetilde{N}} &= \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1 + |x|)^{\widetilde{N}} |\partial^\alpha \varphi(x + y)| \\ &= \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1 + |x - y|)^{\widetilde{N}} |\partial^\alpha \varphi(x)| \\ &\le 2^{\widetilde{N}} \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1 + |x|)^{\widetilde{N}} |\partial^\alpha \varphi(x)| = 2^{\widetilde{N}} \|\varphi\|_{N,\widetilde{N}}.\end{aligned}$$

Combining the above, we have

$$M_{N,\widetilde{N}} f(x) \le \sup\{M^\circ_\phi f(x) : \phi \in \mathcal{S}, \|\phi\|_{N,\widetilde{N}} \le 2^{\widetilde{N}}\} \le 2^{\widetilde{N}} M^\circ_{N,\widetilde{N}} f(x). \qquad \square$$

Recall that by Theorem 2.28, for any continuous cover, there exists an equivalent pointwise continuous cover. For the most part of this chapter, we will assume this pointwise continuity property, as we will require "maximal sets" to be open.

Theorem 6.7. *Let Θ be a pointwise continuous cover. Then for any $f \in \mathcal{S}'$, $N, \widetilde{N} \in \mathbb{N}$, and $\lambda > 0$, the set*

$$\Omega = \{x \in \mathbb{R}^n : M^\circ_{N,\widetilde{N}} f(x) > \lambda\}$$

is open.

Proof. Let $f \in \mathcal{S}'$. We first observe that for any fixed $\varphi \in \mathcal{S}$ and $t \in \mathbb{R}$,

$$x \mapsto f * \varphi_{x,t}(x)$$

is a continuous function on $\mathbb{R}^n$. Indeed, let $x' \to x$. Then under the assumption that Θ is pointwise continuous, $\|M_{x',t} - M_{x,t}\| \to 0$. This implies that $\varphi_{x',t} \to \varphi_{x,t}$ in $\|\cdot\|_{N,\widetilde{N}}$ for any $N, \widetilde{N} \in \mathbb{N}$. Hence $f * \varphi_{x',t}(x') \to f * \varphi_{x,t}(x)$ as $x' \to x$.

Now, for any $x \in \Omega$, there exist $\varphi \in \mathcal{S}_{N,\widetilde{N}}$ and $t \in \mathbb{R}$ such that

$$|f * \varphi_{x,t}(x)| > \lambda.$$

Since $f * \varphi_{\cdot,t}(\cdot)$ is continuous, we deduce that for x' in a sufficiently small neighborhood of x, $|f * \varphi_{x',t}(x')| > \lambda$. This implies that $x' \in \Omega$, and hence Ω is open. $\square$

The next result is a pointwise variable anisotropic variant of Lemma 3.1.2 in [61]. It enables to relate maximal functions constructed over different Schwartz functions.

Theorem 6.8 ([65]). *Let Θ be a continuous cover of $\mathbb{R}^n$, and let $\varphi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \varphi \neq 0$. Then, for any $\psi \in \mathcal{S}$, $x \in \mathbb{R}^n$, and $t \in \mathbb{R}$, there exists a sequence $\{\eta^k\}_{k=0}^\infty$, $\eta^k \in \mathcal{S}$, such* *that*

$$\psi = \sum_{k=0}^{\infty} \eta^k * \varphi^k \tag{6.14}$$

converges in $\mathcal{S}$, where

$$\varphi^k := \left|\det(M_{x,t+kJ}^{-1} M_{x,t})\right| \varphi(M_{x,t+kJ}^{-1} M_{x,t} \cdot), \quad k \geq 0,$$

where $J > 0$ is given by (2.30). *Furthermore, for any positive integers N, $\widetilde{N}$, and L, there exists a constant $c > 0$, depending on φ, L, N, $\widetilde{N}$, $\mathbf{p}(\Theta)$ but not on ψ, such that*

$$\|\eta^k\|_{N,\widetilde{N}} \leq c 2^{-kL} \|\psi\|_{N+n+1+\lceil L/(a_6 J)\rceil, \widetilde{N}+n+1}. \tag{6.15}$$

Proof. By scaling φ we can assume without loss of generality that $|\widehat{\varphi}(\xi)| \geq 1/2$ for $|\xi| \leq 2$. This assumption only impacts the constant in (6.15). Let $\zeta \in \mathcal{S}$ be such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on B^*, and $\operatorname{supp}(\zeta) \subseteq 2B^*$. We fix $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, denote $M_k := M_{x,t+kJ}$, and define the sequence of functions $\{\zeta_k\}_{k=0}^{\infty}$, where $\zeta_0 := \zeta$ and

$$\zeta_k := \zeta\big((M_{x,t}^{-1} M_k)^T \cdot\big) - \zeta\big((M_{x,t}^{-1} M_{k-1})^T \cdot\big), \quad k \geq 1,$$

where M^T denotes the transpose of a matrix M. We claim that

$$\operatorname{supp}(\zeta_k) \subseteq \{\xi \in \mathbb{R}^n : \; a_5^{-1} 2^{-a_6 J} 2^{a_6 kJ} \leq |\xi| \leq 2 a_3^{-1} 2^{a_4 kJ}\}. \tag{6.16}$$

Indeed, by the properties of ζ, (2.30), and the “shape condition” (2.14) we have

$$\begin{aligned} \xi \in \operatorname{supp}(\zeta_k) &\Rightarrow (M_{x,t}^{-1} M_k)^T(\xi) \in 2B^* \vee (M_{x,t}^{-1} M_{k-1})^T(\xi) \in 2B^* \\ &\Rightarrow \xi \in 2(M_k^{-1} M_{x,t})^T(B^*) \vee \xi \in 2(M_{k-1}^{-1} M_{x,t})^T(B^*) \\ &\Rightarrow \xi \in 2a_3^{-1} 2^{a_4 kJ} B^*. \end{aligned}$$

In the other direction, (2.30) and the properties of ζ yield

$$\begin{aligned} \xi \in (M_{k-1}^{-1} M_{x,t})^T(B^*) &\Rightarrow (M_{x,t}^{-1} M_k)^T(\xi), (M_{x,t}^{-1} M_{k-1})^T(\xi) \in B^* \\ &\Rightarrow \zeta_k(\xi) = 0. \end{aligned}$$

Applying (2.14), we have

$$\xi \notin (M_{k-1}^{-1} M_{x,t})^T(B^*) \Rightarrow |\xi| \geq a_5^{-1} 2^{a_6(k-1)J}.$$

This proves (6.16). Also, by (2.14), for any $\xi \in \mathbb{R}^n$,

$$\left|(M_{x,t}^{-1} M_k)^T \xi\right| \leq \|M_{x,t}^{-1} M_k\| |\xi| \leq a_5 2^{-a_6 kJ} |\xi| \to 0, \quad k \to \infty.$$

From this we deduce that for any $\xi \in \mathbb{R}^n$ and for k large enough, $(M_{x,t}^{-1}M_k)^T\xi \in B^*$. This implies that

$$\sum_{k=0}^{\infty} \zeta_k(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Thus, formally, a Fourier transform of (6.14) is given by

$$\widehat{\psi} = \sum_{k=0}^{\infty} \widehat{\eta^k}\widehat{\varphi}\big((M_{x,t}^{-1}M_k)^T\cdot\big), \quad \widehat{\eta^k} := \frac{\zeta_k}{\widehat{\varphi}((M_{x,t}^{-1}M_k)^T\cdot)}\widehat{\psi}.$$

Observe that η^k is well defined and is in $\mathcal{S}$. Indeed, $\widehat{\eta^k}$ is well defined with $0/0 := 0$, since by our assumption on φ

$$\begin{aligned} \xi \in \operatorname{supp}(\zeta_k) &\Rightarrow \xi \in 2\big(M_k^{-1}M_{x,t}\big)^T(B^*) \\ &\Rightarrow \big|\big(M_{x,t}^{-1}M_k\big)^T\xi\big| \le 2 \\ &\Rightarrow \big|\widehat{\varphi}\big(\big(M_{x,t}^{-1}M_k\big)^T\xi\big)\big| \ge \frac{1}{2}. \end{aligned}$$

From this it is obvious that $\widehat{\eta^k} \in \mathcal{S}$, and so $\eta^k \in \mathcal{S}$. We now proceed to prove (6.15). First, observe that for any $\eta \in \mathcal{S}$ and $N, \widetilde{N} \in \mathbb{N}$,

$$\|\eta\|_{N,\widetilde{N}} \le C(N,\widetilde{N},n)\|\widehat{\eta}\|_{\widetilde{N},N+n+1}. \tag{6.17}$$

Next, we claim that for any $K \in \mathbb{N}$,

$$\max_{|\alpha|\le K}\big\|\partial^\alpha\big(\zeta_k/\widehat{\varphi}\big(\big(M_{x,t}^{-1}M_k\big)^T\cdot\big)\big)\big\|_\infty \le C(K,n,\varphi). \tag{6.18}$$

Indeed, on its support, any partial derivative of $\zeta_k/\widehat{\varphi}((M_{x,t}^{-1}M_k)^T\cdot)$ has a representation of a denominator whose absolute value is bounded from below and a numerator that is a superposition of compositions of partial derivatives of ζ and $\widehat{\varphi}$ with contractive matrices of the type $(M_{x,t}^{-1}M_k)^T$. Using (6.16), (6.17), and (6.18), we obtain

$$\begin{aligned} \|\eta^k\|_{N,\widetilde{N}} &\le C\|\widehat{\eta^k}\|_{\widetilde{N},N+n+1} \\ &\le C \sup_{|\xi|\ge a_5^{-1}2^{-a_6J}2^{a_6kJ}} \max_{|\alpha|\le\widetilde{N}} \big|\partial^\alpha\widehat{\eta^k}(\xi)\big|\big(1+|\xi|\big)^{N+n+1} \\ &\le C \sup_{|\xi|\ge a_5^{-1}2^{-a_6J}2^{a_6kJ}} \max_{|\alpha|\le\widetilde{N}} \big|\partial^\alpha\widehat{\psi}(\xi)\big|\big(1+|\xi|\big)^{N+n+1} \\ &= C \sup_{|\xi|\ge a_5^{-1}2^{-a_6J}2^{a_6kJ}} \max_{|\alpha|\le\widetilde{N}} \big|\partial^\alpha\widehat{\psi}(\xi)\big|\big(1+|\xi|\big)^{N+n+1+\lceil L/(a_6J)\rceil}\big(1+|\xi|\big)^{-\lceil L/(a_6J)\rceil} \end{aligned}$$

$$\le C2^{-kL}\|\widehat{\psi}\|_{\widetilde{N},N+n+1+\lceil L/(a_6J)\rceil}$$
$$\le C2^{-kL}\|\psi\|_{N+n+1+\lceil L/(a_6J)\rceil,\widetilde{N}+n+1}. \qquad \square$$

The next lemma is needed to show that (up to a constant) the grand radial maximal function can be defined using Schwartz functions supported on B^*.

Lemma 6.9. *Let Θ be a continuous cover, and let $N \ge 1$. Denote by $\widetilde{N}$ the minimal integer that satisfies $\widetilde{N} > (a_4N+1)/a_6$, where a_4, a_6 are defined by* (2.14). *Then there exist constants $c_1, c_2 > 0$, which depend on $\mathbf{p}(\Theta)$, the dimension n, and N, such that for any $\psi \in \mathcal{S}_{N,\widetilde{N}}$, $x \in \mathbb{R}^n$, and $s \in \mathbb{R}$, there exists a representation*

$$\psi_{x,s} = \sum_{i=0}^{\infty} \phi^i_{x,s_i},$$

where
(i) $s_0 = s$ *and* $s_{i+1} = s_i - J$, $i = 0,1,2,\dots$, *for* $J(\mathbf{p}(\Theta)) > 0$ *defined by* (2.30),
(ii) $\phi^i \in \mathcal{S}$ *and* $\operatorname{supp}(\phi^i) \subseteq B^*$,
(iii) $\|\phi^i\|_{N,\widetilde{N}} \le c_1\|\psi\|_{N,\widetilde{N}}2^{-c_2 i}$, *where* $c_2 := J(a_6\widetilde{N} - a_4N - 1) > 0$.

Proof. Without loss of generality, by applying an affine transform argument we may assume that $x = 0$, $s = 0$, and $\theta(x,s) = B^*$. By (2.30) there exists a constant $J(\mathbf{p}(\Theta)) > 0$ such that

$$2M_{0,t}(B^*) \subseteq \theta(0, t-J), \quad \forall t \in \mathbb{R}.$$

Let $\varphi \in \mathcal{S}$ be radial such that $\operatorname{supp}(\varphi) = B^*$, $0 \le \varphi \le 1$, and $\varphi = 1$ on $2^{-1}B^*$. Then $\phi^0 := \psi\varphi$, satisfies the following properties:
(i) $\phi^0 \in \mathcal{S}$, $\operatorname{supp}(\phi^0) \subseteq B^*$,
(ii) $\phi^0(y) = \psi(y)$ on $2^{-1}B^*$ and therefore by (2.30) also on $\theta(0,J) \subseteq 2^{-1}B^*$,
(iii) $\|\phi^0\|_{N,\widetilde{N}} \le \tilde{c}\|\psi\|_{N,\widetilde{N}}$.

Assume by induction that for $k \ge 0$, we have constructed a series $\psi_k := \sum_{i=0}^k \phi^i_{0,-iJ}$ with the following properties:
(i) $\phi^i \in \mathcal{S}$, $\operatorname{supp}(\phi^i) \subseteq B^*$, $0 \le i \le k$,
(ii) $\operatorname{supp}(\psi_k) \subseteq \theta(0,-kJ)$,
(iii) $\psi_k = \psi$ on $\theta(0,-(k-1)J)$,
(iv) $\|\phi^i\|_{N,\widetilde{N}} \le c_1\|\psi\|_{N,\widetilde{N}}2^{-c_2 i}$, $0 \le i \le k$.

Let

$$g^{k+1}(y) := \begin{cases} (\psi-\psi_k)(y), & y \in \theta(0,-kJ), \\ \psi(y), & y \in \theta(0,-(k+1)J) \setminus \theta(0,-kJ), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $g^{k+1}(y) = 0$ for $y \in \theta(0, -(k-1)J)$, since by our induction process $\psi = \psi_k$ on this ellipsoid. Let

$$h^{k+1}(y) := \left|\det(M_{0,-(k+1)J})\right| g^{k+1}(M_{0,-(k+1)J}y).$$

For $\phi^{k+1} := h^{k+1}\varphi$, we have the following:
(i) $\phi^{k+1} \in \mathcal{S}$, and $\operatorname{supp}(\phi^{k+1}) \subseteq B^*$,
(ii) $\phi^{k+1}(y) = h^{k+1}(y)$ for $y \in M^{-1}_{0,-(k+1)J} M_{0,-kJ}(B^*)$,
(iii) $\|\phi^{k+1}\|_{N,\widetilde{N}} \le \tilde{c}\|h^{k+1}\|_{N,\widetilde{N}}$.

Case I: $y \in B^* \setminus M^{-1}_{0,-(k+1)J} M_{0,-kJ}(B^*)$. In this case,

$$\phi^{k+1}(y) = \left|\det(M_{0,-(k+1)J})\right| \psi(M_{0,-(k+1)J}y)\varphi(y).$$

With $c_2 := J(a_6\widetilde{N} - a_4N - 1) > 0$, for any $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \le N$, using (2.13) and (2.23), we estimate

$$\begin{aligned}
\left|\partial^\alpha \phi^{k+1}(y)\right| &\le C\left|\det(M_{0,-(k+1)J})\right| \max_{|\gamma|\le N}\left|\partial^\gamma [\psi(M_{0,-(k+1)J}\cdot)](y)\right| \\
&\le C2^{J(k+1)(1+a_4N)} \max_{|\gamma|\le N}\left|\partial^\gamma \psi(M_{0,-(k+1)J}y)\right| \\
&\le C2^{J(k+1)(1+a_4N)}\left(1 + |M_{0,-(k+1)J}y|\right)^{-\widetilde{N}} \|\psi\|_{N,\widetilde{N}} \\
&\le C\|\psi\|_{N,\widetilde{N}} 2^{J(k+1)(1+a_4N-a_6\widetilde{N})} \\
&\le c_1\|\psi\|_{N,\widetilde{N}} 2^{-c_2(k+1)}.
\end{aligned}$$

Case II: $y \in M^{-1}_{0,-(k+1)J} M_{0,-kJ}(B^*) \setminus M^{-1}_{0,-(k+1)J} M_{0,-(k-1)J}(B^*)$. This case is similar to case I.
Case III: $y \in M^{-1}_{0,-(k+1)J} M_{0,-(k-1)J}(B^*)$. In this case, $h^{k+1}(y) = 0$, which implies $\phi^{k+1}(y) = 0$.

Note that $\phi^{k+1}_{0,-(k+1)J}$ is supported on $\theta(0, -(k+1)J) \setminus \theta(0, -(k-1)J)$ with $\phi^{k+1}_{0,-(k+1)J} = \psi - \psi_k$ on $\theta(0, -kJ) \setminus \theta(0, -(k-1)J)$. Therefore for

$$\psi_{k+1} = \sum_{i=0}^{k+1} \phi^i_{0,-iJ} = \psi_k + \phi^{k+1}_{0,-(k+1)J},$$

we have that $\psi_{k+1} = \psi$ on $\theta(0, -kJ)$. This proves the induction and concludes the proof of the lemma. □

Theorem 6.10. *For any cover* Θ, $N \ge 1$, *and* $\widetilde{N} > (a_4N + 1)/a_6$, *there exist constants* $c_1, c_2 > 0$, *depending on the parameters of the cover* $N, \widetilde{N}$, *such that for any* $f \in \mathcal{S}'$,

$$M^\circ_{N,\widetilde{N}} f(x) \le c_1 \sup_{\psi\in\mathcal{S}_{N,\widetilde{N}}, \operatorname{supp}(\psi)\subseteq B^*} M^\circ_\psi f(x), \quad x \in \mathbb{R}^n, \tag{6.19}$$

and for any $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$*,*

$$M^{\circ}_{N,\widetilde{N}}f(x) \le c_2 M_{\Theta} f(x), \quad x \in \mathbb{R}^n. \tag{6.20}$$

Therefore the maximal theorem (Theorem 6.3) also holds for $M^{\circ}_{N,\widetilde{N}}$*.*

Proof. To prove (6.19), let $M^{\circ} := M^{\circ}_{N,\widetilde{N}}$, and let M°_C be the restriction of M° defined by only using functions in $\mathcal{S}_{N,\widetilde{N}}$ with support in B^*. For any $\psi \in \mathcal{S}_{N,\widetilde{N}}$, $s \in \mathbb{R}$, and $x \in \mathbb{R}^n$, let $\psi_{x,s} = \sum_{j=1}^{\infty} \phi^j_{x,s_j}$ be the representation of Lemma 6.9, where ϕ^j are supported on B^*. Thus

$$\begin{aligned}
|f * \psi_{x,s}(x)| &\le \sum_{j=1}^{\infty} |f * \phi^j_{x,s_j}(x)| \\
&\le M^{\circ}_C f(x) \sum_{j=1}^{\infty} \|\phi^j\|_{N,\widetilde{N}} \\
&\le c_1 M^{\circ}_C f(x).
\end{aligned}$$

Therefore

$$M^{\circ} f(x) = \sup_{\psi \in \mathcal{S}_{N,\widetilde{N}}} \sup_{s \in \mathbb{R}} |f * \psi_{x,s}(x)| \le c_1 M^{\circ}_C f(x).$$

Inequality (6.20) is a simple consequence of (6.19), and the maximal theorem for M° is a direct application of (6.20) and Theorem 6.3. □

Our next goal is providing some results relating to the "approximation of the identity" of the pointwise anisotropic convolutions.

Theorem 6.11. *Let* $\varphi \in L_1(\mathbb{R}^n)$ *with* $\int \varphi = 1$.

(i) *For any* $f \in L_\infty(\mathbb{R}^n)$*,*

$$f * \varphi_{x,t}(x) \to f(x) \quad \text{as } t \to \infty$$

at each point $x \in \mathbb{R}^n$ *where* f *is continuous.*

(ii) *For any continuous* f *and compact domain* $\Omega \subset \mathbb{R}^n$*,*

$$\|f * \varphi_{\cdot,t}(\cdot) - f\|_{L_\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

(iii) *Assume further that* $\varphi \in \mathcal{S}$*. Then, for any* $f \in L_p(\mathbb{R}^n)$*,* $1 < p < \infty$*,*

$$\|f * \varphi_{\cdot,t}(\cdot) - f\|_p \to 0 \quad \text{as } t \to \infty.$$

Proof. (i) Let $x \in \mathbb{R}^n$ be a continuity point of f, and let $\varepsilon > 0$. If $\|f\|_\infty = 0$ we are done. Otherwise, since $\varphi \in L_1$ and f is bounded, there exists $R > 0$ such that

$$\int_{|y|>R} |\varphi| \le \frac{\varepsilon}{4\|f\|_\infty}.$$

Since f is continuous at x, there exists $\delta > 0$ such that

$$\left|f(x) - f(y)\right| \le \frac{\varepsilon}{2\|\varphi\|_1}, \quad \forall y \in \mathbb{R}^n,\ |x-y| \le \delta.$$

By (2.14)

$$\|M_{x,t}\| \le a_5 \|M_{x,0}\| 2^{-a_6 t}, \quad t \ge 0. \tag{6.21}$$

This implies that there exists t_0 such that for $t \ge t_0$,

$$|M_{x,t} y| \le \delta, \quad y \in \mathbb{R}^n,\ |y| \le R.$$

Using the above estimates and also $\int \varphi = 1$ gives, for $t \ge t_0$,

$$\begin{aligned}
\left|f * \varphi_{x,t}(x) - f(x)\right| &\le \int_{\mathbb{R}^n} \left|f(x - M_{x,t}y) - f(x)\right|\left|\varphi(y)\right| dy \\
&\le \int_{|y|\le R} \left|f(x - M_{x,t}y) - f(x)\right|\left|\varphi(y)\right| dy + \int_{|y|>R} \left|f(x - M_{x,t}y) - f(x)\right|\left|\varphi(y)\right| dy \\
&\le \|\varphi\|_1 \sup_{z\in\mathbb{R}^n, |x-z|\le\delta} \left|f(z) - f(x)\right| + 2\|f\|_\infty \int_{|y|>R} \left|\varphi(y)\right| dy \\
&\le \varepsilon.
\end{aligned}$$

(ii) Since f is uniformly continuous on Ω, the proof is similar to that of (i), where we use (2.26) to replace (6.21) by

$$\|M_{x,t}\| \le c\big(\Omega, \mathbf{p}(\Theta)\big) 2^{-a_6 t}, \quad \forall x \in \Omega,\ t \ge 0.$$

(iii) By (6.20) and (ii), for any continuous compactly supported $g \in C_0(\mathbb{R}^n)$, we have

$$\left|g(x)\right| \le \sup_t \left|g * \varphi_{x,t}(x)\right| \le C M_\Theta g(x), \quad \forall x \in \mathbb{R}^n,$$

and so $|g * \varphi_{x,t}(x) - g(x)|$ is dominated by $cM_\Theta g(x)$ with $M_\Theta g \in L_p$, $1 < p < \infty$. By Lebesgue's dominated convergence theorem from (ii) we get

$$\left\|g - g * \varphi_{\cdot,t}(\cdot)\right\|_p \to 0 \quad \text{as } t \to 0. \tag{6.22}$$

Now, for any $f \in L_p$ and $\varepsilon > 0$, there exists $g \in C_0(\mathbb{R}^n)$ such that $\|f - g\|_p < \varepsilon$. By (6.22) there also exists $t_0 > 0$ such that $\|g - g * \varphi_{\cdot,t}(\cdot)\|_p < \varepsilon$ for all $t \geq t_0$. Applying also the maximal function inequality (6.4), using $t \geq t_0$, we conclude

$$\begin{aligned}\left\|f - f * \varphi_{\cdot,t}(\cdot)\right\|_p &\leq \|f - g\|_p + \left\|g - g * \varphi_{\cdot,t}(\cdot)\right\|_p + \left\|(g - f) * \varphi_{\cdot,t}(\cdot)\right\|_p \\ &\leq 2\varepsilon + C\left\|M_\Theta(g - f)\right\|_p \\ &\leq 2\varepsilon + C\|g - f\|_p \\ &\leq C\varepsilon. \end{aligned}$$

□

We have seen that $M^\circ_{N,\widetilde{N}}$ satisfies the maximal inequality, which also implies that for any $\varphi \in \mathcal{S}_{N,\widetilde{N}}$, we also have the maximal inequality, e. g., $\|M^\circ_\varphi f\|_p \leq C\|f\|_p$ for any $f \in L_p$, $1 < p \leq \infty$. The following is a converse.

Theorem 6.12. *Suppose $\varphi \in \mathcal{S}$, $\int \varphi \neq 0$, and $1 \leq p \leq \infty$. If*
(i) *Θ is a continuous cover and $f \in C(\mathbb{R}^n)$, or*
(ii) *Θ is a t-continuous cover and $f \in \mathcal{S}'(\mathbb{R}^n)$,*

then $M^\circ_\varphi f \in L_p(\mathbb{R}^n) \Rightarrow f \in L_p(\mathbb{R}^n)$.

Proof. Without loss of generality, we may assume that $\int \varphi = 1$. Let us first prove the theorem under condition (i). Using Theorem 6.11(ii), we have that on any compact domain $\Omega \subset \mathbb{R}^n$,

$$\left\|f * \varphi_{\cdot,t}(\cdot) - f\right\|_{L_\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{6.23}$$

With this, the proof for the case $p = \infty$ is obvious. Let $1 < p < \infty$. Since for any $t \in \mathbb{R}$, $\|f * \varphi_{\cdot,t}(\cdot)\|_p \leq \|M^\circ_\varphi f\|_p$, the set $\{f * \varphi_{\cdot,t}(\cdot)\}_{t\in\mathbb{R}}$ is uniformly bounded in L_p. By the Banach–Alaoglu theorem there exists a subsequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that $f * \varphi_{\cdot,t_k}(\cdot)$ converges weakly-$*$ in L_p to $\tilde{f} \in L_p$. Let $\Omega \subset \mathbb{R}^n$ be compact. Using (6.23) and applying both f and $\tilde{f}$ as functionals in $L_p(\Omega)$ to appropriately selected test functions in C_0^∞ (and hence in $L_{p'}$), supported in Ω, we may deduce that $f = \tilde{f}$ a. e. on Ω. Since Ω is arbitrary, we may deduce that $f = \tilde{f}$ also in $L_p(\mathbb{R}^n)$. The proof of the case $p = 1$ is similar, where by the Banach–Alaoglu theorem there exists a subsequence $\{t_k\}$, $t_k \to \infty$, such that $f * \varphi_{\cdot,t_k}(\cdot)$ converges weakly-$*$ to an absolutely continuous measure. This concludes the proof for case (i).

Now assume condition (ii). In this case, due to the strong assumption of a t-continuous cover, the proof is similar to that of [7, Theorem 3.9]. We sketch the case $1 < p < \infty$. For a t-continuous cover, we have that for fixed $t \in \mathbb{R}$, $\varphi_{x,t}$ is constant for $x \in \mathbb{R}^n$, which implies $\varphi_t := \varphi_{x,t} \in \mathcal{S}$. Since for a t-continuous cover, the matrices $\{M_{x,t}\}$ are constant for fixed t and $\operatorname{diam}(\theta(x,t)) \to 0$ as $t \to \infty$, this allows us to show that for any sequence $\{t_k\}$, $t_k \to \infty$, and for $f \in \mathcal{S}'$, $f * \varphi_{t_k} \to f$ in $\mathcal{S}'$ (see [7, Lemma 3.8]). At the same time, as in the previous case, there exists a sequence $\{t_k\}$,

$t_k \to \infty$ as $k \to \infty$, such that $f * \varphi_t$ converges weakly-$*$ in L_p to $\tilde{f} \in L_p$. Thus $f = \tilde{f}$ in $\mathcal{S}'$ and so also in L_p. □

6.2 Anisotropic Hardy spaces defined by maximal functions

Let Θ be a continuous cover of $\mathbb{R}^n$ with parameters $\mathbf{p}(\Theta) = (a_1, \dots, a_6)$, and let $0 < p \le \infty$. We define $N_p := N_p(\Theta)$ as the minimal integer satisfying

$$N_p > \frac{\max(1, a_4)n + 1}{a_6 \min(p, 1)}, \tag{6.24}$$

and then $\tilde{N}_p := \tilde{N}_p(\Theta)$ as the minimal integer satisfying

$$\tilde{N}_p > \frac{a_4 N_p(\Theta) + 1}{a_6}. \tag{6.25}$$

Definition 6.13. Let Θ be a continuous ellipsoid cover, and let $0 < p \le \infty$. Denoting $M^\circ := M^\circ_{N_p, \tilde{N}_p}$, we define the *anisotropic Hardy space* corresponding to Θ as

$$H^p(\Theta) := \{f \in \mathcal{S}' : M^\circ f \in L_p\}$$

with the (quasi-)norm $\|f\|_{H^p(\Theta)} := \|M^\circ f\|_p$.

Remarks

(i) Theorem 6.10 implies that the maximal theorem holds for M°, and Theorem 6.12 gives a converse. Therefore, for any cover Θ and $1 < p \le \infty$, $H^p(\Theta) \sim L_p(\mathbb{R}^n)$. Thus, exactly as in the classical isotropic case, we focus our attention on the range $0 < p \le 1$. Moreover, in Section 6.5, we show that in contrast to the case $1 < p \le \infty$, the equivalence $H^1(\Theta) \sim H^1(\Theta')$ holds if and only if Θ and Θ' induce equivalent quasi-distances.

(ii) We note again that in the general case of spaces of homogeneous type, we can only define and analyze atomic Hardy spaces (see Section 6.3) for values of p "close" to 1.

(iii) We will obtain in Section 6.3, through the equivalence with the anisotropic atomic Hardy spaces, that

$$\left\|M^\circ_{N,\tilde{N}} f\right\|_p \sim \left\|M^\circ f\right\|_p, \quad \forall N \ge N_p, \tilde{N} \ge \tilde{N}_p, \ \forall f \in \mathcal{S}', \tag{6.26}$$

where the constants of equivalency do not depend on f.

Theorem 6.14. *For a continuous cover* Θ, $H^p(\Theta)$, $0 < p \le 1$, *is continuously embedded in* $\mathcal{S}'$.

Proof. For $\psi \in \mathcal{S}$ and $x \in \theta(0,0)$, denote $\psi^x(y) := |\det(M_{x,0})|\psi(-M_{x,0}y + x)$. Since by (2.14) all of the ellipsoids $\theta(x,0)$, $x \in \theta(0,0)$, have "equivalent" shapes, it is not difficult to see that there exists a constant $c(\mathbf{p}(\Theta))$ such that

$$\|\psi^x\|_{N,\tilde{N}} \le c\|\psi\|_{N,\tilde{N}}, \quad \forall x \in \theta(0,0).$$

Observe that (using notation (6.6)) for any $f \in \mathcal{S}'$,

$$|\langle f, \psi\rangle| = |f * \psi^x_{x,0}(x)| \le c\|\psi\|_{N,\tilde{N}} M^\circ f(x), \quad \forall x \in \theta(0,0).$$

Therefore, if $f \in H^p(\Theta)$, for any $\psi \in \mathcal{S}$

$$\begin{aligned} |\langle f, \psi\rangle|^p &\le C\|\psi\|^p_{N,\tilde{N}} \int_{\theta(0,0)} M^\circ f(x)^p dx \\ &\le C\|\psi\|^p_{N,\tilde{N}} \|f\|^p_{H^p(\Theta)}. \end{aligned}$$

□

Theorem 6.15. *For a continuous cover* Θ, $H^p(\Theta)$ *is complete.*

Proof. The proof is identical to that of [7, Proposition 3.12]. To prove that $H^p(\Theta)$ is complete, it suffices to show that for any sequence $\{f_i\}$, $\|f_i\|_{H^p(\Theta)} < 2^{-i}$, $i \in \mathbb{N}$, the series $\sum_i f_i$ converges in $H^p(\Theta)$. Theorem 6.14 implies that $f_i \in \mathcal{S}'$ for all i and that the partial sums are Cauchy in $\mathcal{S}'$. Since $\mathcal{S}'$ is complete, $\sum_i f_i$ converges in $\mathcal{S}'$ to some $f \in \mathcal{S}'$. Therefore

$$\begin{aligned} \left\| f - \sum_{i=1}^{M} f_i \right\|^p_{H^p(\Theta)} &= \left\| \sum_{i=M+1}^{\infty} f_i \right\|^p_{H^p(\Theta)} \\ &\le \sum_{i=M+1}^{\infty} \|f_i\|^p_{H^p(\Theta)} \\ &\le \sum_{i=M+1}^{\infty} 2^{-ip} \to 0 \quad \text{as } M \to \infty. \end{aligned}$$

□

The main result of this section is the following generalization of the isotropic case [41, 61] to the pointwise variable anisotropic case. It essentially determines that the Hardy space $H^p(\Theta)$ can be equivalently determined using anisotropic convolutions with a single Schwartz function. However, we formulate and prove a partial result for general covers and the full equivalency in the stricter setting of t-continuous covers (see Definition 2.12).

Theorem 6.16 ([65]). *Let* $0 < p \le 1$, $\varphi \in \mathcal{S}$ *with* $\int \varphi \ne 0$, *and* $f \in \mathcal{S}'$. *Then*

(i) *If* Θ *is a continuous cover, then there exist constants* $c_1, c_2 > 0$, *which do not depend on* f, *such that*

$$\|M^\circ_\varphi f\|_p \le \|M_\varphi f\|_p \le c_1\|f\|_{H^p(\Theta)} \le c_2\|T^N_\varphi f\|_p, \quad N > \frac{1}{a_6 p}.$$

(ii) *If* Θ *is a t-continuous ellipsoid cover, then*

$$M_\varphi^\circ f \in L_p \iff M_\varphi f \in L_p \iff f \in H^p(\Theta) \iff T_\varphi^N f \in L_p, \quad N > \frac{1}{a_6 p},$$

where the constants do not depend on f.

The proof is rather technical and requires several other auxiliary maximal functions with truncations, decay terms and apertures.

Lemma 6.17. *Let* Θ *be a pointwise continuous cover. Let* $F : \mathbb{R}^n \times \mathbb{R} \to (0, \infty)$ *be a Lebesgue-measurable function. Then, for fixed aperture* $k \in \mathbb{Z}$ *and truncation* $t_0 \in \mathbb{R}$, *the maximal function of* F,

$$F_k^{t_0}(x) := \sup_{t \geq t_0} \sup_{y \in \theta(x, t-kJ)} F(y,t), \tag{6.27}$$

is lower semicontinuous, i. e.,

$$\{x \in \mathbb{R}^n : F_k^{t_0}(x) > \lambda\}$$

is open for any $\lambda > 0$. *Here* J *is defined by* (2.30).

Proof. If $F_k^{t_0}(x) > \lambda$ for $x \in \mathbb{R}^n$, then there exist $t \geq t_0$ and $y \in \theta(x, t - kJ)$ such that $F(y,t) > \lambda$. Under the assumption that the cover Θ is pointwise continuous (see Definition 2.11), there exists $\delta > 0$ such that if $x' \in B(x, \delta)$, then $y \in \theta(x', t - kJ)$. Therefore $F_k^{t_0}(x') \geq F(y,t) > \lambda$. We conclude that $\{x \in \mathbb{R}^n : F_k^{t_0}(x) > \lambda\}$ is open. □

Next, we show some estimates for $F_k^{t_0}$.

Lemma 6.18. *Let* Θ *be a pointwise continuous cover, and let* $F_k^{t_0}$ *be as in* (6.27). *There exists a constant* $c > 0$ *such that for any* $k' < k$, $t_0 < 0$, *and* $\lambda > 0$, *we have*

$$\left|\{x \in \mathbb{R}^n : F_k^{t_0}(x) > \lambda\}\right| \leq c 2^{(k-k')J} \left|\{x \in \mathbb{R}^n : F_{k'}^{t_0}(x) > \lambda\}\right| \tag{6.28}$$

and

$$\int_{\mathbb{R}^n} F_k^{t_0} \leq c 2^{(k-k')J} \int_{\mathbb{R}^n} F_{k'}^{t_0}. \tag{6.29}$$

Proof. Let $\Omega := \{x \in \mathbb{R}^n : F_{k'}^{t_0}(x) > \lambda\}$. We claim that there exists $c_1(\mathbf{p}(\Theta)) > 0$ such that

$$\{x \in \mathbb{R}^n : F_k^{t_0}(x) > \lambda\} \subseteq \{x \in \mathbb{R}^n : M_\Theta(\mathbf{1}_\Omega)(x) \geq c_1 2^{(k'-k)J}\}. \tag{6.30}$$

Under this assumption, applying the weak-L_1 maximal inequality (6.3) gives (6.28) by

$$\begin{aligned}\left|\{x \in \mathbb{R}^n : F_k^{t_0}(x) > \lambda\}\right| &\le \left|\{x \in \mathbb{R}^n : M_\Theta(\mathbf{1}_\Omega)(x) \ge c_1 2^{(k'-k)J}\}\right| \\ &\le C c_1^{-1} 2^{(k-k')J} \|\mathbf{1}_\Omega\|_1 \\ &= C c_1^{-1} 2^{(k-k')J} \left|\{x \in \mathbb{R}^n : F_{k'}^{t_0}(x) > \lambda\}\right|.\end{aligned}$$

Integrating (6.28) on $(0, \infty)$ with respect to λ yields (6.29).

Thus it remains to prove (6.30). Let $x \in \mathbb{R}^n$ with $F_k^{t_0}(x) > \lambda$. Then there exist $t \ge t_0$ and $y \in \theta(x, t - kJ)$ such that $F(y, t) > \lambda$. We claim that

$$a_5^{-1} \cdot \theta(y, t - k'J) = y + a_5^{-1} M_{y,t-k'J}(B^*) \subseteq \theta(x, t - kJ - \gamma) \cap \Omega, \tag{6.31}$$

where γ is given by Lemma 2.18. Since $a_5 \ge 1$, if $z \in a_5^{-1} \cdot \theta(y, t-k'J)$, then $z \in \theta(y, t-k'J)$. Since $k' < k$, using Lemma 2.18 gives that $z \in \theta(x, t - kJ - \gamma)$. This also means that $\theta(z, t - k'J) \cap \theta(y, t - k'J) \ne \emptyset$, and we may apply (2.14) to obtain $\|M_{z,t-k'J}^{-1} M_{y,t-k'J}\| \le a_5$. From this we have

$$a_5^{-1} M_{y,t-k'J}(B^*) \subseteq M_{z,t-k'J}(B^*)$$

and

$$y \in z + a_5^{-1} M_{y,t-k'J}(B^*) \subseteq z + M_{z,t-k'J}(B^*) = \theta(z, t - k'J).$$

We may deduce that $F_{k'}^{t_0}(z) \ge F(y, t) > \lambda$, which implies that $z \in \Omega$ and proves (6.31). By (6.31) we have

$$\left|\theta(x, t - kJ - \gamma) \cap \Omega\right| \ge a_5^{-n} \left|\theta(y, t - k'J)\right| \ge \frac{a_1}{a_5^n} 2^{k'J-t}.$$

We apply this to conclude (6.30) by

$$\begin{aligned}M_\Theta(\mathbf{1}_\Omega)(x) &\ge \frac{1}{|\theta(x, t - kJ - \gamma)|} \int_{\theta(x,t-kJ-\gamma)} \mathbf{1}_\Omega(y) dy \\ &\ge a_2^{-1} 2^{t-kJ-\gamma} \left|\theta(x, t - kJ - \gamma) \cap \Omega\right| \\ &\ge \frac{a_1}{a_2 a_5^n 2^\gamma} 2^{(k'-k)J} \\ &=: c_1 2^{(k'-k)J}.\end{aligned}$$

□

Next, we define pointwise variable anisotropic maximal functions that are truncated by $t \ge t_0$ and contain additional decay terms with a decay parameter L:

$$M_\varphi^{\circ(t_0,L)} f(x) := \sup_{t \ge t_0} \left|f * \varphi_{x,t}(x)\right| \left(1 + \left|M_{x,t_0}^{-1} x\right|\right)^{-L} \left(1 + 2^{t+t_0}\right)^{-L},$$

$$
\begin{aligned}
M_\varphi^{(t_0,L)} f(x) &:= \sup_{t\ge t_0} \sup_{y\in\theta(x,t)} \left|f * \varphi_{x,t}(y)\right| \left(1+\left|M_{x,t_0}^{-1} y\right|\right)^{-L} \left(1+2^{t+t_0}\right)^{-L},\\
T_\varphi^{N(t_0,L)} f(x) &:= \sup_{t\ge t_0} \sup_{y\in\mathbb{R}^n} \frac{|f * \varphi_{x,t}(y)|}{(1+|M_{x,t}^{-1}(x-y)|)^N (1+2^{t+t_0})^L (1+|M_{x,t_0}^{-1} y|)^L},\\
M_{N,\widetilde{N}}^{\circ(t_0,L)} f(x) &:= \sup_{\varphi\in\mathcal{S}_{N,\widetilde{N}}} M_\varphi^{\circ(t_0,L)} f(x),\\
M_{N,\widetilde{N}}^{(t_0,L)} f(x) &:= \sup_{\varphi\in\mathcal{S}_{N,\widetilde{N}}} M_\varphi^{(t_0,L)} f(x).
\end{aligned}
$$

Lemma 6.19. *Let Θ be a t-continuous cover, $p > 0$, $N > 1/(a_6 p)$, and $\varphi \in \mathcal{S}$. There exists a constant $c > 0$ such that for any $t_0 < 0$, $L \ge 0$, and $f \in \mathcal{S}'$,*

$$\left\|T_\varphi^{N(t_0,L)} f\right\|_p \le c \left\|M_\varphi^{(t_0,L)} f\right\|_p.$$

Proof. Under the strict assumption of a t-continuous cover, we may assume that $M_{x,t} := M_t$ is constant for all $x \in \mathbb{R}^n$ and denote $\varphi_{x,t} := \varphi_t$. Then we consider the function

$$F(y,t) := \left|f * \varphi_t(y)\right|^p \left(1+\left|M_{t_0}^{-1} y\right|\right)^{-pL} \left(1+2^{t+t_0}\right)^{-pL}.$$

Let $F_0^{t_0}$ be as in (6.27). Observe that $F_0^{t_0} = (M_\varphi^{(t_0,L)} f)^p$. Then for $t \ge t_0$ and $y \in \theta(x,t)$,

$$F(y,t)\left(1+\left|M_t^{-1}(x-y)\right|\right)^{-pN} \le F(y,t) \le F_0^{t_0}(x).$$

When $y \in \theta(x, t-kJ) \setminus \theta(x, t-(k-1)J)$, for some $k \ge 1$, we have

$$M_t^{-1}(x-y) \notin M_t^{-1} M_{t-(k-1)J}(B^*). \tag{6.32}$$

By the shape condition (2.14)

$$M_{t-(k-1)J}^{-1} M_t(B^*) \subseteq a_5 2^{-a_6(k-1)J} B^* \Rightarrow a_5^{-1} 2^{a_6(k-1)J} B^* \subseteq M_t^{-1} M_{t-(k-1)J}(B^*).$$

Combining this with (6.32) gives

$$\left|M_t^{-1}(x-y)\right| \ge a_5^{-1} 2^{a_6(k-1)J}.$$

Therefore, for any $t \ge t_0$,

$$F(y,t)\left(1+\left|M_t^{-1}(x-y)\right|\right)^{-pN} \le a_5^{pN} 2^{-pNa_6(k-1)J} F_k^{t_0}(x).$$

Taking the supremum over all $y \in \mathbb{R}^n$ and $t \ge t_0$ yields

$$\left(T_\varphi^{N(t_0,L)} f(x)\right)^p \le C \sum_{k=0}^{\infty} 2^{-pNa_6 kJ} F_k^{t_0}(x).$$

We combine this with (6.29), the condition $N > (pa_6)^{-1}$, and the observation that $F_0^{t_0} = (M_\varphi^{(t_0,L)} f)^p$ to conclude

$$\begin{aligned} \|T_\varphi^{N(t_0,L)}\|_p^p &\le C \sum_{k=0}^{\infty} 2^{-pNa_6 kJ} \int_{\mathbb{R}^n} F_k^{t_0} \\ &\le C \sum_{k=0}^{\infty} 2^{-pNa_6 kJ} 2^{kJ} \int_{\mathbb{R}^n} F_0^{t_0} \\ &\le C \|M_\varphi^{(t_0,L)} f\|_p^p. \end{aligned}$$

□

Lemma 6.20. *Let Θ be a continuous cover, $\varphi \in \mathcal{S}$, $\int_{\mathbb{R}^n} \varphi \ne 0$, and $f \in \mathcal{S}'$. Then for any N, L, there exist $0 < U \le \widetilde{U}$, $U \ge N_p$, $\widetilde{U} \ge \widetilde{N}_p$, large enough and a constant $c > 0$ such that for any $t_0 < 0$,*

$$M_{U,\widetilde{U}}^{\circ(t_0,L)} f(x) \le c T_\varphi^{N(t_0,L)} f(x), \quad \forall x \in \mathbb{R}^n, \tag{6.33}$$

and

$$M_{U,\widetilde{U}}^{\circ} f(x) \le c T_\varphi^{N} f(x), \quad \forall x \in \mathbb{R}^n. \tag{6.34}$$

Proof. We will prove (6.33). The proof of (6.34) is almost identical and simpler. By Theorem 6.8, for any $\psi \in \mathcal{S}$, $x \in \mathbb{R}^n$, and $t \in \mathbb{R}$, there exists a sequence $\{\eta^k\}_{k=0}^{\infty}$, $\eta^k \in \mathcal{S}$, that satisfies (6.14),

$$\psi = \sum_{k=0}^{\infty} \eta^k * \varphi^k,$$

where

$$\varphi^k := \left|\det(M_{x,t+kJ}^{-1} M_{x,t})\right| \varphi(M_{x,t+kJ}^{-1} M_{x,t} \cdot), \quad k \ge 0.$$

Furthermore, for any positive integers U, $\widetilde{U}$, and V,

$$\|\eta^k\|_{U,\widetilde{U}} \le C 2^{-kV} \|\psi\|_{U+n+1+\lceil V/(a_6 J)\rceil, \widetilde{U}+n+1}, \tag{6.35}$$

where the constant depends on φ, U, $\widetilde{U}$, V, $\mathbf{p}(\Theta)$ but not on ψ. Denoting $M_k := M_{x,t+kJ}$ for $t \ge t_0$, this implies

$$\begin{aligned} &|f * \psi_{x,t}(x)| \\ &\quad = \left| \left[f * \sum_{k=0}^{\infty} (\eta^k * \varphi^k)_{x,t} \right](x) \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \left[f * \sum_{k=0}^{\infty} \left|\det(M_k^{-1})\right| \int_{\mathbb{R}^n} \eta^k(y)\varphi(M_k^{-1}(\cdot - M_{x,t}y))dy \right](x) \right| \\
&= \left| \left[f * \sum_{k=0}^{\infty} \left|\det(M_k^{-1})\right| \int_{\mathbb{R}^n} (\eta^k)_{x,t}(y)\varphi(M_k^{-1}(\cdot - y))dy \right](x) \right| \\
&= \left| \sum_{k=0}^{\infty} [f * (\eta^k)_{x,t} * \varphi_{x,t+kJ}](x) \right| \\
&\le \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |f * \varphi_{x,t+kJ}(x-y)| |(\eta^k)_{x,t}(y)| dy \\
&\le T_\varphi^{N(t_0,L)} f(x) \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} (1 + |M_k^{-1}y|)^N (1 + |M_{x,t_0}^{-1}(x-y)|)^L (1 + 2^{t+t_0+kJ})^L |(\eta^k)_{x,t}(y)| dy.
\end{aligned}
$$

Thus we derive

$$
\begin{aligned}
&M_\psi^{\circ(t_0,L)} f(x) \\
&\quad \le T_\varphi^{N(t_0,L)} f(x) \sup_{t \ge t_0} \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \frac{(1 + |M_k^{-1}y|)^N (1 + |M_{x,t_0}^{-1}(x-y)|)^L (1 + 2^{t+t_0+kJ})^L}{(1 + |M_{x,t_0}^{-1}x|)^L (1 + 2^{t+t_0})^L} |(\eta^k)_{x,t}(y)| dy \\
&\quad =: T_\varphi^{N(t_0,L)} f(x) \sup_{t \ge t_0} \sum_{k=0}^{\infty} I_{t,k}.
\end{aligned}
$$

Let us now estimate $I_{t,k}$ for $t \ge t_0$ and $k \ge 0$. We begin with simple observations that

$$
\frac{(1 + 2^{t+t_0+kJ})^L}{(1 + 2^{t+t_0})^L} \le 2^{kJL},
$$

and

$$
\frac{(1 + |M_{x,t_0}^{-1}(x-y)|)^L}{(1 + |M_{x,t_0}^{-1}x|)^L} \le (1 + |M_{x,t_0}^{-1}y|)^L, \quad \forall x, y \in \mathbb{R}^n.
$$

Therefore, using also (2.14) for $t \ge t_0$, we get

$$
\begin{aligned}
I_{t,k} &\le C2^{t+kJL} \int_{\mathbb{R}^n} (1 + |M_k^{-1}y|)^N (1 + |M_{x,t_0}^{-1}y|)^L |\eta^k(M_{x,t}^{-1}y)| dy \\
&\le C2^{kJL} \int_{\mathbb{R}^n} (1 + \|M_k^{-1}M_{x,t}\| |y|)^N (1 + \|M_{x,t_0}^{-1}M_{x,t}\| |y|)^L |\eta^k(y)| dy \\
&\le C2^{kJ(L+a_4N)} \int_{\mathbb{R}^n} (1 + |y|)^{N+L} |\eta^k(y)| dy \\
&\le C2^{kJ(L+a_4N)} \|\eta^k\|_{0,N+L+n+1}.
\end{aligned}
$$

We now apply (6.35) with $V := \lceil J(L + a_4 N)\rceil + 1$, which gives

$$I_{t,k} \le C2^{-k}\|\psi\|_{n+1+\lceil V/(a_6 J)\rceil, N+L+2n+2}.$$

This yields that for any $\psi \in \mathcal{S}_{U,\widetilde{U}}$ where $U := \max(N_p, n+1+\lceil V/(a_6 J)\rceil)$ and $\widetilde{U} := \max(\widetilde{N}_p, N+L+2n+2)$,

$$M_{\psi}^{\circ(t_0,L)} f(x) \le C T_{\varphi}^{N(t_0,L)} f(x), \quad \forall x \in \mathbb{R}^n,$$

and taking the supremum over $\psi \in \mathcal{S}_{U,\widetilde{U}}$ allows us to get (6.33). □

The next lemma shows the technical role of the decay parameter L. It is required to ensure the integrability in L_p of $M_{\varphi}^{(t_0,L)} f$ for a given pair $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$.

Lemma 6.21. *Let Θ be a t-continuous cover. Then, for any $\varphi \in \mathcal{S}$, $f \in \mathcal{S}'$, $N > 0$, and $t_0 < 0$, there exist $L > 0$ and $N' > 0$ large enough such that*

$$M_{\varphi}^{(t_0,L)} f(x) \le c2^{-t_0(2a_4 N' + 2L + a_4 L)} \big(1 + |x|\big)^{-N}, \quad \forall x \in \mathbb{R}^n,$$

where c depends on $\mathbf{p}(\Theta)$, φ, N', and f.

Proof. Since $f \in \mathcal{S}'$, there exist a constant $c(f)$ and a parameter N' such that for any $\varphi \in \mathcal{S}$,

$$\big|f * \varphi(y)\big| \le c(f)\|\varphi\|_{N',N'} \big(1 + |y|\big)^{N'}.$$

Under the strict assumption of a t-continuous cover, we may again use the notation $M_{x,t} := M_t$ and $\varphi_{x,t} := \varphi_t$ for $x \in \mathbb{R}^n$. Thus, for any $t_0 < 0$, $t \ge t_0$, and $y \in \mathbb{R}^n$,

$$\big|f * \varphi_t(y)\big| \big(1 + \big|M_{t_0}^{-1} y\big|\big)^{-L} \big(1 + 2^{t+t_0}\big)^{-L} \le C2^{-L(t+t_0)} \|\varphi_t\|_{N',N'} \big(1 + |y|\big)^{N'} \big(1 + \big|M_{t_0}^{-1} y\big|\big)^{-L}. \tag{6.36}$$

We first estimate $\|\varphi_t\|_{N',N'}$:

$$\begin{aligned}
\|\varphi_t\|_{N',N'} &\le \big|\det(M_t^{-1})\big| \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \le N'} \big(1 + |z|\big)^{N'} \big|\partial^{\alpha}[\varphi(M_t^{-1} \cdot)](z)\big| \\
&\le C2^t \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \le N'} \big(1 + |M_t z|\big)^{N'} \big\|M_t^{-1}\big\|^{|\alpha|} \big|\partial^{\alpha} \varphi(z)\big|.
\end{aligned}$$

We consider two cases.
Case I: $t \ge 0$. By (2.14) we have

$$\big\|M_t^{-1}\big\| \le \big\|M_t^{-1} M_0\big\| \big\|M_0^{-1}\big\| \le a_3^{-1} 2^{a_4 t} \big\|M_0^{-1}\big\| \le C2^{a_4 t}.$$

Also,

$$|M_t z| \le \|M_0\| \big\|M_0^{-1} M_t\big\| |z| \le \|M_0\| 2^{-a_6 t} |z| \le C|z|. \tag{6.37}$$

So in this case,

$$\|\varphi_t\|_{N',N'} \le C2^t 2^{a_4 t N'}\|\varphi\|_{N',N'}. \tag{6.38}$$

Case II: $t_0 \le t < 0$. Appealing to (2.14) we have

$$\|M_t^{-1}\| \le \|M_t^{-1}M_0\|\|M_0^{-1}\| \le a_5 2^{a_6 t}\|M_0^{-1}\| \le C$$

and

$$|M_t z| \le \|M_0\| a_3^{-1} 2^{-a_4 t}|z| \le C2^{-a_4 t_0}|z|. \tag{6.39}$$

We combine these two estimates for the case $t_0 \le t < 0$ to derive

$$\|\varphi_t\|_{N',N'} \le C2^t 2^{-a_4 t_0 N'}\|\varphi\|_{N',N'}. \tag{6.40}$$

For any $N \ge 1$, let $L := N + N'$. Using estimates (6.38) and (6.40), for $t_0 < 0$ and $t \ge t_0$, we have

$$2^{-L(t+t_0)}\|\varphi_t\|_{N',N'} \le C2^{-t_0(a_4 N' + 2L)}\|\varphi\|_{N',N'}.$$

Inserting this into (6.36), we get

$$\left|f*\varphi_t(y)\right|\left(1+\left|M_{t_0}^{-1}y\right|\right)^{-L}\left(1+2^{t+t_0}\right)^{-L} \le C2^{-t_0(a_4N'+2L)}\|\varphi\|_{N',N'}(1+|y|)^{N'}\left(1+\left|M_{t_0}^{-1}y\right|\right)^{-L}. \tag{6.41}$$

Now for any $y \in \theta(x,t)$, there exists $z \in B^*$ such that $y = M_t z + x$. We may use (6.37) and (6.39) to bound $|M_t z| \le C2^{-a_4 t_0}$, and so

$$1 + |y| \le (1 + |x|)(1 + |M_t z|) \le C2^{-a_4 t_0}(1 + |x|). \tag{6.42}$$

We also need a sort of inverse using $x = y - M_t z$ and $t \ge t_0$:

$$\begin{aligned} 1 + |x| &\le 1 + \|M_0\|\left\|M_0^{-1}M_{t_0}\right\|\left|M_{t_0}^{-1}x\right| \\ &\le C2^{-a_4 t_0}\left(1 + \left|M_{t_0}^{-1}x\right|\right) \\ &\le C2^{-a_4 t_0}\left(1 + \left|M_{t_0}^{-1}y\right|\right)\left(1 + \left\|M_{t_0}^{-1}M_t\right\||z|\right) \\ &\le C2^{-a_4 t_0}\left(1 + \left|M_{t_0}^{-1}y\right|\right)\left(1 + a_5 2^{-a_6(t-t_0)}\right) \\ &\le C2^{-a_4 t_0}\left(1 + \left|M_{t_0}^{-1}y\right|\right). \end{aligned}$$

So we obtain

$$1 + \left|M_{t_0}^{-1}y\right| \ge C2^{a_4 t_0}(1 + |x|). \tag{6.43}$$

We now plug (6.42) and (6.43) into (6.41) and use $L := N + N'$ to obtain

$$\begin{aligned}|f * \varphi_t(y)|(1 + |M_{t_0}^{-1} y|)^{-L}(1 + 2^{t+t_0})^{-L} &\leq C 2^{-t_0(a_4 N' + 2L)} 2^{-a_4 t_0 N'} (1 + |x|)^{N'} 2^{-a_4 t_0 L} (1 + |x|)^{-L} \\ &\leq C 2^{-t_0(2a_4 N' + 2L + a_4 L)} (1 + |x|)^{-N},\end{aligned}$$

where the constant depends on f, φ, N', and $\mathbf{p}(\Theta)$. Taking the supremum over all $y \in \theta(x, t)$ and $t \geq t_0$ provides the required estimate for $M_\varphi^{(t_0, L)} f(x)$ and concludes the proof. □

Proof of Theorem 6.16. Let $\varphi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \varphi \neq 0$, and let $f \in \mathcal{S}'$. We first assume the general case where Θ is a continuous cover. Using the simple pointwise estimates (6.12) and (6.13), we have that

$$\|M_\varphi^\circ f\|_p \leq \|M_\varphi f\|_p \leq 2^N \|T_\varphi^N f\|_p,$$

and if $f \in H^p(\Theta)$, then

$$\|M_\varphi^\circ\|_p \leq \|M_\varphi f\|_p \leq \|M_{N_p, \widetilde{N}_p} f\|_p \leq c \|M^\circ f\|_p = c \|f\|_{H^p(\Theta)} < \infty.$$

Then, applying first (6.26) and then (6.34), we also get that for sufficiently large $U \geq N_p$ and $\widetilde{U} \geq \widetilde{N}_p$,

$$\|f\|_{H^p(\Theta)} \leq c_1 \|M_{U, \widetilde{U}}^\circ f\|_p \leq c_2 \|T_\varphi^N f\|_p, \quad N > \frac{1}{a_6 p}.$$

This concludes the proof of Theorem 6.16(i). From this point we assume the particular case of a t-continuous cover and proceed to prove (ii). By Lemma 6.19 applied with $L = 0$ we have

$$\|T_\varphi^{N(t_0, 0)} f\|_p \leq C \|M_\varphi^{(t_0, 0)} f\|_p.$$

Taking the limit as $t_0 \to -\infty$, by the monotone convergence theorem we get

$$\|T_\varphi^N f\|_p \leq C \|M_\varphi f\|_p.$$

We now apply Lemma 6.20 with $N > 1/(a_6 p)$, and $L = 0$ and Lemma 6.19 with $L = 0$, to conclude there exist $0 < U \leq \widetilde{U}$, $U \geq N_p$, $\widetilde{U} \geq \widetilde{N}_p$, large enough, such that for any $f \in \mathcal{S}'$ and $t_0 < 0$,

$$\|M_{U, \widetilde{U}}^{\circ(t_0, 0)} f\|_p \leq C \|M_\varphi^{(t_0, 0)} f\|_p.$$

Taking the limit as $t_0 \to -\infty$, by the monotone convergence theorem we get

$$\|M_{U, \widetilde{U}}^\circ f\|_p \leq C \|M_\varphi f\|_p.$$

From this and (6.26) we derive that

$$\begin{aligned}\|f\|_{H^p(\Theta)} &= \|M^\circ_{N_p,\widetilde{N}_p}\|_p \\ &\le C\|M^\circ_{U,\widetilde{U}}f\|_p \\ &\le C\|M_\varphi\|_p.\end{aligned}$$

It remains to show that

$$\|M_\varphi f\|_p \le C\|M^\circ_\varphi f\|_p.$$

Assume that $M^\circ_\varphi f \in L_p$. Note that at this point, we do not know if $M_\varphi f \in L_p$. That is why we now must proceed (exactly as in the classical isotropic case) with the truncated maximal functions with the decay terms. Thus, taking $0 < U \le \widetilde{U}$, $U \ge N_p$, $\widetilde{U} \ge \widetilde{N}_p$, large enough, by applying the pointwise estimate of Lemma 6.20 and then Lemma 6.19, for any given $t_0 < 0$, we have

$$\|M^{\circ(t_0,L)}_{U,\widetilde{U}}f\|_p \le C_1\|M^{(t_0,L)}_\varphi f\|_p, \tag{6.44}$$

where $L > 0$ is chosen large enough (not depending on t_0) to fulfill the conditions of Lemma 6.21 for $N > n/p$, ensuring that the right-hand side of (6.44) is finite. With $C_2 := 2^{1/p}C_1$, denote

$$\Omega_{t_0} := \{x \in \mathbb{R}^n : M^{\circ(t_0,L)}_{U,\widetilde{U}}f(x) \le C_2 M^{(t_0,L)}_\varphi f(x)\}. \tag{6.45}$$

We claim that

$$\int_{\mathbb{R}^n} (M^{(t_0,L)}_\varphi f)^p \le 2\int_{\Omega_{t_0}} (M^{(t_0,L)}_\varphi f)^p. \tag{6.46}$$

Indeed, on $\mathbb{R}^n \setminus \Omega_{t_0}$, by (6.44) and (6.45) we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\Omega_{t_0}} (M^{(t_0,L)}_\varphi f)^p &\le C_2^{-p}\int_{\mathbb{R}^n\setminus\Omega_{t_0}} (M^{\circ(t_0,L)}_{U,\widetilde{U}}f)^p \\ &\le \frac{1}{2}\int_{\mathbb{R}^n} (M^{(t_0,L)}_\varphi f)^p,\end{aligned}$$

and so we may obtain (6.46) by the standard "trick"

$$\int_{\mathbb{R}^n} \cdot = \int_{\Omega_{t_0}} \cdot + \int_{\mathbb{R}^n\setminus\Omega_{t_0}} \cdot \le \int_{\Omega_{t_0}} \cdot + \frac{1}{2}\int_{\mathbb{R}^n} \cdot \Rightarrow \int_{\mathbb{R}^n} \cdot \le 2\int_{\Omega_{t_0}} \cdot.$$

Our next step is showing that for $0 < q < p$, there exists a constant $C_3 > 0$ such that for any $x \in \Omega_{t_0}$ and $t_0 < 0$,

$$M_\varphi^{(t_0,L)} f(x) \le C_3 \big[M_\Theta \big(M_\varphi^{\circ(t_0,L)} f\big)^q (x)\big]^{1/q}, \tag{6.47}$$

where M_Θ is the anisotropic maximal function (6.1). To accomplish, we first recall that under the strict assumption of a t-continuous cover, we may assume that $M_{x,t} := M_t$ is constant for all $x \in \mathbb{R}^n$ and denote $\varphi_{x,t} := \varphi_t$. Now we define

$$F(y,t) := \big|f * \varphi_t(y)\big| \big(1 + \big|M_{t_0}^{-1} y\big|\big)^{-L} \big(1 + 2^{t+t_0}\big)^{-L}, \quad t \ge t_0,\ y \in \mathbb{R}^n.$$

Let $F_0^{t_0}(x)$ be as in (6.27) with $k = 0$, and let $x \in \mathbb{R}^n$. Then there exists $t' \in \mathbb{R}$ with $t' \ge t_0$ and $y' \in \theta(x, t')$ such that

$$F(y', t') \ge \frac{F_0^{t_0}(x)}{2} = \frac{M_\varphi^{(t_0,L)} f(x)}{2}. \tag{6.48}$$

Let $x' \in \theta(y', t' + kJ)$ for some sufficiently large $k \ge 1$ to be specified later. This implies that

$$M_{t'}^{-1}(x' - y') \in M_{t'}^{-1} M_{t'+kJ}(B^*). \tag{6.49}$$

Denote

$$\Phi(z) := \varphi\big(z + M_{t'}^{-1}(x' - y')\big) - \varphi(z).$$

Obviously, we have

$$f * \Phi_{t'}(y') = f * \varphi_{t'}(x') - f * \varphi_{t'}(y'). \tag{6.50}$$

Using (6.49) and the mean value theorem, we may estimate

$$\begin{aligned}
\|\Phi\|_{U,\widetilde{U}} &\le \sup_{h \in M_{t'}^{-1} M_{t'+kJ}(B^*)} \|\varphi(\cdot + h) - \varphi\|_{U,\widetilde{U}} \\
&= \sup_{h \in M_{t'}^{-1} M_{t'+kJ}(B^*)} \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \le U} (1 + |z|)^{\widetilde{U}} \big|\partial^\alpha \varphi(z + h) - \partial^\alpha \varphi(z)\big| \\
&\le C \big\|M_{t'}^{-1} M_{t'+kJ}\big\| \sup_{w \in M_{t'}^{-1} M_{t'+kJ}(B^*)} \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \le U+1} (1 + |z|)^{\widetilde{U}} \big|\partial^\alpha \varphi(z + w)\big|.
\end{aligned}$$

The shape condition (2.14) gives that

$$\big\|M_{t'}^{-1} M_{t'+kJ}\big\| \le a_5 2^{-a_6 kJ} \Rightarrow |w| \le a_5 2^{-a_6 kJ},$$

which for $k \geq 0$ also implies

$$1 + |z| = 1 + |z + w - w| \leq \big(1 + |z + w|\big)\big(1 + |w|\big) \leq C\big(1 + |z + w|\big).$$

We now plug these estimates

$$\begin{aligned}\|\Phi\|_{U,\widetilde{U}} &\leq C2^{-a_6 kJ} \sup_{z,w\in\mathbb{R}^n} \sup_{|\alpha|\leq U+1} \big(1 + |z + w|\big)^{\widetilde{U}} \big|\partial^\alpha \varphi(z + w)\big| \\ &\leq C2^{-a_6 kJ} \|\varphi\|_{U+1,\widetilde{U}} \\ &\leq C_4 2^{-a_6 kJ},\end{aligned}$$

where C_4 depends on $\mathbf{p}(\Theta)$, U, $\widetilde{U}$, and φ.

Let $z \in B^*$ be such that $x' = y' + M_{t'+kJ} z$. Applying (2.14), for $t' \geq t_0$ and $k \geq 0$, we have

$$\begin{aligned}1 + \big|M_{t_0}^{-1} x'\big| &= 1 + \big|M_{t_0}^{-1}(y' + M_{t'+kJ} z)\big| \\ &\leq (1 + \big|M_{t_0}^{-1} y'\big|)(1 + \big\|M_{t_0}^{-1} M_{t'+kJ}\big\| |z|) \\ &\leq (1 + \big|M_{t_0}^{-1} y'\big|)(1 + a_5 2^{-a_6(t'-t_0+kJ)}) \\ &\leq (1 + a_5)(1 + \big|M_{t_0}^{-1} y'\big|).\end{aligned}$$

Let $x \in \Omega_{t_0}$. Using these last two estimates together with (6.50), (6.48), and (6.45), we obtain

$$\begin{aligned}(1 + a_5)^L F(x', t') &= (1 + a_5)^L \big|f * \varphi_{t'}(x')\big| (1 + \big|M_{t_0}^{-1} x'\big|)^{-L} (1 + 2^{t'+t_0})^{-L} \\ &= (1 + a_5)^L \big|f * \varphi_{t'}(y') + f * \Phi_{t'}(y')\big| (1 + \big|M_{t_0}^{-1} x'\big|)^{-L} (1 + 2^{t'+t_0})^{-L} \\ &\geq (\big|f * \varphi_{t'}(y')\big| - \big|f * \Phi_{t'}(y')\big|)(1 + \big|M_{t_0}^{-1} y'\big|)^{-L} (1 + 2^{t'+t_0})^{-L} \\ &\geq \frac{M_\varphi^{(t_0,L)} f(x)}{2} - M_{U,\widetilde{U}}^{(t_0,L)} f(x) \|\Phi\|_{U,\widetilde{U}} \\ &\geq \frac{M_\varphi^{(t_0,L)} f(x)}{2} - 2^{\widetilde{U}} M_{U,\widetilde{U}}^{\circ(t_0,L)} f(x) C_4 2^{-a_6 kJ} \\ &\geq \frac{M_\varphi^{(t_0,L)} f(x)}{2} - 2^{\widetilde{U}} C_2 M_\varphi^{(t_0,L)} f(x) C_4 2^{-a_6 kJ}.\end{aligned}$$

Now choose k large enough such that $2^{\widetilde{U}} C_2 C_4 2^{-a_6 kJ} \leq 1/4$. This yields that for $x \in \Omega_{t_0}$, $x' \in y' + M_{t'+kJ}(B^*)$, and $y' \in \theta(x, t')$,

$$(1 + a_5)^L F(x', t') \geq \frac{M_\varphi^{(t_0,L)} f(x)}{4}.$$

We also have

$$x' \in y' + M_{t'+kJ}(B^*) \subseteq x + M_{t'}(B^*) + M_{t'+kJ}(B^*) \subseteq x + 2M_{t'}(B^*) \subseteq \theta(x, t' - J).$$

Thus we are able to conclude (6.47) from

$$\begin{aligned}
[M_\varphi^{(t_0,L)} f(x)]^q &\le \frac{C}{|M_{t'+kJ}(B^*)|} \int\limits_{y'+M_{t'+kJ}(B^*)} [F_x(x', t')]^q dx' \\
&\le C \frac{2^{kJ}}{|\theta(x, t' - J)|} \int\limits_{\theta(x,t'-J)} [M_\varphi^{\circ(t_0,L)} f(z)]^q dz \\
&\le C_3 M_\Theta([M_\varphi^{\circ(t_0,L)} f]^q)(x).
\end{aligned}$$

Consequently, (6.46), (6.47), and the maximal inequality for $p/q > 1$ yield

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} [M_\varphi^{(t_0,L)} f(x)]^p dx &\le 2 \int\limits_{\Omega_{t_0}} [M_\varphi^{(t_0,L)} f(x)]^p dx \\
&\le C \int\limits_{\Omega_{t_0}} [M_\Theta([M_\varphi^{\circ(t_0,L)} f]^q)(x)]^{p/q} dx \\
&\le C \int\limits_{\mathbb{R}^n} [M_\varphi^{\circ(t_0,L)} f(x)]^p dx.
\end{aligned}$$

Recalling that L does not depend on $t_0 < 0$, we may now take the limit as $t_0 \to -\infty$. Observe that as $t_0 \to -\infty$, the decay terms of $M_\varphi^{(t_0,L)}$ and $M_\varphi^{\circ(t_0,L)}$ converge pointwise to 1. Indeed, for any $y \in \mathbb{R}^n$ and $t_0 < 0$, using (2.14), we have

$$\begin{aligned}
|M_{t_0}^{-1} y| &= |M_{t_0}^{-1} M_0 M_0^{-1} y| \\
&\le \|M_{t_0}^{-1} M_0\| \|M_0\| |y| \\
&\le a_5 2^{a_6 t_0} \|M_0\| |y| \to 0, \quad t_0 \to -\infty.
\end{aligned}$$

This gives $\|M_\varphi f\|_p \le C \|M_\varphi^\circ f\|_p$, where the constant does not depend on $f \in \mathcal{S}'$. This concludes the proof of (ii). □

6.3 Anisotropic atomic spaces

As in the classical case, the anisotropic Hardy spaces can be characterized and then investigated through the atomic decompositions [31]. In the general setting of a space of homogeneous type X, equipped with a quasi-distance ρ and measure μ, a $(p, \infty, 1)$-atom a is a function with the following properties:

(i) $\operatorname{supp}(a) \subseteq B_\rho$, where B_ρ is a ball subject to the quasi-distance ρ,

(ii) $\|a\|_\infty \le \mu(B_p)^{-1/p}$,
(iii) $\int a d\mu = 0$.

We then may proceed (in the same manner we do below) to define the atomic space $H^1_{\infty,1}(X)$ through decompositions of atoms of the type $\sum_i \lambda_i a_i$, where $\{a_i\}$ are atoms, and $\sum_i |\lambda_i| < \infty$ [20]. However, in this general framework, for smaller values of p, the atoms lack the power of higher vanishing moments, which come into play in the classic setting of $\mathbb{R}^n$ and the Euclidean distance. In the setting of ellipsoid covers, we are able to generalize the Euclidean case to the full range $0 < p \le 1$.

Definition 6.22. For a cover Θ, we say that a triple (p,q,l) is *admissible* if $0 < p \le 1$, $1 \le q \le \infty$, $p < q$, and $l \in \mathbb{N}$ is such that $l \ge N_p(\Theta)$ (see (6.24)). A (p,q,l)-*atom* is a function $a : \mathbb{R}^n \to \mathbb{R}$ such that
(i) $\operatorname{supp}(a) \subseteq \theta$ for some $\theta \in \Theta$,
(ii) $\|a\|_q \le |\theta|^{1/q-1/p}$,
(iii) $\int_{\mathbb{R}^n} a(y) y^\alpha dy = 0$ for all $\alpha \in \mathbb{Z}^n_+$ such that $|\alpha| \le l$.

Definition 6.23. Let Θ be a continuous ellipsoid cover, and let (p,q,l) be an admissible triple. We define the *atomic Hardy space* $H^p_{q,l}(\Theta)$ associated with Θ as the set of all tempered distributions $f \in \mathcal{S}'$ of the form $f = \sum_{i=1}^\infty \lambda_i a_i$, where $\sum_{i=1}^\infty |\lambda_i|^p < \infty$, and a_i is a (p,q,l)-atom for every $i \in \mathbb{N}$. The atomic quasi-norm of f is defined as

$$\|f\|_{H^p_{q,l}(\Theta)} := \inf\left\{\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{1/p} : f = \sum_{i=1}^\infty \lambda_i a_i\right\}.$$

Our main goal is to prove the following:

Theorem 6.24 ([31]). *Let Θ be a pointwise continuous ellipsoid cover, and let (p,q,l) be an admissible triple. Then*

$$H^p(\Theta) \sim H^p_{q,l}(\Theta).$$

The proof of the theorem is composed of two inclusions proved in Theorems 6.26 and 6.43.

6.3.1 The inclusion $H^p_{q,l}(\Theta) \subseteq H^p(\Theta)$

First, we prove that each admissible atom is in $H^p(\Theta)$.

Theorem 6.25. *Suppose (p,q,l) is admissible for a continuous cover Θ. Then there exists a constant $c(p,q,l,n,\mathbf{p}(\Theta)) > 0$ such that for any (p,q,l)-atom a, $\|M^\circ a\|_p \le c$.*

Proof. Let $\theta(z,t)$ be the ellipsoid associated with an atom a, where $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We estimate the integral of the function $(M^\circ a)^p$ separately on $\theta(z,t-J)$ and on $\theta(z,t-J)^c$, where J is from (2.30).

We begin with the estimate of $\int_{\theta(z,t-J)}(M^\circ a)^p$. There are two cases, $q>1$ and $q=1$. We start with $1<q<\infty$. Since $p \le 1$, we have $q/p>1$, and by the Hölder inequality we have

$$\int\limits_{\theta(z,t-J)} (M^\circ a)^p \le \Bigg(\int\limits_{\theta(z,t-J)} (M^\circ a)^q\Bigg)^{p/q} \big|\theta(z,t-J)\big|^{1-p/q}. \tag{6.51}$$

Applying Theorem 6.10 and then property (ii) in Definition 6.22 gives

$$\begin{aligned}\Bigg(\int\limits_{\theta(z,t-J)} (M^\circ a)^q\Bigg)^{p/q} &\le \|M^\circ a\|^p_{L_q(\mathbb{R}^n)}\\ &\le C\|a\|^p_q\\ &\le C\big|\theta(z,t-J)\big|^{p/q-1},\end{aligned}$$

which, combined with (6.51), gives $\int_{\theta(z,t-J)}(M^\circ a)^p \le C$. The case $q=\infty$ is simpler.

The second case is $q=1$. Since $p<q$, we have $p<1$. Let us denote $\omega_\lambda := \{x \in \mathbb{R}^n : M^\circ a(x) > \lambda\}$ for $\lambda>0$. By the maximal theorem we have that

$$|\omega_\lambda| \le \frac{C}{\lambda}\|a\|_1,$$

which, combined with property (ii) in Definition 6.22, gives

$$\big|\omega_\lambda \cap \theta(z,t-J)\big| \le \frac{C}{\lambda}\big|\theta(z,t-J)\big|^{1-1/p}.$$

We use this estimate and $p<1$ to obtain

$$\begin{aligned}\int\limits_{\theta(z,t-J)} \big(M^\circ a(x)\big)^p dx &= \int\limits_0^\infty \big|\omega_\lambda \cap \theta(z,t-J)\big| p\lambda^{p-1} d\lambda\\ &\le \int\limits_0^{|\theta(z,t-J)|^{-1/p}} \big|\theta(z,t-J)\big| p\lambda^{p-1} d\lambda\\ &\quad + C\int\limits_{|\theta(z,t-J)|^{-1/p}}^\infty \big|\theta(z,t-J)\big|^{1-1/p} p\lambda^{p-2} d\lambda\\ &\le C.\end{aligned}$$

We now estimate $\int_{\theta(z,t-J)^c}(M^\circ a)^p$. For every $k \geq 1$, we have that $\theta(z,t-kJ+J) \subset \theta(z,t-kJ)$, where J is from (2.30). Applying (2.13) gives

$$\int\limits_{\theta(z,t-J)^c} (M^\circ a(x))^p dx = \sum_{k=2}^{\infty} \int\limits_{\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)} (M^\circ a(x))^p dx$$
$$\leq C \sum_{k=2}^{\infty} 2^{-t} 2^{kJ} \sup_{x\in\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)} (Ma(x))^p.$$

Therefore, to prove the theorem, it is sufficient to show that

$$\sup_{x\in\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)} (M^\circ a(x))^p \leq c_1 2^t 2^{-c_2 k} \tag{6.52}$$

for every $k \geq 2$, where $c_2 > J$. Furthermore, by (6.19) it is sufficient to prove

$$\sup_{\psi\in\mathcal{S}_{N_p,\widetilde{N}_p},\operatorname{supp}(\psi)\subseteq(B^*)} \sup_{s\in\mathbb{R}} \sup_{x\in\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)} |a * \psi_{x,s}(x)|^p \leq c_1 2^t 2^{-c_2 k}. \tag{6.53}$$

Therefore let $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p}$ with $\operatorname{supp}(\psi) \subseteq B^*$, $s \in \mathbb{R}^n$, and $x \in \theta(z,t-kJ) \setminus \theta(z,t-kJ+J)$. Since $\operatorname{supp}(a) \subseteq \theta(z,t)$ and $\operatorname{supp}(\psi_{x,s}(x-\cdot)) \subseteq \theta(x,s)$, if $\theta(z,t) \cap \theta(x,s) = \emptyset$, then $a * \psi_{x,s}(x) = 0$. Thus we may assume that

$$\theta(z,t) \cap \theta(x,s) \neq \emptyset. \tag{6.54}$$

Suppose P is a polynomial (to be chosen later) of order $N_p(\Theta)$. Applying (2.13), the vanishing moments property of atoms (Definition 6.22), and the Hölder inequality, for $1 < q \leq \infty$, we have

$$|a * \psi_{x,s}(x)| \leq C2^s \left| \int\limits_{\mathbb{R}^n} a(y)\psi(M_{x,s}^{-1}(x-y))dy \right|$$
$$\leq C2^s \left| \int\limits_{\mathbb{R}^n} a(y)(\psi(M_{x,s}^{-1}(x-y)) - P(M_{x,s}^{-1}(x-y)))dy \right|$$
$$\leq C2^s \int\limits_{\theta(z,t)} |a(y)||\psi(M_{x,s}^{-1}(x-y)) - P(M_{x,s}^{-1}(x-y))|dy$$
$$\leq C2^s \|a\|_q \left(\int\limits_{\theta(z,t)} |\psi(M_{x,s}^{-1}(x-y)) - P(M_{x,s}^{-1}(x-y))|^{q'} dy \right)^{1/q'}$$
$$\leq C2^s \|a\|_q 2^{-s/q'} \left(\int\limits_{F(\theta(z,t))} |\psi(y) - P(y)|^{q'} dy \right)^{1/q'},$$

where $1/q + 1/q' = 1$, and

$$F(\theta(z,t)) := M_{x,s}^{-1}(x - [M_{z,t}(B^*) + z]) = M_{x,s}^{-1}(x - z) - M_{x,s}^{-1}M_{z,t}(B^*).$$

Therefore

$$\begin{aligned}|a * \psi_{x,s}(x)|^p &\le C2^{sp/q}\|a\|_q^p |F(\theta(z,t))|^{p/q'} \sup_{y \in F(\theta(z,t))} |\psi(y) - P(y)|^p \\ &\le C2^t 2^{(s-t)p/q} |M_{x,s}^{-1}M_{z,t}(B^*)|^{p/q'} \sup_{y \in F(\theta(z,t))} |\psi(y) - P(y)|^p.\end{aligned}$$

A similar and simpler calculation for $q = 1$ provides the corresponding estimate

$$|a * \psi_{x,s}(x)|^p \le C2^t 2^{(s-t)p} \sup_{y \in F(\theta(z,t))} |\psi(y) - P(y)|^p.$$

We now analyze the set $F(\theta(z,t))$. We know that

$$F(\theta(z,t)) = M_{x,s}^{-1}(x - z) - M_{x,s}^{-1}M_{z,t}(B^*),$$

where

$$x \in \theta(z, t - kJ) \setminus \theta(z, t - kJ + J) = M_{z,t-kJ}(B^*) \setminus M_{z,t-kJ+J}(B^*) + z,$$

which implies that

$$x - z \in M_{z,t-kJ}(B^*) \setminus M_{z,t-kJ+J}(B^*).$$

Therefore

$$F(\theta(z,t)) \subset [M_{x,s}^{-1}M_{z,t-kJ}(B^*) \setminus M_{x,s}^{-1}M_{z,t-kJ+J}(B^*)] - M_{x,s}^{-1}M_{z,t}(B^*). \tag{6.55}$$

Since (2.30) gives for $k \ge 2$,

$$M_{x,s}^{-1}M_{z,t}(B^*) \subseteq (1/2)M_{x,s}^{-1}M_{z,t-kJ+J}(B^*),$$

we have

$$F(\theta(z,t)) \subseteq ((1/2)M_{x,s}^{-1}M_{z,t-kJ+J}(B^*))^c. \tag{6.56}$$

Case 1: $t \le s$. We choose $P = 0$ and estimate the term $|M_{x,s}^{-1}M_{z,t}(B^*)|^{p/q'}$ for $q > 1$. From (2.14) and (6.54) we induce that

$$M_{x,s}^{-1}M_{z,t}(B^*) \subset a_3^{-1}2^{a_4(s-t)}B^*,$$

which implies that

$$\left|M_{x,s}^{-1}M_{z,t}(B^*)\right|^{p/q'} \le C2^{a_4(s-t)np/q'}. \tag{6.57}$$

Using (6.56), (6.57), and $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p} \subset \mathcal{S}_{N_p,N_p}$, we derive

$$\left|a * \psi_{x,s}(x)\right|^p \le C2^t 2^{(s-t)(p/q+na_4p/q')} \sup_{y\in F(\theta(z,t))} (1+|y|)^{-pN_p}. \tag{6.58}$$

We now estimate the term

$$\sup_{y\in F(\theta(z,t))} (1+|y|)^{-pN_p}.$$

Since $2 \le k$ and $t \le s$, we have $t - kJ + J \le s$, which implies by (6.56) and (2.14) that for $y \in F(\theta(z,t))$,

$$|y| \ge (2a_5)^{-1}2^{a_6(s-t)}2^{a_6Jk}2^{-a_6J} \Rightarrow (1+|y|)^{-pN_p} \le C2^{-a_6(s-t)pN_p}2^{-a_6JpN_pk}.$$

Combining this with (6.58) and applying $s - t \ge 0$ and (6.24), we conclude that

$$\begin{aligned}\left|a * \psi_{x,s}(x)\right|^p &\le C2^t 2^{(s-t)[p/q+pna_4/q'-a_6pN_p]}2^{-(a_6JpN_p)k}\\ &\le C2^t 2^{-(a_6JpN_p)k}.\end{aligned}$$

Since N_p satisfies (6.24), setting $c_2 := a_6JpN_p > J$ provides the desired estimate (6.52) for the first case.

Case 2: $s \le t$. From (6.54) and (2.14) we have that for $q > 1$,

$$\left|M_{x,s}^{-1}M_{z,t}(B^*)\right|^{p/q'} \le C2^{(s-t)a_6pn/q'},$$

which yields for $1 \le q \le \infty$ and $p < q$

$$\begin{aligned}\left|a * \psi_{x,s}(x)\right|^p &\le C2^t 2^{(s-t)(p/q+a_6pn/q')} \sup_{y\in F(\theta(z,t))} \left|\psi(y) - P(y)\right|^p\\ &\le C2^t \sup_{y\in F(\theta(z,t))} \left|\psi(y) - P(y)\right|^p.\end{aligned}$$

We now choose P to be the Taylor polynomial of order N_p (degree $N_p - 1$) of ψ expanded at point $M_{x,s}^{-1}(x-z)$ and estimate $\sup_{y\in F(\theta(z,t))} |\psi(y) - P(y)|^p$. From (2.14) we have $M_{x,s}^{-1}M_{z,t}(B^*) \subset a_5 2^{-a_6(t-s)}B^*$. The Taylor remainder theorem gives

$$\begin{aligned}\sup_{y\in F(\theta(z,t))} \left|\psi(y) - P(y)\right| &\le C \sup_{u\in M_{x,s}^{-1}M_{z,t}(B^*)} \sup_{|\alpha|=N_p} \left|\partial^\alpha \psi(M_{x,s}^{-1}(x-z)+u)\right| |u|^{N_p}\\ &\le C2^{-a_6(t-s)N_p} \sup_{y\in F(\theta(z,t))} (1+|y|)^{-N_p}.\end{aligned}$$

This allows us to estimate

$$\left|a * \psi_{x,s}(x)\right|^p \le C2^t 2^{-a_6(t-s)pN_p} \sup_{y \in F(\theta(z,t))} (1+|y|)^{-pN_p}. \tag{6.59}$$

We have two cases to consider. The first one is where $t - kJ + J \le s \le t$, and the second one is where $s \le t - kJ + J$. We start with the first case. From (2.14) we have

$$a_5^{-1} 2^{-a_6 J} 2^{a_6(s-t)} 2^{a_6 kJ} B^* \subset M_{x,s}^{-1} M_{z,t-kJ+J}(B^*),$$

which, combined with (6.56), leads to

$$(1+|y|)^{-pN_p} \le C 2^{a_6 pN(t-s)} 2^{-a_6 pN_p kJ}, \quad \forall y \in F(\theta(z,t)).$$

Using this estimate with (6.59) allows us to conclude with $c_2 := a_6 JpN_p > J$ that

$$\left|a * \psi_{x,s}(x)\right|^p \le C2^t 2^{c_2 k}.$$

For the case where $s \le t - kJ + J$, from (6.59) we proceed using the estimate $(1+|y|)^{-pN_p} \le C$ for all $y \in \mathbb{R}^n$, the fact that $J(k-1) \le t - s$, and assumption (6.24) to obtain

$$\begin{aligned} \left|a * \psi_{x,s}(x)\right|^p &\le C2^t 2^{-a_6(t-s)pN_p} \\ &\le C2^t 2^{-a_6 J(k-1)pN_p} \\ &\le C2^t 2^{-c_2 k}. \end{aligned}$$

Thus we get (6.52) for the case $s \le t$, which completes the proof. □

Theorem 6.26. *Let* Θ *be a continuous cover and suppose* (p, q, l) *is admissible (see Definition* 6.22*). Then*

$$H_{q,l}^p(\Theta) \subseteq H^p(\Theta).$$

Proof. Let $f \in H_{q,l}^p(\Theta)$. For $\varepsilon > 0$, assume that $f = \sum_{i=1}^\infty \lambda_i a_i$, where $\sum_{i=1}^\infty |\lambda_i|^p \le \|f\|_{H_{q,l}^p(\Theta)}^p + \varepsilon$. Then by Theorem 6.25

$$\begin{aligned} \|f\|_{H^p(\Theta)}^p &= \int_{\mathbb{R}^n} \left[M^\circ\left(\sum_{i=1}^\infty \lambda_i a_i \right) \right]^p \\ &\le \sum_{i=1}^\infty |\lambda_i|^p \int_{\mathbb{R}^n} [M^\circ(a_i)]^p \\ &\le C(\|f\|_{H_{q,l}^p(\Theta)}^p + \varepsilon). \end{aligned}$$ □

6.3.2 The Calderón–Zygmund decomposition

To show the converse inclusion $H^p(\Theta) \subseteq H^p_{q,l}(\Theta)$, we need to carefully construct, for each given distribution, an appropriate atomic decomposition. We achieve this by using a pointwise variable Calderón–Zygmund decomposition. For a given pointwise continuous cover Θ, we consider a tempered distribution f such that for every $\lambda > 0$, $|\{x : M^\circ f(x) > \lambda\}| < \infty$. For fixed $\lambda > 0$, we define

$$\Omega := \{x : M^\circ f(x) > \lambda\}.$$

By Theorem 6.7 we know that Ω is an open set, and thus we may apply the Whitney lemma 2.21. This implies that there exist constants $\gamma(\mathbf{p}(\Theta))$ and $L(\mathbf{p}(\Theta))$ such that for $m := J + \gamma$, where J is from (2.30), there exist sequences $\{x_i\}_{i\in\mathbb{N}} \subset \Omega$ and $\{t_i\}_{i\in\mathbb{N}}$ such that

$$\Omega = \bigcup_{i\in\mathbb{N}} \theta(x_i, t_i), \tag{6.60}$$

$$\theta(x_i, t_i + \gamma) \cap \theta(x_j, t_j + \gamma) = \emptyset, \quad i \neq j, \tag{6.61}$$

$$\theta(x_i, t_i - J - 2\gamma) \cap \Omega^c = \emptyset, \quad \text{but} \quad \theta(x_i, t_i - J - 2\gamma - 1) \cap \Omega^c \neq \emptyset, \quad \forall i \in \mathbb{N}, \tag{6.62}$$

$$\theta(x_i, t_i - J - \gamma) \cap \theta(x_j, t_j - J - \gamma) \neq \emptyset \Rightarrow |t_i - t_j| \leq \gamma + 1, \tag{6.63}$$

$$\#\Lambda_i \leq L, \quad \Lambda_i := \{j \in \mathbb{N} : \theta(x_j, t_j - J - \gamma) \cap \theta(x_i, t_i - J - \gamma) \neq \emptyset\}, \quad \forall i \in \mathbb{N}. \tag{6.64}$$

Fix $\phi \in \mathcal{S}$ such that $\operatorname{supp}(\phi) \subset 2B^*$, $0 \leq \phi \leq 1$, and $\phi \equiv 1$ on B^*. For every $i \in \mathbb{N}$, we define

$$\widetilde{\phi}_i(x) := \phi(M^{-1}_{x_i,t_i}(x - x_i)).$$

By (2.30)

$$\operatorname{supp}(\widetilde{\phi}_i) \subseteq x_i + 2M_{x_i,t_i}(B^*) \subseteq \theta(x_i, t_i - J).$$

We define a partition of unity of Ω by

$$\varphi_i(x) := \begin{cases} \frac{\widetilde{\phi}_i(x)}{\sum_j \widetilde{\phi}_j(x)} & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases} \tag{6.65}$$

Observe that by the properties of the Whitney cover

(i) φ_i is well defined since for any $x \in \Omega$, $1 \leq \sum_i \widetilde{\phi}_i(x) \leq L$,

(ii) $\varphi_i \in \mathcal{S}$ and $\operatorname{supp}(\varphi_i) \subseteq \theta(x_i, t_i - J)$.

(iii) for every $x \in \mathbb{R}^n$,

$$\sum_i \varphi_i(x) = \mathbf{1}_\Omega(x),$$

which implies that the family $\{\varphi_i\}$ forms a smooth partition of unitary subordinate to the covering of Ω by the ellipsoids $\{\theta(x_i, t_i - J)\}$. Let Π_l denote the space of polynomials of n variables of degree $\le l$, where $l \ge N_p(\Theta)$ (see Definition 6.22). For each $i \in \mathbb{N}$, we introduce a Hilbert space structure on the space Π_l by setting

$$\langle P, Q\rangle_i := \frac{1}{\int \varphi_i} \int_{\mathbb{R}^n} PQ\varphi_i, \quad \forall P, Q \in \Pi_l. \tag{6.66}$$

The distribution $f \in \mathcal{S}'$ induces a linear functional on Π_l by

$$Q \to \langle f, Q\rangle_i, \quad \forall Q \in \Pi_l,$$

which by the Riesz lemma is represented by a unique polynomial $P_i \in \Pi_l$ such that

$$\langle f, Q\rangle_i = \langle P_i, Q\rangle_i, \quad \forall Q \in \Pi_l. \tag{6.67}$$

Obviously, P_i is the orthogonal projection of f with respect to the norm (6.66).

For every $i \in \mathbb{N}$, we define the locally "good part" $P_i\varphi_i$ and "bad part" $b_i := (f - P_i)\varphi_i$. We will show that the series $\sum_i b_i$ converges in $\mathcal{S}'$, which will allow us to define the global "good part" $g := f - \sum_i b_i$. Moreover, we will show that for $f \in H^p(\Theta)$, the series $\sum_i b_i$ converges in $H^p(\Theta)$.

Definition 6.27. The representation $f = g + \sum_i b_i$, where g and b_i are as above, is a *Calderón–Zygmund decomposition* of degree l and height λ associated with M°.

Lemma 6.28. *For any $i \in \mathbb{N}$, let $z_i \in \theta(x_i, t_i - K_1)$ and $s_i \in \mathbb{R}$ be such that $t_i \le s_i + K_2$, where $K_1, K_2 > 0$. Then there exists a constant $c > 0$, depending on the parameters of the cover, N, K_1, K_2, and choice of φ, such that*

$$\max_{|\alpha|\le N} \left\| \partial^\alpha [\varphi_i(z_i - M_{z_i,s_i}(\cdot))] \right\|_\infty \le c.$$

Proof. Observe that it is sufficient to bound the derivatives of $\varphi_i(M_{z_i,s_i}(\cdot))$. Recall that $\operatorname{supp}(\varphi_i) \subseteq \theta(x_i, t_i - J)$ for $i \in \mathbb{N}$. Also, (6.64) ensures that for $U_i := \{j \in \mathbb{N} : \theta(x_j, t_j - J) \cap \theta(x_i, t_i - J) \ne \emptyset\}$, we have that $\#U_i \le \#\Lambda_i \le L$. Thus we may write

$$\begin{aligned} \varphi_i((M_{z_i,s_i}(y)) &= \frac{\widetilde{\phi}_i(M_{z_i,s_i}(y))}{\sum_{j\in\mathbb{N}} \widetilde{\phi}_j(M_{z_i,s_i}(y))} \\ &= \frac{\phi(M^{-1}_{x_i,t_i} M_{z_i,s_i}(y) - M^{-1}_{x_i,t_i}(x_i))}{\sum_{j\in U_i} \phi(M^{-1}_{x_j,t_j} M_{z_i,s_i}(y) - M^{-1}_{x_j,t_j}(x_j))}. \end{aligned}$$

The desired estimate follows from iterative application of quotient rule combined with

$$\max_{|\alpha|\le N} \left\| \partial^\alpha [\varphi(M^{-1}_{x_j,t_j} M_{z_i,s_i}(\cdot))] \right\|_\infty \le C, \quad \forall j \in U_i, \tag{6.68}$$

where $c > 0$ depends on the parameters of the cover, N, K_1, K_2, and choice of φ. Indeed, (6.68) holds, since by (6.63) $|t_i - t_j| \le \gamma + 1$ for every $j \in U_i$, and so application of (2.14) yields that $\|M_{x_j,t_j}^{-1} M_{x_i,t_i}\| \le c_1$ and $\|M_{x_i,t_i}^{-1} M_{z_i,s_i}\| \le c_2$ for some constants $c_1, c_2 > 0$. Thus we also have $\|M_{x_j,t_j}^{-1} M_{z_i,s_i}\| \le c_1 c_2$. □

For a fixed $i \in \mathbb{N}$, let $\{\pi_\beta : \beta \in \mathbb{N}_+^n, |\beta| \le l\}$ be an orthonormal basis for Π_l with respect to the Hilbert space structure (6.66). For $|\beta| \le l$ and a point $z \in \theta(x_i, t_i - J - 2\gamma - 1) \cap \Omega^c$ (whose existence is guaranteed by (6.62)), we define

$$\Phi_\beta(y) := \frac{|\det(M_{z,t_i})|}{\int \varphi_i} \pi_\beta(z - M_{z,t_i}(y)) \varphi_i(z - M_{z,t_i}(y)). \tag{6.69}$$

Lemma 6.29. *For any $N, \tilde{N}$, there exists $c(N, \tilde{N}, l, \mathbf{p}(\Theta)) > 0$ such that*

$$\|\Phi_\beta\|_{N,\tilde{N}} \le c, \quad \forall \beta \in \mathbb{N}_+^n,\ |\beta| \le l.$$

Proof. We have

$$\begin{aligned} \operatorname{supp}(\Phi_\beta) &= \operatorname{supp}(\varphi_i(z - M_{z,t_i}(\cdot))) \\ &\subseteq \{y \in \mathbb{R}^n : y \in M_{z,t_i}^{-1}(z - x_i) + M_{z,t_i}^{-1} M_{x_i,t_i-J}(B^*)\}. \end{aligned}$$

Since $z \in \theta(x_i, t_i - J - 2\gamma - 1) = x_i + M_{x_i,t_i-J-2\gamma-1}(B^*)$, by (2.14)

$$M_{z,t_i}^{-1}(z - x_i) \in M_{z,t_i}^{-1} M_{x_i,t_i-J-2\gamma-1}(B^*) \subseteq c_1 B^*.$$

Also, by Lemma 2.18, for any $s > 0$, $\theta(x_i, t_i - J) \subset \theta(x_i, t_i - J - s - \gamma)$, and so

$$M_{z,t_i}^{-1} M_{x_i,t_i-J}(B^*) \subset M_{z,t_i}^{-1} M_{x_i,t_i-J-2\gamma-1}(B^*) \subseteq c_2 B^*.$$

Therefore, combining the last two estimates, we conclude that for some $c_3 > 0$, $\operatorname{supp}(\Phi_\beta) \subset c_3 B^*$. Thus, to prove the Lemma, it remains to show that the partial derivatives of Φ_β up to the order N are bounded. We begin with the estimate of the first term in (6.69). We know that

$$\int_{\mathbb{R}^n} \varphi_i = \int_{\theta(x_i,t_i-J)} \varphi_i \ge \int_{\theta(x_i,t_i)} \frac{1}{L} = \frac{1}{L} |\theta(x_i, t_i)|.$$

Applying (2.13) gives

$$\frac{|\det(M_{z,t_i})|}{\int \varphi_i} \le L \frac{|\theta(z, t_i)|}{|\theta(x_i, t_i)|} \le L a_1^{-1} a_2.$$

For the third term in (6.69), we get from Lemma 6.28 with the choice $K_1 = J + 2\gamma + 1$ and $K_2 = 0$ that

$$\max_{|\alpha|\le N} \|\partial^\alpha[\varphi_i(z - M_{z,t_i}(\cdot))]\|_\infty \le c.$$

We now estimate the partial derivatives of the second term. Since Π_l is finite vector space, all the norms are equivalent, and there exists a constant $c_4(c_3, N, l, n) > 0$ such that for every $P \in \Pi_l$,

$$\max_{|\alpha|\le N} \|\partial^\alpha P\|_{L_\infty(c_3 B^*)} \le c_4 \|P\|_{L_2(B^*)}.$$

By Lemma 1.23, since $\theta(x_i, t_i) \subset \theta(x_i, t_i - J - 3\gamma - 1)$ with $|\theta(x_i, t_i - J - 3\gamma - 1)| \le c|\theta(x_i, t_i)|$, we also have that

$$\|P\|_{L_2(\theta(x_i, t_i - J - 3\gamma - 1))} \le c\|P\|_{L_2(\theta(x_i, t_i))}.$$

By Lemma 2.18

$$\theta(z, t_i) \subset \theta(x_i, t_i - J - 3\gamma - 1).$$

Applying the last three estimates together with $\varphi_i \ge 1/L$ on $\theta(x_i, t_i)$ and the fact that π_β is normalized with respect to (6.66), we get

$$\begin{aligned}
\max_{|\alpha|\le N} \|\partial^\alpha[\pi_\beta(z - M_{z,t_i}\cdot)]\|_{L_\infty(c_3 B^*)} &\le c_4 \|\pi_\beta(z - M_{z,t_i}\cdot)\|_{L_2(B^*)} \\
&= c_4 |\det M_{z,t_i}|^{-1/2} \|\pi_\beta\|_{L_2(\theta(z,t_i))} \\
&\le c_4 |\det M_{z,t_i}|^{-1/2} \|\pi_\beta\|_{L_2(\theta(x_i, t_i - J - 3\gamma - 1))} \\
&\le C |\det M_{z,t_i}|^{-1/2} \|\pi_\beta\|_{L_2(\theta(x_i,t_i))} \\
&\le C |\det M_{z,t_i}|^{-1/2} \|\pi_\beta \varphi_i\|_{L_2(\theta(x_i,t_i))} \\
&\le C\left(\int \varphi_i\right)^{-1/2} \|\pi_\beta \varphi_i\|_{L_2} = C.
\end{aligned}$$

Now since Φ_β is supported on $c_3 B^*$ and we have bounded the $\mathcal{S}_{N,\tilde{N}}$ norm of the three terms in (6.69) by absolute constants, we can apply the product rule to conclude the lemma. □

Now we can estimate the local good parts of f.

Lemma 6.30. *There exists a constant $c > 0$ such that*

$$\|P_i \varphi_i\|_\infty \le \|P_i\|_{L_\infty(\theta(x_i, t_i - J))} \le c\lambda,$$

where φ_i is defined in (6.65), *and P_i is defined by* (6.67). *If $M^\circ f \in L_\infty$, then we also have*

$$\|P_i\varphi_i\|_\infty \le c\|M^\circ f\|_\infty. \tag{6.70}$$

Proof. Combining $\mathrm{supp}(\varphi_i) \subseteq \theta(x_i, t_i - J)$ and $\|\varphi_i\|_\infty \le 1$, we have

$$\|P_i\varphi_i\|_\infty \le \|P_i\|_{L_\infty(\theta(x_i,t_i-J))}.$$

For the function Φ_β defined in (6.69) and the point $z \in \theta(x_i, t_i - J - 2\gamma - 1) \cap \Omega^c$, Lemma 6.29 yields

$$\left|f * (\Phi_\beta)_{z,t_i}(z)\right| \le \|\Phi_\beta\|_{N_p,\widetilde{N}_p} M^\circ f(z) \le c\lambda,$$

where N_p and $\widetilde{N}_p$ are defined by (6.24) and (6.25). Note that for the case $M^\circ f \in L_\infty$, we also have

$$\left|f * (\Phi_\beta)_{z,t_i}(z)\right| \le c\|M^\circ f\|_\infty.$$

Next, using definition (6.69) and then (6.66), we have

$$\begin{aligned}\left|f * (\Phi_\beta)_{z,t_i}(z)\right| &= \left|\det(M_{z,t_i}^{-1})\right|\left|\int f(y)\Phi_\beta\big(M_{z,t_i}^{-1}(z-y)\big)dy\right| \\ &= \left|\frac{1}{\int \varphi_i}\int_{\mathbb{R}^n} f(y)\pi_\beta(y)\varphi_i(y)dy\right| \\ &= \left|\langle f, \pi_\beta\rangle_i\right|.\end{aligned}$$

Therefore for all $|\beta| \le l$,

$$\left|\langle f, \pi_\beta\rangle_i\right| \le C\lambda, \tag{6.71}$$

and if $M^\circ f \in L_\infty$, then

$$\left|\langle f, \pi_\beta\rangle_i\right| \le C\|M^\circ f\|_\infty. \tag{6.72}$$

By Lemma 1.24, Lemma 1.23, and (6.66) we have

$$\begin{aligned}\|\pi_\beta\|_{L_\infty(\theta(x_i,t_i-J))} &\le C\left|\theta(x_i, t_i - J)\right|^{-1/2}\|\pi_\beta\|_{L_2(\theta(x_i,t_i-J))} \\ &\le C\frac{1}{(\int \varphi_i)^{1/2}}\|\pi_\beta\|_{L_2(\theta(x_i,t_i))} \\ &\le C\langle \pi_\beta, \pi_\beta\rangle_i \\ &\le C.\end{aligned}$$

Recall that by (6.67) we have that

$$P_i = \sum_{|\beta|\le l} \langle f, \pi_\beta \rangle_i \pi_\beta. \tag{6.73}$$

Combining with this (6.71), we get

$$\sup_{y\in\theta(x_i,t_i-J)} |P_i(y)| \le \sum_{|\beta|\le l} |\langle f, \pi_\beta\rangle_i| |\pi_\beta(y)| \le C\lambda.$$

Combining this with (6.72) for the case $M^\circ f \in L_\infty$ gives

$$\sup_{y\in\theta(x_i,t_i-J)} |P_i(y)| \le C\|M^\circ f\|_\infty. \qquad \square$$

Lemma 6.31. *There exists a constant $c > 0$ such that*

$$M^\circ b_i(x) \le cM^\circ f(x), \quad \forall x \in \theta(x_i, t_i - J - \gamma).$$

Proof. Let $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p}$, $x \in \theta(x_i, t_i - J - \gamma)$, and $s \in \mathbb{R}$. Using (6.19), we can further assume that $\operatorname{supp}(\psi) \subseteq B^*$. We get

$$\begin{aligned} |b_i * \psi_{x,s}(x)| &= |((f - P_i)\varphi_i) * \psi_{x,s}(x)| \\ &\le |((f\varphi_i) * \psi_{x,s}(x)| + |(P_i\varphi_i) * \psi_{x,s}(x)| \\ &=: I_1 + I_2. \end{aligned}$$

We first estimate I_2. Since $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p}$ and $\widetilde{N}_p > n$, we have that $\|\psi\|_{L_1} \le c$, where $c > 0$ does not depend on ψ. For $x \in \theta(x_i, t_i - J - \gamma) \subset \Omega$, we have that $M^\circ f(x) > \lambda$, and combining this with Lemma 6.30, we have

$$I_2 \le |\det(M_{x,s}^{-1})| \int_{\mathbb{R}^n} |P_i(y)\varphi_i(y)||\psi_{x,s}(x-y)|dy \le C\lambda\|\psi\|_{L_1} \le CM^\circ f(x).$$

For the estimate of I_1, there are two cases.
Case 1: $t_i \le s$. For $\Phi(y) := \varphi_i(x - M_{x,s}(y))\psi(y)$, we have

$$I_1 = |f * \Phi_{x,s}(x)| \le \|\Phi\|_{N_p,\widetilde{N}_p} M^\circ f(x).$$

Let us estimate the term $\|\Phi\|_{N_p,\widetilde{N}_p}$. First, observe that $\operatorname{supp}(\Phi) \subseteq \operatorname{supp}(\psi) \subseteq B^*$. Now, since $t_i \le s$ and $x \in \theta(x_i, t_i - J - \gamma)$, Lemma 6.28 with $K_1 = J + \gamma$ and $K_2 = 0$ yields

$$\max_{|\alpha|\le N_p} \|\partial^\alpha[\varphi_i(x - M_{x,s}(\cdot))]\|_\infty \le C.$$

Therefore by the product rule $\|\Phi\|_{N_p,\widetilde{N}_p} \le C\|\psi\|_{N_p,\widetilde{N}_p} \le C$, and hence $I_1 \le CM^\circ f(x)$.

Case 2: $s < t_i$. We define

$$\widetilde{\Phi}(y) := \frac{|\det(M_{x,t_i})|}{|\det(M_{x,s})|}\varphi_i(x - M_{x,t_i}(y))\psi(M_{x,s}^{-1}M_{x,t_i}(y)).$$

Observe that

$$I_1 = |f * \widetilde{\Phi}_{x,t_i}(x)| \leq \|\widetilde{\Phi}\|_{N_p,\widetilde{N}_p} M^\circ f(x). \tag{6.74}$$

Therefore it suffices to show that $\|\widetilde{\Phi}\|_{N_p,\widetilde{N}_p} \leq C$. Since $s < t_i$, the first constant term of $\widetilde{\Phi}$ is bounded by

$$\frac{|\det(M_{x,t_i})|}{|\det(M_{x,s})|} \leq a_1^{-1}a_2 2^{s-t_i} \leq C.$$

For the second term, because $x \in \theta(x_i, t_i - J - \gamma)$, we get

$$\operatorname{supp}(\widetilde{\Phi}) \subset \operatorname{supp}(\varphi_i(x - M_{x,t_i}(\cdot))) \subset cB^*.$$

Lemma 6.28 gives

$$\max_{|\alpha|\leq N}\|\partial^\alpha[\varphi_i(x - M_{x,t_i}(\cdot))]\|_\infty \leq C.$$

Since $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p}$ and $\|M_{x,s}^{-1}M_{x,t_i}\| \leq C$ for $s \leq t_i$, we have

$$\|\psi(M_{x,s}^{-1}M_{x,t_i}\cdot)\|_{N_p,\widetilde{N}_p} \leq C\|\psi\|_{N_p,\widetilde{N}_p} \leq C.$$

Collecting the three estimates and applying the chain rule, we conclude that

$$\|\widetilde{\Phi}|\|_{N_p,\widetilde{N}_p} \leq C\|\psi\|_{N_p,\widetilde{N}_p} \leq C. \qquad \square$$

Lemma 6.32. *There exists a constant $c > 0$ such that for all $i \in \mathbb{N}$,*

$$M^\circ b_i(x) \leq c\lambda\nu^{-k}, \quad k \geq 0,$$

for all $x \in \theta(x_i, t_i - J(k+2) - \gamma) \setminus \theta(x_i, t_i - J(k+1) - \gamma)$, where $\nu := 2^{a_6 J N_p}$.

Proof. Fix $x \in \theta(x_i, t_i - J(k+2) - \gamma) \setminus \theta(x_i, t_i - J(k+1) - \gamma)$ for some $k \geq 0$, and let $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p}$ and $s \in \mathbb{R}$. Using (6.19), we may again assume that $\operatorname{supp}(\psi) \subseteq B^*$. Since $\operatorname{supp}(b_i) \subset \theta(x_i, t_i - J)$, if $\theta(x_i, t_i - J) \cap \theta(x, s) = \emptyset$, then $b_i * \psi_{x,s}(x) = 0$. Hence we assume that

$$\theta(x_i, t_i - J) \cap \theta(x, s) \neq \emptyset. \tag{6.75}$$

We then consider two cases.

Case 1: $s \geq t_i$. By Lemma 2.18, if $s \geq t_i$, then (6.75) further implies that $\theta(x,s) \subset \theta(x_i, t_i - J - \gamma)$, and so $x \in \theta(x_i, t_i - J - \gamma)$. However, in this particular case the point x does not satisfy the assumptions of the lemma, and Lemma 6.31 already yields $M^\circ b_i(x) \leq c\lambda$.

Case 2: $s < t_i$. For $w \in \theta(x_i, t_i - J - 2\gamma - 1) \cap \Omega^c$, denote $z := M_{w,t_i}^{-1}(x - w)$ and $\psi_{w,x} := \psi(M_{x,s}^{-1} M_{w,t_i} \cdot)$, and let $R_z := R_z^{N_p} \psi_{w,x}$ be the Taylor remainder (1.30) of $\psi_{w,x}$ about z of order N_p. We define the following Schwartz function, which essentially depends on i and x:

$$\Phi(y) := \frac{|\det(M_{x,s}^{-1})|}{|\det(M_{w,t_i}^{-1})|} \varphi_i(w - M_{w,t_i} y) R_z(z + y).$$

By (6.67) our construction of the local bad part b_i ensures that it has $l \geq N_p$ vanishing moments. Thus

$$\begin{aligned} |b_i * \psi_{x,s}(x)| &= |\det(M_{x,s}^{-1})| \left| \int b_i(y) R_z(M_{w,t_i}^{-1}(x - y)) dy \right| \\ &\leq |f * \Phi_{w,t_i}(w)| + |\det(M_{x,s}^{-1})| |P_i \varphi_i * R_z(M_{w,t_i}^{-1} \cdot)(x)| \\ &=: I_1 + I_2. \end{aligned}$$

We begin with the estimate of I_1. Since $w \in \Omega^c$, we have

$$I_1 \leq \|\Phi\|_{N_p, \widetilde{N}_p} M^\circ f(w) \leq \lambda \|\Phi\|_{N_p, \widetilde{N}_p}.$$

Thus, to complete the estimate of I_1, it is sufficient to show that $\|\Phi\|_{N_p, \widetilde{N}_p} \leq C v^{-k}$. We first note that $\operatorname{supp}(\Phi) \subseteq \operatorname{supp}(\varphi_i(w - M_{w,t_i} \cdot)) \subseteq c_1 B^*$. Thus it is sufficient to prove that $\|\partial^\alpha \Phi\|_\infty \leq C v^{-k}$, $\forall \alpha \in \mathbb{N}_+^n$, $|\alpha| \leq N_p$. We now estimate by (2.13) the first factor using $s < t_i$:

$$\frac{|\det(M_{x,s}^{-1})|}{|\det(M_{w,t_i}^{-1})|} \leq a_1^{-1} a_2 2^{s - t_i} \leq C.$$

Appealing to Lemma 6.28 gives a bound for the second factor,

$$\max_{|\alpha| \leq N_p} \|\partial^\alpha [\varphi_i(w - M_{w,t_i} \cdot)]\|_\infty \leq C.$$

We now deal with the derivatives of the third factor. Observe that our assumption (6.75) allows us to use (2.14) to obtain

$$\begin{aligned} \|M_{x,s}^{-1} M_{w,t_i}\| &\leq \|M_{x,s}^{-1} M_{x_i, t_i - 2J - \gamma - 1}\| \|M_{x_i, t_i - 2J - \gamma - 1}^{-1} M_{w,t_i}\| \\ &\leq C 2^{-a_6 (t_i - s)}. \end{aligned}$$

Next, since $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p} \subset \mathcal{S}_{N_p,N_p}$, we see by (1.34) that for any $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le N_p$, and $u \in B(z, c_1)$

$$\begin{aligned}|\partial^\alpha R_z(u)| &= |\partial^\alpha [R_z^{N_p}\psi_{w,x}](u)| \\ &= |R_z^{N_p-|\alpha|}\partial^\alpha \psi_{w,x}(u)| \\ &\le C2^{-a_6(t_i-s)N_p}\|\psi\|_{N_p,N_p}\big(1+|M_{x,s}^{-1}M_{w,t_i}u|\big)^{-N_p} \\ &\le C2^{-a_6(t_i-s)N_p}\big(1+|M_{x,s}^{-1}M_{w,t_i}u|\big)^{-N_p}.\end{aligned}$$

Therefore there exists $c_2(\mathbf{p}(\Theta)) > 0$ for which

$$\max_{|\alpha|\le N_p}\sup_{y\in c_1B^*}|\partial^\alpha R_z(z+y)| \le \sup_{y\in c_2B^*} C2^{-a_6(t_i-s)N_p}\big(1+|M_{x,s}^{-1}(x-w)+y|\big)^{-N_p}.$$

Let $\tilde{k} := k - \lceil(2\gamma+1)/J\rceil \ge 0$ for $k \ge k_0(\mathbf{p}(\Theta))$, which implies $J(\tilde{k}+1)+2\gamma+1 \le J(k+1)$. We get

$$\begin{aligned}&\theta(w,t_i-J-2\gamma-1)\cap\theta(x_i,t_i-J-2\gamma-1)\neq\emptyset \\ &\quad\Rightarrow \theta\big(w,t_i-J(\tilde{k}+1)-2\gamma-1\big)\cap\theta\big(x_i,t_i-J(\tilde{k}+1)-2\gamma-1\big)\neq\emptyset \\ &\quad\Rightarrow \theta\big(w,t_i-J(\tilde{k}+1)-2\gamma-1\big)\subseteq\theta\big(x_i,t_i-J(k+1)-\gamma\big) \\ &\quad\Rightarrow x\notin\theta\big(w,t_i-J(\tilde{k}+1)-2\gamma-1\big).\end{aligned}$$

This means that there exists a constant $c_3(\mathbf{p}(\Theta)) > 0$ for which $M_{x,s}^{-1}(x-w) \notin c_3M_{x,s}^{-1}M_{w,t_i-Jk}(B^*)$.

We need to consider two subcases, $t_i - Jk \le s < t_i$ and $s < t_i - Jk$. For the first subcase, application of assumption (6.75) and (2.14) yields

$$2^{a_6(s-t_i)}2^{a_6Jk}B^* \subseteq c_3M_{x,s}^{-1}M_{w,t_i-Jk}(B^*).$$

This gives

$$\begin{aligned}2^{-a_6(t_i-s)N_p}\sup_{v\in M_{x,s}^{-1}(x-w)+c_2B^*}\big(1+|v|\big)^{-N_p} &\le C2^{-a_6(t_i-s)N_p}2^{a_6(t_i-s)N_p}\big(2^{a_6JN_p}\big)^{-k} \\ &\le C\nu^{-k}.\end{aligned}$$

To show that the bound also holds for the other subcase, $s < t_i - Jk$, we simply proceed by

$$\begin{aligned}2^{-a_6(t_i-s)N_p}\sup_{v\in M_{x,s}^{-1}(x-w)+c_2B^*}\big(1+|v|\big)^{-N_p} &\le C2^{-a_6(t_i-s)N_p} \\ &\le C2^{-a_6JkN_p} \\ &= C\nu^{-k}.\end{aligned}$$

From the two subcases for the third factor of Φ, we obtain

$$\max_{|\alpha|\le N_p} \sup_{y\in cB^*} \left|\partial^\alpha R_z(z+y)\right| \le C\nu^{-k}. \tag{6.76}$$

Therefore collecting the bounds for all three terms gives $\|\Phi\|_{N_p,\widetilde{N}_p} \le C\nu^{-k}$, which implies $I_1 \le C\lambda\nu^{-k}$.

To complete the proof, we estimate I_2 by combining Lemma 6.30 and (6.76):

$$\begin{aligned} I_2 &\le \left|\det(M_{x,s}^{-1})\right| \int_{\theta(x_i.t_i-J)} \left|P_i(y)\varphi_i(y)\right|\left|R_z(M_{w,t_i}^{-1}(x-y))\right|dy \\ &\le C\lambda\left|\det(M_{x,s}^{-1})\right|\left|\theta(x_i.t_i-J)\right| \sup_{y\in\theta(x_i,t_i-J)} \left|R_z(M_{w,t_i}^{-1}(x-y))\right| \\ &\le C\lambda 2^{s-t_i} \sup_{y\in cB^*}\left|R_z(z+y)\right| \\ &\le C\lambda\nu^{-k}. \end{aligned}$$

□

Lemma 6.33. *Suppose $f \in H^p(\Theta)$, $0 < p \le 1$. Then, for any $\lambda > 0$, the series $\sum_i b_i$ converges in $H^p(\Theta)$, and there exist constants $c_1, c_2 > 0$, independent of f, $i \in \mathbb{N}$, and $\lambda > 0$, such that*
(i) $\|b_i\|_{H^p(\Theta)}^p \le c_1 \int_{\theta(x_i,t_i-J-\gamma)} (M^\circ f)^p$,
(ii) $\|\sum_i b_i\|_{H^p(\Theta)}^p \le c_2 \int_\Omega (M^\circ f)^p$.

Proof. First, observe that assumption (6.24) implies that for $\nu := 2^{a_6 JN_p}$, from Lemma 6.32 we have $\nu^{-p}2^J = 2^{(1-a_6pN_p)J} < 1$. Then recall that since $\theta(x_i, t_i - J - \gamma) \subset \Omega$, $M^\circ f(x) > \lambda$ for all $x \in \theta(x_i, t_i - J - \gamma)$. We use these two observations and further apply Lemmas 6.31 and 6.32 to obtain (i):

$$\begin{aligned} \int_{\mathbb{R}^n} (M^\circ b_i)^p &= \int_{\theta(x_i,t_i-J-\gamma)} (M^\circ b_i)^p + \sum_{k=0}^{\infty} \int_{\theta(x_i,t_i-J(k+2)-\gamma)\setminus\theta(x_i,t_i-J(k+1)-\gamma)} (M^\circ b_i)^p \\ &\le C \int_{\theta(x_i,t_i-J-\gamma)} (M^\circ f)^p + C\lambda^p \sum_{k=0}^{\infty} \left|\theta(x_i, t_i - J(k+2) - \gamma)\right|\nu^{-kp} \\ &\le C \int_{\theta(x_i,t_i-J-\gamma)} (M^\circ f)^p + C2^{-t_i}\lambda^p \sum_{k=0}^{\infty} (\nu^{-p}2^J)^k \\ &\le C \int_{\theta(x_i,t_i-J-\gamma)} (M^\circ f)^p. \end{aligned}$$

Since by Theorem 6.15 $H^p(\Theta)$ is complete, from (i) and (6.64) we have

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(M^\circ\Big(\sum_i b_i\Big)\Big)^p &\le \sum_i \int_{\mathbb{R}^n}(M^\circ b_i)^p\\ &\le C\sum_i \int_{\theta(x_i,t_i-J-\gamma)}(M^\circ f)^p\\ &\le C\int_\Omega (M^\circ f)^p.\end{aligned}$$

□

Lemma 6.34. *If $f \in L_q(\mathbb{R}^n)$, $1 \le q < \infty$, then the series $\sum_{i\in\mathbb{N}} b_i$ converges in $L_q(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$, independent of f, i, and λ, such that $\|\sum_{i\in\mathbb{N}}|b_i|\|_q \le c\|f\|_q$.*

Proof. From the definition of $\{b_i\}$ and Lemma 6.30 we have

$$\begin{aligned}\int_{\mathbb{R}^n}|b_i|^q &= \int_{\mathbb{R}^n}\big|(f-P_i)\varphi_i\big|^q\\ &\le C\int_{\theta(x_i,t_i-J)}|f\varphi_i|^q + C\int_{\theta(x_i,t_i-J)}|P_i\varphi_i|^q\\ &\le C\int_{\theta(x_i,t_i-J)}|f|^q + C\lambda^q\big|\theta(x_i,t_i-J)\big|.\end{aligned}$$

The construction of the Whitney cover of Ω gives that $\Omega = \bigcup_i \theta(x_i,t_i)$ and also that $\theta(x_i,t_i-J) \subseteq \theta(x_i,t_i-J-2\gamma) \Rightarrow \theta(x_i,t_i-J) \subset \Omega$, which in turns means that we also have $\Omega = \bigcup_i \theta(x_i,t_i-J)$. Therefore property (6.64), (6.20) with constant $c_2 > 0$, and the maximal theorem (Theorem 6.3) yield, for $q = 1$,

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_i|b_i| &\le \sum_i\int_{\mathbb{R}^n}|b_i|\\ &\le \sum_i\int_{\theta(x_i,t_i-J)}|f| + C\lambda\sum_i\big|\theta(x_i,t_i-J)\big|\\ &\le C\int_\Omega|f| + C\lambda|\Omega|\\ &= C\int_\Omega|f| + C\lambda\big|\{x\in\mathbb{R}^n : M^\circ f(x) > \lambda\}\big|\\ &\le C\int_\Omega|f| + C\lambda\big|\{x\in\mathbb{R}^n : M_\Theta f(x) > c_2^{-1}\lambda\}\big|\\ &\le C\|f\|_1.\end{aligned}$$

This completes the proof for $q = 1$. Using the same technique for the case $1 < q < \infty$ gives

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_i |b_i|^q &\le C \int_\Omega |f|^q + C\lambda^q |\Omega| \\ &\le C \int_\Omega |f|^q + C\|M^\circ f\|_q^q \\ &\le C \int_\Omega |f|^q + C\|M_\Theta f\|_q^q \\ &\le C\|f\|_q^q.\end{aligned}$$

To complete the proof for the case $1 < q < \infty$, we observe that the bound $\| \sum_i |b_i| \|_q^q \le C \int_{\mathbb{R}^n} \sum_i |b_i|^q$ holds due to the fact that for each i, $\operatorname{supp}(b_i) \subseteq \theta(x_i, t_i - J)$, and by property (6.64) there are at most L locally bad parts b_j whose supports intersect $\operatorname{supp}(b_i)$. □

Lemma 6.35. *Suppose $\sum_i b_i$ converges in $\mathcal{S}'$. Then there exist a constant c, independent of $f \in \mathcal{S}'$ and $\lambda > 0$, such that*

$$M^\circ g(x) \le c\lambda \sum_i \nu^{-k_i(x)} + M^\circ f(x) \mathbf{1}_{\Omega^c}(x), \tag{6.77}$$

where the "good" part g is per Definition 6.27, $\nu := 2^{a_6 J N_p}$ is from Lemma 6.32, and

$$k_i(x) = \begin{cases} k & \text{if for } k \ge 0,\ x \in \theta(x_i, t_i - J(k+2) - \gamma) \setminus \theta(x_i, t_i - J(k+1) - \gamma), \\ 0, & x \in \theta(x_i, t_i - J - \gamma). \end{cases} \tag{6.78}$$

Proof. If $\sum_i b_i$ converges in $\mathcal{S}'$ and $x \in \Omega^c$, then from Lemma 6.32 we know that

$$M^\circ g(x) \le M^\circ f(x) + \sum_i M^\circ b_i(x) \le M^\circ f(x) \mathbf{1}_{\Omega^c}(x) + c\lambda \sum_i \nu^{-k_i(x)}.$$

For any $x \in \Omega$, there exists $j \in \mathbb{N}$ such that $x \in \theta(x_j, t_j - J)$. Recall from (6.64) that $\#\Lambda(j) \le L$, where

$$\Lambda(j) := \{ i \in \mathbb{N} : \theta(x_j, t_j - J - \gamma) \cap \theta(x_i, t_i - J - \gamma) \ne \emptyset \}.$$

We have that

$$M^\circ g(x) \le M^\circ \Big(f - \sum_{i \in \Lambda(j)} b_i \Big)(x) + M^\circ \Big(\sum_{i \notin \Lambda(j)} b_i \Big)(x). \tag{6.79}$$

By Lemma 6.32

$$M^{\circ}\Big(\sum_{i\notin\Lambda(j)} b_i\Big)(x) \le C\lambda \sum_{i\notin\Lambda(j)} \nu^{-k_i(x)}.$$

So to prove (6.77), it suffices to bound for $x \in \theta(x_j, t_j - J)$

$$M^{\circ}\Big(f - \sum_{i\in\Lambda(j)} b_i\Big)(x) \le C\lambda = C\lambda\nu^{-k_j(x)}.$$

We need to bound $|(f - \sum_{i\in\Lambda(j)} b_i) * \psi_{x,s}(x)|$ for any $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p}$ and $s \in \mathbb{R}$. Using (6.19), we may again assume that $\operatorname{supp}(\psi) \subseteq B^*$. There are again two cases.
Case 1: $s \ge t_j$. Defining $\eta := 1 - \sum_{i\in\Lambda(j)} \varphi_i$, we have

$$\begin{aligned}\Big|\Big(f - \sum_{i\in\Lambda(j)} b_i\Big) * \psi_{x,s}(x)\Big| &\le \big|f\eta * \psi_{x,s}(x)\big| + \Big|\Big(\sum_{i\in\Lambda(j)} P_i\varphi_i\Big) * \psi_{x,s}(x)\Big| \\ &=: I_1 + I_2.\end{aligned}$$

Since by (6.65) $\{\varphi_i\}$ are a partition of unity on Ω; in particular, $\eta \equiv 0$ on $\theta(x_j, t_j - J - \gamma)$. On the other hand, for $s \ge t_j$, $\operatorname{supp}(\psi_{x,s}) \subseteq \theta(x,s) \subseteq \theta(x_j, t_j - J - \gamma)$. This means that in this case, $I_1 = 0$.

We continue with the estimate of I_2. Since $\psi \in \mathcal{S}_{N_p,\widetilde{N}_p}$ and $\#\Lambda(j) \le L$, application of Lemma 6.30 yields

$$\begin{aligned} I_2 &\le \sum_{i\in\Lambda(j)} \big|P_i\varphi_i * \psi_{x,s}(x)\big| \\ &\le C\lambda\|\psi\|_1 \\ &\le C\lambda = C\lambda\nu^{-k_j(x)}.\end{aligned}$$

Case 2: $s < t_j$. For $w \in \theta(x_j, t_j - J - 2\gamma - 1) \cap \Omega^c$, define

$$\Phi(y) := \frac{|\det(M_{x,s}^{-1})|}{|\det(M_{w,t_j}^{-1})|}\psi\big(M_{x,s}^{-1}(x-w) + M_{x,s}^{-1}M_{w,t_j}y\big).$$

As in previous proofs, we obtain that there exist constants $c_1, c_2 > 0$, depending only on $\mathbf{p}(\Theta)$, such that $\operatorname{supp}(\Phi) \subseteq c_1B^*$ and $\|\Phi\|_{N_p,\widetilde{N}_p} \le c_2$. We apply $M^{\circ}f(w) < \lambda$, Lemmas 6.31 and 6.32, and $\#\Lambda(j) \le L$ to conclude

$$\begin{aligned}\Big|\Big(f - \sum_{i\in\Lambda(j)} b_i\Big) * \psi_{x,s}(x)\Big| &= \Big|\Big(f - \sum_{i\in\Lambda(j)} b_i\Big) * \Phi_{w,t_j}(w)\Big| \\ &\le \big|f * \Phi_{w,t_j}(w)\big| + \sum_{i\in\Lambda(j)} \big|b_i * \Phi_{w,t_j}(w)\big|\end{aligned}$$

$$\leq C\|\Phi\|_{N_p,\widetilde{N}_p}\lambda$$
$$\leq C\lambda \nu^{-k_j(x)}. \qquad \square$$

Lemma 6.36. *If $f \in H^p(\Theta)$, $0 < p \leq 1$, then $M^\circ g \in L_q$ for all $1 \leq q < \infty$, and there exists a constant $c_1 > 0$, independent of f and λ, such that*

$$\int_{\mathbb{R}^n} (M^\circ g)^q \leq c_1 \lambda^{q-p} \int_{\mathbb{R}^n} (M^\circ f)^p. \tag{6.80}$$

If $f \in L_q$, then $g \in L_\infty$, and there exists $c_2 > 0$, independent of f and λ, such that

$$\|g\|_\infty \leq c_2 \lambda. \tag{6.81}$$

Proof. Since $f \in H^p(\Theta)$, by Lemma 6.33 $\sum_i b_i$ converges in $\mathcal{S}'$, and we may apply Lemma 6.35 to obtain

$$\int_{\mathbb{R}^n} (M^\circ g(x))^q dx \leq C\lambda^q \int_{\mathbb{R}^n} \Big(\sum_{i\in\mathbb{N}} \nu^{-k_i(x)}\Big)^q dx + C \int_{\Omega^c} (M^\circ f(x))^q dx, \tag{6.82}$$

where $k_i(x)$ are defined in (6.78). We start with the case $q = 1$. Recalling that $\nu := 2^{a_6 J N_p}$ and $N_p > a_6^{-1}$, for a fixed $i \in \mathbb{N}$, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \nu^{-k_i(x)} dx &= \int_{\theta(x_i,t_i-J-\gamma)} dx + \sum_{k=0}^{\infty} \int_{\theta(x_i,t_i-J(k+2)-\gamma)\setminus\theta(x_i,t_i-J(k+1)-\gamma)} \nu^{-k_i(x)} dx \\
&\leq |\theta(x_i, t_i - J - \gamma)| + \sum_{k=0}^{\infty} |\theta(x_i, t_i - J(k+2) - \gamma)| \nu^{-k} \\
&\leq C2^{-t_i}\Big(1 + \sum_{k=0}^{\infty} 2^{Jk}\nu^{-k}\Big) \\
&\leq C|\theta(x_i, t_i)|.
\end{aligned}$$

Recall that $\Omega := \{x \in \mathbb{R}^n : M^\circ f(x) > \lambda\}$. Therefore from (6.60) and (6.82) we can derive (6.80) for $q = 1$ by

$$\begin{aligned}
\int_{\mathbb{R}^n} M^\circ g &\leq C\lambda \sum_{i\in\mathbb{N}} |\theta(x_i, t_i)| + \int_{\Omega^c} M^\circ f \\
&\leq C\lambda|\Omega| + \int_{\Omega^c} M^\circ f \\
&\leq C\lambda^{1-p} \int_{\mathbb{R}^n} (M^\circ f)^p.
\end{aligned}$$

Now let $1 < q < \infty$. For any $i \in \mathbb{N}$ and $x \in \theta(x_i, t_i - J(k+2) - \gamma) \setminus \theta(x_i, t_i - J(k+1) - \gamma)$, $k \geq 0$, by (6.2) we have

$$\begin{aligned} 2^{-kJ} &\leq C \frac{1}{|\theta(x_i, t_i - J(k+2) - \gamma)|} \int_{\theta(x_i, t_i - J(k+2) - \gamma)} \mathbf{1}_{\theta(x_i, t_i)} \\ &\leq C M_\Theta \mathbf{1}_{\theta(x_i, t_i)}(x) \\ &\leq C M_B \mathbf{1}_{\theta(x_i, t_i)}(x). \end{aligned}$$

Since $a_6 N_p > 1$, we may apply the Fefferman–Stein vector-valued maximal function inequality (2.12) and then (6.60) to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\sum_i \nu^{-k_i(x)} \Big)^q dx &= \int_{\mathbb{R}^n} \Big(\sum_i 2^{-k_i(x) J a_6 N_p} \Big)^q dx \\ &\leq C \int_{\mathbb{R}^n} \Big\{ \Big[\sum_i \big(M_B \mathbf{1}_{\theta(x_i, t_i)}(x) \big)^{a_6 N_p} \Big]^{1/(a_6 N)} \Big\}^{a_6 N_p q} dx \\ &\leq C \int_{\mathbb{R}^n} \Big[\sum_i \big(\mathbf{1}_{\theta(x_i, t_i)}(x) \big)^{a_6 N_p} \Big]^q dx \\ &\leq C|\Omega|. \end{aligned}$$

Plugging into (6.82) now gives (6.80) for $1 < q < \infty$:

$$\begin{aligned} \int_{\mathbb{R}^n} (M^\circ g)^q &\leq C\lambda^q |\Omega| + C \int_{\Omega^c} (M^\circ f)^q \\ &\leq C\lambda^{q-p} \int_{\Omega} (M^\circ f)^p + \lambda^{q-p} \int_{\Omega^c} (M^\circ f)^p \\ &= C\lambda^{q-p} \int_{\mathbb{R}^n} (M^\circ f)^p. \end{aligned}$$

We now turn to prove (6.81). If $f \in L_q$, then by Lemma 6.34 we have that g and b_i, $i \in \mathbb{N}$, are functions and $\sum_{i \in \mathbb{N}} b_i$ converges in L_q. Thus, in L_q

$$g = f - \sum_i b_i = f \mathbf{1}_{\Omega^c} + \sum_i P_i \varphi_i.$$

By Lemma 6.30 and (6.64), for every $x \in \Omega$, we have $|g(x)| \leq C\lambda$. Also, $|g(x)| = |f(x)| \leq M^\circ f(x) \leq \lambda$ for a. e. $x \in \Omega^c$. Therefore $\|g\|_\infty \leq c\lambda$. □

Corollary 6.37. *$H^p(\Theta) \cap L_q$, $1 < q < \infty$, is dense in $H^p(\Theta)$.*

Proof. Let $f \in H^p(\Theta)$ and $\lambda > 0$. Consider the Calderón–Zygmund decomposition of f of degree $l \geq N_p(\Theta)$ and height λ,

$$f = g^\lambda + \sum_{i \in \mathbb{N}} b_i^\lambda.$$

By Lemma 6.33 we have

$$\|f - g^\lambda\|_{H^p(\Theta)} = \Big\| \sum_{i \in \mathbb{N}} b_i^\lambda \Big\|_{H^p(\Theta)} \to 0 \quad \text{as } \lambda \to \infty,$$

which implies that $g^\lambda \to f$ in $H^p(\Theta)$. Now Lemma 6.36 gives that $M^\circ g^\lambda \in L_q(\mathbb{R}^n)$. We apply (6.20) and then the maximal inequality for $1 < q < \infty$ to conclude that $g^\lambda \in L_q(\mathbb{R}^n)$. □

Remark 6.38. Corollary 6.37 is limited to the case $1 < q < \infty$, since it leverages on the maximal function inequality. Once we complete our proof of the equivalence of the atomic Hardy spaces with Hardy spaces, we will be able to establish the density of $H^p(\Theta) \cap L_q$, in $H^p(\Theta)$ for the full range $1 \leq q \leq \infty$ (see Corollary 6.44).

6.3.3 The inclusion $H^p(\Theta) \subseteq H^p_{q,l}(\Theta)$

Let $f \in H^p(\Theta)$ for some $0 < p \leq 1$. For each $k \in \mathbb{Z}$, we consider the Calderón–Zygmund decomposition of f of degree $l \geq N_p(\Theta)$ at height 2^k associated with $\Omega_k := \{x : M^\circ f(x) > 2^k\}$. The sequences $\{x_i^k\}_i$, $x_i^k \in \Omega_k$, and $\{t_i^k\}_i$, $t_i^k \in \mathbb{R}$, satisfy (6.60)–(6.64) with respect to Ω_k. We then have

$$f = g^k + \Sigma_i b_i^k, \quad k \in \mathbb{Z},$$

where

$$b_i^k := (f - P_i^k)\varphi_i^k,$$

$\{\varphi_i^k\}$ are defined by (6.65) with $\operatorname{supp}(\varphi_i^k) = \theta_i^k := \theta(x_i^k, t_i^k - J)$, and $\{P_i^k\}$ are defined by (6.67).

We now define P_{ij}^{k+1} as the orthogonal projection of $(f - P_j^{k+1})\varphi_i^k$ with respect to the inner product

$$\langle P, Q \rangle_j := \frac{1}{\int \varphi_j^{k+1}} \int_{\mathbb{R}^n} PQ\varphi_j^{k+1}, \quad \forall P, Q \in \Pi_l, \tag{6.83}$$

that is, if $\theta(x_i^k, t_i^k - J) \cap \theta(x_j^{k+1}, t_j^{k+1} - J) \neq \emptyset$, then P_{ij}^{k+1} is the unique polynomial in Π_l such that

$$\int_{\mathbb{R}^n} (f - P_j^{k+1})\varphi_i^k Q \varphi_j^{k+1} = \int_{\mathbb{R}^n} P_{ij}^{k+1} Q \varphi_j^{k+1}, \quad \forall Q \in \Pi_l;$$

otherwise, we may take $P_{i,j}^{k+1} = 0$.

Lemma 6.39. *Suppose* $\theta(x_i^k, t_i^k - J) \cap \theta(x_j^{k+1}, t_j^{k+1} - J) \neq \emptyset$. *Then*
(i) $t_j^{k+1} \geq t_i^k - 2\gamma - 1$,
(ii) $\theta(x_j^{k+1}, t_j^{k+1} - J) \subset \theta(x_i^k, t_i^k - J - 3\gamma - 1)$, *and*
(iii) *there exists* $L' > 0$ *such that for every* $j \in \mathbb{N}$, $\#I(j) \leq L'$ *with*

$$I(j) := \{i \in \mathbb{N} : \theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) \neq \emptyset\}.$$

Proof. To prove (i), assume by contradiction that $t_j^{k+1} < t_i^k - 2\gamma - 1$. Then by Lemma 2.18, $\theta(x_i^k, t_i^k - J) \cap \theta(x_j^{k+1}, t_j^{k+1} - J) \neq \emptyset$ implies that

$$\theta(x_i^k, t_i^k - J - 2\gamma - 1) \subseteq \theta(x_j^{k+1}, t_j^{k+1} - J - \gamma).$$

Since $\Omega_{k+1} \subseteq \Omega_k$, we have $(\Omega_k)^c \subseteq (\Omega^{k+1})^c$. Hence from (6.62) we have

$$\emptyset \neq (\Omega^k)^c \cap \theta(x_i^k, t_i^k - J - 2\gamma - 1) \subset (\Omega^{k+1})^c \cap \theta(x_j^{k+1}, t_j^{k+1} - J - \gamma) = \emptyset,$$

which is contradiction. Property (ii) is a consequence of (i) and Lemma 2.18. We continue with (iii). For a fixed j, let $I_1(j) := \{i \in I(j) : t_i^k \leq t_j^{k+1}\}$. Then for each such i, $\theta(x_j^{k+1}, t_j^{k+1} - J) \subseteq \theta(x_i^k, t_i^k - J - \gamma)$. Since x_j^{k+1} is contained in each $\theta(x_i^k, t_i^k - J - \gamma)$, $i \in I_1(j)$, we obtain by (6.64) that $\#I_1(j) \leq L$. Now denote $I_2(j) := \{i \in I(j) : t_i^k > t_j^{k+1}\}$. Observe that

$$\theta(x_i^k, t_i^k + \gamma) \subseteq \theta(x_i^k, t_i^k - J) \subseteq \theta(x_j^{k+1}, t_j^{k+1} - J - \gamma).$$

At the same time, by (i) we have that $t_i^k - 2\gamma - 1 \leq t_j^{k+1}$, and therefore all the ellipsoids $\theta(x_i^k, t_i^k + \gamma)$, $i \in I_2(j)$, are pairwise disjoint, are all contained in the ellipsoid $\theta(x_j^{k+1}, t_j^{k+1} - J - \gamma)$, but also have their volume proportional to it by a multiple constant. Therefore $\#I_2(j) \leq L''$. We conclude that (iii) is satisfied with $L' := L + L''$. □

Lemma 6.40. *There exist a constant* $c > 0$*, independent of* $i, j \in \mathbb{N}$ *and* $k \in \mathbb{Z}$*, such that*

$$\|P_{ij}^{k+1} \varphi_j^{k+1}\|_\infty \leq c 2^{k+1}.$$

Furthermore, if $M^\circ f \in L_\infty$*, then*

$$\|P_{ij}^{k+1} \varphi_j^{k+1}\|_\infty \leq c \|M^\circ f\|_\infty. \tag{6.84}$$

Proof. Let $\{\pi_\beta : \beta \in \mathbb{N}_+^n, |\beta| \le l\}$ be an orthonormal basis with respect to the inner product (6.83). Since P_{ij}^{k+1} is the orthogonal projection of $(f - P_j^{k+1})\varphi_i^k$, for $x \in \operatorname{supp}(\varphi_j^{k+1}) = \theta(x_j^{k+1}, t_j^{k+1} - J)$, we have

$$\begin{aligned}\left|P_{ij}^{k+1}(x)\varphi_j^{k+1}(x)\right| &\le \left|P_{ij}^{k+1}(x)\right| \\ &= \left|\sum_{|\beta|\le l}\left(\frac{1}{\int \varphi_j^{k+1}}\int_{\mathbb{R}^n}(f - P_j^{k+1})\varphi_i^k \pi_\beta \varphi_j^{k+1}\right)\pi_\beta(x)\right| \\ &\le C\max_{|\beta|\le l}\|\pi_\beta\|_{L_\infty(\theta(x_j^{k+1}, t_j^{k+1}-J))}(I_1 + I_2),\end{aligned}$$

where

$$I_1 := \frac{1}{\int \varphi_j^{k+1}}\sum_{|\beta|\le l}\left|\int_{\mathbb{R}^n} f\varphi_i^k \pi_\beta \varphi_j^{k+1}\right|, \quad I_2 := \frac{1}{\int \varphi_j^{k+1}}\sum_{|\beta|\le l}\left|\int_{\mathbb{R}^n} P_j^{k+1}\varphi_i^k \pi_\beta \varphi_j^{k+1}\right|.$$

For β, $|\beta| \le l$, we have using Lemma 1.24, Lemma 1.23, the properties of φ_j^{k+1}, and (6.83) that

$$\begin{aligned}\|\pi_\beta\|_{L_\infty(\theta(x_j^{k+1}, t_j^{k+1}-J))} &\le C|\theta(x_j^{k+1}, t_j^{k+1} - J))|^{-1/2}\|\pi_\beta\|_{L_2(\theta(x_j^{k+1}, t_j^{k+1}-J))} \\ &\le C\frac{1}{(\int \varphi_j^{k+1})^{1/2}}\|\pi_\beta\|_{L_2(\theta(x_j^{k+1}, t_j^{k+1}))} \\ &\le C\langle \pi_\beta, \pi_\beta\rangle_j^{1/2} \\ &\le C.\end{aligned}$$

We note in passing that we could also use the equivalence of finite-dimensional Banach spaces for this last argument. From this point we may assume that

$$\theta(x_i^k, t_i^k - J) \cap \theta(x_j^{k+1}, t_j^{k+1} - J) \ne \emptyset, \tag{6.85}$$

else $P_{ij}^{k+1} = 0$, and we are done.

We now estimate I_1. Let $w \in (\Omega^{k+1})^c \cap \theta(x_j^{k+1}, t_j^{k+1} - J - 2\gamma - 1)$, and for each β, $|\beta| \le l$, define

$$\Phi^\beta(y) := \frac{|\det(M_{w,t_j^{k+1}})|}{\int \varphi_j^{k+1}}(\varphi_i^k \cdot \pi_\beta \cdot \varphi_j^{k+1})(w - M_{w,t_j^{k+1}}(y)).$$

Under assumption (6.85), we may apply Lemma 6.39 to see that $\operatorname{supp}(\Phi^\beta) \subseteq c_1 B^*$ for all β, $|\beta| \le l$, for some fixed constant $c_1(\mathbf{p}(\Theta))$. Using the method of proof of Lemma 6.29, we can then show that $\max_{|\beta|\le l}\|\Phi^\beta\|_{N_p,\widetilde{N}_p} \le c_2$ for a fixed constant $c_2(\mathbf{p}(\Theta))$. Using also the bound $M^\circ f(w) < 2^{k+1}$, we obtain

$$\begin{aligned} I_1 &\le \sum_{|\beta|\le l} \left| f * \Phi^\beta_{w,t_j^{k+1}}(w) \right| \\ &\le C \max_{|\beta|\le l} \|\Phi^\beta\|_{N_p,\widetilde{N}_p} M^\circ f(w) \\ &\le C 2^{k+1}. \end{aligned}$$

Note that if $M^\circ f \in L_\infty$, then

$$I_1 \le C\|M^\circ f\|_\infty.$$

We now estimate I_2. Since $\operatorname{supp}(\varphi_j^{k+1}) \subset \theta(x_j^{k+1}, t_j^{k+1} - J)$, for each β, $|\beta| \le l$, we have

$$\frac{1}{\int \varphi_j^{k+1}} \int_{\mathbb{R}^n} P_j^{k+1} \varphi_i^k \pi_\beta \varphi_j^{k+1} = \frac{1}{\int \varphi_j^{k+1}} \int_{\theta(x_j^{k+1}, t_j^{k+1}-J)} P_j^{k+1} \varphi_i^k \pi_\beta \varphi_j^{k+1}.$$

From Lemma 6.30 we have

$$\|P_j^{k+1}\|_{L_\infty(\theta(x_j^{k+1}, t_j^{k+1}-J))} \le C 2^{k+1},$$

and if $M^\circ f \in L_\infty$, then

$$\|P_j^{k+1}\|_{L_\infty(\theta(x_j^{k+1}, t_j^{k+1}-J))} \le C\|M^\circ f\|_\infty.$$

We previously showed that

$$\|\pi_\beta\|_{L_\infty(\theta(x_j^{k+1}, t_j^{k+1}-J))} \le C, \quad \forall \beta, |\beta| \le l.$$

This leads to

$$\begin{aligned} I_2 &\le C 2^{k+1} \frac{1}{\int \varphi_j^{k+1}} \int_{\mathbb{R}^n} |\varphi_i^k \varphi_j^{k+1}| \\ &\le C 2^{k+1}, \end{aligned}$$

and if $M^\circ f \in L_\infty$, then

$$I_2 \le C\|M^\circ f\|_\infty. \qquad \square$$

Lemma 6.41. *Let $k \in \mathbb{Z}$. Then $\sum_{i\in\mathbb{N}} (\sum_{j\in\mathbb{N}} P_{ij}^{k+1} \varphi_j^{k+1}) = 0$, where the series converges pointwise and in $\mathcal{S}'$.*

Proof. By (6.64) we have that for any $x \in \mathbb{R}^n$, $\#\{j \in \mathbb{N} : \varphi_j^{k+1}(x) \neq 0\} \leq L$. Also, since P_{ij}^{k+1} is the orthogonal projection of $(f - P_j^{k+1})\varphi_i^k$ with respect to (6.83), we have $P_{ij}^{k+1} = 0$ if $\theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) = \emptyset$. For a fixed $j \in \mathbb{N}$, let $I(j) := \{i \in \mathbb{N} : \theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) \neq \emptyset\}$. Lemma 6.39 gives that $\#I(j) \leq L'$. Combining this with Lemma 6.40, we get

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \left|P_{ij}^{k+1}(x)\varphi_j^{k+1}(x)\right| \leq C2^{k+1}.$$

By the Lebesgue dominated convergence theorem $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\varphi_j^{k+1}$ converges unconditionally in $\mathcal{S}'$.

To conclude the proof, it suffices to show that

$$\sum_{i\in\mathbb{N}} P_{ij}^{k+1} = \sum_{i\in I(j)} P_{ij}^{k+1} = 0, \quad \forall j \in \mathbb{N}.$$

Indeed, $\sum_{i\in I(j)} P_{ij}^{k+1}$ is an orthogonal projection of $(f - P_j^{k+1})\sum_{i\in I(j)} \varphi_i^k$ onto Π_l with respect to the inner product (6.83). Since $\sum_{i\in I(j)} \varphi_i^k(x) = 1$ for $x \in \theta(x_j^{k+1}, t_j^{k+1} - J)$, $\sum_{i\in I(j)} P_{ij}^{k+1}$ is the orthogonal projection of $(f - P_j^{k+1})$ onto Π_l with respect to the inner product (6.83), which is zero by the definition of P_j^{k+1} in (6.67). □

Lemma 6.42. *Let Θ be a pointwise continuous cover and suppose (p, ∞, l) is admissible (see Definition 6.22). Then there exists a constant $c > 0$ such that for any $f \in H^p(\Theta) \cap L_q$, $1 \leq q < \infty$, there exist a sequence of (p, ∞, l)-atoms $\{a_i^k\}_{k\in\mathbb{Z}, i\in\mathbb{N}}$ and coefficients $\{\lambda_i^k\}_{k\in\mathbb{Z}, i\in\mathbb{N}}$ such that*

$$\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \left|\lambda_i^k\right|^p \leq c\|f\|_{H^p(\Theta)}^p \tag{6.86}$$

and

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \quad \text{converges in } H^p(\Theta) \text{ and } L_q. \tag{6.87}$$

Additionally, recalling the Whitney-type decomposition (6.60)–(6.64)

$$\Omega_k := \{x \in \mathbb{R}^n : M^\circ f(x) > 2^k\} = \bigcup_{i\in\mathbb{N}} \theta(x_i^k, t_i^k),$$

the atomic decomposition satisfies the following properties:

$$\operatorname{supp}(a_i^k) \subseteq \theta(x_i^k, t_i^k - J - 3\gamma - 1) \cap \Omega_k, \tag{6.88}$$

$$\|\lambda_i^k a_i^k\|_\infty \leq c2^k. \tag{6.89}$$

Proof. Consider the Calderón–Zygmund decomposition $f = g^k + \sum_i b^k$ of degree l at height 2^k associated with M°. By Lemma 6.33

$$\|f - g^k\|^p_{H^p(\Theta)} = \Big\|\sum_i b_i^k\Big\|^p_{H^p(\Theta)} \leq C \int_{\Omega_k} (M^\circ f)^p \to 0, \quad k \to \infty.$$

Also, using the assumption that $f \in L_q$, by (6.81) we have that $\|g^k\|_\infty \to 0$ as $k \to -\infty$. Therefore

$$f = \sum_{k\in\mathbb{Z}} (g^{k+1} - g^k) \quad \text{in } \mathcal{S}'.$$

From Lemma 6.41 and the fact that $\sum_{i\in\mathbb{N}} \varphi_i^k b_j^{k+1} = \mathbf{1}_{\Omega_k} b_j^{k+1} = b_j^{k+1}$ we have

$$\begin{aligned}
g^{k+1} - g^k &= \Big(f - \sum_{j\in\mathbb{N}} b_j^{k+1}\Big) - \Big(f - \sum_{j\in\mathbb{N}} b_j^k\Big) \\
&= \sum_{j\in\mathbb{N}} b_j^k - \sum_{j\in\mathbb{N}} b_j^{k+1} + \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\varphi_j^{k+1} \\
&= \sum_{i\in\mathbb{N}} b_i^k - \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \varphi_i^k b_j^{k+1} + \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\varphi_j^{k+1} \\
&= \sum_{i\in\mathbb{N}}\Big(b_i^k - \sum_{j\in\mathbb{N}}[\varphi_i^k b_j^{k+1} - P_{ij}^{k+1}\varphi_j^{k+1}]\Big) \\
&=: \sum_{i\in\mathbb{N}} h_i^k.
\end{aligned}$$

Since $b_i^k = (f - P_i^k)\varphi_i^k$, we have

$$h_i^k = (f - P_i^k)\varphi_i^k - \sum_{j\in\mathbb{N}} [\varphi_i^k(f - P_j^{k+1}) - P_{ij}^{k+1}]\varphi_j^{k+1}. \tag{6.90}$$

We now prove that $h_i^k = \lambda_i^k a_i^k$, where a_i^k and λ_i^k are the required atoms and coefficients, respectively, that satisfy all the required properties and claims of the lemma. We start with the vanishing moments property of atoms. By the construction of P_i^k (see (6.67)) and P_{ij}^{k+1} (see (6.83)) we have

$$\int_{\mathbb{R}^n} h_i^k Q = 0, \quad \forall Q \in \Pi_l. \tag{6.91}$$

The partition of unity $\sum_{j\in\mathbb{N}} \varphi_j^{k+1} = \mathbf{1}_{\Omega^{k+1}}$ allows us to write

$$h_i^k = f\mathbf{1}_{(\Omega_{k+1})^c}\varphi_i^k - P_i^k\varphi_i^k + \sum_{j\in\mathbb{N}} P_j^{k+1}\varphi_j^{k+1}\varphi_i^k + \sum_{j\in\mathbb{N}} P_{ij}^{k+1}\varphi_j^{k+1}. \tag{6.92}$$

Since $\operatorname{supp}(\varphi_i^k) \subset \Omega_k$ for all $i \in \mathbb{N}$ and $\operatorname{supp}(\varphi_j^{k+1}) \subset \Omega_{k+1} \subseteq \Omega_k$ for all $j \in \mathbb{N}$, we have that $\operatorname{supp}(h_i^k) \subset \Omega_k$. This is the first claim of (6.88). From the definition of P_{ij}^{k+1} we know that $\theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) = \emptyset \Rightarrow P_{ij}^{k+1} = 0$. We also know that $\operatorname{supp}(\varphi_j^{k+1}) \subset \theta(x_j^{k+1}, t_j^{k+1} - J)$, and hence from Lemma 6.39 we come to the conclusion that $\operatorname{supp}(\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\varphi_j^{k+1}) \subset \theta(x_i^k, t_i^k - J - 3\gamma - 1)$, which implies the second claim of (6.88),

$$\operatorname{supp}(h_i^k) \subset \theta(x_i^k, t_i^k - J - 3\gamma - 1). \tag{6.93}$$

From (6.92) we have

$$\|h_i^k\|_\infty \le \|f\mathbf{1}_{(\Omega^{k+1})^c}\varphi_i^k\|_\infty + \|P_i^k\varphi_i^k\|_\infty + \Big\|\sum_{j\in\mathbb{N}} P_j^{k+1}\varphi_i^k\varphi_j^{k+1}\Big\|_\infty + \Big\|\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\varphi_j^{k+1}\Big\|_\infty.$$

We know that $|f(x)| \le cM^\circ f(x) \le c2^{k+1}$ for almost every $x \in (\Omega^{k+1})^c$. Also from Lemma 6.30 we have $\|P_i^k\varphi_i^k\|_\infty \le c2^k$, and from Lemmas 6.39 and 6.40 we conclude that

$$\Big\|\sum_{j\in\mathbb{N}} P_j^{k+1}\varphi_i^k\varphi_j^{k+1}\Big\|_\infty \le c2^{k+1} \quad \text{and} \quad \Big\|\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\varphi_j^{k+1}\Big\|_\infty \le c2^{k+1}.$$

Collecting these last estimates yields

$$\|h_i^k\|_\infty \le c2^k, \tag{6.94}$$

which gives (6.89). From (6.91), (6.93), and (6.94), h_i^k is a multiple of a (p, ∞, l)-atom a_i^k, meaning that

$$h_i^k = \lambda_i^k a_i^k, \quad \lambda_i^k \sim 2^k 2^{-t_i^k/p},$$

where $\{a_i^k\}$ and $\{\lambda_i^k\}$ satisfy (6.88) and (6.89). From (6.60) and (6.61) we may conclude (6.86):

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\sum_{i\in\mathbb{N}}|\lambda_i^k|^p &\le C\sum_{k=-\infty}^{\infty} 2^{kp}\sum_{i\in\mathbb{N}}|\theta(x_i^k, t_i^k + \gamma)| \\ &\le C\sum_{k=-\infty}^{\infty} 2^{kp}|\Omega_k|\end{aligned}$$

$$\begin{aligned}&\le C\sum_{k=-\infty}^{\infty} p(2^k)^{p-1}|\Omega_k|2^{k-1}\\&\le C\int_0^{\infty} p\lambda^{p-1}\big|\{x\in\mathbb{R}^n : M^\circ f(x)>\lambda\}\big|d\lambda\\&= C\|M^\circ f\|_p^p\\&= C\|f\|_{H^p(\Theta)}^p.\end{aligned}$$

Therefore $f=\sum_{k=-\infty}^{\infty}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k$ in $\mathcal{S}'$ is an atomic decomposition of f, which implies

$$\|f\|_{H^p_{q,l}(\Theta)}\le C\|f\|_{H^p(\Theta)}.$$

We also get the convergence in $H^p(\Theta)$:

$$\begin{aligned}\Big\|f-\sum_{k=-\infty}^{k'}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k\Big\|_{H^p(\Theta)}^p &= \Big\|\sum_{k=k'}^{\infty}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k\Big\|_{H^p(\Theta)}^p\\&\le C\sum_{k=k'}^{\infty}\sum_{i\in\mathbb{N}}|\lambda_i^k|^p\to 0,\quad k'\to\infty.\end{aligned}$$

To see the convergence in L_q, observe that since $f\in L_q$ for a. e. $x\in\mathbb{R}^n$, there exists $k(x)\in\mathbb{Z}$ such that $2^{k(x)}<M^\circ f(x)\le 2^{k(x)+1}$. From this it follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}|h_i^k(x)| &\le C\sum_{k\le k(x)} 2^k\mathbf{1}_{\Omega_k}(x)\\&\le C2^{k(x)}\le CM^\circ f(x).\end{aligned}$$

Therefore the series $\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h_i^k$ converges absolutely pointwise a. e. to some function $\tilde{f}\in L_q$. By the Lebesgue dominated convergence theorem we deduce that $\tilde{f}=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k$ converges unconditionally in L_q. Since the same atomic decomposition converges in $\mathcal{S}'$ to f, we necessarily have $f=\tilde{f}\in L^q$, which yields (6.87). □

Theorem 6.43. *Let Θ be a pointwise continuous cover and suppose (p,q,l) is admissible (see Definition 6.22). Then $H^p(\Theta)\subseteq H^p_{q,l}(\Theta)$.*

Proof. By Lemma 6.42 we have for any $f\in H^p(\Theta)\cap L_2$ an atomic representation (6.87) with (p,∞,l)-atoms satisfying (6.86). Observe that a (p,∞,l)-atom is also a (p,q,l)-atom for any admissible $1\le q<\infty$. Applying the density of $H^p(\Theta)\cap L^2$ in H^p (Corollary 6.37), we complete the proof. □

Corollary 6.44. *If Θ is a pointwise continuous cover, then $H^p(\Theta)\cap L_q$ is dense in $H^p(\Theta)$ for $1\le q\le\infty$.*

Proof. Theorem 6.43 implies that every $f\in H^p(\Theta)$ has an atomic decomposition $f=\sum_i\lambda_i a_i$, converging in the $H^p(\Theta)$ quasi-norm, where $\sum_i|\lambda_i|^p\le C\|f\|_{H^p(\Theta)}^p$, and $\{a_i\}$ are

(p, ∞, l)-atoms. The partial finite atomic sums are compactly supported L_∞ functions and thus also L_q functions for any $1 \le q < \infty$. Since the partial finite sums converge to f in the Hardy norm, we obtain the denseness of $H^p(\Theta) \cap L_q$ in $H^p(\Theta)$. □

6.4 The space BMO(Θ)

Definition 6.45. Let Θ be a cover, and let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. Denote the means over the ellipsoids by

$$f_\theta := \frac{1}{|\theta|} \int_\theta f, \quad \theta \in \Theta.$$

Then f is said to belong to the space of *Bounded Mean Oscillation* BMO(Θ) if there exists a constant $0 < M < \infty$ such that

$$\sup_{\theta \in \Theta} \frac{1}{|\theta|} \int_\theta |f(x) - f_\theta| dx \le M.$$

We denote by $\|f\|_{\mathrm{BMO}(\Theta)}$ the infimum over all such constants.

It is standard to extend the above definition to allow arbitrary constants c_θ in place of the means f_θ, $\theta \in \Theta$. Indeed, if for given $\{c_\theta\}_{\theta \in \Theta}$, we have

$$\sup_{\theta \in \Theta} \frac{1}{|\theta|} \int_\theta |f(x) - c_\theta| dx \le M',$$

then $|c_\theta - f_\theta| \le M'$ for all $\theta \in \Theta$, and $\|f\|_{\mathrm{BMO}(\Theta)} \le 2M'$.

It is obvious that $L_\infty(\mathbb{R}^n) \subset \mathrm{BMO}(\Theta)$ for any cover. The following is a typical example for a nonbounded function in BMO(Θ).

Example 6.46. For any continuous cover Θ of $\mathbb{R}^n$, we have that $\log(\rho(\cdot, 0)) \in \mathrm{BMO}(\Theta)$, where ρ is the induced quasi-distance (2.35).

Proof. For any $\theta \in \Theta$, let $a := \inf_{y \in \theta} \rho(y, 0)$.
Case I: $|\theta| \le a$. Let $\{y_m\}_{m \ge 1}$, $y_m \in \theta$, $\rho(y_m, 0) \to a$ as $m \to \infty$. Since for any $x, y_m \in \theta$, $\rho(x, 0) \le \kappa(\rho(x, y_m) + \rho(y_m, 0))$, where $\kappa \ge 1$ is defined in (2.1). We have, as $m \to \infty$,

$$\frac{1}{|\theta|} \int_\theta (\log(\rho(x,0)) - \log a) dx \le \frac{1}{|\theta|} \int_\theta (\log \kappa(\rho(x, y_m) + \rho(y_m, 0)) - \log a) dx$$
$$\le \log \kappa + \frac{1}{|\theta|} \int_\theta \log\left(\frac{|\theta| + \rho(y_m, 0)}{a}\right) dx$$

$$\to \log \kappa + \log\left(\frac{|\theta| + a}{a}\right)$$
$$\le \log \kappa + \log 2.$$

Case II: $a \le |\theta|$. By the triangle inequality (2.1) $\theta \subset B_\rho(0, 2\kappa|\theta|)$, which in particular implies that $\rho(x, 0) \le 2\kappa|\theta|$ for all $x \in \theta$. Combining with Theorem 2.23, we get

$$\frac{1}{|\theta|}\int_\theta (\log(2\kappa|\theta|) - \log(\rho(x,0)))dx$$
$$\le C\frac{1}{|B_\rho(0,2\kappa|\theta|)|}\int_{B_\rho(0,2\kappa|\theta|)} (\log(2\kappa|\theta|) - \log(\rho(x,0)))dx.$$

Applying Theorem 2.23 again, we have

$$\frac{1}{|B_\rho(0,2\kappa|\theta|)|}\int_{B_\rho(0,2\kappa|\theta|)} (\log(2\kappa|\theta|) - \log\rho(x,0))dx$$
$$= \log(2\kappa|\theta|) + \frac{1}{|B_\rho(0,2\kappa|\theta|)|}\sum_{j=1}^\infty \int_{B_\rho(0,2\kappa|\theta|2^{-j+1})\setminus B_\rho(0,2\kappa|\theta|2^{-j})} \log\rho(x,0)^{-1}dx$$
$$\le \log(2\kappa|\theta|) + \frac{1}{|B_\rho(0,2\kappa|\theta|)|}\sum_{j=1}^\infty \left|B_\rho(0,2\kappa|\theta|2^{-j+1})\setminus B_\rho(0,2\kappa|\theta|2^{-j})\right| \log\left((2\kappa|\theta|)^{-1}2^j\right)$$
$$\le \log(2\kappa|\theta|) - \log(2\kappa|\theta|) + c'\frac{1}{2\kappa|\theta|}\sum_{j=1}^\infty 2\kappa|\theta|2^{-j+1}j$$
$$\le c'\sum_{j=1}^\infty 2^{-j+1}j = c''.$$

□

Recall that a given ellipsoid cover induces a natural quasi-distance ρ and a space of homogeneous type $X = (\mathbb{R}^n, \rho, dx)$. The space $\mathrm{BMO}(X)$ is defined [33] using averages over balls. So here with

$$f_{B_\rho} := \frac{1}{|B_\rho|}\int_{B_\rho} f, \quad \forall B_\rho = B_\rho(x,r),\ x \in \mathbb{R}^n,\ r > 0,$$

f is said to belong to the space of *Bounded Mean Oscillation* $\mathrm{BMO}(X)$ if there exists a constant $0 < M < \infty$ such that

$$\sup_{B_\rho} \frac{1}{|B_\rho|}\int_{B_\rho} |f(x) - f_{B_\rho}|dx \le M.$$

Naturally, $\|f\|_{\mathrm{BMO}(X)}$ is defined as the infimum over all such constants.

Theorem 6.47. $\mathrm{BMO}(\Theta) \sim \mathrm{BMO}(X)$.

Proof. The proof is a simple application of Theorem 2.23, which says that for any ball $B_\rho(x,r)$, there exist ellipsoids $\theta', \theta'' \in \Theta$ with centers at x such that

$$\theta' \subseteq B_\rho \subseteq \theta'', \quad |\theta'| \sim |B_\rho| \sim |\theta''|.$$

Conversely, for any $\theta \in \Theta$, there exist balls B'_ρ, B''_ρ such that

$$B'_\rho \subseteq \theta \subseteq B''_\rho, \quad |B'_\rho| \sim |\theta| \sim |B''_\rho|. \tag{6.95}$$

Thus averaging on the anisotropic balls and ellipsoids is equivalent. Namely, for any $\theta \in \Theta$, let B''_ρ satisfy (6.95), and let $c_\theta := f_{B''_\rho}$. Then there exists $c(\mathbf{p}(\Theta)) > 0$ such that $|B''_\rho| \le c|\theta|$, which gives

$$\frac{1}{|\theta|}\int_\theta |f - c_\theta| \le c\frac{1}{|B''_\rho|}\int_{B''_\rho} |f - f_{B''_\rho}| \le c\|f\|_{\mathrm{BMO}(X)}.$$

Therefore $\|f\|_{\mathrm{BMO}(\Theta)} \le 2c\|f\|_{\mathrm{BMO}(X)}$. The proof of the inverse embedding is similar. □

Using the method of proof of Theorem 6.47, we can also show that for equivalent covers (see Definition 2.27), we obtain equivalent BMO spaces.

Next, we recall the definition of the atomic $H^1(X)$ space for $X = (\mathbb{R}^n, \rho, dx)$ [20]. In the general setting of spaces of homogeneous type, we define a $(1,\infty,1)$-atom a as a function with the following properties:

(i) $\mathrm{supp}(a) \subseteq B_\rho$ for some ball B_ρ,
(ii) $\|a\|_\infty \le |B_\rho|^{-1}$,
(iii) $\int a = 0$.

Then the atomic Hardy space $H^1_{\infty,1}(X)$ is defined through atomic decompositions of such atoms.

Theorem 6.48. *We have that*
(i) $H^1(\Theta)$ *and* $\mathrm{BMO}(\Theta)$ *are dual spaces,*
(ii) $H^1(\Theta) \sim H^1_{\infty,1}(X)$.

Proof. The proof of (i) is a mere repetition of the proof of classic case of the isotropic BMO and H^1 spaces over $\mathbb{R}^n$ (see, e. g., [61]), where atoms supported over ellipsoids replace atoms supported on Euclidean balls. We note that the anisotropic finite atomic spaces of Section 6.7 and in particular Corollary 6.63 replace the classic isotropic finite atomic spaces. The proof of (ii) is immediate from (i), since using Theorem 6.47, these spaces are duals of the same space $\mathrm{BMO}(\Theta) \sim \mathrm{BMO}(X)$. □

Remark 6.49. We recall that the Hardy spaces $H^p(\Theta)$ can be defined and characterized using atomic spaces for arbitrarily small $p > 0$, whereas this is not possible in the general framework of spaces of homogeneous type. Furthermore, in Section 6.8, we prove that the anisotropic Campanato spaces presented in Section 5.5 are (modulo polynomials of fixed degree) the dual spaces of $H^p(\Theta)$ for any $0 < p < 1$.

6.5 Classification of anisotropic Hardy spaces

Since for any cover Θ, the anisotropic Hardy space $H^p(\Theta) \sim L_p(\mathbb{R}^n)$, $1 < p < \infty$, an important question is to what extent are various Hardy spaces over different covers different for the range $0 < p \le 1$? Theorem 6.51, which is the main result of this section, shows that for the range $0 < p \le 1$, two anisotropic Hardy spaces are equivalent if and only if the associated covers induce an equivalent quasi-distance.

We begin by showing that the anisotropic Hardy spaces are invariant under affine transformations.

Lemma 6.50. *Let Θ be a pointwise continuous cover, let $Ax = Mx + b$ be a non-singular affine transformation, and let (p, q, l) be an admissible triplet. Then:*
(i) *a is a (p, q, l)-atom in $H^p(\Theta)$ iff $|\det M|^{-1/p} a(A^{-1}\cdot)$ is a (p, q, l)-atom in $H^p(A(\Theta))$.*
(ii) *For any $f \in \mathcal{S}'$, $f \in H^p(\Theta)$ iff $f(A^{-1}\cdot) \in H^p(A(\Theta))$.*

Proof. To prove (i), let a be a (p, q, l)-atom in $H^p(\Theta)$ and denote $\tilde{a} := |\det M|^{-1/p} a(A^{-1}\cdot)$. We verify that $\tilde{a}$ satisfies the three properties of an atom in $H^p(A(\Theta))$:
(i′) $\operatorname{supp}(a) \subseteq \theta \Rightarrow \operatorname{supp}(\tilde{a}) \subseteq A(\theta)$.
(ii′) For $1 \le q \le \infty$,

$$\begin{aligned}\|\tilde{a}\|_q &= |\det M|^{-1/p} \|a(A^{-1}\cdot)\|_q \\ &= |\det M|^{1/q-1/p} \|a\|_q \\ &\le |\det M|^{1/q-1/p} |\theta|^{1/q-1/p} \\ &= \left|A(\theta)\right|^{1/q-1/p}.\end{aligned}$$

(iii′) For any $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le l$, we have the vanishing moment property by

$$\begin{aligned}\int_{\mathbb{R}^n} \tilde{a}(x) x^\alpha dx &= |\det M|^{-1/p} \int_{\mathbb{R}^n} a(A^{-1}x) x^\alpha dx \\ &= |\det M|^{1-1/p} \int_{\mathbb{R}^n} a(y)(Ay)^\alpha dy = 0.\end{aligned}$$

Claim (ii) follows directly from the atomic decomposition. If $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j|^p < 2\|f\|^p_{H^p_{q,l}(\Theta)}$, then $f(A^{-1}\cdot) = \sum_j \tilde{\lambda}_j \tilde{a}_j$, where using (i), $\tilde{a}_j := |\det M|^{-1/p} a_j(A^{-1}\cdot)$ are (p, q, l)

atoms in $H^p(A(\Theta))$, and $\tilde{\lambda}_j := |\det M|^{1/p}\lambda_j$. Thus

$$\begin{aligned}\|f(A^{-1}\cdot)\|_{H^p(A(\Theta))} &\le C\|f(A^{-1}\cdot)\|_{H^p_{q,l}(A(\Theta))}\\ &\le C\Big(\sum_j |\tilde{\lambda}_j|^p\Big)^{1/p}\\ &= C|\det M|^{1/p}\Big(\sum_j |\lambda_j|^p\Big)^{1/p}\\ &\le C|\det M|^{1/p}\|f\|_{H^p(\Theta)}.\end{aligned}$$

□

Theorem 6.51 ([31]). *Let Θ_1 and Θ_2 be two pointwise continuous covers, and let ρ_1 and ρ_2 be the corresponding induced quasi-distances. Then the following statements are equivalent:*

(i) $\rho_1 \sim \rho_2$,
(ii) $H^1(\Theta_1) \sim H^1(\Theta_2)$,
(iii) $H^p(\Theta_1) \sim H^p(\Theta_2)$ *for all* $0 < p \le 1$.

Notice that, in fact, Theorem 6.51 characterizes only the case $p = 1$. Further generalization of the proof is needed to show that the quasi-distances are equivalent iff the Hardy spaces are equivalent for some $0 < p_0 \le 1$. The proof of the theorem requires some preparation. First, we recall some basic definitions from convex analysis.

Definition 6.52. Let $K \subset \mathbb{R}^n$ be a bounded convex domain. Let $L \subset \mathbb{R}^n$ be a hyperplane through the origin with normal N. For each $x \in L$, let the perpendicular line through $x \in L$ be $G_x := \{x + yN : y \in \mathbb{R}\}$, and let $l_x := \text{length}(K \cap G_x)$. The *Steiner symmetrization* of K with respect to L is

$$S_L(K) = \{x + yN : x \in L, K \cap G_x \ne \emptyset, -(1/2)l_x \le y \le (1/2)l_x\}.$$

It is not hard to see that whenever K is convex, so is $S_L(K)$ and that the Steiner symmetrization preserves volume, i. e., $|S_L(K)| = |K|$ (see [6]).

For any hyperplane of the form $H := \{(y_1, \dots, y_{n-1}, h) : y_i \in \mathbb{R}\}$ with fixed h, we denote $H^+ := \{(y_1, \dots, y_{n-1}, y_n) : y_n \ge h\}$ and $H^- := \{(y_1, \dots, y_{n-1}, y_n) : y_n \le h\}$.

Lemma 6.53. *Let θ be an ellipsoid in $\mathbb{R}^n$. For $1 \le i \le n-1$, let L_i be the hyperplane $L_i := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0\}$. Let $H_i = \{(y_1, \dots, y_{n-1}, h_i)\}$, $i = 1, 2$, be two hyperplanes, where $h_1 > h_2$. Then the following hold:*

(a) *The convex body $S_{L_1} \circ S_{L_2} \circ \cdots \circ S_{L_{n-1}}(\theta)$ is symmetric with respect the x_i-axis for every* $1 \le i \le n-1$.
(b) $|H_1^- \cap H_2^+ \cap \theta| = |H_1^- \cap H_2^+ \cap S_{L_1} \circ \cdots \circ S_{L_{n-1}}(\theta)|$.

(c) *with* $\tilde{x}_n := \inf_{(y_1,\dots,y_n)\in\theta} y_n$ *and* $\tilde{z}_n := \sup_{(y_1,\dots,y_n)\in\theta} y_n$, *we have that*

$$\left|H_1^- \cap H_2^+ \cap \theta\right| \leq n!\big((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n)\big)|\theta|.$$

Proof. Statements (a) and (b) follow from the construction of $S_{L_1} \circ S_{L_2} \circ \dots \circ S_{L_{n-1}}(\theta)$. We now prove (c). First, we show that for any bounded convex domain $K \subset \mathbb{R}^n$, symmetric with respect the x_i-axis for every $1 \leq i \leq n-1$,

$$|K_2| \leq n!|K|, \tag{6.96}$$

where K_2 is the minimal (with respect to volume) box that contains K. By the symmetry, without loss of generality, we may assume that for $a_1, \dots, a_{n-1} > 0$, the points $(\pm a_1, 0, \dots, 0)$, $(0, \pm a_2, 0, \dots, 0)$, …, $(0, \dots, 0, \pm a_{n-1})$ belong to ∂K, the boundary of K, as well as $a^- = (0, 0, \dots, a_n^-)$, $a^+ = (0, 0, \dots, a_n^+)$, $a_n^- < a_n^+$. Let K_1 denote the convex hull of

$$\big\{(\pm a_1, 0 \dots, 0), (0, \pm a_2, 0, \dots, 0), \dots, (0, \dots, 0, \pm a_{n-1}), a^-, a_+\big\},$$

and let K_2 be the box

$$[-a_1, a_1] \times [-a_2, a_2] \times \dots \times [-a_{n-1}, a_{n-1}] \times \big[a_n^-, a_n^+\big].$$

Obviously,

$$K_1 \subseteq K \subseteq K_2.$$

A simple integral calculation shows that $|K_1| = (a_n^+ - a_n^-)(\prod_{i=1}^{n-1} a_i)2^{n-1}/n!$ and $|K_2| = (a_n^+ - a_n^-)(\prod_{i=1}^{n-1} a_i)2^{n-1}$, which implies (6.96). Therefore, with $K = S_{L_1} \circ S_{L_2} \circ \dots \circ S_{L_{n-1}}(\theta)$, we get

$$|K_2| \leq n!\left|S_{L_1} \circ S_{L_2} \circ \dots \circ S_{L_{n-1}}(\theta)\right|,$$

where K_2 is the minimal box that contains $S_{L_1} \circ S_{L_2} \circ \dots \circ S_{L_{n-1}}(\theta)$. Thus from (6.96) and (b) we have

$$\begin{aligned}
\left|H_1^- \cap H_2^+ \cap \theta\right| &= \left|H_1^- \cap H_2^+ \cap S_{L_1} \circ \dots \circ S_{L_{n-1}}(\theta)\right| \\
&\leq \left|H_1^- \cap H_2^+ \cap K_2\right| \\
&\leq \big((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n)\big)|K_2| \\
&\leq n!\big((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n)\big)\left|S_{L_1} \circ \dots \circ S_{L_{n-1}}(\theta)\right| \\
&= n!\big((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n)\big)|\theta|.
\end{aligned}$$

□

Lemma 6.54. *Let* Θ *be a cover of* $\mathbb{R}^n$ *such that* $B^* \in \Theta_0$. *For* $1 \le i \le n$, *define*

$$g_i(x_1,\ldots,x_n) := \begin{cases} \log|x_i|, & (x_1,\ldots,x_n) \in B^*, \\ 0, & (x_1,\ldots,x_n) \notin B^*. \end{cases} \tag{6.97}$$

Then $g_i \in \mathrm{BMO}(\Theta)$ *with* $0 < c_1 \le \|g_i\|_{\mathrm{BMO}(\Theta)} \le c_2(\mathbf{p}(\Theta))$.

Proof. Without loss of generality, we assume that $n > 1$ (the univariate case is known [61]) and $i = n$. Thus, for the rest of the proof, we denote $g := g_n$. By the definition of the BMO space

$$\|g\|_{\mathrm{BMO}(\Theta)} \ge \frac{1}{|B^*|}\int_{B^*} \big|g(x) - c_{B^*}\big|dx =: c_1,$$

where $c_{B^*} = \frac{1}{|B^*|}\int_{B^*} g(y)dy$.

In the other direction, if $\theta \cap B^* = \emptyset$, then $g(x) = 0$ on θ, and we are done. Otherwise, $\theta \cap B^* \ne \emptyset$. Assume that $\theta = \theta(x,t)$. If $t \le 0$, then

$$\frac{1}{|\theta|}\int_{\theta} \big|g(x) - c_\theta\big|dx \le \frac{1}{|\theta|}\int_{B^*} \big|g(x)\big|dx \le c.$$

We now deal with the case $\theta \cap B^* \ne \emptyset$, $\theta = \theta(x,t)$ with $t \ge 0$. Let $a := \inf_{y\in\theta}|y_n|$. There are two cases.

Case I: $\sup_{(y_1,\ldots,y_n)\in\theta}|y_n - a| \le a$. Here we have by the monotonicity of the log function

$$\begin{aligned}\frac{1}{|\theta|}\int_{\theta}(\log|y_n| - \log a)dy &\le \frac{1}{|\theta|}\int_{\theta}(\log(|y_n - a| + a) - \log a)dy \\ &= \frac{1}{|\theta|}\int_{\theta}\log\Big(\frac{|y_n - a|}{a} + 1\Big)dy \le \log 2.\end{aligned}$$

Case II: $\sup_{(y_1,\ldots,y_n)\in\theta}|y_n - a| > a$. This condition implies that for $\theta = \theta(x,t)$, $3\cdot\theta = x + 3M_{x,t}(B^*)$ intersects the hyperplane $\{y = (y_1,\ldots,y_{n-1},0)\}$. Let $z := (z_1,\ldots,z_{n-1},0)$ be a point in the intersection. Using (2.28), $3\cdot\theta \subseteq \theta(x,t-3J_1)$, and by Lemma 2.18, $\theta(x,t-3J_1) \subseteq \theta(z,t-3J_1-\gamma)$. Therefore

$$\theta = \theta(x,t) \subset \theta(x,t-3J_1) \subset \theta(z,t-3J_1-\gamma) =: \eta.$$

Let $b := \sup_{(y_1,\ldots,y_n)\in\eta}|y_n|$. With this definition,

$$\frac{1}{|\theta|}\int_{\theta}(\log b - \log|y_n|)dy \le C\frac{1}{|\eta|}\int_{\eta}(\log b - \log|y_n|)dy.$$

Denoting

$$H_j := \{(y_1, \dots, y_n) \in \eta : |y_n| \le 2^{-j} b\}, \quad j \ge 0,$$

we may apply Lemma 6.53(c) with $\tilde{z}_n = b$ and $\tilde{x}_n = 0$ to conclude

$$\begin{aligned}
\frac{1}{|\eta|} \int_\eta (\log b - \log |y_n|) dy &= \log b + \frac{1}{|\eta|} \sum_{j=1}^{\infty} \int_{H_{j-1} \setminus H_j} \log |y_n|^{-1} dy \\
&\le \log b + \frac{1}{|\eta|} \sum_{j=1}^{\infty} |H_{j-1} \setminus H_j| \log(b^{-1} 2^j) \\
&\le \log b - \log b + n! \log 2 \frac{1}{|\eta|} \sum_{j=1}^{\infty} 2^{-j} |\eta| j \\
&\le n! \log 2 \sum_{j=1}^{\infty} 2^{-j} j = c(n).
\end{aligned}$$

□

Proof of Theorem 6.51. It is obvious that (i) ⇒ (iii) ⇒ (ii), and so it remains to show that (ii) ⇒ (i). First, observe that for $n = 1$, any cover induces a quasi-distance equivalent to the Euclidean distance, so the result is obvious. For $n \ge 2$, assume to the contrary that (ii) holds but (i) does not. Then without loss of generality there exists a sequence of pairs of points $u_m, v_m \in \mathbb{R}^n$, $u_m \ne v_m$, $m \ge 1$, such that

$$\frac{\rho_1(u_m, v_m)}{\rho_2(u_m, v_m)} \to 0 \quad \text{as } m \to \infty. \tag{6.98}$$

Assuming that (6.98) holds, we will construct a sequence of compactly supported piecewise constant functions $\{f_m\}$ such that

$$\frac{\|f_m\|_{H^1(\Theta_1)}}{\|f_m\|_{H^1(\Theta_2)}} \to 0 \quad \text{as } m \to \infty,$$

thereby contradicting our assumption that $H^1(\Theta_1) \sim H^1(\Theta_2)$.

Let $0 < \varepsilon < 1$, and let $m \ge 1$ be such that $\rho_1(u_m, v_m)/\rho_2(u_m, v_m) \le \varepsilon/2$. Let $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ be such that

$$\begin{aligned}
&u_m, v_m \in \theta_1, \quad \rho_1(u_m, v_m) \le |\theta_1| \le (1+\varepsilon)\rho_1(u_m, v_m), \\
&u_m, v_m \in \theta_2, \quad \rho_2(u_m, v_m) \le |\theta_2| \le (1+\varepsilon)\rho_2(u_m, v_m).
\end{aligned}$$

This implies

$$\frac{|\theta_1|}{|\theta_2|} \le \varepsilon.$$

We now choose three ellipsoids centered at $z_m := (u_m + v_m)/2$ as follows:

(i) $\tilde{\theta}_1 := \theta(z_m, t_1) \in \Theta_1$ such that $|\tilde{\theta}_1| \sim |\theta_1|$ and $u_m, v_m \in \tilde{\theta}_1$,
(ii) $\tilde{\theta}_2 := \theta(z_m, t_2) \in \Theta_2$ such that $|\tilde{\theta}_2| \sim |\theta_2|$ with $u_m, v_m \in (\tilde{\theta}_2)^c$,
(iii) $\hat{\theta}_2 := \theta(z_m, t_2 + J_2) \in \Theta_2$, where $J_2 := J(\mathbf{p}(\Theta_2))$ is the constant of (2.30) related to Θ_2, satisfying $2M_{z_m,t_2+J_2}(B^*) \subset M_{z_m,t_2}(B^*)$.

Take an affine transformation A_m incorporating a rotational element that satisfies
(i) $A_m(B^*) = \hat{\theta}_2$,
(ii) $A_m^{-1}(\tilde{\theta}_1)$ is symmetric with respect to the $x_n = 0$ hyperplane.

We define new covers $\Theta_1' := A_m^{-1}\Theta_1$ and $\Theta_2' := A_m^{-1}\Theta_2$ with equivalent parameters to Θ_1 and Θ_2, respectively, and new points $\tilde{u}_m := A_m^{-1}(u_m)$ and $\tilde{v}_m := A_m^{-1}(v_m)$. We now have the following geometric objects "at the origin" with the following properties:
(i) $B^* = A_m^{-1}(\hat{\theta}_2) \in \Theta_2'$,
(ii) $\tilde{\theta}_1' := A_m^{-1}(\tilde{\theta}_1) \in \Theta_1'$ with $\tilde{u}_m, \tilde{v}_m \in \tilde{\theta}_1'$ and $|\tilde{\theta}_1'| < c\varepsilon$,
(iii) $\theta_2' := A_m^{-1}(\tilde{\theta}_2) \in \Theta_2'$ with $2B^* \subset \theta_2'$, $\tilde{u}_m, \tilde{v}_m \in (\theta_2')^c \subset (2B^*)^c$ and $|\theta_2'| \sim 1$.

We write $\tilde{\theta}_1' = \tilde{\theta}_1'(0, \tilde{t}_1') = M_{0,\tilde{t}_1'}(B^*)$, where $\tilde{t}_1' \in \mathbb{R}$. Since $\tilde{\theta}_1' \cap (2B^*)^c \neq \emptyset$, we may define

$$s' := \sup\{s \geq 0 : (2B^*)^c \cap M_{0,\tilde{t}_1'+s}(B^*) \neq \emptyset\}, \quad \theta_1' := M_{0,\tilde{t}_1'+s'}(B^*) \in \Theta_1'.$$

The newly constructed ellipsoid θ_1' may no longer contain the points $\tilde{u}_m, \tilde{v}_m$, but it has a center at the origin and the following properties:
(i) $(2B^*)^c \cap \theta_1' \neq \emptyset$,
(ii) $|\theta_1'| \leq c\varepsilon$,
(iii) $|B^* \cap \theta_1'| \sim |(2B^* \setminus B^*) \cap \theta_1'| \sim |\theta_1'|$.
(iv) By rotation about the origin of the entire construction of the covers Θ_1', Θ_2' we may assume that θ_1' has its longest axis along the x_1-axis.

Therefore there exist two boxes Ω_1 and Ω_2, identical up to a shift, that are symmetric to the main axes and of dimensions $d_1 \times \cdots \times d_n$, with the following properties:
(i) $\Omega_1 = [0, d_1] \times \cdots \times [0, d_n] \subset B^* \cap \theta_1'$,
(ii) $\Omega_2 \subset (2B^* \setminus B^*) \cap \theta_1'$,
(iii) $d_1 \sim 1$, and there exists $2 \leq i \leq n$ such that $d_i \leq c\sqrt[n-1]{\varepsilon}$,
(iv) $|\Omega_1| = |\Omega_2| \sim |\theta_1'|$, which implies that $1/d_i \sim \frac{d_1 \times \cdots \times d_{i-1} \times d_{i+1} \times \cdots \times d_n}{|\theta_1'|}$.

We will now construct a function $f_m' \in H^1(\Theta_1')$ with $\|f_m'\|_{H^1(\Theta_1')} \leq c$ for which $\|f_m'\|_{H^1(\Theta_2')} \geq c' \log(c''\varepsilon^{-1})$. This will mean that for $f_m := f_m'(A_m^{-1}\cdot)$, we have

$$\frac{\|f_m\|_{H^1(\Theta_1)}}{\|f_m\|_{H^1(\Theta_2)}} \leq c'''\left(\log(c''\varepsilon^{-1})\right)^{-1},$$

which is a contradiction to the assumption $H^1(\Theta_1) \sim H^1(\Theta_2)$, since ε can be chosen arbitrarily small.

We define $f'_m \in H^1(\Theta'_1)$ by $f'_m := |\theta'_1|^{-1}(\mathbf{1}_{\Omega_1} - \mathbf{1}_{\Omega_2})$. Now f'_m is not necessarily an atom in $H^1(\Theta'_1)$, since it may not have the sufficient $N_1(\Theta'_1)$ vanishing moments as per Definition 6.22. However, f'_m is a constant multiple of an atom in $H^1_{\infty,1}(X)$, where $X = (\mathbb{R}^n, \rho'_1, dx)$ is the space of homogeneous type induced by Θ'_1, and therefore, based on Theorem 6.48, we may deduce that $\|f'_m\|_{H^1(\Theta'_1)} \le c$. By Lemma 6.54 the function g_i defined by (6.97) is in $\mathrm{BMO}(\Theta'_2)$ with $\|g_i\|_{\mathrm{BMO}(\Theta'_2)} \sim 1$. From the properties of g_i and the boxes Ω_1 and Ω_2, for sufficiently small ε, we have

$$\begin{aligned}
\|f'_m\|_{H^1(\Theta'_2)} &\ge C \sup_{\varphi\in\mathrm{BMO}(\Theta'_2)} \frac{|\langle f'_m, \varphi\rangle|}{\|\varphi\|_{\mathrm{BMO}(\Theta'_2)}} \\
&\ge C\big|\langle f'_m, g_i\rangle\big| \\
&\ge -C\frac{1}{|\theta'_1|}\int_{\Omega_1} \log|x_i|dx \\
&= -C\frac{d_1\times\cdots\times d_{i-1}\times d_{i+1}\times\cdots\times d_n}{|\theta'_1|}\int_0^{d_i}\log(x_i)dx_i \\
&\ge -C\frac{1}{d_i}\int_0^{d_i}\log(x_i)dx_i \\
&\ge c'\log(c''\varepsilon^{-1}).
\end{aligned}$$

□

6.6 Anisotropic molecules

Definition 6.55. Let Θ be a continuous cover, let (p, q, l) be admissible, and let $\delta > a_4 l + 1$. Suppose that g is a measurable function on $\mathbb{R}^n$ such that for $\tilde{c} > 0$, $\theta = \theta(z,t) \in \Theta$, $z \in \mathbb{R}^n$, and $t \in \mathbb{R}$,

$$\|g\|_{L_q(\theta)} \le \tilde{c}|\theta|^{1/q-1/p}, \tag{6.99}$$

$$\big|g(x)\big| \le \tilde{c}\big|\theta(z,t)\big|^{-\frac{1}{p}}2^{-kJ\delta}, \quad \forall x \in \theta\big(z, t-(k+1)J\big)\setminus\theta(z,t-kJ),\ k\ge 0, \tag{6.100}$$

$$\int_{\mathbb{R}^n} g(x)x^\alpha dx = 0, \quad \forall\alpha\in\mathbb{Z}^n_+,\ |\alpha|\le l. \tag{6.101}$$

Then, we say that g is a *molecule localized around* θ.

Theorem 6.56. *Let* Θ *be a pointwise continuous cover, let* (p, q, l) *be admissible, and let* $\delta > a_4 l + 1$. *If* g *is a molecule, then* $g \in H^p(\Theta)$ *and* $\|g\|_{H^p(\Theta)} \le \tilde{c}c(\mathbf{p}(\Theta), p, q, l, \delta)$.

Before we prove the theorem, we need the following definition and result. For any $l \in \mathbb{N}$ and bounded convex domain $\Omega \in \mathbb{R}^n$, define $\pi_\Omega : L_1(\Omega) \to \Pi_l$ as the natural Riesz representation of the action of $f \in L_1(\Omega)$ on Π_l,

$$\int_\Omega \pi_\Omega(f)Q = \int_\Omega fQ, \quad \forall Q \in \Pi_l. \tag{6.102}$$

Lemma 6.57. *For any $l \in \mathbb{N}$, there exist a positive constant $c(n,l) > 0$ such that for any ellipsoid $\theta \in \mathbb{R}^n$ and $f \in L_1(\theta)$,*

$$\|\pi_\theta f\|_{L_\infty(\theta)} \le c|\theta|^{-1}\|f\|_{L_1(\theta)}. \tag{6.103}$$

Proof. Let $\{P_\beta\}_{|\beta|\le l}$ be an orthonormal basis of Π_l in $L_2(B^*)$. For $f \in L_1(B^*)$, we have

$$\pi_{B^*}f = \sum_{|\beta|\le l}\Big(\int_{B^*} fP_\beta\Big)P_\beta.$$

Therefore, since $\|P_\beta\|_{L_\infty(B^*)} \sim \|P_\beta\|_{L_2(B^*)} = 1$ for all $|\beta| \le l$, for any $y \in B^*$, we have

$$|\pi_{B^*}f(y)| \le \sum_{|\beta|\le l}\Big(\int_{B^*} |f||P_\beta|\Big)|P_\beta(y)| \le C|B^*|^{-1}\int_{B^*}|f|, \tag{6.104}$$

which proves the case $\theta = B^*$. Now let θ be an arbitrary ellipsoid in $\mathbb{R}^n$, and let A_θ be an affine transform such that $\theta = A_\theta(B^*)$, $Ax = Mx+v$. Then by (6.102), for any $f \in L_1(\theta)$ and $Q \in \Pi_l$,

$$\begin{aligned}\int_\theta \pi_{B^*}(f(A_\theta\cdot))(A_\theta^{-1}x)Q(x)dx &= |\det(M)|\int_{B^*}\pi_{B^*}(f(A_\theta\cdot))(y)Q(A_\theta y)dy\\ &= |\det(M)|\int_{B^*} f(A_\theta y)Q(A_\theta y)dy\\ &= \int_\theta f(x)Q(x)dx\\ &= \int_\theta \pi_\theta f(x)Q(x)dx.\end{aligned}$$

This provides the affine transformation identity

$$\pi_\theta f(x) = \pi_{B^*}(f(A_\theta\cdot))(A_\theta^{-1}x). \tag{6.105}$$

From (6.105) and the bound on B^* (6.104), for any $x \in \theta$, we have

$$\begin{aligned} |\pi_\theta f(x)| &= |\pi_{B^*}(f(A_\theta \cdot))(A_\theta^{-1}x)| \\ &\le \sup_{y \in B^*} |\pi_{B^*}(f(A_\theta \cdot))(y)| \\ &\le C \int_{B^*} |f(A_\theta y)| dy \\ &\le C |\det(M)|^{-1} \int_\theta |f| \\ &= C|\theta|^{-1} \|f\|_{L_1(\theta)}. \end{aligned}$$

□

Proof of Theorem 6.56. We follow the proof in [12] (see also [3]). For any ellipsoid η and $f \in L_1(\eta)$, define $\tilde{\pi}_\eta f := f - \pi_\eta f$. By (6.103), for $1 \le q \le \infty$, we get

$$\begin{aligned} \|\tilde{\pi}_\eta f\|_{L_q(\eta)} &\le \|f\|_{L_q(\eta)} + \|\pi_\eta f\|_{L_q(\eta)} \\ &\le \|f\|_{L_q(\eta)} + |\eta|^{1/q} \|\pi_\eta f\|_{L_\infty(\eta)} \\ &\le \|f\|_{L_q(\eta)} + C|\eta|^{1/q-1} \|f\|_{L_1(\eta)} \\ &\le \|f\|_{L_q(\eta)} + C\|f\|_{L_q(\eta)} \le C\|f\|_{L_q(\eta)}. \end{aligned}$$

Furthermore,

$$\int_\eta \tilde{\pi}_\eta f(x) x^\alpha dx = 0, \quad \forall |\alpha| \le l. \tag{6.106}$$

Let g be a molecule localized around $\theta := \theta(z,t)$. We want to represent g as a combination of atoms supported on $\theta(z, t - kJ)$, $k \ge 0$. Define the sequence of function $\{g_k\}_{k=0}^\infty$ by

$$g_k := \mathbf{1}_{\theta(z,t-kJ)} \tilde{\pi}_{\theta(z,t-kJ)} g.$$

Clearly, $\operatorname{supp}(g_k) \subset \theta(z, t - kJ)$. By (6.106), g_k, $k \ge 0$, has vanishing moments up to order l. For $k = 0$, applying property (6.99) yields with a constant $c_1 > 0$

$$\|g_0\|_q = \|\tilde{\pi}_{\theta(z,t)} g\|_{L_q(\theta(z,t))} \le C\|g\|_{L_q(\theta(z,t))} \le \tilde{c}c_1 |\theta|^{1/q-1/p}.$$

Therefore g_0 is a $\tilde{c}c_1$ multiple of a (p, q, l)-atom. The goal now is showing that

$$g = g_0 + \sum_{j=0}^\infty (g_{k+1} - g_k), \tag{6.107}$$

where the convergence is both in L_1 and $H^p(\Theta)$, with $g_{k+1} - g_k$ being appropriate multiples of (p, ∞, l)-atoms supported on $\theta(z, t - (k+1)J)$.

We claim that $g_k \to g$ in L_1 (and hence in $\mathcal{S}'$) as $k \to \infty$. It suffices to show that $\|\pi_{\theta(z,t-kJ)} g\|_{L_1(\theta(z,t-kJ))} \to 0$ as $k \to \infty$. Indeed, let $\{P_\beta : |\beta| \le l\}$ be an orthonormal basis of Π_l with respect to the $L_2(B^*)$ norm. Using (6.105), for $x \in \theta(z, t-kJ)$, we have

$$\begin{aligned}\pi_{\theta(z,t-kJ)} g(x) &= \pi_{B^*}\big(g(A_{\theta(z,t-kJ)}\cdot)\big)(A^{-1}_{\theta(z,t-kJ)}x)\\ &= \sum_{|\beta|\le l}\Big(\int_{B^*} g(A_{\theta(z,t-kJ)}\cdot)P_\beta\Big)P_\beta(A^{-1}_{\theta(z,t-kJ)}x)\\ &= \big|\det(M_{z,t-kJ})\big|^{-1}\sum_{|\beta|\le l}\Big(\int_{\theta(z,t-kJ)} gP_\beta(A^{-1}_{\theta(z,t-kJ)}\cdot)\Big)P_\beta(A^{-1}_{\theta(z,t-kJ)}x).\end{aligned}$$

From the above we obtain the L_∞ estimate

$$\|\pi_{\theta(z,t-kJ)} g\|_{L_\infty(\theta(z,t-kJ))} \le C\big|\det(M_{z,t-kJ})\big|^{-1}\sum_{|\beta|\le l}\Big|\int_{\theta(z,t-kJ)} gP_\beta(A^{-1}_{\theta(z,t-kJ)}\cdot)\Big|. \tag{6.108}$$

To obtain an L_1 estimate, we use

$$\big\|P_\beta(A^{-1}_{\theta(z,t-kJ)}\cdot)\big\|_{L_1(\theta(z,t-kJ))} \le C\big|\det(M_{z,t-kJ})\big|$$

with the vanishing moments property of the molecule g (6.101), its decay (6.100), and the uniform bound $\|P_\beta\|_\infty \le c$ for all $|\beta| \le l$ to obtain

$$\int_{\theta(z,t-kJ)} gP_\beta(A^{-1}_{\theta(z,t-kJ)}\cdot) = -\int_{\theta(z,t-kJ)^c} gP_\beta(A^{-1}_{\theta(z,t-kJ)}\cdot) \to 0 \quad \text{as } k \to \infty.$$

From this we conclude that

$$\|\pi_{\theta(z,t-kJ)} g\|_{L_1(\theta(z,t-kJ))} \to 0 \quad \text{as } k \to \infty,$$

which shows that $g_k \to g$ in L_1. Next, we estimate

$$\begin{aligned}\|g_{k+1} - g_k\|_\infty &= \|\mathbf{1}_{\theta(z,t-(k+1)J)}\tilde{\pi}_{\theta(z,t-(k+1)J)}g - \mathbf{1}_{\theta(z,t-kJ)}\tilde{\pi}_{\theta(z,t-kJ)}g\|_\infty\\ &= \|\mathbf{1}_{\theta(z,t-(k+1)J)\setminus\theta(z,t-kJ)}g - \mathbf{1}_{\theta(z,t-(k+1)J)}\pi_{\theta(z,t-(k+1)J)}g + \mathbf{1}_{\theta(z,t-kJ)}\pi_{\theta(z,t-kJ)}g\|_\infty\\ &\le \|\mathbf{1}_{\theta(z,t-(k+1)J)\setminus\theta(z,t-kJ)}g\|_\infty + \|\mathbf{1}_{\theta(z,t-(k+1)J)}\pi_{\theta(z,t-(k+1)J)}g\|_\infty\\ &\quad + \|\mathbf{1}_{\theta(z,t-kJ)}\pi_{\theta(z,t-kJ)}g\|_\infty\\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

For the estimate of the term I, we apply (6.100) to get

$$\begin{aligned} \mathrm{I} &= \|\mathbf{1}_{\theta(z,t-(k+1)J)\setminus\theta(z,t-kJ)}g\|_\infty \\ &\le \tilde{c}\big|\theta(z,t)\big|^{-1/p}2^{-kJ\delta} \\ &\le \tilde{c}C\big|\theta(z,t-kJ)\big|^{-\frac{1}{p}}2^{-kJ(\delta-1/p)}. \end{aligned}$$

We now bound the term III (the bound of the term II is similar). Notice that $|P_\beta(x)| \le c|x|^l$ for any $x \in (B^*)^c$ and some constant $c > 0$. By (6.101) and (6.100) we have

$$\begin{aligned} \Big|\int_{\theta(z,t-kJ)} gP_\beta(A^{-1}_{\theta(z,t-kJ)}\cdot)\Big| &= \Big|\int_{\theta(z,t-kJ)^c} gP_\beta(A^{-1}_{\theta(z,t-kJ)}\cdot)\Big| \\ &\le C\int_{\theta(z,t-kJ)^c} \big|g(x)\big|\big|M^{-1}_{z,t-kJ}(x-z)\big|^l dx \\ &\le \tilde{c}C\big|\theta(z,t)\big|^{-\frac{1}{p}}\sum_{i=k}^{\infty}2^{-iJ\delta}\int_{\theta(z,t-(i+1)J)\setminus\theta(z,t-iJ)} \big|M^{-1}_{z,t-kJ}(x-z)\big|^l dx \\ &\le \tilde{c}C\big|\theta(z,t)\big|^{-\frac{1}{p}}\big|\det(M_{z,t-kJ})\big|\sum_{i=k}^{\infty}2^{-iJ\delta}\int_{M^{-1}_{z,t-kJ}M_{z,t-(i+1)J}(B^*)} |y|^l dy \\ &\le \tilde{c}C\big|\theta(z,t)\big|^{-\frac{1}{p}}2^{-t+kJ}\sum_{i=k}^{\infty}2^{-iJ\delta}\big\|M^{-1}_{z,t-kJ}M_{z,t-(i+1)J}\big\|^l \\ &\quad\times\big|\det(M^{-1}_{z,t-kJ}M_{z,t-(i+1)J})\big| \\ &\le \tilde{c}C\big|\theta(z,t)\big|^{-\frac{1}{p}}2^{-t-kJ(\delta-1)}\sum_{i=k}^{\infty}2^{-J\delta(i-k)}2^{a_4 lJ(i-k)}2^{J(i-k)} \\ &\le \tilde{c}C\big|\theta(z,t-kJ)\big|^{-\frac{1}{p}}2^{-t-kJ(\delta-1-1/p)}. \end{aligned}$$

The last series converges since $\delta > a_4 l + 1$. Therefore from (6.108) we may bound III $\le \tilde{c}C|\theta(z,t-kJ)|^{-\frac{1}{p}}2^{-kJ(\delta-1/p)}$. As already noted, we have a similar bound for II. Combining the estimates of I, II, and III, we conclude that for some constant $c_2 > 0$, we have

$$\|g_{k+1} - g_k\|_\infty \le \tilde{c}c_2\big|\theta(z,t-(k+1)J)\big|^{-\frac{1}{p}}2^{-kJ(\delta-1/p)}, \quad k \ge 0.$$

Since each g_k also has vanishing moments up to order l, $g_{k+1} - g_k$ is a λ_k multiple of a (p,∞,l)-atom a_k supported on $\theta(z,t-(k+1)J)$, that is, $g_{k+1} - g_k = \lambda_k a_k$, where a_k is an atom, and $\lambda_k = \tilde{c}c_2 2^{-kJ(\delta-1/p)}$. Since any (p,∞,l)-atom is a (p,q,l)-atom, by (6.107) we have

$$\|g\|^p_{H^p_{q,l}(\Theta)} \le \tilde{c}^p c_1^p + \tilde{c}^p c_2^p\sum_{k=0}^{\infty}2^{-kpJ(\delta-1/p)} \le \tilde{c}^p C.$$

The last series converges since $l \geq N_p(\Theta)$, $a_4 \geq a_6$, and hence

$$\delta - 1/p > a_4 N_p(\Theta) - 1/p > a_4 \frac{\max(1, a_4)n + 1}{a_6 p} - 1/p > 0.$$

This finishes the proof of the theorem. □

6.7 Finite atomic spaces

In this section, we follow [66] (see also [14]) and analyze pointwise variable anisotropic finite atomic spaces. Our main application is the characterization in Section 6.8 of the dual spaces of the anisotropic Hardy spaces using anisotropic Campanato spaces.

Definition 6.58. Let Θ be a continuous cover, and let (p, q, l) be admissible as in Definition 6.22. We define $H^p_{\mathrm{fin},q,l}(\Theta)$ as the space of all finite combinations of (p, q, l)-atoms with the quasi-norm

$$\|f\|_{H^p_{\mathrm{fin},q,l}(\Theta)} := \inf\left\{\left(\sum_{i=1}^{k} |\lambda_i|^p\right)^{1/p} : f = \sum_{i=1}^{k} \lambda_i a_i,\ \{a_i\} \text{ are } (p, q, l)\text{-atoms}\right\}.$$

Theorem 6.59. *Let Θ be a pointwise continuous cover. For any admissible (p, q, l), $1 < q \leq \infty$,*
(i) *for $1 < q < \infty$, $\|\cdot\|_{H^p_{\mathrm{fin},q,l}(\Theta)}$ and $\|\cdot\|_{H^p(\Theta)}$ are equivalent quasi-norms on $H^p_{\mathrm{fin},q,l}(\Theta)$.*
(ii) *$\|\cdot\|_{H^p_{\mathrm{fin},\infty,l}(\Theta)}$ and $\|\cdot\|_{H^p(\Theta)}$ are equivalent quasi-norms on $H^p_{\mathrm{fin},\infty,l}(\Theta) \cap C(\mathbb{R}^n)$.*

Proof. It is obvious from Definitions 6.23 and 6.58 and Theorem 6.24 that $H^p_{\mathrm{fin},q,l}(\Theta) \subset H^p_{q,l}(\Theta) \sim H^p(\Theta)$, $1 < q \leq \infty$, and that for any $f \in H^p_{\mathrm{fin},q,l}(\Theta)$,

$$\|f\|_{H^p_{q,l}(\Theta)} \leq \|f\|_{H^p_{\mathrm{fin},q,l}(\Theta)}.$$

Hence, to prove (i), it is sufficient to show that there exists $c > 0$ such that when $1 < q < \infty$, for all $f \in H^p_{\mathrm{fin},q,l}(\Theta)$,

$$\|f\|_{H^p_{\mathrm{fin},q,l}(\Theta)} \leq c\|f\|_{H^p(\Theta)}. \tag{6.109}$$

A similar claim holds for case (ii). We prove (6.109) in five steps.

Step 1. For any $f \in H^p_{\mathrm{fin},q,l}(\Theta)$, by homogeneity we can assume that $\|f\|_{H^p(\Theta)} = 1$. Since f may be represented by a finite combination of atoms, it has compact support, and by (2.25) there exists $t_0 \in \mathbb{R}$ such that $\mathrm{supp}(f) \subset \theta(0, t_0)$. Recall that for each $k \in \mathbb{Z}$, we have $\Omega_k := \{x : M^\circ f(x) > 2^k\}$. Since f has a finite atomic representation, it is easy to see that $f \in H^p(\Theta) \cap L_q(\mathbb{R}^n)$ for $1 < q < \infty$. It is also easy to see that for the case $q = \infty$, $f \in H^p(\Theta) \cap L_2(\mathbb{R}^n)$. Therefore by Lemma 6.42 there exists a (possibly infinite)

(p, ∞, l)-atomic representation $f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k$, which holds in L_q (L_2 for $q = \infty$), the sequence converges to f a. e., and there exists a constant $C_2 > 0$ such that

$$\sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} |\lambda_i^k|^p \le C\|f\|^p_{H^p(\Theta)} \le C_2. \tag{6.110}$$

Step 2. We claim that there exists $\widetilde{c} > 0$, which does not depend on f, such that

$$M^\circ f(x) \le \widetilde{c}\left|\theta(0, t_0)\right|^{-1/p}, \quad \forall x \in \theta(0, t_0 - \gamma)^c. \tag{6.111}$$

Let $x \in \theta(0, t_0 - \gamma)^c$. We claim that $\theta(x, t) \cap \theta(0, t_0) = \emptyset$ for all $t > t_0$. Otherwise, by Lemma 2.18 $\theta(x, t) \subset \theta(0, t_0 - \gamma) \Rightarrow x \in \theta(0, t_0 - \gamma)$, which is a contradiction. Applying also (6.19), we have

$$M^\circ f(x) \le C \sup_{\varphi\in\mathcal{S}_{N_p,\widetilde{N}_p}, \operatorname{supp}(\varphi)\subseteq B^*} \sup_{t\le t_0} \left|f * \varphi_{x,t}(x)\right|. \tag{6.112}$$

Therefore it is sufficient to bound $M^\circ f(x)$ by taking the supremum over $\operatorname{supp}(\varphi) \subseteq B^*$ and $t \le t_0$. Let $\psi \in \mathcal{S}$ be such that $\operatorname{supp}(\psi) \subset 2B^*$, $0 \le \psi \le 1$, and $\psi \equiv 1$ on B^*. Letting $z \in \theta(0, t_0)$, we have $M^{-1}_{0,t_0} z \subset B^*$, and hence $\psi(M^{-1}_{0,t_0} z) \equiv 1$. From this and $\operatorname{supp}(f) \subset \theta(0, t_0)$ we deduce that

$$f * \varphi_{x,t}(x) = \left|\det(M^{-1}_{x,t})\right| \int_{\theta(0,t_0)} f(z)\varphi(M^{-1}_{x,t}(x - z))\psi(M^{-1}_{0,t_0} z)\, dz.$$

We assume that $\theta(x, t) \cap \theta(0, t_0) \ne \emptyset$. Otherwise, $f * \varphi_{x,t}(x) = 0$, and were are done. We now define

$$\Phi(z) := \frac{|\det(M^{-1}_{x,t})|}{|\det(M^{-1}_{0,t_0})|}\varphi(M^{-1}_{x,t}x + M^{-1}_{x,t}M_{0,t_0}z)\psi(-z).$$

Then we have

$$\begin{aligned} f * \Phi_{0,t_0}(0) &= \left|\det(M^{-1}_{0,t_0})\right| \int_{\mathbb{R}^n} f(z)\Phi(M^{-1}_{0,t_0}(-z))\, dz \\ &= \left|\det(M^{-1}_{x,t})\right| \int_{\mathbb{R}^n} f(z)\varphi(M^{-1}_{x,t}(x - z))\psi(M^{-1}_{0,t_0} z)\, dz \\ &= f * \varphi_{x,t}(x). \end{aligned} \tag{6.113}$$

By (2.13), since $t \le t_0$,

$$\frac{|\det(M^{-1}_{x,t})|}{|\det(M^{-1}_{0,t_0})|} \le C2^{t-t_0} \le C.$$

For any $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le N_p$, $t \le t_0$, by (2.14) we have

$$\begin{aligned}
\left|\partial^\alpha \Phi(z)\right| &\le C\left|\partial_z^\alpha\left[\varphi(M_{x,t}^{-1}x + M_{x,t}^{-1}M_{0,t_0}z)\psi(-z)\right]\right| \\
&\le C \max_{|\beta|\le N_p} \left\|M_{x,t}^{-1}M_{0,t_0}\right\|^{|\beta|}\left|(\partial^\beta\varphi)(M_{x,t}^{-1}x + M_{x,t}^{-1}M_{0,t_0}z)\right| \\
&\le C \max_{|\beta|\le N_p} \left[a_5 2^{-a_6(t_0-t)}\right]^{|\beta|}\left|(\partial^\beta\varphi)(M_{x,t}^{-1}x + M_{x,t}^{-1}M_{0,t_0}z)\right| \\
&\le C \max_{|\beta|\le N_p} \left|(\partial^\beta\varphi)(M_{x,t}^{-1}x + M_{x,t}^{-1}M_{0,t_0}z)\right|,
\end{aligned}$$

which, together with $\|\varphi\|_{N_p,\widetilde{N}_p} \le 1$ and $\mathrm{supp}(\Phi) \subseteq \mathrm{supp}(\psi) \subseteq 2B^*$, implies that

$$\begin{aligned}
\|\Phi\|_{N_p,\widetilde{N}_p} &\le C \max_{|\alpha|\le N_p} \sup_{z\in 2B^*} \left|\partial^\alpha\Phi(z)\right| \\
&\le C \max_{|\beta|\le N_p} \sup_{z\in 2B^*} \left|(\partial^\beta\varphi)(M_{x,t}^{-1}x + M_{x,t}^{-1}M_{0,t_0}z)\right|\left(1 + \left|M_{x,t}^{-1}x + M_{x,t}^{-1}M_{0,t_0}z\right|\right)^{\widetilde{N}_p} \\
&\le C\|\varphi\|_{N_p,\widetilde{N}_p} \le C.
\end{aligned}$$

For any $u \in \theta(0, t_0)$ and any $y \in \theta(u, t_0)$, define

$$\widetilde{\Phi}(z) := \frac{|\det(M_{0,t_0}^{-1})|}{|\det(M_{u,t_0}^{-1})|}\Phi\big(M_{0,t_0}^{-1}(M_{u,t_0}z - y)\big).$$

By (2.13) it follows that

$$\frac{|\det(M_{0,t_0}^{-1})|}{|\det(M_{u,t_0}^{-1})|} \le C.$$

By $\mathrm{supp}(\Phi) \subseteq 2B^*$ and $y \in \theta(u, t_0)$ we obtain that for some $c(\mathbf{p}(\Theta)) > 0$, $\mathrm{supp}(\widetilde{\Phi}) \subseteq cB^*$. Combining this with $\|\Phi\|_{N_p,\widetilde{N}_p} \le C$ gives

$$\begin{aligned}
\|\widetilde{\Phi}\|_{N_p,\widetilde{N}_p} &\le C \max_{|\alpha|\le N_p} \sup_{z\in\mathrm{supp}(\widetilde{\Phi})} \left|\partial^\alpha\widetilde{\Phi}(z)\right| \\
&\le C \max_{|\alpha|\le N_p} \sup_{z\in\mathrm{supp}(\widetilde{\Phi})} \left|(\partial^\alpha\Phi)(M_{0,t_0}^{-1}(M_{u,t_0}z - y))\right|\left(1 + \left|M_{0,t_0}^{-1}(M_{u,t_0}z - y)\right|\right)^{\widetilde{N}_p} \\
&\le C\|\Phi\|_{N_p,\widetilde{N}_p} \le C.
\end{aligned}$$

Therefore, noticing that $(\|\widetilde{\Phi}\|_{N_p,\widetilde{N}_p})^{-1}\widetilde{\Phi} \in \mathcal{S}_{N_p,\widetilde{N}_p}$ and applying Lemma 6.6, for any $u \in \theta(0,t_0)$, we obtain

$$\begin{aligned}(\|\widetilde{\Phi}\|_{N_p,\widetilde{N}_p})^{-1}|f * \Phi_{0,t_0}(0)| &\le C(\|\widetilde{\Phi}\|_{N_p,\widetilde{N}_p})^{-1}|f * \widetilde{\Phi}_{u,t_0}(y)| \\ &\le C \sup_{y\in\theta(u,t_0)} \left|f * ((\|\widetilde{\Phi}\|_{N_p,\widetilde{N}_p})^{-1}\widetilde{\Phi})_{u,t_0}(y)\right| \\ &\le CM_{N_p,\widetilde{N}_p} f(u) \le CM^{\circ} f(u),\end{aligned}$$

which, together with $\|f\|_{H^p(\Theta)} = 1$, for $t \le t_0$, yields

$$\begin{aligned}|f * \varphi_{x,t}(x)| &= |f * \Phi_{0,t_0}(0)| \\ &\le C \inf_{u\in\theta(0,t_0)} M^{\circ} f(u) \\ &\le C|\theta(0,t_0)|^{-1/p}\|M^{\circ} f\|_{L_p(\theta(0,t_0))} \\ &\le C|\theta(0,t_0)|^{-1/p}\|f\|_{H^p(\Theta)} \\ &= C|\theta(0,t_0)|^{-1/p}.\end{aligned}$$

From this and (6.112) we deduce (6.111).

Step 3. Denote by k' the largest integer such that $2^{k'} < \widetilde{c}|\theta(0,t_0)|^{-1/p}$, where $\widetilde{c}$ is as in Step 2. Then by (6.111) we have

$$\Omega_k \subset \theta(0, t_0 - \gamma) \quad \text{for } k > k'. \tag{6.114}$$

Using the atomic decomposition satisfying (6.110), let $h := \sum_{k\le k'} \sum_i \lambda_i^k a_i^k$ and $g := \sum_{k>k'} \sum_i \lambda_i^k a_i^k$. Now let us show that h is a multiple of a (p, ∞, l)-atom.

By (6.88) $\operatorname{supp}(a_i^k) \subset \Omega_k$. Together with (6.114), this implies $\operatorname{supp}(g) \subset \bigcup_{k>k'} \Omega_k \subset \theta(0,t_0-\gamma)$. Since we also have $\operatorname{supp}(f) \subset \theta(0,t_0) \subset \theta(0,t_0-\gamma)$, we obtain that $\operatorname{supp}(h) = \operatorname{supp}(f-g) \subset \theta(0,t_0-\gamma)$.

Since $2^{k'} < \widetilde{c}|\theta(0,t_0-\gamma)|^{-1/p}$, using also (6.89), we have with a fixed constant $C_1 > 0$

$$|h(x)| \le \sum_{k\le k'} \sum_i |\lambda_i^k a_i^k(x)| \le C \sum_{k\le k'} 2^k \le C_1|\theta(0,t_0-\gamma)|^{-1/p}.$$

The third required property from h of vanishing moments is obvious from the representation of h by (p,∞,l)-atoms and the previous two properties. Thus h is a C_1-multiple of a (p,∞,l)-atom and hence also for $q < \infty$, a C_1-multiple of a (p,q,l)-atom for any admissible triplet (p,q,l).

Step 4. We now focus on the case $1 < q < \infty$. By Lemma 6.42 $\sum_{k>k'} \sum_i \lambda_i^k a_i^k$ converges to g in L_q. For any positive integer K, let $F_K := \{(i,k) : k > k', |i| + |k| \le K\}$ and $g_K := \sum_{(i,k)\in F_K} \lambda_i^k a_i^k$. If K is large enough, then by the Lebesgue dominated convergence theorem and $g \in L_q$ we have $\|g - g_K\|_q \le |\theta(0,t_0-\gamma)|^{1/q-1/p}$. Since $\operatorname{supp}(g - g_K) \subset$

$\theta(0, t_0 - \gamma)$ and $g - g_k$ has l vanishing moments, we deduce that $g - g_K$ is a (p, q, l)-atom. Therefore $f = h + g_K + (g - g_K)$ is a finite atomic decomposition of f. Consequently, applying Step 3, $\|f\|_{H^p(\Theta)} = 1$, and (6.110), we have

$$\|f\|^p_{H^p_{\mathrm{fin},q,l}(\Theta)} \le \|h\|^p_{H^p_{\mathrm{fin},q,l}(\Theta)} + \sum_{(i,k)\in F_K} |\lambda_i^k|^p + \|g - g_K\|^p_{H^p_{\mathrm{fin},q,l}(\Theta)}$$
$$\le C_1^p + C_2 + 1 = C = C\|f\|^p_{H^p(\Theta)},$$

which proves (6.109) for $1 < q < \infty$ and ends the proof of (i).

Step 5. We now proceed to prove (ii). Let $f \in H^p_{\mathrm{fin},\infty,l}(\Theta) \cap C(\mathbb{R}^n)$. Recall from Step 3 the decomposition $f = h + g$, where h is a multiple of a (p, ∞, l)-atom. So it remains to decompose g to a finite superposition of (p, ∞, l)-atoms. Since we assumed f to be continuous in $\mathbb{R}^n$ and showed that its support is in the closure of $\theta(0, t_0 - \gamma)$, it is bounded. By Theorem 6.10 there exists a constant $c > 0$ such that $\|M^\circ f\|_\infty \le c\|f\|_\infty$. Let k'' be the largest integer such that $2^{k''} \le c\|f\|_\infty$. For any $k > k''$, we have that $2^k \ge c\|f\|_\infty$, and so $\Omega_k = \emptyset$. This implies that in this case, g constructed in Step 3 has a representation $g = \sum_{k'}^{k''} \sum_i \lambda_i^k a_i^k$. Recall that for $k > k'$,

$$\mathrm{supp}(a_i^k) \subseteq \theta(x_i^k, t_i^k - J - 3\gamma - 1) \cap \Omega_k \subset \theta(0, t_0 - \gamma).$$

For a given $\delta > 0$, to be chosen momentarily, we decompose $g = g_1 + g_2$ so that $g_1 = \sum_{(k,i)\in F_1} \lambda_i^k a_i^k$ and $g_2 = \sum_{(k,i)\in F_2} \lambda_i^k a_i^k$ with

$$F_1 := \{(k, i) : |\theta(x_i^k, t_i^k - J)| \ge \delta, k' < k \le k''\},$$
$$F_2 := \{(k, i) : |\theta(x_i^k, t_i^k - J)| < \delta, k' < k \le k''\}.$$

Next, we claim that the set F_1 is finite. Indeed, for each fixed $k' < k_0 \le k''$, by property (6.61) of the Whitney decomposition the cores $\{\theta(x_i^{k_0}, t_i^{k_0} + \gamma)\}_{(k_0,i)\in F_1}$ are pairwise disjoint. Furthermore, they are all contained in $\theta(0, t_0 - \gamma)$ and have volume $\ge c\delta$ for some fixed $c(\mathbf{p}(\Theta)) > 0$. Thus

$$\#F_1 \le (k'' - k')c^{-1}\delta^{-1}|\theta(0, t_0 - \gamma)| \le (k'' - k')c^{-1}\delta^{-1}a_2 2^{-(t_0-\gamma)}.$$

Therefore g_1 is a finite superposition of (p, ∞, l)-atoms, which by (6.110) satisfies

$$\sum_{(k,i)\in F_1} |\lambda_i^k|^p \le C_2.$$

We now turn to prove that for sufficiently small δ, we can ensure that g_2 is constant multiple of a (p, ∞, l)-atom. Since f is continuous, for

$$\varepsilon := (k'' - k')^{-1}|\theta(0, t_0 - \gamma)|^{-1/p}, \tag{6.115}$$

there exists $\delta' > 0$ such that if $|x-y| \leq \delta'$, then $|f(x)-f(y)| < \varepsilon$. By Lemma 2.26, for $\theta(0, t_0 - y)$, there exists a constant $c(\mathbf{p}(\Theta), t_0) > 0$ such that if $x \in \theta(0, t_0 - y)$ or $y \in \theta(0, t_0 - y)$ and $\rho(x,y) < 1$, then $|x-y| \leq c\rho(x,y)^{a_6}$. This implies that with the choice

$$\delta := \min\left(1, \left(\frac{\delta'}{c}\right)^{a_6^{-1}}\right),$$

we obtain

$$\rho(x,y) < \delta \Rightarrow |x-y| < \delta' \Rightarrow \left|f(x) - f(y)\right| < \varepsilon, \quad x \in \theta(0, t_0 - y) \vee y \in \theta(0, t_0 - y).$$

For any $(k,i) \in F_2$ and $x \in \theta(x_i^k, t_i^k - J)$, we have that $\rho(x, x_i^k) \leq |\theta(x_i^k, t_i^k - J)| < \delta$, which means $|f(x) - f(x_i^k)| < \varepsilon$. Write

$$\tilde{f}(x) := \left(f(x) - f\left(x_i^k\right)\right)\mathbf{1}_{\theta(x_i^k, t_i^k - J)}(x), \quad \tilde{P}_i^k(x) := P_i^k(x) - f\left(x_i^k\right),$$

where P_i^k is defined by (6.67). We see that for any $Q \in \Pi_l$,

$$\frac{1}{\int \varphi_i^k} \int_{\mathbb{R}^n} (\tilde{f} - \tilde{P}_i^k) Q \varphi_i^k = \frac{1}{\int \varphi_i^k} \int_{\mathbb{R}^n} (f - P_i^k) Q \varphi_i^k = 0.$$

Since $\|\tilde{f}\|_\infty < \varepsilon$, we have by the maximal theorem that $\|M^\circ \tilde{f}\|_\infty \leq C\varepsilon$, which in turn allows us to apply (6.70) to obtain

$$\|\tilde{P}_i^k \varphi_i^k\|_\infty \leq \|M^\circ \tilde{f}\|_\infty \leq C\varepsilon.$$

We also have $\tilde{P}_{i,j}^{k+1} = P_{i,j}^{k+1}$, where $P_{i,j}^{k+1}$ is defined by (6.83), and so using (6.84), we get

$$\|\tilde{P}_{i,j}^k \varphi_j^{k+1}\|_\infty \leq \|M^\circ \tilde{f}\|_\infty \leq C\varepsilon.$$

So, for $x \in \theta(x_i^k, t_i^k - J)$ and $(k,i) \in F_2$, recalling formula (6.90) for $h_i^k = \lambda_i^k a_i^k$ and using Lemma 6.39(iii), we get

$$\begin{aligned}
\left|\lambda_i^k a_i^k(x)\right| &= \left|h_i^k(x)\right| \\
&= \left|\left(f(x) - P_i^k(x)\right)\varphi_i^k(x) - \sum_{j \in \mathbb{N}} \left[\varphi_i^k(x)\left(f(x) - P_j^{k+1}(x)\right) - P_{ij}^{k+1}(x)\right]\varphi_j^{k+1}(x)\right| \\
&= \left|\left(\tilde{f}(x) - \tilde{P}_i^k(x)\right)\varphi_i^k(x) - \sum_{j \in \mathbb{N}} \left[\varphi_i^k(x)\left(\tilde{f}(x) - \tilde{P}_j^{k+1}(x)\right) - \tilde{P}_{ij}^{k+1}(x)\right]\varphi_j^{k+1}(x)\right| \\
&\leq \left|\tilde{f}(x)\mathbf{1}_{\Omega_{k+1}}(x)\right| + \left|\tilde{P}_i^k(x)\varphi_i^k(x)\right| + \sum_{j \in \mathbb{N}} \left|\varphi_i^k(x)\tilde{P}_j^{k+1}(x)\varphi_j^{k+1}(x)\right| + \sum_{j \in \mathbb{N}} \left|\tilde{P}_{ij}^{k+1}(x)\varphi_j^{k+1}(x)\right| \\
&\leq C\varepsilon.
\end{aligned}$$

Using (6.115), we get that with a fixed constant $C_3 > 0$,

$$|g_2(x)| \le \sum_{(k,i)\in F_2} |\lambda_i^k a_i^k(x)| \le C(k'' - k')\varepsilon = C_3 |\theta(0, t_0 - \gamma)|^{-1/p},$$

which implies that g_2 is a C_3-multiple of a (p, ∞, l)-atom.

Finally, we conclude that f has a finite (p, ∞, l)-atomic decomposition $f = h + \sum_{(k,i)\in F_1} \lambda_i^k a_i^k + g_2$ with

$$\|f\|^p_{H^p_{\mathrm{fin},\infty,l}(\Theta)} \le C_1^p + C_2 + C_3^p = C = C\|f\|_{H^p(\Theta)}. \qquad \square$$

As an application of Theorem 6.59, we establish the boundedness in $H^p(\Theta)$ of quasi-Banach-valued sublinear operators. This will be useful when we will characterize the dual spaces of the anisotropic Hardy spaces using anisotropic Campanato spaces in Section 6.8. Let us demonstrate with an example where the difficulty may arise. Assume that for a linear functional F on $H^p(\Theta)$, which is not known a priori to be bounded, we prove a uniform bound $|F(a)| \le c$ for all admissible (p, q, l)-atoms a. This does not automatically guarantee the boundedness of the functional on $H^p(\Theta)$. Indeed, Bownik [9] provided a proof of the existence of a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$ that maps all $(1, \infty, 0)$-atoms to uniformly bounded scalars but yet cannot be extended to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. To this end, we follow [66] to generalize the careful analysis of [7] and [10].

Definition 6.60. Let $\gamma \in (0, 1]$. A quasi-Banach space $\mathcal{B}_\gamma$ with quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is said to be a *γ-quasi-Banach space* if for all $f, g \in \mathcal{B}_\gamma$,

$$\|f + g\|^\gamma_{\mathcal{B}_\gamma} \le \|f\|^\gamma_{\mathcal{B}_\gamma} + \|g\|^\gamma_{\mathcal{B}_\gamma}.$$

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces l^p, L^p, and $H^p(\Theta)$ with $p \in (0, 1]$ are typical p-quasi-Banach spaces.

Definition 6.61. For any given γ-quasi-Banach space $\mathcal{B}_\gamma$ with $\gamma \in (0, 1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is said to be $\mathcal{B}_\gamma$-sublinear if for any $f, g \in \mathcal{Y}$ and $a, b \in \mathbb{C}$,

(i) $\|Tf - Tg\|_{\mathcal{B}_\gamma} \le \|T(f - g)\|_{\mathcal{B}_\gamma}$,
(ii) $\|T(af + bg)\|^\gamma_{\mathcal{B}_\gamma} \le |a|^\gamma \|Tf\|^\gamma_{\mathcal{B}_\gamma} + |b|^\gamma \|Tg\|^\gamma_{\mathcal{B}_\gamma}$.

Theorem 6.62. *Let (p, q, l) be an admissible triplet as in Definition 6.22, let $\gamma \in (0, 1]$, and let $\mathcal{B}_\gamma$ be a γ-quasi-Banach space. Assume that either of the following two statements holds:*

(i) *$1 < q < \infty$, and $T : H^p_{\mathrm{fin},q,l}(\Theta) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator satisfying*

$$\|Tf\|_{\mathcal{B}_\gamma} \le c\|f\|_{H^p_{\mathrm{fin},q,l}(\Theta)}, \quad \forall f \in H^p_{\mathrm{fin},q,l}(\Theta). \tag{6.116}$$

(ii) $T : H^p_{\mathrm{fin},\infty,l}(\Theta) \cap C(\mathbb{R}^n) \to \mathcal{B}_\gamma$ *is a* $\mathcal{B}_\gamma$*-sublinear operator satisfying*

$$\|Tf\|_{\mathcal{B}_\gamma} \le c\|f\|_{H^p_{\mathrm{fin},\infty,l}(\Theta)}, \quad \forall f \in H^p_{\mathrm{fin},\infty,l}(\Theta) \cap C(\mathbb{R}^n).$$

Then T *is uniquely extendable to a bounded* $\mathcal{B}_\gamma$*-sublinear operator from* $H^p(\Theta)$ *to* $\mathcal{B}_\gamma$.

Proof. We first show (i). Since $H^p_{\mathrm{fin},q,l}(\Theta)$ is dense in $H^p(\Theta)$, for any $f \in H^p(\Theta)$, there exists a Cauchy sequence $\{f_j\}_{j=1}^\infty$, $f_j \in H^p_{\mathrm{fin},q,l}(\Theta)$, such that $\|f - f_j\|_{H^p(\Theta)} \to 0$ as $j \to \infty$. By this, Definition 6.61(i), (6.116), and Theorem 6.59(i) we conclude that, as $j, k \to \infty$,

$$\begin{aligned}\|Tf_k - Tf_j\|_{\mathcal{B}_\gamma} &\le \|T(f_k - f_j)\|_{\mathcal{B}_\gamma} \\ &\le C\|f_k - f_j\|_{H^p_{\mathrm{fin},q,l}(\Theta)} \\ &\le C\|f_k - f_j\|_{H^p(\Theta)} \to 0.\end{aligned}$$

Thus $\{Tf_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{B}_\gamma$. By the completeness of $\mathcal{B}_\gamma$ we find that there exists $g \in \mathcal{B}_\gamma$ such that $g = \lim_{j\to\infty} Tf_j$ in $\mathcal{B}_\gamma$. Here g is independent of the choice of $\{f_j\}_{j=1}^\infty$. Indeed, suppose another sequence $\{f'_j\}_{j=1}^\infty \subset H^p_{\mathrm{fin},q,l}(\Theta)$ satisfies $f'_j \to f$ as $j \to \infty$ in $H^p(\Theta)$. Then by Definition 6.61(i), (6.116), and Theorem 6.59(i), as $j \to \infty$,

$$\begin{aligned}\|Tf'_j - g\|^\gamma_{\mathcal{B}_\gamma} &\le \|Tf'_j - Tf_j\|^\gamma_{\mathcal{B}_\gamma} + \|Tf_j - g\|^\gamma_{\mathcal{B}_\gamma} \\ &\le C\|f'_j - f_j\|^\gamma_{H^p(\Theta)} + \|Tf_j - g\|^\gamma_{\mathcal{B}_\gamma} \to 0.\end{aligned}$$

Thus we denote $Tf := g$. From this, (6.116), and Theorem 6.59(i) again we further deduce that

$$\begin{aligned}\|Tf\|^\gamma_{\mathcal{B}_\gamma} &\le \limsup_{j\to\infty}[\|Tf - Tf_j\|^\gamma_{\mathcal{B}_\gamma} + \|Tf_j\|^\gamma_{\mathcal{B}_\gamma}] \\ &\le C \limsup_{j\to\infty} \|Tf_j\|^\gamma_{\mathcal{B}_\gamma} \\ &\le C \limsup_{j\to\infty} \|f_j\|^\gamma_{H^p_{\mathrm{fin},q,l}(\Theta)} \\ &\le C \lim_{j\to\infty} \|f_j\|^\gamma_{H^p(\Theta)} \\ &\le C\|f\|^\gamma_{H^p(\Theta)},\end{aligned}$$

which completes the proof of (i).

To prove (ii), we first need to prove that $H^p_{\mathrm{fin},\infty,l}(\Theta) \cap C(\mathbb{R}^n)$ is dense in $H^p(\Theta)$. Since $H^p_{\mathrm{fin},\infty,l}(\Theta)$ is dense in $H^p(\Theta)$, it suffices to prove that $H^p_{\mathrm{fin},\infty,l}(\Theta) \cap C(\mathbb{R}^n)$ is dense in $H^p_{\mathrm{fin},\infty,l}(\Theta)$ with respect to the quasi-norm $\|\cdot\|_{H^p(\Theta)}$.

To see this, let $f \in H^p_{\mathrm{fin},\infty,l}(\Theta)$. Then f may be represented by a finite combination of (p, ∞, l)-atoms, $f = \sum_{i=1}^k \lambda_i a_i$. Also, it has compact support, and by (2.25) there exists $t_0 \in \mathbb{R}$ such that $\mathrm{supp}(f) \subset \theta(0, t_0)$.

Take $\phi \in \mathcal{S}$ such that $\phi \geq 0$, $\operatorname{supp}(\phi) \subset B^*$, and $\int_{\mathbb{R}^n} \phi = 1$, and denote its dilations $\phi_h := h^{-n}\phi(h^{-1}\cdot)$, $h > 0$.

For $i \in \{1, 2, \dots, k\}$, assume that $\operatorname{supp}(a_i) \subset \theta(x_i, t_i)$. Using (2.22), let

$$\begin{aligned} h &\leq \min\{\|M_{0,t_0}^{-1}\|^{-1}, \|M_{x_1,t_1}^{-1}\|^{-1}, \dots, \|M_{x_k,t_k}^{-1}\|^{-1}\} \\ &= \min\{\sigma_{\min}(\theta(0, t_0)), \sigma_{\min}(\theta(x_1, t_1)), \dots, \sigma_{\min}(\theta(x_k, t_k))\}. \end{aligned}$$

Then $\phi_h * f \in C^\infty(\mathbb{R}^n)$, and by (2.30) $\operatorname{supp}(\phi_h * f) \subset \theta(0, t_0 - J)$, and

$$\operatorname{supp}(\phi_h * a_i) \subset \theta(x_i, t_i - J), \quad 1 \leq i \leq k.$$

Next, we see that

$$\|\phi_h * a_i\|_\infty \leq \|a_i\|_\infty \leq |\theta(x_i, t_i)|^{-1/p}, \quad 1 \leq i \leq k.$$

Furthermore, since each atom a_i has l vanishing moments, for any $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq l$,

$$\int_{\mathbb{R}^n} (\phi_h * a_i)(x) x^\alpha dx = 0, \quad 1 \leq i \leq k.$$

Thus, for each $1 \leq i \leq k$,

$$|\theta(x_i, t_i - J)|^{-1/p} |\theta(x_i, t_i)|^{1/p} \phi_h * a_i$$

is a (p, ∞, l)-atom. We conclude that $\phi_h * f = \sum_{i=1}^k \lambda_i \phi_h * a_i$ is a finite combination of smooth (p, ∞, l)-atoms.

Next, observe that $f - \phi_h * f$ has also l vanishing moments and $\operatorname{supp}(f - \phi_h * f) \subset \theta(0, t_0 - J)$. We have $\|f - \phi_h * f\|_2 \to 0$ as $h \to 0$. Let

$$\lambda_h := \|f - \phi_h * f\|_2 |\theta_{0,t_0-J}|^{-(1/2-1/p)}, \quad a_h := (f - \phi_h * f)/\lambda_h.$$

Then a_h is a $(p, 2, l)$-atom, $f - \phi_h * f = \lambda_h a_k$, and $\lambda_h \to 0$ as $h \to 0$. From this we deduce that, as $h \to 0$,

$$\|f - \phi_h * f\|_{H^p(\Theta)} \leq C\lambda_h \to 0,$$

which shows that $H^p_{\mathrm{fin},\infty,l}(\Theta) \cap C(\mathbb{R}^n)$ is dense in $H^p_{\mathrm{fin},\infty,l}(\Theta)$ with respect to the quasi-norm $\|\cdot\|_{H^p(\Theta)}$. From this and an argument similar to that used in the proof of (i) it follows that (ii) holds. □

Corollary 6.63. *Let (p, q, l) be an admissible triplet as in Definition 6.22 with $0 < p \leq \gamma \leq 1$, and let $\mathcal{B}_\gamma$ be a γ-quasi-Banach space. Assume that either of the following two statements holds:*

(i) $1 < q < \infty$, *and* T *is a* $\mathcal{B}_\gamma$*-sublinear operator from* $H^p_{\mathrm{fin},q,l}(\Theta)$ *to* $\mathcal{B}_\gamma$ *satisfying*

$$\sup\{\|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any } (p,q,l)\text{-atom}\} < \infty. \tag{6.117}$$

(ii) T *is a* $\mathcal{B}_\gamma$*-sublinear operator defined on all continuous* (p,∞,l)*-atoms satisfying*

$$\sup\{\|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any continuous } (p,\infty,l)\text{-atom}\} < \infty.$$

Then T *is uniquely extendable to a bounded* $\mathcal{B}_\gamma$*-sublinear operator from* $H^p(\Theta)$ *into* $\mathcal{B}_\gamma$*.*

Proof. By similarity we only prove (i). To this end, by Theorem 6.62 it suffices to show that, for any $f \in H^p_{\mathrm{fin},q,l}(\Theta)$, (6.116) holds. Indeed, by definition there exist coefficients $\{\lambda_i\}_{i=1}^k$, $\lambda_i \in \mathbb{C}$, and (p,q,l)-atoms $\{a_i\}_{i=1}^k$ such that $f = \sum_{i=1}^k \lambda_i a_i$ and

$$\sum_{i=1}^k |\lambda_i|^p \le 2\|f\|^p_{H^p_{\mathrm{fin},q,l}(\Theta)}.$$

From this, Definition 6.61(ii), $p \le \gamma$, and (6.117) we deduce that

$$\|Tf\|_{\mathcal{B}_\gamma} \le \left[\sum_{i=1}^k |\lambda_i|^\gamma \|Ta_i\|^\gamma_{\mathcal{B}_\gamma}\right]^{1/\gamma} \le C\left[\sum_{i=1}^k |\lambda_i|^p\right]^{1/p} \le C\|f\|_{H^p_{\mathrm{fin},q,l}(\Theta)}.$$

Combined with Theorem 6.62, this provides the proof of the corollary. □

6.8 The anisotropic dual Campanato spaces

As noted in Section 6.4, the dual of $H^1(\Theta)$ is $\mathrm{BMO}(\Theta)$. Thus, our analysis of dual spaces in this section is focused on the case $0 < p < 1$, where we provide a generalization of the classic isotropic case. The anisotropic dual spaces were analyzed in [32], but here we use a different approach, which applies the finite atomic spaces from Section 6.7. The main result of this section is the following:

Theorem 6.64. *Let* Θ *be a pointwise continuous ellipsoid cover, and let* $0 < p < 1 \le q < \infty$ *and* $l \ge N_p(\Theta)$*. Then*

$$\left(H^p(\Theta)\right)^* = \mathcal{C}^{1/p-1/q}_{q',l+1}(\Theta)/\Pi_l,$$

where $\mathcal{C}^\alpha_{q',r}(\Theta)$ *are the Campanato spaces of Section* 5.5*, and* $1/q' + 1/q = 1$*.*

Corollary 6.65. *For any* $1 < q' \le \infty$*,* $\alpha > 1/q'$*, and* $r_1, r_2 \ge N_p(\Theta) + 1$*,*

$$\mathcal{C}^\alpha_{q',r_1}(\Theta)/\Pi_{r_1-1} \sim \mathcal{C}^\alpha_{q',r_2}(\Theta)/\Pi_{r_2-1}.$$

Proof. For fixed α, q', choose q as the dual of q' and then $0 < p < 1$ by $1/p = \alpha + 1/q > 1$. Since by Theorem 6.64 both spaces $\mathcal{C}^{\alpha}_{q',r_1}(\Theta)/\Pi_{r_1-1}$ and $\mathcal{C}^{\alpha}_{q',r_2}(\Theta)/\Pi_{r_2-1}$ are duals of $H^p(\Theta)$, they are equivalent. □

The proof of Theorem 6.64 requires the following lemma.

Lemma 6.66. *Let (p, q, l) be an admissible triple. Then, for any $g \in \mathcal{C}^{1/p-1/q}_{q',l+1}(\Theta)$ and any (p, q, l)-atom a,*

$$\left|\int ga\right| \leq c\|g\|_{\mathcal{C}^{1/p-1/q}_{q',l+1}(\Theta)}. \tag{6.118}$$

Proof. In the case $q > 1$, for a (p, q, l)-atom a associated with an ellipsoid θ, using the vanishing moments of the atom and the Whitney theorem (Theorem 1.34), we have

$$\begin{aligned}\left|\int ga\right| &= \inf_{P\in\Pi_l}\left|\int_\theta (g-P)a\right| \\ &\leq \|a\|_q \left(\inf_{P\in\Pi_l}\int_\theta |g-P|^{q'}\right)^{1/q'} \\ &\leq C|\theta|^{1/q-1/p}\omega_{l+1}(g,\theta)_{q'} \\ &\leq C\|g\|_{\mathcal{C}^{1/p-1/q}_{q',l+1}(\Theta)}.\end{aligned}$$

The case $q = \infty$ is similar. □

Let $g \in \mathcal{C}^{1/p-1/q}_{q',l+1}(\Theta)$. Since by the previous lemma the action of g on atoms is uniformly bounded, we may be tempted to define $F_g f := \sum_i \lambda_i \int ga_i$ for $f \in H^p(\Theta)$ with atomic decomposition $f = \sum_i \lambda_i a_i$. However, since we do not a priori know that F_g is a bounded functional on $H^p(\Theta)$, we may not immediately apply this argument. We will see that the proof of Theorem 6.64 requires an application of the finite atomic spaces from Section 6.7. Meanwhile, we remark here in passing that if it is known a priori that a functional is bounded, then its norm may be determined by the action on atoms.

Remark 6.67. Let F be a bounded linear functional on $H^p_{q,l}(\Theta)$, where (p, q, l) is an admissible triple. Then

$$\begin{aligned}\|F\|_{(H^p_{q,l}(\Theta))^*} &:= \sup\{|Ff| : \|f\|_{H^p_{q,l}(\Theta)} \leq 1\} \\ &= \sup\{|Fa| : a \text{ is a } (p,q,l)\text{-atom}\}.\end{aligned}$$

Proof. By definition 6.23, for every (p, q, l)-atom a, we have $\|a\|_{H^p_{q,l}(\Theta)} \leq 1$. Thus

$$\sup\{|Fa| : a \text{ is a } (p,q,l)\text{-atom}\} \leq \sup\{|Ff| : \|f\|_{H^p_{q,l}(\Theta)} \leq 1\}.$$

In the other direction, consider $f \in H^p(\Theta)$ such that $\|f\|_{H^p_{q,l}(\Theta)} \le 1$. Then, for every $\varepsilon > 0$, there exists an atomic representation $f = \sum_i \lambda_i a_i$, in the sense of $H^p_{q,l}(\Theta)$, such that $(\sum_i |\lambda_i|^p)^{1/p} < 1 + \varepsilon$. Since F is a bounded linear functional, $Ff = \sum_i \lambda_i F a_i$, and therefore

$$\begin{aligned}|Ff| &\le \sum_i |\lambda_i||Fa_i| \\ &\le \left(\sum_i |\lambda_i|^p\right)^{1/p} \sup\{|Fa| : a \text{ is a } (p,q,l)\text{-atom}\} \\ &\le (1+\varepsilon)\sup\{|Fa| : a \text{ is a } (p,q,l)\text{-atom}\}.\end{aligned}$$ □

We are now ready to prove our main result.

Proof of Theorem 6.64. We begin with $(H^p(\Theta))^* \subseteq C^{1/p-1/q}_{q',l+1}(\Theta)$. To this end, we prove that for any bounded linear functional $F_g \in (H^p(\Theta))^*$, there exists $g \in L^{\mathrm{loc}}_{q'}(\mathbb{R}^n)$ such that

$$\|g\|_{C^{1/p-1/q}_{q',l+1}(\Theta)} \le C\|F_g\|_{(H^p(\Theta))^*},$$

and for any $f \in H^p(\Theta) \cap L_q$,

$$F_g f = \int_{\mathbb{R}^n} fg.$$

Since by Corollary 6.44 $H^p(\Theta) \cap L_q$ is dense in $H^p(\Theta)$, this can be extended to provide adequate representation of F_g. Let (p, q, l) be an admissible triplet, $0 < p < 1 \le q < \infty$. For any $\theta \in \Theta$, let $L^0_q(\theta) := \{f \in L_q(\theta) : P_{\theta,q} f = 0\}$, where $P_{\theta,q}$ is the polynomial approximation (3.33) of degree l. Here we assume that $f \in L^0_q(\theta)$ vanishes outside of θ, and therefore we can identify its normalized version, $|\theta|^{1/q-1/p}\|f\|^{-1}_q f$ as an (p, q, l)-atom with $H^p_{q,l}(\Theta)$-norm ≤ 1. Consequently, since F_g is assumed to be a priori a bounded operator, for all $f \in L^0_q(\theta)$,

$$|F_g f| \le \|F_g\|_{(H^p_{q,l}(\Theta))^*} |\theta|^{1/p-1/q} \|f\|_q. \tag{6.119}$$

Recall that by (2.30) there exists $J(p(\Theta)) > 0$ such that $\theta(x, t) \subset \theta(x, t - J)$ for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. By (6.119), for any $m \ge 0$ and $f \in L^0_q(\theta(0, -Jm))$, we have that

$$|F_g f| \le \|F_g\|_{(H^p_{q,l}(\Theta))^*} \left|\theta(0, -Jm)\right|^{1/p-1/q} \|f\|_q.$$

By the Hahn–Banach theorem F_g can be extended to the space $L_q(\theta(0, -Jm))$ without increasing its norm. By the Riesz representation theorem for finite measure spaces and $1 \le q < \infty$ there exists a unique function $g_m \in L_{q'}(\theta(0, -Jm))$ (up to a set of measure zero and a polynomial of degree l) such that $F_g f = \int_{\theta(0,-Jm)} g_m f$ for all $f \in L_q^0(\theta(0,-Jm))$. We readily see that $g_{m+1}|_{\theta(0,-Jm)} = g_m$, and, consequently, we may identify the action of the functional F_g using $g \in L_{q'}^{\mathrm{loc}}(\mathbb{R}^n)$ that is set as $g(x) := g_m(x)$ if $x \in \theta(0,-Jm)$. By (6.119), for any $\theta \in \Theta$, the norm of g as a functional on $L_q^0(\theta)$ satisfies

$$\|g\|_{L_q^0(\theta)^*} \leq \|F_g\|_{(H_{q,l}^p(\Theta))^*} |\theta|^{1/p-1/q}.$$

Also, as $L_q^0(\theta)^* = L_{q'}(\theta)/\Pi_l$, we have that

$$\|g\|_{L_q^0(\theta)^*} = \inf_{P\in\Pi_l} \|g-P\|_{L_{q'}(\theta)} \geq 2^{-(l+1)}\omega_{l+1}(g,\theta)_{q'}.$$

We may now conclude that

$$\begin{aligned}\|g\|_{C_{q',l+1}^{1/p-1/q}(\Theta)} &= \sup_{\theta\in\Theta} |\theta|^{1/q-1/p}\omega_{l+1}(g,\theta)_{q'} \\ &\leq C\sup_{\theta\in\Theta} |\theta|^{1/q-1/p}\|g\|_{L_q^0(\theta)^*} \\ &\leq C\|F_g\|_{(H^p(\Theta))^*}.\end{aligned}$$

We now prove the second direction. For $g \in C_{q',l+1}^{1/p-1/q}(\Theta)$, denote $F_g f := \int fg$ for $f \in H_{\mathrm{fin},q,l}^p$. Obviously, F_g is a linear functional on $H_{\mathrm{fin},q,l}^p$, and by (6.118) it is uniformly bounded on atoms. Thus Corollary 6.63 implies that F_g can be uniquely extended to bounded linear functional on $H^p(\Theta)$, where for all $f \in H^p(\Theta)$,

$$|F_g f| \leq C\|g\|_{C_{q',l+1}^{1/p-1/q}(\Theta)}\|f\|_{H^p(\Theta)}.$$ □

7 Anisotropic singular operators

Calderón–Zygmund (CZ) operators play an important role in harmonic analysis. They are bounded not only on the $L^p(\mathbb{R}^n)$ spaces for $1 < p < \infty$, but also on their natural extensions for $0 < p \le 1$, the Hardy spaces $H^p(\mathbb{R}^n)$. In the classical isotropic setting of $\mathbb{R}^n$, we consider a CZ operator $T : L_2 \to L_2$ of regularity s with a measurable kernel $K(x,y)$ satisfying

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad \forall x \notin \operatorname{supp}(f), \quad \forall f \in C_c^\infty(\mathbb{R}^n),$$

and

$$\left|\partial_y^\alpha K(x,y)\right| \le C|x-y|^{-n-|\alpha|}, \quad \forall x \ne y,\ |\alpha| \le s. \tag{7.1}$$

It is well known [61] that a CZ operator T is bounded on the isotropic Hardy spaces $H^p(\mathbb{R}^n)$, provided that $s > n(1/p - 1)$ and T preserves vanishing moments $T^*(x^\alpha) = 0$ for $|\alpha| < s$.

The study of CZ operators on spaces of homogeneous type began with Coifman and Weiss [19]. Since ellipsoid covers generate spaces of homogeneous type, any result in the general setting holds here [33, 61] (see, e. g., Theorem 7.3). However, again, the lack of higher-order regularity and vanishing moments in the general setting of spaces of homogeneous type limits the analysis. Bownik [7] introduced anisotropic CZ operators associated with expansive dilations and has shown their boundedness on anisotropic Hardy spaces, where the anisotropy is fixed and global on $\mathbb{R}^n$. In this chapter, we provide a generalization to the setting of pointwise variable anisotropic ellipsoid covers. In Section 7.1, we show that an anisotropic singular integral operator maps $H^p(\Theta)$, $0 < p \le 1$, to itself, provided that it has sufficient regularity and vanishing moments [12]. In Section 7.2, we cover some basic definitions and results concerning anisotropic CZ operators acting on spaces of anisotropic smooth molecules.

Let Θ be a continuous cover inducing the quasi-distance ρ defined by (2.35). A pointwise variable anisotropic analogue of an isotropic CZ kernel operator takes the following form.

Definition 7.1. A locally square-integrable function K on $\Omega := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \ne y\}$ is called a *variable anisotropic singular integral kernel* with respect to a continuous ellipsoid cover Θ if there exist two positive constants $c_1 > 1$ and $c_2 > 0$ such that

$$\int_{B_\rho(y,c_1 r)^c} \left|K(x,y) - K(x,y')\right| dx \le c_2, \quad \forall y \in \mathbb{R}^n,\ y' \in B_\rho(y,r), \tag{7.2}$$

where $B_\rho(\cdot,\cdot)$ are the anisotropic balls defined in (2.37).

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We say that T is a *variable anisotropic singular integral operator (VASIO)* of order 0 if $T : L^2 \to L^2$ is a bounded linear operator and there exists a kernel K satisfying (7.2) such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad \forall x \notin \operatorname{supp}(f), \quad \forall f \in C_c^\infty(\mathbb{R}^n).$$

We say that T with kernel K such that $K(x,\cdot) \in C^s$ for all $x \in \mathbb{R}^n$ is a *VASIO of order s* if there exists a constant $c_3 > 0$ such that for any $x \neq y$ and $\alpha \in \mathbb{N}_+^n$, $|\alpha| \leq s$, we have

$$\left|\partial_y^\alpha [K(\cdot, M_{y,m}\cdot)](x, M_{y,m}^{-1}y)\right| \leq \frac{c_3}{\rho(x,y)}, \tag{7.3}$$

where $m = -\log_2 \rho(x,y)$. More precisely, the left-hand side of (7.3) means $|\partial_y^\alpha \widetilde{K}(x, M_{y,m}^{-1}y)|$, where $\widetilde{K}(x,y) := K(x, M_{y,m}y)$. The smallest constant c_3 satisfying (7.3) is called the Calderón–Zygmund norm of T, which is denoted by $\|T\|_{(s)}$. In Section 7.2, we will also require the following "symmetric condition" for any $x \neq y$ and $\alpha, \beta \in \mathbb{N}_+^n$, $|\alpha|, |\beta| \leq s$:

$$\left|\partial_x^\alpha \partial_y^\beta [K(M_{x,m}\cdot, M_{y,m}\cdot)](M_{x,m}^{-1}x, M_{y,m}^{-1}y)\right| \leq \frac{c_4}{\rho(x,y)}. \tag{7.4}$$

Next, we show that the definition of singular integral operators in the setting of ellipsoid covers is consistent with the CZ operators on spaces of homogeneous type [19, 33].

Theorem 7.2. *Let K be kernel of a VASIO of order* 1. *Then there exists a positive constant c such that for all $x \neq y \in \mathbb{R}^n$, we have*

$$\left|K(x,y)\right| \leq \frac{c}{\rho(x,y)}, \tag{7.5}$$

$$\left|K(x,y) - K(x,y')\right| \leq c\frac{\rho(y,y')^{a_6}}{\rho(x,y)^{1+a_6}} \quad \text{if } \rho(y,y') \leq \frac{1}{2\kappa}\rho(x,y). \tag{7.6}$$

In particular, the kernel K satisfies (7.2).

Proof. Estimate (7.3) with $\alpha = 0$ implies (7.5). Next, we prove (7.6). For fixed $x, y \in \mathbb{R}^n$ with $x \neq y$, let $r := (\kappa+1)\rho(x,y)$, where κ is defined in (2.1). By Theorem 2.23 there exists $m \in \mathbb{R}$ such that

$$B_\rho(x,r) \subset \theta(x,m) \quad \text{and} \quad 2^{-m} \sim \left|\theta(x,m)\right| \sim r. \tag{7.7}$$

Define the rescaled kernel $\widetilde{K}(u,v) := K(u, M_{y,m}v)$, $u, v \in \mathbb{R}^n$.

Take any $y' \in \mathbb{R}^n$ such that $\rho(y,y') \le \frac{1}{2\kappa}\rho(x,y)$. By Lagrange's mean value theorem there exists $\xi \in [y,y']$ such that

$$\begin{aligned}|K(x,y) - K(x,y')| &= |\widetilde{K}(x, M_{y,m}^{-1}y) - \widetilde{K}(x, M_{y,m}^{-1}y')| \\ &= \Big|\sum_{|\alpha|=1} \partial_y^\alpha \widetilde{K}(x, M_{y,m}^{-1}\xi)(M_{y,m}^{-1}y - M_{y,m}^{-1}y')^\alpha\Big| \\ &\le C \max_{|\alpha|=1} |\partial_y^\alpha [K(\cdot, M_{y,m}\cdot)](x, M_{y,m}^{-1}\xi)||M_{y,m}^{-1}(y-y')|.\end{aligned}$$

Let $l := -\log_2 \rho(x,\xi)$. By (7.3) we have

$$\begin{aligned}&|K(x,y) - K(x,y')| \\ &\quad\le C \max_{|\alpha|=1} |\partial_y^\alpha [K(\cdot, M_{\xi,l}M_{\xi,l}^{-1}M_{y,m}\cdot)](x, (M_{\xi,l}^{-1}M_{y,m})^{-1}(M_{\xi,l})^{-1}\xi)||M_{y,m}^{-1}(y-y')| \\ &\quad\le C\|M_{\xi,l}^{-1}M_{y,m}\| \max_{|\alpha|=1} |\partial_y^\alpha [K(\cdot, M_{\xi,l}\cdot)](x, M_{\xi,l}^{-1}\xi)||M_{y,m}^{-1}(y-y')| \\ &\quad\le C\|M_{\xi,l}^{-1}M_{y,m}\| \frac{1}{\rho(x,\xi)} |M_{y,m}^{-1}(y-y')|.\end{aligned}$$

Observe that by the convexity of ellipsoids $y, y' \in \theta \Rightarrow [y,y'] \subset \theta \Rightarrow \xi \in \theta$ for any $\theta \in \Theta$. This implies that $\rho(y,\xi) \le \rho(y,y')$, and so

$$\begin{aligned}\rho(x,y) &\le \kappa(\rho(x,\xi) + \rho(y,\xi)) \\ &\le \kappa(\rho(x,\xi) + \rho(y,y')) \\ &\le \kappa\Big(\rho(x,\xi) + \frac{1}{2\kappa}\rho(x,y)\Big),\end{aligned}$$

which gives $\rho(x,y) \le 2\kappa\rho(x,\xi)$. Likewise,

$$\begin{aligned}\rho(x,\xi) &\le \kappa(\rho(x,y) + \rho(y,\xi)) \\ &\le \kappa\rho(x,y) + \kappa\rho(y,y') \\ &\le (\kappa + 1/2)\rho(x,y).\end{aligned}$$

Hence, using absolute constants that do not depend on the points, we have

$$\rho(x,y) \sim \rho(x,\xi). \tag{7.8}$$

Since $\rho(x,\xi) \le (\kappa + 1/2)\rho(x,y) < r$, we have that $\xi \in B_\rho(x,r) \subset \theta(x,m)$. This implies $\theta(x,m) \cap \theta(\xi,l) \ne \emptyset$. Since $\rho(x,y) \sim \rho(x,\xi)$, we have $2^{-m} \sim 2^{-l}$, and hence $\|M_{\xi,l}^{-1}M_{y,m}\| \le C$ by (2.14). Combined with (7.8), this gives

$$|K(x,y) - K(x,y')| \le C\frac{|M_{y,m}^{-1}(y-y')|}{\rho(x,y)}. \tag{7.9}$$

Let $k \in \mathbb{Z}$ be such that $y' \in \theta(y, kJ) \setminus \theta(y, (k+1)J)$, where J is given in (2.30). By (7.7) we have

$$2^{-kJ} \sim \rho(y, y') \leq C\rho(x, y) \sim 2^{-m}.$$

This implies that there exists a constant $c > 0$ such that $m - kJ \leq c$. Since $M_{y,kJ}^{-1}(y - y') \in B^*$, the shape condition (2.14) implies that

$$\begin{aligned}
|M_{y,m}^{-1}(y - y')| &= |M_{y,m}^{-1} M_{y,kJ} M_{y,kJ}^{-1}(y - y')| \\
&\leq \|M_{y,m}^{-1} M_{y,kJ}\| |M_{y,kJ}^{-1}(y - y')| \\
&\leq \|M_{y,m}^{-1} M_{y,kJ+c} M_{y,kJ+c}^{-1} M_{y,kJ}\| \\
&\leq \|M_{y,m}^{-1} M_{y,kJ+c}\| \|M_{y,kJ+c}^{-1} M_{y,kJ}\| \\
&\leq C 2^{-a_6(kJ-m)} \sim \frac{\rho(y, y')^{a_6}}{\rho(x, y)^{a_6}}.
\end{aligned}$$

Combining this with (7.9) yields (7.6):

$$|K(x, y) - K(x, y')| \leq C \frac{1}{\rho(x, y)} \frac{\rho(y, y')^{a_6}}{\rho(x, y)^{a_6}} = C \frac{\rho(y, y')^{a_6}}{\rho(x, y)^{1+a_6}}.$$

Finally, from general results for spaces of homogeneous type it follows that K satisfies (7.2). More precisely, we claim that (7.2) holds with the constant $c_1 = 2\kappa$. Indeed, take $y' \in B_\rho(y, r)$ for some $r > 0$. For any $x \in B_\rho(y, 2\kappa r)^c$,

$$\rho(y, y') \leq r \leq \frac{1}{2\kappa} \rho(x, y).$$

This allows us to apply (7.6) and Theorem 2.23 to obtain (7.2):

$$\begin{aligned}
\int_{B_\rho(y, 2\kappa r)^c} |K(x, y) - K(x, y')| dx &\leq C \int_{B_\rho(y, 2\kappa r)^c} \frac{r^{a_6}}{\rho(x, y)^{1+a_6}} dx \\
&= C r^{a_6} \sum_{i=1}^{\infty} \int_{B_\rho(y, 2^{i+1}\kappa r) \setminus B_\rho(y, 2^i \kappa r)} \frac{1}{\rho(x, y)^{1+a_6}} dx \\
&\leq C r^{a_6} \sum_{i=1}^{\infty} \frac{1}{(2^i \kappa r)^{(1+a_6)}} |B_\rho(y, 2^{i+1}\kappa r)| \\
&\leq C \sum_{i=1}^{\infty} 2^{-i a_6} \leq C.
\end{aligned}$$

□

Since our anisotropic spaces are a particular case of spaces of homogeneous type, we have the following (see, e. g., [61, Section I.5]):

Theorem 7.3. *Let $T : L^2 \to L^2$ be a VASIO of order* 0. *Then:*

(i) *T is bounded from* L_1 *to weak-*L_1*;*
(ii) *T can be extended to a bounded linear operator on* L_q, $1 < q \le 2$*;*
(iii) *If the kernel K further satisfies the symmetric condition*

$$\int_{B_\rho(x,c_1r)^c} |K(x,y) - K(x',y)|dy \le c_2, \quad \forall x \in \mathbb{R}^n, \quad \forall x' \in B_\rho(x,r), \tag{7.10}$$

then T can also be extended by duality to a bounded linear operator on L_q, $2 < q < \infty$.

The following lemma proves a useful formulation of the property of VASIO operators of higher orders.

Lemma 7.4. *Suppose that T is a VASIO of order s as in Definition* 7.1. *Then there exists a constant* $c > 0$ *such that for any* $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, $k \in \mathbb{N}$, $x \in \theta(z,\ t-(k+1)J) \setminus \theta(z,\ t-kJ)$, $k \ge 0$, *and* $y \in \theta(z,\ t)$, *for all* $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le s$, *we have*

$$\left|\partial_y^\alpha [K(\cdot, M_{z,\,t-kJ}\cdot)](x, M_{z,\,t-kJ}^{-1}y)\right| \le c2^{t-kJ}. \tag{7.11}$$

Here J is given by (2.30), *and the constant c depends only on* $\|T\|_{(s)}$ *and* $\mathbf{p}(\Theta)$. *Furthermore, if T satisfies* (7.4), *then for all* $\beta \in \mathbb{Z}_+^n$, $|\alpha|, |\beta| \le s$, *we have*

$$\left|\partial_x^\alpha \partial_y^\beta [K(M_{z,t-kJ}\cdot, M_{z,t-kJ}\cdot)](M_{z,t-kJ}^{-1}x, M_{z,t-kJ}^{-1}y)\right| \le c2^{t-kJ}. \tag{7.12}$$

Proof. Using Theorem 2.23, it is easy to see that $x \in \theta(y,\ t-(k+1)J-\gamma) \setminus \theta(y,\ t-kJ+\gamma)$ and

$$\rho(x,z) \sim \rho(x,y) \sim 2^{-t+kJ}. \tag{7.13}$$

By Definition 7.1 we have

$$\left|\partial_y^\alpha [K(\cdot, M_{y,m}\cdot)](x, M_{y,m}^{-1}y)\right| \le \frac{C}{\rho(x,y)} \le C2^m, \tag{7.14}$$

where $m = -\log_2 \rho(x,y)$. From (7.13) it follows that $2^{-m} = \rho(x,y) \sim 2^{-t+kJ}$. Hence there exists a constant $c_1 > 0$ such that

$$\left|m - (t-kJ)\right| \le c_1. \tag{7.15}$$

Define $M := M_{y,m}^{-1} M_{z,t-kJ}$. As $y \in \theta(z,t) \subset \theta(z,t-kJ)$ for $k \ge 1$, we have that $\theta(y,m) \cap \theta(z,t-kJ) \ne \emptyset$. Application of (7.15) and the shape condition (2.14) give $\|M\| \le c$. Hence by (7.14), and (7.15) we conclude that for $|\alpha| \le s$,

$$\begin{aligned}\left|\partial_y^\alpha[K(\cdot, M_{z,t-kJ}\cdot)](x, M_{z,t-kJ}^{-1}y)\right| &= \left|\partial_y^\alpha[K(\cdot, M_{y,m}M_{y,m}^{-1}M_{z,t-kJ}\cdot)](x, (M_{y,m}^{-1}M_{z,t-kJ})^{-1}M_{y,m}^{-1}y)\right| \\ &\le C\|M_{y,m}^{-1}M_{z,t-kJ}\|^{|\alpha|}\left|\partial_y^\alpha[K(\cdot, M_{y,m}\cdot)](x, M_{y,m}^{-1}y)\right| \\ &\le C2^m \le C2^{t-kJ}.\end{aligned}$$

This proves (7.11). The proof of (7.12) is similar. □

7.1 Anisotropic singular operators on $H^p(\Theta)$

Our goal is to show that anisotropic CZ operators are bounded on $H^p(\Theta)$. Generally, as in the classical isotropic case, we cannot expect this unless we also assume that T preserves vanishing moments.

Definition 7.5. We say that a VASIO T of order s has l vanishing moments, $l < a_6 s/a_4$, if for some $1 < q < \infty$ and all $f \in L^q$ with compact support with vanishing moments

$$\int_{\mathbb{R}^n} f(x)x^\alpha dx = 0, \quad \forall \alpha \in \mathbb{Z}_+^n,\ |\alpha| < s,$$

we also have

$$\int_{\mathbb{R}^n} Tf(x)x^\alpha dx = 0, \quad \forall \alpha \in \mathbb{Z}_+^n,\ |\alpha| \le l.$$

This definition generalizes the case of covers constructed through expansive dilations [7] and the isotropic case [55]. The actual value of q is not relevant in Definition 7.5, as we merely need that $T : L^q \to L^q$ is bounded. The next result justifies the integrability of $\int_{\mathbb{R}^n} Tf(x)x^\alpha dx$ in Definition 7.5.

Lemma 7.6. *Let T be a VASIO of order s. Suppose that $f \in L_q$, $1 < q < \infty$, satisfies $\operatorname{supp}(f) \subset \theta(z,t)$ for some $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$ and that $\int_{\mathbb{R}^n} f(x)x^\alpha dx = 0$ for all $|\alpha| < s$. Then, for some $c > 0$ depending only $\|T\|_{(s)}$ and $\mathbf{p}(\Theta)$, for any $x \in \theta(z, t-(k+1)J) \setminus \theta(z, t-kJ)$, $k \in \mathbb{N}$,*

$$\left|Tf(x)\right| \le c\|f\|_q\left|\theta(z,t)\right|^{-1/q}2^{-kJ(1+a_6 s)}. \tag{7.16}$$

In particular, if $l < a_6 s/a_4$, then

$$\int_{\mathbb{R}^n} \left|Tf(x)\right|(1 + |x|^l)\, dx < \infty. \tag{7.17}$$

Proof. Take any $x \in \theta(z, t-(k+1)J) \setminus \theta(z, t-kJ)$, $k \in \mathbb{N}$, and $y \in \theta(z,t)$. Define the rescaled kernel $\widetilde{K}(u,v) := K(u, M_{z,t-kJ}v)$, $u, v \in \mathbb{R}^n$. By Lemma 7.4 we have, for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le s$,

$$\left|\partial_y^\alpha \widetilde{K}(x, M_{z,t-kJ}^{-1}y)\right| \le C2^{t-kJ}. \tag{7.18}$$

Since $\operatorname{supp}(f) \subset \theta(z,t)$, we can write

$$Tf(x) = \int_{\theta(z,t)} K(x,y)f(y)dy = \int_{\theta(z,t)} \widetilde{K}(x, M_{z,t-kJ}^{-1}y)f(y)dy. \tag{7.19}$$

Now we expand $\widetilde{K}$ into the Taylor polynomial of degree $s-1$ (only in y variable) at the point $(x, M_{z,t-kJ}^{-1}z)$, that is,

$$\begin{aligned}\widetilde{K}(x, M_{z,t-kJ}^{-1}y) = &\sum_{|\alpha|\le s-1} \frac{\partial_y^\alpha \widetilde{K}(x, M_{z,t-kJ}^{-1}z)}{\alpha!}\left(M_{z,t-kJ}^{-1}y - M_{z,t-kJ}^{-1}z\right)^\alpha \\ &+ R^s_{M_{z,t-kJ}^{-1}z}\widetilde{K}(x,\cdot)(M_{z,t-kJ}^{-1}y).\end{aligned} \tag{7.20}$$

Then using (7.18) and (2.14), we see that the remainder term satisfies

$$\begin{aligned}\left|R^s_{M_{z,t-kJ}^{-1}z}\widetilde{K}(x,\cdot)(M_{z,t-kJ}^{-1}y)\right| &\le C \sup_{\xi\in\theta(z,t)} \sup_{|\alpha|=s} \left|\partial_y^\alpha \widetilde{K}(x, M_{z,t-kJ}^{-1}\xi)\right|\left|M_{z,t-kJ}^{-1}(y-z)\right|^s \\ &\le C2^{t-kJ} \sup_{w\in B^*}\left|M_{z,t-kJ}^{-1}M_{z,t}w\right|^s \\ &\le C2^{t-kJ(1+a_6 s)}.\end{aligned} \tag{7.21}$$

Moreover, by Hölder's inequality we have

$$\int_{\theta(z,t)} |f| \le \|f\|_q \left|\theta(z,t)\right|^{1/q'} \le C2^{-t/q'}\|f\|_q, \tag{7.22}$$

where $1/q + 1/q' = 1$. Finally, using (7.19), (7.20), the vanishing moments of f, (7.21), and (7.22), we obtain that

$$\begin{aligned}\left|Tf(x)\right| &\le \int_{\theta(z,t)} \left|R^s_{M_{z,t-kJ}^{-1}z}\widetilde{K}(x,\cdot)(M_{z,t-kJ}^{-1}y)\right|\left|f(y)\right|dy \\ &\le C2^{-kJ(1+a_6 s)}2^{t/q}\|f\|_q,\end{aligned}$$

which implies (7.16).

To show the second part (7.17), we first choose $k_0 \in \mathbb{N}$ large enough such that for any $x \in \theta(z, t - k_0 J)^c$, we have $\rho(x, z) > 1$. Then we split the integral into two parts:

$$\int_{\mathbb{R}^n} |Tf(x)|(1 + |x|^l)dx = \int_{\theta(z,t-k_0J)} |Tf(x)|(1 + |x|^l)dx + \int_{\theta(z,t-k_0J)^c} |Tf(x)|(1 + |x|^l)dx$$
$$=: I + II.$$

The first integral is bounded by Hölder's inequality and the boundedness of $T : L^q \to L^q$ (assuming further (7.10) for $2 < q < \infty$),

$$I \le C \int_{\theta(z,t-k_0J)} |Tf|$$
$$\le C\Bigg(\int_{\theta(z,t-k_0J)} |Tf|^q\Bigg)^{1/q} \big|\theta(z, t - k_0 J)\big|^{1/q'} < \infty.$$

We now estimate the second integral. By Theorem 2.26 there exists a constant $c_1(z, \mathbf{p}(\Theta)) > 0$ such that for any $x \in \mathbb{R}^n$ satisfying $\rho(x, z) > 1$,

$$|x - z| \le c_1 \rho(x, z)^{a_4}.$$

Furthermore, for $x \in \theta(z, t - (k + 1)J)$ with $k \ge k_0$, we have that $\rho(x, z) \le c_2 2^{-t+kJ}$. Combining the two estimates gives

$$|x - z| \le C2^{(-t+kJ)a_4}.$$

Hence, for $k > k_0$,

$$\int_{\theta(z,t-(k+1)J)\setminus\theta(z,t-kJ)} |x - z|^l \, dx \le C\big|\theta(z, t - (k + 1)J)\big|2^{(-t+kJ)la_4}$$
$$\le C2^{(-t+kJ)(la_4+1)}.$$

Applying now (7.16) and the assumption $la_4 - a_6 s < 0$ gives (with a constant that also depends on $|z|$)

$$II = \sum_{k=k_0}^{\infty} \int_{\theta(z,t-(k+1)J)\setminus\theta(z,t-kJ)} |Tf(x)|(1 + |x|^l)\, dx$$
$$\le C\|f\|_q \big|\theta(z, t)\big|^{-1/q} \sum_{k=k_0}^{\infty} 2^{-kJ(1+a_6 s)} \int_{\theta(z,t-(k+1)J)\setminus\theta(z,t-kJ)} (1 + |z|^l + |x - z|^l)\, dx$$

$$\leq C\|f\|_q \left|\theta(z,t)\right|^{-1/q} \sum_{k=k_0}^{\infty} 2^{-kJ(1+a_6 s)}\left(2^{-t+kJ} + 2^{(-t+kJ)(la_4+1)}\right)$$

$$\leq C\|f\|_q \left|\theta(z,t)\right|^{-1/q} \sum_{k=k_0}^{\infty} \left(2^{-t-kJa_6 s} + 2^{-t(la_4+1)+kJ(la_4-sa_6)}\right) < \infty. \qquad \square$$

The main two results of this section are the following theorems.

Theorem 7.7 ([12]). *Let Θ be an ellipsoid cover, and let $0 < p \leq 1$. Suppose that T is a* VASIO *of order s that satisfies the vanishing moment property, such that*

$$s > \frac{a_4}{a_6} N_p(\Theta), \tag{7.23}$$

where $N_p(\Theta)$ is defined in (6.24). *Then T extends to a bounded linear operator from $H^p(\Theta)$ to itself.*

Theorem 7.8 ([12]). *Let Θ be an ellipsoid cover, and let $0 < p \leq 1$. Suppose T is a* VASIO *of order s with*

$$s > \frac{1/p - 1}{a_6}. \tag{7.24}$$

Then T extends to a bounded linear operator from $H^p(\Theta)$ to L_p.

To prove Theorems 7.7 and 7.8, we need the following lemma, which shows that a VASIO preserving vanishing moments maps atoms (Definition 6.22) to molecules (Definition 6.55).

Lemma 7.9. *Under the conditions of Theorem 7.7, Ta is a molecule for any $(p, q, s-1)$-atom a, $1 < q < \infty$. Furthermore, there exists a constant $c > 0$, depending also on the CZ norm $\|T\|_{(s)}$ of T, but not on a, such that $\|Ta\|_{H^p(\Theta)} \leq c$.*

Proof. Let a be a $(p, q, s-1)$-atom, $1 < q < \infty$, $\operatorname{supp}(a) \subset \theta(z,t)$, $z \in \mathbb{R}^n$, and $t \in \mathbb{R}$. Since by Theorem 7.3 T is bounded on L_q (we need to further assume (7.10) if $2 < q < \infty$), by property (ii) of a we have

$$\begin{aligned}\left(\int_{\theta(z,t-J)} |Ta|^q\right)^{1/q} &\leq C\|a\|_q \\ &\leq C\left|\theta(z,t)\right|^{1/q-1/p} \\ &\leq C\left|\theta(z,t-J)\right|^{1/q-1/p}.\end{aligned}$$

Hence Ta satisfies the first property of a molecule (6.99) with respect to $\theta(z, t-J)$. By Lemma 7.6 for $x \in \theta(z, t-(k+1)J) \setminus \theta(z, t-kJ)$, $k \in \mathbb{N}$, we have

$$\left|Ta(x)\right| \leq C\|a\|_q \left|\theta(z,t)\right|^{-1/q} 2^{-kJ(1+a_6 s)} \leq C\left|\theta(z,t-J)\right|^{-1/p} 2^{-kJ(1+a_6 s)}. \tag{7.25}$$

Condition (7.23) ensures $\delta := 1 + a_6 s > 1 + a_4 N_p(\Theta)$, and so (7.25) implies that Ta satisfies the second condition of a molecule (6.100). Next, by the vanishing moments property of T (see Definition 7.5) and condition (7.23) Ta has

$$l := \left\lfloor \frac{a_6 s}{a_4} \right\rfloor \geq N_p(\Theta)$$

vanishing moments. Thus Ta satisfies all the conditions of Definition 6.55 and is a molecule. This allows us to apply Theorem 6.56 and conclude there exists a constant $c > 0$ independent of a such that $\|Ta\|_{H^p(\Theta)} \leq c$. □

Proof of Theorem 7.7. Let $f \in H^p(\Theta) \cap L_q$. By Lemma 6.42 there exists an atomic decomposition

$$f = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}_0} \lambda_i^k a_i^k, \tag{7.26}$$

which converges in L_q, such that a_i^k are (p, ∞, s)-atoms and hence also (p, q, s)-atoms. Furthermore,

$$\sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}_0} |\lambda_i^k|^p \leq C \|f\|_{H^p(\Theta)}^p. \tag{7.27}$$

Since by Theorem 7.3 T is bounded on L_q, $1 < q < \infty$, it follows that $Tf = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}_0} \lambda_i^k T a_i^k$ in L^q, and hence

$$Tf = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}_0} \lambda_i^k T a_i^k \quad \text{in } \mathcal{S}'. \tag{7.28}$$

Since T is a VASIO of order s with vanishing moments, by Lemma 7.9 we obtain $\|T a_i^k\|_{H^p(\Theta)} \leq C'$. Thus by (7.27) and (7.28) we have

$$\begin{aligned} \|Tf\|_{H^p(\Theta)}^p &= \|M^\circ(Tf)\|_p^p \\ &\leq \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}_0} |\lambda_i^k|^p \|M^\circ(T a_i^k)\|_p^p \\ &= \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}_0} |\lambda_i^k|^p \|T a_i^k\|_{H^p(\Theta)}^p \\ &\leq C' \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}_0} |\lambda_i^k|^p \\ &\leq C' C \|f\|_{H^p(\Theta)}^p. \end{aligned}$$

By Corollary 6.44 $L_q \cap H^p(\Theta)$ is dense in $H^p(\Theta)$, which by Theorem 6.15 is complete. Thus we deduce that T extends to a bounded linear operator from $H^p(\Theta)$ to $H^p(\Theta)$. □

Proof of Theorem 7.8. Let a be a (p,q,s)-atom, $1 < q < \infty$, with $\operatorname{supp}(a) \subset \theta(z,t)$, where $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We first show that

$$\|Ta\|_p \le C'. \tag{7.29}$$

By the boundedness of T on L_q, $1 < q < \infty$ (assuming further (7.10) for $2 < q < \infty$), and Hölder's inequality

$$\begin{aligned}\int_{\theta(z,t-J)} |Ta|^p &\le \left(\int_{\theta(z,t-J)} |Ta|^q\right)^{p/q} \left|\theta(z,t-J)\right|^{1-p/q} \\ &\le C\|a\|_q^p \left|\theta(z,t-J)\right|^{1-p/q} \le C.\end{aligned}$$

Next, by Lemma 7.6 we deduce that (7.25) holds for $x \in \theta(z,t-(k+1)J) \setminus \theta(z,t-kJ)$, $k \in \mathbb{N}$. Hence

$$\begin{aligned}\int_{\theta(z,t-J)^c} \left|Ta(x)\right|^p dx &= \sum_{k=1}^{\infty} \int_{\theta(z,t-(k+1)J)\setminus\theta(z,t-kJ)} \left|Ta(x)\right|^p dx \\ &\le C\left|\theta(z,t-J)\right|^{-1} \sum_{k=1}^{\infty} 2^{-pkJ(1+a_6 s)} \left|\theta(z,t-(k+1))J\right| \\ &\le C\sum_{k=1}^{\infty} 2^{-pkJ(1+a_6 s-1/p)} \le C.\end{aligned}$$

The last series converges by assumption (7.24). Combining the last two estimates yields (7.29).

Now we proceed exactly as in the proof of Theorem 7.7. Any $f \in L_q \cap H^p(\Theta)$ admits an atomic decomposition (7.26) such that (7.27) holds and $Tf = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}_0} \lambda_i^k T a_i^k$ in L_q. As $p < q$, it is easy to see that we have this equality also in L_p. By (7.27) and (7.29) we deduce that for $f \in L_q \cap H^p(\Theta)$,

$$\begin{aligned}\|Tf\|_p^p &= \left\|\sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}_0} \lambda_i^k T a_i^k\right\|_p^p \\ &\le \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}_0} |\lambda_i^k|^p \|T a_i^k\|_p^p \\ &\le C' \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}_0} |\lambda_i^k|^p \\ &\le C' C \|f\|_{H^p(\Theta)}^p.\end{aligned}$$

The density of $L_q \cap H^p(\Theta)$ in $H^p(\Theta)$ implies that T extends to a bounded linear operator from $H^p(\Theta)$ to L^p. □

7.2 Anisotropic singular operators on smooth molecules

In this section, we touch upon the subject of analysis of smooth anisotropic singular operators and molecules. The isotropic case is well covered in [40] using "direct" pointwise estimates techniques, which are independent of L_2 and Fourier theories. One of the goals in [40] is showing that, under certain conditions, a singular operator may preserve the fundamental properties of a family of building blocks. Here we will see that, under certain conditions, anisotropic singular operators do map smooth building blocks to smooth building blocks. However, our results present some quantifiable loss of the regularity. Achieving a generalization of the unifying theory of [40] to the anisotropic setting is still an ongoing challenge (see also Remark 4.10).

Definition 7.10. Let Θ be a continuous cover inducing a quasi-distance ρ. A function $f \in C^s(\mathbb{R}^n)$, $r \geq 1$, is said to belong to the *anisotropic test function space* $\mathcal{M}(s, \delta, x_0, t)$, $\delta > a_6 n - 1$, $x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}$, if there exists a constant $c > 0$ such that

$$\left|\partial^\alpha [f(A_{x_0,m}\cdot)](A_{x_0,m}^{-1}x)\right| \leq c\frac{2^{-t\delta}}{(2^{-t} + \rho(x, x_0))^{1+\delta}}, \quad \forall \alpha \in \mathbb{Z}_+^n,\ |\alpha| \leq s, \tag{7.30}$$

where $m = \min(t, -\log_2 \rho(x, x_0))$.

It is evident that x_0 is the center of the test function f and t is the scale with "width" 2^{-t}. We may verify that $\mathcal{M}(s, \delta, x_0, t)$ is a Banach space with $\|f\|_{\mathcal{M}} := \|f\|_{\mathcal{M}(s,\delta,x_0,t)}$ defined by the infimum over all constants c satisfying (7.30).

Definition 7.11. An anisotropic test function $f \in \mathcal{M}(s, \delta, x_0, t)$ is said to be a *smooth molecule* in $\mathcal{M}_0(s, \delta, x_0, t)$ if $\int_{\mathbb{R}^n} f(x)x^\alpha dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| < a_6^{-1}(1+\delta) - n$. A *smooth atom a* is a smooth molecule in $\mathcal{M}_0(s, \delta, x_0, t)$ with support in $\theta(x_0, t)$. We observe that for a smooth atom, we can impose the condition

$$\left|\partial^\alpha [a(A_{x_0,t}\cdot)](A_{x_0,t}^{-1}x)\right| \leq c\frac{2^{-t\delta}}{(2^{-t} + \rho(x, x_0))^{1+\delta}}, \tag{7.31}$$

which is equivalent to (7.30), since $a(x) = 0$ whenever $\rho(x, x_0) > a_2 2^{-t}$.

Example 7.12. Let Θ be a continuous cover, and let $\widehat{\Theta}$ be the equivalent "sampled" discrete cover guaranteed by Theorem 2.32. Let $\{S_k\}_{k\in\mathbb{Z}}$ be the multiresolution kernels of order r defined by (3.46), and let $\{D_k\}_{k\in\mathbb{Z}}$ be the wavelet kernels defined by (4.15) corresponding to $\widehat{\Theta}$. Then, for sufficiently large r, any $x_0 \in \mathbb{R}^n$, and any $\delta > 0$,

$$S_k(x_0, \cdot), S_k(\cdot, x_0) \in \mathcal{M}(r, \delta, x_0, k), \quad D_k(x_0, \cdot), D_k(\cdot, x_0) \in \mathcal{M}_0(r, \delta, x_0, k).$$

Proof. It is sufficient to show that $S_k(x_0, \cdot) \in \mathcal{M}(r, \delta, x_0, k)$, since the proof for $S_k(\cdot, x_0)$ is symmetric. Also, the wavelet kernels $\{D_k\}_{k\in\mathbb{Z}}$ are difference kernels, which inherit their regularity from the multiresolution kernels and also possess, in both variables, the vanishing moments property of molecules (see (4.16)).

For $x \in \mathbb{R}^n$, let $m = \min(k, -\log_2 \rho(x, x_0))$. For any $\beta \in \mathbb{Z}_+^n$, $|\beta| \le r$, using (3.55) and (3.43), we estimate

$$\begin{aligned}
\left|\partial^\beta [S_k(\cdot, A_{x_0,m}\cdot)](x_0, A_{x_0,m}^{-1}x)\right| &\le \sum_{x_0\in\theta_\lambda, x\in\theta_{\lambda'}, \lambda,\lambda'\in\Lambda_k} |B_{\lambda,\lambda'}|\left|\varphi_\lambda(x_0)\right|\left|\partial^\beta[\varphi_{\lambda'}(A_{x_0,m}\cdot)](A_{x_0,m}^{-1}x)\right| \\
&\le C2^k q_*^{(2^k\rho(x_0,x))^\alpha} \sum_{x\in\theta_{\lambda'}} \left\|M_{\theta_{\lambda'}}^{-1}M_{x_0,m}\right\|^{|\beta|} \\
&\le C2^k q_*^{(2^k\rho(x_0,x))^\alpha} \left\|M_{x,k}^{-1}M_{x_0,m}\right\|^{|\beta|},
\end{aligned}$$

where the constants and $0 < q_*, \alpha < 1$, depend on $\mathbf{p}(\Theta)$ (since $\mathbf{p}(\widehat{\Theta})$ depend on $\mathbf{p}(\Theta)$). There are two cases.

Case I: $m > k$. In this case, by (2.14) we have

$$\left\|M_{x,k}^{-1}M_{x_0,m}\right\| \le a_5 2^{-a_6(m-k)} \le C,$$

and so, as in the proof of Corollary 3.14, for any $\delta > 0$,

$$\begin{aligned}
\left|\partial^\beta [S_k(\cdot, A_{x_0,m}\cdot)](x_0, A_{x_0,m}^{-1}x)\right| &\le C2^k q_*^{(2^k\rho(x_0,x))^\alpha} \\
&\le C\frac{2^{-k\delta}}{(2^{-k}+\rho(x,x_0))^{1+\delta}}.
\end{aligned}$$

Case II: $m \le k$. In this case, using (2.14) and recalling that in this case $m = -\log_2 \rho(x, x_0)$, we have

$$\begin{aligned}
\left|\partial^\beta [S_k(\cdot, A_{x_0,m}\cdot)](x_0, A_{x_0,m}^{-1}x)\right| &\le C2^k q_*^{(2^k\rho(x_0,x))^\alpha} \left\|M_{x,k}^{-1}M_{x_0,m}\right\|^{|\beta|} \\
&\le C2^k q_*^{(2^k\rho(x_0,x))^\alpha} 2^{a_4(k-m)|\beta|} \\
&= C2^k q_*^{(2^k\rho(x_0,x))^\alpha} (2^k\rho(x_0,x))^{a_4|\beta|}.
\end{aligned}$$

We now proceed similarly to the proof of Corollary 3.14. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$, $h(z) := q_*^{z^\alpha} z^\gamma$ for some $0 < q_*, \alpha < 1$, $\gamma > 0$. Then there exists a constant $c(q_*, \alpha, \gamma, \delta) > 0$ such that

$$h(z) = q_*^{z^\alpha} z^\gamma \le c\left(\frac{1}{1+z}\right)^{1+\delta+\gamma} z^\gamma \le c\left(\frac{1}{1+z}\right)^{1+\delta}.$$

Therefore application with $z = 2^k\rho(x_0, x)$ and $\gamma = a_4|\beta|$ gives

$$\begin{aligned} \left|\partial^\beta[S_k(\cdot, A_{x_0,m}\cdot)](x_0, A^{-1}_{x_0,m}x)\right| &\le C2^k q_*^{(2^k\rho(x_0,x))^\alpha}(2^k\rho(x_0,x))^{a_4|\beta|} \\ &\le C2^k\left(\frac{1}{1+2^k\rho(x,x_0)}\right)^{1+\delta} \\ &= C\frac{2^{-k\delta}}{(2^{-k}+\rho(x,x_0))^{1+\delta}}. \end{aligned}$$ □

Definition 7.13. We say that a VASIO kernel operator T with kernel K is a *smooth VASIO* of order and vanishing moments s if it satisfies the regularity "symmetric" condition (7.4) and also the additional vanishing moments condition

$$\int_{\mathbb{R}^n} \partial_x^\alpha K(x,y)y^{\tilde{\alpha}}dy = 0, \quad \forall|\alpha|, |\tilde{\alpha}| \le s. \tag{7.32}$$

The vanishing moments condition (7.32) is stronger than the condition of Definition 7.5. It can be interpreted in the following sense:

(i) There exists a sequence of operators T_j with kernels $\{K_j\}_{j\ge1}$ such that $K_j(x,\cdot)$, $K_j(\cdot,y) \in \mathcal{S}$ for all $x, y \in \mathbb{R}^n$.

(ii) $\int_{\mathbb{R}^n} \partial^\alpha K_j(x,y)y^{\tilde{\alpha}}dy = 0$, $\forall x \in \mathbb{R}^n$, $|\alpha|, |\tilde{\alpha}| \le s$,

(iii) For any $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\partial^\alpha T_j f \to \partial^\alpha T f$ pointwise for any $f \in \mathcal{M}(s, \delta, x_0, t)$.

We now show that a smooth VASIO with vanishing moments maps smooth atoms to smooth molecules, however, with some quantifiable reduced regularity.

Theorem 7.14. *Let T be a smooth VASIO of order and vanishing moments s. Then there exists a constant $c > 0$ such that for any atom $a \in \mathcal{M}_0(s, \delta, x_0, t)$,*

$$\|Ta\|_{\mathcal{M}_0(\tilde{s},\tilde{\delta},x_0,t)} \le c\|a\|_{\mathcal{M}_0(s,\delta,x_0,t)},$$

where

$$\tilde{s} < \frac{a_6}{a_4}s, \quad \tilde{\delta} < a_6 s.$$

The constant does not depend on $x_0 \in \mathbb{R}^n$ or $t \in \mathbb{R}$.

Proof. We first prove that Ta is a test function, i. e., $Ta \in \mathcal{M}(\tilde{s}, \tilde{\delta}, x_0, t)$. Let $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le \tilde{s}$. For $x \in \mathbb{R}^n$, we have two cases.

Case I: $x \in \theta(x_0, t - J)$. In this case, $\rho(x, x_0) \le C2^{-t}$, and thus it is sufficient to prove that

$$\left|\partial^\alpha[Ta(M_{x_0,t}\cdot)](M^{-1}_{x_0,t}x)\right| \le C2^t, \quad \forall|\alpha| \le \tilde{s}.$$

In the estimate below we use the notation $m = m(y) := -\log_2(x,y)$, noting that for $y \in \theta(x_0,t)$, $m \geq t - \tilde{c}$. We also apply the vanishing moments of the singular kernel and (2.5) with $a_6 s - a_4|\alpha| \geq a_6 s - a_4\tilde{s} > 0$:

$$\begin{aligned}
&\left|\partial^\alpha[Ta(M_{x_0,t}\cdot)](M_{x_0,t}^{-1}x)\right| \\
&\quad = \left|\int_{\theta(x_0,t)} \partial_x^\alpha[K(M_{x_0,t}\cdot,\cdot)](M_{x_0,t}^{-1}x,y) R^s_{A_{x_0,t}^{-1}x}[a(A_{x_0,t}\cdot)](A_{x_0,t}^{-1}y)dy\right| \\
&\quad \leq \int_{\theta(x_0,t)} \left|\partial_x^\alpha[K(M_{x,m}M_{x,m}^{-1}M_{x_0,t}\cdot,\cdot)]((M_{x,m}^{-1}M_{x_0,t})^{-1}M_{x,m}^{-1}x,y)\right| \\
&\qquad \times \left|R^s_{A_{x_0,t}^{-1}x}[a(A_{x_0,t}\cdot)](A_{x_0,t}^{-1}y)\right|dy \\
&\quad \leq C\|T\|\|a\|_{\mathcal{M}(r,\delta,x_0,t)} \int_{\theta(x_0,t)} \rho(x,y)^{-1}\left\|M_{x,m}^{-1}M_{x_0,t}\right\|^{|\alpha|}2^t\left|A_{x_0,t}^{-1}x - A_{x_0,t}^{-1}y\right|^s dy \\
&\quad \leq C2^t\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} \int_{\theta(x_0,t)} \rho(x,y)^{-1}2^{a_4(m-t)|\alpha|}\left(|\theta_{x_0,t}|^{-1}\rho(x,y)\right)^{a_6 s}dy \\
&\quad \leq C2^t\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} \int_{\theta(x_0,t)} \rho(x,y)^{-1-a_4|\alpha|+a_6 s}2^{t(a_6 s - a_4|\alpha|)}dy \\
&\quad \leq C2^t\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)}2^{t(a_6 s - a_4|\alpha|)} \int_{\rho(x,y)\leq 2^{-t+J}} \rho(x,y)^{-1-a_4|\alpha|+a_6 s}dy \\
&\quad \leq C2^t\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)}.
\end{aligned}$$

Case II: $x \in \theta(x_0, t-J)^c$. Now fix $m := -\log_2\rho(x,x_0)$ and assume that $x \in \theta(x_0, t-J(k+1)) \setminus \theta(x_0, t-Jk)$ for some $k \geq 0$. This implies that $\rho(x,x_0) \geq c2^{-t+Jk}$. We apply (7.12), the vanishing moments of the atom, and $\tilde{\delta} < a_6 s$ to obtain

$$\begin{aligned}
&\left|\partial^\alpha[Ta(M_{x_0,m}\cdot)](M_{x_0,m}^{-1}x)\right| \\
&\quad = \left|\int_{\theta(x_0,t)} \partial_x^\alpha[K(M_{x_0,m}\cdot,\cdot)](M_{x_0,m}^{-1}x,y)a(y)dy\right| \\
&\quad = \left|\int_{\theta(x_0,t)} \partial_x^\alpha[K(M_{x_0,m}\cdot,M_{x_0,m}\cdot)](M_{x_0,m}^{-1}x,M_{x_0,m}^{-1}y)a(y)dy\right| \\
&\quad \leq \int_{\theta(x_0,t)} \left|R^s_{A_{x_0,m}^{-1}x_0}[\partial_x^\alpha[K(M_{x_0,m}\cdot,M_{x_0,m}\cdot)](A_{x_0,m}^{-1}x,\cdot)](A_{x_0,m}^{-1}y)\right||a(y)|dy \\
&\quad \leq C2^t\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} \int_{\theta(x_0,t)} 2^{t-Jk}\left|A_{x_0,m}^{-1}x_0 - A_{x_0,m}^{-1}y\right|^s dy \\
&\quad \leq C2^t\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} \int_{\theta(x_0,t)} 2^{t-Jk}\left(|\theta_{x_0,m}|^{-1}\rho(x_0,y)\right)^{a_6 s}dy
\end{aligned}$$

$$\begin{aligned}
&\le C2^t \|T\| \|a\|_{\mathcal{M}(s,\delta,x_0,t)} \int\limits_{\theta(x_0,t)} 2^{t-kJ}\left(2^{t-kJ}2^{-t}\right)^{a_6 s} dy \\
&\le C\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} 2^{t-kJ(1+a_6 s)} \\
&= C\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} 2^{-ta_6 s} 2^{(t-kJ)(1+a_6 s)} \\
&\le C\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} 2^{-ta_6 s} \rho(x,x_0)^{-(1+a_6 s)} \\
&\le C\|T\|\|a\|_{\mathcal{M}(s,\delta,x_0,t)} \frac{2^{-ta_6 s}}{(2^{-t} + \rho(x,x_0))^{1+a_6 s}} \\
&\le C\|T\|\|a\|_{\mathcal{M}(r,\delta,x_0,t)} \frac{2^{-t\tilde{\delta}}}{(2^{-t} + \rho(x,x_0))^{1+\tilde{\delta}}}.
\end{aligned}$$

Once we have established that Ta has sufficient regularity, then for any $|\alpha| < a_6^{-1}(1 + \tilde{\delta}) - n$, we may apply the vanishing moments condition (7.32), which is stronger and implies the vanishing moments property of Definition 7.5:

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} Ta(x) x^\alpha dx &= \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} K(x,y) a(y) dy\, x^\alpha dx \\
&= \int\limits_{\mathbb{R}^n} a(y) \left(\int\limits_{\mathbb{R}^n} K(x,y) x^\alpha dx \right) dy = 0.
\end{aligned}$$

□

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