DE GRUYTER

GRADUATE

Nikolai Saveliev

LECTURES ON THE TOPOLOGY OF 3-MANIFOLDS

AN INTRODUCTION TO THE CASSON INVARIANT

2ND EDITION

De Gruyter Textbook

Saveliev · Lectures on the Topology of 3-Manifolds

Nikolai Saveliev

Lectures on the Topology of 3-Manifolds

An Introduction to the Casson Invariant

2nd revised edition

De Gruyter

Mathematics Subject Classification 2010: Primary: 57M27; Secondary: 57M25, 57N10, 57N13, 57R58.

ISBN: 978-3-11-025035-0
e-ISBN: 978-3-11-025036-7

Library of Congress Cataloging-in-Publication Data

Saveliev, Nikolai, 1966–
Lectures on the topology of 3-manifolds : an introduction to the Casson invariant / by Nikolai Saveliev. – 2nd ed.
p. cm.
Includes bibliographical references and index.
ISBN 978-3-11-025035-0 (alk. paper)
1. Three-manifolds (Topology) I. Title.
QA613.2.S28 2012
514′.34–dc23
2011037009

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the internet at http://dnb.d-nb.de.

Typesetting: Da-TeX Gerd Blumenstein, Leipzig, www.da-tex.de
Printing and binding: Hubert & Co. GmbH & Co. KG, Göttingen
♾ Printed on acid-free paper
Printed in Germany

www.degruyter.com

Preface

This short book grew out of lectures the author gave at the University of Michigan in the Fall of 1997. The purpose of the course was to introduce second year graduate students to the theory of 3-dimensional manifolds and its role in the modern 4-dimensional topology and gauge theory. The course assumed only familiarity with the basic concepts of topology including: the fundamental group, the (co)homology theory of manifolds, and the Poincaré duality.

Progress in low-dimensional topology has been very fast over the last two decades, leading to the solution of many difficult problems. One of the consequences of this "acceleration of history" is that many results have only appeared in professional journals and monographs. Among these are Casson's results on the Rohlin invariant of homotopy 3-spheres, as well as his λ-invariant. The monograph "Casson's invariant for oriented homology 3-spheres: an exposition" by S. Akbulut and J. McCarthy, though beautifully written, is hardly accessible to students who have completed only a basic course in algebraic topology. The purpose of this book is to provide a much-needed bridge to these topics.

Casson's construction of his λ-invariant is rather elementary compared to further developments related to gauge theory. This book is in no way intended to explore this subject, as it requires an extensive knowledge of Riemannian geometry and partial differential equations.

The book begins with topics that may be considered standard for a book in 3-manifolds: existence of Heegaard splittings, Singer's theorem about the uniqueness of a Heegaard splitting up to stable equivalence, and the mapping class group of a closed surface. Then we introduce Dehn surgery on framed links, give a detailed description of the Kirby calculus of framed links in S^3, and use this calculus to prove that any oriented closed 3-manifold bounds a smooth simply-connected parallelizable 4-manifold.

The second part of the book is devoted to Rohlin's invariant and its properties. We first review some facts about 4-manifolds and their intersection forms, then we do some knot theory. The latter includes Seifert surfaces and matrices, the Alexander polynomial and Conway's formula, and the Arf-invariant and its relation to the Alexander polynomial. Our approach differs from the common one in that we work in a homology sphere rather than in S^3, though the difference here is more technical than conceptual. This part concludes with a geometric proof of the Rohlin Theorem (after M. Freedman and R. Kirby), and with the surgery formula for the Rohlin invariant.

The last part of the book deals with Casson's invariant and its applications, mostly along the lines of Akbulut and McCarthy's book. We employ a more intuitive approach here to emphasize the ideas behind the construction, and refer the reader to the aforementioned book for technical details.

The book is full of examples. Seifert fibered manifolds appear consistently among these examples. We discuss their Heegaard splittings, Dehn surgery description, classification, Rohlin invariant, SU(2)-representation spaces, twisted cohomology, Casson invariant, etc.

Throughout the book, we mention the latest developments whenever it seems appropriate. For example, in the section on 4-manifold topology, we give a review of recent results relating 4-manifolds and unimodular forms, including the "10/8-conjecture" and Donaldson polynomials. The Rohlin invariant gives restrictions on the genus of surfaces embedded in a smooth 4-manifold. When describing this old result, we also survey the results that follow from the Thom conjecture, proved a few years ago by Kronheimer and Mrowka with the help of Seiberg–Witten theory.

The topology of 3-manifolds includes a variety of topics not discussed in this book, among which are hyperbolic manifolds, Thurston's geometrization conjecture, incompressible surfaces, prime decompositions of 3-manifolds, and many others.

The book has brief notes on further developments, and a list of exercises at the end of each lecture.

The book is closely related, in several instances, both in content and method, to the books Akbulut–McCarthy [2] and Fomenko–Matveev [49], from which I have borrowed quite shamelessly. However, it is hoped that the present treatment will serve its purpose of providing an accessible introduction to certain topics in the topology of 3-manifolds. Other major sources I relied upon while writing this book include Browder [24], Fintushel–Stern [45], Freedman–Kirby [52], Guillou–Marin [64], Kirby [84], Livingston [105], Matsumoto [107], McCullough [110], Neumann–Raymond [122], Rolfsen [137] and Taubes [152].

Figures 1.3, 1.6, 1.10, 3.4, 3.9, 4.3 were reproduced, with kind permission, from "Algorithmic and Computer Methods for Three-Manifolds" by A. T. Fomenko and S. V. Matveev, ©1997 Kluwer Academic Publishers.

I am indebted to Boris Apanasov, Olivier Collin, John Dean, Max Forester, Slawomir Kwasik, Walter Neumann, Liviu Nicolaescu, Frank Raymond, Thang T. Q. Le, and Vladimir Turaev for sharing their expertise and advice, and for their help and support during my work on this book. I would also like to thank the graduate students who took my course at the University of Michigan. I wish to express my gratitude to John Dean who read the manuscript to polish the English usage. I was partially supported by NSF Grant DMS-97-04204 and by Max-Planck-Institut für Mathematik in Bonn, Germany, during my work on this book.

Comments on this edition

In the twelve years since the publication of this book, the face of low-dimensional topology has been profoundly changed by the proof of the three-dimensional Poincaré conjecture. The effect this had on the Casson invariant was that its original application to proving that the Rohlin invariant of a homotopy 3-sphere must vanish was rendered moot. Despite this, Casson's contribution remains as relevant as ever: in fact, a lot of the modern day low-dimensional topology, including a number of Floer homology theories, can be traced back to his λ-invariant. These Floer homology theories have been also linked to contact topology and Khovanov homology, and together they constitute a very active area of research.

I did not attempt to cover any of these new topics in the second edition. However, I added a couple of brief sections, where it seemed appropriate, to indicate how the material in this book is relevant to Heegaard Floer homology and open book decompositions. Other than that, I added a few updates and exercises, and corrected a number of typos.

I am thankful to everyone who has commented on the book, and especially to Ken Baker, Ivan Dynnikov, Jochen Kroll, and Marina Prokhorova.

Miami, August 2011 Nikolai Saveliev

Contents

Introduction

A topological space M is called a (topological) n-dimensional *manifold*, or n-manifold, if each point of M has an open neighborhood homeomorphic to $\mathbb{R}^n$. In other words, a manifold is a locally Euclidean space. To avoid pathological examples, it is standard to assume that all manifolds are Hausdorff and have a countable base of topology, and we will follow this convention. Most manifolds we consider will also be compact and connected.

Let U and V be two open sets in an n-manifold M each homeomorphic to $\mathbb{R}^n$ via homeomorphisms $\phi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^n$. Then

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V) \tag{1}$$

is a homeomorphism of open sets in Euclidean space $\mathbb{R}^n$. A manifold M is *smooth* if there is an open covering $\mathcal{U}$ of M such that for any open sets $U, V \in \mathcal{U}$ the map (1) is a diffeomorphism. A manifold M is called *piecewise linear* or simply *PL* if there is an open covering $\mathcal{U}$ of M such that for any open sets $U, V \in \mathcal{U}$ the map (1) is a piecewise linear homeomorphism. Another way to describe PL manifolds is as follows.

A triangulation of a polyhedron is called *combinatorial* if the link of each its vertex is PL-homeomorphic to a PL-sphere. Every PL-manifold admits a combinatorial triangulation. Any polyhedron which admits a combinatorial triangulation is a PL-manifold.

A Hausdorff topological space M whose topology has a countable base is called an *n-manifold with boundary* if each point of M has an open neighborhood homeomorphic to either Euclidean space $\mathbb{R}^n$ or closed upper half-space $\mathbb{R}^n_+$. The union of points of the second type is either empty or an $(n-1)$-dimensional manifold, which is denoted by ∂M and called the *boundary* of M. Note that the boundary of ∂M is empty. A manifold M is called *closed* if it is compact and its boundary is empty. Analogous definitions hold for smooth and PL manifolds.

The following fact is very important for us: if $n \le 3$ then the concepts of topological, smooth, and PL manifolds coincide, see Bing [15] and Moise [116]. More precisely, any topological manifold M of dimension less than or equal to 3 admits a smooth and a PL-structure. These are unique in that there is a diffeomorphism or a PL-homeomorphism between any two smooth or PL-manifolds that are homeomorphic to M. Moreover, if a PL-manifold of dimension $n \le 3$ is homeomorphic to a smooth manifold then there is a homeomorphism between them whose restriction to each simplex of a certain triangulation is a smooth embedding.

In dimension 4, every PL-manifold has a unique smooth structure, and vice versa, see Cairns [27] and Hirsch [75]. However, there exist topological manifolds in dimension 4 that admit no smooth structure, and there are topological 4-manifolds with more than one smooth structure. These questions will be discussed in more detail in Lecture 5. Furthermore, there exists a closed 4-dimensional topological manifold which is not homeomorphic to *any* simplicial complex, much less a combinatorial one. A key ingredient in the construction of such a manifold is the Casson invariant, which is defined later in these lectures.

The relationships between topological, smooth, and PL-manifolds are more complicated in dimensions 5 and higher. They will be briefly discussed in Lecture 18.

Glossary

We explain some standard geometric and topological background material used in the book. Shown in *italic* are terms whose meaning is explained somewhere in the glossary text.

CW-complexes. A topological space X is called a CW-complex if X can be represented as a union

$$X = \bigcup_{q=0}^{\infty} X^{(q)}$$

where the 0-skeleton $X^{(0)}$ is a countable (possibly finite) discrete set of points, and each $(q+1)$-skeleton $X^{(q+1)}$ is obtained from the q-skeleton $X^{(q)}$ by attaching $(q+1)$-cells. More explicitly, for each q there is a collection $\{e_j \mid j \in J_{q+1}\}$ where

(1) each e_j is a subset of $X^{(q+1)}$ such that if $e'_j = e_j \cap X^{(q)}$, then $e_j \setminus e'_j$ is disjoint from $e_k \setminus e'_k$ if $j, k \in J_{q+1}$ with $j \neq k$,

(2) for each $j \in J_{q+1}$, there is a characteristic map $g_j : (D^{q+1}, \partial D^{q+1}) \to (X^{(q+1)}, X^{(q)})$ such that g_j is a quotient map from D^{q+1} to e_j, which maps $D^{q+1} \setminus \partial D^{q+1}$ homeomorphically onto $e_j \setminus e'_j$,

(3) a subset of X is closed if and only if its intersection with each skeleton $X^{(q)}$ is closed.

Each $e_j \setminus e'_j$ is called a $(q+1)$-cell. When all characteristic maps are embeddings, the CW-complex is called regular.

Cellular homology. Let X be a *CW-complex*, and R a commutative ring with an identity element. For each q, let $C_q(X, R)$ be the free R-module with basis the q-cells. We will define the boundary homomorphism $\partial_{q+1}\colon C_{q+1}(X, R) \to C_q(X, R)$. To define $\partial_{q+1}(c)$, where c is a fixed $(q+1)$-cell, fix an orientation for D^{q+1}, thus determining an orientation for the q-sphere ∂D^{q+1}, and look at how the characteristic map g of c carries ∂D^{q+1} to $X^{(q)}$. For each e_k in $X^{(q)}$, fix a point z_k in $c_k = e_k \setminus e'_k$. One can show that g is homotopic to a map such that for each k, the preimage of z_k is a finite set of points $p_{k,1}, \ldots, p_{k,n_k}$. Moreover g takes a neighborhood of each $p_{k,j}$ homeomorphically to a neighborhood of z_k (by compactness, the preimage of z_k is empty for all but finitely many k). For each j with $1 \leq j \leq n_k$, let $\varepsilon_{k,j} = \pm 1$

according to whether g restricted to the neighborhood of $p_{k,j}$ preserves or reverses orientation. Let

$$\varepsilon_k = \sum_{j=1}^{n_k} \varepsilon_{k,j} \quad \text{and} \quad \partial_{q+1}(c) = \sum_{k=1}^{\infty} \varepsilon_k c_k$$

where all but finitely many ε_k are equal to zero.

The numbers ε_k can also be described as follows. The quotient space $X^{(q)}/X^{(q-1)}$ is homeomorphic to a one-point union of q-dimensional spheres, one for each q-cell $c_k = e_k \setminus e'_k$. Given a $(q+1)$-cell c, its characteristic map $g : (D^{q+1}, \partial D^{q+1}) \to (X^{(q+1)}, X^{(q)})$ induces the map

$$\varphi_k : \partial D^{q+1} \to X^{(q)} \to X^{(q)}/X^{(q-1)} \to S^q,$$

where the last arrow maps the sphere S^q corresponding to the cell c_k identically to itself, while contracting all other spheres to a point. The *degree* of φ_k is ε_k. This description of ε_k ensures that $\partial_{q+1}(c)$ is well-defined.

This defines the homomorphism ∂_{q+1} on the generators, and the definition extends by linearity to the entire free R-module $C_{q+1}(X, R)$. One can prove that $\partial_q \partial_{q+1} = 0$. The reason is that algebraically, the q-sphere ∂D^{q+1} acts as though it were a regular CW-complex with one q-cell corresponding to each preimage point of a z_k. Since ∂D^{q+1} is a manifold, the boundaries of these q-cells form a collection of $(q-1)$-cells, each appearing as part of the boundary of two q-cells, but with opposite orientations. Consequently, the algebraic sum of the boundaries of these q-cells is 0. Applying ∂_q to $\partial_{q+1}(c)$ simply adds up the images of the boundaries of those q-cells in $C_{q-1}(X, R)$, and the pairs with opposite signs cancel out, giving 0.

An element of $C_q(X, R)$ is a formal finite sum $\sum r_k c_k$, where each c_k is a q-cell; such a sum is called a q-chain. Now form a sequence of R-modules and homomorphisms

$$\cdots \to C_{q+1}(X, R) \xrightarrow{\partial_{q+1}} C_q(X, R) \xrightarrow{\partial_q} C_{q-1}(X, R) \to \cdots \to C_0(X, R) \to 0. \quad (2)$$

This is called a chain complex, since $\partial_q \partial_{q+1} = 0$ for all q. This implies that the image of ∂_{q+1} is contained in the kernel of ∂_q for each q. If the image of ∂_{q+1} equals the kernel of ∂_q for each q, the sequence is called exact. If not, we measure its deviation from exactness by defining cellular homology groups

$$H_q(X; R) = \ker(\partial_q)/\operatorname{im}(\partial_{q+1}).$$

Elements of $\ker(\partial_q)$ are called cycles, and elements of $\operatorname{im}(\partial_{q+1})$ are called boundaries. Explicitly, an element of $H_q(X; R)$ is a coset $a_q + \partial_{q+1}(C_{q+1}(X, R))$, where $\partial_q a_q = 0$, but it is usually written as $[a_q]$. Note that $[a_q] = [a'_q]$ if and only if $a_q = a'_q + \partial_{q+1}(b_{q+1})$ for some $(q+1)$-chain b_{q+1}.

To complete the definition of H_* as a *homology theory*, we need to define f_* for all continuous maps $f: X \to Y$. We first define $C_q(f): C_q(X, R) \to C_q(Y, R)$. By the Cellular Approximation Theorem, f may be changed within its homotopy class so that $f(X^{(q)}) \subset Y^{(q)}$ for all q. Then, define $C_q(f)(c)$ similarly to the way that $\partial_q(c)$ was defined above. Then $f_*([c]) = [C_q(f)(c)]$.

It is not easy to prove that this is well-defined and satisfies the *Eilenberg–Steenrod axioms*, but it can be done. In particular, $H_*(X; R)$ does not depend on the *CW-complex* structure chosen for X since the identity map induces an isomorphism on the homologies defined using two different *CW-complex* structures on X, and f_* depends only on the homotopy class of f.

When A is a subcomplex of X define the relative homology groups $H_q(X, A; R)$ by setting $C_q(X, A, R) = C_q(X, R)/C_q(A, R)$ and noting that ∂_q induces ∂_q : $C_q(X, A, R) \to C_{q-1}(X, A, R)$. Then, $H_q(X, A; R)$ is defined as the homology of the chain complex $C_*(X, A, R)$. The long exact sequence of the second *Eilenberg–Steenrod axiom* is then a purely algebraic consequence of the existence of short exact sequences

$$0 \to C_q(A, R) \to C_q(X, R) \to C_q(X, A, R) \to 0.$$

Note that every element of $H_q(X, A; R)$ can be represented by a q-chain whose boundary lies in A.

Cohomology of spaces. Once *cellular*, *simplicial*, or *singular* homology is defined, cohomology can be defined algebraically. This is based on the following fact. If A and B are R-modules, and $\varphi: A \to B$ is an R-module homomorphism, then there is an R-module homomorphism $\varphi^*: \mathrm{Hom}(B, R) \to \mathrm{Hom}(A, R)$ defined by $\varphi^*(\alpha) = \alpha \circ \varphi$. Clearly $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$, so if we define the coboundary homomorphism by $\delta_q = \partial_q^*$, then $\delta_{q+1}\delta_q = \partial_{q+1}^*\partial_q^* = (\partial_q\partial_{q+1})^* = 0^* = 0$. Therefore, abbreviating $\mathrm{Hom}(C_q(X), R)$ to $C^q(X, R)$, we have a cochain complex

$$0 \to C^0(X, R) \to \cdots \to C^{q-1}(X, R) \xrightarrow{\delta_q} C^q(X, R) \xrightarrow{\delta_{q+1}} C^{q+1}(X, R) \to \cdots \tag{3}$$

whose deviation from exactness is measured by the cohomology groups

$$H^q(X, R) = \ker \delta_{q+1} / \operatorname{im} \delta_q.$$

A continuous map $f: X \to Y$ induces homomorphisms $f^*: H^q(Y, R) \to H^q(X, R)$ with $(f \circ g)^* = g^* \circ f^*$, and there are corresponding versions of the *Eilenberg–Steenrod axioms* and *Mayer–Vietoris exact sequence* for cohomology.

An important case is when $R = F$ is a field. Then it can be proved that $H^q(X; F) \cong \mathrm{Hom}(H_q(X; F), F)$, the dual vector space of $H_q(X, F)$. Hence $H^q(X; F)$ and $H_q(X; F)$ are vector spaces of the same rank, although there is no natural isomorphism between them.

Connected sums. Let M_1 and M_2 be closed oriented manifolds of dimension n, and $D^n \subset M_k$, $k = 1, 2$, a pair of n-discs embedded in M_1 and M_2. A connected sum of M_1 and M_2 is defined as the manifold $M_1 \# M_2 = (M_1 \setminus \operatorname{int} D^n) \cup (M_2 \setminus \operatorname{int} D^n)$ obtained by *gluing* the manifolds $M_k \setminus \operatorname{int}(D^n)$ along their common boundary S^{n-1} via an orientation reversing homeomorphism $r: S^{n-1} \to S^{n-1}$. The manifold $M_1 \# M_2$ inherits an orientation from those on M_1 and M_2. The manifolds $M_1 \# M_2$ and $M_1 \# (-M_2)$, where $-M_2$ stands for the manifold M_2 with reversed orientation, need not be homeomorphic. Note also that if the manifolds M_1 and M_2 are smooth, a choice of smoothly embedded discs in M_1 and M_2 and a smooth identification map provides us with a smooth manifold $M_1 \# M_2$.

If the manifolds M_1 and M_2 have non-empty boundaries, one can still form their connected sum by choosing the n-discs in their interiors. One can also form their boundary connected sum, $M_1 \natural M_2$, by identifying $(n-1)$-discs $D^{n-1} \subset \partial M_k$, $k = 1, 2$, via an orientation reversing homeomorphism. The boundary of $M_1 \natural M_2$ is $(\partial M_1) \# (\partial M_2)$.

Cutting open. This is an operation which is "inverse" to the *gluing* of spaces. Let Y be a closed subspace of a connected space X such that the closure of $X \setminus Y$ coincides with X. Suppose that $X \setminus Y$ consists of a finite number of connected components, $X_1, \dots, X_n$. Consider the space

$$X' = \bigcup X_i \times \{i\} \subset X \times \mathbb{R},$$

that is, move the components apart from each other. The closure of X' in the product topology on $X \times \mathbb{R}$ is the result of cutting X open along Y.

Degree of a map. Let $f: (M, \partial M) \to (N, \partial N)$ be a continuous map between oriented connected compact manifolds of identical dimension n. The degree of f is an integer $\deg f$ satisfying $f_*[M, \partial M] = \deg f \cdot [N, \partial N]$, where $[M, \partial M]$ and $[N, \partial N]$ are the *fundamental classes* of the manifolds M and N, and $f_*: H_n(M, \partial M) \to H_n(N, \partial N)$ the induced map. If $f: M \to N$ is a smooth map between smooth closed oriented manifolds, choose any point $y \in N$ such that f is *transversal* to y. Then the degree of f coincides with the integer

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sign}(\det d_x f),$$

where $d_x f: T_x M \to T_y N$ is the derivative of f at a point $x \in M$, and is independent of the choice of y.

Eilenberg–MacLane spaces. The Eilenberg–MacLane spaces $K(\pi, n)$ are the fundamental building blocks of homotopy theory. They are *CW-complexes* characterized uniquely up to homotopy equivalence as having a single non-trivial homotopy group:

$$\pi_i(K(\pi,n)) = \begin{cases} \pi, & \text{if } i = n, \\ 0, & \text{if } i \neq n. \end{cases}$$

Of course, the group π is required to be Abelian if $n \geq 2$. Standard examples of Eilenberg–MacLane spaces include $K(\mathbb{Z},1) = S^1$ and $K(\mathbb{Z},2) = \mathbb{C}P^\infty$, where $\mathbb{C}P^\infty$ is defined as the limiting space of the tower of complex projective spaces $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \mathbb{C}P^3 \subset \cdots$ with respect to the natural inclusions. Eilenberg–MacLane spaces are classifying spaces for cohomology in that

$$H^n(X;\pi) = [X, K(\pi,n)] \tag{4}$$

for any space X and Abelian group π, where the brackets denote the set of all homotopy classes of continuous maps. The isomorphism (4) is obtained as follows. From the *Hurewicz theorem* and the *Universal Coefficient theorem*, it is easy to see that $H^n(K(\pi,n);\pi) \cong \mathrm{Hom}(\pi,\pi)$. Let $\iota\colon \pi \to \pi$ be the identity map. Associate to any $f\colon X \to K(\pi,n)$ the cohomology class $f^*\iota \in H^n(X;\pi)$; this is the correspondence (4).

Gluing construction. Let X and Y be topological spaces, and $f\colon Z \to Y$ a continuous map where $Z \subset X$ is a subspace of X. Consider the disjoint union $X \cup Y$ and introduce the equivalence relation generated by $z \sim f(z)$ whenever $z \in Z$. The space $X \cup_f Y = (X \cup Y)/\sim$ with the quotient topology is said to be obtained by gluing X and Y along f. In most cases we consider, the map f will be a homeomorphism of Z onto its image $f(Z) \subset Y$.

Handles. Let X be a smooth n-manifold with boundary, and $0 \leq k \leq n$. An n-dimensional k-handle is a copy of $D^k \times D^{n-k}$, attached to the boundary of X along $(\partial D^k) \times D^{n-k}$ using an embedding $f : (\partial D^k) \times D^{n-k} \to \partial X$. The corners that arise can be smoothed out, see for instance Chapter 1 of Conner–Floyd [31], hence $X \cup_f (D^k \times D^{n-k})$ is again a smooth manifold. For example, a 1-handle is a product $D^1 \times D^{n-1}$ attached along a pair of $(n-1)$-balls, $S^0 \times D^{n-1}$. A 2-handle is a product $D^2 \times D^{n-2}$ attached along $S^1 \times D^{n-2}$. For more details see Gompf–Stipsicz [61] or Rourke–Sanderson [138].

Homology theory. Let R be a commutative ring with an identity element. Sometimes R will be required to be a principal ideal domain. By a homology theory we mean a functor from the category of pairs of spaces and continuous maps to the category of graded R-modules and graded homomorphisms. That is, for each pair (X, A), where A is a subspace of X, there is an R-module

$$H_*(X,A;R) = \bigoplus_{q=0}^{\infty} H_q(X,A;R),$$

and for each continuous map of pairs $f: (X, A) \to (Y, B)$ there are homomorphisms $f_*: H_q(X, A; R) \to H_q(Y, B; R)$ for every q, so that $(f \circ g)_* = f_* \circ g_*$. We abbreviate $H_q(X, A; R)$ to $H_q(X, A)$ and $H_q(X, \emptyset)$ to $H_q(X)$. It will be clear from the context what the ring R is. The following *Eilenberg–Steenrod axioms* must hold:

(1) (homotopy invariance) If $f, g: (X, A) \to (Y, B)$ are homotopic then $f_* = g_*$.

(2) (long exact sequence) For every pair (X, A) and every q there are homomorphisms $\partial: H_q(X, A) \to H_{q-1}(A)$ fitting into a long exact sequence

$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots \to H_0(X, A) \to 0,$$

where $i: A \to X$ and $j: (X, \emptyset) \to (X, A)$ are inclusion maps.

(3) (excision axiom) If U is an open subspace of X whose closure is contained in the interior of A, then the inclusion map $j: (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms $j_*: H_q(X \setminus U, A \setminus U) \to H_q(X, A)$ for all q.

(4) (coefficient module) If P is a one point space, then $H_0(P) = R$ and $H_q(P) = 0$ for $q \geq 1$.

The module in axiom (4) is called the coefficient ring for the homology theory. We often refer to $H_q(X, A)$ as homology groups. Strictly speaking, one should say homology modules, but for the common cases $R = \mathbb{Z}$ and $R = \mathbb{Z}/n$, the homology modules are Abelian groups.

There are many ways to define homology groups. For a fixed ring R, all the standard ways produce the same results when X is a *simplicial* or a *CW-complex* and A is a subcomplex. The most widely used theories are *simplicial*, *singular*, and *cellular homology*. We will be working with the latter most of the time.

The Eilenberg–Steenrod axioms imply the *Mayer–Vietoris exact sequence*, which is very powerful for computation of homology. It applies in quite general situations, but we will only state it for *CW-complexes*. Suppose that A and B are subcomplexes of a CW-complex X, with $X = A \cup B$. Then there are homomorphisms $\partial: H_q(X) \to H_{q-1}(A \cap B)$ fitting into a long exact sequence

$$\cdots \to H_q(A \cap B) \xrightarrow{(i_*, -j_*)} H_q(A) \oplus H_q(B) \xrightarrow{I_* + J_*} H_q(X) \xrightarrow{\partial} H_{q-1}(A \cap B) \to \cdots$$

where $i: A \cap B \to A$, $j: A \cap B \to B$, $I: A \to X$, and $J: B \to X$ are inclusion maps.

Here are some consequences of the axioms and the Mayer–Vietoris sequence. Assume that K is a CW-complex and L is a subcomplex, possibly empty. Then, if K is n-dimensional, or more generally if every cell of $K \setminus L$ has dimension less than or equal to n, then $H_q(K, L) = 0$ for all $q > n$. Moreover, $H_0(K) = \oplus R$ with one summand for each path component of K.

A cohomology theory is defined similarly, together with cohomological versions of the Eilenberg–Steenrod axioms and the Mayer–Vietoris exact sequence.

Homotopy lifting property. A map $p\colon E \to B$ has the homotopy lifting property with respect to a space X if, for every two maps $f\colon X \to E$ and $G\colon X \times I \to B$ for which $pf = Gi$ (where $I = [0,1]$ and $i\colon X \to X \times I$ is the map $x \mapsto (x,0)$), there exists a continuous map $\tilde{G}\colon X \times I \to E$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"] \arrow[ur, "\tilde{G}"] & B \end{tikzcd} \tag{5}$$

A map $p\colon E \to B$ is called a fibration if it has the homotopy lifting property with respect to every space X. If $b \in B$, then $p^{-1}(b) = F$ is called a fiber. Different fibers of a fibration need not be homeomorphic, however, they all are homotopy equivalent. A map $p\colon E \to B$ is called a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes X. *Locally trivial bundles* are Serre fibrations, and in fact fibrations if the base B is paracompact.

Let $p : E \to B$ be a fibration, and $\tilde{G}_0$, $\tilde{G}_1$ two maps making the above diagram commute. Then $\tilde{G}_0$ and $\tilde{G}_1$ are fiberwise homotopic rel $X \times \{0\}$, see for instance Spanier [150], Corollary 2.8.11.

Homotopy theory. We refer the reader to Hatcher [71] or Spanier [150] for the basics of the homotopy theory, including homotopy, homotopy equivalences, the fundamental group $\pi_1(X, x_0)$, van Kampen's theorem, covering spaces, higher homotopy groups $\pi_n(X, x_0)$ etc.

Hurewicz Theorem. Suppose $\sigma\colon (S^n, s_0) \to (X, x_0)$ is a map representing an element of $\pi_n(X, x_0)$. Let γ_n be a fixed generator of $H_n(S^n; \mathbb{Z}) = \mathbb{Z}$. The Hurewicz homomorphism $\rho\colon \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$ is defined by $\rho([\sigma]) = \sigma_*(\gamma_n)$. One can show that this homomorphism is natural, that is, if $f\colon X \to Y$ is a continuous map, the diagram

$$\begin{tikzcd} \pi_n X \arrow[r, "\rho"] \arrow[d, "f_*"] & H_n X \arrow[d, "f_*"] \\ \pi_n Y \arrow[r, "\rho"] & H_n Y \end{tikzcd}$$

commutes. The basic relationship between homotopy groups and homology groups is given by the Hurewicz theorem, which in its simplest form asserts the following. Let X be a topological space such that $\pi_0 X = \pi_1 X = \ldots = \pi_{n-1} X = 0$ for some $n \geq 1$.

(1) If $n = 1$ then $\rho\colon \pi_1 X \to H_1 X$ is given by Abelianization and is surjective.

(2) If $n \geq 2$ then $\tilde{H}_0 X = H_1 X = \ldots = H_{n-1} X = 0$ and $\rho\colon \pi_n X \to H_n X$ is an isomorphism.

Immersions and embeddings. Let M and N be smooth manifolds of dimensions m and n respectively, such that $m \leq n$. A smooth map $f: M \to N$ is called an immersion if $d_x f: T_x M \to T_{f(x)} N$ is injective for all $x \in M$. The map f is called an embedding if it is an immersion which is injective and proper, meaning that the preimage of every compact set in N is compact in M. In the case of manifolds $M \subset N$, we say that M is immersed in N if the inclusion map $i: M \to N$ is an immersion. We say that M is a (smoothly embedded) submanifold of N if $i: M \to N$ is an embedding. In the latter case, $i: M \to N$ is a diffeomorphism of M onto its image.

All of the above definitions extend verbatim to manifolds with boundary. In addition, a submanifold $M \subset N$ is called properly embedded if $\partial M = M \cap \partial N$.

In the case of manifolds without a smooth structure, we say that M is an (embedded) submanifold of N if the inclusion map $i: M \to N$ is a homeomorphism of M onto its image.

Isotopy. Two homeomorphisms $f_0, f_1: X \to X$ are called isotopic if there is a homotopy $f_t : X \to X$, $0 \leq t \leq 1$, between f_0 and f_1 such that each f_t is a homeomorphism. An isotopy is said to have compact support if the maps f_t are all equal to the identity map outside some fixed compact set $K \subset X$. Two (topological) embeddings $f_0, f_1: X \to Y$ are said to be isotopic if there is a level preserving embedding $F: X \times [0,1] \to Y \times [0,1]$ which agrees with f_0 on $X \times \{0\}$ and with f_1 on $X \times \{1\}$.

Künneth Formula. We will need a version of the Künneth formula which relates the cohomology of the product $X \times Y$ of two finite CW-complexes to the cohomology of the factors. It asserts that there is the following split short exact sequence for each $n \geq 0$,

$$0 \to \sum_{i+j=n} H^i(X) \otimes H^j(Y) \to H^n(X \times Y) \to \sum_{p+q=n+1} \operatorname{Tor}(H^p(X), H^q(Y)) \to 0.$$

Locally trivial bundles. A locally trivial bundle with fiber F is a continuous map $p: E \to B$ for which there exist an open cover $\mathcal{V}$ of B and homeomorphisms $\varphi_V: V \times F \to p^{-1}(V)$ for all $V \in \mathcal{V}$ such that $p\varphi_V(b,x) = b$ for all $(b,x) \in V \times F$. In a locally trivial bundle, all fibers $p^{-1}(b)$ are homeomorphic. Every locally trivial bundle $p: E \to B$ is a Serre fibration (that is, it has the *homotopy lifting property* with respect to every CW-complex), and a fibration if the base B is paracompact.

A vector bundle with fiber a vector space F is a locally trivial bundle $p : E \to B$ all of whose fibers are vector spaces and all the restrictions $\varphi_V : \{b\} \times F \to p^{-1}(b)$ are linear isomorphisms. The tangent bundle to a smooth manifold is an example of a vector bundle; another example is the normal bundle to a submanifold of a smooth manifold.

Every (real) vector bundle $p : E \to B$ over a paracompact base B admits a positive definite inner product. Associated with each such bundle are the locally trivial disc bundle (whose fiber over $b \in B$ is the closed unit disc in $p^{-1}(b)$) and the sphere bundle (whose fiber over $b \in B$ is the unit sphere in $p^{-1}(b)$).

Orientation. A closed n-manifold M is called orientable if $H_n(M;\mathbb{Z}) = \mathbb{Z}$. The choice of a generator $[M]$ in $\mathbb{Z}$ is called an orientation, and the generator is called the fundamental class of M. A manifold together with a choice of orientation is called oriented. A compact n-manifold M with boundary is called orientable if $H_n(M, \partial M;\mathbb{Z}) = \mathbb{Z}$. The choice of a generator $[M, \partial M]$ in $\mathbb{Z}$ is called an orientation, and $[M, \partial M]$ is referred to as the fundamental class of M. A smooth manifold M is orientable if and only if the restriction of its tangent bundle to every smooth curve is trivial. The boundary of an orientable manifold is orientable. If, in place of $\mathbb{Z}$, we use a commutative ring R with an identity element, we will get manifolds orientable over R, etc.

PL-homeomorphisms. For two *simplicial complexes* K_1, K_2, we say that a map $f : |K_1| \to |K_2|$ is piecewise-linear, or PL for short, if f defines a simplicial map $K_1' \to K_2'$ under suitable simplicial subdivisions K_1' and K_2' of K_1 and K_2. If the map f is a homeomorphism, it is referred to as a PL-homeomorphism.

Poincaré duality. Let R be a commutative ring with an identity element, and X a space. There exist several natural pairings between the homology and cohomology of X, among which we will mention the obvious pairing between homology and cohomology groups,

$$\langle\ ,\ \rangle : H^p(X;R) \otimes H_p(X;R) \to R,$$

and the so-called cup- and cap-products,

$$\smile : H^p(X;R) \otimes H^q(X;R) \to H^{p+q}(X;R),$$
$$\frown : H^p(X;R) \otimes H_n(X;R) \to H_{n-p}(X;R),$$

related by the formula $\langle x \smile y, \mu\rangle = \langle x, y \frown \mu\rangle$ whenever $x \in H^p(X;R)$, $y \in H^q(X;R)$, and $\mu \in H_{p+q}(X;R)$.

The following is the simplest form of the Poincaré duality for (topological) manifolds. Let M be a closed n-manifold oriented over a ring R. Then the homomorphism

$$\mathrm{PD} : H^q(M;R) \to H_{n-q}(M;R)$$

defined by the formula $x \mapsto x \frown [M]$, is an isomorphism, where $[M] \in H_n(M;R)$ is the *fundamental class* of M.

Another way to formulate Poincaré duality is as follows. Let M be a closed n-manifold oriented over a field $R = F$. Then the bilinear form

$$Q\colon H^q(M;F) \otimes H^{n-q}(M;F) \to F,$$

defined by the formula $u \otimes v \mapsto \langle u \smile v, [M]\rangle$ is non-degenerate. To see this we note that the relation $\langle u \smile v, [M]\rangle = \langle u, v \frown [M]\rangle$ makes the following diagram commute

$$\begin{tikzcd} H^q M \otimes H^{n-q} M \arrow[rr, "1 \otimes \mathrm{PD}"] \arrow[dr, "Q"'] & & H^q M \otimes H_q M \arrow[dl, "\langle\, ,\, \rangle"] \\ & F & \end{tikzcd}$$

The bilinear form $\langle\, ,\, \rangle$ is non-degenerate because of the isomorphism $\mathrm{Hom}(H_q(M;F), F) = H^q(M;F)$ over the field F. Since PD is an isomorphism, it follows that the bilinear form Q is also non-degenerate.

The latter reformulation of Poincaré duality is not valid over the integers because the groups $H^q(M;\mathbb{Z})$ and $\mathrm{Hom}(H_q(M;\mathbb{Z}), \mathbb{Z})$ are not isomorphic, in general. This problem can be fixed as follows. Let M be a closed n-manifold oriented over the integers, and let Tor denote the torsion subgroup. Then the bilinear form

$$H^q(M)/\operatorname{Tor} \otimes H^{n-q}(M)/\operatorname{Tor} \to \mathbb{Z}$$

defined by $u \otimes v \mapsto \langle u \smile v, [M]\rangle$ is non-degenerate over the integers, and induces an isomorphism of $H^q(M)/\operatorname{Tor}$ to $\mathrm{Hom}(H^{n-q}(M), \mathbb{Z})$ for all q.

Among numerous generalizations of Poincaré duality, a particularly useful one in low-dimensional topology is *Poincaré–Lefschetz duality*. Let M be a compact n-manifold with boundary ∂M which is oriented over R. Then for each q, $H^q(M;R)$ is isomorphic to $H_{n-q}(M, \partial M; R)$, and $H^q(M, \partial M; R)$ is isomorphic to $H_{n-q}(M;R)$. Explicitly, the isomorphisms can be established by taking a cap-product with the relative fundamental class $[M, \partial M] \in H_n(M, \partial M; R)$. We also have a pairing

$$H^q(M;R) \otimes H^{n-q}(M, \partial M; R) \to R,$$

which is automatically non-degenerate when R is a field, and which becomes non-degenerate over the integers after factoring out the torsion subgroups. For more general duality theorems, see Hatcher [71], Section 3.3, and Spanier [150], Chapter VI.

The duality theorems have strong naturality properties in relation to the induced homomorphisms in homology and cohomology. For example, the following diagram is commutative

$$\begin{tikzcd} H^{q-1}(M) \arrow[r] \arrow[d, "\cong"] & H^{q-1}(\partial M) \arrow[r] \arrow[d, "\cong"] & H^q(M, \partial M) \arrow[r] \arrow[d, "\cong"] & H^q(M) \arrow[d, "\cong"] \\ H_{n-q+1}(M, \partial M) \arrow[r] & H_{n-q}(\partial M) \arrow[r] & H_{n-q}(M) \arrow[r] & H_{n-q}(M, \partial M) \end{tikzcd}$$

where the horizontal lines are the long exact sequences of the pair $(M, \partial M)$, and the vertical isomorphisms are given by either *Poincaré* or *Poincaré–Lefschetz duality*.

The following observation is useful when dealing with smooth manifolds; see [18], Section 6. Let $f : X \to Y$ be a smooth orientation preserving map between smooth closed oriented manifold X and Y. Suppose that $S \subset Y$ is a closed oriented submanifold of Y and that f is *transversal* to S. If $\omega \in H^*(Y; \mathbb{R})$ is Poincaré dual to S then $f^*\omega \in H^*(X; \mathbb{R})$ is Poincaré dual to the submanifold $f^{-1}(S)$. In short, under Poincaré duality, the pull back in cohomology corresponds to taking preimage in geometry.

Reduced homology. Let us consider the chain complex (2) together with the augmentation $\varepsilon: C_0(X, R) \to R$ defined as the homomorphism taking each generating 0-cell c_k of $C_0(X, R)$ to $1 \in R$, so that

$$\varepsilon\left(\sum r_k c_k\right) = \sum r_k.$$

The reduced chain complex $\tilde{C}_q(X, R)$ is defined to be the chain complex such that $\tilde{C}_q = C_q$ if $q \neq 0$, $\tilde{C}_0 = \ker \varepsilon$, and $\tilde{\partial}_q = \partial_q$. Note that $\varepsilon\partial_1 = 0$ hence $\partial_1(\tilde{C}_1) \subset \tilde{C}_0$. The homology groups of the reduced chain complex $\tilde{C}_*(X, R)$ are called reduced homology groups of X and denoted $\tilde{H}_*(X; R)$. If $R = \mathbb{Z}$ or R is a field, then $H_q(X; R) = \tilde{H}_q(X; R)$ for $q \neq 0$ and $H_0(X; R) = \tilde{H}_0(X; R) \oplus R$. Reduced simplicial and singular homology groups are defined similarly starting with the corresponding chain complex.

Simplicial complexes. Points $x_0, x_1, \ldots, x_k$ in $\mathbb{R}^m$ are called independent if there is no $(k-1)$-dimensional plane in $\mathbb{R}^m$ passing through them all (of course, m should be greater than or equal to k). The convex hull of $(k+1)$ independent points $x_0, x_1, \ldots, x_k$ is called a k-simplex. It consists of all points of the form $x = a_0 x_0 + \cdots + a_k x_k$ such that $a_0, a_1, \ldots, a_k$ are non-negative and $a_0 + a_1 + \cdots + a_k = 1$. A point, an interval, a triangle, and a tetrahedron are respectively 0-, 1-, 2-, and 3-simplices. The points $x_0, x_1, \ldots x_k$ are called vertices of the k-simplex they define. Any subset of the vertices defines a simplex which is a face of the original k-simplex.

Let $\mathbb{R}^m$ be a fixed Euclidean space. A (finite) geometric simplicial complex in $\mathbb{R}^m$ is a finite collection, K, of simplices of $\mathbb{R}^m$ such that, for any simplex in K all its faces also belong to K, and any two simplices of K are either disjoint or intersect in a face of each. The underlying space of a complex K, that is, the set of points of $\mathbb{R}^m$ belonging to some simplex of K, with the topology induced by that in $\mathbb{R}^m$, is called the polyhedron of K and written $|K|$, and K is called a triangulation of $|K|$. We define a (finite) simplicial complex as a topological space X homeomorphic to a polyhedron $|K|$. The combinatorial structure on X induced by that of K is called a triangulation of X.

The boundary of a $(k+1)$-simplex A consists of all proper faces of A; it is a simplicial complex whose polyhedron is homeomorphic to a k-sphere and is usually referred to as a k-dimensional PL-sphere. For each simplex A in a simplicial complex K, the link of A in K consists of those simplices in K not meeting A which are faces of simplices containing A.

Given two simplicial complexes, K and L, a map $f: K \to L$ is called simplicial if, for any collection of vertices in K spanning a simplex, their images in L span a simplex. For more details, see Hilton–Wylie [74].

Simplicial homology. An important special case of *cellular homology* is simplicial homology, where X is a *simplicial complex* and each q-simplex is regarded as a q-cell. Because of the large number of simplices needed to triangulate even a fairly simple space, simplicial homology is not very useful for explicit computation. However, because the characteristic maps are embeddings, simplicial homology is much easier to use in proofs. For example, the definition of the boundary homomorphisms ∂_q is much more transparent.

Singular homology. Singular homology is an abstraction of *simplicial homology* where simplices are replaced by singular simplices. A singular q-simplex in X is a continuous map $\sigma: \Delta_q \to X$ where Δ_q is a fixed standard q-simplex. The singular q-simplices form a basis for the R-module of singular chains $C_q(X, R)$, which is uncountably generated for most spaces X. This is a computational disadvantage, but note that singular homology is defined for any space X; the rather nice structure of *CW-complex* or *simplicial complex* need not be present.

Transversality. Let M and N be smooth manifolds of dimensions m and n respectively, and let L be a k-dimensional submanifold of N. A smooth map $f: M \to N$ is said to be transversal to L if for any point $x \in M$ with $f(x) \in L$, the subspaces $T_{f(x)}L$ and $(d_x f)(T_x M)$ span the tangent space $T_{f(x)}N$. This captures the idea that $f(M)$ and L cut across each other as much as possible. If $m + k < n$ then f is transversal to L only when $f(M) \cap L$ is empty. It follows from the implicit function theorem that if f is transversal to L, then $f^{-1}(L)$ is a smoothly *embedded submanifold* of M, of codimension $n - k$. If M is compact, then $f^{-1}(L)$ is also a compact submanifold of M.

A point $y \in N$ is called a regular value of a smooth map $f: M \to N$ if f is transversal to y. In other words, y is regular if, for any point $x \in M$ such that $f(x) = y$, the linear map $d_x f: T_x M \to T_y N$ is onto. Each x is then called a regular point of the map f.

Here is a transversality theorem proved by R. Thom, see Bröcker–Jänich [23] or Guillemin–Pollack [63]. Let $f: M \to N$ be a smooth map, and let L be a smooth submanifold of N. Then f can be arbitrarily closely approximated by maps $g: M \to$

N which are transversal to L. If f is a continuous map it still can be approximated by transversal maps, by combining the transversality theorem with the following smooth approximation theorem. Let $f: M \to N$ be a continuous map which is smooth on an open neighborhood U of a closed set A. Then, arbitrarily close to f, there exists a smooth map $h: M \to N$ with $h|_A = f|_A$.

All of these results have analogues for manifolds with boundary, which can be found in the books cited above.

Tubular neighborhood. Let $M \subset N$ be a submanifold, and let m and n be respectively the dimensions of M and N. A tubular neighborhood of M in N is the image of an embedding $t: \nu(M) \to N$, where $\nu(M) \to M$ is a closed disc bundle associated with a *vector bundle* over M of rank $n - m$, and $t(x, 0) = x$ whenever $x \in M$. The tubular neighborhood theorem, see for instance Guillemin–Pollack [63], asserts that every smoothly embedded submanifold $M \subset N$ has a tubular neighborhood, and that the bundle in question can be chosen to be a normal bundle of $M \subset N$.

Universal Coefficient Theorem. The version of the universal coefficient theorem that we will use asserts that, for every space X and every Abelian group A, there are the following exact sequences for all $n \geq 0$:

$$0 \to \mathrm{Ext}(H_{n-1}(X), A) \to H^n(X; A) \to \mathrm{Hom}(H_n(X), A) \to 0.$$

Each of these sequences splits, that is, there are isomorphisms for all $n \geq 0$,

$$H^n(X; A) \cong \mathrm{Hom}(H_n(X), A) \oplus \mathrm{Ext}(H_{n-1}(X), A),$$

which however need not be natural. As a corollary, $H^1(X; \mathbb{Z}) = \mathrm{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z})$. Any textbook in algebraic topology has a treatment of this theorem, see for instance Hatcher [71] or Spanier [150].

Whitehead Theorem. Let X and Y be connected *CW-complexes*. If a continuous map $f: X \to Y$ induces isomorphisms on all homotopy groups, then f is a homotopy equivalence. An important special case occurs when X is a connected CW-complex and its homotopy groups $\pi_q(X)$ vanish for all $q \geq 1$. Taking Y to be a single point, and f a constant map, the Whitehead theorem shows that X is homotopy equivalent to Y and hence X is contractible. In fact, one can show with the help of the *Hurewicz theorem* that if X is a CW-complex such that $\pi_1(X) = 0$ and $H_q(X; \mathbb{Z}) = 0$ for all $q \geq 1$ then X is contractible.

Lecture 1

Heegaard splittings

1.1 Introduction

Let M_1 and M_2 be compact 3-dimensional manifolds with homeomorphic boundaries, and $f: \partial M_1 \to \partial M_2$ a homeomorphism. By gluing M_1 to M_2 along f we get a new compact 3-dimensional manifold, $M = M_1 \cup_f M_2$, with empty boundary.

Example. Two 3-dimensional balls glued by any homeomorphism of their boundaries produce the 3-dimensional sphere (for a proof, see Theorem 1.4).

In general, the result of this gluing operation depends on the homeomorphism f.

Example. Gluing of two copies of the solid torus $S^1 \times D^2$ results in $S^1 \times S^2$ if f is the identity. However, if $f: S^1 \times S^1 \to S^1 \times S^1$ is a homeomorphism of the boundary torus interchanging the two copies of S^1 the resulting manifold is S^3. The latter is established with the help of the formula $\partial(X_1 \times X_2) = (\partial X_1 \times X_2) \cup (X_1 \times \partial X_2)$. If $X_1 = X_2 = D^2$ are 2-dimensional discs, we get

$$S^3 = \partial D^4 = \partial(D^2 \times D^2) = (S^1 \times D^2) \cup (D^2 \times S^1),$$

which is a decomposition of the 3-sphere into two solid tori glued along their common boundary, the torus $S^1 \times S^1$.

It is useful to keep in mind the following 3-dimensional picture of the 3-sphere decomposition we just constructed.

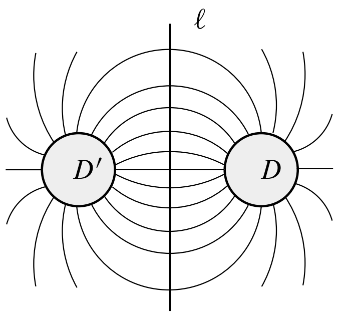


Figure 1.1

In Figure 1.1, the sphere S^3 is represented as the result of revolving the 2-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ about the circle $\ell \cup \{\infty\}$ where ℓ is a straight line in $\mathbb{R}^2$. Under this revolution, the disc $D \subset \mathbb{R}^2 \setminus \ell$ generates a solid torus M_1. Each of the arcs connecting the discs D and D' generates a 2-dimensional disc in S^3, the set of all such discs being parametrized by the points of the circle $\ell \cup \infty$. Therefore, the complement of the solid torus M_1 is another solid torus M_2, and $S^3 = M_1 \cup M_2$.

1.2 Existence of Heegaard splittings

A *handlebody* is an orientable 3-dimensional manifold obtained from the 3-ball D^3 by attaching g copies of 1-handles $D^2 \times [-1, 1]$. The gluing homeomorphisms match the $2g$ discs $D^2 \times \{\pm 1\}$ with $2g$ disjoint 2-discs in $\partial D^3 = S^2$ so that the resulting manifold is orientable, see Figure 1.2.

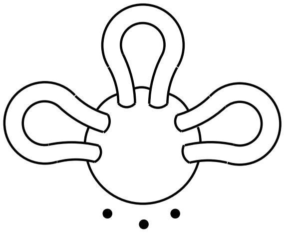


Figure 1.2

The integer g is called the *genus* of a handlebody. The boundary of a handlebody of genus g is homeomorphic to a Riemann surface of genus g.

It turns out that any closed orientable 3-manifold M can be obtained by gluing together two handlebodies. In other words, M can be represented as $M = H \cup H'$ where H and H' are handlebodies such that $H \cap H' = \partial H = \partial H'$. Obviously, the handlebodies must have the same genus, say g. Such a decomposition of the manifold M is called a *Heegaard splitting* of M of genus g.

Theorem 1.1. *Any closed orientable* 3*-manifold admits a Heegaard splitting.*

Proof. Let T be a triangulation of a closed orientable 3-manifold M. We associate with T a Heegaard splitting of M as follows. Let us replace each vertex of T by a ball, each edge by a cylinder, each side of a tetrahedron by a "plate", and each tetrahedron by a ball, see Figure 1.3.

The union $H(T)$ of the vertex balls and the cylinders is a handlebody, and so is the union $H'(T)$ of the tetrahedra balls and plates. Therefore, $M = H(T) \cup H'(T)$ is a Heegaard splitting of the manifold M. □

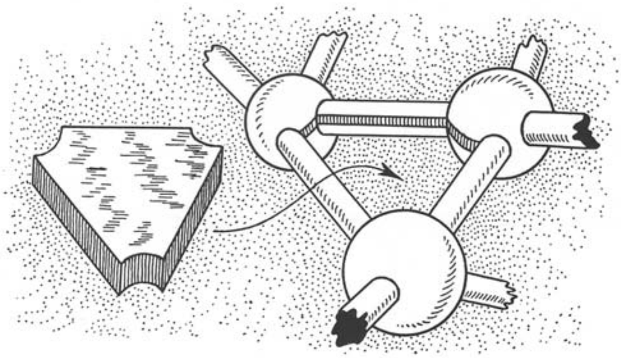


Figure 1.3

1.3 Stable equivalence of Heegaard splittings

Given a Heegaard splitting $M = H_g \cup H'_g$ of genus g, one can easily construct another Heegaard splitting of M of genus $g + 1$ as follows. Add an unknotted 1-handle B to H_g to get a handlebody H_{g+1} of genus $g + 1$. Here, we call a handle unknotted if there is a 2-disc D in M such that $D \cap H_{g+1} = \partial D$ and the curve ∂D goes along B only once, see Figure 1.4.

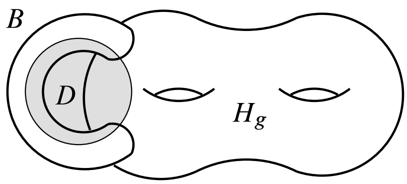


Figure 1.4

Next, we thicken the disc D to get $C = D \times I$. Note that $B \cup C$ is homeomorphic to a 3-ball, hence

$$M \cong H_g \cup (B \cup C) \cup H'_g = (H_g \cup B) \cup (C \cup H'_g),$$

where $H_g \cup B = H_{g+1}$. The thickened disc C intersects the handlebody H'_g in two discs, therefore, $C \cup H'_g = H'_{g+1}$ is a handlebody of genus $g + 1$, and $M = H_{g+1} \cup H'_{g+1}$ is a Heegaard splitting of genus $g + 1$.

The operation described above is called *stabilization*. For instance, the genus 1 splitting of the 3-sphere described in Figure 1.1 can be obtained by stabilization from a genus 0 splitting. Two Heegaard splittings of a manifold M are called *equivalent* if there exists a homeomorphism of M onto itself taking one splitting into the other, and *stably equivalent* if they are equivalent after applying stabilization to each of them a certain number of times.

The following result was proved by Singer [149], see also Reidemeister [133]. Our proof follows closely the Fomenko–Matveev book [49].

Theorem 1.2. *Any two Heegaard splittings of a closed orientable* 3*-manifold* M *are stably equivalent.*

Proof. We will prove that (1) any two Heegaard splittings associated to triangulations as in the proof of Theorem 1.1 are stably equivalent, and (2) any Heegaard splitting is stably equivalent to a Heegaard splitting associated to a triangulation.

Let T be a triangulation of M. A triangulation T' of M is called a subdivision of T if each simplex of T' is contained in a certain simplex of T. A simple way to construct subdivisions is as follows: we pick a point a in M, leave the simplices not containing a unchanged, and subdivide each simplex containing a as shown in Figure 1.5. In the picture on the bottom of Figure 1.5, the point a sits in the interior of the tetrahedron, while in the top right picture it belongs to its bottom face. Such a subdivision is called a star subdivision of T with the vertex a. According to Alexander [4], any two triangulations of M have a common subdivision obtained from each of them by a sequence of star subdivisions.

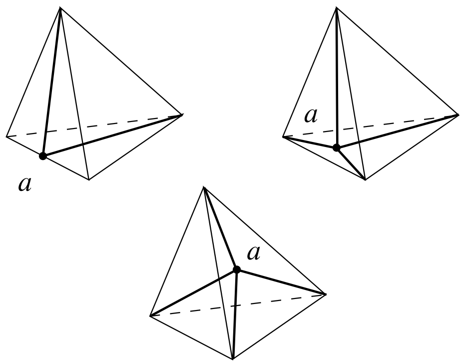


Figure 1.5

Observe that if a triangulation T' is obtained by a star subdivision from a triangulation T, then the corresponding Heegaard splitting $M = H(T') \cup H'(T')$ is obtained from $M = H(T) \cup H'(T)$ by a sequence of stabilizations. One can see in Figure 1.6 that if the point a sits inside a tetrahedron, the handlebody $H(T')$ is obtained from the handlebody $H(T)$ by adding three unknotted handles. If a is on a side, we need four stabilizations, and if a is on an edge, the number of stabilization operations equals the number of tetrahedra containing that edge.

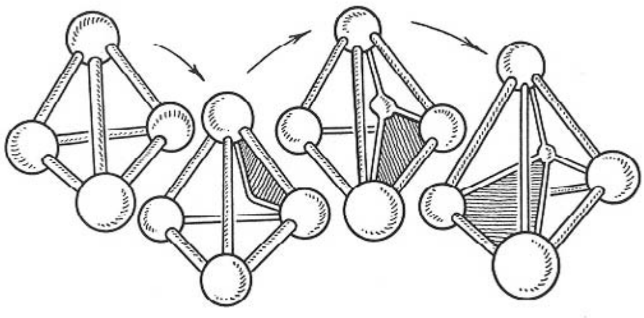


Figure 1.6

Together with the fact that any two triangulations are related by a sequence of star subdivisions, this observation proves claim (1).

To prove claim (2) we will need a couple of technical results. Let K be a one-dimensional subcomplex of a triangulation of some 3-manifold. We denote by $U(K)$ the union of those balls and cylinders which correspond to the vertices and edges of K. The space $U(K)$ is a handlebody.

Let H_g be a handlebody of genus g as shown in Figure 1.2, and $\Gamma \subset H_g$ its axial graph. By definition, the graph Γ is a collection of g circles in H_g intersecting in exactly one point. It is obtained by contracting each of the handles of H_g to an arc, and then contracting the central ball to the point to which all the arcs are attached. We claim that there is a triangulation τ of H_g such that Γ is a subcomplex and the following two conditions are satisfied:

(a) the handlebody $U(\tau^{(1)})$, where $\tau^{(1)}$ is the 1-skeleton of τ with $\Gamma \subset \tau^{(1)}$, is obtained from $U(\Gamma)$ by adding unknotted handles;

(b) the handlebody $U(\tau^{(1)})$ is obtained by adding unknotted handles to $U(\partial\tau^{(1)})$ where $\partial\tau^{(1)}$ is the 1-skeleton of the restriction of τ to ∂H_g.

It is very easy to construct such a triangulation – almost any will do. For instance, one can represent H_g as a 2-disc with g holes times an interval and use the product triangulation. It should be mentioned that properties (a) and (b) of a triangulation are preserved when one passes to a star subdivision. It follows then from the existence of a common star subdivision that any triangulation has a subdivision satisfying (a) and (b). However, in what follows, we will not use this fact in its full generality.

Let $M = H \cup H'$ be an arbitrary Heegaard splitting of M. Choose a triangulation T of M in which both H and H' are subcomplexes. Let τ and τ' be the restrictions of T on H and H', respectively. Let Γ be an axial graph of H. By subdividing T if necessary, we may assume that τ satisfies (a) and τ' satisfies (b). Then $U((\tau')^{(1)})$ is obtained from $U((\partial\tau')^{(1)})$ by adding unknotted handles. By adding the same handles to $U(\tau^{(1)})$ we get $U(T^{(1)})$. The handlebody $U(\tau^{(1)})$ in turn is obtained by adding unknotted handles to $U(\Gamma)$. In short, we have the following diagram

$$U(\Gamma) \to U(\tau^{(1)}) \to U(T^{(1)}) = H(T)$$

where $H(T)$ is a handlebody in the Heegaard splitting of $M = H(T) \cup H'(T)$ constructed from the triangulation T as in the proof of Theorem 1.1, and the arrows represent adding unknotted handles. The original Heegaard splitting $M = H \cup H'$ is equivalent to the Heegaard splitting of M into $U(\Gamma)$ and its complementary handlebody. The latter is stably equivalent to the splitting $M = H(T) \cup H'(T)$. □

1.4 The mapping class group

In a Heegaard splitting $H \cup_f H'$, handlebodies H and H' are glued along their common boundary F by a homeomorphism $f : F \to F$. One can always orient H and H' in such a way that the homeomorphism f is either orientation preserving or orientation reversing. We choose the latter, keeping in mind the following example.

Example. Let $S^3 = H \cup H'$ be the Heegaard splitting of S^3 of genus 1 shown in Figure 1.1. The standard orientation of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ given by the basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$, induces orientations on both H and H'. Orient the boundary $\partial H = T^2$ of H by choosing a basis a, b in its tangent space so that the triple a, b, n is positively oriented where n is an inner normal vector to T^2 in H. Similarly, orient $\partial H'$ by its inner normal vector with respect to H'. Since n is an outer normal vector with respect to H', the orientations that T^2 gets as the boundary of H and H' are opposite to each other. Therefore, the gluing map is orientation reversing.

Recall that two homeomorphisms $f_0, f_1 : F \to F$ are called *isotopic* if there is a homotopy f_t, $0 \le t \le 1$, between them such that each f_t is a homeomorphism. Note that if f preserves (or reverses) orientation, so do all homeomorphisms isotopic to f. Gluing H and H' by isotopic homeomorphisms produces homeomorphic manifolds. This observation justifies the following definition.

Let $\text{Homeo}(F)$ be the group of all orientation preserving homeomorphisms of a closed oriented surface F, and let $\text{Homeo}_0(F)$ be the normal subgroup of homeomorphisms isotopic to the identity. The quotient group

$$H(F) = \text{Homeo}(F) / \text{Homeo}_0(F)$$

is called the *mapping class group* of the surface F. This group is a subgroup of the larger group of all homeomorphisms of F modulo isotopy. As a subgroup, it has index 2 because the composition of any two orientation reversing homeomorphisms is orientation preserving. Next, we will describe a set of generators in $H(F)$.

Let c be a simple closed curve, i.e. an embedded circle, in F and pick an annulus $U(c)$ one of whose boundary components is c, see Figure 1.7. Let us identify $U(c)$ with the annulus $\{z \mid 1 \leq |z| \leq 2\}$ in the complex plane, and define a *Dehn twist* $\tau_c \colon F \to F$ along c as the homeomorphism given by the formula

$$r \cdot e^{i\phi} \mapsto r \cdot e^{i(\phi + 2\pi(r-1))} \tag{1.1}$$

inside $U(c)$, and equal to the identity outside. A less formal way to think of τ_c is as follows. Cut F along c, twist one of the ends through 360° in one of the possible two directions, and glue the ends back together.

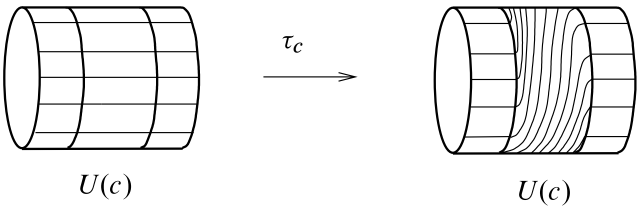


Figure 1.7

Another choice of $U(c)$ or replacing the curve c by an isotopic curve give isotopic twists. It is worth mentioning here that any two non-trivial homotopic simple closed curves on a surface F are isotopic, see Baer [10] and [11], and also Epstein [42]. The choice of the twist direction is essential – twists in opposite directions define elements in $H(F)$ which are the inverses of each other.

Dehn twists were introduced by Dehn [34]. A proof of the following theorem can be found in Lickorish [104].

Theorem 1.3. *Let F_g be a closed orientable surface of genus g. Then the group $H(F_g)$ is generated by the Dehn twists along the curves α_i, β_j, γ_k, $1 \leq i, j \leq g$, $1 \leq k \leq g-1$, pictured in Figure* 1.8.

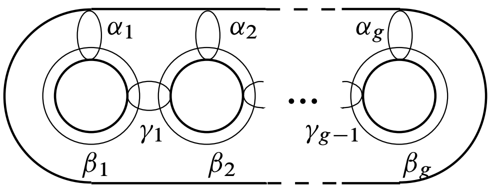


Figure 1.8

1.5 Manifolds of Heegaard genus ≤ 1

Let M be a closed orientable 3-manifold. We say that M has *Heegaard genus* g if it admits a Heegaard splitting of genus g and does not admit Heegaard splittings of smaller genus.

The cases $g = 0$ and $g = 1$ deserve special treatment. If $g = 0$, the splitting surface is a 2-dimensional sphere. The following theorem is due to Alexander [4].

Theorem 1.4. *The only closed* 3*-manifold of Heegaard genus* 0 *is* S^3.

Proof. Let us think of D^3 as the unit ball $|\mathbf{r}| \leq 1$ in $\mathbb{R}^3$, and S^2 its boundary. Any homeomorphism $f: S^2 \to S^2$ can be extended to a homeomorphism $F: D^3 \to D^3$ with the help of the formula $F(t \cdot \mathbf{r}) = t \cdot f(\mathbf{r}),\ 0 \leq t \leq 1$ (this result is usually referred to as Alexander's lemma). Now, let M be a manifold of Heegaard genus 0, so that $M = D_1 \cup D_2$ with $D_1 = D_2 = D^3$. There exists a homeomorphism carrying D_1 to the upper hemisphere of S^3. This homeomorphism can be extended to a homeomorphism from M onto S^3 with the help of Alexander's lemma. □

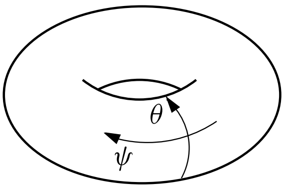


Figure 1.9

Let $T^2 = S^1 \times S^1$ be a 2-torus, and choose generators in $\pi_1 T^2 = \mathbb{Z} \oplus \mathbb{Z}$ as follows. Think of T^2 as the boundary of a solid torus $S^1 \times D^2$ embedded in $\mathbb{R}^3$ as shown in Figure 1.9. Let θ and ψ be the standard angle coordinates on T^2. The curves μ and λ given respectively by the equations $\psi = 0$ and $\theta = 0$ are called the *meridian* and *longitude*. They play a different role with respect to the solid torus in that μ bounds a disc in $S^1 \times D^2$ while λ does not. These curves together give a set of generators of the group $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. Orient the torus by choosing the basis $(\partial/\partial\psi, \partial/\partial\theta)$ in the tangent space.

The mapping class group of the torus can be described explicitly as follows. The π_1 functor converts a homeomorphism f of T^2 into a group automorphism f_* of $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. A homeomorphism of T^2 isotopic to identity induces the identity map on $\pi_1(T^2)$. The automorphisms of $\mathbb{Z} \oplus \mathbb{Z}$ are given by the integral 2×2-matrices invertible over the integers. A matrix is invertible over the integers if and only if its determinant is ± 1. The matrix of f_* corresponds to an orientation preserving homeomorphism f if and only if its determinant equals 1. Therefore, we have a well-defined homomorphism

$$\Pi\colon H(T^2) \to \mathrm{SL}(2,\mathbb{Z})$$

into the group $\mathrm{SL}(2,\mathbb{Z})$ of integral 2×2-matrices with determinant 1. Each matrix in $\mathrm{SL}(2,\mathbb{Z})$ can be reduced to the identity by elementary transformations on its rows and columns. Hence, each $A \in \mathrm{SL}(2,\mathbb{Z})$ is a product of matrices of the form

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}.$$

These matrices are realized by the twists along the curves μ and λ. Thus, Π is surjective. One can show, see e.g. Rolfsen [137], Theorem 2.D.4, that Π is also injective. Therefore, we obtain the following result.

Theorem 1.5. *The map* $\Pi\colon H(T^2) \to \mathrm{SL}(2,\mathbb{Z})$ *is an isomorphism.*

The isotopy classes of orientation reversing homeomorphisms $f\colon T^2 \to T^2$ are in one-to-one correspondence with the integral 2×2-matrices of determinant -1. Such matrices are of the form $\tau \cdot A$, where $A \in \mathrm{SL}(2,\mathbb{Z})$ and

$$\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

is realized by the torus homeomorphism $(\psi,\theta) \mapsto (\psi,-\theta)$.

Now we can describe the 3-manifolds of Heegaard genus 1. Let M be a manifold obtained by gluing two solid tori by an orientation reversing homeomorphism $f\colon T^2 \to T^2$ of their boundaries. In the meridian-longitude bases on the two tori, (μ_1,λ_1) and (μ_2,λ_2), the homeomorphism f corresponds to a matrix

$$A = \begin{pmatrix} -q & s \\ p & r \end{pmatrix} \quad \text{with} \quad qr + ps = 1. \tag{1.3}$$

In particular, the image of the meridian μ_1 of the first torus is isotopic to the curve $-q\cdot\mu_2 + p\cdot\lambda_2$, which winds $-q$ times in the θ_2-direction and p times in the ψ_2-direction on the second torus.

Lemma 1.6. *The image of* μ_1 *completely determines the manifold* M.

Proof. To see this we notice that the solid torus $D^2\times S^1$ can be attached in two steps. First we glue in $D^2 \times J$ where J is a small segment of S^1, see Figure 1.10. The entire solid torus can be represented as

$$D^2 \times S^1 = (D^2 \times J) \cup D^3,$$

hence to get M we only need to attach the 3-ball along its boundary $\partial D^3 = S^2$. All orientation preserving homeomorphisms of S^2 are isotopic to identity, which completes the proof. □

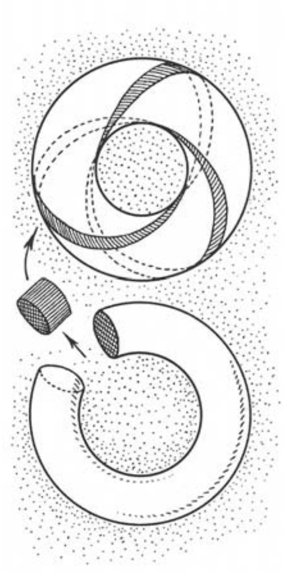


Figure 1.10

The manifold M is thus completely determined by p and q. This manifold is called the *lens space* $L(p,q)$. The condition $qr + ps = 1$ on the matrix (1.3) implies that p and q are relatively prime. One can easily check that $\pi_1 L(p,q) = \mathbb{Z}/p$.

Different pairs (p,q) may give homeomorphic lens spaces $L(p,q)$. This is due in the first place to an ambiguity in the choice of basis curves on T^2. On the one hand, the meridians μ_1 and μ_2 are determined uniquely (up to isotopy and change of orientation) by the condition that they bound a 2-disc. If we change the orientation of μ_1, we will have to change the orientation of λ_1 as well. This operation replaces A by $-A$. Therefore, we may assume that $p \geq 0$.

On the other hand, the choice of longitude is very far from being unique - any curve of the form $n \cdot \mu_1 + \lambda_1$ is as good as λ_1 since it maps to λ_1 by n Dehn twists along μ_1. The effect of replacing λ_1 by $n \cdot \mu_1 + \lambda_1$ is in adding n times first column of A to its second column; similarly, the effect of replacing λ_2 by $n \cdot \mu_2 + \lambda_2$ is in subtracting n times second row of A from its first row.

Now, if $p = 0$, one may assume that $A = \tau$ is the matrix (1.2). The corresponding lens space $L(0,1)$ is simply $S^1 \times S^2$. Suppose that $p \neq 0$, then one can make q non-negative and less than p, i.e. $0 \leq q \leq p - 1$. If $p = 1$ then $q = 0$, so one can assume that

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $L(1,0) = S^3$, which is a manifold of Heegaard genus 0. Finally, if $p \geq 2$, then $1 \leq q \leq p - 1$, and we obtain the following result.

Theorem 1.7. *Any* 3*-dimensional manifold of genus* 1 *is either* $S^1 \times S^2$ *or a lens space* $L(p,q)$ *with* p *and* q *relatively prime,* $p \geq 2$*, and* $1 \leq q \leq p-1$.

To complete the picture we note that lens spaces with different p's are not homeomorphic – not even homotopy equivalent – since their fundamental groups are not isomorphic. At the same time, lens spaces $L(p,q)$ and $L(p,q')$ with different q and q' may be homotopy equivalent and even homeomorphic.

Example. For any relatively prime p and q the lens spaces $L(p,q)$ and $L(p,-q)$ are homeomorphic (via an orientation reversing homeomorphism). To see this, we simply change the orientations of the both solid tori in the construction of $L(p,q)$, e.g. by reversing the orientations of the longitude λ_1 and the meridian μ_2. In this new basis, the matrix (1.3) is replaced by the matrix

$$\begin{pmatrix} q & s \\ p & -r \end{pmatrix}.$$

Example. If we exchange the roles of the two solid tori in the construction of $L(p,q)$ the matrix (1.3) will be replaced by its inverse,

$$\begin{pmatrix} -r & s \\ p & q \end{pmatrix},$$

with $qr = 1 \bmod p$. Together with the result from the preceding example this implies that lens spaces $L(p,q)$ and $L(p,q')$ with $qq' = \pm 1 \bmod p$ are homeomorphic. For example, $L(7,2)$ and $L(7,3)$ are homeomorphic. In fact, it is also true that the lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic if and only if $q' = \pm q^{\pm 1} \bmod p$; this was first proved in Reidemeister [132].

Example. It is worth mentioning in conclusion, even without a proof, that $L(p,q)$ and $L(p,q')$ are homotopy equivalent if and only if $qq' = \pm m^2 \bmod p$ for some integer m, see Whitehead [159]. For example, $L(5,1)$ and $L(5,2)$ are not homotopy equivalent, and $L(7,1)$ and $L(7,4)$ are homotopy equivalent but not homeomorphic.

Remark. In these notes we have tried to follow the orientation conventions for lens spaces of Raymond [131] and Hirzebruch–Neumann–Koch [76]. However, the opposite orientation conventions are often used. Some authors, for instance Rolfsen [137], take care of this problem by working with manifolds which are orientable but not oriented.

1.6 Seifert manifolds

The construction of lens spaces can be generalized as follows. Let $F = S^2 \setminus \operatorname{int}(D_1^2 \cup \dots \cup D_n^2)$ be a 2-sphere with the interiors of n disjoint discs removed. The product

$F \times S^1$ is a compact orientable 3-manifold whose boundary consists of n tori $(\partial D_i^2) \times S^1$, $i = 1, \ldots, n$. The fundamental group of $F \times S^1$ has a presentation

$$\langle\, x_1, \ldots, x_n, h \mid hx_i = x_i h,\ x_1 \ldots x_n = 1 \,\rangle,$$

where the generators x_i are represented by the curves ∂D_i^2 oriented as the boundary curves of F. Suppose we are given n pairs of relatively prime integers, (a_i, b_i), $i = 1, \ldots, n$, with $a_i \geq 2$. We glue in n solid tori so that the meridian of the i-th solid torus is glued to a curve on $(\partial D_i^2) \times S^1$ isotopic to $a_i \cdot x_i + b_i \cdot h$. For each i, the image under this gluing of the curve $\{0\} \times S^1 \subset D^2 \times S^1$ is called the i-th *singular fiber*.

The closed manifold that we obtain by this construction is called the *Seifert manifold* $M((a_1, b_1), \ldots, (a_n, b_n))$ of genus 0 with n singular fibers. The "genus 0" here refers not to the Heegaard genus but rather to the fact that the genus of the 2-sphere used in the construction is zero, and that the construction can be generalized by replacing S^2 with any closed orientable surface of genus g.

We do not make an attempt at fixing an orientation on a Seifert manifolds – this will be done in Lecture 2.

Example. A Seifert manifold $M(a, b)$ with one singular fiber is a lens space $L(b, a)$. A Seifert manifold $M((a_1, b_1), (a_2, b_2))$ with two singular fibers is also a lens space.

If a Seifert manifold M has at least three singular fibers, it is not homeomorphic to a lens space. This can be seen, for example, from the fact that the fundamental group of M is not Abelian. Therefore, the Heegaard genus of M is at least 2.

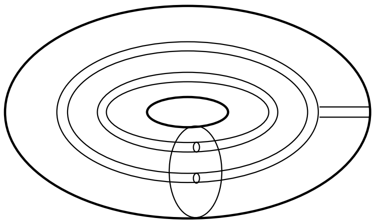


Figure 1.11

In fact, the Heegaard genus of any Seifert manifold $M((a_1, b_1), (a_2, b_2), (a_3, b_3))$ with three singular fibers equals 2. A Heegaard splitting of genus 2 can be constructed as follows. Choose first two of the three solid tori that were glued into $F \times S^1$ in the construction above, and connect them by an unknotted solid tube inside M to get a handlebody of genus 2. Its complement in M is shown in Figure 1.11. It is a solid torus with two parallel circular tunnels drilled, plus a short tunnel connecting one of these tunnels with the "outside world". The other tunnel is filled according to the (a_3, b_3)-rule. The short tunnel can be stretched so that the picture will look like a filled thickened torus with a handle. The filled torus is homeomorphic to a regular solid torus. Thus, we get a Heegaard splitting of M of genus 2. For more details of this construction see Lecture 19.

A similar construction shows that the Heegaard genus of a Seifert manifold with $n \geq 2$ singular fibers is at most $n - 1$.

Remark. In conclusion we mention that Seifert manifolds admit a fixed point free action of the circle S^1 so that the singular fibers as defined above are the only orbits of this action with non-trivial isotropy group: the circle wraps a_i times in the direction of the i-th singular fiber. For a general theory of Seifert manifolds we refer the reader to Neumann–Raymond [122] or Lee–Raymond [98], see also Lecture 19.

1.7 Heegaard diagrams

Lemma 1.6 can be extended to Heegaard splittings $M = H \cup H'$ of arbitrary genus $g \geq 1$ as follows. Let H be a handlebody obtained by attaching g copies of 1-handles $D_i^2 \times [-1, 1]$ to the 3-ball as shown in Figure 1.2. The central discs $D_i^2 \times \{0\}$ of these handles have collars $D_i^2 \times J$ whose removal from H leaves us with a 3-ball. Attach H to H' by attaching these collars first so that the curves $\partial D_i^2 \times \{0\}$, $i = 1, \dots, g$, are identified with g disjoint simple closed curves $\alpha_1, \dots, \alpha_g$ in the boundary of H'. What is left to attach is the 3-ball; this can be done in a way which is unique up to isotopy.

This construction shows that the collection of curves $\alpha_1, \dots, \alpha_g$ in $\partial H'$ as above completely determines M. This collection, together with the handlebody H', is referred to as a *Heegaard diagram* for M. Note that the homology classes of curves $\alpha_1, \dots, \alpha_g$ are linearly independent in $H_1(\partial H')$, being the images of g linearly independent classes $[\partial D_i^2 \times \{0\}] \in H_1(\partial H)$ under an isomorphism. This linear independence is equivalent to the open surface $\partial H' \setminus \{\alpha_1 \cup \dots \cup \alpha_g\}$ being connected, see Exercise 10.

Example. Figure 1.12 shows a Heegaard diagram of genus two. It consists of disjoint curves α_1 and α_2 on the surface $\partial H'$ and a choice of handlebody to call H' (there are two choices for H', the inner and the outer one with respect to the embedding $\partial H' \subset S^3$ as shown).

In the previous example, a particular embedding of H' in S^3 enabled us to draw a Heegaard diagram on its surface. Lacking such an embedding, one can rely on the following description of an abstract handlebody.

Let F be a closed orientable surface of genus g, and $\beta_1, \dots, \beta_g \subset F$ a family of disjoint simple closed curves whose homology classes are linearly independent in $H_1(F)$. Attach g copies of 2-handles to $F \times [0, 1]$ using curves $\beta_1 \times \{1\}, \dots, \beta_g \times \{1\}$. This gives us a 3-manifold with two boundary components. One boundary component

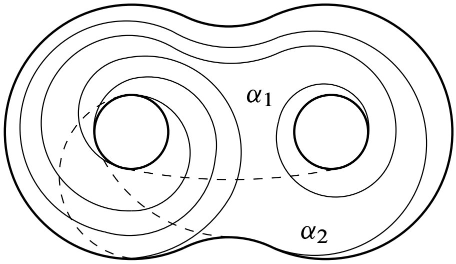


Figure 1.12

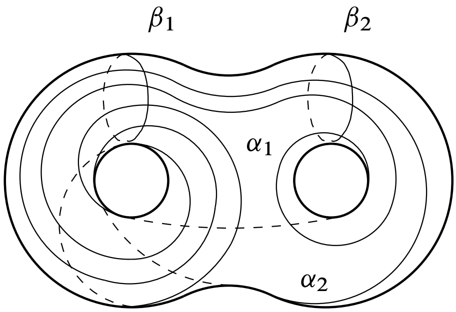


Figure 1.13

is F, and the other has Euler characteristic two and hence is a 2-sphere. Attaching the 3-ball results in a handlebody with boundary F.

Using this description, a Heegaard diagram for a 3-manifold M will consist of a surface F of genus g and two g-tuples of curves $(\alpha_1, \dots, \alpha_g)$ and $(\beta_1, \dots, \beta_g)$ in F called respectively *α-curves* and *β-curves*. Each g-tuple must consist of disjoint simple closed curves whose homology classes are linearly independent in $H_1(F)$, but there are no conditions on how the two g-tuples interact. The manifold $M = H \cup H'$ is recovered from this data by using α-curves as the attaching curves for H and β-curves as the attaching curves for H'.

Example. The choice of the inner handlebody in Figure 1.12 gives rise to the Heegaard diagram as shown in Figure 1.13.

Example. Figures 1.14, 1.15 and 1.16 show Heegaard diagrams of genus 1 which are drawn, for the sake of convenience, in a plane from which two disjoint open discs have been removed. To get back the surface of genus one from such a plane, we compactify it by one point at infinity and glue the two boundary circles together using the obvious mirror reflection. Figure 1.14 shows two Heegaard diagrams of S^3, Figure 1.15 five Heegaard diagrams of $S^1 \times S^2$, and Figure 1.16 two Heegaard diagrams of $L(3,1)$ (with unspecified orientation).

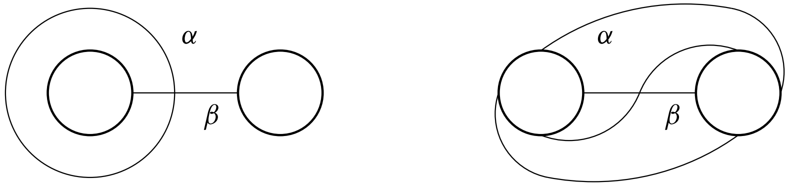


Figure 1.14

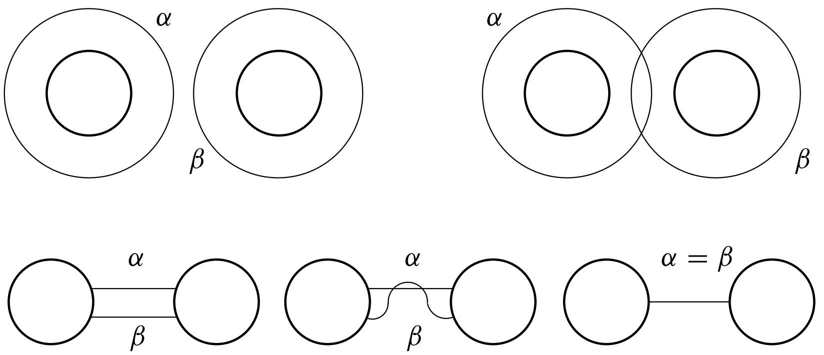


Figure 1.15

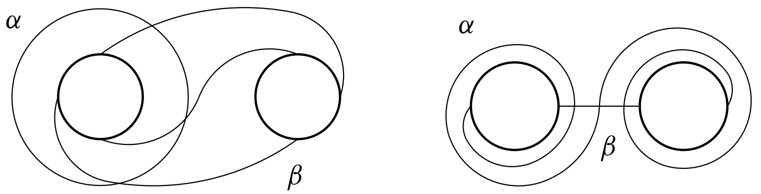


Figure 1.16

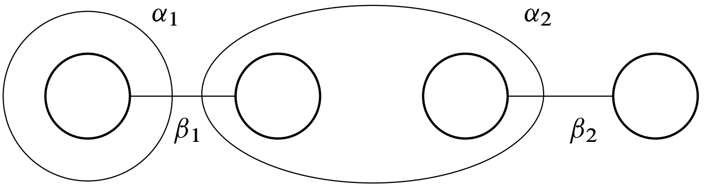


Figure 1.17

Example. Shown in Figure 1.17 is a Heegaard diagrams of S^3 of genus two, with the same convention about gluing the two boundary circles on the left together using the mirror reflection, and doing the same for the two circles on the right.

Heegaard diagrams of the type described above, with α- and β-curves, feature prominently in the theory of Heegaard Floer homology, see Ozsváth–Szabó [127] for an introduction. They will not be essential, however, for the rest of this book.

1.8 Exercises

1. Determine the homeomorphism classes of compact 3-manifolds obtained from D^3 by identifying finitely many pairs of disjoint discs in its boundary.
2. Let F be an orientable compact surface of genus g with $\partial F = S^1$. Show that $F \times I$ is a handlebody. What is its genus?
3. Use Poincaré–Lefschetz duality to prove the following about the Euler characteristic χ:
   (a) $\chi(M) = 0$ for any closed orientable 3-manifold M;
   (b) $2\chi(M) = \chi(\partial M)$ for any compact orientable 3-manifold M with boundary ∂M.
4. Prove that if a closed orientable 3-manifold M has Heegaard genus g then its fundamental group has a presentation with g generators and g relations.
5. Prove that for any integer $g \geq 0$ there is a closed orientable 3-manifold of Heegaard genus g.
6. Show that Heegaard genus of $S^1 \times S^1 \times S^1$ equals three.
7. Use van Kampen's theorem to calculate $\pi_1 L(p,q)$.
8. What is the Heegaard genus of a connected sum of two lens spaces?
9. Real projective space is the 3-manifold $\mathbb{R}P^3$ obtained from the 3-ball by identifying antipodal points on the boundary. Draw a picture showing that $\mathbb{R}P^3$ is homeomorphic to the lens space $L(2,1)$.
10. Let F be a closed orientable surface of genus g, and $\alpha_1, \dots, \alpha_g$ a collection of disjoint simple closed curves on F. Prove that the homology classes $[\alpha_1], \dots, [\alpha_g] \in H_1(F)$ are linearly independent if and only if the open surface $F \setminus \{\alpha_1 \cup \dots \cup \alpha_g\}$ is connected.
11. Prove that a Seifert manifold $M((a_1,b_1),(a_2,b_2))$ of genus zero with two singular fibers is a lens space, and use van Kampen's theorem to calculate its fundamental group.
12. Let $M = M((a_1,b_1),\dots,(a_n,b_n))$ be a Seifert manifold of genus zero with $n \geq 3$ singular fibers. Calculate $\pi_1 M$ and $H_1 M$.

Lecture 2

Dehn surgery

2.1 Knots and links in 3-manifolds

A finite collection of smoothly embedded disjoint closed curves in a closed orientable 3-manifold M is called a *link*. A one-component link is called a *knot*. We will not distinguish between equivalent knots and links: two links, $\mathcal{L}$ and $\mathcal{L}'$, in M are said to be equivalent if there is a smooth orientation preserving automorphism $h\colon M \to M$ such that $h(\mathcal{L}) = \mathcal{L}'$. In case the links have two or more components, we also assign a fixed ordering of the components and require that h respect the orderings. Every link $\mathcal{L} \subset M$ can be thickened to get its *tubular neighborhood* $N(\mathcal{L})$ which is a collection of smoothly embedded disjoint solid tori, $D^2 \times S^1$, one for each link component, whose cores $\{0\} \times S^1$ form the link $\mathcal{L}$.

Links in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ can be thought of as links in $\mathbb{R}^3$. The requirement that each of the curves of a link be smoothly embedded avoids pathological examples like the one pictured in Figure 2.1.

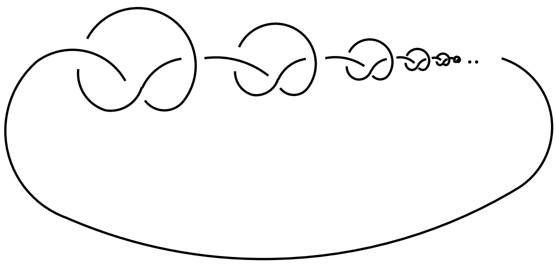


Figure 2.1

Let P be a plane in $\mathbb{R}^3$ and $p\colon \mathbb{R}^3 \to P$ the orthogonal projection. Given a link $\mathcal{L}$ in $\mathbb{R}^3$, we will say that p is a *regular projection* for $\mathcal{L}$ if every line $p^{-1}(x)$, $x \in P$, intersects $\mathcal{L}$ in $0, 1$ or 2 points and the Jacobian $d_y p$ has rank 1 at every intersection point $y \in p^{-1}(x)$. Every link admits a regular projection, see Crowell–Fox [32]. Thus links in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ are often described by their regular projections, and drawn as smooth curves in $\mathbb{R}^2$ with marked undercrossings and overcrossings at each double point.

Any knot in S^3 equivalent to the knot $(\cos t, \sin t, 0)$, $0 \le t \le 2\pi$, is called a *trivial knot* or an *unknot*.

2.2 Surgery on links in S^3

Let k be a knot in a closed orientable 3-manifold M, and $N(k)$ its tubular neighborhood. By cutting the manifold M open along the 2-torus $\partial N(k)$ we get two manifolds – one is the *knot exterior* K which is $M \setminus \operatorname{int} N(k)$, and the other is the solid torus $N(k)$ which we will identify with the standard solid torus $D^2 \times S^1$. Thus K is a manifold with boundary $\partial K = T^2$ and $M = K \cup (D^2 \times S^1)$. One can use an arbitrary homeomorphism $h: \partial D^2 \times S^1 \to \partial K$ to glue $D^2 \times S^1$ back in K. The space we obtain by this construction, $Q = K \cup_h (D^2 \times S^1)$, is a closed orientable 3-manifold. We say that Q is obtained from M by *surgery* along k.

The manifold Q depends on the choice of homeomorphism h. In fact, the manifold Q is completely determined by the image under h of the meridian $\partial D^2 \times \{*\}$ of the solid torus $D^2 \times S^1$, i.e. by the curve $c = h(\partial D^2 \times \{*\})$ on the boundary of K. To see this, one simply repeats the argument that proved Lemma 1.6 from Lecture 1.

If $M = S^3$ then a curve on ∂K is given, up to isotopy, by a pair of relatively prime integers (p, q). The construction is as follows. The space K has integral homology groups $H_0(K) = H_1(K) = \mathbb{Z}$ and $H_i(K) = 0$ if $i \geq 2$. Any meridian of $N(k)$ represents a generator of $H_1(K)$; this is a curve on ∂K which we call m. Up to isotopy, there is a unique longitude which is homologically trivial in K; this gives another curve, ℓ, on ∂K. These two form a basis for $H_1(\partial K)$ which is unique up to isotopy and reversing the orientations of m and ℓ. The longitude ℓ is called a *canonical longitude* to distinguish it from the longitude defined in Lecture 1.

We fix the orientations as follows. Choose the standard orientation on $S^3 = \mathbb{R}^3 \cup \{\infty\}$; it induces an orientation on K. We choose directions on the curves m and ℓ so that the triple $\langle m, \ell, n\rangle$ is positively oriented. Here, n is a normal vector to ∂K pointing inside K, see Figure 2.2.

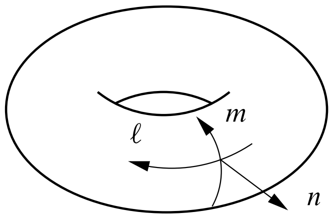


Figure 2.2

Any simple closed curve c on ∂K is now isotopic to a curve of the form $c = p \cdot m + q \cdot \ell$. The pairs (p, q) and $(-p, -q)$ define the same curve c since the orientation of c is of no importance to us. One can conveniently think of a pair (p, q) as a reduced fraction p/q. Then there is a one-to-one correspondence between the set of isotopy classes of non-trivial simple closed curves on the torus ∂K and the set of reduced fractions p/q. This set should be completed by $1/0 = \infty$, which corresponds to the meridian m. The result of $1/0$-surgery on any knot $k \subset S^3$ is again S^3.

Surgeries of the type described above are called *rational*. A surgery is called *integral* if $q = \pm 1$. Similarly, one defines rational and integral surgeries along a link $\mathcal{L} \subset M$: the surgery along each link component should be rational, respectively, integral. In general, surgery along a knot $k \subset M$ cannot be described by a rational number since there is no canonical choice of the longitude (such a choice exists, however, for a homology 3-sphere M, see Section 6.1). Nevertheless, the concept of integral surgery still makes sense: the curve $\partial D^2 \times \{*\}$ on $D^2 \times S^1$ should be attached to a curve on ∂K running exactly once along *a* longitude.

Theorem 2.1 (Lickorish [104] and Wallace [157]). *Every closed orientable 3-manifold M can be obtained from S^3 by an integral surgery on a link $\mathcal{L} \subset S^3$.*

Lemma 2.2. *Let $h_1, h_2 \colon \partial H \to \partial H'$ be homeomorphisms of the surfaces of two handlebodies such that $h_1 = h_2 \tau_c$ where τ_c is a twist along a simple closed curve $c \subset \partial H$. Then the manifold $M_2 = H \cup_{h_2} H'$ is obtained from the manifold $M_1 = H \cup_{h_1} H'$ by an integral surgery along a knot $k \subset M_1$ isotopic to the image of c.*

Proof of Lemma 2.2. We push the curve c inside the handlebody H to get a knot $k \subset H$. Let $N(k)$ be its tubular neighborhood, and $A \cong S^1 \times I$ an annulus connecting c and $\partial N(k)$, see Figure 2.3.

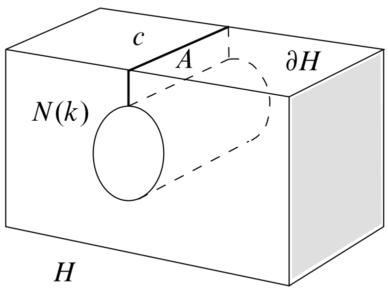


Figure 2.3

Let $\varphi \colon H \setminus \operatorname{int} N(k) \to H \setminus \operatorname{int} N(k)$ be a homeomorphism which cuts the space $H \setminus \operatorname{int} N(k)$ open along the annulus A, twists one of the rims by 360°, and glues it back in. The restriction of the homeomorphism φ to ∂H is the twist τ_c while its restriction to $\partial N(k)$ is a twist along the longitude $A \cap N(k)$ of the knot k. Let $M_i' = (H \setminus \operatorname{int} N(k)) \cup_{h_i} H'$, $i = 1, 2$. The formula

$$\Phi(x) = \begin{cases} \varphi(x), & \text{if } x \in H \setminus \operatorname{int} N(k), \\ x, & \text{if } x \in H', \end{cases} \tag{2.1}$$

defines a homeomorphism of M_1' to M_2'. The conditions $h_1 = h_2\tau_c$ and $\varphi|_{\partial H} = \tau_c$ assure that the two parts of the formula (2.1) agree on the boundary, see Figure 2.4.

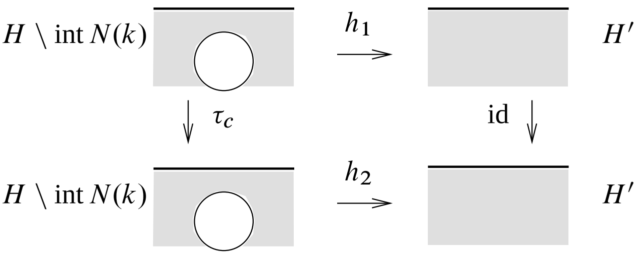


Figure 2.4

Thus, if we remove the solid tori corresponding to $N(k)$ from the manifolds M_1 and M_2, they become homeomorphic. This implies that M_2 is obtained from M_1 by surgery along the knot k. Since Φ maps the meridian m of the torus $\partial N(k)$ to the curve $m \pm \ell$, this surgery is integral. □

Proof of Theorem 2.1. Every manifold M can be represented as $M = H \cup_{h_2} H'$, where H and H' are handlebodies of genus g, and h_2 is an orientation reversing homeomorphism of their boundaries. Similarly, $S^3 = H \cup_{h_1} H'$. Therefore, $h_2^{-1}h_1$ is an orientation preserving homeomorphism so $h_2^{-1}h_1 = \tau_{c_1}\tau_{c_2}\cdots\tau_{c_n}$ where τ_{c_i} is a twist along a curve c_i. According to Lemma 2.2, multiplying the gluing homeomorphism by a Dehn twist has the same effect as performing an integral surgery along a knot. A sequence of such multiplications gives a sequence of surgeries on knots, or a surgery on a link. □

Thus, any closed orientable 3-manifold can be obtained by an integral surgery along a link $\mathcal{L} \subset S^3$. It should be emphasized again that the result of the surgery depends not only on $\mathcal{L}$ but also on the choice of simple closed curves in the boundary $\partial N(k)$ of each component k of the link $\mathcal{L}$. As we have seen, such a curve is uniquely determined by a reduced fraction p/q, including $1/0$. A choice of such a fraction for each component of $\mathcal{L}$ is called a *framing* of $\mathcal{L}$. A link $\mathcal{L}$ with a fixed framing will be called a *framed link*. An integral surgery corresponds to a link framed by integers.

2.3 Surgery description of lens spaces and Seifert manifolds

Let $p \geq 2$. The lens space $L(p,1)$ can be obtained by gluing together two solid tori by the homeomorphism

$$\begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix}$$

of their boundaries which attaches the meridian μ_1 of the first torus to the curve $-\mu_2 + p \cdot \lambda_2$ on the second, see Figure 2.5 where $p = 3$.

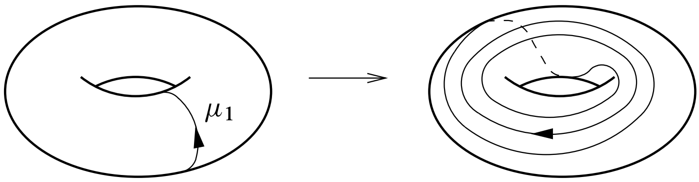


Figure 2.5

If we turn the second solid torus inside out and think of it as a trivial knot exterior, the meridian μ_1 will be attached to the curve $\ell - p \cdot m$. Thus $L(p, 1)$ has surgery description shown in Figure 2.6.

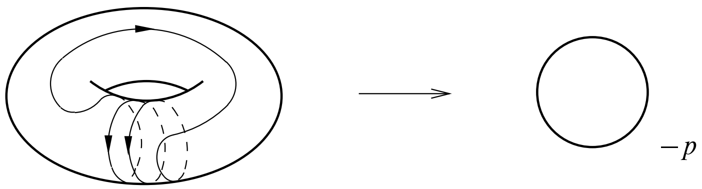


Figure 2.6

Similarly, any $L(p, q)$ will be a rational surgery on a trivial knot with framing $-p/q$. To produce $L(p, q)$ by an integral surgery, replace one of the solid tori $S^1 \times D^2$ by $S^1 \times \Delta^2$ where Δ^2 is an annulus. The construction above which produced $L(p, 1)$, will then give a manifold with boundary a torus. The latter can be pictured as a surgered solid torus as shown in Figure 2.7; the lens space $L(p, 1)$ can be obtained from it by gluing in a solid torus by the homeomorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

Repeat the construction with p replaced by any integer q relatively prime to p. Glue these two surgered solid tori together along their boundary by the homeomorphism (2.2). We obtain S^3 surgered along the link pictured in Figure 2.7. On the other hand,

$$\begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} -q & -1 \\ pq - 1 & p \end{pmatrix},$$

therefore, the link in Figure 2.7 represents $L(pq - 1, q)$.

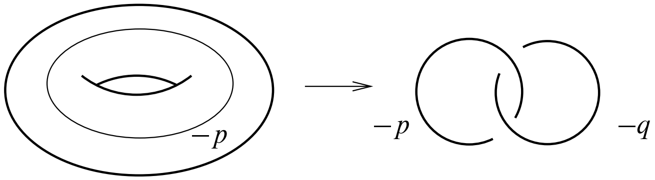


Figure 2.7

Theorem 2.3. *Any lens space $L(p,q)$ has a surgery description as in Figure 2.8, where $p/q = [x_1, \dots, x_n]$ is a continued fraction decomposition,*

$$[x_1, \dots, x_n] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\dots - \cfrac{1}{x_n}}} \tag{2.3}$$

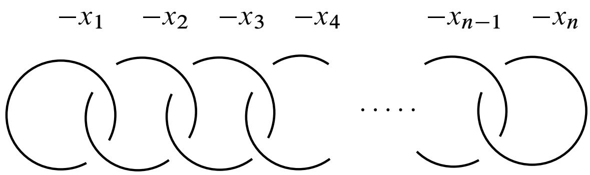


Figure 2.8

Proof. The construction for $L(pq-1,q)$ can be repeated sufficiently many times to produce the link in Figure 2.8. The only thing we need to check is that, if $p/q = [x_1, \dots, x_n]$, then

$$\begin{pmatrix} -q & s \\ p & r \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ x_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ x_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} -1 & 0 \\ x_n & 1 \end{pmatrix}$$

for some r and s. This is true for $n = 1$ and $n = 2$ because

$$\frac{p}{1} = [p] \quad \text{and} \quad \frac{pq-1}{q} = [p,q].$$

By induction, suppose that $p'/q' = [x_2, \dots, x_n]$, then

$$\begin{pmatrix} -1 & 0 \\ x_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -q' & s' \\ p' & r' \end{pmatrix} = \begin{pmatrix} -p' & -r' \\ x_1 p' - q' & x_1 r' + s' \end{pmatrix},$$

so that

$$\frac{x_1 p' - q'}{p'} = x_1 - \frac{q'}{p'} = x_1 - \frac{1}{[x_2, \dots, x_n]} = [x_1, \dots, x_n].$$

Since every rational number has a continued fraction of the form described, we are finished. □

The link in Figure 2.8 is usually drawn as the weighted graph shown in Figure 2.9 where each vertex corresponds to an unknot, and two vertices are connected by an edge if the corresponding unknots are linked.

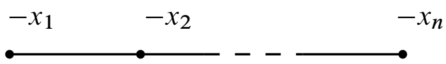


Figure 2.9

Example. The lens space $L(7,3)$ is a surgery on each of the following links in Figure 2.10 according to the continued fraction decompositions $7/3 = [3,2,2]$ and $7/3 = [2,-3]$.

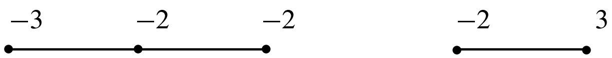


Figure 2.10

A Seifert manifold $M((a_1,b_1),\dots,(a_n,b_n))$ has a rational surgery description shown in Figure 2.11. This description fixes an orientation of the manifold M. From now on, we will refer to $M((a_1,b_1),\dots,(a_n,b_n))$ as an oriented 3-manifold with this particular orientation.

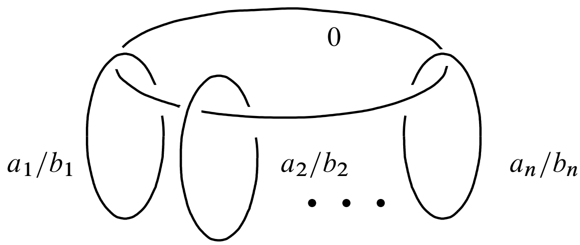


Figure 2.11

With the graph notations as above, the manifold $M((a_1,b_1),\dots,(a_n,b_n))$ can be described as shown in Figure 2.12 where $a_i/b_i = [x_{i1},\dots,x_{im_i}]$.

Example. The manifold $M((3,2),(4,-1),(5,-2))$ has the surgery description shown in Figure 2.13.

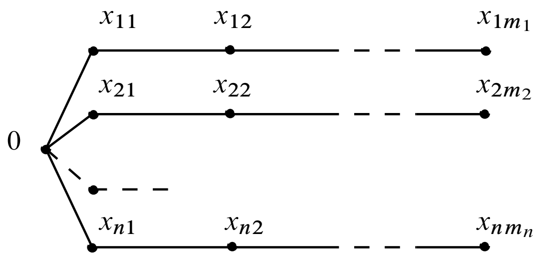


Figure 2.12

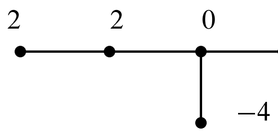


Figure 2.13

2.4 Surgery and 4-manifolds

An oriented compact smooth 4-dimensional manifold W is called an (oriented) *cobordism* between two closed oriented 3-manifolds M_1 and M_2 if $\partial W = -M_1 \cup M_2$ where $-M_1$ stands for M_1 with reversed orientation. If M_1 is empty, one says that M_2 is *cobordant to zero*.

There is a close relationship between surgeries on framed links and cobordisms. Let k be a knot in M with an integral framing defined by a curve c in ∂K such that $[c] = [k] \in H_1(N(k))$. Let a be a point on the boundary of D^2. Then there exists a unique (up to isotopy) diffeomorphism $h\colon S^1 \times D^2 \to N(k)$ such that $h(S^1 \times \{0\}) = k$ and $h(S^1 \times \{a\}) = c$. Glue a 2-handle $D^2 \times D^2$ to the 4-manifold $M \times [0,1]$ with the help of the embedding $h\colon S^1 \times D^2 = (\partial D^2) \times D^2 \to N(k) \subset M = M \times \{1\}$. What we get is a 4-manifold $W = (M \times [0,1]) \cup_h (D^2 \times D^2)$. It is called the *trace* of surgery on k.

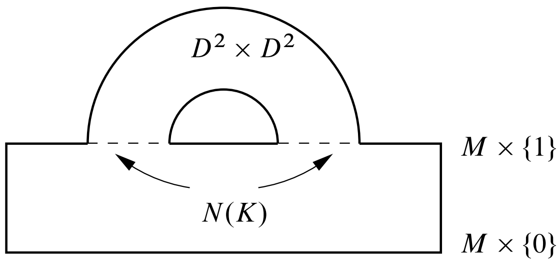


Figure 2.14

Theorem 2.4. *The manifold W is a cobordism between M and the manifold obtained from M by surgery on k.*

Proof. The boundary of W consists of two components. One of these, namely $M \times \{0\}$, is homeomorphic to M. Gluing $D^2 \times D^2$ to $M \times [0,1]$ changes $M \times \{1\}$ as follows: the solid torus $N(K) = h(\partial D^2 \times D^2)$ is removed and replaced by the solid torus $D^2 \times \partial D^2$ (which is a "free" portion of the boundary $\partial(D^2 \times D^2)$). Note that the meridian $\partial D^2 \times \{a\}$ is identified with the curve $c = h(\partial D^2 \times \{a\})$. This means that an integral surgery is performed on $M \times \{1\}$ along k with the framing given by c. Formally speaking, the manifold W is not smooth as it has "corners" after gluing in the handle. However, there is a canonical way to provide W with the structure of a smooth manifold. One "smoothes out" the corners using techniques described, e.g., in Chapter 1 of Conner–Floyd [31]. $\square$

Corollary 2.5. *Any closed oriented* 3*-manifold is cobordant to zero.*

Proof. Any closed oriented 3-manifold M can be obtained by an integral surgery on a link in S^3. Theorem 2.4 then implies that M is cobordant to S^3 which, in its turn, bounds a 4-ball. Therefore, M is cobordant to zero. $\square$

Example. For any p, the lens space $L(p,1)$ is a surgery on the link shown in Figure 2.6. The corresponding 4-manifold $E_p = D^4 \cup (D^2 \times D^2)$ with boundary $\partial E_p = L(p,1)$ can be thought of as a union of $D^4 \cong D^2 \times D^2$ and a 2-handle $D^2 \times D^2$ glued along $S^1 \times D^2 \subset \partial(D^2 \times D^2)$ by a certain homeomorphism $h\colon S^1 \times D^2 \to S^1 \times D^2$. The homeomorphism h attaches $S^1 \times \{0\}$ to $S^1 \times \{0\}$ and twists a copy of D^2 p times in the counter-clockwise direction as one completes one circle along S^1. Schematically, this can be pictured as in Figure 2.15.

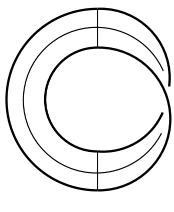
$D^4 = D^2 \times D^2$  $D^2 \times D^2$

Figure 2.15

The central discs of both handles $D^2 \times D^2$ are glued along S^1 to produce a copy of S^2 inside E_p. The manifold E_p is then a locally trivial bundle over S^2 with the fiber D^2.

Example. The manifold E_0 is a trivial bundle, i.e. a product $E_0 = S^2 \times D^2$. Its boundary is $\partial E_0 = \partial(S^2 \times D^2) = S^2 \times S^1$.

Example. The manifold E_1 can be identified with $\mathbb{C}P^2 \setminus \operatorname{int} D^4$, where $\mathbb{C}P^2$ is the complex projective plane, so that $\partial E_1 = \partial(\mathbb{C}P^2 \setminus \operatorname{int} D^4) = S^3$. Before we prove this, we recall that, by definition,

$$\mathbb{C}P^2 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 \setminus 0\}/\mathbb{C}^*$$

where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers acting by the rule $(z_0, z_1, z_2) \mapsto (cz_0, cz_1, cz_2),\ c \in \mathbb{C}^*$. The equivalence class of (z_0, z_1, z_2) is usually denoted by $[z_0 : z_1 : z_2]$.

The complex projective plane $\mathbb{C}P^2$ is covered by three coordinate charts $U_i = \{z_i \neq 0\},\ i = 0, 1, 2$, each of which is homeomorphic to $\mathbb{C}^2$ via homeomorphisms

$$h_0\colon U_0 \to \mathbb{C}^2, \quad [z_0 : z_1 : z_2] \mapsto (z_1/z_0, z_2/z_0),$$
$$h_1\colon U_1 \to \mathbb{C}^2, \quad [z_0 : z_1 : z_2] \mapsto (z_0/z_1, z_2/z_1),$$
$$h_2\colon U_2 \to \mathbb{C}^2, \quad [z_0 : z_1 : z_2] \mapsto (z_0/z_2, z_1/z_2).$$

The charts U_0 and U_1 together cover all of $\mathbb{C}P^2$ but the point $[0 : 0 : 1]$. Therefore, $U_0 \cup U_1$ is a punctured $\mathbb{C}P^2$. Both $h_0(U_0 \cap U_1)$ and $h_1(U_0 \cap U_1)$ as subsets of $\mathbb{C}^2$ consist of all points (z, w) with $z \neq 0$ so that $h_0(U_0 \cap U_1) = h_1(U_0 \cap U_1) = \mathbb{C}^* \times \mathbb{C}$. The gluing map

$$h_1(U_0 \cap U_1) \xrightarrow{h_1^{-1}} U_0 \cap U_1 \xrightarrow{h_0} h_0(U_0 \cap U_1)$$

is given by the formula $(z, w) \mapsto (z^{-1}, wz^{-1})$. The points (z, w) with $|z| \geq 1$ are mapped by this map to the points (z, w) with $|z| \leq 1$. Thus one can truncate $h_0(U_0)$ and $h_1(U_1)$ by the condition $|z| \leq 1$, and think of the gluing operation as happening along $S^1 \times \mathbb{C}$, where S^1 is given by $|z| = 1$, according to the map $(z, w) \mapsto (z^{-1}, wz^{-1})$. This is the map describing the manifold E_1, after one replaces $\mathbb{C}$ with the unit complex disc D^2 and hence the punctured $\mathbb{C}P^2$ with $\mathbb{C}P^2 \setminus \operatorname{int} D^4$.

As a complex manifold, $\mathbb{C}P^2$ comes with a canonical orientation. More careful analysis shows that in fact E_1 is diffeomorphic via an orientation preserving diffeomorphism to $\overline{\mathbb{C}P}^2 \setminus \operatorname{int} D^4$ where $\overline{\mathbb{C}P}^2$ stands for the complex projective plane with reversed orientation, and that $E_{-1} = \mathbb{C}P^2 \setminus \operatorname{int} D^4$. The results of the last two examples can be summarized as in Figure 2.16.

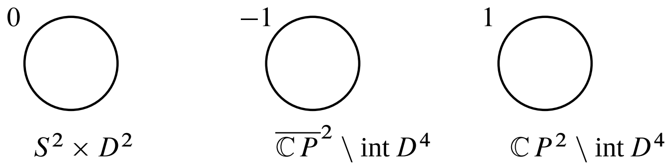


Figure 2.16

2.5 Exercises

1. Let M be a compact 3-manifold with $\partial M = S^1 \times S^1$. Prove that M cannot be simply-connected.
2. Let M be an integral homology 3-sphere, and let k be a knot in M. Prove that $H_*(M \setminus k) = H_*(S^1)$.
3. Regard D^2 as $\{(x,y) \mid x^2 + y^2 \leq 1\}$. Let $\varphi\colon D^2 \to D^2$ be rotation about the origin through the angle of $2\pi/n$, where n is a positive integer. Let E be a small disc centered at $(1/2, 0)$, small enough so that $E, \varphi(E), \ldots, \varphi^{n-1}(E)$ are disjoint. Finally, let D_n be the disc with n holes,

$$D_n = D^2 \setminus \bigcup_{i=0}^{n-1} \varphi^i\,(\mathrm{int}\,E),$$

and let $X_n = D_n \times I/(x,0) \sim (\varphi(x),1)$. Describe X_n as a link exterior. Calculate $\pi_1(X_n)$ and find a 2-generator presentation for this group.
4. Let $\mathcal{L} \subset S^3$ be a link separated by a smoothly embedded 2-sphere into two non-empty sublinks. Prove that surgery on $\mathcal{L}$ yields a 3-manifold which is a connected sum of the manifolds obtained by surgery on the two sublinks.
5. Let k be a knot in S^3. Prove that the 3-manifold obtained by a rational p/q-surgery on k can also be obtained by an integral surgery on the link which consists of k and a chain of unknots as shown in Figure 2.17 with framings determined by a continued fraction decomposition $p/q = [x_1, \ldots, x_n]$.

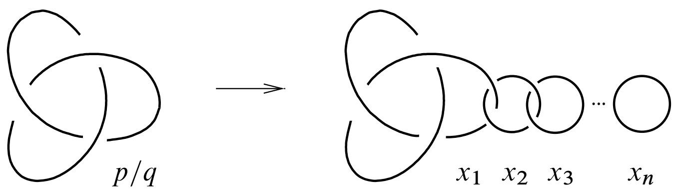


Figure 2.17

6. Prove that the 4-manifold obtained by surgery on the link pictured in Figure 2.18 is $(S^2 \times S^2) \setminus \mathrm{int}\, D^4$.

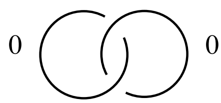


Figure 2.18

7. Prove that the total space of the disc bundle associated with the tangent bundle of S^2 is diffeomorphic to $E_{\pm 2}$ up to a choice of orientation (in particular, the total space of the associated circle bundle is a copy of $\mathbb{R}P^3$).

Lecture 3
Kirby calculus

3.1 The linking number

Let L_1 and L_2 be two disjoint *oriented* knots in S^3 or $\mathbb{R}^3$. Their *linking number* $\mathrm{lk}(L_1, L_2)$ is defined in one of the following equivalent ways.

(1) Since $H_1(S^3) = 0$, the curve L_1 bounds a surface F. Without loss of generality, one may assume that F is smoothly embedded and orientable, see Theorem 7.1. Orient F by a normal vector n so that the triple $\langle \tau, \nu, n \rangle$ is positively oriented where τ is a tangent vector to L_1, and ν is an inner normal vector to L_1 in F. After a small perturbation if necessary, we may assume that L_2 meets F transversally in a finite number of points. At each of them L_2 passes locally through F in one of the possible two directions, n or $-n$. Weight the intersections of the first type by $+1$ and those of the second type by -1. The sum of these numbers is $\mathrm{lk}(L_1, L_2)$. This number is independent of the choice of F and perturbations, compare with the definition **(3)** below.

(2) Consider a regular projection of $L_1 \cup L_2$. Each point at which L_1 crosses under L_2 counts as shown in Figure 3.1. The sum of these numbers, over all crossings of L_1 under L_2, is called $\mathrm{lk}(L_1, L_2)$.

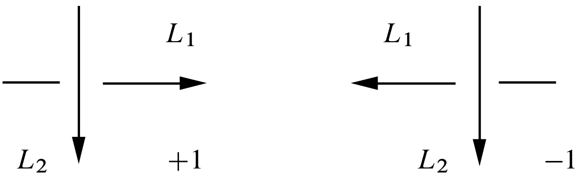


Figure 3.1

(3) Let $[L_1]$ be the homology class in $H_1(S^3 \setminus L_2)$ carried by L_1. The group $H_1(S^3 \setminus L_2) = \mathbb{Z}$ is generated by the homology class $[m]$ of a meridian m of L_2. With the choice of orientation on m as shown in Figure 2.2 we define $\mathrm{lk}(L_1, L_2)$ by the equation $[L_1] = \mathrm{lk}(L_1, L_2) \cdot m$.

Note that $\mathrm{lk}(L_1, L_2) = \mathrm{lk}(L_2, L_1)$ and $\mathrm{lk}(-L_1, L_2) = -\mathrm{lk}(L_1, L_2)$ where $-L_1$ is L_1 with reversed orientation.

Example. The pairs of oriented knots pictured in Figure 3.2 have linking numbers $+2$ and -2, respectively.

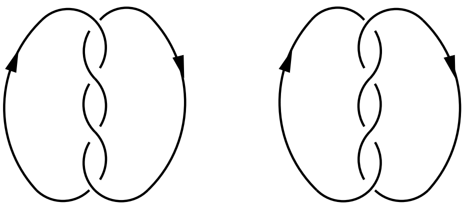


Figure 3.2

With the help of the linking number one can easily describe the canonical meridian-longitude pair (m, ℓ) for a knot $k \subset S^3$. Recall from Lecture 2 that m and ℓ are simple closed curves on ∂K such that $[m] \in H_1(K) = \mathbb{Z}$ is a generator, and ℓ is a longitude such that $0 = [\ell] \in H_1(K)$. Comparing with the definition **(3)** of the linking number we see that the condition $[\ell] = 0$ is equivalent to $\mathrm{lk}(\ell, k) = 0$. Also, the orientations of both m and ℓ were chosen in Lecture 2 so that $\mathrm{lk}(k, m) = +1$, assuming that the orientations of k and ℓ are consistent.

Example. Consider the trefoil knot pictured on the left in Figure 3.3. The "obvious" choice of ℓ as a longitude "parallel" to k does not give the canonical longitude since $\mathrm{lk}(k, \ell) = -3$. The canonical meridian-longitude pair is shown in Figure 3.3 on the right.

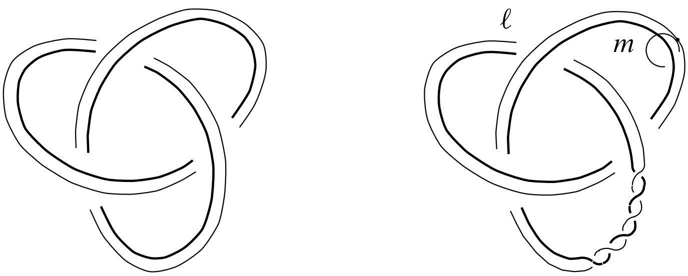


Figure 3.3

With the above said, the integral n framing of a knot k corresponds to the choice of a longitude ℓ such that $\mathrm{lk}(\ell, k) = n$. Another convenient way of representing a framed knot is in the form of a closed band (a homeomorphic image of $S^1 \times I$). One component of its boundary (no matter which) represents the knot, and the other the longitude, see Figure 3.4.

Shown in Figure 3.5 is a band representing a trivial knot with framing n – it has n full right twists if $n \geq 0$, and $-n$ full left twists if $n \leq 0$. One can easily see that the orientation of the knot does not matter so there is no sign ambiguity here.

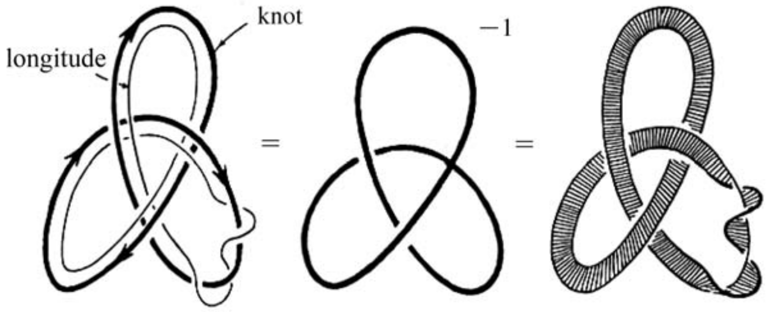


Figure 3.4

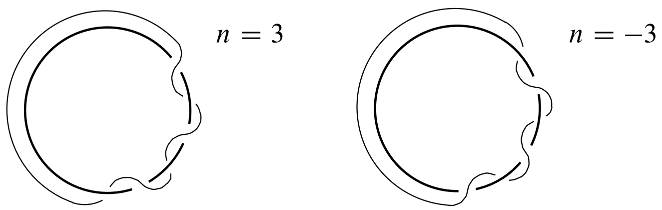


Figure 3.5

3.2 Kirby moves

As we have seen, any closed orientable 3-manifold can be obtained by an integral surgery on a link in S^3. The question is how to determine if two framed links in S^3 give the same 3-manifold. The following are two elementary operations on a framed link $\mathcal{L}$ called *Kirby moves* which do not change the 3-manifold.

Move $K1$. Add or delete an unknotted circle with framing ± 1 which belongs to a 3-ball D^3 that does not intersect the other components of $\mathcal{L}$, see Figure 3.6.

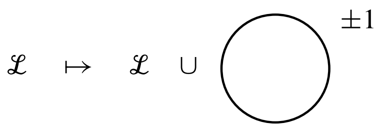


Figure 3.6

Move $K2$. Slide one component of the link $\mathcal{L}$ over another. Namely, let L_1 and L_2 be two link components framed by integers n_1 and n_2, respectively, and L_2' a longitude defining the framing n_2 of the knot L_2 (the latter, as we know, means that $\mathrm{lk}(L_2, L_2') = n_2$). Now, replace the pair $L_1 \cup L_2$ by $L_\# \cup L_2$ where $L_\# = L_1 \#_b L_2'$

and b is any band connecting L_1 to L_2' and disjoint from the other link components. The rest of the link $\mathcal{L}$ remains unchanged, see Figure 3.7. We will say that L_1 was slid over L_2, and write $L_\# = L_1 + L_2$ when no confusion can arise.

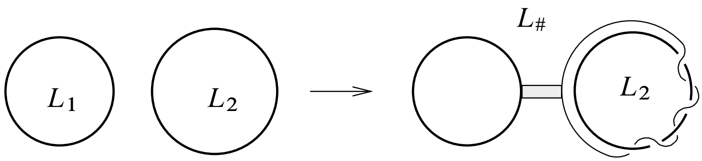


Figure 3.7

The framings of all components but L_1 are preserved; the framing of the new component $L_\#$ is equal to

$$n_1 + n_2 + 2\,\mathrm{lk}(L_1, L_2). \tag{3.1}$$

To compute $\mathrm{lk}(L_1, L_2)$ we need to orient both L_1 and L_2, which have not been oriented so far. We orient them in such a way that together they define an orientation on $L_\#$, see Figure 3.8. As this figure shows, the choice of orientations depends on how the band b is glued in.

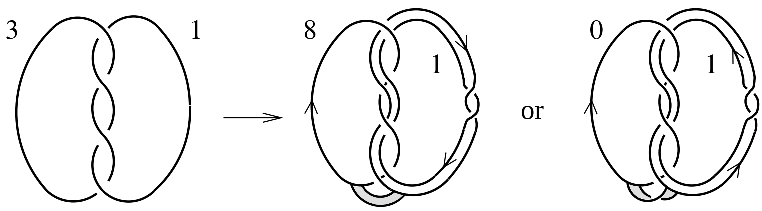


Figure 3.8

Theorem 3.1 (Kirby [82]). *The closed oriented manifolds obtained by integral surgery on framed links $\mathcal{L}$ and $\mathcal{L}'$ are homeomorphic by an orientation preserving homeomorphism if and only if $\mathcal{L}'$ can be obtained from $\mathcal{L}$ by a sequence of moves of types $K1$ and $K2$.*

The moves $K1$ and $K2$ have been known long before Kirby's theorem. Kirby proved the hard part of the theorem: if the manifolds are homeomorphic then the links are related by the moves $K1$ and $K2$. We will only prove the easy part: the moves $K1$ and $K2$ do not change the manifold.

Proof. Adding a disjoint unknotted ± 1 component to the link $\mathcal{L}$ corresponds to taking connected sum of the manifold obtained by surgery along $\mathcal{L}$ with S^3 surgered along the unknot. Since the surgery of S^3 on a (± 1)-unknot produces S^3 again, we are finished.

As for the second move, without loss of generality one may assume that the link $\mathcal{L}$ consists of two components, $\mathcal{L} = L_1 \cup L_2$. Let $L_\# \cup L_2$ be obtained from $\mathcal{L}$ by move $K2$, and let M be the manifold obtained by surgery on L_2. The manifold M is obtained by gluing a solid torus in the exterior of L_2 so that the longitude of L_2 defined by its framing is being glued to the boundary of a disc $D = D^2 \times \{*\} \subset D^2 \times S^1$. Since the knots L_1 and $L_\#$ are disjoint from L_2, one can think of them as sitting in M. The knot $L_\#$ can then be isotoped into L_1 (inside M) by pushing a portion of it along the disc D. □

As we have seen before, integral surgery on a link represents a 3-manifold M as the boundary of a 4-manifold W obtained by adding 2-handles to D^4. The first Kirby move replaces W by $W\#\mathbb{C}P^2$ or $W\#\overline{\mathbb{C}P}^2$; the second slides one 2-handle over another without changing W, see Figure 3.9.

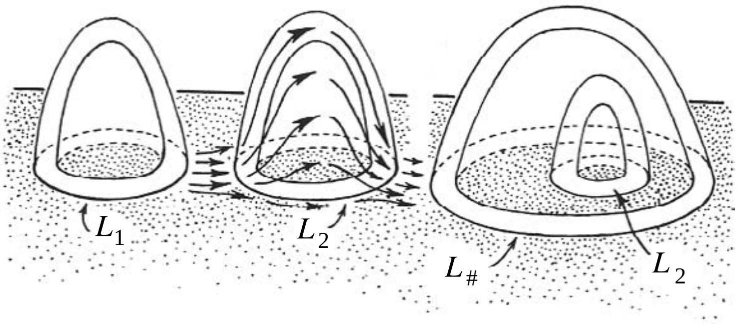


Figure 3.9

At this point there are a number of elementary examples that should be understood.

Proposition 3.2. *An unknot with framing* ± 1 *can always be moved away from the rest of the link* $\mathcal{L}$ *with the effect of giving all arcs going through the unknot a full left/right twist and changing the framings by adding* ∓ 1 *to each arc, assuming they represent different components of* $\mathcal{L}$ *(in general, the framing changes according to the rule* (3.1)*), see Figure* 3.10.

Proof. Slide each arc once over the unknot and keep track of framings using (3.1). □

The cases $n = 1$ and $n = 2$ are shown in Figures 3.11 and 3.12. In Figure 3.12, if the arcs belong to different components of $\mathcal{L}$, their framings increase by ± 1 each; if the arcs belong to the same component, the framing changes by either 0 or ± 4.

The operation shown in Figure 3.10 together with discarding the unknotted, unlinked component is called *blow-down*. The inverse operation in called *blow-up*.

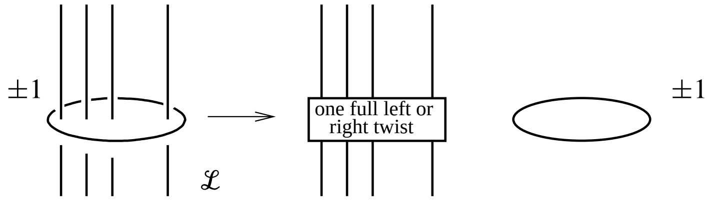


Figure 3.10

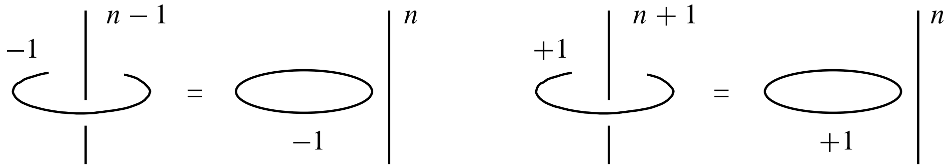


Figure 3.11

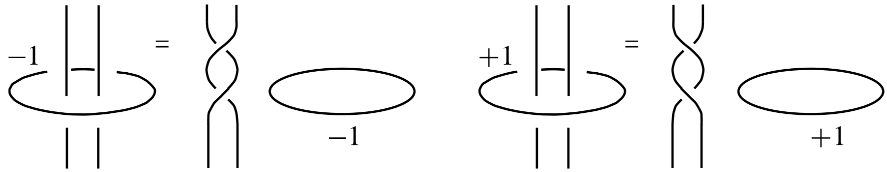


Figure 3.12

Example. The framed links in Figure 3.13 all represent S^3 as the boundary of various 4-dimensional manifolds.

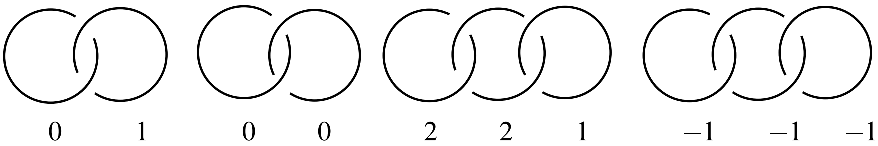


Figure 3.13

Example. If one keeps track of all the 2-handles added, one can prove that certain 4-dimensional manifolds, and not just their boundaries, are diffeomorphic. For instance, the diagram in Figure 3.14 proves that $(S^2 \times S^2)\#\mathbb{C}P^2 = \mathbb{C}P^2\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, compare with Exercise 6 of Lecture 2.

Example. The two links in Figure 3.15 describe homeomorphic 3-dimensional manifolds, both homeomorphic to the Seifert manifold $M((2,-1),(3,1),(5,1))$, which is

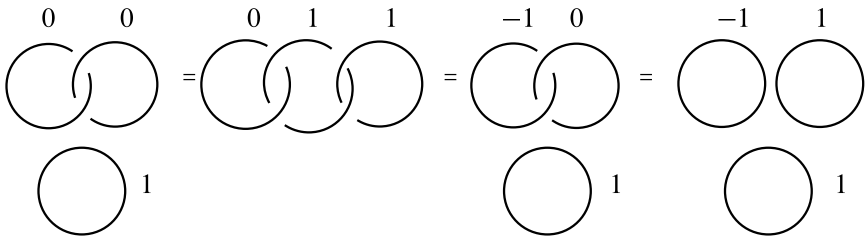


Figure 3.14

also called the *Poincaré homology sphere* and denoted $\Sigma(2,3,5)$. The knot pictured on the right in Figure 3.15 is called a left-handed *trefoil*. Its mirror image is called a right-handed trefoil.

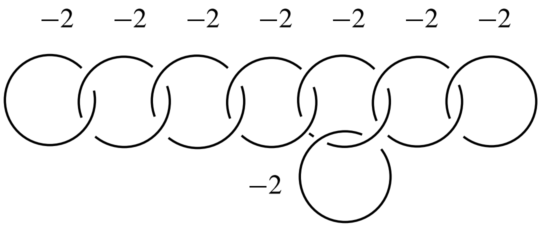


Figure 3.15

To show the first part of the statement, introduce three unknots with framing $+1$ and slide the endmost circles over them (compare to Figure 3.11). Now apply the move in Figure 3.12 to the three circles with framing -1. Iterate this process, discarding unknotted, unlinked components, to get the links in Figure 3.16.

Apply Proposition 3.2 to a succession of unknots with $+1$ framing to finish the argument, see Figure 3.17. Finally, to prove that this manifold is $M((2,-1),(3,1),(5,1))$, we proceed as shown in Figure 3.18.

Example. For every $m = 1, 2, \ldots$, consider the Seifert manifold $\Sigma(2,3,6m+1)$ given by the surgery graph shown in the upper left corner of Figure 3.19. One can show as in the example above that $\Sigma(2,3,6m+1)$ is a $(-1/m)$-surgery on a right-handed trefoil.

On the other hand, the sequence of Kirby moves shown in Figure 3.19 proves that $\Sigma(2,3,6m+1)$ can also be obtained by 1-surgery along the knot k pictured on the bottom of Figure 3.19 – compare with Akbulut–Kirby [1], Figure 23. The knot k is called a *twist knot* of type $(2m+2)_1$ where $2m+2$ stands for the minimal number of crossings in its regular projection shown on the bottom of Figure 3.19.

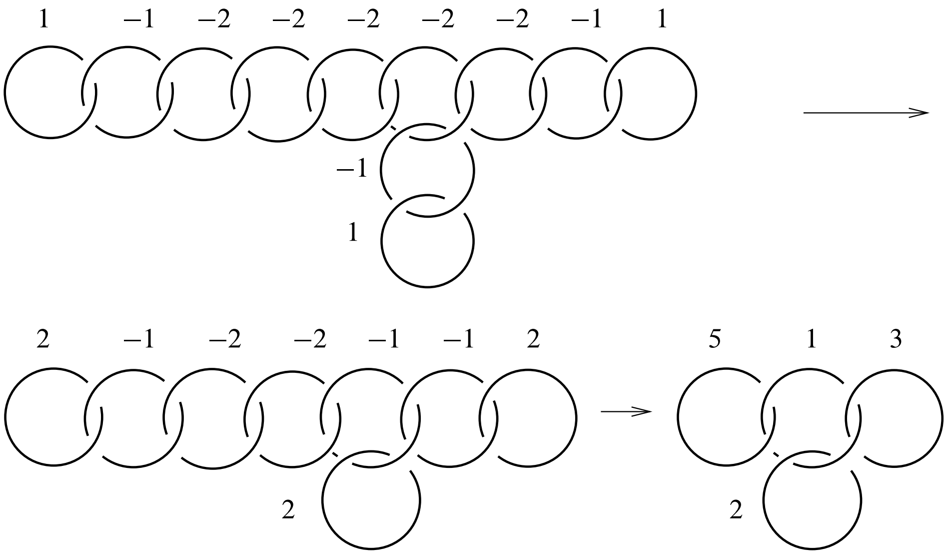


Figure 3.16

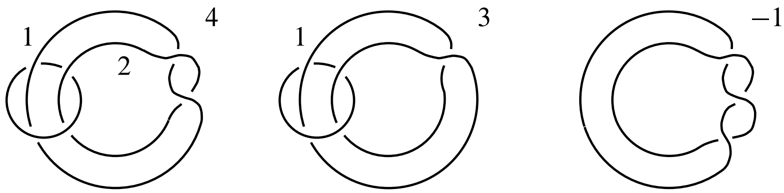


Figure 3.17

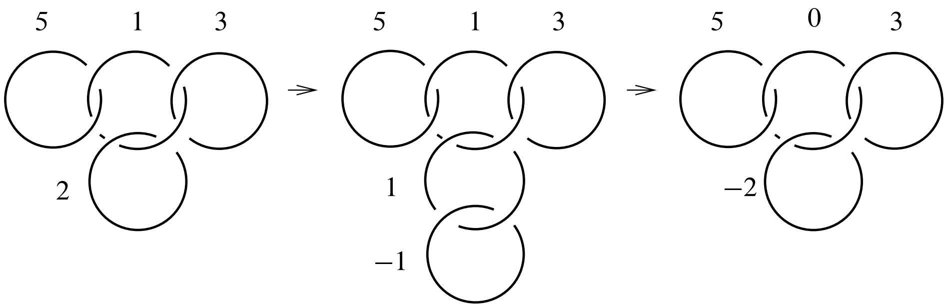


Figure 3.18

Example. Another example of a Seifert manifold which can be obtained by a surgery on a single knot is the manifold $M((3,1),(4,1),(7,-4))$, see Figure 3.20. The knot has 10 crossings in its regular projection, and has the name 10_{132} in the knot table of Rolfsen [137].

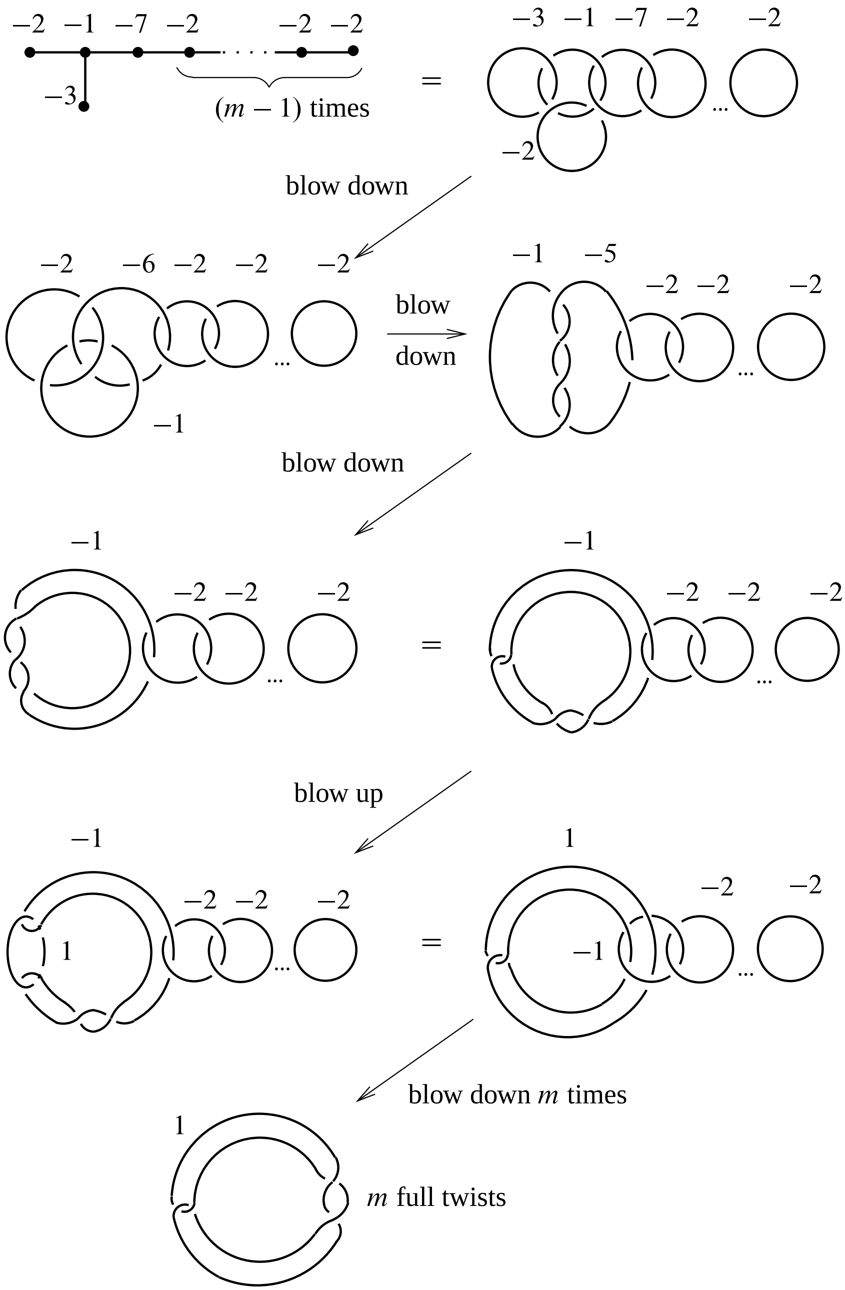

−2 −1 −7 −2 −2 −2
−3
(m − 1) times
=
−3 −1 −7 −2 −2
−2
blow down
−2 −6 −2 −2 −2
−1
blow
down
−1 −5
−2 −2 −2
blow down
−1
−2 −2 −2
=
−1
−2 −2 −2
blow up
−1
−2 −2 −2
1
=
1
−2 −2
−1
blow down m times
1
m full twists

Figure 3.19

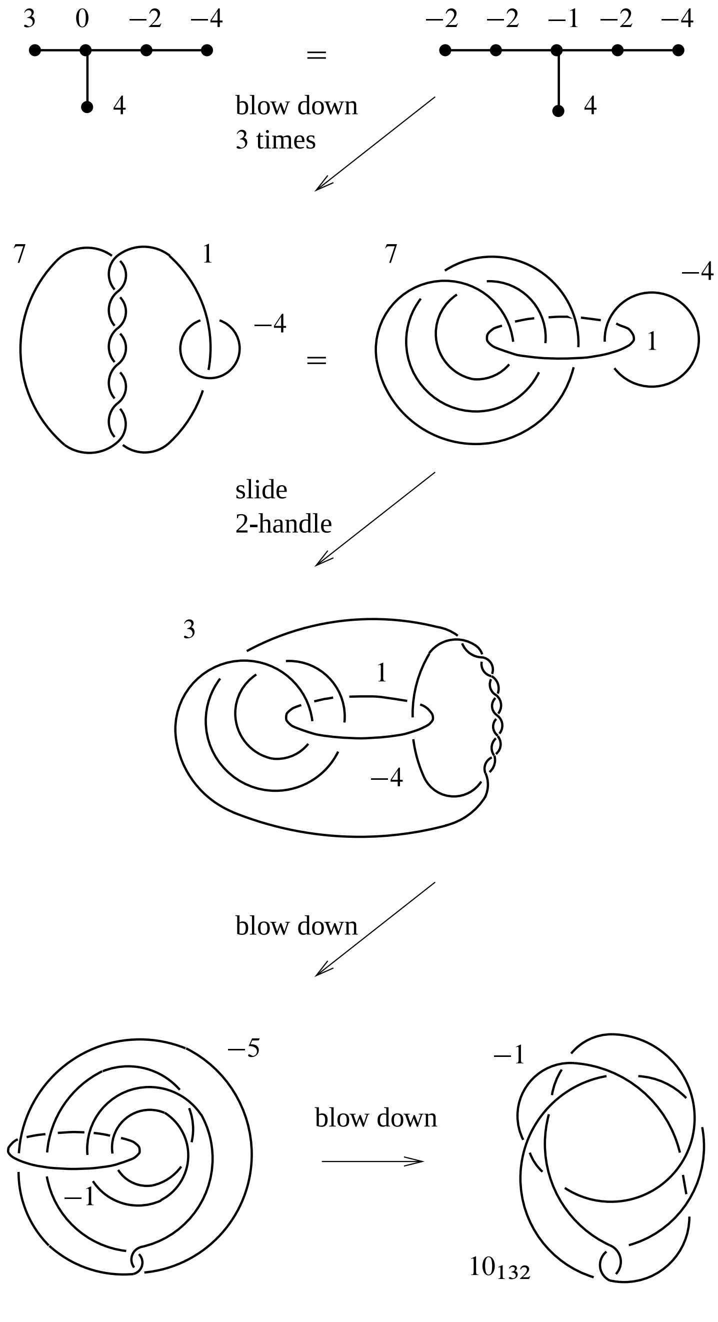

3 0 −2 −4
4
=
−2 −2 −1 −2 −4
4
blow down
3 times
7
1
−4
=
7
−4
1
slide
2-handle
3
1
−4
blow down
−5
−1
blow down
−1
10_{132}

Figure 3.20

Proposition 3.3. *If in a framed link $\mathcal{L}$ a component L_0 is an unknot with framing zero which links only one other component L_1 geometrically once, then $L_0 \cup L_1$ may be moved away from the link $\mathcal{L}$ without changing framings, and canceled.*

Proof. If a strand of L_i crosses L_1, the component L_0 can be used to change the crossing without changing framings as in Figure 3.21.

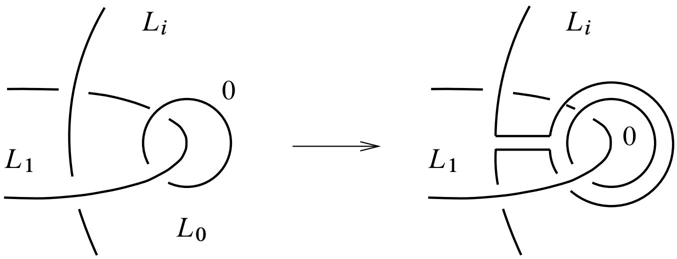


Figure 3.21

An iteration of this move proves the first statement. The same move changes crossings of L_1 itself, thereby unknotting L_1 and changing its framing by an even integer. We end up with the link pictured on the left in Figure 3.22.

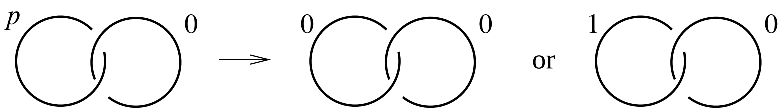


Figure 3.22

Each time the left circle is slid over the right, the framing changes by ± 2, so that eventually we arrive at a link representing S^3. □

For two *oriented* knots $k_1 \subset M_1$ and $k_2 \subset M_2$, let P_i be a point on k_i and (D_i^3, D_i^1) a ball neighborhood of P_i in (M_i, k_i), $i = 1, 2$. The *connected sum* of the knots k_1 and k_2, denoted by $k_1 \# k_2$, is an oriented knot in the manifold $M_1 \# M_2$ obtained by pasting together the pairs

$$(M_1 \setminus \operatorname{int} D_1^3,\ k_1 \setminus \operatorname{int} D_1^1) \quad \text{and} \quad (M_2 \setminus \operatorname{int} D_2^3,\ k_2 \setminus \operatorname{int} D_2^1)$$

along an orientation-reversing homeomorphism $(\partial D_1^3, \partial D_1^1) \to (\partial D_2^3, \partial D_2^1)$. The construction for knots in S^3 can be described as follows: $k_1 \# k_2$ is a knot obtained by taking diagrams of k_1 and k_2 separated by a 2-sphere and connecting them as shown in Figure 3.23 to match the orientations of the knots. In general, the connected sum operation is not well-defined for non-oriented knots.

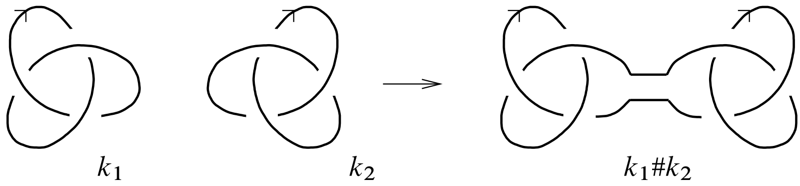


Figure 3.23

Example. The Figure 3.24 demonstrates that a surgery on a connected sum of knots, $k_1\#k_2$, is equivalent to a surgery on the link consisting of the knots k_1 and k_2 linked once by a 0-framed unknot. To see the equivalence, we simply slide k_1 over k_2 with the help of the second Kirby move, and then use Proposition 3.3 to cancel the pair $\{0\text{-circle}\} \cup k_2$.

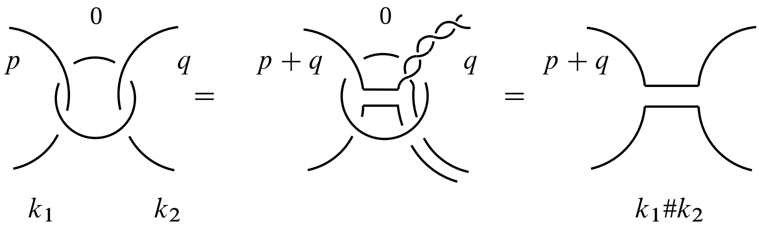


Figure 3.24

3.3 The linking matrix

Let $\mathcal{L} = L_1 \cup \cdots \cup L_n$ be an *oriented* framed link in S^3, the i-th component being framed by $e_i \in \mathbb{Z}$. The matrix $A = (a_{ij}),\ i, j = 1, \ldots, n$, with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j, \\ \operatorname{lk}(L_i, L_j), & \text{if } i \neq j, \end{cases}$$

is called the *linking matrix* of $\mathcal{L}$. It is symmetric since

$$a_{ij} = \operatorname{lk}(L_i, L_j) = \operatorname{lk}(L_j, L_i) = a_{ji}.$$

The effect of the Kirby moves on A is follows. The move $K1$ replaces A by

$$\begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \pm 1 \end{pmatrix}.$$

Suppose that move $K2$ slides L_i over L_j to produce the pair $(L_i + L_j) \cup L_j$. The new linking matrix is obtained from A by adding (or subtracting) the j-th row to (from) the i-th row and the j-th column to (from) the i-th column.

Example. For the links in Figure 3.8,

$$\begin{pmatrix} 3 & \pm 2 \\ \pm 2 & 1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

3.4 Reversing orientation

Let M be a closed oriented manifold obtained by surgery on a framed link $\mathcal{L} = L_1 \cup \cdots \cup L_n$ in S^3, the i-th component L_i being framed by e_i, $i = 1, \ldots, n$. Let us fix an orientation on the link $\mathcal{L}$ and let A be the linking matrix of $\mathcal{L}$. Let $-M$ be the manifold M with reversed orientation. To get a surgery description of $-M$, we reverse the orientation of the link exterior K by taking its mirror image with respect to any plane in $\mathbb{R}^3$. This operation results in reversing orientations of the boundary components of K so that each framing e_i turns into $-e_i$. Thus, the manifold $-M$ will be obtained by surgery on the link $\mathcal{L}^* = L_1^* \cup \cdots \cup L_n^*$ which is a mirror image of $\mathcal{L}$, the component L_i^* being framed by $-e_i$. Moreover, the orientation of $\mathcal{L}$ induces an orientation on $\mathcal{L}^*$ so that $\operatorname{lk}(L_i^*, L_j^*) = -\operatorname{lk}(L_i, L_j)$ for all $i \neq j$. Therefore, the linking matrix of $\mathcal{L}^*$ is $-A$.

Example. We already know from Lecture 2 that the lens spaces $L(7,3)$ and $L(7,4)$ are homeomorphic by an orientation reversing homeomorphism. This can also be seen with the help of Kirby calculus as follows. Let us first represent the lens spaces in question by their surgery diagrams, see Figure 3.25. The manifold $-L(7,3)$ then has the surgery description shown in Figure 3.26, which identifies it with $L(7,4)$.

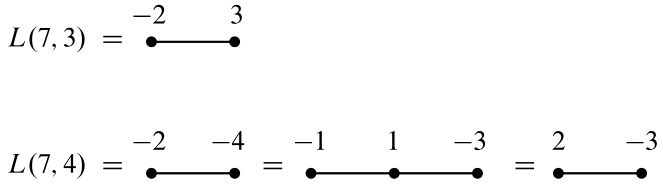


Figure 3.25

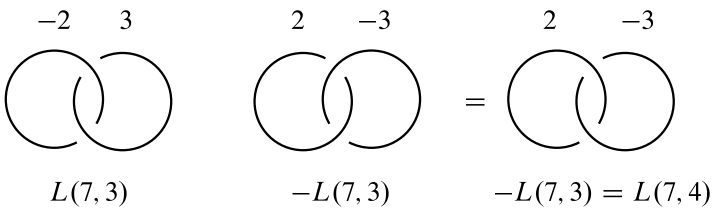


Figure 3.26

3.5 Exercises

1. Prove that surgery on each of the framed links in Figure 3.27 yields $\Sigma(2,3,5)$.

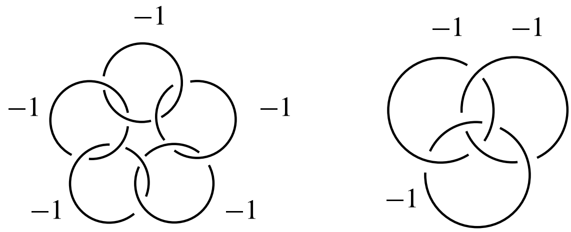


Figure 3.27

2. Show that surgery on the knots shown in Figure 3.28 yields the Seifert homology sphere $\Sigma(2,3,7)$.

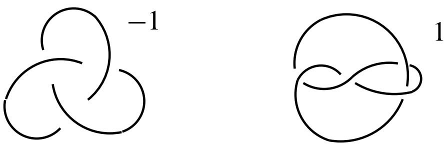


Figure 3.28

3. Show that the 3-torus $S^1 \times S^1 \times S^1$ can be described as the result of surgery on the link pictured in Figure 3.29.

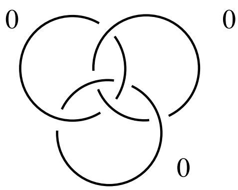


Figure 3.29

4. Let $k_1 \subset \Sigma_1$ and $k_2 \subset \Sigma_2$ be oriented knots in oriented homology spheres, K_1 and K_2 their exteriors, and (m_1, ℓ_1) and (m_2, ℓ_2) the canonical meridian-longitude pairs on ∂K_1 and ∂K_2, respectively. By the *splice* of Σ_1 and Σ_2 along k_1 and k_2 we will mean the manifold $\Sigma = K_1 \cup K_2$ obtained by gluing K_1 and K_2 along their boundaries by an orientation reversing homeomorphism matching m_1 to ℓ_2 and ℓ_1 to m_2.

   (a) Prove that Σ is a homology sphere.

   (b) Define a trivial knot in a homology sphere Σ as an unknot in a copy of $D^3 \subset \Sigma$. Prove that the splice of Σ_1 and Σ_2 along trivial knots is simply the connected sum $\Sigma_1 \# \Sigma_2$.

   (c) Let homology spheres Σ_1' and Σ_2' be obtained from Σ_1 and Σ_2 by (-1)-surgery on, respectively, k_1 and k_2. Let $k_1^* \subset \Sigma_1'$ and $k_2^* \subset \Sigma_2'$ be the images of the canonical longitudes ℓ_1 and ℓ_2. Prove that the splice of Σ_1 and Σ_2 along k_1 and k_2 is homeomorphic to the homology sphere obtained from $\Sigma_1' \# \Sigma_2'$ by $(+1)$-surgery on the knot $k_1^* \# k_2^*$.

Lecture 4
Even surgeries

A framed link $\mathcal{L} = L_1 \cup \cdots \cup L_n$ in S^3 is called *even* if all its framings are even integers.

Theorem 4.1. *Any closed orientable* 3*-dimensional manifold can be obtained from* S^3 *by surgery along an even link.*

Example. The Poincaré homology sphere $\Sigma(2,3,5)$ can be obtained by surgery on each of the links in Figure 3.15. The link on the left is even while the link on the right is not.

The idea of our proof of Theorem 4.1 is to reduce a given framed link by Kirby moves to an even link. A straightforward attempt at reducing the number of odd framed components in the link is usually not a success. The right approach is to kill the so-called characteristic sublink.

Let $A = (a_{ij}),\ i, j = 1, \dots, n$, be the linking matrix of $\mathcal{L}$ reduced modulo 2. Note that A modulo 2 is well-defined for unoriented links while A itself requires a choice of orientation of $\mathcal{L}$, see Section 3.3. Consider the linear system over $\mathbb{Z}/2$,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = a_{ii}, \quad i = 1, \dots, n. \tag{4.1}$$

The matrix A of this system is symmetric, and the right-hand side of the system is the column of the diagonal elements of A. Such a system always has a solution. Let $(x_1, \dots, x_n),\ x_j = 0$ or 1, be a solution of (4.1). The sublink

$$\mathcal{L}' = \{L_j \subset \mathcal{L} \mid x_j = 1\}$$

is called a *characteristic sublink*. A characteristic sublink always exists but it is not unique if $\det A = 0 \bmod 2$. A link is even if and only if it has an empty characteristic sublink. Once a characteristic sublink $\mathcal{L}'$ is fixed, each component of $\mathcal{L}$ can be thought of as a pair (L_k, x_k) where $x_k = 1$ if and only if $L_k \subset \mathcal{L}'$ (x_k is not to be confused with the framing on L_k).

Example. Let us consider the link $\mathcal{L}$ shown in Figure 4.1. The equation (4.1) for this link takes the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and this system has the unique solution $x_1 = 0$, $x_2 = 1$, $x_3 = 0$. Therefore, there is only one characteristic sublink, which consists of the middle circle of the link $\mathcal{L}$. By blowing the central curve down we obtain an even link describing the same 3-manifold as the original link $\mathcal{L}$.

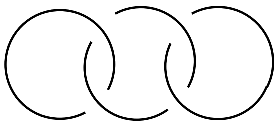


Figure 4.1

The Kirby moves have the following effect on a characteristic sublink $\mathcal{L}'$. One can easily see that the first move replaces $\mathcal{L}$ by $\mathcal{L} \cup (L_{n+1}, 1)$ where L_{n+1} is the unknotted link component in Figure 3.6. Suppose that a component L_i is slid over a component L_j in the second move. In the resulting link the components (L_k, x_k), $k \neq i, j$, remain unchanged while the pair $(L_i, x_i) \cup (L_j, x_j)$ is replaced by $(L_i + L_j, x_i) \cup (L_j, x_i + x_j)$. To see the latter we notice that the effect of the second move on the matrix A is in adding the j-th row to the i-th row and the j-th column to the i-th column. It can be easily checked that $(\dots, x_i, \dots, x_i + x_j, \dots)$ is a solution of the new system. For instance, if both L_i and L_j were characteristic, and L_i is slid over L_j, then L_j is no longer characteristic while $L_i + L_j$ is.

Proof of Theorem 4.1. Any closed oriented 3-manifold is an integral surgery on a framed link $\mathcal{L}$. Let $\mathcal{L}'$ be its characteristic sublink. If $\mathcal{L}'$ has more than one component, we can slide one of its components over another by using the second Kirby move. The result is a new link with fewer characteristic components. Thus one can assume that the characteristic sublink consists of just one component, a knot k.

If the knot k is trivial, its framing can be changed to ± 1 by repeating the move in Figure 3.10 of Lecture 3 sufficiently many times, after which the knot can be blown down. This operation does not create any new characteristic components, therefore, the characteristic sublink of the new link is empty.

In general, the knot k can be unknotted by Kirby moves so that it is still the only characteristic component. It can be done as follows.

Step 1. Let us consider the transformation P which replaces one fragment of k with another as shown in Figure 4.2, keeping the rest of it unchanged. We will prove that the knot k can be turned into a trivial knot by a sequence of transformations P.

Step 2. The knot k bounds an embedded compact orientable surface F in S^3, see Theorem 7.1. This surface, of course, may intersect the other components of the link. The classification theorem for such surfaces implies that F is isotopic to a surface

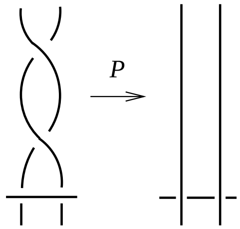


Figure 4.2

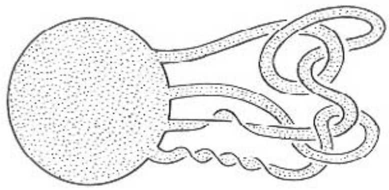


Figure 4.3

which is a 2-disc D with several bands attached, see Figure 4.3. Here is the overall idea. Fix a triangulation of the surface. A small neighborhood of each vertex forms a disc. Thin neighborhoods of the edges form bands joining the discs together. Hence, a neighborhood of the edges is homeomorphic to a union of discs with bands added. One checks that adding the faces has the same effect as not attaching certain of the bands, and that the number of discs can be reduced to one.

Step 3. The bands may be knotted and linked with each other, and each of them may be twisted an even number of times (since the surface is orientable). With the help of transformation P, the bands can be unknotted and unlinked.

Let us denote the points at which the i-th band is attached to the disc D by P_i and Q_i. For any i, there exists a j such that the pairs (P_i, Q_i) and (P_j, Q_j) are linked in $\partial D = S^1$ (otherwise the boundary of F would consist of more that one component).

unlinked pairs of points on a circle

linked pairs of points on a circle

Figure 4.4

Let (P_i, Q_i) and (P_j, Q_j) be linked pairs of points on ∂D as shown in Figure 4.5. Whenever there are points P_k or Q_k, $k \neq i, j$, between P_j and Q_i, they can be slid over the i-th band to the interval $[P_i, P_j]$. In turn, any points P_k or Q_k in $[P_i, P_j]$ can be slid along the j-th band to the right of Q_j, and the points P_k or Q_k in $[Q_i, Q_j]$ can be slid along the i-th band to the left of P_i.

Thus, we can isotope the surface F so that there are no attaching points between P_i and Q_j other than P_j and Q_i. It may happen that some bands become linked again under this procedure – if necessary, we unlink them by the transformation P.

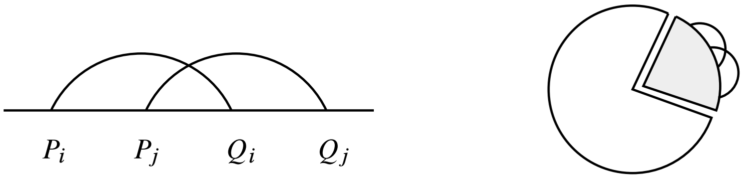


Figure 4.5

Step 4. We proceed by induction and show that the knot k is a connected sum of knots, each of which is obtained as the boundary of a disc with only two bands. After that, the number of full twists on each of the bands can be reduced to at most 1 since double full twists can be eliminated by P, see Figure 4.6.

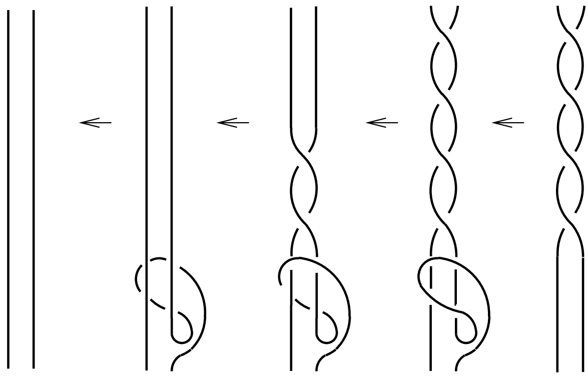


Figure 4.6

One can also assume that all twists are full right-hand twists so that the knot k is in fact a connected sum of the knots shown in Figure 4.7 (if at least one of the bands is not twisted at all, the corresponding knot is trivial).

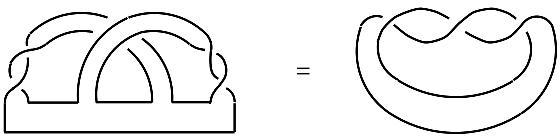


Figure 4.7

The knot in Figure 4.7 is a left handed trefoil. It can be unknotted by undoing a double full twist with the help of the transformation P as shown in Figure 4.8.

Step 5. We only need to prove that the transformation P can be realized by Kirby moves preserving k as the only characteristic component. This can be seen from

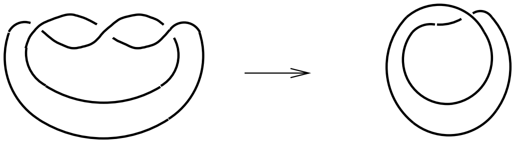


Figure 4.8

Figure 4.9. Note that it is essential that the number of strands is odd so that we do not introduce new characteristic components by the second Kirby move. Note also that every time we apply the transformation P we add a new component to the link $\mathcal{L}$.

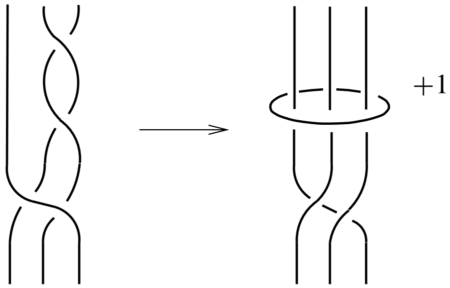


Figure 4.9 □

Originally, Theorem 4.1 was proved by J. Milnor [112] by different techniques. Our proof follows the lines of the proof given in the Fomenko–Matveev book [49], which in turn modifies the proof of Kaplan [79].

4.1 Exercises

1. Prove that the linear system (4.1) always has a solution.
2. Kill the characteristic sublink in the link shown in Figure 4.10.

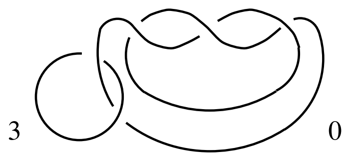


Figure 4.10

Lecture 5
Review of 4-manifolds

The main object of our interest in this lecture will be a connected, closed (compact without boundary), oriented 4-manifold. Such a manifold will be either a topological or a smooth manifold. To avoid the group theoretic problems arising from the fact that any finitely presented group can occur as the fundamental group of a (smooth) closed 4-manifold, see [106], page 143, we assume that our manifolds are simply-connected. All homology and cohomology groups are assumed to be with integral coefficients unless otherwise stated.

5.1 Definition of the intersection form

Let M be a closed, oriented, connected, and simply-connected 4-manifold. By Poincaré duality, $H_4(M) = H^0(M) = \mathbb{Z}$. The choice of a generator in $H_4(M)$ corresponds to the choice of orientation of M. As soon as M is oriented, the generator of $H_4(M)$ is called the *fundamental class* of M and is denoted by $[M]$. Since M is simply-connected, $H_1(M) = 0$ and, by Poincaré duality, $H_3(M) = 0$. Therefore, all homology information about M is contained in the second (co)homology.

Lemma 5.1. *The groups $H^2(M)$ and $H_2(M)$ are torsion free.*

Proof. The universal coefficient theorem in cohomology gives

$$H^2(M) = \mathrm{Hom}(H_2(M), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(M), \mathbb{Z}),$$

and since $H_1(M) = 0$, it follows that $H^2(M)$ is torsion free. By Poincaré duality, $H_2(M) = H^2(M)$. □

Let us consider the bilinear form on the free Abelian group $H^2(M)$,

$$Q_M : H^2(M) \otimes H^2(M) \to \mathbb{Z}, \tag{5.1}$$

given by the cup product, $(a, b) \mapsto \langle a \smile b, [M] \rangle$. This is a symmetric integral form called the *intersection form* of the manifold M. By Poincaré duality, the form Q_M is non-degenerate over the integers, which means that in a certain basis of $H^2(M)$ (and therefore in any basis) its matrix is invertible over the integers. Note that an integral matrix Q is invertible over the integers if and only if it is *unimodular*, that is, $\det Q = \pm 1$.

Here is another way to define the intersection form for a *smooth* manifold M (a similar definition for topological manifolds is more subtle, and we will not attempt it here). Poincaré duality provides an isomorphism PD: $H^2(M) \to H_2(M)$. We will say that a class $\alpha \in H_2(M)$ is *represented by* a smoothly embedded closed oriented surface $F_\alpha \subset M$ if $i_*([F_\alpha]) = \alpha$ in homology, where i is the inclusion map and $[F_\alpha] \in H_2(F_\alpha)$ is the fundamental class of F_α. We will abuse notations and write $i_*([F_\alpha]) = [F_\alpha] \in H_2(M)$.

Lemma 5.2. *Let M be a closed oriented smooth* 4*-manifold. Any class $\alpha \in H_2(M)$ can be represented by a smoothly embedded closed oriented surface F_α.*

Proof. Let $a \in H^2(M)$ be Poincaré dual to $\alpha \in H_2(M)$, and consider a natural bijection

$$H^2(M;\mathbb{Z}) = [\,M, K(\mathbb{Z},2)\,] = [\,M, \mathbb{C}P^\infty\,],$$

where the brackets denote homotopy classes of maps. Since $\dim M = 4$, we further have that $[\,M, \mathbb{C}P^\infty\,] = [\,M, \mathbb{C}P^2\,]$. Under this bijection, the cohomology class a corresponds to the homotopy class of a map $f_a : M \to \mathbb{C}P^2$. Since $H^2(\mathbb{C}P^2) = \mathbb{Z}$ is freely generated by the Poincaré dual $\mathrm{PD}^{-1}[\mathbb{C}P^1]$ of the complex projective line $\mathbb{C}P^1 \subset \mathbb{C}P^2$, the correspondence between a and f_a can be described as

$$a = f_a^*(\mathrm{PD}^{-1}[\mathbb{C}P^1]).$$

Choose f_a within its homotopy class to be smooth and transversal to $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Then $F_\alpha = f_a^{-1}(\mathbb{C}P^1)$ is a closed oriented surface smoothly embedded in M, and its homology class $[F_\alpha] \in H_2(M)$ is Poincaré dual to a, after possibly reversing the orientation of F_α. Since α is also Poincaré dual to a, we conclude that $[F_\alpha] = \alpha$. □

Let M be smooth. Represent $\alpha, \beta \in H_2(M)$ by smoothly embedded, oriented surfaces F_α and F_β. Perturb either one so that F_α meets F_β transversally in n points $P_1, \dots, P_n$, schematically shown in Figure 5.1.

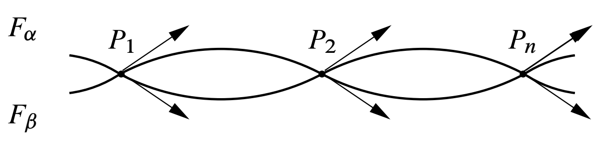


Figure 5.1

Each point P_i can be assigned a sign $\varepsilon(P_i) = \pm 1$, according to whether $T_{P_i} F_\alpha \oplus T_{P_i} F_\beta$ has the same or opposite orientation as $T_{P_i} M$. Then define the intersection form on the second homology,

$$H_2(M) \otimes H_2(M) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \alpha \cdot \beta = \sum_{i=1}^{n} \varepsilon(P_i). \tag{5.2}$$

Lemma 5.3. *The form* (5.2) *is well-defined. Let* $a, b \in H^2(M)$ *be cohomology classes and* $\alpha = \mathrm{PD}(a)$, $\beta = \mathrm{PD}(b)$ *their Poincaré duals. Then* $\alpha \cdot \beta = \langle a \smile b, [M] \rangle = Q_M(a, b)$.

Proof. For a proof, we refer the reader to the book Bott–Tu [18], Section 6. □

To evaluate $\alpha \cdot \alpha$ for $\alpha \in H_2(M)$, one should represent α by two homologous surfaces, F_α and F'_α, meeting transversally, and count the intersection points as above. A convenient way to think of the surface F'_α is as a surface obtained from F_α by a "small perturbation": schematically, this is shown in two dimensions in Figure 5.2.

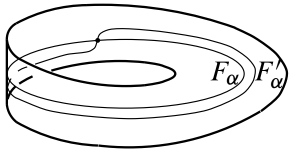


Figure 5.2

From now on, we will not distinguish between the forms (5.1) and (5.2) and when referring to an intersection form will mean either of them. Given a basis $e_1, \ldots, e_n$ in $H_2(M)$, the intersection form determines the intersection matrix $e_i \cdot e_j$. Standard examples of (smooth) 4-manifolds and their intersection forms are as follows.

Example. $H_2(S^2 \times S^2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ with generators α and β represented, respectively, by the surfaces $S^2 \times \{q\}$ and $\{p\} \times S^2$ where p is a point in the first copy of S^2 and q a point in the second. Since $\alpha \cdot \alpha = \beta \cdot \beta = 0$, and, with appropriate orientations, $\alpha \cdot \beta = 1$, the intersection form is given by the so-called hyperbolic matrix,

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Example. $H_2(\mathbb{C}P^2)$ is isomorphic to $\mathbb{Z}$ with a generator represented by $\mathbb{C}P^1$. This follows, for example, from the Mayer–Vietoris exact sequence for $\mathbb{C}P^2 = E_{-1} \cup_{S^3} D^4$, see Lecture 2, and the fact that $S^2 = \mathbb{C}P^1 \subset E_{-1}$ is a deformation retract of E_{-1}. The intersection form of $\mathbb{C}P^2$ is $(+1)$; the intersection form of $\mathbb{C}P^2$ with opposite orientation, $\overline{\mathbb{C}P}^2$, is (-1).

Example. $H_2(M \# N) = H_2(M) \oplus H_2(N)$ for any 4-manifolds M and N, and $Q_{M\#N} = Q_M \oplus Q_N$. In particular, the intersection form of $p \cdot \mathbb{C}P^2 \,\#\, q \cdot \overline{\mathbb{C}P}^2$

is

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

with p ones and q negative ones on the diagonal.

Example. The *Kummer surface*

$$K3 = \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\}$$

is a simply-connected closed oriented 4-manifold with intersection form $E_8 \oplus E_8 \oplus 3H$ where E_8 is the following 8×8 unimodular matrix

$$E_8 = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 0 & 1 \\ & & & & 1 & -2 & 1 & 0 \\ & & & & 0 & 1 & -2 & 0 \\ & & & & 1 & 0 & 0 & -2 \end{pmatrix}$$

(empty entries mean zeroes). It is the linking matrix of the framed link in Figure 5.3, whose components should be properly oriented. As we know from Lecture 3, the surgery on this link produces the Poincaré homology sphere $\Sigma(2,3,5)$.

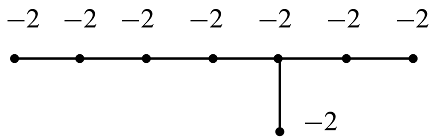


Figure 5.3

The framed link in Figure 5.4 gives an explicit description of $K3$, see Harer, Kas, and Kirby [68]. It has 22 components, two of which have zero framings as shown. The remaining 20 components are framed by -2. The surgery on this link gives a 4-manifold with boundary S^3. To get a closed manifold one simply attaches a 4-ball to its boundary.

It is worth mentioning that not every 4-manifold can be obtained by adding 2- and 4-handles to D^4. Handles of other indices may be needed. The complete version of Kirby calculus in dimension 4 can be found in Kirby [84] or Gompf–Stipsicz [61].

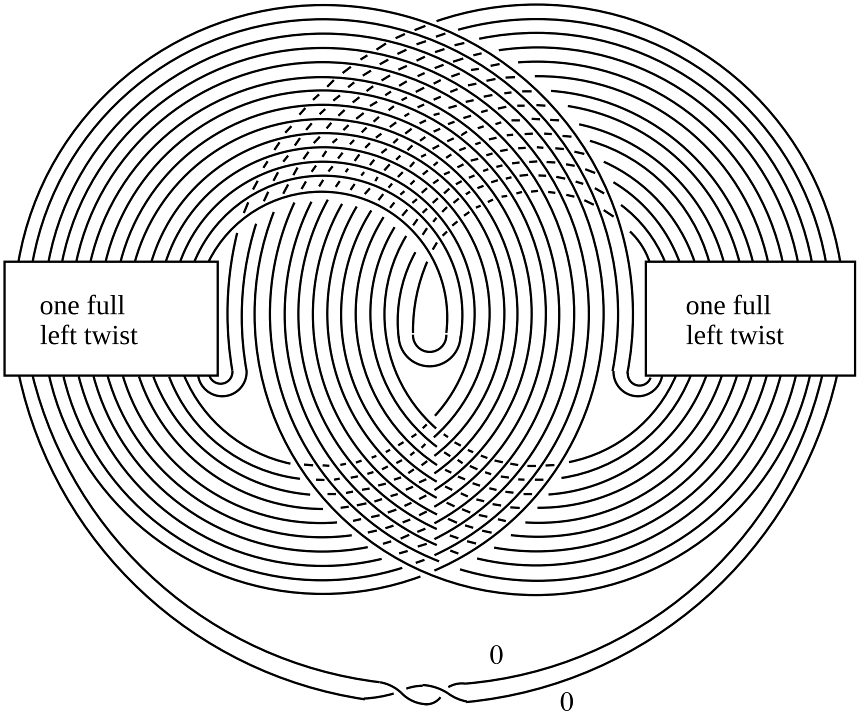


Figure 5.4

5.2 The unimodular integral forms

Let L be a lattice (i.e. a finitely generated, free Abelian group), and let $Q: L \otimes L \to \mathbb{Z}$ be a unimodular, symmetric, bilinear, integral form on L. An example of such a form is the intersection form (5.1) or (5.2). There are three basic invariants of Q. The first is the *rank*, defined by

$$\operatorname{rank} Q = \operatorname{rank}_{\mathbb{Z}} L = \dim_{\mathbb{R}}(L \otimes \mathbb{R}).$$

The second is the signature, defined as follows. Tensoring with $\mathbb{R}$ gives a real symmetric form Q on $L \otimes \mathbb{R}$. There is a basis for $L \otimes \mathbb{R}$ in which the form is diagonal, i.e. there is a basis $\{e_1, \dots, e_n\}$ with $Q(e_i, e_j) = \lambda_i \delta_{ij}$. Suppose that we have b_+ positive λ_i's and b_- negative λ_i's so that $b_+ + b_-$ is the rank of Q, then we define the *signature* of Q as

$$\operatorname{sign} Q = b_+ - b_-.$$

The form Q is called *definite* if b_+ or b_- vanishes (in the former case Q is said to be negative definite, and in the latter positive definite), and *indefinite* otherwise.

The rank and the signature of Q are invariants of the associated real bilinear form. The third invariant, called the *type*, is not. The type is said to be *even* if $Q(x,x) = 0 \bmod 2$ for all $x \in L$, otherwise the type is said to be *odd*.

Two forms, $Q_1 \colon L_1 \otimes L_1 \to \mathbb{Z}$ and $Q_2 \colon L_2 \otimes L_2 \to \mathbb{Z}$, are called isomorphic, $Q_1 \cong Q_2$, if there is an isomorphism $\varphi \colon L_1 \to L_2$ of lattices making the following diagram commute:

$$\begin{tikzcd} L_1 \otimes L_1 \arrow[rr, "\varphi\otimes\varphi"] \arrow[dr, "Q_1"'] & & L_2 \otimes L_2 \arrow[dl, "Q_2"] \\ & \mathbb{Z} & \end{tikzcd}$$

The rank, signature, and type are all invariants of the isomorphism. If Q is the intersection form (5.1) of a manifold M, its rank, signature and type are referred to as the rank, signature and type of the manifold M.

The following are some basic facts about unimodular, symmetric, bilinear, integral forms Q, see Milnor-Husemoller [115] and Serre [145].

- If Q is odd and indefinite then $Q \cong b_+ \cdot (+1) \oplus b_- \cdot (-1)$ where $b_+ + b_- = \operatorname{rank} Q$ and $b_+ - b_- = \operatorname{sign} Q$.
- If Q is even then $\operatorname{sign} Q = 0 \bmod 8$.
- Suppose that Q is even and indefinite. If $\operatorname{sign} Q \le 0$ then $Q \cong a \cdot E_8 \oplus b \cdot H$ where $a = -(\operatorname{sign} Q)/8$ and $b = (\operatorname{sign} Q + \operatorname{rank} Q)/2$. If $\operatorname{sign} Q \ge 0$ then $Q \cong a \cdot (-E_8) \oplus b \cdot H$, where $a = (\operatorname{sign} Q)/8$ and $b = (\operatorname{rank} Q - \operatorname{sign} Q)/2$.

Example. The form E_8 is even and negative definite with $\operatorname{sign} E_8 = -8$. The form H is even and indefinite with $\operatorname{sign} H = 0$.

5.3 Four-manifolds and intersection forms

The first question that we will address is the extent to which the intersection form determines a 4-manifold.

Theorem 5.4 (Whitehead [160]). *If M and N are simply-connected, closed, oriented 4-manifolds, then they are orientation preserving homotopy equivalent if and only if their intersection forms are isomorphic.*

Example. The intersection forms of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $S^2 \times S^2$ are not isomorphic over the integers because one is odd and the other is even (they are isomorphic over the reals, though). Therefore, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $S^2 \times S^2$ are not homotopy equivalent, either orientation preserving or orientation reversing.

Theorem 5.5 (Wall [155]). *Let M and N be simply-connected, closed, oriented, smooth 4-manifolds. If their intersection forms are isomorphic then there is $k \geq 0$ such that $M \# k(S^2 \times S^2)$ is diffeomorphic to $N \# k(S^2 \times S^2)$.*

The number k in this theorem is not specified. There exist, however, closed simply-connected oriented smooth 4-manifolds with isomorphic intersection forms which are not diffeomorphic, so that k in the Wall's theorem is not always zero. Such examples can be constructed with the help of the *Donaldson polynomials*, see Donaldson [37]. The Donaldson polynomial of degree d,

$$\mathbf{D}_d \colon H^2(M;\mathbb{R}) \to \mathbb{R}$$

is defined with the help of the gauge theory for smooth 4-manifolds M satisfying certain conditions. These polynomials are diffeomorphism invariants. For convenience, they are usually organized into a Donaldson series,

$$\mathbf{D}_M(x) = \sum_d \mathbf{D}_d(x)/d!$$

Example. Let $M = K3 \# \overline{\mathbb{C}P}^2$ be a connected sum of the Kummer surface $K3$ with $\overline{\mathbb{C}P}^2$. The intersection form of M is $Q = 2 \cdot E_8 \oplus 3 \cdot H \oplus (-1)$, so rank $Q = 23$ and sign $Q = -17$ with $b^+ = 3$ and $b^- = 20$. This form is odd and indefinite and therefore isomorphic over the integers to the form $3 \cdot (+1) \oplus 20 \cdot (-1)$. Let $N = 3 \cdot \mathbb{C}P^2 \# 20 \cdot \overline{\mathbb{C}P}^2$. The intersection forms of M and N are isomorphic. On the other hand (see e.g. Kronheimer and Mrowka [94])

$$\mathbf{D}_M(x) = e^{Q(x,x)/2} \cdot \cosh Q(E,x), \quad \text{while } \mathbf{D}_N(x) = 0$$

as analytic functions. Here, $E = \mathrm{PD}^{-1}[\mathbb{C}P^1]$ is the cohomology class Poincaré dual to the generator $\mathbb{C}P^1$ of $H_2(\overline{\mathbb{C}P}^2)$. Therefore, M and N are not diffeomorphic. It should be mentioned that the situation is completely different if one uses $\mathbb{C}P^2$ instead of $\overline{\mathbb{C}P}^2$ in the construction of the manifold M. It turns out that $K3 \# \mathbb{C}P^2$ *is* diffeomorphic to $4 \cdot \mathbb{C}P^2 \# 19 \cdot \overline{\mathbb{C}P}^2$, which can be shown directly with the help of Kirby calculus.

Freedman's theorem which is stated below implies that, in fact, if M and N are closed simply-connected smooth 4-manifolds with isomorphic intersection forms then they are *homeomorphic*.

Next we address the question of which forms can be realized as the intersection forms of 4-manifolds.

Theorem 5.6 (Rohlin [135]). *If M is a simply-connected, closed, smooth, oriented 4-manifold with even intersection form then* $\operatorname{sign} M = 0 \bmod 16$.

This theorem prohibits many forms from being intersection forms of smooth simply-connected closed 4-manifolds – this list includes, for example, E_8 and $E_8 \oplus H$. Simple-connectivity is essential: Habegger [66] constructed a smooth closed 4-manifold M with $Q_M \cong E_8 \oplus H$ which is not simply-connected, see also Fintushel–Stern [44]. Rohlin's theorem will be proved in Lecture 10 in a more general form.

Theorem 5.7 (Freedman [51]). *Given a unimodular, symmetric, bilinear, integral form which is even (odd), there exists, up to homeomorphism, exactly one (two) simply-connected, closed, topological* 4*-manifold(s) representing that form. In the odd case, one of the manifolds is never smooth.*

This theorem implies, for example, that there exists a simply-connected closed topological manifold M with $Q_M = E_8$; this M cannot be smooth by Rohlin's theorem. Also, the manifolds $M = K3 \# \overline{\mathbb{C}P}^2$ and $N = 3 \cdot \mathbb{C}P^2 \# 20 \cdot \overline{\mathbb{C}P}^2$ in the example above are both smooth and have odd indefinite intersection forms that are isomorphic over the integers. Therefore, M and N are homeomorphic but not diffeomorphic. A very important special case of Freedman's theorem is as follows.

Corollary 5.8. *If a topological* 4*-manifold M is homotopy equivalent to S^4 then M is homeomorphic to S^4.*

This proves the topological 4-dimensional Poincaré conjecture, a 4-dimensional cousin of the 3-dimensional Poincaré conjecture recently proved by Perelman, see for instance [119]. The question whether every smooth 4-manifold which is homeomorphic to S^4 is also diffeomorphic to S^4 remains unanswered; it is known as the smooth 4-dimensional Poincaré conjecture.

Theorem 5.9 (Donaldson [35] and [38]). *If the intersection form of a smooth, closed, oriented* 4*-manifold is positive definite then the form is isomorphic to $p \cdot (+1)$.*

It is worth mentioning that, although the number of positive definite unimodular integral symmetric bilinear forms of any rank is finite, it grows rapidly. For example, there are more than 10^{51} positive definite forms of rank 40. However, Donaldson's theorem implies that none of these forms but the diagonalizable ones can be the intersection forms of smooth closed 4-manifolds.

Theorem 5.10 (Furuta [56]). *If the intersection form Q of a smooth, simply-connected, closed, oriented* 4*-manifold is even then* $\operatorname{rank} Q > 5/4 \cdot |\operatorname{sign} Q|$.

In particular, if an even form $Q \cong a \cdot E_8 \oplus b \cdot H$ is an intersection form of a smooth, simply-connected, closed, oriented 4-manifold then $a < b$. Conjecturally, a stronger conclusion holds for the latter theorem, namely, that $\operatorname{rank} Q \geq 11/8 \cdot |\operatorname{sign} Q|$. This is known as the $11/8$-*conjecture*. The equality $\operatorname{rank} Q = 11/8 \cdot |\operatorname{sign} Q|$ is realized, for example, by the Kummer surface K3.

5.4 Exercises

1. Prove that any integral symmetric bilinear form of rank 2 which is unimodular and indefinite has a non-zero vector of square zero. Use this observation to classify all such forms.
2. Verify directly that the form E_8 is unimodular and has signature -8.
3. Prove that the form $E_8 \oplus (-1)$ is not diagonalizable over the integers. (Hint: count the number of vectors of square -1).
4. Prove that any closed, oriented, simply-connected 4-manifold with even intersection form and vanishing signature is homeomorphic to either S^4 or a connected sum of several copies of $S^2 \times S^2$.

Lecture 6
Four-manifolds with boundary

6.1 The intersection form

Let M be a compact oriented connected simply-connected smooth 4-manifold with $\partial M \neq \emptyset$. Every class in $H_2(M)$ can still be represented by a smoothly embedded closed oriented surface (see for instance [61], Remark 1.2.4) so the intersection form

$$Q_M\colon H_2(M) \otimes H_2(M) \to \mathbb{Z}, \quad (a,b) \mapsto a \cdot b, \tag{6.1}$$

over the integers can still be defined as in (5.2). However, this form is not necessarily unimodular. For instance, the intersection form of $M = S^2 \times D^2$ is $Q_M = 0$.

Let R be a commutative ring with an identity element. An oriented 3-manifold Σ is called an *R-homology sphere* if it has the same R-homology as S^3, that is, $H_*(\Sigma; R) = H_*(S^3; R)$. If $R = \mathbb{Z}$, we refer to Σ as an integral homology sphere, or simply a *homology sphere*. If $R = \mathbb{Q}$, we talk about rational homology spheres. For example, every lens space $L(p,q)$ with $p \geq 1$ is a rational homology sphere.

Theorem 6.1. *The intersection form* (6.1) *is unimodular if and only if ∂M is a disjoint union of integral homology spheres.*

Proof. Assume for simplicity that ∂M is connected. We will first show that, if the form (6.1) is unimodular, then ∂M is a homology sphere. Let $i_*\colon H_2(\partial M) \to H_2(M)$ be the homomorphism induced by the inclusion $i\colon \partial M \to M$. If $\operatorname{im} i_* \neq 0$ then the form (6.1) is degenerate because $a \cdot b = 0$ for any $a \in \operatorname{im} i_*$ and every $b \in H_2 M$. Since M is simply-connected, Poincaré duality, together with the universal coefficient theorem, implies that $H_3(M, \partial M) = H^1(M) = \operatorname{Hom}(H_1(M), \mathbb{Z}) = 0$. Therefore, we have the following exact sequence

$$0 \to H_2(\partial M) \xrightarrow{i_*} H_2(M) \xrightarrow{j_*} H_2(M, \partial M) \to H_1(\partial M) \to 0.$$

Then $\operatorname{im}\{H_2(\partial M) \to H_2(M)\} = \operatorname{im} i_*$ is trivial if and only if $H_2(\partial M) = 0$. By Poincaré duality the latter means that ∂M is an integral homology sphere.

If ∂M is a homology sphere, $H^2(M) = H^2(M, \partial M)$, and the non-degeneracy of (6.1) follows from Poincaré–Lefschetz duality, which in our case states that the form $H^2(M) \otimes H^2(M, \partial M) \to \mathbb{Z}$ given by $(a,b) \mapsto \langle a \smile b, [M, \partial M]\rangle$ is non-degenerate. $\square$

Theorem 6.2. *Let M be a 4-manifold with boundary obtained by integral surgery on a framed link $\mathcal{L}$ in S^3. Then Q_M is isomorphic to the linking matrix of $\mathcal{L}$.*

Note that the linking matrix of $\mathcal{L}$ depends on how the components of $\mathcal{L}$ are oriented, however, different choices lead to isomorphic linking matrices. The same is true about the intersection form Q_M. In the proof, we will choose an oriented basis in $H_2(M)$ and will orient $\mathcal{L}$ so that Q_M will be *equal* to the linking matrix of $\mathcal{L}$.

Example. The intersection form of E_{-1}, obtained by $(+1)$-surgery on an unknot in S^3 is (1), see Lecture 2. This is consistent with the fact that $E_{-1} \cup_{S^3} D^4 = \mathbb{C}P^2$ and, as we know, $Q_{\mathbb{C}P^2} = (1)$.

Before we prove Theorem 6.2 we describe the following construction. Let $k \subset S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ be a knot. It bounds a smooth surface F_k inside $D^4 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 \le 1\}$ such that $F_k \cap S^3 = k$ and the intersection is transversal, see Figure 6.1. For example, take a Seifert surface of k in S^3, see Lecture 7, and slightly push its interior radially into D^4. Given such an embedding $F_k \subset D^4$, define $f\colon F_k \to \mathbb{R}$ as a restriction to F_k of the distance function measuring the distance from a point $x \in D^4$ to the center of D^4. One may assume without loss of generality that $f(x) \neq 0$ for all $x \in F_k$.

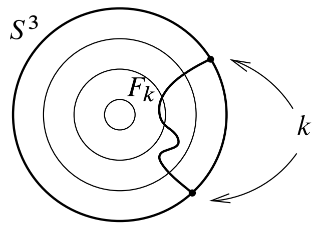


Figure 6.1

Let $S^3_t = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = t^2\}$ be the t-level set of the distance function. Recall that the intersection $F_k \cap S^3_t$ is called transversal at a point $p \in F_k \cap S^3_t$ if $T_pF_k + T_pS^3_t = T_pD^4$. The points $p \in F_k \cap S^3_t$ at which the intersection is not transversal are the critical points of f (where the gradient of f vanishes). In local coordinates x^i on F_k, this means that $(\partial f/\partial x^i)(p) = 0$, so the Taylor expansion of f near such a point p takes the form

$$f(x) = f(p) + \sum_{i,j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)(p) \cdot (x^i - p^i)(x^j - p^j) + \cdots .$$

A critical point p is called *non-degenerate* if

$$\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)(p) \neq 0.$$

Non-degenerate critical points are isolated. A function f whose critical points are all non-degenerate is called a *Morse function*. A Morse function has only finitely many critical points (since F_k is compact). Whether f is a Morse function or not depends on the embedding $F_k \subset D^4$. However, after a small perturbation of the embedding, if necessary, one may assume that f is a Morse function, see Milnor [113]. From now on, when talking about F_k, we will assume that the intersection $F_k \cap S^3_t$ is transversal at all but finitely many points, which are non-degenerate.

The surface F_k can be conveniently thought of as a "movie": at each "moment of time" t, one can draw the intersection $F_k \cap S^3_t$, inside S^3 or $\mathbb{R}^3$. Each intersection is a (perhaps singular) curve in the 3-space, and together they span a 2-dimensional surface in D^4. For example, the surface F_k schematically pictured in Figure 6.1 might have the movie description shown in Figure 6.2. In this example the knot k, which appears in the $t = 1$ slice, is trivial.

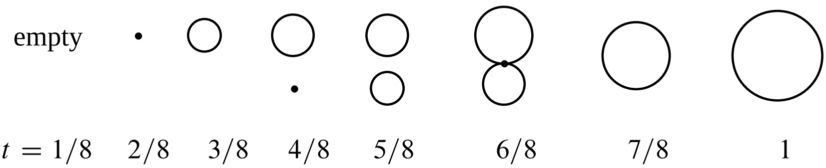


Figure 6.2

The slices S^3_t are 3-dimensional, so the intersection curves $F_k \cap S^3_t$ may in general be knotted. Figure 6.3 shows the movie of a surface bounded by the trefoil.

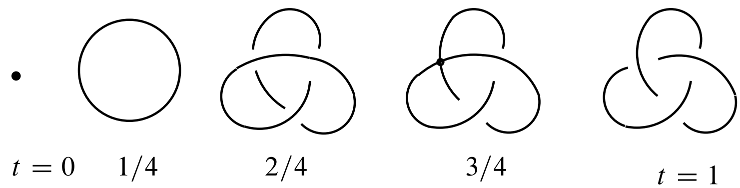


Figure 6.3

Let k and ℓ form a link in S^3, and $F_k, F_\ell \subset D^4$ be the surfaces as above such that $F_k \cap S^3 = k$ and $F_\ell \cap S^3 = \ell$. Suppose that F_k and F_ℓ are oriented. Then one can define the intersection number $F_k \cdot F_\ell$ as follows. First, perturb the embeddings (if necessary) to make the intersection $F_k \cap F_\ell$ transversal, keeping $F_k \cap S^3$ and $F_\ell \cap S^3$ fixed, then count the (± 1)'s associated to the intersection points as in (5.2).

Lemma 6.3. *Given an oriented link $k \cup \ell$ in S^3, there is a canonical choice of orientations on F_k and F_ℓ such that* $\mathrm{lk}(k, \ell) = F_k \cdot F_\ell$.

Proof. Suppose that the link components k and ℓ are oriented, that the surfaces F_k and F_ℓ intersect transversally, and consider a movie of F_k and F_ℓ. Each time the surfaces intersect, the linking number changes by $1 \bmod 2$, see Figure 6.4.

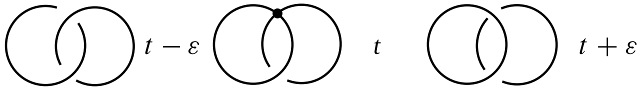


Figure 6.4

The movie connects the original link $k \cup \ell$ with an empty link. Therefore, $F_k \cdot F_\ell$ equals $\mathrm{lk}(k,\ell)$ modulo 2. The choice of orientations described in the rest of the proof will give $F_k \cdot F_\ell = \mathrm{lk}(k,\ell)$.

Orient $S^3 = \mathbb{R}^3 \cup \{\infty\}$ by the standard basis $\langle e_1, e_2, e_3\rangle$ in $\mathbb{R}^3$. The ball D^4 will be oriented by $\langle n, e_1, e_2, e_3\rangle$ where n is the normal vector to S^3 looking outside with respect to the ball D^4. Now, each of the curves $F_k \cap S^3_t$ inherits an orientation from k. Suppose that $F_k \cap S^3_t$ is oriented by its tangent vector, t_1. At a point $p \in F_k \cap S^3_t$, choose a vector n_1 in the plane $T_p F_k$ which is normal to t_1 and whose projection onto the t-direction equals 1 (this projection is not zero due to the transversality condition). The pair $\langle n_1, t_1\rangle$ orients F_k. The surface F_ℓ can be oriented similarly by $\langle n_2, t_2\rangle$.

Let $p \in F_k \cap F_\ell$. In the movie of F_k and F_ℓ, the point p corresponds to a crossing change like the one in Figure 6.5 (here, k' belongs to the movie of F_k, and ℓ' to the movie of F_ℓ).

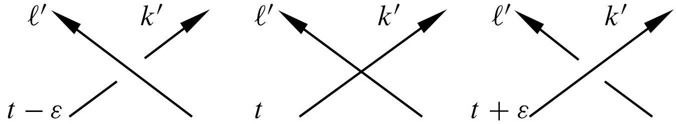


Figure 6.5

In the situation shown in Figure 6.5, as we pass through time t, the linking number $\mathrm{lk}(k',\ell')$ increases by 1. On the other hand, we assign ± 1 to the intersection point p according to whether the orientations of $T_p F_k \oplus T_p F_\ell$ and $T_p D^4$ agree or differ. Thus we need to compare the orientations of the bases $\langle n_1, t_1, n_2, t_2\rangle$ and $\langle n, e_1, e_2, e_3\rangle$. The first basis is orientation-preservingly equivalent to $\langle n_1 + n_2, t_1, t_2, n_1 - n_2\rangle$. The vectors t_1, t_2, and $n_1 - n_2$ are all tangent to S^3_t, and $n_1 + n_2$ has positive projection onto n. Therefore, one only needs to show that $\langle t_1, t_2, n_1 - n_2\rangle$ defines the correct orientation on $\mathbb{R}^3$. But as the strand k' moves upward with respect to the strand ℓ', the vector $n_1 - n_2$ must look upward, see Figure 6.6, so we are finished. The remaining case, in which the linking number decreases by 1 as we pass through an intersection point, is dealt with similarly. □

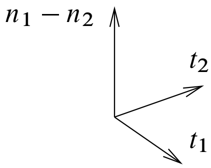


Figure 6.6

Proof of Theorem 6.2. Suppose that M is obtained by p-surgery on a single knot k, so that $M = D^4 \cup_{S^1 \times D^2} (D^2 \times D^2)$. By the Mayer–Vietoris exact sequence, the group $H_2(M) = H_1(S^1 \times D^2) = \mathbb{Z}$ is generated by the homology class of a surface A obtained by gluing together surfaces $F_k \subset D^4$ and $G_k \subset D^2 \times D^2$ such that $F_k \cap S^3 = G_k \cap S^3 = k$, see Figure 6.7.

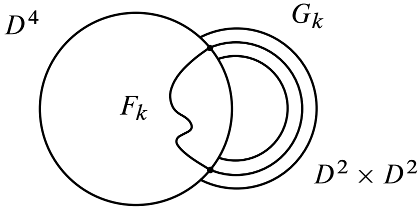


Figure 6.7

We choose F_k to be an embedded surface in D^4 as above. The surface G_k can be chosen to be the central disc of $D^2 \times D^2$. The homological self-intersection $A \cdot A$ can now be computed as follows. Let us consider a parallel copy G'_k of G_k inside $D^2 \times D^2$, then $k' = G'_k \cap S^3$ is a longitude of $k \subset S^3$ with linking number $\mathrm{lk}(k, k') = p$. Let F'_k be a perturbation of F_k inside its homology class, transversal to F_k, and such that $F'_k \cap S^3 = k'$. Then the surface $A' = F'_k \cup_{k'} G'_k$ is homologous to A, and $A \cdot A = A \cdot A' = (F_k \cup G_k) \cdot (F'_k \cup G'_k) = F_k \cdot F'_k = \mathrm{lk}(k, k') = p$. Here we used the fact that $G_k \cap G'_k = \emptyset$.

In general, if M is a surgery on a framed link, the result follows from the previous argument, repeated for each link component, and Lemma 6.3. □

Corollary 6.4. *Let a* 3*-manifold X be obtained by surgery on a framed link $\mathcal{L}$ with linking matrix A. Then X is a homology sphere if and only if* $\det A = \pm 1$.

Corollary 6.5. *Every integral symmetric bilinear form Q is the intersection form of a smooth simply-connected* 4*-manifold M with boundary. If Q is unimodular, then ∂M is a homology sphere.*

Proof. Take a framed link $\mathcal{L}$ with linking matrix Q; the manifold M is the surgery on $\mathcal{L}$. Since $\pi_1 D^4$ is trivial, and adding 2-handles does not change the fundamental group, M is simply-connected. □

Example. The form E_8 is realized by a smooth 4-manifold as in Figure 3.15 with boundary the Poincaré homology sphere. The form H is realized by $(S^2 \times S^2) \setminus \operatorname{int} D^4$.

Corollary 6.6. *Any closed oriented* 3*-manifold bounds a smooth oriented simply-connected* 4*-manifold whose intersection form is even.*

Proof. This follows from Theorem 6.2 and Theorem 4.1 of Lecture 4. □

6.2 Homology spheres via surgery on knots

A lens space $L(p,q)$ is the $(-p/q)$-surgery on an unknot in S^3. One can easily check that $H_1(L(p,q)) = \pi_1(L(p,q)) = \mathbb{Z}/p$. Therefore $L(p,q)$ is a homology sphere iff $p = \pm 1$, in which case it is S^3.

Any manifold obtained by a p/q-surgery on a knot has the first homology of $L(p,q)$; it is a homology sphere iff $p = \pm 1$. For example, for any m, the $(-1/m)$-surgery on right handed trefoil is a homology sphere; as we have seen in Lecture 3, it is the Seifert manifold $M((2,1),(3,-1),(6m+1,-m))$.

6.3 Seifert homology spheres

Let $M = M((a_1,b_1),\dots,(a_n,b_n))$, each $a_i \geq 2$ and $n \geq 3$, be a Seifert manifold, see Lecture 1. One can see from the definition of M that

$$\pi_1(M) = \langle\, x_1,\dots,x_n,h \mid [h,x_i] = 1, x_i^{a_i} h^{b_i} = 1, x_1 \dots x_n = 1 \,\rangle.$$

Moreover, $H_1(M) = \operatorname{coker} A$ where $A\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ is given by the matrix

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & 0 & b_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & a_n & b_n \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

in the natural basis $x_1,\dots,x_n,h$ of $H_1(M)$ obtained by Abelianizing $\pi_1(M)$. In particular, M is a homology sphere if and only if $\det A = \pm 1$, that is,

$$a_1 \cdots a_n \cdot \sum_{i=1}^{n} b_i/a_i = \pm 1.$$

The choice of the plus or minus sign here corresponds to the choice of orientation of M. Note that if M is a homology sphere, the integers $a_1,\dots,a_n$ must be pairwise relatively prime.

Theorem 6.7. *For any pairwise relatively prime integers $a_1, \dots, a_n$, with each $a_i \geq 2$, there exists a unique (up to orientation preserving homeomorphism) Seifert manifold $M((a_1, b_1), \dots, (a_n, b_n))$ such that*

$$a_1 \cdots a_n \cdot \sum_{i=1}^{n} \frac{b_i}{a_i} = 1. \tag{6.2}$$

This oriented manifold is a homology sphere; it is usually denoted by $\Sigma(a_1, \dots, a_n)$.

Proof. The equation (6.2) reduced modulo a_i takes the form $b_i a_1 \cdots \hat{a}_i \cdots a_n = 1 \mod a_i$ ("hat" stands for the missing factor). We see that it determines b_i uniquely modulo a_i for each i. The only freedom we have in choosing b_i is in replacing b_i by $b_i' = b_i + k_i a_i$ so that $\sum_{i=1}^n k_i = 0$. The latter is implied by the equation (6.2). Thus, we only need to show that the manifolds $M = M((a_1, b_1), \dots, (a_n, b_n))$ and $M' = M((a_1, b_1'), \dots, (a_n, b_n'))$ are homeomorphic. The manifold M' has the surgery description shown in Figure 6.8.

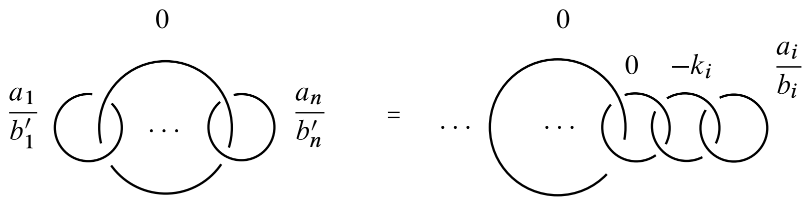


Figure 6.8

The two links in Figure 6.8 describe homeomorphic manifolds because, for each i,

$$\frac{a_i}{b_i'} = \frac{a_i}{b_i + k_i a_i} = 0 - \frac{1}{-\dfrac{b_i + k_i a_i}{a_i}} = \left[0, -k_i, \frac{a_i}{b_i}\right].$$

Now, one can slide the central 0-framed handle once over each of the $(-k_i)$-framed handles, and then eliminate, with the help of Kirby calculus, all the pairs of $(0, -k_i)$-framed handles. Since $\sum_{i=1}^n k_i = 0$, the central handle still has framing 0, so we arrive at the link diagram in Figure 6.9 which describes the manifold M. Therefore, M and M' are homeomorphic by an orientation preserving homeomorphism. □

Example. To find a link description of $\Sigma(5, 6, 7)$, first find b_1, b_2, b_3 such that $42b_1 + 35b_2 + 30b_3 = 1$. For example, $b_1 = 3, b_2 = -1, b_3 = -3$ will do. Then calculate the continued fractions

$$\frac{5}{3} = [2, 3], \quad -\frac{6}{1} = [-6], \quad -\frac{7}{3} = [-2, 3].$$

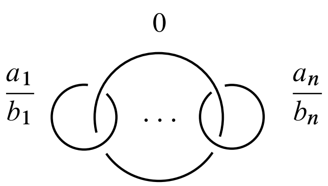


Figure 6.9

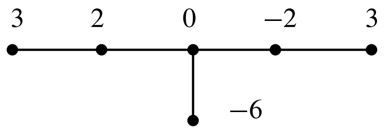


Figure 6.10

The linking diagram for $\Sigma(5, 6, 7)$ is now described by the graph in Figure 6.10. The choice $b_1 = 3, b_2 = -1, b_3 = 4$ would give $42b_1 + 35b_2 + 30b_3 = 1 + 1 \cdot 210$ and the diagram in Figure 6.11.

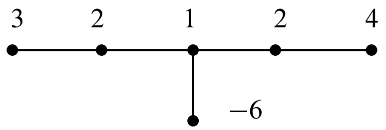


Figure 6.11

Remark. Theorem 6.7 explains the notation $\Sigma(2, 3, 5)$ for the Poincaré homology sphere in Lecture 3. The homology spheres in Figure 3.19 are Seifert homology spheres $\Sigma(2, 3, 6m + 1)$, and the one in Figure 3.20 is $\Sigma(3, 4, 7)$.

6.4 The Rohlin invariant

All 4-manifolds in this subsection are assumed to be connected, simply-connected, oriented, smooth, and compact, with or without boundary. By the signature of such a manifold we mean the signature of its intersection form.

Lemma 6.8. *Let M_1 and M_2 be 4-manifolds with boundaries ∂M_1 and ∂M_2 such that $H_*(\partial M_1) = H_*(\partial M_2) = H_*(S^3)$, and let $\varphi \colon \partial M_1 \to \partial M_2$ be an orientation-reversing diffeomorphism of their boundaries. Then $M = M_1 \cup_\varphi M_2$ is a smooth manifold with* $\operatorname{sign} M = \operatorname{sign} M_1 + \operatorname{sign} M_2$.

Remark. This theorem holds without the requirement that ∂M_1 and ∂M_2 be homology spheres, and is known as the Novikov additivity of the signature, see for instance Kirby [84].

Proof of Lemma 6.8. Let N (unoriented) denote the submanifold of M equal to the identification of ∂M_1 with ∂M_2. Let y be an element of $H_2(M)$ and intersect y with N. Since $H_1(N) = 0$, the 1-cycle $y \cap N$ bounds in N, and y splits as a sum of elements in $H_2(M_1)$ and $H_2(M_2)$. Therefore, $Q_M = Q_{M_1} \oplus Q_{M_2}$, and $\operatorname{sign} M = \operatorname{sign} M_1 + \operatorname{sign} M_2$. □

Let Σ be an oriented homology sphere, and W a 4-manifold with boundary Σ whose intersection form Q_W is even. Since Q_W is unimodular, its signature is divisible by 8, and we can define the *Rohlin invariant* by the formula

$$\mu(\Sigma) = \frac{1}{8} \operatorname{sign} W \mod 2.$$

A different choice W' gives the same value of $\mu(\Sigma)$ because $\operatorname{sign} W - \operatorname{sign} W' = \operatorname{sign}(W \cup_\Sigma -W') = 0 \mod 16$ by Rohlin's theorem, see Theorem 5.6.

The invariant μ has the following properties. If $-\Sigma$ is Σ with opposite orientation then $\mu(-\Sigma) = \mu(\Sigma)$. Also, if Σ_1 and Σ_2 are homology spheres then their connected sum $\Sigma_1 \# \Sigma_2$ is a homology sphere, and $\mu(\Sigma_1 \# \Sigma_2) = \mu(\Sigma_1) + \mu(\Sigma_2)$.

Example. The Poincaré homology sphere $\Sigma(2,3,5)$ bounds a smooth 4-manifold with the intersection form isomorphic to E_8, see Figure 3.15. Since $\operatorname{sign} E_8 = -8$ and E_8 is even, we find that $\mu(\Sigma(2,3,5)) = 1 \mod 2$.

Some deeper properties of the Rohlin invariant will be discussed in Lecture 11.

6.5 Exercises

1. Prove that any $\mathbb{Z}/2$-homology 3-sphere is orientable.
2. Given a positive integer s, find a link descriptions of $\Sigma(2, 2s-1, 2s+1)$.
3. Prove that any Seifert homology sphere $\Sigma(a_1, \ldots, a_n)$ with even a_1 can be obtained by a surgery according to an even-weighted star-shaped graph.

Lecture 7

Invariants of knots and links

7.1 Seifert surfaces

A *Seifert surface* for an oriented link in S^3 is a connected compact oriented surface smoothly embedded in S^3 with oriented boundary equal to the link. As usual, we draw links in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ as links in $\mathbb{R}^3$.

Example. The left-handed trefoil pictured in Figure 7.1 bounds a Möbius strip, M_0, which is not orientable, and also a Seifert surface, M_1.

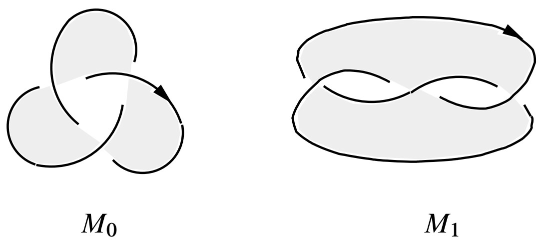


Figure 7.1

The surface M_1 is homeomorphic to a punctured torus, which can be seen as follows. We first push in a narrow strip through the center band, then deform the resulting surface M_2 by an isotopy to appear as M_3, see Figure 7.2. The surfaces M_3 and M_4 are homeomorphic (though we had to change the embedding), and M_4 is obviously homeomorphic to a punctured torus.

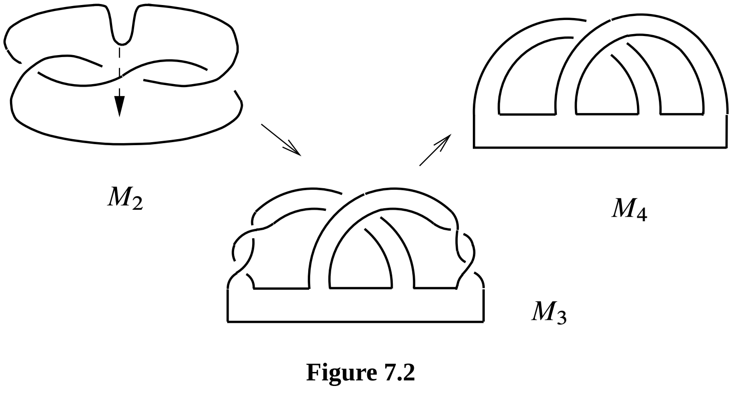


Figure 7.2

Theorem 7.1. *Every oriented link in S^3 bounds a Seifert surface.*

Proof. The proof consists of an explicit construction. Let us fix a regular projection for the link. Near each crossing point, delete the over- and undercrossing and replace them by “short-cut” arcs keeping track of the orientations. The result of this procedure is a collection of circles, called Seifert circles, drawn over the link diagram.

In Figure 7.3, there are two Seifert circles on the left, one inside the other, and two Seifert circles on the right, one under the other.

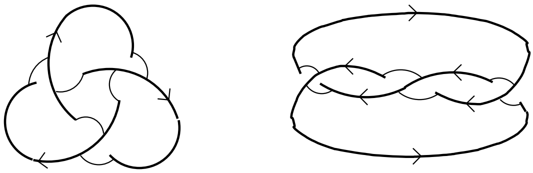


Figure 7.3

These circles can be used to construct a Seifert surface as follows. Each of the circles is the boundary of a disc lying in the plane, see Figure 7.4 where the discs are shaded. If any of the circles are nested, the inner discs may need to be lifted above outer discs, according to the nesting. To form the Seifert surface connect the discs together by attaching twisted bands at the points corresponding to crossing points in the original link diagram. These bands should be twisted to correspond to the direction of the crossing in the link. If the surface is connected, it is a Seifert surface. Otherwise, join the components by tubes. □

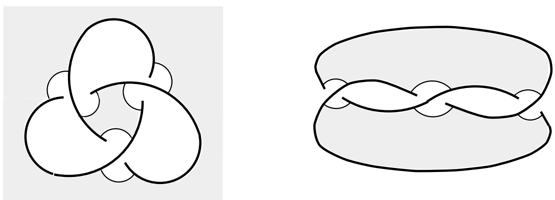


Figure 7.4

Given a regular projection of a link of n components, let c be the number of crossings and s be the number of Seifert circles. The Seifert surface constructed by the algorithm above has genus g, where $2g = 2 - s - n + c$. The least genus of all Seifert surfaces of a given knot is called the *genus* of the knot; note that it is independent of the knot’s orientation. For instance, the genus of the trefoil is at most 1 since it bounds a Seifert surface of genus 1. The genus of an oriented link is defined similarly.

We saw in Lecture 4 that every embedded connected surface in $\mathbb{R}^3$ with non-empty boundary is isotopic to a surface constructed by attaching bands to a disc. The surfaces

M_3 and M_4 pictured above give an example of surfaces constructed by attaching bands to a disc. They are homeomorphic but not isotopic.

If a Seifert surface is presented as a disc with bands, that surface can be deformed by sliding one of the points at which a band is attached over another band without changing the isotopy type of its boundary. The resulting surface is again a disc with bands. A Seifert surface can also be modified by adding two new bands, as illustrated in Figure 7.5.

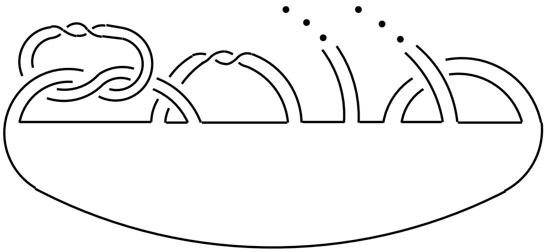


Figure 7.5

One of the bands added is untwisted and unknotted; the other can be twisted (an even number of times), or knotted, and can link the other bands. It is clear that the boundary of the new surface is the same link as for the original Seifert surface. The operation of adding such a pair of new bands is called *stabilization*. Two Seifert surfaces for an oriented link are called *stably equivalent* if there is a sequence of stabilizations that can be applied to each so that the resulting surfaces can be deformed into each other.

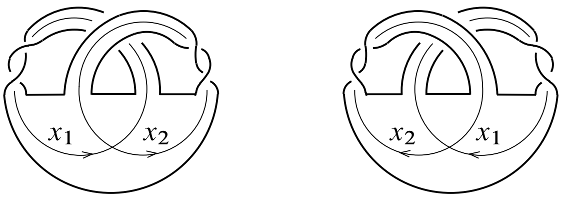
Theorem 7.2. *Any two Seifert surfaces for an oriented link are stably equivalent.*

A proof of this theorem can be found in Levine [101] and also in the Kauffman's book [80], Theorem 7.7.

7.2 Seifert matrices

Given an oriented link $\mathcal{L}$, fix a Seifert surface F for $\mathcal{L}$. Since F is oriented, it is possible to distinguish one side as a "top" side (formally, this means picking a non-zero normal vector field on F which orients F). Given any simple closed oriented curve x on F, one can form its *positive push off*, x^+, which runs parallel to x, and lies just above F in the direction of the normal field. Let simple closed curves $x_1, \dots, x_n$ generate a basis in $H_1(F; \mathbb{Z})$. The associated *Seifert matrix* is the $n \times n$ matrix S with $S_{ij} = \mathrm{lk}(x_i, x_j^+)$. It is not difficult to see that the Seifert matrix gets transposed when one changes the orientation of F.

Example. Figure 7.6 shows Seifert surfaces for the left-handed and right-handed trefoil knots together with bases for the first homology.



Left-handed trefoil Right-handed trefoil

Figure 7.6

Orient the surfaces so that the normal field points towards us at the bottom part of the surface. The corresponding Seifert matrices are

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Example. If a Seifert surface F is formed from a single disc by attaching bands, there naturally arises a family of curves on F which produces a basis for $H_1(F;\mathbb{Z})$, see Figure 7.7.

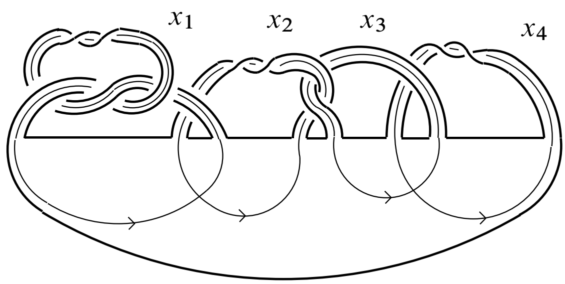


Figure 7.7

Orient the surface in Figure 7.7 by the normal vector field pointing towards us at the bottom part of the surface. Then the boundary knot has Seifert matrix

$$\begin{pmatrix} -4 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Seifert matrix clearly depends on the choices made in its definition. As for the choice of a basis, any two bases are related by an invertible integral matrix U; the corresponding Seifert matrices are S and $U^\top SU$. Any deformation of the surface results in a change of basis. The effect of stabilization on the Seifert matrix is to add two new columns and rows, with entries as indicated:

$$S \to \begin{pmatrix} & & & * & 0 \\ & S & & \vdots & \vdots \\ & & & * & 0 \\ * & \dots & * & * & 1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix} \tag{7.1}$$

Two integral matrices, S_1 and S_2, are called *S-equivalent* if there is a sequence of stabilization operations (7.1) that can be applied to each so that the resulting matrices, S_1' and S_2', are related by the equation $S_1' = U^\top S_2' U$ for some invertible integral matrix U. A consequence of Theorem 7.2 and the discussion above is the following:

Theorem 7.3. *Any two Seifert matrices of an oriented link are S-equivalent.*

7.3 The Alexander polynomial

Let S be a Seifert matrix for an oriented link $\mathcal{L}$, and $S^\top$ its transpose. The polynomial

$$\Delta_{\mathcal{L}}(t) = \det(t^{1/2}S - t^{-1/2}S^\top) \tag{7.2}$$

in $t^{1/2}$ and $t^{-1/2}$ is called the *Alexander polynomial* of $\mathcal{L}$.

Corollary 7.4. *The polynomial $\Delta_{\mathcal{L}}(t)$ is a well-defined invariant of an oriented link $\mathcal{L}$.*

Proof. One needs to prove that $\Delta_{\mathcal{L}}(t)$ does not depend on the choice of S. According to Theorem 7.3, one only needs to check that $\det(t^{1/2}S - t^{-1/2}S^\top)$ is not affected by the stabilization (7.1) on S. This is a matter of elementary algebra. □

Note that $\Delta_{\mathcal{L}}(t) = (-1)^{\mathrm{rank}(S)}\Delta_{\mathcal{L}}(t^{-1})$. In particular, if k is a knot, then $\mathrm{rank}(S)$ is even, so $\Delta_k(t)$ is in fact a polynomial in t and t^{-1}, and $\Delta_k(t) = \Delta_k(t^{-1})$. In addition, $\Delta_k(t)$ is independent of the choice of orientation on k. The Alexander polynomial of an unknot is 1.

Example. Both left- and right-handed trefoil knots have the Alexander polynomial $t^{-1} - 1 + t$. This proves in particular that both trefoil knots are not trivial; therefore, their genus is 1.

Example. The Alexander polynomial of the knot in Figure 7.7 is $4t^{-2} - 12t^{-1} + 17 - 12t + 4t^2$.

Example. Let $p, q \geq 2$ be relatively prime integers, and consider the polynomial $f(z,w) = z^p + w^q$ of two complex variables $z, w \in \mathbb{C}$. It has a singular point ($\partial f/\partial z = \partial f/\partial w = 0$) at the origin. The intersection $k_{p,q}$ of $V = f^{-1}(0)$ with a 3-sphere S_ε of radius $\varepsilon > 0$ centered at the origin is a knot in $S_\varepsilon \cong S^3$ called the right-handed (p,q)-*torus knot*. The left-handed (p,q)-torus knot is a mirror image of $k_{p,q}$. It is easily verified that $k_{p,q}$ lies in the torus consisting of all (z,w) with $|z| = a$ and $|w| = b$ where a and b are positive constants (if $\varepsilon = \sqrt{2}$ then $a = b = 1$). In fact, $k_{p,q}$ consists of all the pairs $(a\, e^{iq\theta}, b\, e^{ip\theta + i\pi/q})$ as the parameter θ ranges from 0 to 2π. Thus $k_{p,q}$ sweeps around the torus q times in one coordinate and p times in the other.

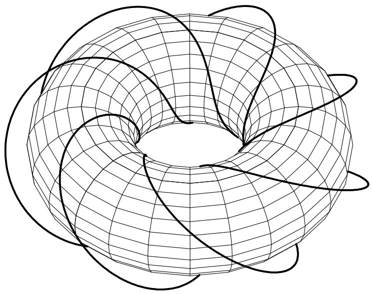


Figure 7.8

The right-handed trefoil is the knot $k_{2,3}$. Figure 7.8 shows the knot $k_{3,7}$. It is slightly pushed off the torus surface for a better view (the picture was created with the help of `Maple`). We will show in Lecture 8 that

$$\Delta_{k_{p,q}}(t) = t^{-(p-1)(q-1)/2} \cdot \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}. \tag{7.3}$$

Example. Let k be a knot in S^3 and k^* its mirror image. Then

$$\Delta_{k^*}(t) = \Delta_k(t). \tag{7.4}$$

This can be seen as follows. Let $\tau\colon S^3 \to S^3$ be an orientation reversing diffeomorphism of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ induced by reflection with respect to a 2-plane in $\mathbb{R}^3$. Then $k^* = \tau(k)$. Let F be a Seifert surface of k of genus g, then $\tau(F)$ is a Seifert surface of k^*. If the orientation of $\tau(F)$ is induced via τ by that of F then, for any simple closed curve x in F,

$$\mathrm{lk}(\tau(x), \tau(x)^+) = \mathrm{lk}(\tau(x), \tau(x^+)) = -\,\mathrm{lk}(x, x^+),$$

and the Seifert matrices S and S^* of the knots k and k^*, respectively, are related by $S^* = -S$. Since the size of both S and S^* is $2g \times 2g$, we see that $\Delta_{k^*}(t) = \Delta_k(t)$.

Example. The right-handed and the left-handed (p, q)-torus knots are mirror images of each other, therefore, their Alexander polynomials coincide.

Example. For a link $\mathcal{L}$ with more than one components, $\Delta_{\mathcal{L}}(t)$ may depend on the orientation of $\mathcal{L}$. Here is an example. Pictured in Figure 7.9 are two Seifert surfaces of the Hopf link $\mathcal{L}$ with different orientations of the link components. Each of them is a band with one full left- or right-handed twist. The bases in the first homology in both cases are represented by the middle curves of the bands. The corresponding Seifert matrices are (-1) and (1), respectively. Hence the Alexander polynomial $\Delta_{\mathcal{L}}(t)$ of the link $\mathcal{L}$ is either $-t^{1/2} + t^{-1/2}$ or $t^{1/2} - t^{-1/2}$, depending on the choice of its orientation.

Figure 7.9

Theorem 7.5 (Conway formula). *Let $\mathcal{L}_+$, $\mathcal{L}_0$, and $\mathcal{L}_-$ be three oriented links which coincide away from the ball B, and intersect this ball in two unknotted arcs, each as in Figure* 7.10. *Then*

$$\Delta_{\mathcal{L}_+}(t) - \Delta_{\mathcal{L}_-}(t) + (t^{1/2} - t^{-1/2})\, \Delta_{\mathcal{L}_0}(t) = 0.$$

Proof. Let F_0 be a Seifert surface of the link $\mathcal{L}_0$ which intersects B in two disjoint discs, each of which is bounded by an arc on ∂B and a component of $\mathcal{L}_0 \cap B$. Denote by F_+ and F_- Seifert surfaces of the links $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively, which coincide with F_0 outside of B, and intersect B in the twisted bands shown in Figure 7.10.

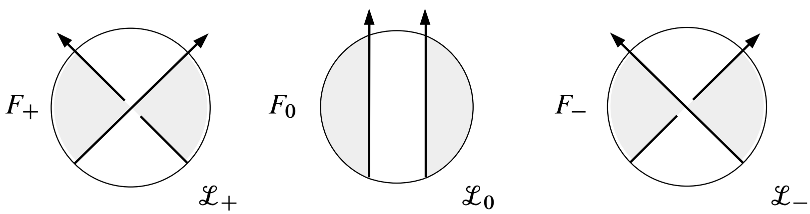


Figure 7.10

The surfaces F_+ and F_- are homeomorphic. Let $a_1, \dots, a_n$ be curves on F_0 forming a basis for $H_1(F_0; \mathbb{Z})$, and let a_0 be a curve on $F_\pm$ such that the curves

$a_0, a_1, \ldots, a_n$ together give a basis for $H_1(F_\pm; \mathbb{Z})$. If S_+, S_-, S_0 are Seifert matrices corresponding to these bases, then

$$S_- = \begin{pmatrix} * & * & \ldots & * \\ * & & & \\ \vdots & & S_0 & \\ * & & & \end{pmatrix}, \quad S_- = S_+ + \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix},$$

therefore,

$$t^{1/2} S_- - t^{-1/2} S_-^\top = t^{1/2} S_+ - t^{-1/2} S_+^\top + \begin{pmatrix} t^{1/2} - t^{-1/2} & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}$$

$$= \begin{pmatrix} * & * & \ldots & * \\ * & & & \\ \vdots & & t^{1/2} S_0 - t^{-1/2} S_0^\top & \\ * & & & \end{pmatrix}.$$

The result follows by expanding $\det(t^{1/2} S_+ - t^{-1/2} S_+^\top)$ and $\det(t^{1/2} S_- - t^{-1/2} S_-^\top)$ along the first column. □

Example. Let us compute the Alexander polynomial of the twist knot k_m of type $(2m+2)_1$ shown in Figure 7.11 (where $m = 1$), compare with Figure 3.19.

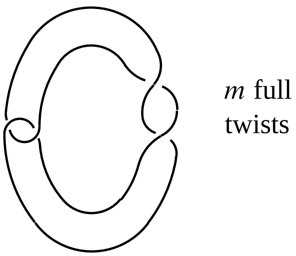


Figure 7.11

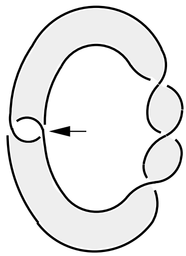


Figure 7.12

The knot k_m bounds the Seifert surface shaded in Figure 7.12 (we added a half twist to the knot projection to make sure that the obvious choice for a Seifert surface, the twisted band, is orientable). Next we apply Conway's formula with $k_m = \mathcal{L}_+$ at the intersection marked with an arrow in Figure 7.12. The resulting links are shown in Figure 7.13.

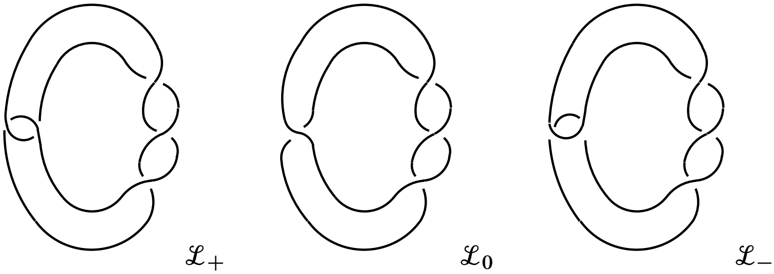


Figure 7.13

The knot $\mathcal{L}_-$ is a trivial knot with $\Delta_{\mathcal{L}_-}(t) = 1$, the link $\mathcal{L}_0$ has Seifert matrix (m) so that $\Delta_{\mathcal{L}_0}(t) = m\,(t^{1/2} - t^{-1/2})$, therefore

$$\Delta_{k_m}(t) = 1 - m\,(t^{1/2} - t^{-1/2})^2 = (1 + 2m) - m\,(t + t^{-1}).$$

The knot k_1 is called the *figure-eight* knot. Its Alexander polynomial is $3 - t - t^{-1}$. One can easily see that the knot k_1 has genus 1. If we allow m to be negative, the choice $m = -1$ will provide us with a trefoil knot with the already familiar Alexander polynomial $-1 + t + t^{-1}$.

7.4 Other invariants from Seifert surfaces

The *intersection form* of a compact oriented surface F is the bilinear integral form I defined on the first homology of F as follows. Any two classes $x, y \in H_1(F;\mathbb{Z})$ can be represented by simple closed oriented curves which intersect transversally in finitely many points. Each intersection point is weighted by ± 1 according to the convention shown in Figure 7.14, where the normal vector is assumed to point towards us. The sum of these ± 1 is the intersection number $I(x, y) = x \cdot y$.

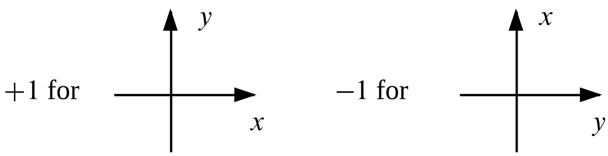


Figure 7.14

If F is closed, the intersection form I is isomorphic to the form defined by the cup-product in the first cohomology of F,

$$I\colon H^1(F;\mathbb{Z}) \otimes H^1(F;\mathbb{Z}) \to \mathbb{Z}, \quad (a, b) \mapsto \langle a \smile b, [F]\rangle. \tag{7.5}$$

The isomorphism $H_1(F;\mathbb{Z}) = H^1(F;\mathbb{Z})$ is provided by Poincaré Duality. The form I is skew-symmetric and unimodular, that is, its determinant is ± 1. The following is a result from elementary linear algebra.

Lemma 7.6. *Let F be a closed oriented surface and I its intersection form represented by an integral matrix with respect to a choice of basis in $H^1(F;\mathbb{Z})$. Then there is a real matrix U such that $I = U^\top J U$ where J is the block-diagonal matrix*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Since I is non-degenerate, there exist vectors $x, y \in H^1(F;\mathbb{R})$ such that $I(x,y) \neq 0$. These vectors are automatically linearly independent. We set $e_1 = x$ and $e_2 = y/I(x,y)$. Then $I(e_1,e_2) = 1$ (and also $I(e_2,e_1) = -1$, $I(e_1,e_1) = I(e_2,e_2) = 0$). Let P be the linear subspace of $H^1(F;\mathbb{R})$ spanned by the vectors e_1 and e_2. Denote by Q the linear subspace consisting of all the vectors $u \in H^1(F;\mathbb{R})$ such that $I(u,e_1) = I(u,e_2) = 0$. Then $H^1(F;\mathbb{R}) = P \oplus Q$. This can be checked as follows. First, any vector $v \in H^1(F;\mathbb{R})$ can be represented in the form $v = ae_1 + be_2 + u$ with $u \in Q$. We simply choose $a = I(v,e_2)$ and $b = -I(v,e_1)$, then $u = v - I(v,e_2)e_1 + I(v,e_1)e_2$ belongs to Q. Second, the intersection $P \cap Q$ is zero due to the fact that, for any $u = ae_1 + be_2$, the coefficients a and b can be found by the formulas $a = I(u,e_2)$ and $b = -I(u,e_1)$. Thus, $H^1(F;\mathbb{R}) = P \oplus Q$, and the matrix of I splits as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus I'.$$

The obvious induction completes the proof. □

Corollary 7.7. *The determinant of the intersection form I of a closed oriented surface F equals* 1.

If F is a Seifert surface of a knot, the intersection form I is still unimodular with determinant 1. If F is a Seifert surface of a link with two or more components, the form I is degenerate because any boundary component of F gives rise to a class in $H_1(F;\mathbb{Z})$ which is non-zero but has zero intersection with every class in $H_1(F;\mathbb{Z})$. Therefore, any matrix representing this intersection form will have the determinant zero.

One can easily check from the definitions that, for any choice of a basis in $H_1(M;\mathbb{Z})$, the Seifert form S and the intersection form I are related by the formula

$$I = S^\top - S.$$

Corollary 7.8. *If k is a knot,* $\Delta_k(1) = 1$*; if $\mathcal{L}$ is a link with two or more components,* $\Delta_{\mathcal{L}}(1) = 0$.

Here is another bilinear integral form associated with an oriented link. For any link $\mathcal{L}$, the form $Q = S + S^\top$ is symmetric. Since $Q = I \bmod 2$ the form Q is even. If $\mathcal{L} = k$ is actually a knot then $\det Q$ is odd and, in particular, Q is non-degenerate over the real numbers.

Let k be a knot in the 3-sphere, S its Seifert matrix, and $Q = S + S^\top$. By Theorem 7.3 any invariant of Q which does not change under stabilization and change of basis will be a knot invariant. For instance, the determinant of Q only changes by a factor of (± 1) as S is stabilized, hence $|\det Q| = |\Delta_k(-1)|$ is a knot invariant. It is called the *knot determinant*.

A more interesting invariant of a knot can be obtained if we consider the signature of Q. As Q is symmetric, it can be diagonalized over the reals. The number of positive entries in its diagonal form minus the number of negative entries is the signature of Q, denoted by $\operatorname{sign} Q$.

Theorem 7.9. *The value of* $\operatorname{sign} Q$ *is independent of the choice of Seifert matrix, and hence is a well-defined knot invariant called the* knot signature, $\operatorname{sign} k$.

Proof. One only needs to prove that stabilization of S does not change the signature of $Q = S + S^\top$:

$$Q' = \begin{pmatrix} & & & * & 0 \\ & Q & & \vdots & \vdots \\ & & & * & 0 \\ * & \dots & * & * & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} & & & 0 & 0 \\ & Q & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \dots & 0 & * & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

where $\rightsquigarrow$ stands for a sequence of elementary transformations on rows and columns. Therefore,

$$\operatorname{sign} Q' = \operatorname{sign} Q + \operatorname{sign} \begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix} = \operatorname{sign} Q. \qquad \square$$

Example. The left- and the right-handed trefoils are not equivalent. They can be distinguished by their signatures, since one has signature 2 and the other -2.

7.5 Knots in homology spheres

Let $k \subset \Sigma$ be a knot in an oriented homology sphere, $H_*(\Sigma) = H_*(S^3)$. Let $N(k)$ be its tubular neighborhood and $K = \Sigma \setminus \operatorname{int} N(k)$ the knot exterior, compare with Lecture 2. Then $H_*(K) = H_*(S^1)$; in particular, $H_1(K) = \mathbb{Z}$, and this group is generated by a curve $m \subset \partial K$ which we call the *meridian* of k. The canonical choice of a *longitude* $\ell \subset \partial K$ is given by the requirement that $0 = [\ell] \in H_1(K)$. The orientations of m and ℓ should be chosen to be compatible with the orientation of Σ, see Figure 2.2. This choice of m and ℓ identifies ∂K with $S^1 \times \partial D^2 = S^1 \times S^1$. Let $p_0 \colon \partial K \to \partial D^2$ be the projection of $\partial K = S^1 \times \partial D^2$ onto the second factor. It maps the longitude to a single point and, when restricted to the meridian, is a homeomorphism.

Lemma 7.10. *The projection $p_0 \colon \partial K \to \partial D^2$ extends to a continuous map $p \colon K \to \partial D^2$.*

Proof. The projection p_0 defines a homotopy class $[p_0] \in [\partial K, \partial D^2] = [\partial K, S^1] = [\partial K, K(\mathbb{Z}, 1)] = H^1(\partial K)$. The inclusion $i : \partial K \to K$ induces a homomorphism $i^* : [K, S^1] \to [\partial K, S^1]$, which coincides with the homomorphism $i^* : H^1(K) \to H^1(\partial K)$, and p_0 extends to $p : K \to S^1$ if and only if $[p_0] \in \operatorname{im} i^*$. The map i^* can be included in the long cohomology exact sequence as in the commutative diagram below, whose vertical arrows represent Poincaré–Lefschetz duality isomorphisms:

$$\begin{tikzcd}
H_2(K, \partial K) \arrow[r] & H_1(\partial K) \arrow[r, "i_*"] & H_1(K) \\
H^1(K) \arrow[u, "\mathrm{PD}"] \arrow[r, "i^*"] & H^1(\partial K) \arrow[u, "\mathrm{PD}"] \arrow[r, "\delta"] & H^2(K, \partial K) \arrow[u, "\mathrm{PD}"]
\end{tikzcd}$$

The element in $H_1(\partial K)$ corresponding to $[p_0] \in H^1(\partial K)$ under the Poincaré duality isomorphism is $[\ell]$. Therefore, $\delta[p_0] = \mathrm{PD}^{-1}\, i_*[\ell] = 0$, and $[p_0] \in \ker \delta = \operatorname{im} i^*$. $\square$

A similar construction works for oriented links in Σ of more than one component. One should simply repeat the argument above using the projection

$$p_0 \colon \bigcup_{i=1}^{n} (S^1_i \times \partial D^2) \to \partial D^2.$$

Let $\mathcal{L} \subset \Sigma$ be an oriented link in a homology sphere Σ. Its *Seifert surface* is a connected compact oriented surface smoothly embedded in Σ with oriented boundary the link $\mathcal{L}$.

Theorem 7.11. *Every oriented link $\mathcal{L} \subset \Sigma$ bounds a Seifert surface in Σ.*

Proof. Consider the link exterior K together with the projection $p_0 \colon \partial K \to \partial D^2$, and extend it to a map $p \colon K \to \partial D^2$. After a small perturbation inside the homotopy class of p, if necessary, we may assume that p is smooth and transversal to a point $x \in \partial D^2$. Then $p^{-1}(x) = F'$ is a properly embedded surface in K with boundary $\mathcal{L}' \subset \partial K$. Connect $\mathcal{L}$ to $\mathcal{L}'$ by annuli inside the normal neighborhoods of the link components to get a surface $F \subset \Sigma$ with boundary $\mathcal{L}$. This surface may have several connected components; to make it into a Seifert surface, join these components (if more than one) by tubes. □

Using such a Seifert surface, all the usual invariants of classical knot theory can be defined in this more general context – the Seifert matrix, Alexander polynomial, quadratic and intersection forms, and the knot signature. Note that the linking number in Σ can be defined with the help of definitions **(1)** or **(3)** of Lecture 3 but not definition **(2)** (the one with regular projections).

7.6 Boundary links and the Alexander polynomial

The problem of computing the Alexander polynomial for knots in a general homology sphere is more involved than that for knots in S^3. Here, we describe a way to do the calculations.

Lemma 7.12. *Let $k \cup \ell$ be a link in a homology sphere Σ with $\operatorname{lk}(k, \ell) = 0$. Then there is a Seifert surface F_ℓ for ℓ such that $F_\ell \cap k = \emptyset$.*

Example. The following link is called the *Whitehead link*. The Seifert surface F_ℓ of ℓ shown in Figure 7.15 is disjoint from the knot k. It is obtained by attaching a tube to the shaded disc with holes.

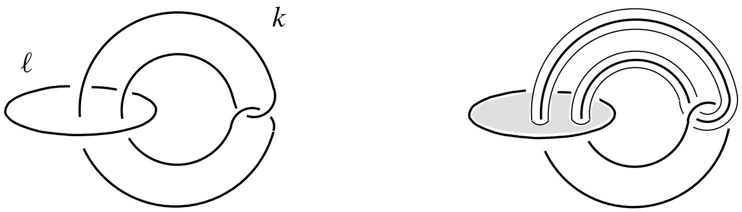


Figure 7.15

Proof of Lemma 7.12. Our proof is a customized version of the proof of Theorem 7.10. Another proof can be obtained by first choosing an arbitrary Seifert surface for ℓ and then getting rid of the intersections by adding tubes as for the Whitehead link above. The condition $\operatorname{lk}(k, \ell) = 0$ ensures that this can be done.

Let $K = \Sigma \setminus \operatorname{int}(N(k) \cup N(\ell))$. This is a 3-manifold with two boundary components each of which is a 2-torus. Its first homology $H_1(K)$ is generated by the canonical meridians on $\partial N(k)$ and $\partial N(\ell)$. Let ℓ' be the canonical longitude on $\partial N(\ell)$. The choice of ℓ' identifies $\partial N(\ell)$ with $S^1 \times \partial D^2$. Consider the map $p_0 \colon \partial K \to S^1$ that is defined as the projection $S^1 \times \partial D^2 \to \partial D^2$ on $\partial N(\ell)$, and as a constant map to a point $x_0 \in \partial D^2$ on $\partial N(k)$. The map p_0 can be extended to a continuous map $p \colon K \to S^1$.

To see this, we consider the inclusion-induced homomorphism $i^*\colon H^1(K) \to H^1(\partial K)$ and include it into the following commutative diagram with exact rows:

$$\begin{tikzcd}
H_2(K,\partial K) \arrow[r] & H_1(\partial K) \arrow[r, "i_*"] & H_1(K) \\
H^1(K) \arrow[u, "\mathrm{PD}"'] \arrow[r, "i^*"] & H^1(\partial K) \arrow[u, "\mathrm{PD}"'] \arrow[r, "\delta"] & H^2(K,\partial K) \arrow[u, "\mathrm{PD}"']
\end{tikzcd}$$

The vertical rows in this diagram represent Poincaré–Lefschetz duality isomorphisms. The element in $H_1(\partial K)$ corresponding to $[p_0] \in H^1(\partial K) = \mathbb{Z}^2 \oplus \mathbb{Z}^2$ under the isomorphism PD is $([\ell'], 0)$. Note that ℓ' is the canonical longitude, and that ℓ' is homologous to zero in the exterior of k since $\mathrm{lk}(k,\ell) = \mathrm{lk}(k,\ell') = 0$. It follows that $i_*[\ell] = 0$. Because of the commutativity of the diagram, $\delta[p_0] = 0$ and $[p_0] \in \operatorname{im} i^*$. The latter means that p_0 extends over K.

To finish the proof, we use a small homotopy to make p into a smooth map transversal to a point $x_1 \in S^1$ different from $x_0 \in S^1$, and make $F'_\ell = p^{-1}(x_1)$ into a Seifert surface F_ℓ as in the proof of Theorem 7.10. □

A link $k \cup \ell$ in a homology sphere Σ is called a *boundary link* if the knots k and ℓ bound disjoint Seifert surfaces. In particular, k and ℓ have zero linking number. The condition of having zero linking number is weaker than that of being a boundary link. For example, the Whitehead link in Figure 7.15 has $\mathrm{lk}(k,\ell) = 0$ but it is not boundary, which can be shown using the following observation.

Lemma 7.13. *Let $k \cup \ell$ be a boundary link in a homology sphere Σ, and $\Sigma' = \Sigma + \varepsilon \cdot k$ a surgery of Σ along k with $\varepsilon = \pm 1$. Then $\Delta_{\ell\subset\Sigma}(t) = \Delta_{\ell\subset\Sigma'}(t)$ where $\ell \subset \Sigma'$ is the image of $\ell \subset \Sigma$ under the surgery.*

Proof. Let F_k and F_ℓ be disjoint Seifert surfaces for k and ℓ. Choose cycles $x, y \subset F_\ell$ and compute $\mathrm{lk}(x, y^+)$, which is the intersection number of y^+ and a Seifert surface F_x of x. Since x and F_k are disjoint, we have $\mathrm{lk}(x,k) = 0$, hence the surface F_x can be chosen to be disjoint from k. Both y^+ and F_x are now away from the knot k, hence the intersection $y^+ \cap F_x$ is not affected by the surgery along k. This is true for any curves x and y representing basis elements in $H_1(F_\ell)$, therefore, the Seifert matrices and the Alexander polynomials of $\ell \subset \Sigma$ and $\ell \subset \Sigma'$ coincide. □

Example. The Whitehead link shown in Figure 7.15 is not boundary. Suppose it is, and perform the (-1)-surgery of S^3 along ℓ. The result of this surgery is again S^3, however, the image k' of the trivial knot k in the surgered manifold is the figure-eight knot. The knots k and k' have different Alexander polynomials, which contradicts Lemma 7.13.

Example. Let us compute the Alexander polynomial of the knot ℓ in the homology sphere Σ obtained by surgery along the trefoil knot shown in Figure 7.16. The $(+1)$-surgery on the knot k shown in Figure 7.17 unknots the trefoil and turns Σ into S^3.

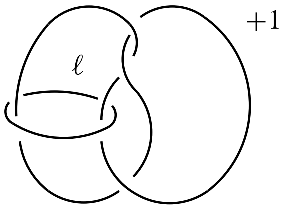


Figure 7.16

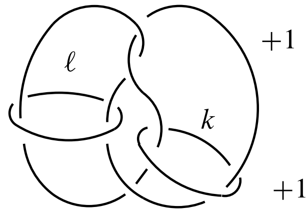


Figure 7.17

Since the knots ℓ and k have disjoint Seifert surfaces, see Figure 7.18, the Alexander polynomials of $\ell \subset \Sigma$ and of its image in S^3 are the same. The image of ℓ in S^3 is isotopic to the figure-eight knot shown in Figure 7.19, whose Alexander polynomial is equal to $3 - t - t^{-1}$. Therefore, $\Delta_{\ell \subset \Sigma}(t) = 3 - t - t^{-1}$.

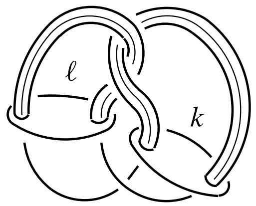


Figure 7.18

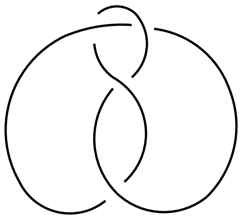


Figure 7.19

The following lemma demonstrates that the method we used in the preceding example to compute the Alexander polynomial of a knot in a homology sphere has a general nature.

Lemma 7.14. *Let k be a knot in a homology 3-sphere Σ. Then there exists a knot ℓ in S^3 such that $\Delta_{k \subset \Sigma}(t) = \Delta_{\ell \subset S^3}(t)$.*

Proof. It follows from Lemma 12.2 that the 3-sphere S^3 can be obtained by doing (± 1)-surgeries on the components of a link $c_1 \cup \dots \cup c_m$ in Σ with $\operatorname{lk}(c_i, c_j) = 0$ for all $i \neq j$,

$$S^3 = \Sigma + \varepsilon_1 \cdot c_1 + \dots + \varepsilon_m \cdot c_m, \quad \varepsilon_i = \pm 1.$$

The proof is by induction on m. It is obvious for $m = 0$. Next, one can always choose c_m in its isotopy class so that the link $k \cup c_m$ is a boundary link. This can be seen as follows. Choose Seifert surfaces, F_k and F_c, for k and c_m. By general position, the surfaces F_k and F_c can be isotoped with the help of *separate* isotopies into discs with thin bands that are disjoint from each other. Hence, we may assume that F_k and F_c are disjoint. Of course, this may change the link $k \cup c_m$, but both the knot k and the knot c_m are preserved.

As soon as the link $k \cup c_m$ is boundary, $\Delta_{k \subset \Sigma}(t) = \Delta_{k \subset \Sigma + \varepsilon_m \cdot c_m}(t)$ where $k \subset \Sigma + \varepsilon_m \cdot c_m$ is the knot k considered as a knot in the surgered manifold $\Sigma + \varepsilon_m \cdot c_m$. If we denote the manifold $\Sigma + \varepsilon_m \cdot c_m$ by Σ' we have

$$S^3 = \Sigma' + \varepsilon_1 \cdot d_1 + \cdots + \varepsilon_{m-1} \cdot d_{m-1},$$

where $d_1 \cup \cdots \cup d_{m-1}$ is the link $c_1 \cup \cdots \cup c_{m-1}$ considered as a link in Σ'. By induction, there is a knot ℓ in S^3 such that $\Delta_{k \subset \Sigma}(t) = \Delta_{\ell \subset S^3}(t)$. $\square$

7.7 Exercises

1. Given a regular projection of a link of n components, let c be the number of crossings and s the number of Seifert circles. Prove that the Seifert surface constructed in the proof of Theorem 7.1 has genus $1 - (s + n - c)/2$.

2. Prove that the genus of the torus knot $k_{p,q}$ is no greater than $(p-1)(q-1)/2$.

3. Use the formula (7.3) for the Alexander polynomial of a torus knot to prove that the genus of $k_{p,q}$ equals $(p-1)(q-1)/2$.

4. Let $p, q > 1$ be integers and p', q' integers such that $pq' + p'q = 1$. Prove that the link in Figure 7.20 provides a surgery description for the right-handed (p,q)-torus knot $k_{p,q}$, that is, surgery on the circles labeled 0, p/p', and q/q' turns the knot k into $k_{p,q} \subset S^3$.

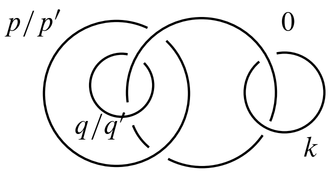


Figure 7.20

5. Compute the Alexander polynomial of the figure-eight knot. Prove that the figure-eight knot has genus one.

6. A Whitehead double of a knot k is constructed by replacing k with the curve shown in Figure 7.21 on the left. The picture on the right illustrates a double of a trefoil knot. The number of twists between two parallel strands is arbitrary. Show that a Whitehead double of a knot k has genus at most one.

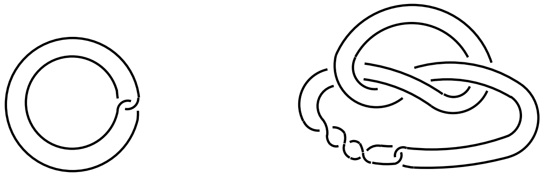


Figure 7.21

7. A knot in S^3 is called *amphicheiral* if it is isotopic to its mirror image. Prove that amphicheiral knots have zero signature.

8. Let $k_1 \# k_2$ be a connected sum of two oriented knots, k_1 and k_2, in homology spheres. Prove that $\Delta_{k_1 \# k_2}(t) = \Delta_{k_1}(t) \cdot \Delta_{k_2}(t)$ and $\operatorname{sign}(k_1 \# k_2) = \operatorname{sign}(k_1) + \operatorname{sign}(k_2)$.

9. A knot k in $S^3 = \partial D^4$ is called *slice* if there is a smoothly embedded proper disc $D^2 \subset D^4$ such that $\partial D^2 = k$.

   (a) Prove that the connected sum of an oriented knot k with the knot $-k^*$, which is the mirror image of k with reversed orientation, is a slice knot.

   (b) Prove that the connected sum of slice knots is a slice knot.

   Oriented knots $k_1, k_2 \subset S^3$ are called *concordant* if the knot $k_1 \# (-k_2^*)$ is slice. The two properties above ensure that the concordance classes of oriented knots in S^3 form an Abelian group with respect to connected sums of knots, the zero element being the class of a trivial knot. This group is called the (smooth) *knot concordance group*.

10. Prove that slice knots have zero signature.

11. Let k_1 and k_2 be oriented knots in an oriented homology sphere Σ. Prove that $\operatorname{lk}(k_1, k_2) = \operatorname{lk}(k_2, k_1)$.

12. Calculate the Alexander polynomial of the knot k in the homology sphere Σ obtained by surgery on the two-component link shown in Figure 7.22.

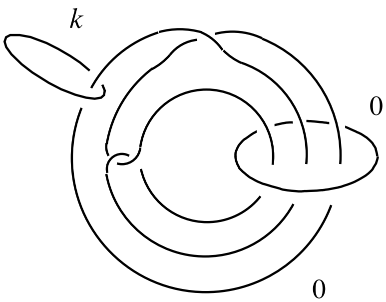


Figure 7.22

Lecture 8

Fibered knots

8.1 The definition of a fibered knot

A knot k in a homology 3-sphere Σ is called a *fibered knot* of genus g if its complement $\Sigma \setminus k$ is the total space of a locally trivial bundle $p : \Sigma \setminus k \to S^1$ whose fiber F is an orientable surface of genus g. We require further that k have a neighborhood framed as $S^1 \times D^2$, with $k = S^1 \times \{0\}$, in such a way that the restriction of p to $S^1 \times (D^2 \setminus \{0\})$ is the map to S^1 of the form $(x, y) \mapsto y/|y|$. It follows that the closure of each fiber F is a compact orientable surface of genus g, and that these surfaces fit around k in the manner shown in Figure 8.1.

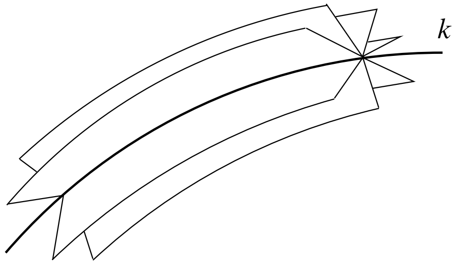


Figure 8.1

Lemma 8.1. *Let $k \subset \Sigma$ be a fibered knot. Then the closure $\bar{F}$ of each fiber F is a Seifert surface for k.*

Proof. We only need to prove that both the closure of F and its boundary are connected or, given the knot exterior $K = \Sigma \setminus \operatorname{int} N(k)$, that the compact surface $F_0 = F \cap K$ and its boundary $\partial F_0 \subset \partial K$ are connected. Both K and ∂K are fibered over S^1, with fibers F_0 and ∂F_0, respectively. Consider the homotopy exact sequences of these fibrations:

$$\begin{array}{ccccccccc} 1 \to & \pi_1(\partial F_0) & \longrightarrow & \pi_1(\partial K) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_0(\partial F_0) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow = & & \downarrow & \\ 1 \to & \pi_1(F_0) & \longrightarrow & \pi_1(K) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_0(F_0) & \to 1 \end{array}$$

This diagram commutes, and $\pi_1(\partial K) \to \pi_1(S^1)$ is surjective. Hence $\pi_1(K) \to \pi_1(S^1)$ is surjective, that is, both ∂F_0 and F_0 are connected. $\square$

Note that in our proof of Theorem 7.10, we constructed a map $p\colon \Sigma\setminus\operatorname{int} N(k) \to S^1$ for any knot k. This map would be a fibration map for the knot k exterior if we could make it transversal to every point $x \in S^1$. However, this is not always possible.

Example. Let k be a trivial knot in S^3. In Figure 8.2, the sphere S^3 is represented by revolving the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$ about the circle $\ell \cup \{\infty\}$, compare with Figure 1.1.

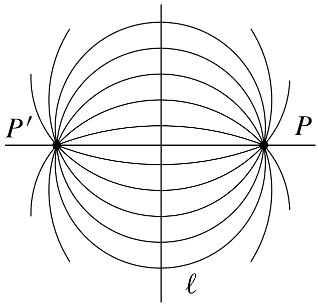


Figure 8.2

Under this revolution, the point P generates the knot k. Each of the open arcs connecting P and P' generates an open 2-dimensional disc F in S^3. These discs exhaust the knot k complement, they are disjoint, and they are parametrized by the points of the circle $\ell \cup \{\infty\}$. Therefore, $S^3 \setminus k = S^1 \times F$, and the projection onto the first factor is a trivial F-bundle over S^1. Thus k is a fibered knot in S^3 of genus 0. The closure of each of the discs F in S^3 is a closed disc in S^3 with boundary k.

Example. Any (p,q)-torus knot in S^3 is a fibered knot of genus $(p-1)(q-1)/2$; in particular, a trefoil is a fibered knot of genus 1. A proof of this fact, which is based on singularity theory, is sketched later in this Lecture. The paper Zeeman [161] has an explicit construction of a genus 1 fibration of the trefoil complement, see also Rolfsen [137], pages 327–333.

Example. Let $\Sigma(a_1,\dots,a_n)$ be a Seifert homology sphere, and k a singular fiber, see Lecture 1. The knot k is fibered, see e.g. Eisenbud–Neumann [41].

Example. The figure-eight knot is a fibered knot of genus 1, see for example Burde–Zieschang [26], page 71.

It is worth mentioning that, given a fibered knot, the result of a 0-framed surgery along it is a locally trivial bundle over S^1 with the fiber homeomorphic to $\bar{F} \cup D^2$, a closed Riemann surface.

8.2 The monodromy

The locally trivial bundle $p : \Sigma \setminus k \to S^1$ can be thought of as follows. View S^1 as the interval $[0, 2\pi]$ with the ends identified. Since $[0, 2\pi]$ is contractible, every bundle over it with the fiber F is isomorphic to the trivial bundle with the total space $F \times [0, 2\pi]$. The original bundle $p : \Sigma \setminus k \to S^1$ over the circle is then obtained by identifying the surfaces $F \times \{0\}$ and $F \times \{2\pi\}$ by a homeomorphism $h : F \to F$, called the *monodromy homeomorphism*. The induced automorphism

$$h_* : H_1(F) \to H_1(F) \tag{8.1}$$

is called the *monodromy transformation*.

Lemma 8.2. *The monodromy transformation* (8.1) *is well-defined.*

Proof. We will proceed by making precise the above construction of $h : F \to F$. Consider the map $\gamma : [0, 2\pi] \to S^1$ given by $\gamma(t) = e^{it}$. The pull-back of the bundle $p : \Sigma \setminus k \to S^1$ via γ is a trivial bundle over $[0, 2\pi]$. Therefore, there is a continuous family of homeomorphisms $h_t : F_{\gamma(0)} \to F_{\gamma(t)}$, where $F_{\gamma(t)} = p^{-1}(\gamma(t))$ is the fiber over $\gamma(t) \in S^1$, such that $h_0 = \mathrm{id}$ and $\Sigma \setminus k$ is obtained from $F_{\gamma(0)} \times [0, 2\pi]$ by identifying the surfaces $F_{\gamma(0)}$ and $F_{\gamma(2\pi)} = F_{\gamma(0)}$ using $h_{2\pi}$. This is the homeomorphism that we referred to as $h : F \to F$.

Let us define $\tilde{G} : F_{\gamma(0)} \times I \to \Sigma \setminus k$ by the formula $\tilde{G}(x, t) = h_t(x)$. This map makes the following diagram commute:

$$\begin{tikzcd}
F_{\gamma(0)} \arrow[r, "f"] \arrow[d, "i"'] & \Sigma \setminus k \arrow[d, "p"] \\
F_{\gamma(0)} \times I \arrow[r, "G"] \arrow[ur, "\tilde{G}"] & S^1
\end{tikzcd}$$

where $G(x, t) = \gamma(t)$ and f is the inclusion of $F_{\gamma(0)}$ into $\Sigma \setminus k$ as a fiber over $\gamma(0)$. In other words, $\tilde{G}$ is a lift of G in the sense of diagram (5). Any two such lifts are fiberwise homotopic; in particular, $h_{2\pi}$ is defined uniquely up to homotopy, and hence the induced map $h_* = (h_{2\pi})_*$ in homology is well-defined. $\square$

The monodromy transformation and a Seifert matrix of a fibered knot k are closely related as explained in the following lemma.

Lemma 8.3. *Let k be a fibered knot with a fiber F. Let us fix a basis in $H_1(F)$ and let S be the Seifert matrix of k and M the matrix of the monodromy transformation $h_*\colon H_1(F) \to H_1(F)$ with respect to this basis. Then $M^\top S = S^\top$.*

Proof. Let $x, y \in H_1(F)$ be basis vectors in $H_1(F)$. If we think of them as columns, we have $S(x, y) = x^\top S y$, $S^\top(x, y) = x^\top S^\top y = y^\top S x$ and $h_*(y) = My$, so we only need to prove that $x^\top M^\top S y = y^\top S x$. Since $S(x, y) = \mathrm{lk}(x, (h_\pi)_* y)$ and $Mx = (h_{2\pi})_* x$, we need to show that

$$\mathrm{lk}((h_{2\pi})_* x, (h_\pi)_* y) = \mathrm{lk}(y, (h_\pi)_* x).$$

The latter is obviously true,

$$\mathrm{lk}((h_{2\pi})_* x, (h_\pi)_* y) = \mathrm{lk}((h_\pi)_* x, y) = \mathrm{lk}(y, (h_\pi)_* x). \qquad \square$$

Corollary 8.4. *The Alexander polynomial of a fibered knot k equals the symmetrized characteristic polynomial of its monodromy transformation h_*.*

Remember that the Alexander polynomial $\Delta_k(t)$ of a knot k is symmetric in that $\Delta_k(t^{-1}) = \Delta_k(t)$ and $\Delta_k(1) = 1$. Therefore, the corollary asserts that the characteristic polynomial of h_*, after possibly multiplying it by a factor of $\pm t^s$, is equal to the Alexander polynomial of k; in particular, it is symmetric.

Example. Let k be a right-handed trefoil. Then, in an appropriate basis,

$$S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad M = (S^\top)^{-1} S = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

so that the characteristic polynomial of M is $t^2 - t + 1$. It becomes symmetric after multiplying by t^{-1} to get $-1 + t + t^{-1}$, which, as we know, is the Alexander polynomial of k.

Proof of Corollary 8.4. The Alexander polynomial of k is defined as

$$\Delta_k(t) = \det(t^{1/2} S - t^{-1/2} S^\top).$$

Use Lemma 8.3 to obtain

$$\Delta_k(t) = \det(t^{1/2} S - t^{-1/2} M^\top S) = \det(tE - M^\top) \cdot \det(t^{-1/2} S),$$

which is the characteristic polynomial of h_* up to a factor of $\pm t^s$ (killed by the symmetrization). $\square$

Corollary 8.5. *If $k \subset \Sigma$ is a fibered knot with a fiber F, then the genus of k equals the genus of $\bar{F}$, i.e. the closure $\bar{F}$ is a Seifert surface of k of minimal genus.*

Proof. Since $\bar{F}$ is a Seifert surface, the genus of k cannot exceed the genus of $\bar{F}$. On the other hand, the degree of the Alexander polynomial of k equals the highest degree of t in the symmetrized characteristic polynomial of h_*, which in turn equals the genus of $\bar{F}$. Since the genus of k cannot be less than the degree of its Alexander polynomial, we are finished. $\square$

Corollary 8.6. *If a knot k is fibered then its Alexander polynomial is monic, i.e. its top degree coefficient equals* ± 1.

Proof. The top degree coefficient of $\Delta_k(t)$ equals $\pm \det M$ where the matrix M of $h_*\colon H_1(F) \to H_1(F)$ is invertible over the integers since h is a homeomorphism. Therefore, $\det M = \pm 1$. □

Example. The knot shown in Figure 7.7 is not fibered since its Alexander polynomial, $4t^{-2} - 12t^{-1} + 17 - 12t + 4t^2$, has the top degree coefficient 4 and therefore is not monic.

Given a fibered knot in Σ, one obtains a Heegaard splitting $\Sigma = M_1 \cup_F M_2$ as follows. We think of S^1 as the interval $[0, 2\pi]$ with the ends identified. Let M_1 be the closure in Σ of the preimage under p of the upper half-circle, $p^{-1}([0,\pi]) = F \times [0, \pi]$. Similarly, let M_2 be the closure of $p^{-1}([\pi, 2\pi]) = F \times [\pi, 2\pi]$. Both M_1 and M_2 are handlebodies of genus equal twice the genus of F, and the splitting surface is $\partial M_1 = \partial M_2 = \bar{F} \cup \bar{F}$, the union of two copies of the closure $\bar{F}$ of F along the knot k. The gluing map $\partial M_1 \to \partial M_2$ is an extension to the closure of the map given by the formula

$$\begin{aligned} (x, \pi) &\mapsto (x, \pi), && x \in F, \\ (x, 0) &\mapsto (h(x), 2\pi), && x \in F, \end{aligned} \tag{8.2}$$

where $h = h_{2\pi}\colon F \to F$ is the monodromy homeomorphism.

8.3 More about torus knots

We give here a rough sketch of Milnor's [114] proof of the fact that the complement of a (p, q)-torus knot admits a fibration over S^1 by surfaces of genus $(p-1)(q-1)/2$.

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be given by the formula $f(z, w) = z^p + w^q$ where p and q are relatively prime positive integers. The intersection of the (singular) surface $V = \{(z, w) \mid f(z, w) = 0\}$ with the sphere $S^3 = \{(z, w) \mid |z|^2 + |w|^2 = 1\}$ is a torus knot k of type (p, q), see Lecture 7. The formula

$$\varphi(z, w) = \frac{f(z, w)}{|f(z, w)|} \tag{8.3}$$

defines a map $\varphi\colon S^3 \setminus k \to S^1$ onto the circle S^1 of unit complex numbers. Milnor shows in Section 4 of his book that the map φ is transversal to every point in S^1 and therefore is the projection of a locally trivial bundle such that each fiber $F_{\exp(i\theta)} = \varphi^{-1}(e^{i\theta})$ is a smooth open Riemann surface. All fibers $F_{\exp(i\theta)}$ are naturally homeomorphic via homeomorphisms that can be described explicitly: the map $h_t\colon S^3 \setminus k \to S^3 \setminus k$ defined by the formula

$$h_t(z, w) = (e^{it/p} z, e^{it/q} w)$$

carries each fiber $\varphi^{-1}(y)$ homeomorphically onto the fiber $\varphi^{-1}(e^{it}y)$ for all t between 0 and 2π. The monodromy homeomorphism $h = h_{2\pi}$ is given by the formula

$$h_{2\pi}(z, w) = (e^{2\pi i/p}z, e^{2\pi i/q}w). \tag{8.4}$$

The closure of each of $F_{\exp(i\theta)}$ in S^3 has the knot k as its boundary. To see this, pick a point $(z_0, w_0) \in k$ and choose local coordinates for S^3 in a neighborhood U of (z_0, w_0) so that $f = u + iv$ where u and v are the real and the imaginary parts of f. A point of U belongs to the fiber $F_1 = \varphi^{-1}(1)$ if and only if $u + iv = |u + iv|$, or $u > 0,\ v = 0$. Hence the closure of F_1 intersects U in the set $u \geq 0,\ v = 0$. Clearly, $\partial \bar{F}_1 \cap U = k \cap U$.

Similarly, a point of U belongs to the fiber $F_{\exp(i\theta)}$ if and only if $u + iv = e^{i\theta}|u + iv|$, or $u_\theta > 0,\ v_\theta = 0$ where $u_\theta = u\,\cos\theta + v\,\sin\theta,\ v_\theta = -u\,\sin\theta + v\,\cos\theta$, and again $\partial \bar{F}_{\exp(i\theta)} \cap U = k \cap U$. Thus all of the fibers $F_{\exp(i\theta)}$ fit around their common boundary k in the manner shown in Figure 8.1.

Further, Milnor shows that any fiber $F_{\exp(i\theta)}$ is naturally diffeomorphic to the surface Γ_θ given by the equation $z^p + w^q = e^{i\theta}$ in $\mathbb{C}^2$. This description allows for an easy computation of the genus of $F_{\exp(i\theta)}$ and Γ_θ. It is enough to do calculations for Γ_0 only. Let us consider the projection $p\colon \Gamma_0 \to \mathbb{C},\ (z, w) \mapsto z$. If $z^p = 1$ then $p^{-1}(z)$ consists of just one point, $(z, 0)$; otherwise, $p^{-1}(z)$ contains q distinct points. Therefore,

$$\chi(\Gamma_0) = q \cdot \chi(\mathbb{C}) - p \cdot (q-1) = 1 - (p-1)(q-1),$$

and the genus of Γ_0 is $(p-1)(q-1)/2$.

To compute the monodromy of a torus knot, we need to digress to some algebraic topology.

8.4 Joins

Let X and Y be CW-complexes, not necessarily connected. Let us form a new complex $X * Y$ by joining every point in X to every point in Y by an interval. This new space will be called *join* of X and Y. More precisely, the space $X * Y$ is defined as the quotient space $X * Y = (X \times Y \times I)/\!\sim$ where $\sim$ is the equivalence relation

$$(x, y, t) \sim (x', y', t') \quad \text{if and only if} \quad \begin{cases} t = t' = 0 & \text{and} \quad x = x' \quad \text{or} \\ t = t' = 1 & \text{and} \quad y = y'. \end{cases}$$

Alternatively, $X * Y$ can be thought of as consisting of all the quadruples (t, x, s, y) where $x \in X,\ y \in Y$, and s, t are real numbers such that $0 \leq s, t \leq 1$ and $s + t = 1$. In addition, the following identifications are made: for every $x \in X$, all the points $(0, x, 1, y),\ y \in Y$, are identified into one point, and so are the points $(1, x, 0, y)$, $x \in X$, for every given y.

The operation $*$ is commutative and associative up to homeomorphism. It is also functorial in that any two continuous maps $f : X \to X'$ and $g : Y \to Y'$ define, in an obvious fashion, a continuous map $f * g : X * Y \to X' * Y'$.

Example. Let $X = Y = S^0$ be a 0-dimensional sphere (a pair of points). One readily sees that $X * Y = S^1$.

Example. In general, $S^0 * X = SX$ is a suspension over X. In particular, $S^0 * S^n = SS^n = S^{n+1}$. By induction it follows that $S^n * S^m = S^{n+m+1}$.

In general, one can show that $X * Y$ is homotopy equivalent to the suspension $S(X \wedge Y)$ over the smash-product $X \wedge Y$, see for instance Hatcher [71], Chapter 0. Recall that $X \wedge Y = (X \times Y)/(X \vee Y)$ where the one-point union $X \vee Y$ is embedded in $X \times Y$ as a pair of coordinate axes.

It now follows from the Mayer–Vietoris sequence that $\tilde{H}_{n+1}(X * Y) = \tilde{H}_n(X \wedge Y)$. The homology of $X \wedge Y$ can be computed using the long exact sequence of the pair $(X \times Y, X \vee Y)$ (for the sake of simplicity we work over $\mathbb{C}$):

$$\cdots \to \tilde{H}_n(X \vee Y) \xrightarrow{i_*} \tilde{H}_n(X \times Y) \to \tilde{H}_n(X \wedge Y) \to \tilde{H}_{n-1}(X \vee Y) \to \cdots \tag{8.5}$$

where $\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$ for all $n \geq 0$. According to the Künneth formula,

$$H_n(X \times Y) = \sum_{i+j=n} H_i X \otimes H_j Y.$$

Suppose that $n \geq 1$. The homomorphism i_* in (8.5) identifies $\tilde{H}_n X = H_n X$ with a copy of $H_n X$ in $H_n X \otimes H_0 Y$. Similarly, it identifies $\tilde{H}_n Y = H_n Y$ with a copy of $H_n Y$ in $H_0 X \otimes H_n Y$. Therefore, i_* is monomorphic. With a little more effort one can show that $i_*\colon \tilde{H}_0 X \oplus \tilde{H}_0 Y \to \tilde{H}_0(X \times Y)$ is also monomorphic, so the long exact sequence (8.5) splits into the short exact sequences, $n \geq 0$,

$$0 \to \tilde{H}_n X \oplus \tilde{H}_n Y \to \tilde{H}_n(X \times Y) \to \tilde{H}_n(X \wedge Y) \to 0,$$

and we have the final result,

$$\tilde{H}_{n+1}(X * Y) = \sum_{i+j=n} \tilde{H}_i X \otimes \tilde{H}_j Y, \quad n \geq 0. \tag{8.6}$$

This formula behaves naturally with respect to continuous maps of X and Y. A general formula for $\tilde{H}_*(X * Y)$ over a principal ideal domain can be found in Milnor [111].

Below, we will be applying the formula (8.6) to calculate $\tilde{H}_1(X * Y)$ where X and Y are finite sets with discrete topology. The result,

$$\tilde{H}_1(X * Y) = \tilde{H}_0(X) \otimes \tilde{H}_0(Y), \tag{8.7}$$

can be double checked as follows. Let $|X|$ and $|Y|$ be the cardinalities of X and Y, respectively. The space $X \wedge Y$ consists of $(|X|-1)(|Y|-1)+1$ points. Since $X * Y$ is homotopy equivalent to $S(X \wedge Y)$, it can be thought of as a one point union of $(|X|-1)(|Y|-1)$ circles. Therefore, we get the isomorphism (8.7) where $\tilde{H}_0 X$ and $\tilde{H}_0 Y$ are complex vector spaces of dimensions $|X|-1$ and $|Y|-1$, respectively.

8.5 The monodromy of torus knots

The surface $V = \{(z,w) \mid z^p + w^q = 0\} \subset \mathbb{C}^2$ admits a $\mathbb{C}^*$-action given by the formula $t \cdot (z,w) = (t^q z, t^p w)$. This allows one to extend $\varphi \colon S^3 \setminus k \to S^1$, see (8.3), to a locally trivial bundle $\psi \colon \mathbb{C}^2 \setminus V \to S^1$ by the formula

$$\psi(z,w) = \frac{z^p + w^q}{|z^p + w^q|},$$

so that the $\mathbb{R}_+$-part of the $\mathbb{C}^*$-action identifies each fiber $\psi^{-1}(y)$ with $\varphi^{-1}(y) \times \mathbb{R}_+$. Thus the fibers of φ and ψ have the same homotopy type.

Let $\mathbb{Z}/p \subset \mathbb{C}$ and $\mathbb{Z}/q \subset \mathbb{C}$ be the finite cyclic groups consisting of all p-th, respectively, q-th roots of unity, and $J = \mathbb{Z}/p * \mathbb{Z}/q$ their join. One can embed J into $\mathbb{C}^2$ by thinking of it as consisting of all vectors $(s\xi, t\eta) \in \mathbb{C}^2$ with $s,t \geq 0$, $s+t=1$, and $\xi \in \mathbb{Z}/p$, $\eta \in \mathbb{Z}/q$. Note that $J \subset \psi^{-1}(1)$.

Theorem 8.7 (Pham, [128]). *The join J is a deformation retract of the fiber $\psi^{-1}(1)$.*

Proof. Given any point $(z,w) \in \psi^{-1}(1)$, first deform the coordinate z along a path in $\mathbb{C}$ which is chosen so that z^p moves in a straight line to the nearest point $\mathrm{Re}(z^p)$ of the real axis. Repeat the same for the coordinate w. The vector (z,w) moves to a vector (z',w') such that $(z')^p,\ (w')^q \in \mathbb{R}$. The value $z^p + w^q > 0$ does not change during this deformation, so that we remain inside the fiber $\psi^{-1}(1)$. Next, if $(z')^p < 0$ move z' along a straight line to zero, leaving z' fixed if $(z')^p \geq 0$, and same for $(w')^q$. Thus the vector (z',w') moves in a straight line to a vector $(z'',w'') \in \psi^{-1}(1)$ which satisfies $(z'')^p,\ (w'')^q \geq 0$. It follows that $z = s\xi$ and $w = t\eta$ for some $s,t \geq 0$ and some $\xi \in \mathbb{Z}/p,\ \eta \in \mathbb{Z}/q$. Finally, move (z'',w'') along a straight line to the point

$$(z'',w'')/(s+t) \in J.$$

Since the points of J remained fixed throughout the deformation, this completes the proof. □

The monodromy homeomorphism (8.4) extends to a homeomorphism $h = h_{2\pi}\colon \psi^{-1}(1) \to \psi^{-1}(1)$ by the formula

$$h(z,w) = (e^{2\pi i/p} z, e^{2\pi i/q} w).$$

It carries J into itself, and the restriction $h|_J$ can be described as the join

$$h = r_p * r_q \colon J \to J$$

where $r_p \colon \mathbb{Z}/p \to \mathbb{Z}/p$ is given by the formula $r_p(\xi) = e^{2\pi i/p}\xi$, and similarly for r_q. Consider the induced homomorphism

$$(r_p)_* \colon \tilde{H}_0(\mathbb{Z}/p; \mathbb{C}) \to \tilde{H}_0(\mathbb{Z}/p; \mathbb{C})$$

of reduced homology groups. For each integer ν between 1 and $p-1$ define the homology class ω_ν which associates the coefficient $\xi^\nu \in \mathbb{C}$ to each point $\xi \in \mathbb{Z}/p$. Note that ω_ν is a reduced 0-homology class since the sum of its coefficients, ξ^ν, over all p-th degree roots of unity ξ is equal to 0. One can easily see that

$$(r_p)_*(\omega_\nu) = e^{-2\pi i\nu/p}\omega_\nu$$

so that ω_ν is an eigenvector of $(r_p)_*$ with eigenvalue $e^{-2\pi i\nu/p}$. Because of (8.7) the eigenvalues of

$$h_* = (r_p)_* * (r_q)_* : \tilde{H}^1(J; \mathbb{C}) \to \tilde{H}^1(J; \mathbb{C})$$

are all the products $\xi\eta \in \mathbb{C}$ where ξ ranges over all p-th degree roots of unity other than 1, and η ranges over all q-th degree roots of unity other than 1. Thus the characteristic polynomial of h_* equals

$$\prod_{\substack{\xi^p=1,\ \eta^q=1 \\ \xi,\eta\neq 1}} (t - \xi\cdot\eta) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)},$$

and the symmetrized polynomial

$$t^{-(p-1)(q-1)/2} \cdot \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$$

is the Alexander polynomial of the (p,q)-torus knot.

8.6 Open book decompositions

Let M be a closed oriented 3-manifold. An *open book decomposition* of M consists of an oriented link $\mathcal{L} \subset M$, called the *binding*, and a locally trivial bundle $p : M \setminus \mathcal{L} \to S^1$ whose fibers are open surfaces F called *pages*. In addition, $\mathcal{L}$ is required to have a neighborhood $\mathcal{L} \times D^2$ so that the restricted map $p : \mathcal{L} \times (D^2 \setminus \{0\}) \to S^1$ is of the form $(x, y) \mapsto y/|y|$. The closure of each page F is then a connected compact orientable surface with boundary $\mathcal{L}$.

Any oriented link $\mathcal{L} \subset M$ that serves as the binding of an open book decomposition of M is called a *fibered link*. This generalizes the concept of fibered knot from Section 8.1 to arbitrary closed oriented 3-manifolds and links with any number of components.

Example. The construction of Section 8.3 can be extended to other singular complex surfaces, such as the surface V given by the equation $z^2 + w^{2n} = 0$ in $\mathbb{C}^2$ with $n \geq 1$. The only singular point of V is the origin, and the intersection of V with the unit sphere $S^3 \subset \mathbb{C}^2$ is a link $\mathcal{L} \subset S^3$ of two components, called the torus link of type $(2, 2n)$. This link lies in a 2-torus; it sweeps around the torus twice in one coordinate and $2n$ times in the other. For example, the torus link of type $(2, 2)$ is the Hopf link. Milnor's construction desribed in Section 8.3 makes $\mathcal{L}$ into a fibered link whose fibers are connected surfaces of genus $n - 1$ with two boundary components.

Theorem 8.8. *Every closed oriented* 3*-manifold M admits an open book decomposition.*

Proof. This is essentially a corollary of Theorem 2.1 which states that every closed oriented 3-manifold M can be obtained by an integral surgery on a link in S^3.

Recall that in order to prove that theorem in Lecture 2, we represented M as $M = H \cup_{h_2} H'$ where H and H' are handlebodies of genus g, and $h_2 : \partial H \to \partial H'$ is an orientation reversing homeomorphism. For the sake of concreteness, we will view the handlebodies H and H' as giving the standard Heegaard splitting of S^3 of genus g so that $S^3 = H \cup_{h_1} H'$ where h_1 is the (orientation reversing) identity homeomorphism $\partial H \to \partial H'$. Then $h = h_2^{-1} h_1 : \partial H \to \partial H$ is an orientation preserving homeomorphism of the standard Riemann surface $\partial H \subset S^3$. According to Theorem 1.3, one can write h as a product $h = \tau_{c_1} \cdots \tau_{c_n}$ of Dehn twists along simple closed curves $c_i \subset \partial H$; these curves can be chosen from the finite collection of $3g - 1$ curves $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}$ shown in Figure 1.8.

To obtain the manifold M from S^3, we first push each of the curves c_i inside the handlebody H along an annulus A as shown in Figure 2.3 to obtain a knot $k_i \subset H$. Then we drill out the interior of a tubular neighborhood $N(k_i)$ of k_i, and glue $N(k_i)$ back by a homeomorphism of its boundary. This can be done simultaneously for all the curves $c_1, \dots, c_n$ by pushing them along the annuli to different depths inside H and choosing tubular neighborhoods of the knots k_i thin enough so that all the $N(k_i)$ are disjoint.

The outcome of this construction is that there exist two collections of disjoint solid tori $N(k_1), \dots, N(k_n) \subset S^3$ and $N(k_1'), \dots, N(k_n') \subset M$, where each of the knots k_i' is the image of k_i under the surgery, and a homeomorphism

$$\varphi \colon S^3 \setminus \operatorname{int}(N(k_1) \cup \cdots \cup N(k_n)) \to M \setminus \operatorname{int}(N(k_1') \cup \cdots \cup N(k_n'))$$

which carries the meridian of k_i to a longitude of k_i' for each $i = 1, \dots, n$. In addition, the knots $k_1, \dots, k_n$ belong to the collection of $3g - 1$ annuli in S^3 as shown in Figure 8.3. All of these annuli are round and planar, their intersections are orthogonal, and each of them contains several of the knots $k_1, \dots, k_n$ represented by round concentric circles. Also shown in Figure 8.3 is an unknot $\ell \subset S^3$ which runs in the complement to the annuli so that each of the annuli wraps geometrically once around ℓ.

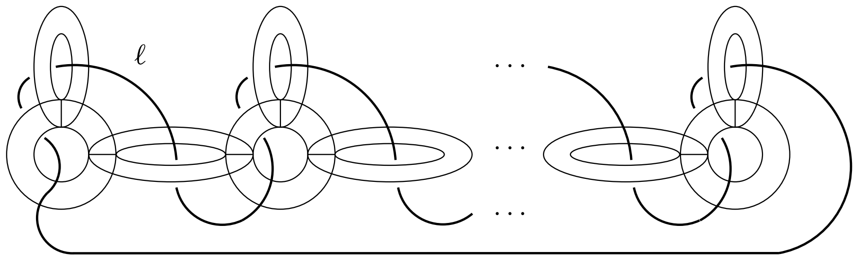


Figure 8.3

The binding of an open book decomposition of M will now consist of the knot $\varphi(\ell) \subset M$ together with the knots $k_1', \dots, k_n'$. The unknot ℓ is fibered, see Figure 8.2, and we can choose a fibration $p : S^3 \setminus \ell \to S^1$ whose fibers intersect each $N(k_i)$ in a meridional disc. Then

$$p \circ \varphi^{-1}\colon M \setminus \operatorname{int}(N(k_1') \cup \dots \cup N(k_n')) \setminus \varphi(\ell) \to S^1$$

is a fibration. Since its fibers intersect $\partial N(k_i')$ in longitudes, this fibration extends to a fibration

$$M \setminus (k_1' \cup \dots \cup k_n' \cup \varphi(\ell)) \to S^1$$

that gives an open book decomposition of M. Note that its pages are actually punctured discs. $\square$

Our discussion of open book decompositions is a very modest introduction to a large and active area of research at the confluence of classical low-dimensional topology with contact geometry and various theories of Floer homology. We highly recommend Etnyre's lectures [43] as further reading.

8.7 Exercises

1. Construct a fibration of a compact orientable 3-manifold M over S^1 such that $\pi_1 M \to \pi_1 S^1$ is not surjective. Is its fiber connected?
2. Let $k \subset S^3$ be a fibered knot of genus one. Prove that the Alexander polynomial of k is either that of a trefoil or the figure-eight knot. (In fact, one can show that k must be one of these knots, see for instance [26]).
3. Prove that the connected sum of two fibered knots is a fibered knot.

Lecture 9
The Arf-invariant

The theories of quadratic and symmetric bilinear forms are identical over a field F in which $0 \neq 2$. In this lecture we are mostly interested in quadratic forms over $\mathbb{Z}/2$, where the two theories differ. The following section on the Arf-invariant of a quadratic form over the field $\mathbb{Z}/2$ follows closely the exposition in Chapter 3 of Browder [24].

9.1 The Arf-invariant of a quadratic form

Let V be a finite dimensional vector space over $\mathbb{Z}/2$. A function $q\colon V \to \mathbb{Z}/2$ is said to be a *quadratic form* if $I(x,y) = q(x+y) - q(x) - q(y)$ is a bilinear form over $\mathbb{Z}/2$. Note that I is symmetric in that $I(x,y) = I(y,x)$. It is clear that $I(x,x) = q(2x) - 2q(x) = 0$ and $q(0) = 0$. A quadratic form q is called *non-degenerate* if its bilinear form I is non-degenerate, i.e. if the determinant of I is not zero in $\mathbb{Z}/2$.

Example. Let $U = (\mathbb{Z}/2)^2$ have a basis a, b. There is only one non-degenerate symmetric bilinear form I on U given by $I(a,a) = I(b,b) = 0$ and $I(a,b) = 1$. Define the quadratic forms $q_0, q_1\colon U \to \mathbb{Z}/2$ by the formulas $q_0(a) = q_0(b) = 0$, $q_0(a+b) = 1$, $q_1(a) = q_1(b) = q_1(a+b) = 1$. The associated bilinear forms of both q_0 and q_1 equal the form I. However, the quadratic forms q_0 and q_1 are not equivalent since the form q_0 sends a majority of vectors of U to 0 while q_1 sends a majority of vectors of U to 1. It turns out that any other non-degenerate quadratic form q on U is equivalent to either q_0 or q_1.

To prove this we only need to consider the form q with $q(a) = 0$ and $q(b) = 1$. We change the basis to $a' = a$, $b' = a + b$ to get $q(a') = 0$ and $q(b') = q(a+b) = I(a,b) + q(a) + q(b) = 0$. Thus q is equivalent to q_0.

Lemma 9.1. *For any non-degenerate quadratic form $q\colon V \to \mathbb{Z}/2$, there exists a* symplectic basis a_i, b_i, $i = 1, \dots, n$, *for V such that $I(a_i, a_j) = I(b_i, b_j) = 0$ and $I(a_i, b_j) = \delta_{ij}$, the Kronecker symbol. In particular,* $\dim V$ *is always even.*

Proof. Choose a basis in V, then the form $I(\ ,\ )$ is given by its matrix I with $\det I = 1$, and $I(x,y) = x \cdot Iy$ where "$\cdot$" stands for the Euclidean dot product. For any $x \neq 0$ there exists u such that $x \cdot u = 1$, and hence $I(x,y) = 1$ for $y = I^{-1}u$. The vectors x and y are linearly independent because $I(x,y) = 1$; in particular, $\dim V \geq 2$. Choose a new basis in V with the first two vectors x and y. The matrix I

in this new basis takes the form

$$\begin{pmatrix} H & * \\ * & I_0 \end{pmatrix} \quad \text{where } H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By elementary transformations it can be turned into

$$\begin{pmatrix} H & 0 \\ 0 & I_1 \end{pmatrix}.$$

The obvious induction completes the proof. □

Let $q\colon V \to \mathbb{Z}/2$ be a non-degenerate quadratic form, and $a_i, b_i,\ i = 1, \ldots, n$, a symplectic basis in V. We define the *Arf-invariant* of q by the formula

$$\operatorname{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{Z}/2.$$

One needs to prove that $\operatorname{Arf}(q)$ is independent of the choice of a symplectic basis. This will follow from the study below of the non-degenerate quadratic forms over $\mathbb{Z}/2$.

Example. The forms $q_0, q_1\colon U \to \mathbb{Z}/2$ from the example above have Arf-invariants $\operatorname{Arf}(q_0) = 0$ and $\operatorname{Arf}(q_1) = 1$. Thus, the Arf-invariant provides a complete classification of non-degenerate quadratic forms on U. Theorem 9.6 below will imply that this is true in general.

Lemma 9.2. *On $U \oplus U$, the forms $q_0 + q_0$ and $q_1 + q_1$ are equivalent.*

Proof. It is clear that the forms $q_0 + q_0$ and $q_1 + q_1$ have the same associated bilinear form I on $U \oplus U$. Let $a_j, b_j,\ j = 1, 2$, be a basis for $U \oplus U$ so that a_j, b_j form a symplectic basis of the j-th copy of U. If $\psi_i = q_i + q_i,\ i = 0, 1$, then $\psi_0(a_j) = \psi_0(b_j) = 0$ and $\psi_1(a_j) = \psi_1(b_j) = 1$, $j = 1, 2$. Choose a new basis for $U \oplus U$,

$$\begin{aligned} a_1' &= a_1 + a_2, & b_1' &= b_1 + a_2, \\ a_2' &= a_2 + b_2 + a_1 + b_1, & b_2' &= b_2 + a_1 + b_1. \end{aligned}$$

One checks easily that this defines a symplectic basis and $\psi_1(a_j') = \psi_0(a_j)$, $\psi_1(b_j') = \psi_0(b_j)$, $j = 1, 2$ so that ψ_1 is equivalent to ψ_0. □

Lemma 9.3. *Let $q\colon V \to \mathbb{Z}/2$ be a non-degenerate quadratic form where* $\dim V = 2m$. *Then q is equivalent to $q_1 + (m-1)q_0$ if, with respect to some basis,* $\operatorname{Arf}(q) = 1$. *The form q is equivalent to mq_0 if* $\operatorname{Arf}(q) = 0$.

Proof. If $a_i, b_i,\ i = 1, \dots, m$, is a symplectic basis for V and if V_i is the subspace spanned by a_i, b_i, let ψ_i denote the restriction of q onto V_i. It is obvious that $q = \sum \psi_i$, where each ψ_i is equivalent to either q_0 or q_1. By the previous lemma, $2q_0 = 2q_1$, so q is equivalent to either mq_0 or $q_1 + (m-1)q_0$. But $\mathrm{Arf}(q_1 + (m-1)q_0) = 1$ and $\mathrm{Arf}(mq_0) = 0$, which implies the result. □

To complete the study of non-degenerate quadratic forms over $\mathbb{Z}/2$, it remains to show that $\varphi_1 = q_1 + (m-1)q_0$ and $\varphi_0 = mq_0$ are not equivalent. We prove this by the following lemma.

Lemma 9.4. *The quadratic form φ_1 sends a majority of elements of V to $1 \in \mathbb{Z}/2$, while φ_0 sends a majority of elements to $0 \in \mathbb{Z}/2$.*

Corollary 9.5. *If q is a non-degenerate quadratic form, then $\mathrm{Arf}(q) = 1$ if and only if q sends a majority of elements of V to $1 \in \mathbb{Z}/2$. In particular, the Arf-invariant is well-defined.*

Proof of Lemma 9.4. We proceed by induction, the case $m = 1$ being trivial. Given a non-degenerate quadratic form φ on V, let $p(\varphi)$ be the number of vectors $x \in V$ such that $\varphi(x) = 1$ and $n(\varphi)$ the number of vectors $x \in V$ such that $\varphi(x) = 0$. Hence $p(\varphi) + n(\varphi) = 2^{2m}$, which is the number of vectors in V, including 0.

The functions p and n satisfy the identities $p(\varphi + q_0) = 3p(\varphi) + n(\varphi)$ and $n(\varphi + q_0) = 3n(\varphi) + p(\varphi)$. This can be seen as follows. Any vector in $V \oplus U$ is of the form (x, u) where $x \in V$ and $u \in U$, and $(\varphi + q_0)(x, u) = \varphi(x) + q_0(u)$. Three of the four vectors in U have $q_0 = 0$ and only one has $q_0 = 1$, so for each vector $x \in V$ such that $\varphi(x) = 1$ we have three vectors (x, u) such that $q_0(u) = 0$, thus $(\varphi + q_0)(x, u) = 1$. Similarly, for each vector $y \in V$ such that $\varphi(y) = 0$ there is one vector (y, v) such that $q_0(v) = 1$, so $(\varphi + q_0)(y, v) = 1$. Hence $p(\varphi + q_0) = 3p(\varphi) + n(\varphi)$, and the other formula follows similarly.

Set $r(\varphi) = p(\varphi) - n(\varphi)$. Then $r(\varphi + q_0) = 2r(\varphi)$, so that if $r(\varphi) > 0$ then $r(\varphi + q_0) > 0$ and if $r(\varphi) < 0$ then $r(\varphi + q_0) < 0$. It follows, since $r(q_1) = 2$ and $r(q_0) = -2$, that $r(q_1 + (m-1)q_0) > 0$ and $r(mq_0) < 0$, which proves the lemma. □

Since r in the above proof is obviously an invariant, it follows that $q_1 + (m-1)q_0$ is not equivalent to mq_0. Thus we have proved the following result.

Theorem 9.6 (C. Arf [5]). *Two non-degenerate quadratic forms on a $\mathbb{Z}/2$-vector space V of finite dimension are equivalent if and only if they have the same Arf-invariant.*

9.2 The Arf-invariant of a knot

An important example of quadratic form arises in knot theory. Let $k \subset \Sigma$ be a knot in an oriented integral homology sphere Σ. Let F be its Seifert surface of genus g and S its Seifert matrix in a fixed basis of the group $H_1(F;\mathbb{Z})$. The skew-symmetric form $I = S^\top - S$ is the intersection form of the surface F; it is unimodular, see Lecture 7. The form $Q = S + S^\top$ is symmetric; it is even and has odd determinant because $Q = I \mod 2$. We define the quadratic form $q\colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$ by the formula

$$q(x) = \frac{1}{2}\, Q(x,x) \mod 2. \tag{9.1}$$

One readily sees that $q(x) = S(x,x) \mod 2$. Its associated bilinear form is $I = Q \mod 2$ since

$$\begin{aligned} q(x+y) - q(x) - q(y) &= S(x+y, x+y) - S(x,x) - S(y,y), \\ &= S(x,y) + S(y,x), \\ &= (S + S^\top)(x,y) = Q(x,y). \end{aligned}$$

One can think of the quadratic form q as follows. Construct a Seifert surface F of k by attaching bands to a disc. There naturally arises a family of curves on F running along the attached bands which give a basis in $H_1(F;\mathbb{Z}/2)$. Let x be one of these curves; then $q(x)$ is the number of full twists mod 2 in a neighborhood of x. For example, the quadratic form of the surface shown in Figure 9.1 is the quadratic form q_1 from the previous section.

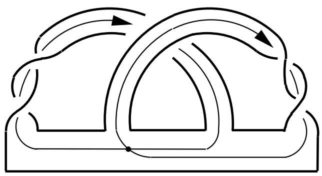


Figure 9.1

Lemma 9.7. *The Arf-invariant* Arf(q) *of the quadratic form* (9.1) *only depends on the knot* $k \subset \Sigma$ *and not on the choices in its definition.*

Thus Arf(q) defines a knot invariant which is usually denoted by Arf(k) and called the *Arf-invariant* of the knot k.

Proof of Lemma 9.7. One only needs to check that Arf(q) does not change under the stabilization operation on Seifert matrices. Stabilization replaces a Seifert matrix S

by

$$S' = \begin{pmatrix} & & & a_1 & 0 \\ & S & & \vdots & \vdots \\ & & & a_{2g} & 0 \\ b_1 & \cdots & b_{2g} & c & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

compare with (7.1). With the help of elementary row and column operations, one can make $c = 0$ and $a_i + b_i = 0$ for all $i = 1, \ldots, 2g$. Then $Q' = S' + (S')^\top$ is of the form

$$Q' = \begin{pmatrix} & & & 0 & 0 \\ & S + S^\top & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

so a symplectic basis for $Q = S + S^\top \bmod 2$ can be completed to a symplectic basis for $Q' = S' + (S')^\top \bmod 2$ so that $\operatorname{Arf}(q') = \operatorname{Arf}(q) + \operatorname{Arf}(q_0) = \operatorname{Arf}(q) \bmod 2$. $\square$

Theorem 9.8. *For any knot $k \subset \Sigma$ in a homology sphere Σ,*

$$\operatorname{Arf}(k) = \frac{1}{2}\,\Delta_k''(1) \bmod 2,$$

where $\Delta_k''(t)$ is the second derivative of the Alexander polynomial of k defined by the formula (7.2).

Proof. Let S be a Seifert matrix of k of size $2g \times 2g$, and $Q = S + S^\top$. There exists an odd integer a and an integral matrix P with odd determinant such that $a^2 \cdot Q = P^\top D P$ where

$$D = \begin{pmatrix} 2p_1 & c_1 \\ c_1 & 2q_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 2p_g & c_g \\ c_g & 2q_g \end{pmatrix}$$

and $c_1, \ldots, c_g$ are odd integers. To prove this claim, let us assume for a moment that we can invert all odd integers. Since the form $Q = \|a_{ij}\|$ is even, $a_{11} = 0 \bmod 2$. At the same time, $\det Q$ is odd, which implies that $a_{1i} = 1 \bmod 2$ for some i. We lose nothing by assuming that $a_{12} = 1 \bmod 2$. The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

has determinant $a_{11}a_{22} - a_{12}^2 = 1 \bmod 2$ and hence is invertible. The matrix Q is of the form

$$Q = \begin{pmatrix} A & L^\top \\ L & B \end{pmatrix},$$

where $A^\top = A$ and $B^\top = B$. If R is the invertible matrix

$$R = \begin{pmatrix} E & A^{-1}L^\top \\ 0 & E \end{pmatrix}$$

where E stands for the identity matrix, then

$$Q = R^\top \begin{pmatrix} A & 0 \\ 0 & -LA^{-1}L^\top + B \end{pmatrix} R, \quad \text{with } \det R = 1.$$

By induction,

$$Q = R^\top DR \tag{9.2}$$

where the matrix R in general is not integral because its elements were allowed to have odd denominators. Let a be the common denominator, then $a = 1 \bmod 2$. To complete the proof of the claim, multiply both sides of (9.2) by a^2 and denote by P the integral matrix $a \cdot R$. Note that $\det P$ is odd.

Thus we have a basis $\{a_j, b_j\}$ in $H_1(F; \mathbb{Z})$ such that

$$a^2 \cdot Q(a_i, a_j) = 2p_i\delta_{ij},$$
$$a^2 \cdot Q(b_i, b_j) = 2q_i\delta_{ij},$$
$$a^2 \cdot Q(a_i, b_j) = c_i\delta_{ij}.$$

Its image under the homomorphism $H_1(F; \mathbb{Z}) \to H_1(F; \mathbb{Z}/2)$ is a symplectic basis for $q \bmod 2$ which we still call $\{a_j, b_j\}$, and

$$q(a_j) = \frac{1}{2}\, Q(a_j, a_j) = p_j \bmod 2,$$
$$q(b_j) = \frac{1}{2}\, Q(b_j, b_j) = q_j \bmod 2.$$

Therefore,

$$\mathrm{Arf}(k) = \sum_{j=1}^{g} p_j q_j \bmod 2.$$

Let us now compute $\Delta_k(-1) = \det(iQ)$. It follows from the claim above that

$$(a^2)^{2g} \det(iQ) = (\det P)^2 \det(iD) = (\det P)^2 \prod_{j=1}^{g} (c_j^2 - 4p_j q_j).$$

Since $x^2 = 1 \bmod 8$ for any odd integer x, we have that modulo 8,

$$\Delta_k(-1) = \det(iQ) = \prod_{j=1}^{g} (1 - 4p_j q_j) = 1 + 4 \sum_{j=1}^{g} p_j q_j = 1 + 4\,\mathrm{Arf}(k). \tag{9.3}$$

Next we will prove that

$$\Delta_k(-1) = 1 + 2\,\Delta_k''(1) \mod 8. \tag{9.4}$$

Since $\Delta_k(1) = 1$ and $\Delta_k(t) = \Delta_k(t^{-1})$, see Lecture 7, the Alexander polynomial $\Delta_k(t)$ can be written in the form ($j \geq 1$)

$$\Delta_k(t) = a_0 + \sum_j a_j(t^j + t^{-j}) \quad \text{where } a_0 = 1 - 2\sum_j a_j.$$

An elementary calculation shows that $\Delta_k''(1) = 2\sum_j j^2 a_j$. Therefore, the right-hand side of (9.4) equals

$$1 + 4\sum_j j^2 a_j.$$

On the other hand,

$$\Delta_k(-1) = a_0 + 2\sum_j (-1)^j a_j = 1 + 2\sum_j a_j((-1)^j - 1) = 1 - 4\sum_{j \text{ odd}} a_j.$$

Now one can easily check that (9.4) holds. Comparing (9.3) and (9.4) completes the proof. □

Example. Let k be a (p,q)-torus knot in S^3, then its Alexander polynomial is

$$\Delta_k(t) = t^{-(p-1)(q-1)/2} \cdot \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}.$$

Therefore, the Arf-invariant of k equals

$$\frac{1}{2}\,\Delta_k''(1) = \frac{(p^2-1)(q^2-1)}{24} \mod 2. \tag{9.5}$$

Example. The Alexander polynomial of a twist knot k of type $(2m+2)_1$ is $(1 + 2m) - m(t + t^{-1})$, therefore, the Arf-invariant of k is $m \mod 2$.

9.3 Exercises

1. Let k be a (p,q)–torus knot in S^3. Verify by a direct calculation with the Alexander polynomial of k that

$$\frac{1}{2}\,\Delta_k''(1) = \frac{(p^2-1)(q^2-1)}{24}.$$

2. Let k_1 and k_2 be oriented knots in homology spheres, and $k_1 \# k_2$ their connected sum. Prove that $\mathrm{Arf}(k_1 \# k_2) = \mathrm{Arf}(k_1) + \mathrm{Arf}(k_2)$.

Lecture 10
Rohlin's theorem

10.1 Characteristic surfaces

Let M be a simply-connected oriented closed smooth 4-manifold, and

$$Q_M \colon H_2(M;\mathbb{Z}) \otimes H_2(M;\mathbb{Z}) \to \mathbb{Z}, \quad a \otimes b \mapsto a \cdot b,$$

its intersection form. A closed oriented surface F smoothly embedded in M is called *characteristic* if

$$F \cdot x = x \cdot x \mod 2 \quad \text{for all } x \in H_2(M;\mathbb{Z}). \tag{10.1}$$

We abuse notation here and use the same symbol F to denote a surface and its homology class in $H_2(M) = H_2(M;\mathbb{Z})$. Let $e_1, \ldots, e_n$ be a basis in $H_2(M)$ then Q_M is given by its matrix $a_{ij} = e_i \cdot e_j$. One can easily check that a surface $F = \sum \varepsilon_i e_i$ is characteristic if and only if

$$\sum_{j=1}^{n} a_{ij}\varepsilon_j = a_{ii} \mod 2 \quad \text{for all } i = 1, \ldots, n. \tag{10.2}$$

Example. Let M be a simply-connected oriented closed smooth 4-manifold. It can be represented as $M = (M \setminus \operatorname{int} D^4) \cup D^4$. Suppose that $M \setminus \operatorname{int} D^4$ is the result of a 4-dimensional surgery of D^4 along a link $L = L_1 \cup \cdots \cup L_n$ in S^3. Thus $M \setminus \operatorname{int} D^4$ is obtained from D^4 by gluing 2-handles. A basis in $H_2(M)$ can then be generated by the surfaces $F_i = F_i' \cup D^2$ where F_i' is a Seifert surface of the knot L_i (with its interior pushed radially inside D^4), and D^2 is the central disc of the corresponding 2-handle. The intersection matrix in this basis is isomorphic to the linking matrix of L, see Theorem 6.2 of Lecture 6. The equations (10.1) and (10.2) defining a characteristic surface turn into the equation for a characteristic sublink L' of L, see equation (4.1) of Lecture 4. The sublink L' is defined uniquely since $\det Q_M = \pm 1$. Given a characteristic sublink L', one can associate a characteristic surface F with it by gluing n copies of D^2, one for each component of L', to a Seifert surface of L'.

With each characteristic surface $F \subset M$, one can associate a quadratic form $\tilde{q} \colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$ (see below), and the Arf-invariant $\operatorname{Arf}(M, F) = \operatorname{Arf}(\tilde{q})$. The following is a generalization of Rohlin's theorem 5.6 from Lecture 5.

Theorem 10.1 (Rohlin [136]). *Let M be a simply-connected oriented closed smooth 4-manifold, and F a closed oriented surface smoothly embedded in M. If F is characteristic then*

$$\frac{1}{8}\,(\operatorname{sign} M - F \cdot F) = \operatorname{Arf}(M, F) \bmod 2. \tag{10.3}$$

Corollary 10.2 (Kervaire–Milnor [81]). *If F in Theorem 10.1 is a 2-sphere then* $\operatorname{sign} M - F \cdot F = 0 \bmod 16$.

This follows from Theorem 10.1 because if F is a 2-sphere, $H_1(F; \mathbb{Z}/2)$ vanishes and $\operatorname{Arf}(M, F) = 0$. The following corollary is obtained from Theorem 10.1 by taking F to be empty.

Corollary 10.3 (Rohlin [135]). *If the intersection form of M in Theorem 10.1 is even then* $\operatorname{sign} M = 0 \bmod 16$.

10.2 The definition of $\tilde{q}$

Let F be a closed oriented characteristic surface smoothly embedded in M. Suppose that a homology class $\gamma \in H_1(F; \mathbb{Z}/2)$ is realized by an embedded circle $\gamma \subset F$. Since $H_1(M; \mathbb{Z}) = 0$, γ bounds a connected orientable surface D embedded in M such that int D is transversal to F. We may deform D slightly to a new surface D' so that $\gamma' = \partial D'$ is a curve in F obtained by shifting ∂D inside F so that $\partial D \cap \partial D' = \emptyset$. One may assume that D and D' intersect transversally. We define

$$\tilde{q}(\gamma) = D \cdot D' + D \cdot F \bmod 2, \tag{10.4}$$

where by $D \cdot D'$ and $D \cdot F$ we mean the intersection numbers of int D with int D' and F, respectively.

Lemma 10.4. *The formula* (10.4) *gives a well-defined quadratic form*

$$\tilde{q}\colon H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/2$$

whose associated bilinear form is the mod 2 *intersection form of the surface F.*

Before we go on to prove this lemma, we consider the following important example.

Example. Suppose that an integral homology sphere Σ is embedded in M and separates the surface F into two pieces, $F = F' \cup D^2$, where $F' \subset \Sigma$ is a Seifert surface of a knot $k \subset \Sigma$. Then we have two quadratic forms,

$$\begin{aligned} &q\colon H_1(F'; \mathbb{Z}/2) \to \mathbb{Z}/2, \quad \text{defined in (9.1) of Lecture 9, and} \\ &\tilde{q}\colon H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/2, \quad \text{defined in (10.4).} \end{aligned}$$

We claim that the inclusion-induced isomorphism $\varphi\colon H_1(F';\mathbb{Z}/2) \to H_1(F;\mathbb{Z}/2)$ makes the following diagram commute:

$$\begin{array}{ccc} H_1(F';\mathbb{Z}/2) & \xrightarrow{q} & \mathbb{Z}/2 \\ \varphi\big\downarrow & & \mathrm{id}\big\downarrow \\ H_1(F;\mathbb{Z}/2) & \xrightarrow{\tilde{q}} & \mathbb{Z}/2 \end{array} \tag{10.5}$$

This can be seen as follows.

Let $\gamma \subset F'$ be an embedded circle in F'. Choose an orientable embedded surface D with $\partial D = \gamma$ such that $D \cap D^2 = \emptyset$ – simply take D equal to a Seifert surface of γ inside Σ and push its interior off D^2. Then $D \cdot F = D \cdot F' = \mathrm{lk}(\gamma, k) \bmod 2$. Let $N(\gamma)$ be a tubular neighborhood of γ in Σ. Since F' is a Seifert surface of the knot k, the intersection $\partial N(\gamma) \cap F'$ is homologous to k via the surface $F' \setminus \mathrm{int}(N(\gamma) \cap F')$, see Figure 10.1. This implies that $[k] = [\partial N(\gamma) \cap F'] \in H_1(\Sigma \setminus \mathrm{int}\, N(\gamma);\mathbb{Z}) = \mathbb{Z}$. Therefore, $D \cdot F = \mathrm{lk}(\gamma, k) = \mathrm{lk}(\gamma, \partial N(\gamma) \cap F') = 0 \bmod 2$.

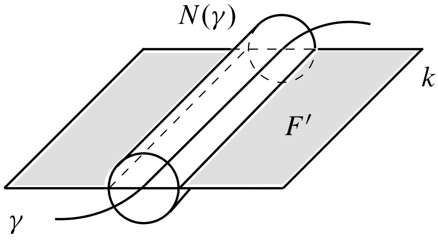


Figure 10.1

Thus $\tilde{q}(\gamma) = D \cdot D' = \mathrm{lk}(\gamma, \gamma') = \mathrm{lk}(\gamma, \gamma^+) = q(\gamma) \bmod 2$ where γ^+ is a (positive) push-off of γ.

Proof of Lemma 10.4. We first check that the number $\tilde{q}(\gamma) \bmod 2$ is independent of the choice of D. Let D_1 and D_2 be two choices for D. If necessary, we may spin D_2 as shown in Figure 10.2 to make $S = D_1 \cup_\gamma D_2$ smoothly embedded (we want the "outside" normal vectors of the surfaces D_1 and D_2 to have opposite directions along their common boundary).

Shown in Figure 10.2 is a movie of D_2 spinning around γ. We choose an interval of γ, represented by the time t axis, so that at a fixed time γ is represented by the center dot (only shown in the first frame of the movie). The horizontal line is a slice of F, normal to γ. The vertical lines represent collars of D_2 and D_2'. Note that this spin changes both $D_2 \cdot D_2'$ and $D_2 \cdot F$ by ± 1, so that $D_2 \cdot D_2' + D_2 \cdot F$ remains unchanged $\bmod 2$.

Let $S' = D_1' \cup D_2'$, then $S \cdot S = S \cdot S' = D_1 \cdot D_1' + D_2 \cdot D_2' \bmod 2$. Since F is characteristic, $S \cdot S = S \cdot F \bmod 2$, so we get $D_1 \cdot D_1' + D_2 \cdot D_2' = D_1 \cdot F + D_2 \cdot F \bmod 2$

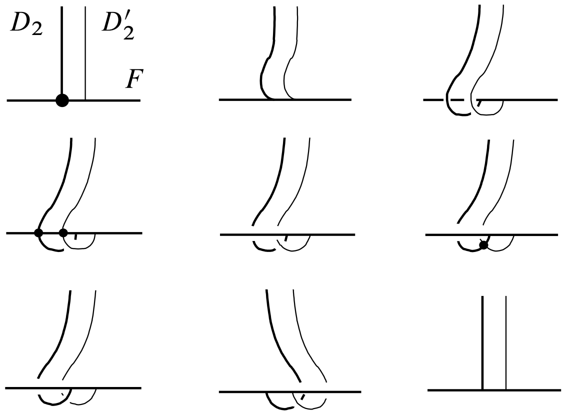


Figure 10.2

and $D_1 \cdot D_1' + D_1 \cdot F = D_2 \cdot D_2' + D_2 \cdot F \mod 2$. Thus $\tilde{q}(\gamma)$ is independent of the choice of D.

Since any two homotopic closed simple curves on F are isotopic (see Lecture 1), $\tilde{q}(\gamma)$ only depends on the homotopy class of γ, and hence defines a map $\tilde{q}\colon \pi_1(F) \to \mathbb{Z}/2$.

Let $\gamma_1 * \gamma_2$ denote a product of loops γ_1 and γ_2, then we claim that

$$\tilde{q}(\gamma_1 * \gamma_2) = \tilde{q}(\gamma_1) + \tilde{q}(\gamma_2) + \gamma_1 \cdot \gamma_2 \mod 2, \tag{10.6}$$

where $\gamma_1 \cdot \gamma_2$ is the intersection modulo 2 of the homology classes represented by γ_1 and γ_2. Since $\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_1 \mod 2$, the formula (10.6) implies that $\tilde{q}(\gamma_1 * \gamma_2) = \tilde{q}(\gamma_2 * \gamma_1)$ and that the map $\tilde{q}\colon \pi_1(F) \to \mathbb{Z}/2$ factors through $H_1(F;\mathbb{Z})$ and $H_1(F;\mathbb{Z}/2)$.

Thus, to complete the proof we only need to check the formula (10.6). For the sake of simplicity, let the curves γ_1 and γ_2 intersect transversely at one point, and let D_1 and D_2 be surfaces that the curves bound as in the definition of $\tilde{q}$. Let γ be a smooth connected sum loop representing $\gamma_1 * \gamma_2$, see Figure 10.3.

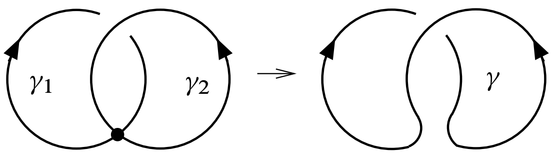


Figure 10.3

We get a bounding surface D for γ from $D_1 \cup D_2$ and the curved triangles T_1 and T_2 shaded in Figure 10.4. Push γ off in the direction of a normal field of γ extending the normal fields on γ_1 and γ_2.

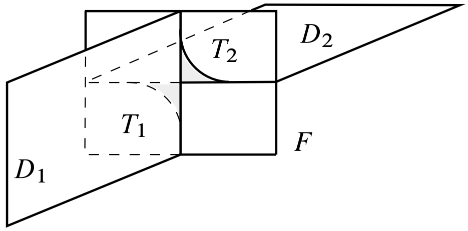


Figure 10.4

Then γ and its push-off will link as shown in Figure 10.5, which indicates that $D \cdot D' = D_1 \cdot D'_1 + D_2 \cdot D'_2 + 1 \bmod 2$.

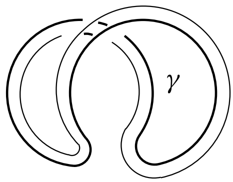


Figure 10.5 □

Lemma 10.5. $\mathrm{Arf}(M, F)$ *only depends on the homology class* $[F] \in H_2(M; \mathbb{Z}/2)$.

Proof. This is a closed surface analogue of Levine's theorem 7.2. For a complete proof see Matsumoto [107]. □

Proof of Theorem 10.1. Let us consider the manifold $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. Its intersection form is odd and indefinite, hence isomorphic to the form $p \cdot (+1) \oplus q \cdot (-1)$ with $p = b_+(M) + 1$ and $q = b_-(M) + 1$. By Wall's theorem, see Theorem 5.5, there is a $k \geq 0$ such that $(M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2) \# k \cdot (S^2 \times S^2)$ is diffeomorphic to $(p \cdot \mathbb{C}P^2 \# q \cdot \overline{\mathbb{C}P}^2) \# k \cdot (S^2 \times S^2)$. Since

$$(S^2 \times S^2) \# \mathbb{C}P^2 = \overline{\mathbb{C}P}^2 \# 2 \cdot \mathbb{C}P^2 \quad \text{and} \quad (S^2 \times S^2) \# \overline{\mathbb{C}P}^2 = \mathbb{C}P^2 \# 2 \cdot \overline{\mathbb{C}P}^2,$$

see Lecture 3, we have that for some ℓ_1 and ℓ_2,

$$M \# \ell_1 \cdot \mathbb{C}P^2 \# \ell_2 \cdot \overline{\mathbb{C}P}^2 = a \cdot \mathbb{C}P^2 \# b \cdot \overline{\mathbb{C}P}^2,$$

where $a = \ell_1 + b_+(M)$ and $b = \ell_2 + b_-(M)$.

Let $\eta \in H_2(\mathbb{C}P^2) = \mathbb{Z}$ and $\bar{\eta} \in H_2(\overline{\mathbb{C}P}^2) = \mathbb{Z}$ be the generators represented by the embedded 2-spheres $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Then $\eta \cdot \eta = 1$ and $\bar{\eta} \cdot \bar{\eta} = -1$. If a class F is characteristic in $H_2(M)$ then the class $F_c = F + \ell_1 \cdot \eta + \ell_2 \cdot \bar{\eta}$ is characteristic in $M \# \ell_1 \cdot \mathbb{C}P^2 \# \ell_2 \cdot \overline{\mathbb{C}P}^2$. The property of being characteristic is preserved under diffeomorphisms, therefore, the image of F_c in $a \cdot \mathbb{C}P^2 \# b \cdot \overline{\mathbb{C}P}^2$ is characteristic.

Both sides of the formula (10.3) are additive with respect to connected sums of manifolds and characteristic surfaces. Therefore, if (10.3) is true for any two of the following three pairs, (M_1, F_1), (M_2, F_2), and $(M_1 \# M_2, F_1 \cup F_2)$, it is true for the third one. Obviously, $\operatorname{sign} \mathbb{C}P^2 - \eta \cdot \eta = 0 = \operatorname{Arf}(\mathbb{C}P^2, \eta)$ and $\operatorname{sign} \overline{\mathbb{C}P}^2 - \bar{\eta} \cdot \bar{\eta} = 0 = \operatorname{Arf}(\overline{\mathbb{C}P}^2, \bar{\eta})$. Moreover, both sides of (10.3) change sign with the change of orientation. Therefore, the formula (10.3) only needs to be proved for characteristic surfaces in $\mathbb{C}P^2$.

If $\eta \in H_2(\mathbb{C}P^2) = \mathbb{Z}$ is a generator represented by the embedded 2-sphere $\mathbb{C}P^1$, then a class $s \cdot \eta \in H_2(\mathbb{C}P^2)$ is characteristic if and only if s is odd. The complex curve

$$C = \{[x_0 : x_1 : x_2] \mid x_0 x_1^{s-1} + x_2^s\} \subset \mathbb{C}P^2$$

is homeomorphic to S^2 and represents the class $s \cdot \eta$, see Lemma 10.6 below. It is smoothly embedded in $\mathbb{C}P^2$ except possibly at the point $[1 : 0 : 0]$. Let B be a 4-ball of radius $\varepsilon > 0$ centered at $[1 : 0 : 0]$. In the affine plane $x_0 = 1$ the intersection $\partial B \cap C$ is given by the equations $x_1^{s-1} + x_2^s = 0$, $|x_1|^2 + |x_2|^2 = \varepsilon^2$. Therefore, $\partial B \cap C \subset \partial B = S^3$ is the $(s, s-1)$-torus knot $k_{s,s-1}$. Let S be a Seifert surface in ∂B with boundary the curve $\partial B \cap C$, then the surface $F = (C \setminus (C \cap \operatorname{int} B)) \cup S$ represents the class $s \cdot \eta$. An easy calculation using the identification of the quadratic forms q and $\tilde{q}$ in the example above shows that

$$\begin{aligned} \operatorname{Arf}(\mathbb{C}P^2, s \cdot \eta) &= \operatorname{Arf}(k_{s,s-1}), \\ &= (s^2 - 1)((s-1)^2 - 1)/24 \bmod 2, \quad \text{see (9.5)}, \\ &= (1 - s^2)/8 \bmod 2, \\ &= (\operatorname{sign} \mathbb{C}P^2 - s\eta \cdot s\eta)/8 \bmod 2. \end{aligned}$$

□

Lemma 10.6. *The complex curve C in $\mathbb{C}P^2$ given by the equation $x_0 x_1^{s-1} + x_2^s = 0$ is homeomorphic to S^2 and represents the homology class $s \cdot [\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$.*

Proof. The formula $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ defines a map $\varphi: C \to \mathbb{C}P^1$. One can easily see that φ is onto, and that for all points $[x_0 : x_1]$ distinct from $[0 : 1]$ and $[1 : 0]$ the preimage $\varphi^{-1}([x_0 : x_1])$ consists of s points, while $\varphi^{-1}([0 : 1]) = [0 : 1 : 0]$ and $\varphi^{-1}([1 : 0]) = [1 : 0 : 0]$. From this information, one can compute the Euler characteristic of C, namely, $\chi(C) = s \cdot \chi(S^2) - 2(s-1) = 2$. Therefore, C is a 2-sphere. The map φ has degree s, therefore, the induced map $\varphi_*: H_2(C) \to H_2(\mathbb{C}P^1)$ is a multiplication by s.

Note that $[0 : 0 : 1] \notin C$. This implies that the inclusion map $i: C \to \mathbb{C}P^2$ factors through $\mathbb{C}P^2 \setminus \{[0 : 0 : 1]\}$, which contains E_{-1} as a deformation retract, see

Lecture 2. Let $\pi\colon E_{-1} \to \mathbb{C}P^1$ be the projection $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ (with fibers D^2); then we have the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "i_0"] \arrow[d, "\varphi"'] & E_{-1} \arrow[r, "i_1"] \arrow[d, "\pi"'] & \mathbb{C}P^2 \\ \mathbb{C}P^1 \arrow[r, equal] & \mathbb{C}P^1 & \end{tikzcd}$$

Here, i_0 and i_1 are the natural inclusions whose composition is the inclusion $i\colon C \to \mathbb{C}P^2$. In the second homology, $(i_1)_*$ and π_* are identity isomorphisms, and by the commutativity of the diagram, the map $i_* = (i_1)_*(\pi)_*^{-1}\varphi_*\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by s since φ_* is. $\square$

10.3 Representing homology classes by surfaces

Let M be a simply-connected oriented closed smooth 4-manifold. As we know from Lemma 5.2, every homology class $u \in H_2(M)$ can be represented by a smoothly embedded surface F. The following is one of the most intriguing problems in 4-dimensional topology: given a class $u \in H_2(M)$, what is the minimal genus of $F \subset M$ representing u? The class u is said to be *spherical* if it can be represented by an embedded 2-sphere.

Example. Let $\eta \in H_2(\mathbb{C}P^2) = \mathbb{Z}$ be the generator represented by the complex line $\mathbb{C}P^1 \subset \mathbb{C}P^2$, so that η is spherical. All other classes in $H_2(\mathbb{C}P^2)$ are then of the form $s\eta$ with $s \in \mathbb{Z}$. It suffices to consider $s \geq 0$. Rohlin's theorem prohibits certain classes $s\eta$ from being spherical. For example, if the class 3η were spherical, we would get a contradiction, since

$$\frac{\operatorname{sign}\mathbb{C}P^2 - 3\eta\cdot 3\eta}{8} = \frac{1-9}{8} \neq \operatorname{Arf}(\mathbb{C}P^2, 3\eta) = 0 \bmod 2.$$

The same argument works for the classes $s\eta$ with $s = \pm 3 \bmod 8$; thereby showing that they are not spherical. On the other hand, the class 2η is spherical – the construction from the proof of Rohlin's theorem that employed the complex curve $C = \{x_0 x_1^{s-1} + x_2^s = 0\}$ produces, in the case of $s = 2$, the surface $\{x_0 x_1 + x_2^2 = 0\}$. Its intersection with a small 3-sphere centered at $[1 : 0 : 0]$ is the torus knot $k_{2,1}$, which is equivalent to an unknot. In particular, it bounds a Seifert surface S of genus 0. Now, the surface $F = D^2 \cup S$ is a smoothly embedded 2-sphere representing 2η.

In general, using a Seifert surface S of minimal possible genus $(s-1)(s-2)/2$ to bound the torus knot $k_{s,s-1}$ is the proof of Rohlin's theorem provides the upper bound of $(s-1)(s-2)/2$ on the minimal genus of embedded surfaces representing $s\eta$ in $\mathbb{C}P^2$. The famous Thom conjecture asserts that this is also a lower bound. This conjecture was proved by Kronheimer and Mrowka [95] with the help of Seiberg–Witten gauge theory.

Lecture 11
The Rohlin invariant

11.1 Definition of the Rohlin invariant

Let Σ be an oriented integral homology 3-sphere. According to Theorem 6.2 and Theorem 4.1, there exists a smooth simply-connected oriented 4-manifold W with even intersection form such that $\partial W = \Sigma$. Then the signature of W is divisible by 8, and

$$\mu(\Sigma) = \frac{1}{8}\,\operatorname{sign} W \mod 2 \tag{11.1}$$

is independent of the choice of W, see Section 6.4 of Lecture 6. We call $\mu(\Sigma)$ the *Rohlin invariant* of Σ. It takes values 0 and 1 mod 2.

Suppose that M is a smooth simply-connected oriented 4-manifold with $\partial M = \Sigma$; we do not assume an even intersection form. Suppose that M has a spherical characteristic surface, that is, a smoothly embedded characteristic surface $F \subset M$ of genus 0. Then

$$\mu(\Sigma) = \frac{1}{8}\,(\operatorname{sign} M - F \cdot F) \mod 2. \tag{11.2}$$

To check the formula (11.2), form a smooth closed manifold $X = M \cup_\Sigma (-W)$. Then F is a spherical characteristic surface in X, and

$$\begin{aligned}
\frac{1}{8}(\operatorname{sign} M - F \cdot F) - \mu(\Sigma) &= \frac{1}{8}(\operatorname{sign} M - F \cdot F) - \frac{1}{8}\operatorname{sign} W, \quad \text{by (11.1)},\\
&= \frac{1}{8}(\operatorname{sign} X - F \cdot F) = 0 \mod 2, \quad \text{by Corollary 10.2.}
\end{aligned}$$

11.2 The Rohlin invariant of Seifert spheres

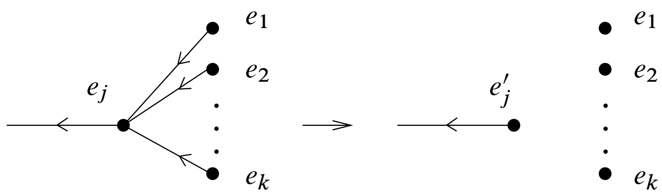
In this section we provide an algorithm for calculating the Rohlin invariant of an arbitrary Seifert homology sphere $\Sigma(a_1, \dots, a_n)$. A closed form formula for $\mu(\Sigma(a_1, \dots, a_n))$ will be given in Lecture 19.

Any Seifert homology sphere $\Sigma(a_1, \dots, a_n)$ is the boundary of a manifold M obtained by surgery according to a star-shaped tree Γ, see Figure 2.12. Each vertex of Γ has an integer weight, say e_i, $i = 1, \dots, s$, attached to it. To calculate the Rohlin invariant of $\Sigma(a_1, \dots, a_n)$ we first need to compute sign Q_M, where Q_M is the intersection form of M, and then describe a spherical characteristic class $F \subset M$, if such exists.

Each vertex of the tree Γ corresponds to a basis vector in the second homology group $H_2(M) = \mathbb{Z}^s$ represented by an embedded 2-sphere. With proper orientations, the intersection form Q_M is isomorphic to the linking matrix $A(\Gamma) = (a_{ij})_{i,j=1,\dots,s}$ with entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j, \\ 1, & \text{if the } i\text{-th and } j\text{-th vertices are connected by an edge}, \\ 0, & \text{otherwise.} \end{cases}$$

There is a simple algorithm to diagonalize matrices of the form $A(\Gamma)$, see Duchon [40] and Eisenbud–Neumann [41]. Let us allow trees weighted by arbitrary rational numbers. Given such a tree, pick a vertex and direct all its edges toward this vertex. Now, the tree can be simplified by performing operations of the two types shown in Figures 11.1 and 11.2 (we re-index vertices if necessary).



$$e'_j = e_j - \frac{1}{e_1} - \cdots - \frac{1}{e_k}$$

Figure 11.1

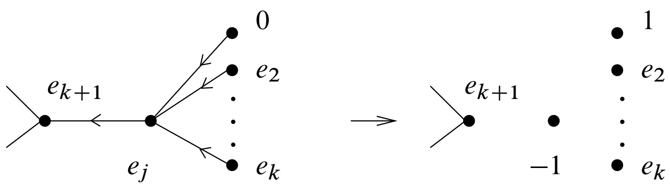


Figure 11.2

We end up with a finite collection of isolated points weighted by rational numbers $d_1, \dots, d_s$. Then $D = \operatorname{diag}(d_1, \dots, d_s)$ is a diagonalization of $A(\Gamma)$, that is, $A(\Gamma) = U^\top D U$ with $\det U = \pm 1$. In particular, $\det A(\Gamma) = \det D$ and $\operatorname{sign} A(\Gamma) = \operatorname{sign} D$.

Example. For the tree Γ in Figure 11.3 we have $\det A(\Gamma) = \det D = 2 \cdot 3 \cdot (1/6) \cdot (-2) \cdot (-1/2) = 1$ and $\operatorname{sign} A(\Gamma) = \operatorname{sign} D = 3 - 2 = 1$.

This algorithm works for any weighted tree (not necessarily star-shaped). As the following example shows, the diagonal entries in D generalize the concept of a continued fraction.

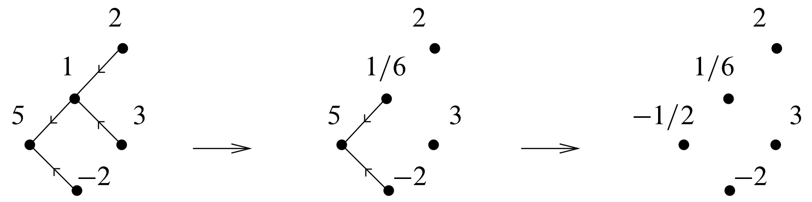


Figure 11.3

Example. The diagonalization D of the matrix $A(\Gamma)$ corresponding to the tree in Figure 11.4 has entries

$$[x_n],\ [x_{n-1}, x_n],\ \ldots,\ [x_2, \ldots, x_{n-1}, x_n],\ [x_1, \ldots, x_{n-1}, x_n],$$

where [,] stands for continued fraction, see (2.3). The determinant of $A(\Gamma)$ equals (up to a sign) the numerator of the reduced fraction $[x_1, \ldots, x_{n-1}, x_n]$.

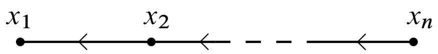


Figure 11.4

Having said this, we can easily compute the signature of Q_M. Now we move on to finding the characteristic surface $F \subset M$. The surface F corresponds to the characteristic sublink, thus it can be described as a collection of vertices in the weighted tree Γ.

Lemma 11.1. *If two vertices of a tree belong to a characteristic sublink, they are not connected by an edge.*

Proof. Suppose that two vertices, u and $v \in H_2(M)$, framed by a and b, respectively, belong to the characteristic sublink and are connected by an edge, see Figure 11.5.

Figure 11.5

Fix an orientation on the link so that $u \cdot v = 1$. Then $F = u + v + \cdots$ and $F \cdot F = u \cdot u + v \cdot v + 2u \cdot v + \cdots = a + b + 2 + \cdots$ Note that sign $M - F \cdot F$ is divisible by 8. Reverse the orientations on the link components to the left of u (including u). This operation preserves the signature and the characteristic sublink but changes $u \cdot v$ from 1 to -1. Thus $F \cdot F = a + b - 2 + \cdots$, which is a change of 4 compared to the calculation of $F \cdot F$ above. This contradicts the divisibility by 8. □

Corollary 11.2. *The characteristic class in the manifold M is spherical.*

Proof. The surface F consists of several disjoint 2-spheres. Connect them by tubes to obtain a connected surface. □

Example. The Seifert homology sphere $\Sigma(2,3,7)$ has the link description as in Figure 11.6.

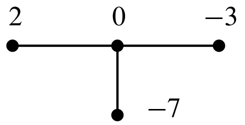


Figure 11.6

By the algorithm above, $\operatorname{sign} M = -2$. To find the characteristic sublink, consider the linking matrix,

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -7 \end{pmatrix},$$

and the system $A\varepsilon = \operatorname{diag} A \mod 2$ that defines the characteristic sublink. This system is of the form

$$\begin{aligned} \varepsilon_1 + \varepsilon_2 + \varepsilon_3 &= 0 \\ \varepsilon_0 \qquad\qquad\qquad &= 0 \\ \varepsilon_0 + \qquad \varepsilon_2 \qquad &= 1 \\ \varepsilon_0 + \qquad\qquad \varepsilon_3 &= 1 \end{aligned}$$

and its only solution is $\varepsilon = (0,0,1,1)$. Therefore, $F \cdot F = -3 - 7 = -10$ and $\mu(\Sigma(2,3,7)) = 1/8\,(-2+10) = 1 \mod 2$.

Remark. Let Σ be a Seifert homology sphere, and M a smooth manifold obtained by surgery according to a star-shaped tree as in the above calculation of the Rohlin invariant of Σ. It turns out that the quantity

$$\bar{\mu}(\Sigma) = \frac{1}{8}\,(\operatorname{sign} M - F \cdot F) \tag{11.3}$$

is independent of the choice of such a manifold M even before it is reduced modulo 2, see Neumann [121]. Thus (11.3) defines an integer valued invariant of Σ known as the $\bar{\mu}$-invariant (of Neumann and Siebenmann). This invariant was actually defined in [121] for the so called plumbed homology spheres, of which Seifert homology spheres are a special case.

11.3 A surgery formula for the Rohlin invariant

Let k be a knot in a homology 3-sphere Σ. We denote the homology sphere obtained by $(1/m)$-surgery of Σ along k by

$$\Sigma + \frac{1}{m} \cdot k, \quad m \in \mathbb{Z}.$$

Let $\Delta_{k\subset\Sigma}(t)$ be the Alexander polynomial of k. Remember that it is symmetric in that $\Delta_{k\subset\Sigma}(1) = 1$ and $\Delta_{k\subset\Sigma}(t^{-1}) = \Delta_{k\subset\Sigma}(t)$.

Theorem 11.3. *Let k be a knot in a homology sphere Σ and $\varepsilon = \pm 1$. Then*

$$\mu(\Sigma + \varepsilon \cdot k) = \mu(\Sigma) + \frac{1}{2} \cdot \Delta''_{k\subset\Sigma}(1) \bmod 2.$$

Proof. Let W and W' be oriented simply-connected smooth 4-manifolds with even intersection forms whose boundaries are $\partial W = \Sigma$ and $\partial W' = \Sigma' = \Sigma + \varepsilon \cdot k$. Then

$$\mu(\Sigma) = \frac{1}{8} \operatorname{sign} W \bmod 2 \quad \text{and} \quad \mu(\Sigma') = \frac{1}{8} \operatorname{sign} W' \bmod 2.$$

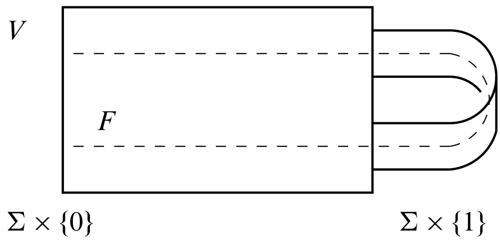


Figure 11.7

Let V be the union of the manifold $\Sigma \times [0,1]$ and a 2-handle attached to $\Sigma \times \{1\}$ along k with framing ε, see Figure 11.7. The boundary of V is $-\Sigma \cup \Sigma'$. The union of a Seifert surface of the knot k in $\Sigma \times \{0\}$ with the cylinder $k \times [0,1] \subset \Sigma \times [0,1]$ and with the axis $D^2 \times \{0\}$ of the 2-handle is a closed orientable surface F smoothly embedded in V with self-intersection $\varepsilon = \pm 1$. The homology class of this surface generates the group $H_2(V;\mathbb{Z}) = \mathbb{Z}$. The union

$$X = W \cup_{\Sigma\times\{0\}} V \cup_{\Sigma'} (-W')$$

is a closed smooth oriented simply-connected 4-manifold with the intersection form $Q_X = Q_W \oplus Q_V \oplus Q_{-W'}$, and with characteristic surface $F \subset X$. Therefore,

$$\begin{aligned}\frac{1}{8}(\operatorname{sign} X - F\cdot F) &= \operatorname{Arf}(X, F) \bmod 2, && \text{by Theorem 10.1,}\\ &= \operatorname{Arf}(k\subset\Sigma) \bmod 2, && \text{by (10.5),}\\ &= \frac{1}{2}\cdot\Delta''_{k\subset\Sigma}(1) \bmod 2, && \text{by Theorem 9.8.}\end{aligned}$$

On the other hand, $\operatorname{sign} X = \operatorname{sign} W + \varepsilon - \operatorname{sign} W'$ and $F\cdot F = \varepsilon$, hence

$$\mu(\Sigma) - \mu(\Sigma') = \frac{1}{8}\,(\operatorname{sign} W - \operatorname{sign} W') = \frac{1}{2}\cdot\Delta''_{k\subset\Sigma}(1) \bmod 2,$$

and the result follows. □

Corollary 11.4. *If k is a knot in S^3 then $\mu(S^3 \pm k) = \frac{1}{2}\cdot\Delta''_k(1) \bmod 2$.*

Recall from Lecture 7 that a link $k\cup\ell$ in a homology sphere Σ is called boundary if the knots k and ℓ bound disjoint Seifert surfaces in Σ.

Example. If k is a knot in a homology sphere Σ and ℓ its canonical longitude then the link $k\cup\ell$ is boundary. Let us perform a $(+1)$-surgery on k to get a homology sphere Σ'. The image of $\ell\subset\Sigma$ in Σ' is well-defined; we again call it ℓ. According to Lemma 7.13,

$$\Delta_{k\subset\Sigma}(t) = \Delta_{\ell\subset\Sigma+k}(t). \tag{11.4}$$

One can further perform a $(+1)$-surgery on $\ell\subset\Sigma'$ to get another homology sphere, Σ'' (it is a homology sphere because $\operatorname{lk}(k,\ell) = 0$ in Σ). We claim that

$$(\Sigma + k) + \ell = \Sigma + \frac{1}{2}\cdot k.$$

To see this, we slide ℓ over k in Σ, the homology sphere Σ being thought of as a surgery along a link in S^3 ($p = 2$ in the picture), and isotope the knot $k+\ell$ into a small circle linking k geometrically once, see Figure 11.8.

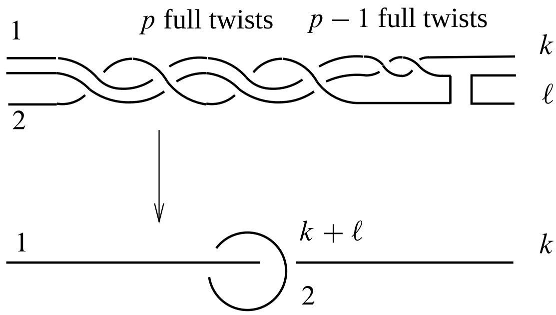


Figure 11.8

The surgery on the resulting framed link $k \cup (k + \ell)$ is equivalent to the rational surgery on k with framing $1/2$, see Figure 11.9.

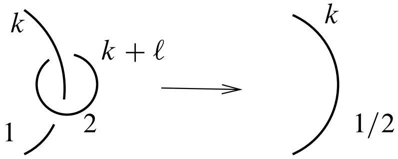


Figure 11.9

Of course, this result can be generalized for any number of parallel copies of the knot $k \subset \Sigma$. Applying the surgery formula for the Rohlin invariant sufficiently many times and keeping in mind the formula (11.4), we obtain the following result.

Corollary 11.5. *Let k be a knot in a homology sphere Σ, and $\Delta_{k\subset\Sigma}(t)$ its Alexander polynomial. Then*

$$\mu\left(\Sigma + \frac{1}{m}\cdot k\right) = \mu(\Sigma) + \frac{m}{2}\cdot \Delta''_{k\subset\Sigma}(1),\ m \in \mathbb{Z}.$$

11.4 The homology cobordism group

Let Σ_0 and Σ_1 be oriented integral homology 3-spheres. They are said to be *homology cobordant*, or *H-cobordant*, if there exists a smooth compact oriented 4-manifold W with boundary $\partial W = -\Sigma_0 \cup \Sigma_1$ such that the inclusion induced homomorphisms $H_*(\Sigma_i) \to H_*(W)$, $i = 0, 1$, are isomorphisms. For example, Σ_0 is H-cobordant to itself via the product cobordism $W = \Sigma_0 \times [0,1]$. As the example below shows, not all H-cobordisms are products.

A homology sphere Σ is said to be *H-cobordant to zero* if it is H-cobordant to S^3. Equivalently, Σ is H-cobordant to zero if there is an oriented smooth compact 4-manifold W such that $\partial W = \Sigma$ and $H_*(W) = H_*(D^4)$. Note that we only require that $H_1(W) = 0$, not that $\pi_1(W) = 0$.

Example. The following construction of a non-trivial H-cobordism is due to Mazur [109]. Let us consider the manifold $S^1 \times D^3$ with boundary $S^1 \times S^2$. One can think of $S^1 \times D^3$ as the result of attaching a 1-handle $D^1 \times D^3$ to D^4. Let us pick a knot in $S^1 \times D^2 \subset S^1 \times S^2$ as pictured in Figure 11.10.

Use this knot to attach a 2-handle to $S^1 \times D^3$ with framing 3. We get a smooth 4-manifold W with boundary. Its homology can be easily calculated with the help of the Mayer–Vietoris sequence for $W = (S^1 \times D^3) \cup_{S^1\times D^2} (D^2 \times D^2)$, where $S^1 \times D^2$ now refers to a tubular neighborhood of the knot in $\partial(S^1 \times D^3)$. The key ingredient in this calculation is that the inclusion induced map $H_1(S^1 \times D^2) \to H_1(S^1 \times D^3)$ is an

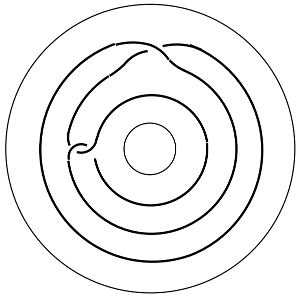


Figure 11.10

isomorphism, due to the fact that the knot is homologous to $S^1 \times \{0\} \subset S^1 \times D^2$. The final result is that $H_*(W) = H_*(D^4)$, hence W is an H-cobordism. In a situation like this we will say that the 1- and 2-handles cancel each other in the homology of W. Moreover, one can show that $\pi_1(W) = 0$ so the manifold W is in fact contractible by the Whitehead Theorem.

The boundary Σ of W can be described as follows. The manifold $S^1 \times S^2$ can be thought of as the boundary of $D^2 \times S^2$. Then Σ is the boundary of the manifold $W' = (D^2 \times S^2) \cup_{S^1 \times S^2} (D^2 \times D^2)$. The 4-manifolds W and W' are distinct but $\partial W = \partial W'$. The manifold $D^2 \times S^2$ can be obtained by gluing a 2-handle to the 4-ball D^4 along an unknot in $S^3 = \partial D^4$ with framing 0 (this manifold was called E_0 in Lecture 2). Thus W' has the link description as shown in Figure 11.11.

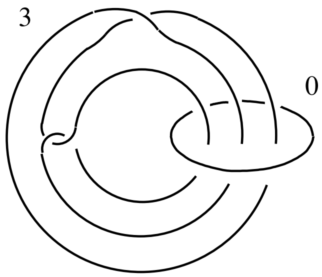


Figure 11.11

Shown in Figure 11.12 is a link description for the original manifold W, with the dot signifying the fact that a 1-handle rather than a 2-handle is attached.

One can use Kirby calculus to show that the homology sphere Σ shown in Figure 11.11 is homeomorphic to $\Sigma(2, 5, 7)$, see Figure 11.13. In particular, $\pi_1(\Sigma)$ is not trivial, hence Σ is homology cobordant to S^3 via a homology cobordism which is not a product.

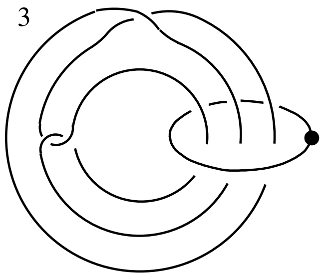


Figure 11.12

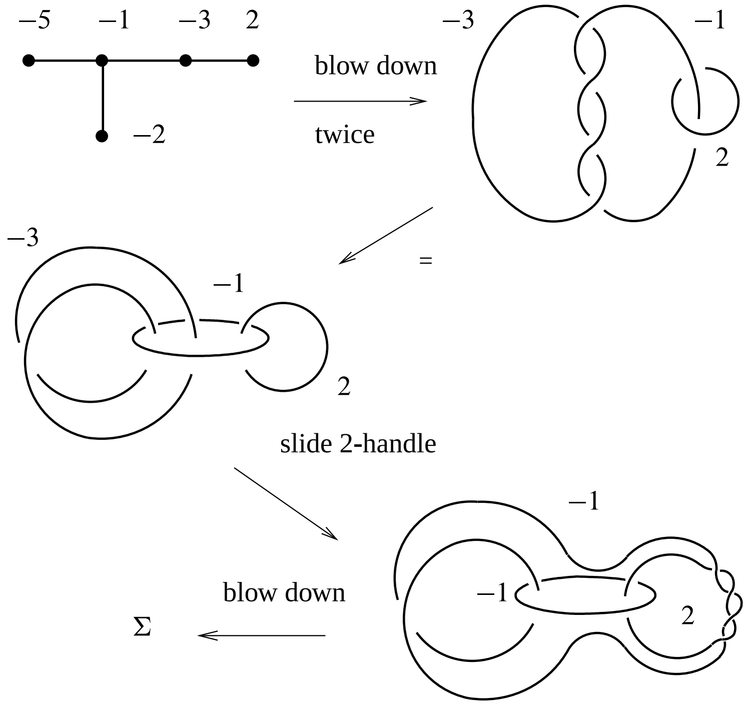


Figure 11.13

One can modify this construction to get other examples of homology spheres homology cobordant to zero. The framing 3 can be replaced by any integer p – among the homology spheres obtained for various p are $\Sigma(3,4,5)$ and $\Sigma(2,3,13)$, see Akbulut and Kirby [1]. The knot we started with can also be replaced by any knot which passes through the 1-handle algebraically once. If it passes only once geometrically, we end up with S^3, see Proposition 3.3. Finally, one can construct homology cobordisms with more that one pair of canceling 1- and 2-handles.

The set of all homology cobordism classes of oriented integral homology 3-spheres forms an Abelian group Θ^3 with the group operation defined by connected sum. Here, the zero element is the homology cobordism class of S^3, and the additive inverse is obtained by reversing the orientation. We call Θ^3 the integral *homology cobordism group*.

Lemma 11.6. *The Rohlin invariant μ defines an epimorphism $\mu\colon \Theta^3 \to \mathbb{Z}/2$.*

Proof. Suppose that a homology sphere Σ is homology cobordant to zero via a smooth *contractible* 4-manifold W. Since the intersection form $Q_W = \emptyset$ is even, we can evaluate the Rohlin invariant as

$$\mu(\Sigma) = \frac{1}{8}\,\operatorname{sign} W = 0 \mod 2.$$

In the case $\pi_1(W) \neq 0$ one needs to generalize Rohlin's theorem to manifolds which are not simply-connected but still have $H_1(W) = 0$. Such a theorem is true, and in fact, our proof in Lecture 10 goes through with only minor modifications. Thus μ is a homology cobordism invariant. It obviously defines a homomorphism, and it is onto since $\mu(\Sigma(2,3,5)) = 1 \mod 2$. □

For a few decades, the existence of the epimorphism μ had been the only known fact about the group Θ^3. It was even conjectured in the 1970's that $\mu\colon \Theta^3 \to \mathbb{Z}/2$ is an isomorphism. In the 1980's, techniques from gauge theory were utilized to prove that the group Θ^3 is infinite.

Example. The Poincaré homology sphere $\Sigma = \Sigma(2,3,5)$ has infinite order in Θ^3. To prove this, recall from Lecture 3 that $\Sigma(2,3,5)$ bounds a smooth compact oriented 4-manifold M with the intersection form $Q_M = E_8$. Let us consider an m-multiple of Σ, the homology sphere $m\,\Sigma = \Sigma\#\cdots\#\Sigma$ (m times). It bounds the manifold X obtained as a boundary connected sum of m copies of M, with the intersection form $Q_X = m\,E_8$. Now suppose that there is an integer $m \geq 1$ such that $m\,\Sigma$ is homology cobordant to zero via a homology cobordism W, and form the manifold $X \cup_{m\,\Sigma} (-W)$. This is a smooth closed oriented manifold with the intersection form $m\,E_8$, which is definite and non-diagonalizable over the integers (since it is even). This contradiction with Donaldson's theorem (Theorem 5.9) proves the result.

Fukumoto–Furuta [54] and Saveliev [141] used the $\bar{\mu}$-invariant (11.3) to prove that any Seifert homology sphere with non-zero Rohlin invariant has infinite order in Θ^3. Many examples of Seifert homology spheres that have infinite order in Θ^3, with both zero and non-zero Rohlin invariants, were given by Fintushel and Stern [46]. Furuta [55] proved that the group Θ^3 is infinitely generated. In fact, he showed that, for any pair of relatively prime integers p, q, the homology 3-spheres $\Sigma(p, q, pqk - 1)$,

$k \geq 1$, are linearly independent over the integers in Θ^3. All of the above results rely on either Donaldson or Seiberg–Witten gauge theory on orbifolds.

The group Θ^3 has a distinguished history in the study of manifolds. Its structure is closely related, for instance, to the following problems.

1. Let Σ be a homotopy 3-sphere, that is, a closed oriented 3-manifold with $\pi_1(\Sigma) = 0$. The now solved Poincaré conjecture asserts that any homotopy sphere is in fact homeomorphic to S^3. In particular, it answers in positive the following question: is it true that $\mu(\Sigma) = 0$ for all homotopy 3-spheres? In fact, this question was answered positively long before the Poincaré conjecture was solved – this was done by A. Casson in 1985 with the help of his λ-invariant. Together with M. Freedman's work on topological 4-manifolds, this led to a construction of a topological 4-manifold which is not homeomorphic to any finite simplicial complex (such manifolds are called simplicially non-triangulable). This construction and the λ-invariant will be discussed in detail below.

2. The following problem is still unsolved: does there exist an element of order two in Θ^3 with non-trivial Rohlin invariant? This is Problem 4.4 on Kirby's list [83]. If this problem has a positive solution, then a theorem of Galewski and Stern [58] and Matumoto [108] will imply that all closed topological n-manifolds are simplicially triangulable if $n \geq 5$, see Lecture 18. It is known, see [54] and [141], that no Seifert homology sphere with non-zero Rohlin invariant, or any homology sphere which is homology cobordant to it, can have order two in Θ^3 (the proof uses the $\bar{\mu}$-invariant). Incidentally, it is not known if Θ^3 has any elements of finite order, with any Rohlin invariant.

3. The following is Problem 4.49 on Kirby's list [83]. Let Σ be an integral homology 3-sphere with Rohlin invariant one. Can Σ be homology cobordant to itself via a simply connected homology cobordism? Note that $\Sigma \times [0,1]$ is a homology cobordism of Σ to itself which however is not simply connected unless $\Sigma = S^3$. Taubes [153] showed that the answer is negative for $\Sigma = \Sigma(2,3,5)$; this result was extended to some other Seifert homology spheres by Fintushel and Stern [45]. The problem remains unsolved in general.

11.5 Exercises

1. Compute the Rohlin and the $\bar{\mu}$-invariants of $\Sigma(2, 4k-1, 8k-3)$ for all integers k.
2. Let Σ be the Brieskorn homology sphere $\Sigma(2,7,13)$.
   (a) Prove that $\mu(\Sigma) = 0$.
   (b) Use Theorem 5.10 to prove that Σ is not homology cobordant to zero.
   (c) Prove that Σ has an infinite order in the homology cobordism group.

3. Prove that $\Sigma\#(-\Sigma)$ is homology cobordant to zero for any homology sphere Σ.
4. Prove that, for any integer n, the integral homology sphere obtained by $(1/n)$-surgery on a slice knot in S^3 is homology cobordant to zero.
5. Prove that if an oriented 3-manifold M bounds a smooth homology ball then M is a homology 3-sphere.
6. Prove that if a homology 3-sphere Σ can be embedded in $\mathbb{R}^4$ then $\mu(\Sigma) = 0$. In particular, the Poincaré homology sphere cannot be embedded in $\mathbb{R}^4$.
7. Let Σ be the splice of homology spheres Σ_1 and Σ_2 along knots k_1 and k_2. Prove that $\mu(\Sigma) = \mu(\Sigma_1) + \mu(\Sigma_2)$.

Lecture 12
The Casson invariant

Let $\mathcal{S}$ be the class of oriented integral homology 3-spheres. A *Casson invariant* is a map $\lambda\colon \mathcal{S} \to \mathbb{Z}$ with properties (0)–(2) listed below.

(0) $\lambda(S^3) = 0$*, and* $\lambda(\mathcal{S})$ *is not contained in any proper subgroup of* $\mathbb{Z}$*.*

(1) *For any homology sphere* Σ *and knot* $k \subset \Sigma$*, the difference*

$$\lambda\left(\Sigma + \frac{1}{m+1}\cdot k\right) - \lambda\left(\Sigma + \frac{1}{m}\cdot k\right), \quad m \in \mathbb{Z}, \tag{12.1}$$

is independent of m*.*

Therefore, the difference (12.1) is an invariant of the knot $k \subset \Sigma$; it is denoted by $\lambda'(k \subset \Sigma)$ or simply $\lambda'(k)$. Let $k \cup \ell$ be a link in a homology sphere Σ with $\mathrm{lk}(k,\ell) = 0$. Then for any integers m, n, the manifold

$$\Sigma + \frac{1}{m}\cdot k + \frac{1}{n}\cdot \ell$$

is a homology sphere. Note that the quantity

$$\begin{aligned}
&\lambda\left(\Sigma + \frac{1}{m+1}\cdot k + \frac{1}{n+1}\cdot \ell\right) - \lambda\left(\Sigma + \frac{1}{m}\cdot k + \frac{1}{n+1}\cdot \ell\right)\\
&- \lambda\left(\Sigma + \frac{1}{m+1}\cdot k + \frac{1}{n}\cdot \ell\right) + \lambda\left(\Sigma + \frac{1}{m}\cdot k + \frac{1}{n}\cdot \ell\right)\\
&\quad = \lambda'\left(k \subset \Sigma + \frac{1}{n+1}\cdot \ell\right) - \lambda'\left(k \subset \Sigma + \frac{1}{n}\cdot \ell\right)\\
&\quad = \lambda'\left(\ell \subset \Sigma + \frac{1}{m+1}\cdot k\right) - \lambda'\left(\ell \subset \Sigma + \frac{1}{m}\cdot k\right)
\end{aligned}$$

is independent of both m and n, and denote it by $\lambda''(k, \ell \subset \Sigma)$ or $\lambda''(k,\ell)$.

(2) $\lambda''(k,\ell \subset \Sigma) = 0$ *for any boundary link* $k \cup \ell$ *in a homology sphere* Σ*.*

The existence and uniqueness of a Casson invariant are given by the following theorem.

Theorem 12.1 (Casson). *There exists a Casson invariant* λ *which is unique up to sign. Moreover, it has the following properties:*

(3) $\lambda'(\text{trefoil}) = \pm 1$.

(3') $\lambda'(k \subset \Sigma) = \frac{1}{2}\Delta''_{k\subset\Sigma}(1) \cdot \lambda'(\text{trefoil})$ *for any knot* $k \subset \Sigma$.

(4) $\lambda(-\Sigma) = -\lambda(\Sigma)$ *where* $-\Sigma$ *stands for* Σ *with reversed orientation.*

(5) $\lambda(\Sigma_1 \# \Sigma_2) = \lambda(\Sigma_1) + \lambda(\Sigma_2)$.

(6) $\lambda(\Sigma) = \mu(\Sigma) \mod 2$ *where* μ *is the Rohlin invariant.*

By a trefoil we mean either a left-handed or a right-handed trefoil; by (3′) their λ'-invariants coincide.

The scheme of the proof is as follows. We first prove that (0), (1), (3), and (3′) imply uniqueness. The next step will be to prove that (0), (1), (3), and (3′) imply (4), (5), and (6), and finally, that (0), (1), and (2) imply (3) and (3′). After that we will only need to prove the existence of a Casson invariant satisfying (0), (1), and (2).

$$\boxed{(0),\ (1),\ (3),\ (3') \Rightarrow \text{uniqueness}}$$

Lemma 12.2. *Let* Σ *be a homology sphere. Then there is a link* $k_1 \cup \dots \cup k_n$ *in* S^3 *such that*

(a) $\Sigma = S^3 + \varepsilon_1 \cdot k_1 + \dots + \varepsilon_n \cdot k_n$,

(b) $\varepsilon_i = \pm 1$ *for all* $i = 1, \dots, n$,

(c) $\operatorname{lk}(k_i, k_j) = 0$ *for all* $i \neq j$.

Proof. The homology sphere Σ is a surgery along a framed link $\mathcal{L}$ in S^3. The linking matrix A of $\mathcal{L}$ is unimodular, and one may assume without loss of generality that it is also odd and indefinite. Then A is diagonalizable over the integers by elementary transformations on the rows and columns. Since all elementary transformations can be realized by the second Kirby move, we are finished. □

Now we can prove uniqueness. Let Σ be a homology sphere, and $k_1 \cup \dots \cup k_n$ a link in S^3 as in Lemma 12.2. Let $\Sigma_i = S^3 + \varepsilon_1 \cdot k_1 + \dots + \varepsilon_i \cdot k_i$ for $i = 0, 1, \dots, n$, so that $\Sigma_0 = S^3$, $\Sigma_n = \Sigma$, and each Σ_i is a homology sphere. Then

$$\lambda(\Sigma) = \lambda(\Sigma_n) = (\lambda(\Sigma_n) - \lambda(\Sigma_{n-1})) + (\lambda(\Sigma_{n-1}) - \lambda(\Sigma_{n-2})) + \dots \\ + (\lambda(\Sigma_1) - \lambda(\Sigma_0)) + \lambda(\Sigma_0),$$

where $\lambda(\Sigma_0) = \lambda(S^3) = 0$. For any $i = 1, \dots, n$,

$$\lambda(\Sigma_i) - \lambda(\Sigma_{i-1}) = \varepsilon_i \cdot \lambda'(k_i \subset \Sigma_{i-1}).$$

Therefore,

$$\lambda(\Sigma) = \sum_{i=1}^{n} \varepsilon_i \cdot \lambda'(k_i \subset \Sigma_{i-1}) = \Big(\sum_{i=1}^{n} \frac{\varepsilon_i}{2} \cdot \Delta''_{k_i\subset\Sigma_{i-1}}(1)\Big) \cdot \lambda'(\text{trefoil}), \qquad (12.2)$$

so λ is unique up to the choice of $\lambda'(\text{trefoil}) = \pm 1$.

$$\boxed{(0),\ (1),\ (3),\ (3') \Rightarrow (4)}$$

If $\Sigma = S^3 + \varepsilon_1 \cdot k_1 + \cdots + \varepsilon_n \cdot k_n$ then $-\Sigma = S^3 - \varepsilon_1 \cdot k_1^* - \cdots - \varepsilon_n \cdot k_n^*$ where the link $k_1^* \cup \cdots \cup k_n^*$ is a mirror image of the link $k_1 \cup \cdots \cup k_n$, see Section 3.4. Note that $-\Sigma_i = S^3 - \varepsilon_1 \cdot k_1^* - \cdots - \varepsilon_i \cdot k_i^*$ so the result will follow from (12.2) after we prove that

$$\Delta_{k_i^* \subset -\Sigma_{i-1}}(t) = \Delta_{k_i \subset \Sigma_{i-1}}(t). \tag{12.3}$$

We abuse notation here in that we denote by k_i both the knot $k_i \subset S^3$ and its image in Σ_{i-1}, and similarly for k_i^*.

The formula (12.3) can be checked as follows. It is true for $i = 1$ by (7.4). Let τ be an orientation reversing diffeomorphism of S^3 with $k_i^* = \tau(k_i)$. The map τ maps the exterior of the knot $k_1 \subset S^3$ to the exterior of the knot $k_1^* \subset S^3$. Therefore, it extends to a homeomorphism $\tau_1 \colon \Sigma_1 \to -\Sigma_1$ satisfying $\tau_1(k_2) = k_2^*$. The proof of (7.4), after a minor modification, shows that

$$\Delta_{k_2 \subset \Sigma_1}(t) = \Delta_{k_2^* \subset -\Sigma_1}(t).$$

An obvious induction on i completes the proof.

$$\boxed{(0),\ (1),\ (3),\ (3') \Rightarrow (5)}$$

Let Σ_1 be obtained by the surgery on a framed link $\mathcal{L}_1 \subset S^3$, and Σ_2 by the surgery on a framed link $\mathcal{L}_2 \subset S^3$, both links being as in Lemma 12.2. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be a union of the links $\mathcal{L}_1$ and $\mathcal{L}_2$ in $S^3 \# S^3 = S^3$. The surgery on $\mathcal{L}$ produces $\Sigma_1 \# \Sigma_2$, and the formula (12.2) can be applied again to prove that $\lambda(\Sigma_1 \# \Sigma_2) = \lambda(\Sigma_1) + \lambda(\Sigma_2)$. The key ingredient here is that the Alexander polynomial of a knot k in a homology sphere Σ is the same no matter whether k is considered a knot in Σ or a knot in a connected sum $\Sigma \# \Sigma'$.

$$\boxed{(0),\ (1),\ (3),\ (3') \Rightarrow (6)}$$

This follows from (12.2) and the surgery formula for the Rohlin invariant, see Theorem 11.3.

$$\boxed{(0),\ (1),\ (2) \Rightarrow (3), (3')}$$

Lemma 12.3. *Let k be a knot in a homology sphere Σ. Then there exists a knot ℓ in S^3 such that $\lambda'(k \subset \Sigma) = \lambda'(\ell \subset S^3)$ and $\Delta_{k \subset \Sigma}(t) = \Delta_{\ell \subset S^3}(t)$.*

Proof. For the Alexander polynomial part, this is the statement of Lemma 7.14 proved in Lecture 7. The same proof goes through with λ' in place of $\Delta''(t)$, after one uses Casson's property (2) for boundary links. □

Thus we only need to prove that $\lambda'(k \subset S^3) = \frac{1}{2}\,\Delta_k''(1)$ for knots k in S^3. Let k be a knot in S^3. By changing crossings in a knot diagram for k, one can make it into an unknot. Since both λ' and $\frac{1}{2}\,\Delta_k''(1)$ are zero for an unknot, we only need to check that both invariants change by the same rule as we change a crossing.

By "change of crossing" we mean the operation on a knot pictured in Figure 12.1, assuming that the rest of the knot remains unchanged.

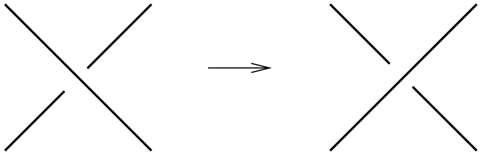


Figure 12.1

This operation can be realized by surgery as follows. Let us orient the knot k and consider a disc D with boundary $\partial D = c$ as shown in Figure 12.2. The disc D intersects the knot k in precisely two points.

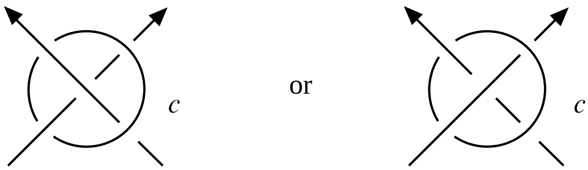


Figure 12.2

A (± 1)-surgery of S^3 along the unknot c yields S^3 again, while turning the knot k into k_c, see Figure 12.3 for the $(+1)$-surgery.

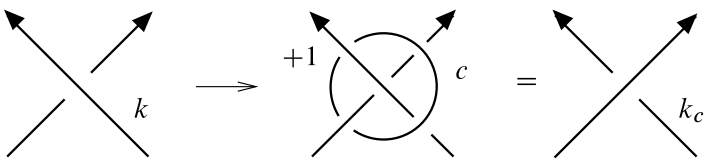


Figure 12.3

We call this operation a *twist across* D. Without loss of generality we consider only the first case in Figure 12.2, so that the change of crossing is obtained by $(+1)$-surgery along c. We find that

$$\lambda'(k_c) - \lambda'(k) = \lambda'(k \subset S^3 + c) - \lambda'(k \subset S^3) = \lambda''(k, c \subset S^3).$$

Two twists on a knot k, one across D and the other across D', are called *disjoint* if $D \cap D'$ is empty and the pairs of points $D \cap k$ and $D' \cap k$ are unlinked in k, see Figure 12.4.

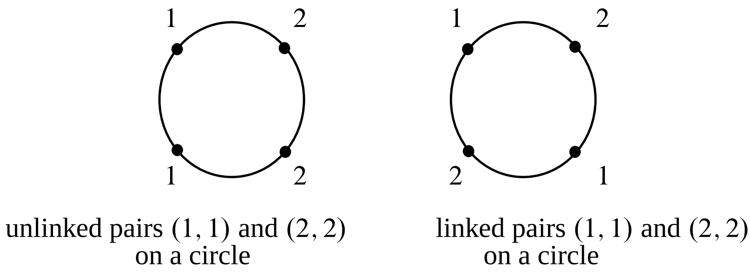


Figure 12.4

Suppose that a twist of the knot k across D' is disjoint from the twist across D, and let $c' = \partial D'$. We get two new knots, $k_{c'}$ and $k_{cc'} = k_{c'c}$, the difference $\lambda'(k_{cc'}) - \lambda'(k_{c'})$ being equal to $\lambda''(k, c \subset S^3 + c')$. We claim that

$$\lambda''(k, c \subset S^3) = \lambda''(k, c \subset S^3 + c'). \tag{12.4}$$

The formula (12.4) is verified as follows:

$$\begin{aligned}
&\lambda''(k, c \subset S^3 + c') - \lambda''(k, c \subset S^3) \\
&\quad = (\lambda'(c \subset S^3 + k + c') - \lambda'(c \subset S^3 + c')) - (\lambda'(c \subset S^3 + k) - \lambda'(c \subset S^3)) \\
&\quad = (\lambda'(c \subset S^3 + k + c') - \lambda'(c \subset S^3 + k)) - (\lambda'(c \subset S^3 + c') - \lambda'(c \subset S^3)) \\
&\quad = \lambda''(c, c' \subset S^3 + k) - \lambda''(c, c' \subset S^3).
\end{aligned}$$

Since the twists across D and D' are disjoint, the link (c, c') is a boundary link in both S^3 and $S^3 + k$, see Figure 12.5. Therefore, $\lambda''(c, c' \subset S^3 + k) = \lambda''(c, c' \subset S^3) = 0$ by Casson's property (2), which proves (12.4).

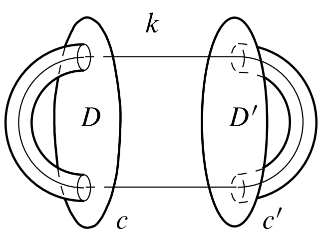


Figure 12.5

A formula similar to (12.4) holds for the second derivative of the Alexander polynomial. Namely, if the knots $k, k_c, k_{c'}$, and $k_{cc'}$ are as above, then

$$\frac{1}{2}\,\Delta''_{k_c}(1) - \frac{1}{2}\,\Delta''_{k}(1) = \frac{1}{2}\,\Delta''_{k_{cc'}}(1) - \frac{1}{2}\,\Delta''_{k_{c'}}(1). \qquad (12.5)$$

This can be checked with the help of Conway's formula, see Theorem 7.5:

$$\Delta_{k_c}(t) - \Delta_k(t) = -(t^{1/2} - t^{-1/2})\Delta_{k_0}(t),$$
$$\Delta_{k_{cc'}}(t) - \Delta_{k_{c'}}(t) = -(t^{1/2} - t^{-1/2})\Delta_{k_{0c'}}(t),$$
$$\Delta_{k_0}(t) - \Delta_{k_{0c'}}(t) = \pm(t^{1/2} - t^{-1/2})\Delta_{k_{00}}(t),$$

where k_{00} is a link of three components (due to the fact that (c, c') is a boundary link). Therefore, $\Delta_{k_{00}}(1) = 0$, see Corollary 7.8. The difference of the left- and the right-hand sides of the formula (12.5) equals one-half the second derivative at $t = 1$ of the function

$$f(t) = \pm(t^{1/2} - t^{-1/2})^2\Delta_{k_{00}}(t).$$

The function $f(t)$ is the product of three functions each of which equals 0 at $t = 1$, therefore, $f''(1) = 0$.

Thus the change in $\lambda'(k)$ due to a twist across D is the same for all knots obtained from k by a twist disjoint from the twist across D, and the same is true for $\frac{1}{2}\,\Delta''_k(1)$.

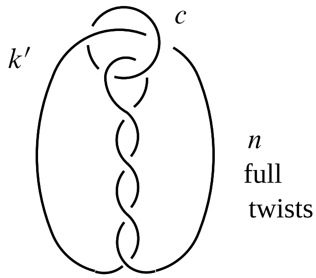


Figure 12.6

Let us fix k, c, and D. By using twists disjoint from twists across D, one can turn the link (k, c) into a link (k', c) shown in Figure 12.6 ($n = 2$ in the picture).

Note that k' is an unknot and that $k'_c = k_n$, where k_n is the knot shown in Figure 12.7. By (12.4), we have

$$\lambda''(k, c \subset S^3) = \lambda'(k'_c) - \lambda'(k') = \lambda'(k_n) - \lambda'(k') = \lambda'(k_n).$$

Now because of (12.5) we only need to compare $\lambda'(k_n)$ with $\frac{1}{2}\,\Delta''_{k_n}(1)$. Let c' be an unknot as in Figure 12.8, and perform the twist across D' with $\partial D' = c'$. Then $(k_n)_{c'} = k_{n+1}$, so that

$$\lambda'(k_{n+1}) - \lambda'(k_n) = \lambda''(k_n, c' \subset S^3).$$

By (12.4) the number $\lambda''(k_n, c' \subset S^3)$ is independent of n. Hence, for any n,

$$\lambda'(k_{n+1}) - \lambda'(k_n) = \lambda''(k_0, c' \subset S^3) = \lambda'(k_1) - \lambda'(k_0) = \lambda'(k_1), \qquad (12.6)$$

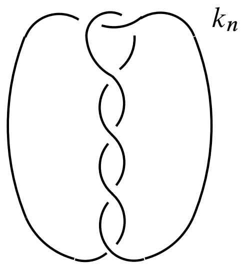


Figure 12.7

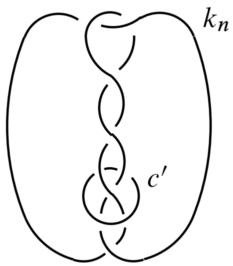


Figure 12.8

where k_1 is a trefoil. Therefore, $\lambda'(k)$ is proportional to λ'(trefoil) for any knot k. By property (1), λ'(trefoil) must be equal to $+1$ or -1.

On the other hand, (12.6) holds with λ' replaced by $\frac{1}{2}\,\Delta''(1)$. Since $\Delta''_{k_1}(1) = 2$ for the trefoil k_1, we have

$$\lambda'(k) = \frac{1}{2}\,\Delta''_k(1)\cdot\lambda'(\text{trefoil}) \quad \text{for any knot } k.$$

This completes the proof of Theorem 12.1 except for the existence part. Existence will be proved in the next several lectures.

12.1 Exercises

1. Prove that any homology sphere Σ that admits an orientation-reversing homeomorphism has $\mu(\Sigma) = 0 \bmod 2$.
2. Let Σ be the splice of homology spheres Σ_1 and Σ_2 along knots k_1 and k_2. Prove that $\lambda(\Sigma) = \lambda(\Sigma_1) + \lambda(\Sigma_2)$.
3. Prove that the Casson invariant of the homology sphere shown in Figure 12.9 in independent of $p \in \mathbb{Z}$. (Hint: apply Casson's surgery formula to the knot k shown in Figure 7.22).

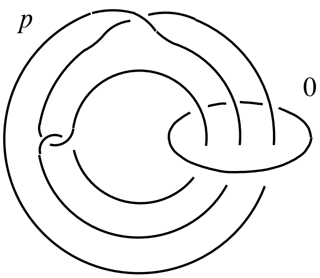


Figure 12.9

Lecture 13
The group SU(2)

The Lie group SU(2) consists of all complex (2×2)-matrices A such that $A\bar{A}^\top = E$ and $\det A = 1$. As a manifold, it can be identified with the 3-sphere S^3 as follows. Any matrix $A \in \mathrm{SU}(2)$ is invertible, therefore, the condition $A\bar{A}^\top = E$ can be rewritten in the form $\bar{A}^\top = A^{-1}$. If

$$A = \begin{pmatrix} a & b \\ u & v \end{pmatrix} \qquad \text{then} \quad \bar{A}^\top = \begin{pmatrix} \bar{a} & \bar{u} \\ \bar{b} & \bar{v} \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} v & -b \\ -u & a \end{pmatrix},$$

so the condition $\bar{A}^\top = A^{-1}$ is equivalent to $v = \bar{a}$ and $u = -\bar{b}$. Thus any $A \in \mathrm{SU}(2)$ is of the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with } a, b \in \mathbb{C},\ \det A = |a|^2 + |b|^2 = 1.$$

This provides the identification

$$\mathrm{SU}(2) = \{(a,b) \in \mathbb{C}^2 \mid |a|^2 + |b|^2 = 1\} = S^3.$$

On the other hand, SU(2) can be identified with the Lie group Sp(1) of unit quaternions. Recall that the norm $|q|$ of a quaternion $q = x + yi + zj + wk \in \mathbb{H}$ is defined by the formula $|q|^2 = x^2 + y^2 + z^2 + w^2$, or by $|q|^2 = q\bar{q}$ where $\bar{q} = x - yi - zj - wk$. One can easily check that $|pq| = |p||q|$. The group Sp(1) consists of all quaternions q with $|q| = 1$. Since $|x + yi + zj + wk|^2 = x^2 + y^2 + z^2 + w^2$, the group Sp(1) is topologically a 3-sphere. The identification $\mathrm{Sp}(1) = \mathrm{SU}(2)$ at the level of Lie groups is given by the formula

$$a + bj \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{13.1}$$

The unit quaternions $1, i, j, k$ are identified via (13.1) with the following matrices:

$$1 = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $\mathrm{U}(1) = S^1$ be the group of unit complex numbers. Then the identification (13.1) provides a canonical inclusion $\mathrm{U}(1) \to \mathrm{SU}(2)$ such that

$$e^{i\varphi} \mapsto \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}.$$

Theorem 13.1. *The group* $\mathrm{U}(1) \subset \mathrm{SU}(2)$ *is a maximal commutative subgroup in* $\mathrm{SU}(2)$*. All other maximal commutative subgroups in* $\mathrm{SU}(2)$ *are of the form* $C^{-1} \cdot \mathrm{U}(1) \cdot C$ *where* $C \in \mathrm{SU}(2)$.

This is an easy exercise involving the quaternions. We will sometimes refer to the elements of $\mathrm{U}(1) \subset \mathrm{SU}(2)$ as complex numbers.

The trace of a matrix defines a function $\mathrm{tr}\colon \mathrm{SU}(2) \to [-2, 2],\ A \mapsto \mathrm{tr}\, A$. Note that $\pm E$ are the only two matrices in $\mathrm{SU}(2)$ with trace ± 2; the rest of the group satisfies $-2 < \mathrm{tr}\, A < 2$.

Theorem 13.2. *Two matrices, A and A', in* $\mathrm{SU}(2)$ *are conjugate if and only if* $\mathrm{tr}\, A = \mathrm{tr}\, A'$.

Proof. The $\Rightarrow$ direction is obvious. Consider a matrix $A \in \mathrm{SU}(2)$ as a linear operator on $\mathbb{C}^2$. Since $\mathbb{C}$ is algebraically closed, A has an eigenspace with eigenvalue $\lambda \in \mathbb{C}$. Choose a unit vector $\psi = (x, y)$ in this eigenspace, and let

$$C = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \in \mathrm{SU}(2) \quad \text{so that } C \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

then

$$C^{-1}AC \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \quad \text{or} \quad C^{-1}AC = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{C}$. Since $C^{-1}AC \in \mathrm{SU}(2)$, we have that

$$C^{-1}AC = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \quad \text{with } \det C^{-1}AC = |\lambda|^2 = 1.$$

Thus any matrix $A \in \mathrm{SU}(2)$ can be conjugated in $\mathrm{SU}(2)$ to a matrix of the form

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \tag{13.2}$$

whose trace equals $2\cos\varphi$. The trace uniquely defines $e^{i\varphi}$ up to complex conjugation. Since

$$\begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we are finished. $\square$

In the quaternionic language, this theorem asserts that any unit quaternion is conjugate to a complex number $e^{i\varphi}$, $0 \le \varphi \le \pi$. Thus, the conjugacy classes in $\mathrm{SU}(2)$ are in one-to-one correspondence with the sets $\mathrm{tr}^{-1}(c)$ with $-2 \le c \le 2$. The equation

$$\mathrm{tr} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = c \tag{13.3}$$

is equivalent to the equation $2\operatorname{Re} a = c$. The latter defines a hyperplane in $\mathbb{R}^4 = \mathbb{H}$ whose intersection with $\mathrm{SU}(2) = S^3 \subset \mathbb{R}^4$ is $\mathrm{tr}^{-1}(c)$. Thus $\mathrm{tr}^{-1}(c) = S^2$ if $-2 < c < 2$, and $\mathrm{tr}^{-1}(-2) = \{-E\}$, $\mathrm{tr}^{-1}(2) = \{E\}$. Schematically, the conjugacy classes in SU(2) can be pictured as the vertical line segments in Figure 13.1, each segment representing a copy of S^2, which intersects the circle $U(1)$ of unit complex numbers in exactly two points, $e^{i\varphi}$ and $e^{-i\varphi}$, unless $e^{i\varphi} = \pm 1$.

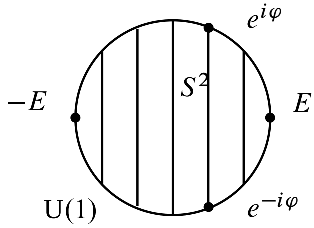


Figure 13.1

At the level of vector spaces, the Lie algebra $\mathfrak{su}(2)$ of the group SU(2) can be identified with the tangent space to $SU(2)$ at $1 = E$, so that $\mathfrak{su}(2) = T_1\,\mathrm{SU}(2)$. To describe $\mathfrak{su}(2)$ in terms of matrices, we consider the exponential map $\exp\colon T_1\,\mathrm{SU}(2) \to \mathrm{SU}(2)$ given by $\alpha \mapsto e^{\alpha}$. Then $\alpha \in \mathfrak{su}(2)$ if and only if $\exp(\alpha) \in \mathrm{SU}(2)$, which is equivalent to the following:

$$\det e^{\alpha} = e^{\operatorname{tr}\alpha} = 1 \qquad \Longleftrightarrow \qquad \operatorname{tr}\alpha = 0,$$
$$\overline{(e^{\alpha})}^{\top} = (e^{\alpha})^{-1} \qquad \Longleftrightarrow \qquad \alpha + \bar{\alpha}^{\top} = 0.$$

Thus, $\mathfrak{su}(2)$ is the 3-dimensional vector space of skew-hermitian matrices with zero trace. All such matrices are of the form

$$\alpha = \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix}, \quad a \in \mathbb{R},\ b \in \mathbb{C}.$$

If $b = u + iv$ with $u, v \in \mathbb{R}$ then

$$\begin{pmatrix} ia & u+iv \\ -u+iv & -ia \end{pmatrix} = a\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + u\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

corresponds to the quaternion $ai + uj + vk$. Therefore, $\mathfrak{su}(2)$ can be thought of as a subspace of $\mathbb{H}$ consisting of purely imaginary quaternions. The Lie algebra structure on $\mathfrak{su}(2)$ is given by the Lie bracket

$$[\alpha, \beta] = \alpha\beta - \beta\alpha. \tag{13.4}$$

Theorem 13.3. *The exponential map provides a diffeomorphism*

$$\exp\colon B_\pi(0) \to \mathrm{SU}(2) \setminus \{-1\}$$

where $B_\pi(0) \subset \mathfrak{su}(2)$ is the open ball of radius π centered at the origin.

Proof. As $e^{g\alpha g^{-1}} = ge^{\alpha}g^{-1}$ we only need to show that the map $\exp\colon (-i\pi, i\pi) \to S^1 \setminus \{-1\}$, $i\varphi \mapsto e^{i\varphi}$, is a diffeomorphism, see Figure 13.2. The latter is obvious.

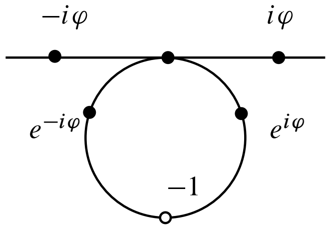


Figure 13.2 $\square$

Remark. The following is a useful formula for evaluating the exponential map. Let q be a purely imaginary quaternion of unit length, so that $\operatorname{Re} q = 0$ and $|q|^2 = 1$. Then, for any real number θ,

$$e^{q\theta} = \cos\theta + q \cdot \sin\theta.$$

This can be seen as follows. Since $\operatorname{Re} q = 0$, there exists a unit quaternion u such that $q = uiu^{-1}$. Then

$$e^{q\theta} = e^{uiu^{-1}\theta} = e^{u(i\theta)u^{-1}} = ue^{i\theta}u^{-1} = u(\cos\theta + i\sin\theta)u^{-1} = \cos\theta + q \cdot \sin\theta.$$

The tangent space $T_g\,\mathrm{SU}(2)$ at $g \in \mathrm{SU}(2)$ can be identified with the image of $\mathfrak{su}(2)$ under left or right translation by g:

$$(L_g)_*(\mathfrak{su}(2)) = T_g\,\mathrm{SU}(2), \quad (R_g)_*(\mathfrak{su}(2)) = T_g\,\mathrm{SU}(2)$$

where $L_g(A) = g \cdot A$ and $R_g(A) = A \cdot g$ for any $A \in \mathrm{SU}(2)$.

Example. The space $T_i\,\mathrm{SU}(2)$ consists of the matrices of the form

$$i\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} = \begin{pmatrix} -a & ib \\ i\bar{b} & -a \end{pmatrix}, \quad a \in \mathbb{R},\ b \in \mathbb{C}.$$

The group SU(2) acts on itself by conjugation,

$$A \mapsto \mathrm{Ad}_A\colon \mathrm{SU}(2) \to \mathrm{SU}(2), \quad \mathrm{Ad}_A(B) = ABA^{-1}.$$

For any A, the derivative $d_1\,\mathrm{Ad}_A$ of $\mathrm{Ad}_A : \mathrm{SU}(2) \to \mathrm{SU}(2)$ at 1 gives an action of SU(2) on its Lie algebra, called again Ad_A,

$$A \mapsto \mathrm{Ad}_A\colon \mathfrak{su}(2) \to \mathfrak{su}(2), \quad \mathrm{Ad}_A(\alpha) = A\alpha A^{-1}.$$

Note that Ad_A is a Lie algebra homomorphism with respect to the bracket (13.4). Thus we have a homomorphism

$$\mathrm{Ad}\colon \mathrm{SU}(2) \to \mathrm{Aut}(\mathfrak{su}(2)), \quad A \mapsto \mathrm{Ad}_A\,. \tag{13.5}$$

The derivative of this map at $1 \in \mathrm{SU}(2)$ can be computed as follows: choose $\alpha \in \mathfrak{su}(2)$, then, up to order ε^2,

$$\mathrm{Ad}_{1+\varepsilon\alpha}(\beta) = (1+\varepsilon\alpha)\beta(1-\varepsilon\alpha) = \beta + \varepsilon(\alpha\beta - \beta\alpha).$$

Denote by $\mathrm{ad}_\alpha\colon \mathfrak{su}(2) \to \mathfrak{su}(2)$ the linear operator $\mathrm{ad}_\alpha(\beta) = [\alpha, \beta]$, then $\mathrm{Ad}_{1+\varepsilon\alpha}(\beta) = (1 + \varepsilon\,\mathrm{ad}_\alpha)(\beta)$ up to order ε^2. Thus, $(d_1\,\mathrm{Ad})(\alpha) = \mathrm{ad}_\alpha$.

Theorem 13.4. *The map* (13.5) *is well-defined as a Lie group homomorphism* $\mathrm{SU}(2) \to \mathrm{SO}(3)$*. It is the universal (double) cover of* $\mathrm{SO}(3)$*; in particular,* $\pi_1 \mathrm{SO}(3) = \mathbb{Z}/2$*.*

Proof. The map (13.5) is a homomorphism; $\mathrm{Ad}(AB) = \mathrm{Ad}(A)\,\mathrm{Ad}(B)$ because $\mathrm{Ad}_{AB}(x) = ABx(AB)^{-1} = A(BxB^{-1})A^{-1} = \mathrm{Ad}_A\,\mathrm{Ad}_B(x)$ for any $x \in \mathfrak{su}(2)$. Since $\mathfrak{su}(2) = \mathbb{R}^3$ as a vector space, one can think of $\mathrm{Aut}(\mathfrak{su}(2))$ as a subgroup of $\mathrm{GL}_3(\mathbb{R})$. Then the map (13.5) is well-defined as a homomorphism $\mathrm{Ad}\colon \mathrm{SU}(2) \to \mathrm{GL}_3(\mathbb{R})$.

The Euclidean dot-product in $\mathbb{R}^3 = \mathfrak{su}(2)$ can be described by the formula

$$u \cdot v = -\frac{1}{2}\,\mathrm{tr}(uv) = -\,\mathrm{Re}(uv)$$

depending on the realization of $\mathfrak{su}(2)$ by either matrices or quaternions. One can easily check that Ad_A preserves the dot-product:

$$\begin{aligned}(\mathrm{Ad}_A\,u)\cdot(\mathrm{Ad}_A\,v) &= -\frac{1}{2}\,\mathrm{tr}(AuA^{-1}\,AvA^{-1}) \\ &= -\frac{1}{2}\,\mathrm{tr}(uv) = u \cdot v.\end{aligned}$$

Therefore, $\mathrm{Ad}(\mathrm{SU}(2)) \subset \mathrm{O}(3)$, the orthogonal group of $\mathbb{R}^3$. Since $\mathrm{SU}(2)$ is connected, the image of $\mathrm{SU}(2)$ should belong to the connected component of the identity in $\mathrm{O}(3)$, that is, to $\mathrm{SO}(3)$. The map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is obviously smooth, hence is a Lie group homomorphism.

The map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective. Each matrix from $\mathrm{SO}(3)$, thought of as acting on $\mathbb{R}^3$, is a product of rotations about the coordinate axes. Thus to show surjectivity we only need to show that the matrix

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

of rotation about the x-axis through an angle ψ belongs to the image of Ad. The rotations with respect to the other two coordinate axes can be handled similarly. Let $\varphi = \psi/2$ and

$$A = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$

then

$$\begin{aligned} \mathrm{Ad}_A(i) &= e^{i\varphi} i e^{-i\varphi} = i, \\ \mathrm{Ad}_A(j) &= e^{i\varphi} j e^{-i\varphi} = e^{2i\varphi} j = \cos\psi \cdot j + \sin\psi \cdot k, \\ \mathrm{Ad}_A(k) &= e^{i\varphi} k e^{-i\varphi} = e^{2i\varphi} k = \cos\psi \cdot k - \sin\psi \cdot j. \end{aligned}$$

Therefore, $\mathrm{Ad}_A = R_x$.

Suppose that $\mathrm{Ad}(A) = \mathrm{Ad}(B)$ then $AxA^{-1} = BxB^{-1}$ for all $x \in \mathfrak{su}(2)$, in other words, $B^{-1}A$ commutes with all $x \in \mathfrak{su}(2)$. This is only possible if $B^{-1}A = \pm E$, hence $B = \pm A$ and (13.5) is a double cover. Since $\pi_1 SU(2) = \pi_1 S^3$ is trivial, this cover is universal. □

Algebraically, the homomorphism SU(2) $\to$ SO(3) can be described as the quotient map of SU(2) by its center $\mathbb{Z}/2 = \{\pm E\}$. Topologically, it is the standard double cover $S^3 \to \mathbb{R}P^3$ after the identification SO(3) $= \mathbb{R}P^3$.

13.1 Exercises

1. Solve the equations $A^2 = 1$, $A^2 = -1$ and $A^3 = 1$ in the group SU(2).
2. Let us view SU(2) as the group of unit quaternions, and denote by U(1) $\subset$ SU(2) the subgroup of unit complex numbers. The group $H = \mathrm{U}(1) \cup j \cdot \mathrm{U}(1) \subset \mathrm{SU}(2)$ is called the binary dihedral group. Verify that H is a group, and find its image in SO(3) under the homomorphism Ad : SU(2) $\to$ SO(3).
3. The group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is an order eight subgroup of SU(2) known as the quaternion group. Find the image of Q_8 in SO(3) under the homomorphism Ad : SU(2) $\to$ SO(3).

Lecture 14
Representation spaces

14.1 The topology of representation spaces

Let M be a compact orientable manifold of dimension less than or equal to 3. Since M is homeomorphic to a finite simplicial complex, its fundamental group admits a finite presentation,

$$\pi_1 M = \langle t_1, \dots, t_n \mid r_1, \dots, r_m \rangle. \tag{14.1}$$

The space of SU(2)-representations of $\pi_1 M$ is defined as the space $R(M) = \mathrm{Hom}(\pi_1 M, \mathrm{SU}(2))$ with the compact open topology, where $\pi_1 M$ has the discrete topology and $\mathrm{SU}(2) = S^3$ has the topology induced from $\mathbb{R}^4$. Remember that the basis of the compact open topology on $R(M)$ consists of the sets $\mathcal{U}_{K,V} = \{f : \pi_1 M \to \mathrm{SU}(2) \mid f(K) \subset V\}$ for all compact $K \subset \pi_1 M$ and all open $V \subset \mathrm{SU}(2)$. Note that a set is compact in the discrete topology if and only if it is finite. Therefore, the topology on $R(M)$ can be conveniently described by its sub-base, which consists of all subsets $\mathcal{U}_{g,V} = \{f : \pi_1 M \to \mathrm{SU}(2) \mid f(g) \in V\}$ of $R(M)$, one for each $g \in \pi_1 M$ and each open subset $V \subset \mathrm{SU}(2)$.

Another way to describe the topology on $R(M)$ is as follows. Consider the space $\mathrm{SU}(2)^{\pi_1 M}$ of all maps $\pi_1 M \to \mathrm{SU}(2)$ (not just the homomorphisms). By comparing the sub-bases of topologies, one can show that the compact open topology on $\mathrm{SU}(2)^{\pi_1 M}$ coincides with the product topology. Therefore, the topology on $R(M)$ can be thought of as the topology induced by the inclusion of $R(M)$ into the product $\mathrm{SU}(2)^{\pi_1 M}$.

The space $R(M)$ can be turned into a real algebraic set as follows. Consider a finite presentation of $\pi_1 M$ as in (14.1). Each SU(2)-representation of $\pi_1 M$ is uniquely determined by the images of the generators t_k, each of which can be viewed as a vector in $\mathbb{R}^4$:

$$t_k \mapsto \begin{pmatrix} x_k + i y_k & u_k + i v_k \\ -u_k + i v_k & x_k - i y_k \end{pmatrix} \mapsto (x_k, y_k, u_k, v_k) \in \mathbb{R}^4.$$

Thus we get an inclusion $R(M) \to \mathbb{R}^{4n}$. All t_k^{-1} are also represented by vectors in $\mathbb{R}^4$. Any product of generators and their inverses is represented by an SU(2)-matrix whose entries are polynomials in x_k, y_k, u_k, and v_k. Therefore, each relation r_ℓ in the presentation (14.1) of $\pi_1 M$ corresponds to a set of polynomial equations in $\mathbb{R}^{4n}$. In addition, for each k, we have the equation $x_k^2 + y_k^2 + u_k^2 + v_k^2 = 1$. Thus, one can think of $R(M)$ as a subset of $\mathbb{R}^{4n}$ described by a system of real polynomial equations, or as a real algebraic set. In fact, $R(M)$ is a closed subset in $\mathrm{SU}(2)^n$, in particular, $R(M)$ is compact.

14.2 Irreducible representations

A representation $\alpha: \pi_1 M \to \mathrm{SU}(2)$ is called *reducible* if there is a proper non-zero $\mathbb{C}$-vector subspace $U \subset \mathbb{C}^2$ such that $\alpha(g)(U) \subset U$ for all $g \in \pi_1 M$ (if such a U exists it has to be 1-dimensional). Otherwise, α is called *irreducible*.

Let α be a reducible representation; then in an appropriate basis,

$$\alpha(g) = \begin{pmatrix} a_g & b_g \\ 0 & c_g \end{pmatrix}, \quad g \in \pi_1 M.$$

Since $\alpha(g) \in \mathrm{SU}(2)$, we find that $b_g = 0,\ c_g = \bar{a}_g$, and $|a_g|^2 = 1$. Therefore, $\alpha(g)$ is of the form

$$\alpha(g) = \begin{pmatrix} e^{i\varphi_g} & 0 \\ 0 & e^{-i\varphi_g} \end{pmatrix}, \quad g \in \pi_1 M.$$

This means that a representation α is reducible if and only if it factors through a copy of $\mathrm{U}(1)$ in $\mathrm{SU}(2)$.

The reducible representations of $\pi_1 M$ in $\mathrm{SU}(2)$ form a closed subset in $R(M)$. Therefore, the space $R^{\mathrm{irr}}(M)$ of irreducible representations is an open subset in $R(M)$. Informally, this means that a small perturbation of an irreducible representation is again irreducible.

The group $\mathrm{SU}(2)$ acts on $R(M)$ by the rule $\alpha \mapsto g\alpha g^{-1}$. The stabilizers of this action can be described as follows.

Among the reducible representations we will distinguish the following three classes: the trivial representation θ defined by the formula $\theta(g) = 1$, the central representations that factor through the center $\mathbb{Z}/2 = \{\pm 1\}$ of $\mathrm{SU}(2)$ but are different from θ, and the other reducible representations, which factor through $\mathrm{U}(1)$ but are not central. If a reducible representation α belongs to one of the first two classes its stabilizer is the entire group $\mathrm{SU}(2)$, otherwise, $\mathrm{Stab}(\alpha) = \mathrm{U}(1)$.

Lemma 14.1. *Let Σ be a homology sphere, then any reducible representation $\pi_1\Sigma \to \mathrm{SU}(2)$ is trivial.*

Proof. Any reducible $\alpha: \pi_1\Sigma \to \mathrm{SU}(2)$ factors through a copy of $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Since $\mathrm{U}(1)$ is an Abelian group, the entire commutator subgroup $[\pi_1\Sigma, \pi_1\Sigma]$ of $\pi_1\Sigma$ is mapped by α to $1 \in \mathrm{SU}(2)$. As $\pi_1\Sigma/[\pi_1\Sigma, \pi_1\Sigma] = H_1\Sigma$, we get a factorization $\alpha: \pi_1\Sigma \to H_1\Sigma \to \mathrm{U}(1) \to \mathrm{SU}(2)$. Since $H_1\Sigma$ is trivial, $\alpha = \theta$. □

Suppose that α is irreducible, then $\mathrm{Stab}(\alpha) = \mathbb{Z}/2$, the center of the group $\mathrm{SU}(2)$. As we see, the action of $\mathrm{SU}(2)$ on $R(M)$ is not free. However, the action of $\mathrm{SO}(3) =$

$\mathrm{SU}(2)/\{\pm 1\}$ is free on the subspace $R^{\mathrm{irr}}(M)$, and we define the *representation space* $\mathcal{R}(M) = R^{\mathrm{irr}}(M)/\,\mathrm{SO}(3)$. The topology on $\mathcal{R}(M)$ is the quotient topology. The full quotient space $R(M)/\,\mathrm{SO}(3)$ will be denoted $\chi(M)$ and called the *character variety* of M. Its elements are called *characters* of representations.

14.3 Representations of free groups

Let M be a handlebody of genus $g \geq 1$, as described in Lecture 1. Since $\pi_1 M$ is a free group on g generators, $R(M) = \mathrm{SU}(2)^g$ as a topological space, and $R(M)$ inherits a smooth structure from $\mathrm{SU}(2)^g$. The subspace $R^{\mathrm{irr}}(M) \subset R(M)$ is open, therefore, it is a smooth *open* (non-compact without boundary) manifold of dimension $3g$. When $g = 1$, this manifold is empty. The representation space $\mathcal{R}(M)$ is then a smooth open manifold of dimension $3g - 3$.

14.4 Representations of surface groups

Let F be a closed oriented Riemann surface of genus $g \geq 1$, and $F_0 = F \setminus \operatorname{int} D^2$ where D^2 is a disc in F. Let $\partial F_0 = \gamma$ be the oriented boundary of F_0; then we have a well-defined map

$$h\colon R(F_0) \to \mathrm{SU}(2), \quad h(\alpha) = \alpha(\gamma),$$

so that $R(F) = h^{-1}(1)$. The group $\pi_1 F_0$ is a free group of rank $2g$. The standard basis $a_1, b_1, \dots, a_g, b_g$ in $\pi_1 F_0$ such that

$$\gamma = \prod_{n=1}^{g} [a_n, b_n] = \prod_{n=1}^{g} a_n b_n a_n^{-1} b_n^{-1}$$

provides the identification

$$R(F_0) = \mathrm{SU}(2)^{2g},$$
$$\alpha \mapsto (A_1, B_1, \dots, A_g, B_g),$$

where $A_n = \alpha(a_n)$ and $B_n = \alpha(b_n)$. The map $h\colon \mathrm{SU}(2)^{2g} \to \mathrm{SU}(2)$ is now given by the formula

$$(A_1, B_1, \dots, A_g, B_g) \mapsto \prod_{n=1}^{g} [A_n, B_n].$$

Theorem 14.2 (Igusa [78], Shoda [147]). *The map h is surjective. It is regular at irreducible representations and only at them; in other words, $d_\alpha h$ is onto if and only if α is irreducible.*

Corollary 14.3. *$\mathcal{R}(F)$ is a smooth open manifold of dimension $6g-6$ (empty if $g=1$).*

Proof of Theorem 14.2. Let $R_\varphi = e^{i\varphi}$ then $h(R_\varphi, j, 1, \ldots, 1) = e^{i\varphi} j e^{-i\varphi} j^{-1} = R_{2\varphi}$. If $A \in \mathrm{SU}(2)$ then $A = CR_\varphi C^{-1}$ for some φ and $C \in \mathrm{SU}(2)$. Hence $A = h(CR_{\varphi/2}C^{-1}, CjC^{-1}, 1, \ldots, 1)$, and h is surjective.

For any representation $\alpha = (A_1, B_1, \ldots, A_g, B_g)$, we have the following natural identification of the tangent spaces

$$T_\alpha R(F_0) = T_{A_1}\,\mathrm{SU}(2) \oplus T_{B_1}\,\mathrm{SU}(2) \oplus \cdots \oplus T_{A_g}\,\mathrm{SU}(2) \oplus T_{B_g}\,\mathrm{SU}(2).$$

By applying appropriate left and right translations to each factor of the splitting of $T_\alpha R(F_0)$ and to $T_{h(\alpha)}\,\mathrm{SU}(2)$, we can construct the following commutative diagram

$$\begin{tikzcd}
T_{A_1}\,\mathrm{SU}(2) \oplus T_{B_1}\,\mathrm{SU}(2) \oplus \cdots \oplus T_{A_g}\,\mathrm{SU}(2) \oplus T_{B_g}\,\mathrm{SU}(2) \arrow[r, "d_\alpha h"] & T_{h(\alpha)}\,\mathrm{SU}(2) \\
T_1\,\mathrm{SU}(2) \oplus T_1\,\mathrm{SU}(2) \oplus \cdots \oplus T_1\,\mathrm{SU}(2) \oplus T_1\,\mathrm{SU}(2) \arrow[r, "D"] \arrow[u, "\cong"', "(L_{A_1})_* \oplus \cdots \oplus (L_{B_g})_*"] & T_1\,\mathrm{SU}(2) \arrow[u, "\cong"', "(R_{h(\alpha)})_*"]
\end{tikzcd}$$

It suffices to prove that D is surjective if and only if α is irreducible.

Let u_n belong to the copy of $T_1\,\mathrm{SU}(2)$ corresponding to $T_{A_n}\,\mathrm{SU}(2)$. Then

$$(L_{A_n})_*(1+\varepsilon u_n) = A_n(1+\varepsilon u_n)$$

and

$$\begin{aligned}
&h(A_1, \ldots, A_n(1+\varepsilon u_n), B_n, \ldots, B_g) \\
&\quad = [A_1, B_1] \cdots [A_{n-1}, B_{n-1}][A_n(1+\varepsilon u_n), B_n][A_{n+1}, B_{n+1}] \cdots [A_g, B_g] \\
&\quad = C_{n-1}[A_n(1+\varepsilon u_n), B_n]C_n^{-1}C_g, \qquad (14.2)
\end{aligned}$$

where $C_k = \prod_{n=1}^{k}[A_n, B_n],\ 0 \le k \le g$, so that $C_0 = 1$ and $C_g = h(\alpha)$. Now,

$$\begin{aligned}
[A_n(1+\varepsilon u_n), B_n] &= A_n(1+\varepsilon u_n)B_n(1-\varepsilon u_n)A_n^{-1}B_n^{-1} \\
&= [A_n, B_n] + \varepsilon(A_n u_n B_n A_n^{-1} B_n^{-1} - A_n B_n u_n A_n^{-1} B_n^{-1}) \qquad (14.3)
\end{aligned}$$

up to order ε^2. To return from $T_{h(\alpha)}\,\mathrm{SU}(2)$ to $T_1\,\mathrm{SU}(2)$, we multiply (14.2) on the right by $h(\alpha)^{-1}$, and taking into account (14.3), we conclude that

$$\begin{aligned}
&D(0, \ldots, 0, u_n, 0, \ldots, 0) \\
&\qquad = C_{n-1} A_n u_n A_n^{-1} C_{n-1}^{-1} - C_{n-1} A_n B_n u_n B_n^{-1} A_n^{-1} C_{n-1}^{-1} \\
&\qquad = x_n - C_{n-1} A_n B_n A_n^{-1} C_{n-1}^{-1} x_n C_{n-1} A_n B_n^{-1} A_n^{-1} C_{n-1}^{-1}
\end{aligned}$$

where $x_n = C_{n-1}A_n u_n A_n^{-1}C_{n-1}^{-1}$. Let v_n belong to the copy of $T_1\,\mathrm{SU}(2)$ corresponding to $T_{B_n}\,\mathrm{SU}(2)$. Then $(L_{B_n})_*(1+\varepsilon v_n) = B_n(1+\varepsilon v_n)$, and

$$h(A_1,\dots,A_n,B_n(1+\varepsilon v_n),\dots,B_g) = C_{n-1}[A_n,B_n(1+\varepsilon v_n)]C_n^{-1}C_g, \quad (14.4)$$

where

$$\begin{aligned}[A_n,B_n(1+\varepsilon v_n)] &= A_nB_n(1+\varepsilon v_n)A_n^{-1}(1-\varepsilon v_n)B_n^{-1}\\ &= [A_n,B_n] + \varepsilon\,(A_nB_nv_nA_n^{-1}B_n^{-1} - A_nB_nA_n^{-1}v_nB_n^{-1})\end{aligned}$$

up to order ε^2. As before, we obtain

$$\begin{aligned}&D(0,\dots,0,v_n,0,\dots,0)\\ &\quad= C_{n-1}A_nB_nv_nB_n^{-1}A_n^{-1}C_{n-1}^{-1} - C_{n-1}A_nB_nA_n^{-1}v_nA_nB_n^{-1}A_n^{-1}C_{n-1}^{-1}\\ &\quad= y_n - C_{n-1}A_nB_nA_n^{-1}B_n^{-1}A_n^{-1}C_{n-1}^{-1}y_nC_{n-1}A_nB_nA_nB_n^{-1}A_n^{-1}C_{n-1}^{-1}\end{aligned}$$

where $y_n = C_{n-1}A_nB_nv_nB_n^{-1}A_n^{-1}C_{n-1}^{-1}$. Therefore, the image of D consists of all vectors of the form

$$\sum_{n=1}^{g}(1-\mathrm{Ad}_{F_n})x_n + \sum_{n=1}^{g}(1-\mathrm{Ad}_{G_n})y_n, \quad x_n,y_n\in T_1\,\mathrm{SU}(2),$$

where $F_n = C_{n-1}A_nB_nA_n^{-1}C_{n-1}^{-1}$ and $G_n = C_{n-1}A_nB_nA_n^{-1}B_n^{-1}A_n^{-1}C_{n-1}^{-1}$. One can easily check that the elements

$$\{c_{n-1}a_nb_na_n^{-1}c_{n-1}^{-1},\ c_{n-1}a_nb_na_n^{-1}b_n^{-1}a_n^{-1}c_{n-1}^{-1}\}_{n=1,\dots,g},$$

where $c_k = \prod_{n=1}^{k}[a_n,b_n]$, form a basis of the free group π_1F_0, hence a representation $(A_1,B_1,\dots,A_g,B_g)$ is reducible if and only if $(F_1,G_1,\dots,F_g,G_g)$ is. Thus, the second statement of the theorem is equivalent to the following claim: a free group representation $(F_1,G_1,\dots,F_g,G_g)$ is irreducible if and only if the map

$$\sum_{n=1}^{g}(1-\mathrm{Ad}_{F_n}) + \sum_{n=1}^{g}(1-\mathrm{Ad}_{G_n})$$

is surjective.

Let us compute the image of $1-\mathrm{Ad}_F$ with $F\in\mathrm{SU}(2)$. Suppose that $F\neq\pm1$; then $\mathrm{Ad}_F\in\mathrm{SO}(3)$ is a non-trivial rotation through an angle φ about an axis $\mathbb{R}_F$. Let us denote by $\mathbb{C}_F$ the plane through the origin perpendicular to $\mathbb{R}_F$, so that $\mathbb{C}_F\oplus\mathbb{R}_F = \mathbb{R}^3$. The axis $\mathbb{R}_F$ is fixed by Ad_F, therefore, $\mathrm{im}(1-\mathrm{Ad}_F) = \mathbb{C}_F$. In the case $F=\pm1$, the operator Ad_F is the identity, so $\mathrm{im}(1-\mathrm{Ad}_F)=0$.

Recall from the proof of Theorem 13.4 that if $F = e^{i\psi}$ then

$$\mathrm{Ad}_F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\psi) & -\sin(2\psi) \\ 0 & \sin(2\psi) & \cos(2\psi) \end{pmatrix}.$$

Therefore, two operators, Ad_F and Ad_G, commute if and only if F and G commute. Non-trivial operators Ad_F and Ad_G commute if and only if their rotation axes coincide, that is $\mathbb{R}_F = \mathbb{R}_G$.

With this understood, suppose that a representation α is reducible. Then it factors through a copy of U(1), which means that all F_n and G_n commute. Hence the operators Ad_{F_n} and Ad_{G_n} have a common rotation axis $\mathbb{R}_F$, so that $\operatorname{im} D = \mathbb{C}_F \neq \mathbb{R}^3$. If α is irreducible, then there are at least two operators among Ad_{F_n} and Ad_{G_n} with different rotation axes, $\mathbb{R}_F$ and $\mathbb{R}_G$. Then $\operatorname{im} D$ contains both $\mathbb{C}_F$ and $\mathbb{C}_G$, which together span $\mathbb{R}^3$. $\square$

14.5 Representations for Seifert homology spheres

In this section we describe the representation spaces of the fundamental groups of Seifert homology spheres following Fintushel–Stern [45]. The fundamental group of $\Sigma = \Sigma(a_1, \dots, a_n)$ has the following presentation

$$\pi_1 \Sigma = \langle x_1, \dots, x_n, h \mid [h, x_k] = 1,\ x_k^{a_k} = h^{-b_k},\ k = 1, \dots, n,\ x_1 \dots x_n = 1 \rangle,$$

where

$$a_1 \dots a_n \sum_{k=1}^{n} \frac{b_k}{a_k} = 1.$$

Lemma 14.4. *If α is an irreducible representation of $\pi_1(\Sigma(a_1, \dots, a_n))$ in* SU(2) *then $\alpha(h) = \pm 1$.*

Proof. If $\alpha(h) \neq \pm 1$ then it belongs to only one copy of U(1) in SU(2). Since h is central, $\alpha(h)$ commutes with $\alpha(g)$ for all $g \in \pi_1 \Sigma$. Therefore, all $\alpha(g)$ lie in the same copy of U(1), and α is reducible. $\square$

Lemma 14.5. *If α is an irreducible representation of $\pi_1(\Sigma(a_1, \dots, a_n))$ in* SU(2) *then at most $n - 3$ of the $\alpha(x_k)$ are equal to ± 1.*

Proof. Suppose that $\alpha(x_k) = \pm 1$ for $k = 3, \dots, n$. Then $\alpha(x_1)\alpha(x_2) = \pm 1$ and $\alpha(x_1)$ commutes with $\alpha(x_2)$. Hence α is reducible, a contradiction. $\square$

First, we will describe the representation space $\mathcal{R}(\Sigma(p, q, r))$ for a Seifert homology sphere with three singular fibers. According to Lemma 14.5, if $\alpha\colon \pi_1 \Sigma(p, q, r) \to$

SU(2) is an irreducible representation, none of the $\alpha(x_k)$ is equal to ± 1. After conjugation, one can assume that $\alpha(x_1) \in \mathbb{C}$, and the condition $\alpha(x_1)^p = \alpha(h)^{-b_1} = \pm 1$ implies that $\alpha(x_1)$ is a degree p root of 1 or -1, depending on $\alpha(h)$ and b_1. More precisely, $\alpha(x_1) = e^{\pi i \ell_1/p}$, where $0 < \ell_1 < p$ and ℓ_1 is even if $\alpha(h)^{b_1} = 1$ and ℓ_1 is odd if $\alpha(h)^{b_1} = -1$. One cannot assume that $\alpha(x_2)$ or $\alpha(x_3)$ are complex numbers, but one can determine their conjugacy classes: $\alpha(x_2) \in S_2$, the conjugacy class of $e^{\pi i \ell_2/q}$, where $0 < \ell_2 < q$ (ℓ_2 is even iff $\alpha(h)^{b_2} = 1$), and $\alpha(x_3) \in S_3$, the conjugacy class of $e^{\pi i \ell_3/r}$, where $0 < \ell_3 < r$ (ℓ_3 is even iff $\alpha(h)^{b_3} = 1$).

Therefore, as soon as we fix $\alpha(h) = \pm 1$, we can associate to each irreducible representation α a triple (ℓ_1, ℓ_2, ℓ_3) with ℓ_1, ℓ_2, ℓ_3 chosen as above. Not all such triples define a representation because, for some of the choices, the last relation $\alpha(x_1)\alpha(x_2)\alpha(x_3) = 1$ may not be satisfied. The product $\alpha(x_1)\alpha(x_2)$ belongs to $\alpha(x_1) \cdot S_2$, the image of S_2 under a rigid motion of S^3 taking 1 to $\alpha(x_1)$. The relation $x_1 x_2 x_3 = 1$ can be written in the form $x_1 x_2 = x_3^{-1}$ with $\alpha(x_3)^{-1}$ conjugated to $\alpha(x_3)$. Therefore, a triple (ℓ_1, ℓ_2, ℓ_3) defines a representation if and only if the intersection $(\alpha(x_1) \cdot S_2) \cap S_3$ is not empty.

Let us consider the projection of SU(2) to the upper half of the complex circle given by the formula $A \mapsto e^{i \arccos(\mathrm{Re}(A))}$. The entire conjugacy class S_3 is mapped to a single point $p = e^{\pi i \ell_3/r}$, while the image of $\alpha(x_1) \cdot S_2$ is an interval I, see Figure 14.1.

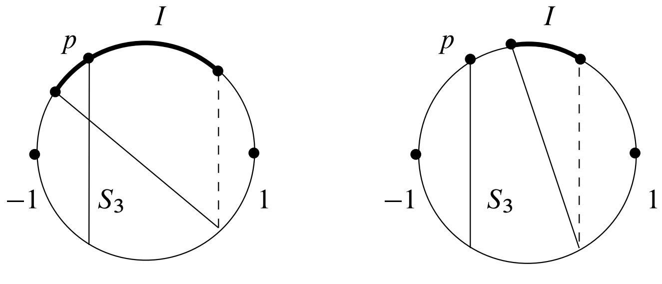


Figure 14.1

The intersection $(\alpha(x_1) \cdot S_2) \cap S_3$ is not empty if and only if $p \in I$. The ends of the interval I are the projections of the intersection points of $\alpha(x_1) \cdot S_2$ with the complex circle, that is, the images of $e^{\pi i(\ell_1/p \pm \ell_2/q)}$ in the upper half-circle. Thus we have the following condition on (ℓ_1, ℓ_2, ℓ_3):

$$\left|\frac{\ell_1}{p} - \frac{\ell_2}{q}\right| < \frac{\ell_3}{r} < 1 - \left|1 - \left(\frac{\ell_1}{p} + \frac{\ell_2}{q}\right)\right|.$$

In case the intersection $(\alpha(x_1) \cdot S_2) \cap S_3$ is not empty, it is a circle. Therefore, together with each representation we have a whole circle of representations. In fact, all representations on this circle are conjugate to each other. One way to prove this is to notice that in our construction one can conjugate everything by a unit complex number, keeping $\alpha(x_1)$ complex, but changing $\alpha(x_2)$ and $\alpha(x_3)$ within their respective conjugacy classes. The circle of unit complex numbers gives rise to the circle $(\alpha(x_1)\cdot S_2)\cap S_3$. Another way to prove that all representations on the circle $(\alpha(x_1)\cdot S_2)\cap S_3$ are conjugate would be to check the following technical result by an explicit calculation.

Lemma 14.6. *Let α and β be irreducible representations of $\pi_1\Sigma(p,q,r)$ in* SU(2) *such that*

(1) $\alpha(h)=\beta(h)$ *and* $\alpha(x_1)=\beta(x_1)\in\mathbb{C}$,

(2) $\operatorname{tr}(\alpha(x_2))=\operatorname{tr}(\beta(x_2))$, *and*

(3) $\operatorname{tr}(\alpha(x_2)\alpha(x_3))=\operatorname{tr}(\beta(x_2)\beta(x_3))$.

Then there is a complex number c such that $\beta=c\,\alpha\,c^{-1}$.

Corollary 14.7. *The space $\mathcal{R}(\Sigma(p,q,r))$ is finite.*

Example. The fundamental group of $\Sigma(2,3,5)$ has the presentation

$$\langle x_1,x_2,x_3,h \mid [h,x_k]=1,\ x_1^2=h,\ x_2^3=h^{-1},\ x_3^5=h^{-1},\ x_1x_2x_3=1\rangle.$$

If $\alpha(h)=1$ then $\alpha(x_1)^2=1$ which means that $\alpha(x_1)=\pm1$ and the representation α is reducible. Let $\alpha(h)=-1$; then we have the following choices: $\ell_1=1,\ \ell_2=1$, and $\ell_3=1,3$. Since $1/5,3/5\in[1/6,5/6]$, both triples $(1,1,1)$ and $(1,1,3)$ are realized as representations. The space $\mathcal{R}(\Sigma(2,3,5))$ consists of two points.

Example. The fundamental group of $\Sigma(3,4,7)$ has the presentation

$$\langle x_1,x_2,x_3,h \mid [h,x_k]=1,\ x_1^3=h^2,\ x_2^4=h^{-1},\ x_3^7=h^{-3},\ x_1x_2x_3=1\rangle.$$

If $\alpha(h)=1$ then the choices for (ℓ_1,ℓ_2,ℓ_3) are $(2,2,2),\ (2,2,4)$, and $(2,2,6)$. The interval $[1/6,5/6]$ contains $2/7$ and $4/7$, but not $6/7$. If $\alpha(h)=-1$ then $\ell_1=2,\ \ell_2=1$ or 3, and $\ell_3=1,3$, or 5. The interval $[5/12,11/12]$ does not contain $1/7$, and the interval $[1/12,7/12]$ does not contain $5/7$. Therefore, the triples coming from representations are $(2,2,2),\ (2,2,4),\ (2,1,3),\ (2,1,5),\ (2,3,1)$, and $(2,3,3)$.

Let us now consider a Seifert homology sphere $\Sigma(p,q,r,s)$ with four singular fibers. The group $\pi_1\Sigma(p,q,r,s)$ has two types of SU(2)-representations: in the first type, one of the generators goes to ±1, and in the second, the images of all generators differ from ±1. Representations of the first type can be handled as before, so we will concentrate on representations of the second type.

Let us choose a quadruple $(\ell_1,\ell_2,\ell_3,\ell_4)$ according to the usual rules, and fix $\alpha(x_1)=e^{\pi i\ell_1/p}$. Let S_k, $k=2,3,4$, be the conjugacy classes of $e^{\pi i\ell_2/q}$, $e^{\pi i\ell_3/r}$, and $e^{\pi i \ell_4/s}$, respectively. We wish to find $\alpha(x_k) \in S_k$ such that $\alpha(x_1)\alpha(x_2)\alpha(x_3) \cdot \alpha(x_4) = 1$. This means that in $\mathrm{SU}(2) = S^3$ we must find radii r_2, r_3, and r_4 of the 2-spheres $\alpha(x_1) \cdot S_2$, $\alpha(x_1)\alpha(x_2) \cdot S_3$, and $\alpha(x_1)\alpha(x_2)\alpha(x_3) \cdot S_4$, respectively, which form a linkage spanning from $\alpha(x_1)$ to 1. If such a linkage can be formed, then we can find a representation with the given $(\ell_1, \ell_2, \ell_3, \ell_4)$. The connected component of this representation is the configuration space of the given linkage modulo rotations leaving the complex circle invariant.

To see the linkage, identify S^3 with $\mathbb{R}^3 \cup \{\infty\}$ using stereographic projection $S^3 \to \mathbb{R}^3$. We may assume that the complex circle is mapped onto the z-axis. The 2-spheres in S^3 correspond to 2-spheres in $\mathbb{R}^3 \cup \{\infty\}$, but the concept of linkage is somewhat changed since spheres of the same radius in S^3 can project to spheres with different radii in $\mathbb{R}^3 \cup \{\infty\}$. The linkage at hand has three radii, r_2, r_3, and r_4, which are distorted by stereographic projection, so we denote them by r_2', r_3', and r_4', see Figure 14.2.

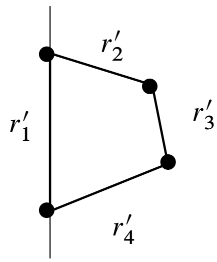


Figure 14.2

Suppose that $r_1' \geq r_2' \geq r_3' \geq r_4'$. Then there are exactly two rigid configurations in the configuration space shown in Figure 14.3.

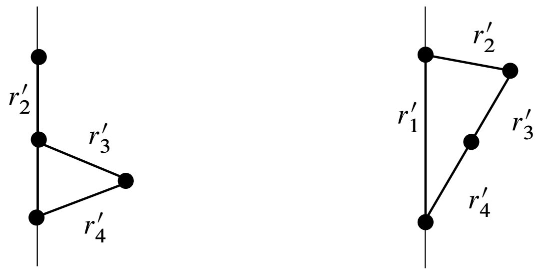


Figure 14.3

If we fix a generic r_2', the configurations (divided out by rotations preserving the z-axis) form a circle, see Figure 14.4. For different choices of r_2', these circles, together with the two points arising from the rigid configurations, form a 2-sphere.

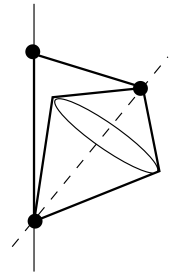


Figure 14.4

Thus we have proved that the representation space $\mathcal{R}(\Sigma(p,q,r,s))$ consists of a finite number of isolated points and a finite number of 2-dimensional spheres.

Example. The fundamental group of $\Sigma(2,3,5,7)$ has the presentation

$$\langle x_1, x_2, x_3, x_4, h \mid [h, x_k] = 1,\ x_1^2 = h^{-1},\ x_2^3 = h^2,\\ x_3^5 = h^2,\ x_4^7 = h^{-4},\ x_1 x_2 x_3 x_4 = 1\rangle.$$

Suppose that $\alpha(h) = 1$ then $\alpha(x_1) = \pm 1$. In the case $\alpha(x_1) = 1$, we have $\ell_1 = 0$ and $\alpha(x_2)\alpha(x_3)\alpha(x_4) = 1$, and we check if a quadruple $(0, \ell_2, \ell_3, \ell_4)$ corresponds to a representation by checking the condition

$$\left|\frac{\ell_2}{3} - \frac{\ell_3}{5}\right| < \frac{\ell_4}{7} < 1 - \left|1 - \left(\frac{\ell_2}{3} + \frac{\ell_3}{5}\right)\right|. \tag{14.5}$$

We end up with 4 representations $(0,2,2,2)$, $(0,2,2,4)$, $(0,2,2,6)$, and $(0,2,4,2)$. If $\alpha(x_1) = -1$ then $\ell_1 = 2$ and $\alpha(x_2)\alpha(x_3)\alpha(x_4) = -1$, so instead of (14.5), the condition

$$\left|\frac{\ell_2}{3} - \frac{\ell_3}{5}\right| < 1 - \frac{\ell_4}{7} < 1 - \left|1 - \left(\frac{\ell_2}{3} + \frac{\ell_3}{5}\right)\right|. \tag{14.6}$$

should be checked. From this, we get 4 more representations, $(2,2,2,2)$, $(2,2,2,4)$, $(2,2,4,4)$, and $(2,2,4,6)$. Similar computations with $\alpha(h) = -1$ and at least one of the $\alpha(x_k)$ equal to ± 1 produce the representations $(1,0,2,2)$, $(1,0,2,4)$, $(1,0,2,6)$, $(1,0,4,4)$, $(1,2,0,2)$, $(1,2,0,4)$, $(1,2,2,0)$, and $(1,2,4,0)$.

Suppose that none of the $\alpha(x_k)$ equals ± 1. Then $\alpha(h) = -1$ and we have the following choices: $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 2$ or 4, and $\ell_4 = 2, 4$, or 6. To determine if any of them defines a representation, we need to check whether the intersection $(\alpha(x_1)\cdot S_2) \cap (\alpha(x_4)\cdot S_3)$ is not empty. The property of this intersection being non-empty does not change if we replace $\alpha(x_4)\cdot S_3$ by its conjugate. Therefore, we may assume that $\alpha(x_4) \in \mathbb{C}$. The computation then reduces to checking whether the intervals on the upper half of the complex circle corresponding to $\alpha(x_1)\cdot S_2$ and

$\alpha(x_4) \cdot S_3$ intersect each other. It turns out that in our example all the combinations of the ℓ_k can be realized, so we get six two-dimensional spheres in $\mathcal{R}(\Sigma(p,q,r,s))$ (in addition to 16 isolated points we already have).

In general, the components of $\mathcal{R}(\Sigma(a_1,\dots,a_n))$ are smooth closed manifolds of even dimensions not exceeding $2(n-3)$, see Fintushel–Stern [45]. For more information on the representation spaces $\mathcal{R}(\Sigma(a_1,\dots,a_n))$ see Bauer–Okonek [13], Furuta–Steer [57], Kirk–Klassen [87] and Saveliev [143].

14.6 Exercises

1. Prove that the SU(2)-character variety $\chi(T^2)$ is homeomorphic to the quotient of the 2-torus $T^2 = S^1 \times S^1$ by the action of $\mathbb{Z}/2$ given by $\tau(z,w) = (\bar{z},\bar{w})$ (where S^1 is viewed as the unit circle in the complex plane). This character variety is commonly referred to as "pillowcase".
2. Let $k \subset \Sigma$ be a knot in a homology sphere Σ, and $K = \Sigma \setminus \operatorname{int} N(k)$ its exterior. Then the inclusion map $i : \partial K \to K$ induces a map
$$i^* : \chi(K) \to \chi(\partial K)$$
of SU(2)–character varieties. Identify $\chi(\partial K)$ with the pillowcase $\chi(T^2)$ by choosing a canonical meridian–longitude pair on ∂K. Calculate the image of i^* in the pillowcase if k is a trivial knot, and if k is a trefoil knot in S^3.
3. Let $k \subset \Sigma$ be a knot in a homology sphere Σ, and $K = \Sigma \setminus \operatorname{int} N(k)$ its exterior. The character variety $\chi(K)$ admits an involution $\tau : \chi(K) \to \chi(K)$ defined on representations by the rule
$$\tau(\alpha)(g) = (-1)^{\omega(g)}\alpha(g) \quad \text{for all } g \in \pi_1(K),$$
where $\omega : \pi_1(K) \to H_1(K) = \mathbb{Z}$ is the Abelianization homomorphism. Prove that the fixed point set of τ consists of the characters of binary dihedral representations, that is, representations with the image in the binary dihedral group $\mathrm{U}(1) \cup j \cdot \mathrm{U}(1) \subset \mathrm{SU}(2)$.
4. Let $k \subset \Sigma$ be a knot in a homology sphere Σ, and $K = \Sigma \setminus \operatorname{int} N(k)$ its exterior. Prove that any irreducible binary dihedral representation $\alpha : \pi_1(K) \to \mathrm{U}(1) \cup j \cdot \mathrm{U}(1)$ sends meridians of k to trace-free matrices, and canonical longitudes to 1.

Lecture 15

The local properties of representation spaces

Let π be a finitely presented discrete group. We wish to analyze the local structure of $\operatorname{Hom}(\pi, \mathrm{SU}(2))$ and $\operatorname{Hom}(\pi, \mathrm{SU}(2))/\mathrm{SO}(3)$ near a representation $\alpha\colon \pi \to \mathrm{SU}(2)$. To do this one normally finds the tangent space at α. Since the space $\operatorname{Hom}(\pi, \mathrm{SU}(2))$ is not a smooth manifold in general, we need to use the more general concept of the *Zariski tangent space*, see for example Chapter 2 of Shafarevich [146].

Let $X \subset \mathbb{R}^n$ be given by a system of equations $F_1 = \cdots = F_m = 0$ in the variables $x_1, \ldots, x_n$. To find the Zariski tangent space to X at a point $x^0 = (x_1^0, \ldots, x_n^0) \in X$ we write $x = x^0 + \varepsilon \cdot a$ with $a \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, and plug this in each polynomial F_i. We get:

$$F_i(x^0 + \varepsilon \cdot a) = F_i(x^0) + \varepsilon \cdot L_i(a) + G_i(\varepsilon a) = \varepsilon \cdot L_i(a) + G_i(\varepsilon a),$$

where $G_i(\varepsilon a)$ is divisible by ε^2. The Zariski tangent space to X at x^0 consists of the vectors $a \in \mathbb{R}^n$ that satisfy the equations $L_1(a) = \cdots = L_m(a) = 0$.

Example. Consider the cone $X \subset \mathbb{R}^3$ given by the equation

$$F(x, y, z) = x^2 + y^2 - z^2 = 0,$$

and find the tangent space to X at a point $(x_0, y_0, z_0) \in X$. We have:

$$\begin{aligned} F(x_0 + \varepsilon a, y_0 + \varepsilon b, z_0 + \varepsilon c) &= (x_0 + \varepsilon a)^2 + (y_0 + \varepsilon b)^2 - (z_0 + \varepsilon c)^2 \\ &= \varepsilon(2x_0 a + 2y_0 b - 2z_0 c) + \varepsilon^2 \cdot (a^2 + b^2 - c^2), \end{aligned}$$

so that the tangent space at (x_0, y_0, z_0) consists of all the vectors (a, b, c) such that $x_0 a + y_0 b - z_0 c = 0$. At any point of X other than the origin, the Zariski tangent space is the (usual) tangent space to X at a smooth point. At the origin, the Zariski tangent space is the entire $\mathbb{R}^3$.

We begin by finding the Zariski tangent space to $\operatorname{Hom}(\pi, \mathrm{SU}(2))$ at α. Let us consider a small perturbation $x \mapsto \alpha(x) + \varepsilon\eta(x)$ of α where $\eta(x) \in T_{\alpha(x)} \mathrm{SU}(2)$ for all $x \in \pi$. By applying right translation, we can write $\eta(x) = \xi(x)\alpha(x)$ with $\xi(x) \in T_1 \mathrm{SU}(2)$. One can think of ξ as a function $\xi\colon \pi \to \mathfrak{su}(2)$. In order for ξ to be in the Zariski tangent space to $\operatorname{Hom}(\pi, \mathrm{SU}(2))$ at α, the map $x \mapsto (1 + \varepsilon\xi(x))\alpha(x)$ should be a representation up to order ε^2,

$$(1 + \varepsilon\xi(xy))\alpha(xy) = (1 + \varepsilon\xi(x))\alpha(x) \cdot (1 + \varepsilon\xi(y))\alpha(y).$$

After dropping $\alpha(xy)$ and the higher order terms on both sides, this takes the form

$$\xi(xy)\alpha(xy) = \xi(x)\alpha(x)\alpha(y) + \alpha(x)\xi(y)\alpha(y).$$

If we multiply this equation on the right by $\alpha(xy)^{-1}$ we will get the so-called cocycle condition

$$\xi(xy) = \xi(x) + \operatorname{Ad}_{\alpha(x)} \xi(y). \tag{15.1}$$

A map $\xi\colon \pi \to \mathfrak{su}(2)$ satisfying the condition (15.1) is called a 1-*cocycle*; all 1-cocycles form a real vector space $Z^1_\alpha(\pi, \mathfrak{su}(2))$. Thus we have proved that

$$T_\alpha \operatorname{Hom}(\pi, \mathrm{SU}(2)) = Z^1_\alpha(\pi, \mathfrak{su}(2)).$$

Next we turn to the action of SO(3) on $\operatorname{Hom}(\pi, \mathrm{SU}(2))$ by conjugation. To compute the tangent space to the orbit of α we need to determine when two 1-cocycles, ξ_1 and ξ_2, are equivalent in the sense that there exists $u \in \mathfrak{su}(2)$ such that

$$(1 + \varepsilon\xi_1(x))\alpha(x) = (1 + \varepsilon u)(1 + \varepsilon\xi_2(x))\alpha(x)(1 + \varepsilon u)^{-1}$$

up to order ε^2. We get the following condition:

$$\xi_1(x) - \xi_2(x) = u - \operatorname{Ad}_{\alpha(x)} u. \tag{15.2}$$

A 1-cocycle $\xi\colon \pi \to \mathrm{SU}(2)$ is called a 1-*coboundary* if there exists $u \in \mathfrak{su}(2)$ such that $\xi(x) = u - \operatorname{Ad}_{\alpha(x)} u$ for all $x \in \pi$. The 1-coboundaries form a vector space $B^1_\alpha(\pi, \mathfrak{su}(2))$.

The space $H^1_\alpha(\pi, \mathfrak{su}(2)) = Z^1_\alpha(\pi, \mathfrak{su}(2))/B^1_\alpha(\pi, \mathfrak{su}(2))$ is called the *first cohomology group* of π with coefficients in the π-module $\mathfrak{su}(2)$ where the action of π is given by the composition $\operatorname{Ad}\circ\alpha\colon \pi \to \mathrm{SU}(2) \to \operatorname{Aut}(\mathfrak{su}(2))$. The result of our computations can now be stated as follows:

$$T_\alpha(\operatorname{Hom}(\pi, \mathrm{SU}(2))/\,\mathrm{SO}(3)) = H^1_\alpha(\pi, \mathfrak{su}(2)).$$

Example. Let us consider the case of $\alpha = \theta$, where θ is the trivial representation, $\theta(g) = 1$. Let M be a topological space with a finitely presented fundamental group. Then $H^1_\theta(\pi_1 M, \mathfrak{su}(2)) = H^1(M, \mathfrak{su}(2))$. This can be seen as follows. The 1-cocycles $\xi\colon \pi_1 M \to \mathfrak{su}(2)$ satisfy the cocycle condition $\xi(xy) = \xi(x) + \operatorname{Ad}_{\theta(x)} \xi(y) = \xi(x) + \xi(y)$. In particular, $\xi(xy) = \xi(yx)$, so the 1-cocycles are simply the $\mathbb{R}$-linear maps $H_1 M \to \mathfrak{su}(2)$. All coboundaries are trivial, therefore, $H^1_\theta(\pi_1 M, \mathfrak{su}(2)) = Z^1_\theta(\pi_1 M, \mathfrak{su}(2)) = \operatorname{Hom}(H_1 M, \mathfrak{su}(2)) = H^1(M, \mathfrak{su}(2))$.

Example. Let M be an oriented handlebody of genus g, hence $\pi_1 M$ is a free group on g generators. As we know from Lecture 14, $\mathcal{R}(M) = R^{\mathrm{irr}}(M)/\,\mathrm{SO}(3)$ is a smooth open manifold of dimension $3g - 3$. Therefore, the Zariski tangent space to $R(M)/\,\mathrm{SO}(3)$ at any orbit $\alpha \in \mathcal{R}(M)$ is the "honest" tangent space to a smooth manifold, so $H^1_\alpha(\pi_1 M, \mathfrak{su}(2)) = \mathbb{R}^{3g-3}$. On the other hand, $H^1_\theta(\pi_1 M, \mathfrak{su}(2)) = H^1(M, \mathfrak{su}(2)) = \mathbb{R}^{3g}$. The "wrong dimension" reflects the fact that θ is a singular point in the quotient $R(M)/\,\mathrm{SO}(3)$. Still, $H^1_\theta(\pi_1 M, \mathfrak{su}(2)) = Z^1_\theta(\pi_1 M, \mathfrak{su}(2))$ is the tangent space to $R(M)$ at θ.

Example. Let F be a closed oriented Riemann surface of genus g, then $H^1_\alpha(\pi_1 F, \mathfrak{su}(2)) = \mathbb{R}^{6g-6}$ for any irreducible α, and $H^1_\theta(\pi_1 F, \mathfrak{su}(2)) = H^1(F, \mathfrak{su}(2)) = \mathbb{R}^{6g}$.

Example. This example is due to Fintushel–Stern [45]. Let $\Sigma = \Sigma(a_1, \dots, a_n)$ be a Seifert homology sphere and $\alpha\colon \pi_1\Sigma \to \mathrm{SU}(2)$ an irreducible representation. Suppose that $\alpha(x_k) \neq \pm 1$ for $k = 1, \dots, m$ and $\alpha(x_k) = \pm 1$ for $k = m+1, \dots, n$. Then $m \geq 3$ according to Lemma 14.5. We will show that $H^1_\alpha(\pi_1\Sigma, \mathfrak{su}(2)) = \mathbb{R}^{2m-6}$.

We first describe the 1-cocycles $\xi\colon \pi_1\Sigma \to \mathfrak{su}(2)$. Since h is a central element in $\pi_1\Sigma$ we have that $\xi(gh) = \xi(hg)$ for all $g \in \pi_1\Sigma$. This implies that $\xi(g) + \mathrm{Ad}_{\alpha(g)}\,\xi(h) = \xi(h) + \xi(g)$ (remember that $\alpha(h) = \pm 1$) and that $\mathrm{Ad}_{\alpha(g)}\,\xi(h) = \xi(h)$ for all $g \in \pi_1\Sigma$. Since α is irreducible this means that $\alpha(h) = 0$. A similar argument shows that $\xi(x_k) = 0$ for $k = m+1, \dots, n$. If $k = 1, \dots, m$, we get

$$(1 + A_k + A_k^2 + \dots + A_k^{a_k-1})\xi(x_k) = 0, \tag{15.3}$$

$$\xi(x_1) + A_1\xi(x_2) + A_1A_2\xi(x_3) + \dots + A_1 \dots A_{m-1}\xi(x_m) = 0, \tag{15.4}$$

where A_k stands for $\mathrm{Ad}_{\alpha(x_k)}$. Each A_k, $k = 1, \dots, m$, considered as an operator on $\mathfrak{su}(2) = \mathbb{R}^3$, has an axis $\mathbb{R}_k$ and perpendicular plane of rotation

$$\mathbb{C}_k = \mathrm{im}(1 - A_k) = \ker(1 + A_k + A_k^2 + \dots + A_k^{a_k-1}).$$

To satisfy (15.3) we need to choose $\xi(x_k) \in \mathbb{C}_k$ for $k = 1, \dots, m$. This gives $2m$ degrees of freedom in the choice of the cocycle ξ, but these are still subject to (15.4). Consider the linear map $L\colon \mathbb{C}_1 \oplus \dots \oplus \mathbb{C}_m \to \mathbb{R}^3$ given by

$$L(z_1, \dots, z_m) = z_1 + A_1 z_2 + A_1A_2 z_3 + \dots + A_1 \dots A_{m-1} z_m.$$

Since α is irreducible, at least two of the axes, say $\mathbb{R}_1$ and $\mathbb{R}_2$, are distinct. The axis $\mathbb{R}_1$ is fixed by A_1, so $\mathbb{R}_1$ and $A_1(\mathbb{R}_2)$ are distinct. Hence their perpendicular planes $\mathbb{C}_1$ and $A_1(\mathbb{C}_2)$ span $\mathbb{R}^3$. Thus L is surjective and the relation (15.4) eliminates three of the degrees of freedom in the choice of ξ. The space of cocycles has dimension $2m - 3$.

The space of coboundaries is spanned by $\mathrm{im}(1 - \mathrm{Ad}_{\alpha(g)})$, $g \in \pi_1\Sigma$. Since α is irreducible, this span is the entire $\mathbb{R}^3$. The space of coboundaries is 3-dimensional, so $H^1_\alpha(\pi_1\Sigma, \mathfrak{su}(2)) = \mathbb{R}^{2m-6}$.

15.1 Exercises

1. Calculate the Zariski tangent spaces $T_\alpha \chi(T^2)$ to the pillowcase $\chi(T^2)$ at all $\alpha \in \chi(T^2)$.
2. Let $k \subset M$ be a knot in a closed oriented 3-manifold M, and $K = M \setminus \operatorname{int} N(k)$ its exterior. The inclusion map $i : \partial K \to K$ induces a map $i^* : \chi(K) \to \chi(\partial K)$ and, for every $\alpha \in \chi(K)$, the linear map of tangent spaces $d_\alpha i^* : T_\alpha \chi(K) \to T_{i^*\alpha} \chi(\partial K)$. Use Poincaré–Lefschetz duality to prove that the image of the latter map is always half-dimensional.

Lecture 16

Casson's invariant for Heegaard splittings

16.1 The intersection product

Let Σ be a homology sphere, $\Sigma = M_1 \cup_F M_2$ a Heegaard splitting, and $F_0 = F \setminus \text{int}\, D^2$. We have the following commutative diagram of inclusions

$$\begin{tikzcd}
 & & M_1 \arrow[dr, "j_1"] & \\
F_0 \arrow[r] & F \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \Sigma \\
 & & M_2 \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Choose a basepoint in F_0 and apply the π_1 functor to obtain a commutative diagram of groups where all homomorphisms are surjective:

$$\begin{tikzcd}
 & & \pi_1 M_1 \arrow[dr] & \\
\pi_1 F_0 \arrow[r] & \pi_1 F \arrow[ur] \arrow[dr] & & \pi_1 \Sigma \\
 & & \pi_1 M_2 \arrow[ur] &
\end{tikzcd}$$

After applying the SU(2)-representation functor R, we obtain a commutative diagram of spaces

$$\begin{tikzcd}
 & & R(M_1) \arrow[dl, "i_1^*"'] & \\
R(F_0) & R(F) \arrow[l] & & R(\Sigma) \arrow[ul, "j_1^*"'] \arrow[dl, "j_2^*"] \\
 & & R(M_2) \arrow[ul, "i_2^*"] &
\end{tikzcd}$$

in which $R(M_1)$, $R(M_2)$ and $R(F_0)$ are actually smooth manifolds. One can easily check that all the maps in this diagram are injective.

Lemma 16.1. *The intersection $R(M_1) \cap R(M_2)$ in $R(F_0)$ is transversal at the trivial representation θ.*

Proof. Transversality of the intersection $R(M_1) \cap R(M_2)$ at θ is equivalent to the assertion that $i_1^* \, T_\theta R(M_1) + i_2^* \, T_\theta R(M_2) = T_\theta R(F_0)$. With the identification $T_\theta R(M_k) = H^1(M_k, \mathfrak{su}(2))$ for $k = 1, 2$, and $T_\theta R(F_0) = H^1(F_0, \mathfrak{su}(2)) =$

$H^1(F, \mathfrak{su}(2))$, the space $i_1^* T_\theta R(M_1) + i_2^* T_\theta R(M_2)$ is the image of the map

$$i_1^* + i_2^* \colon H^1(M_1, \mathfrak{su}(2)) \oplus H^1(M_2, \mathfrak{su}(2)) \to H^1(F, \mathfrak{su}(2)) \tag{16.1}$$

which can be included in the Mayer–Vietoris sequence

$$\cdots \to H^1(\Sigma, \mathfrak{su}(2)) \to H^1(M_1, \mathfrak{su}(2)) \oplus H^1(M_2, \mathfrak{su}(2)) \\ \to H^1(F, \mathfrak{su}(2)) \to H^2(\Sigma, \mathfrak{su}(2)) \to \cdots$$

Since Σ is a homology sphere the map (16.1) is an isomorphism. $\square$

By restricting ourselves to irreducible representations, we get the commutative diagram of inclusions

$$\begin{tikzcd} & & R^{\mathrm{irr}}(M_1) \arrow[dl, "i_1^*"'] & \\ R^{\mathrm{irr}}(F_0) & R^{\mathrm{irr}}(F) \arrow[l] & & R^{\mathrm{irr}}(\Sigma) \arrow[ul, "j_1^*"'] \arrow[dl, "j_2^*"] \\ & & R^{\mathrm{irr}}(M_2) \arrow[ul, "i_2^*"] & \end{tikzcd}$$

All spaces in this diagram except possibly for $R^{\mathrm{irr}}(\Sigma)$ are smooth open manifolds.

Lemma 16.2. *The intersection of $R^{\mathrm{irr}}(M_1)$ and $R^{\mathrm{irr}}(M_2)$ in $R^{\mathrm{irr}}(F)$ is compact.*

Proof. We see from the diagram above that $R^{\mathrm{irr}}(M_1) \cap R^{\mathrm{irr}}(M_2) = R^{\mathrm{irr}}(\Sigma) = R(\Sigma) \setminus \{\theta\}$ since Σ is a homology sphere, see Lemma 14.1. The space $R(\Sigma)$ is compact, and, by Lemma 16.1, the point $\theta \in R(\Sigma)$ is isolated. Therefore, $R^{\mathrm{irr}}(\Sigma)$ is compact. $\square$

By taking the SO(3)-quotient of the representation spaces in the last commutative diagram, we get the commutative diagram of inclusions,

$$\begin{tikzcd} & & \mathcal{R}(M_1) \arrow[dl, "i_1^*"'] & \\ \mathcal{R}(F_0) & \mathcal{R}(F) \arrow[l] & & \mathcal{R}(\Sigma) \arrow[ul, "j_1^*"'] \arrow[dl, "j_2^*"] \\ & & \mathcal{R}(M_2) \arrow[ul, "i_2^*"] & \end{tikzcd}$$

Again, all spaces in this diagram are smooth open manifolds, with the possible exception of $\mathcal{R}(\Sigma)$.

Corollary 16.3. *The intersection of $\mathcal{R}(M_1) \cap \mathcal{R}(M_2) = \mathcal{R}(\Sigma)$ in $\mathcal{R}(F)$ is compact.*

Both $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ are submanifolds of $\mathcal{R}(F)$ but they are not necessarily transversal to each other. Choose any isotopy of $\mathcal{R}(F)$ with compact support that carries $\mathcal{R}(M_2)$ to $\tilde{\mathcal{R}}(M_2)$ where $\tilde{\mathcal{R}}(M_2)$ is transversal to $\mathcal{R}(M_1)$. Since $\dim \mathcal{R}(M_1) =$

$\dim \tilde{\mathcal{R}}(M_2) = 3g - 3$ and $\dim \mathcal{R}(F) = 6g - 6$, the intersection $\mathcal{R}(M_1) \cap \tilde{\mathcal{R}}(M_2)$ is a finite number of points. Given orientations on $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ (see below), we have an induced orientation on $\tilde{\mathcal{R}}(M_2)$, so we can define the algebraic intersection of $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ as the sum

$$\#(\mathcal{R}(M_1) \cap \mathcal{R}(M_2)) \;=\; \sum_{\alpha \in \mathcal{R}(M_1) \cap \tilde{\mathcal{R}}(M_2)} \varepsilon_\alpha, \tag{16.2}$$

where ε_α equals ± 1 depending on whether the orientations of the spaces $T_\alpha \mathcal{R}(M_1) \oplus T_\alpha \tilde{\mathcal{R}}(M_2)$ and $T_\alpha \mathcal{R}(F)$ agree. By a standard homological argument, the number (16.2) is well-defined.

Definition. Given a genus g Heegaard splitting $\Sigma = M_1 \cup_F M_2$ of a homology sphere Σ, its *Casson invariant* is

$$\lambda(\Sigma, M_1, M_2) = \frac{(-1)^g}{2}\, \#(\mathcal{R}(M_1) \cap \mathcal{R}(M_2)). \tag{16.3}$$

It is not immediately clear from this definition that $\lambda(\Sigma, M_1, M_2)$ is always an integer; this will be proved in Lecture 17, see Corollary 17.6.

Theorem 16.4. *Let $\Sigma = M_1 \cup_F M_2$ be a Heegaard splitting of a homology sphere Σ. Then the intersection $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ in $\mathcal{R}(F)$ is transversal at $\alpha \in \mathcal{R}(\Sigma)$ if and only if $H^1_\alpha(\pi_1\Sigma, \mathfrak{su}(2)) = 0$.*

Proof. The group $H^1_\alpha(\pi_1\Sigma, \mathfrak{su}(2))$ can be included in the following Mayer–Vietoris exact sequence in group cohomology, see Brown [25], page 51:

$$\begin{aligned} 0 \to\ & H^1_\alpha(\pi_1\Sigma, \mathfrak{su}(2)) \\ & \to H^1_\alpha(\pi_1 M_1, \mathfrak{su}(2)) \oplus H^1_\alpha(\pi_1 M_2, \mathfrak{su}(2)) \xrightarrow{i_1^* + i_2^*} H^1_\alpha(\pi_1 F, \mathfrak{su}(2)) \to \cdots \end{aligned}$$

We abuse notation here and use the same symbol α to denote the induced representations of $\pi_1 M_1,\ \pi_1 M_2$, and $\pi_1 F$. The intersection in question is transversal at α if and only if the map $i_1^* + i_2^*$ is surjective. Since α is irreducible, $H^1_\alpha(\pi_1 M_k, \mathfrak{su}(2)) = \mathbb{R}^{3g-3}$, $k = 1, 2$, and $H^1_\alpha(\pi_1 F, \mathfrak{su}(2)) = \mathbb{R}^{6g-6}$. Therefore, the map $i_1^* + i_2^*$ is surjective if and only if $H^1_\alpha(\pi_1\Sigma, \mathfrak{su}(2)) = 0$. $\square$

Example. If $\Sigma = \Sigma(p, q, r)$ then $H^1_\alpha(\pi_1\Sigma, \mathfrak{su}(2)) = 0$ for any irreducible α, see Lecture 15, so we get the transversality.

16.2 The orientations

Let F be a closed oriented surface of genus g, and consider the intersection form

$$I\colon H^1(F;\mathbb{R}) \times H^1(F;\mathbb{R}) \to \mathbb{R}, \quad I(a,b) = \langle a \smile b, [F]\rangle,$$

compare with (7.5). This form is unimodular, i.e. $\det I = \pm 1$, by Poincaré duality on the surface F. In fact, Lemma 7.6 implies that $\det I = 1$, and $\det I$ is not affected by a different choice of orientation on F.

The basis we constructed in Lemma 7.6 is called a *symplectic basis* in $H^1(F;\mathbb{R})$. Any two symplectic bases are related by a transition matrix C such that

$$C^\top J C = J. \tag{16.4}$$

Since $\det J = 1$, the equation (16.4) implies that $(\det C)^2 = 1$, and $\det C = \pm 1$.

Lemma 16.5. *Let C be a matrix such that $C^\top J C = J$, then* $\det C = 1$.

Proof. Let $A = (a_{ij})$ be a $2g \times 2g$ skew-symmetric matrix. Its *Pfaffian* $\operatorname{Pf}(A)$ is defined by the formula

$$\operatorname{Pf}(A) = \sum_\sigma \varepsilon(\sigma)\, a_{i_1 j_1} \cdots a_{i_g j_g}$$

where the summation is over all possible partitions σ of the set $\{1, 2, \dots, 2g-1, 2g\}$ into disjoint pairs $\{i_a, j_a\}$, where one supposes that $i_a < j_a$, $a = 1, \dots, g$, and where $\varepsilon(\sigma)$ is the sign of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2g-1 & 2g \\ i_1 & j_1 & \cdots & i_g & j_g \end{pmatrix}.$$

The Pfaffian has the properties that $(\operatorname{Pf}(A))^2 = \det A$ and $\operatorname{Pf}(C^\top A C) = \det C \cdot \operatorname{Pf}(A)$ for any matrix C of order $2g$. One can apply the latter formula to (16.4) to get $\operatorname{Pf}(J) = \det C \cdot \operatorname{Pf}(J)$. Since $\operatorname{Pf}(J) = 1$, we are finished. $\square$

Corollary 16.6. *The vector space $H^1(F;\mathbb{R}) = \mathbb{R}^{2g}$, where g is the genus of F, has a canonical orientation defined by a symplectic basis. This orientation changes by a factor of $(-1)^g$ if the orientation of F is reversed.*

Let $\Sigma = M_1 \cup_F M_2$ be a Heegaard splitting of genus g of a homology sphere Σ. The inclusions $i_k\colon F \to M_k$, $k = 1, 2$, induce homomorphisms $i_k^*\colon H^1(M_k;\mathbb{R}) \to H^1(F;\mathbb{R})$. The Mayer–Vietoris exact sequence

$$\begin{aligned} \cdots &\to H^1(\Sigma;\mathbb{R}) \\ &\to H^1(M_1;\mathbb{R}) \oplus H^1(M_2;\mathbb{R}) \xrightarrow{i_1^* + i_2^*} H^1(F;\mathbb{R}) \to H^2(\Sigma;\mathbb{R}) \to \cdots \end{aligned}$$

with $H^1(\Sigma;\mathbb{R}) = H^2(\Sigma;\mathbb{R}) = 0$ implies that $i_1^* H^1(M_1;\mathbb{R}) + i_2^* H^1(M_2;\mathbb{R}) = H^1(F;\mathbb{R})$. Therefore, $H^1(M_1;\mathbb{R})$ and $H^1(M_2;\mathbb{R})$ can be thought of as direct summands of dimension g in the space $H^1(F;\mathbb{R}) = \mathbb{R}^{2g}$. Since $H^1(F;\mathbb{R})$ is canonically oriented, an orientation of $H^1(M_1;\mathbb{R})$ defines an orientation of $H^1(M_2;\mathbb{R})$ by requiring that $i_1^* + i_2^*$ be orientation preserving.

Example. Consider the standard genus two Heegaard splitting of the 3-dimensional sphere, $S^3 = M_1 \cup_F M_2$, see Figure 16.1, with M_1 being the inner handlebody. The above-described orientation of the cohomology group $H^1(F;\mathbb{R})$ is given by the basis $\langle \alpha_1, \beta_1, \alpha_2, \beta_2 \rangle$ consisting of the Poincaré duals of $a_1, b_1, a_2, b_2 \in H_1(F;\mathbb{R})$ shown in Figure 7.14. If one orients $H^1(M_1;\mathbb{R})$ by the basis $\langle \alpha_1, \alpha_2 \rangle$ then $H^1(M_2;\mathbb{R})$ will be oriented by $\langle \beta_2, \beta_1 \rangle$.

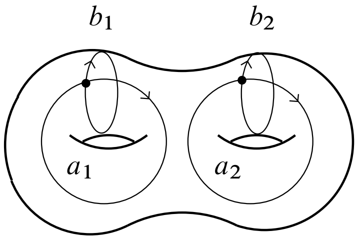


Figure 16.1

Let $F_0 = F \setminus \operatorname{int} D^2$. The orientation of F orients the boundary circle, γ, of F_0. Choose an orientation on SU(2); this fixes, in particular, an orientation on $\mathfrak{su}(2) = T_1\,\mathrm{SU}(2)$.

A choice of basis (consistent with the orientation) for $H^1(F;\mathbb{R})$ identifies the representation space $\operatorname{Hom}(\pi_1(F_0), \mathrm{SU}(2))$ with $\mathrm{SU}(2)^{2g}$. The map $h\colon \mathrm{SU}(2)^{2g} \to \mathrm{SU}(2)$ defined by the formula $\alpha \mapsto \alpha(\gamma)$ together with the orientation of SU(2) orients the nonsingular part of $h^{-1}(1)$, and also $\mathcal{R}(F) \subset h^{-1}(1)/\,\mathrm{SO}(3)$ by the rule that the identification $\mathfrak{su}(2) \oplus T\mathcal{R}(F) = T\mathcal{R}(F_0)$ is orientation preserving. This orientation of $\mathcal{R}(F)$ is independent of the initial choice of orientation of SU(2), but it does depend on the orientation of F via the sign of the intersection form I and the orientation of γ. Change the orientation of F, and the orientation of $\mathcal{R}(F)$ will change by a factor of $(-1)^{g+1}$.

A choice of basis (consistent with the orientation) for $H^1(M_1;\mathbb{R})$ identifies the space $\operatorname{Hom}(\pi_1(M_1), \mathrm{SU}(2))$ with $\mathrm{SU}(2)^g$. The orientation of SU(2) orients this space, and thus $\mathcal{R}(M_1) \subset \mathrm{SU}(2)^g/\,\mathrm{SO}(3)$. If g is even, the choice of SU(2) orientation affects the orientation of $\mathcal{R}(M_1)$. Obviously, the orientation of $H^1(M_1;\mathbb{R})$ affects the orientation of $\mathcal{R}(M_1)$. The orientations of $H^1(M_1;\mathbb{R})$ and SU(2) also orient $\mathcal{R}(M_2)$.

The orientations of both $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ change when the orientation of $H^1(M_1;\mathbb{R})$ is changed; and, in the g even case, when the orientation of SU(2) is changed. In both cases, the orientation of $T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$ at a point of intersection α is insensitive to these choices. However, changing the orientation of F changes the orientation of $T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$ by a factor of $(-1)^g$. Therefore, changing the orientation of F changes the sign of λ.

The handlebody M_1 played a special role above. Switching the roles of M_1 and M_2 in the definition of the orientations on $H^1(M_1;\mathbb{R})$ and $H^1(M_2;\mathbb{R})$ changes the orientation of $T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$ by $(-1)^g$. In addition, the orientation of $T_\alpha\mathcal{R}(M_2) \oplus T_\alpha\mathcal{R}(M_1)$ differs from that of $T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$ by $(-1)^{g+1}$ since $\dim\mathcal{R}(M_1) = \dim\mathcal{R}(M_2) = 3g-3$. Therefore, switching roles of M_1 and M_2 changes the sign of λ.

Thus, the sign ε_α in (16.2) is well-defined given the orientation of F and the specification of which handlebody to call M_1. Changing both choices leaves λ invariant so it is only the induced orientation of Σ which must be specified to avoid ambiguity (deciding which handlebody to call M_1 specifies the normal vector to F – this, together with an orientation of F, orients Σ).

16.3 Independence of Heegaard splitting

Thus, we have defined an invariant $\lambda(\Sigma, M_1, M_2)$ of a Heegaard splitting $\Sigma = M_1 \cup_F M_2$ of a homology sphere Σ. Next, we will show that $\lambda(\Sigma, M_1, M_2)$ only depends on Σ and not on the choice of Heegaard splitting. We will denote this number by $\lambda(\Sigma)$ and call it the *Casson invariant* of Σ.

It is obvious that $\lambda(\Sigma, M_1, M_2)$ is the same for equivalent Heegaard splittings. According to Theorem 1.2 of Lecture 1, any two Heegaard splittings of Σ are stably equivalent, hence, it suffices to show that $\lambda(\Sigma, M_1, M_2)$ is invariant under stabilization.

Let $\Sigma = M_1' \cup_{F'} M_2'$ be a Heegaard splitting obtained from $\Sigma = M_1 \cup_F M_2$ by the addition of an unknotted handle, see Figure 16.2. Then we have the following identifications: $\pi_1 M_1' = \mathbb{Z} * \pi_1 M_1$, and $\pi_1 M_2' = \mathbb{Z} * \pi_1 M_2$, where the factors $\mathbb{Z}$ are generated by the loops a_0 and b_0, respectively, see Figure 16.2.

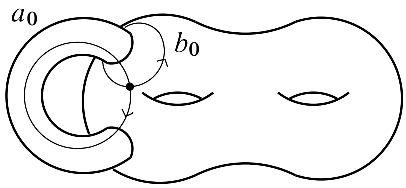


Figure 16.2

Let $F_0' = F' \setminus \operatorname{int} D^2$ be the surface F' with an open disc removed, then $\pi_1 F_0' = \mathbb{Z} * \mathbb{Z} * \pi_1 F_0$, with the group $\mathbb{Z} * \mathbb{Z}$ freely generated by a_0 and b_0. Therefore, we get the following identifications of the representation spaces:

$$R(M_k') = \mathrm{SU}(2) \times R(M_k), \quad k = 1, 2, \qquad R(F_0') = \mathrm{SU}(2) \times \mathrm{SU}(2) \times R(F_0).$$

The induced inclusions are of the form

$$\mathrm{SU}(2) \times R(M_1) \to \mathrm{SU}(2) \times \mathrm{SU}(2) \times R(F_0), \quad (A, \alpha) \mapsto (A, 1, \alpha),$$

$$\mathrm{SU}(2) \times R(M_2) \to \mathrm{SU}(2) \times \mathrm{SU}(2) \times R(F_0), \quad (B, \alpha) \mapsto (1, B, \alpha),$$

and these maps factor through $R(F')$. Inside $R(F_0')$ and $R(F')$ we have the following identifications

$$\begin{aligned} R(M_1') \cap R(M_2') &= (\mathrm{SU}(2) \times 1 \times R(M_1)) \cap (1 \times \mathrm{SU}(2) \times R(M_2)) \\ &= 1 \times 1 \times (R(M_1) \cap R(M_2)) = 1 \times 1 \times R(\Sigma), \end{aligned}$$

see Figure 16.3 where the box represents the product $R(F_0') = \mathrm{SU}(2) \times \mathrm{SU}(2) \times R(F_0)$.

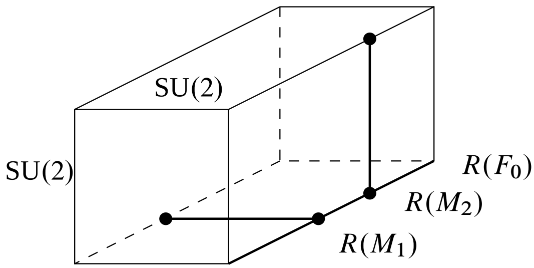


Figure 16.3

Since $R(\Sigma) = R^{\mathrm{irr}}(\Sigma) \cup \{\theta\}$, we see that

$$R^{\mathrm{irr}}(M_1') \cap R^{\mathrm{irr}}(M_2') = 1 \times 1 \times (R^{\mathrm{irr}}(M_1) \cap R^{\mathrm{irr}}(M_2)),$$

and the same holds after factoring out by the SO(3)-action,

$$\mathcal{R}(M_1') \cap \mathcal{R}(M_2') = 1 \times 1 \times (\mathcal{R}(M_1) \cap \mathcal{R}(M_2)).$$

The manifolds $\mathcal{R}(M_2')$ and $\mathcal{R}(M_2)$ may need to be perturbed into $\tilde{\mathcal{R}}(M_2')$ and $\tilde{\mathcal{R}}(M_2)$ respectively so that the intersections $\mathcal{R}(M_1') \cap \tilde{\mathcal{R}}(M_2')$ and $\mathcal{R}(M_1) \cap \tilde{\mathcal{R}}(M_2)$ are transversal. We claim without proof that the perturbations can be chosen so that

$$\mathcal{R}(M_1') \cap \tilde{\mathcal{R}}(M_2') = 1 \times 1 \times (\mathcal{R}(M_1) \cap \tilde{\mathcal{R}}(M_2)), \tag{16.5}$$

see Akbulut–McCarthy [2], pages 70–78. After that, the invariants $\lambda(\Sigma, M_1, M_2)$ and $\lambda(\Sigma, M_1', M_2')$ are given by the formulas

$$\lambda(\Sigma, M_1', M_2') = \frac{(-1)^{g+1}}{2} \sum_{\alpha} \varepsilon_\alpha', \quad \text{and} \quad \lambda(\Sigma, M_1, M_2) = \frac{(-1)^{g}}{2} \sum_{\alpha} \varepsilon_\alpha,$$

where both summations go over the same finite set of points (16.5). Thus to finish the proof it suffices to check that $\varepsilon_\alpha' = -\varepsilon_\alpha$.

For the sake of simplicity, we will assume that the intersections $\mathcal{R}(M_1') \cap \mathcal{R}(M_2')$ and $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ are transversal. All cohomology will have real coefficients.

The space $H^1(F') = \mathbb{R} \oplus \mathbb{R} \oplus H^1(F)$ is oriented by the choice of symplectic basis Poincaré dual to $(a_0, b_0, a_1, b_1, \dots, a_g, b_g)$ where the Poincaré dual of $(a_1, b_1, \dots, a_g, b_g)$ is a symplectic basis in $H^1(F)$, and the cycles a_0 and b_0 are pictured in Figure 16.2. This choice orients $R(F_0')$ so that the identification $TR(F_0') = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus TR(F_0)$ given by the choice of a_0 and b_0 is orientation preserving. To orient $R^{\text{irr}}(F')$, we consider the map

$$h' \colon R(F_0') \to \mathrm{SU}(2), \quad (A_0, B_0, A_1, \dots, B_g) \mapsto \prod_{k=0}^{g} [A_k, B_k]$$

with $R(F') = (h')^{-1}(1)$, and orient $R^{\text{irr}}(F')$ by the rule that the identification $\operatorname{im}(dh') \oplus \ker(dh') = TR(F_0')$ is orientation preserving at any irreducible representation, where $\ker(dh') = TR(F')$ and $\operatorname{im}(dh') = \mathfrak{su}(2)$. At any irreducible representation in $R(F')$ of the form $(1, 1, \alpha)$ we have $dh' = 0 \oplus 0 \oplus dh$. Here $h \colon R(F_0) \to \mathrm{SU}(2)$ is the map that provides orientation for $R(F)$ at an irreducible representation by the rule that $\mathfrak{su}(2) \oplus TR(F) = TR(F_0)$ is orientation preserving. Since $\ker dh' = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \ker dh$ we conclude that the orientation of $TR(F')$ coincides with the direct sum orientation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus TR(F)$. The same is true for the SO(3)-quotients, that is, $T\mathcal{R}(F')$ is oriented as the direct sum $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus T\mathcal{R}(F)$.

Choose an orientation on $H^1(M_1)$ and orient $H^1(M_2)$ so that the identification $H^1(M_1) \oplus H^1(M_2) = H^1(F)$ is orientation preserving. Choose the product orientation on $H^1(M_1') = \mathbb{R} \oplus H^1(M_1)$, and orient $H^1(M_2')$ so that the orientation on $H^1(F') = H^1(M_1') \oplus H^1(M_2')$ is canonical. This orientation of $H^1(M_2')$ differs from the product orientation $H^1(M_2') = \mathbb{R} \oplus H^1(M_2)$ by $(-1)^g$. This is true because the space $\mathbb{R} \oplus H^1(M_2) \oplus \mathbb{R} \oplus H^1(M_2)$, with the product orientations of the factors, gives the canonical orientation on $H^1(F') = \mathbb{R} \oplus \mathbb{R} \oplus H^1(F) = \mathbb{R} \oplus \mathbb{R} \oplus H^1(M_1) \oplus H^1(M_2)$ only after we pull the second $\mathbb{R}$-factor through $H^1(M_1)$, while $\dim H^1(M_1) = g$.

Keeping in mind the factor $(-1)^g$, we orient $H^1(M_2')$ as a product, and compare the orientations of the space $T_\alpha \mathcal{R}(M_1') \oplus T_\alpha \mathcal{R}(M_2') = \mathfrak{su}(2) \oplus T_\alpha \mathcal{R}(M_1) \oplus \mathfrak{su}(2) \oplus T_\alpha \mathcal{R}(M_2)$, which has the $(-1)^{g-1}$ times the orientation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus$

$T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$, with the orientation of $T_\alpha\mathcal{R}(F') = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus T_\alpha\mathcal{R}(F)$. The orientations of $T_\alpha\mathcal{R}(M_1) \oplus T_\alpha\mathcal{R}(M_2)$ and $T_\alpha\mathcal{R}(F)$ differ by ε_α, therefore, $\varepsilon'_\alpha = (-1)^g(-1)^{g-1}\varepsilon_\alpha = -\varepsilon_\alpha$.

16.4 Exercises

1. Verify the properties of the Pfaffian stated without proof in the proof of Lemma 16.5.
2. Given homology spheres Σ_1 and Σ_2, calculate $\mathcal{R}(\Sigma_1 \# \Sigma_2)$ in terms of $\mathcal{R}(\Sigma_1)$ and $\mathcal{R}(\Sigma_2)$. Use this calculation to give another proof that $\lambda(\Sigma_1 \# \Sigma_2) = \lambda(\Sigma_1) + \lambda(\Sigma_2)$. Assume for the sake of simplicity that the Zariski tangent spaces of $\mathcal{R}(\Sigma_1)$ and $\mathcal{R}(\Sigma_2)$ all vanish.
3. Given a homology sphere Σ, verify that $\lambda(-\Sigma) = -\lambda(\Sigma)$ directly from the definition (16.3). Assume for the sake of simplicity that the Zariski tangent spaces of $\mathcal{R}(\Sigma)$ all vanish.

Lecture 17
Casson's invariant for knots

17.1 Preferred Heegaard splittings

Let k be a knot in a homology sphere Σ. In the following two lemmas, we construct Heegaard splittings of Σ compatible with surgery along k.

Lemma 17.1. *There exists a Heegaard splitting $\Sigma = M_1 \cup_F M_2$ such that k is a separating curve on F.*

Proof. Choose a Seifert surface F' for k. Thicken F' to get a handlebody $F' \times [0, 1] \subset \Sigma$. Let K be the closure of the complement of $F' \times [0, 1]$ in Σ. This is a compact 3-manifold, not necessarily a handlebody. Its boundary is $\partial K = \partial(F' \times [0, 1]) = F' \times \{0\} \cup (\partial F' \times [0, 1]) \cup F' \times \{1\}$, a closed surface of genus twice the genus of F'. The knot k is embedded in the surface ∂K as $\partial F' \times \{1/2\} \subset \partial F' \times [0, 1]$ and separates it. Consider a triangulation of K and drill tunnels along its 1-skeleton, adding the corresponding 1-handles to $F' \times [0, 1]$. One can assume that the tunnels start and end away from the knot k. An argument similar to the one in the proof of Theorem 1.1 of Lecture 1 shows that this procedure produces a Heegaard splitting. It may happen however that the knot k no longer separates the surface because we attached some 1-handles starting on one side of the knot k and ending on the other. The situation can be easily fixed by sliding the attaching discs of such 1-handles to $F' \times \{0\}$. □

Lemma 17.2. *There exists a Heegaard splitting $\Sigma = M_1 \cup_F M_2$ such that $M_1 = F' \times [0, 1]$ and $k = \partial F' \times \{1/2\}$ is a separating curve on $F = \partial(F' \times [0, 1])$.*

Proof. Repeat the proof of Lemma 17.1 to construct a Heegaard splitting with all the additional 1-handles attached to $F' \times \{0\}$. Then drill from $F' \times \{1\}$ to $F' \times \{0\}$ and through the cores of those 1-handles, see Figure 17.1. □

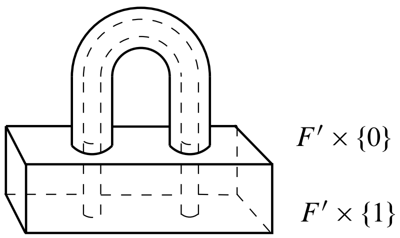


Figure 17.1

The following lemma provides a preferred Heegaard splitting for a boundary link, and it will be used later in this lecture to prove property (2) of the Casson invariant, see Lecture 12.

Lemma 17.3. *Let $k \cup \ell$ be a boundary link in a homology sphere Σ. Then there exists a Heegaard splitting $\Sigma = M_1 \cup_F M_2$ such that $M_1 = F' \times [0,1]$, $\ell = \partial F' \times \{1/2\}$, and k is a separating curve in $F' \times \{0\}$.*

Proof. Let F'_k and F'_ℓ be disjoint Seifert surfaces for k and ℓ. Thicken F'_k to $F'_k \times [0,1]$ so that $F'_k \times [0,1]$ is still disjoint from F'_ℓ. Let F_k be the boundary of $F'_k \times [0,1]$ and form the connected sum of F_k and F'_ℓ by tubing them together. By choosing the tubing away from k and ℓ, we obtain a surface F'' with the properties that $\ell = \partial F''$ and k is a separating curve in F''. The rest of the proof continues along the lines of the proofs of Lemmas 17.1 and 17.2. □

17.2 The Casson invariant for knots

Let k be a knot in a homology sphere Σ, and let $\Sigma = M_1 \cup_F M_2$ be a Heegaard splitting of genus g such that $k \subset F$ separates F as $F = F' \cup_k F''$, see Lemma 17.1. Denote by τ a Dehn twist of F along k. For any n, a Heegaard splitting of the homology sphere

$$\Sigma + \frac{1}{n} \cdot k$$

can be obtained from the splitting $\Sigma = M_1 \cup_F M_2$ by composing the gluing map with τ^n. Introduce the notation $\mathcal{R}_1 = \mathcal{R}(M_1)$ and $\mathcal{R}_2 = \mathcal{R}(M_2)$. By functoriality, the map τ induces the diffeomorphism $\tau^*\colon \mathcal{R}(F) \to \mathcal{R}(F)$ which maps $\mathcal{R}_2 \subset \mathcal{R}(F)$ to $\tau^*\mathcal{R}_2 \subset \mathcal{R}(F)$, and

$$\lambda\left(\Sigma + \frac{1}{n} \cdot k\right) = \frac{(-1)^g}{2} \cdot \#(\mathcal{R}_1 \cap (\tau^n)^*\mathcal{R}_2) = \frac{(-1)^g}{2} \cdot \#(\mathcal{R}_1 \cap (\tau^*)^n\mathcal{R}_2).$$

Therefore,

$$\begin{aligned}
&\lambda\left(\Sigma + \frac{1}{n+1} \cdot k\right) - \lambda\left(\Sigma + \frac{1}{n} \cdot k\right) \\
&\quad = \frac{(-1)^g}{2} \cdot \big[\, \#(\mathcal{R}_1 \cap (\tau^*)^{n+1}\mathcal{R}_2) - \#(\mathcal{R}_1 \cap (\tau^*)^n\mathcal{R}_2) \,\big] \\
&\quad = \frac{(-1)^g}{2} \cdot \big[\, \#(\mathcal{R}_1 \cap (\tau^*)^{n+1}\mathcal{R}_2) - \#(\tau^*\mathcal{R}_1 \cap (\tau^*)^{n+1}\mathcal{R}_2) \,\big] \\
&\quad = -\frac{(-1)^g}{2} \cdot \#\big[(\tau^*\mathcal{R}_1 - \mathcal{R}_1) \cap (\tau^*)^{n+1}\mathcal{R}_2\big].
\end{aligned}$$

We wish to study the difference $\tau^*\mathcal{R}_1 - \mathcal{R}_1$ and to show that the number $\#((\tau^*\mathcal{R}_1 - \mathcal{R}_1) \cap (\tau^*)^{n+1}\mathcal{R}_2)$, and hence

$$\lambda'(k) = \lambda\left(\Sigma + \frac{1}{n+1}\cdot k\right) - \lambda\left(\Sigma + \frac{1}{n}\cdot k\right),$$

are independent of n. This will prove property (1) of the Casson invariant, see Lecture 12.

First of all, we will describe the action of τ^* on $\mathcal{R}(F) = R^{\mathrm{irr}}(F)/\operatorname{SO}(3)$. The space $R(F) = \operatorname{Hom}(\pi_1 F, \operatorname{SU}(2))$ can be identified as

$$R(F) = \{(\alpha', \alpha'') \mid \alpha'(k) = \alpha''(k)\} \subset R(F') \times R(F'').$$

Suppose that the base point of F belongs to F'. Then the action of τ_* on $\pi_1(F)$ is trivial on the loops in $\pi_1 F'$, and is given by the formula $\tau_* x = k^{-1}xk$ on the loops in $\pi_1(F'')$, see Figure 17.2. The map $\tau^*\colon R(F) \to R(F)$ is then given by the formula

$$\tau^*(\alpha', \alpha'') = (\alpha', \alpha'(k)^{-1}\cdot\alpha''\cdot\alpha'(k)),$$

which in turn gives rise to the map $\tau^*\colon \mathcal{R}(F) \to \mathcal{R}(F)$.

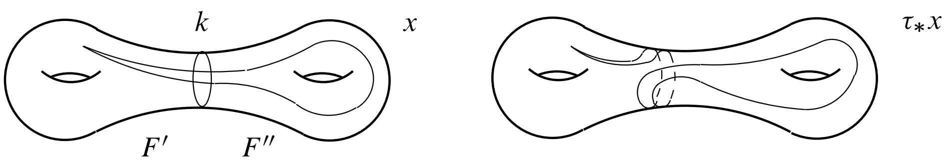


Figure 17.2

As our next step, we define

$$R_-(F) = \{\alpha\colon \pi_1 F \to \operatorname{SU}(2) \mid \alpha(k) = -1\} \subset R(F).$$

All representations in $R_-(F)$ are irreducible: if $\alpha\colon \pi_1 F \to \operatorname{SU}(2)$ is reducible it factors through $H_1 F$, and since k is a separating curve on F, we have $0 = [k] \in H_1 F$, so $\alpha(k) = 1$. One can think of $R_-(F)$ as the product

$$R_-(F) = \{(\alpha', \alpha'') \mid \alpha'(k) = -1,\ \alpha''(k) = -1\} = R_-(F') \times R_-(F''),$$

where $R_-(F') = (h')^{-1}(-1)$ and $R_-(F'') = (h'')^{-1}(-1)$ for the maps $h'\colon R(F') \to \operatorname{SU}(2)$ and $h''\colon R(F'') \to \operatorname{SU}(2)$ given by $h'(\alpha') = \alpha'(k)$ and $h''(\alpha'') = \alpha''(k)$. Since $-1 \in \operatorname{SU}(2)$ is a regular value for both h' and h'' (see Theorem 14.2), $\mathcal{R}_-(F) = R_-(F)/\operatorname{SO}(3)$ gets the structure of a smooth closed manifold of dimension $6g - 9$. The embedding $R_-(F) \subset R^{\mathrm{irr}}(F)$ is SO(3)-equivariant, therefore, we get an induced embedding $\mathcal{R}_-(F) \subset \mathcal{R}(F)$.

The map $\tau^* : R(F) \to R(F)$ restricts to $R_-(F)$ as the map $\tau^*(\alpha', \alpha'') = (\alpha', \alpha'(k)^{-1}\cdot\alpha''\cdot\alpha'(k)) = (\alpha', \alpha'')$. Therefore, $\mathrm{id} = \tau^*\colon \mathcal{R}_-(F) \to \mathcal{R}_-(F)$.

Lemma 17.4. *There exists a canonical isotopy*

$$H_t\colon \mathcal{R}(F) \setminus \mathcal{R}_-(F) \to \mathcal{R}(F) \setminus \mathcal{R}_-(F), \quad t \in [0,1],$$

such that $H_0 = \mathrm{id}$ *and* $H_1 = \tau^*$. *In particular,* H_t *isotopes* $\mathcal{R}_1$ *to* $\tau^*\mathcal{R}_1$ *in the complement of* $\mathcal{R}_-(F)$.

Proof. We showed in Theorem 13.3 that the exponential map $\exp\colon \mathfrak{su}(2) \to SU(2)$, $\alpha \mapsto e^\alpha$, is a diffeomorphism on the ball of radius π about 0,

$$\exp\colon B_\pi(0) \xrightarrow{\cong} SU(2) \setminus \{-1\}.$$

In particular, the natural contraction $B_\pi(0) \times [0,1] \to B_\pi(0)$, $(X,t) \mapsto t \cdot X$, exponentiates to a natural contraction of $SU(2) \setminus \{-1\}$,

$$(SU(2) \setminus \{-1\}) \times [0,1] \to SU(2) \setminus \{-1\}, \quad (A,t) \mapsto A^t.$$

These contractions are equivariant with respect to the action of SO(3). We define an isotopy $H_t\colon R(F) \setminus R_-(F) \to R(F) \setminus R_-(F)$ by the formula

$$H_t(\alpha', \alpha'') = (\alpha', (\alpha'(k)^t)^{-1} \cdot \alpha'' \cdot \alpha'(k)^t). \tag{17.1}$$

This formula makes sense because $\alpha'(k) \neq -1$ for $(\alpha', \alpha'') \in R(F) \setminus R_-(F)$. One can check that the isotopy defined by (17.1) induces a well-defined isotopy $H_t\colon \mathcal{R}(F) \setminus \mathcal{R}_-(F) \to \mathcal{R}(F) \setminus \mathcal{R}_-(F)$. □

Figure 17.3, essentially borrowed from Akbulut–McCarthy [2], schematically shows the space $\mathcal{R}(F)$. Shaded is the trace of the isotopy H_t. Note that the isotopy H_t does not extend over $\mathcal{R}_-(F)$, and that it keeps the reducible representations fixed.

The intersection $R(M_1) \cap (\tau^*)^{n+1} R(M_2)$ is transversal at the trivial representation θ in $R(F_0)$ because $\Sigma + \frac{1}{n+1} \cdot k$ is a homology sphere, see Lemma 16.1. By computing the differential of H_t, one can show that $H_t(R(M_1))$, $0 \le t \le 1$, is tangent to $R(M_1)$ at θ, see Akbulut–McCarthy [2], page 92. Hence one can cut out an open SO(3)-equivariant neighborhood U_n of θ from $R(M_1)$ without changing the intersection $H(\mathcal{R}_1 \times I) \cap (\tau^*)^{n+1}\mathcal{R}_2$ in $\mathcal{R}(F)$.

The only reducible representation in the intersection of $R(M_1)$ and $(\tau^*)^{n+1} R(M_2)$ is θ because $\Sigma + \frac{1}{n+1} \cdot k$ is a homology sphere, see Lemma 14.1. Therefore, one can find an SO(3)-equivariant open neighborhood V_n of the reducible non-trivial representations in $R(M_1)$ disjoint from $(\tau^*)^{n+1} R(M_2)$.

Let $W_n = U_n \cup V_n$. We may assume that $R^{\mathrm{irr}}(M_1) \setminus W_n$ is a smooth compact manifold with boundary, and denote its SO(3)-quotient by $\mathcal{R}_1' \subset \mathcal{R}_1$. The cycle

$$\delta' = \tau^*\mathcal{R}_1' - H(\partial\mathcal{R}_1' \times I) - \mathcal{R}_1'$$

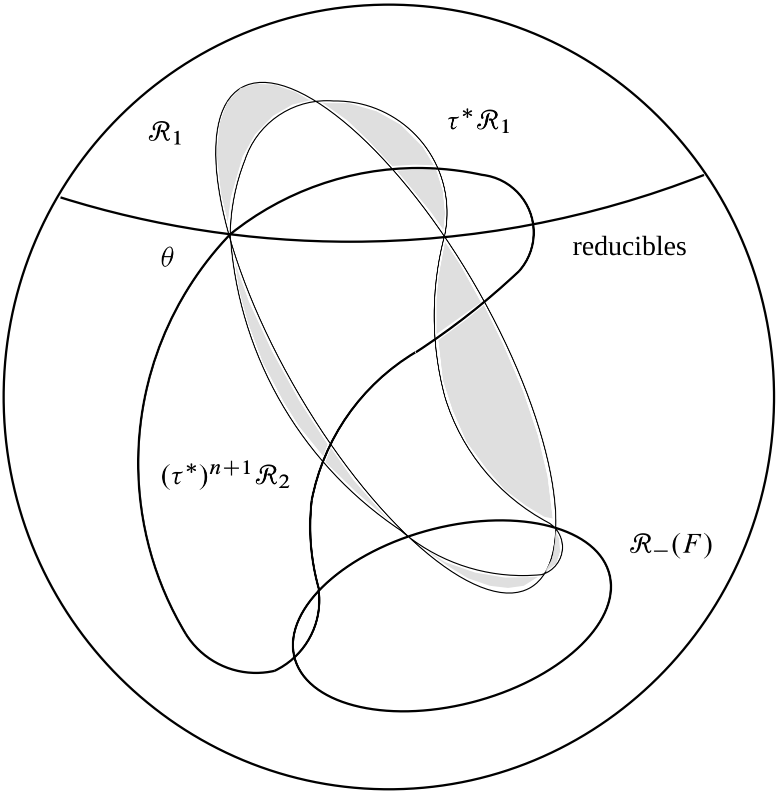


Figure 17.3

is compact, and

$$\#((\tau^*\mathcal{R}_1 - \mathcal{R}_1) \cap (\tau^*)^{n+1}\mathcal{R}_2) = \#(\delta' \cap (\tau^*)^{n+1}\mathcal{R}_2).$$

We may in addition assume that W_n is disjoint from $R_-(F)$. Let $\mathcal{N}$ be a compact manifold neighborhood of $\mathcal{R}_1 \cap \mathcal{R}_-(F)$ in $\mathcal{R}_1$ contained completely in $\mathcal{R}_1'$. Let $\mathcal{R}_1'' = \mathcal{R}_1' \setminus \operatorname{int}(\mathcal{N})$. We choose $\mathcal{N}$ so that $\mathcal{R}_1''$ is a smooth manifold. Let

$$\beta = \tau^*\mathcal{R}_1'' - H(\partial\mathcal{R}_1'' \times I) - \mathcal{R}_1''.$$

Since β is a compact boundary in $\mathcal{R}(F)$, we have that $\#(\beta \cap (\tau^*)^{n+1}\mathcal{R}_2) = 0$. Finally, we define the *difference cycle* $\delta = \delta' - \beta$. Clearly,

$$\#((\tau^*\mathcal{R}_1 - \mathcal{R}_1) \cap (\tau^*)^{n+1}\mathcal{R}_2) = \#(\delta \cap (\tau^*)^{n+1}\mathcal{R}_2) = \#((\tau^*)^{-(n+1)}\delta \cap \mathcal{R}_2).$$

The homology class of δ is independent of the choice of $\mathcal{N}$, and also independent of n, because δ belongs to a tubular neighborhood of $\mathcal{R}_-(F)$ and can be collapsed to a cycle in $\mathcal{R}_-(F)$, see Section 17.3 below. The action of τ^* on $\mathcal{R}_-(F)$ is trivial, therefore, $(\tau^*)^{-(n+1)}\delta = \delta$, and the number $\#((\tau^*\mathcal{R}_1 - \mathcal{R}_1) \cap (\tau^*)^{n+1}\mathcal{R}_2)$ is independent of n. Thus, the knot invariant $\lambda'(k)$ is well-defined, and we have proved property (1) of the invariant λ, see Lecture 12.

17.3 The difference cycle

Let $\Sigma = M_1 \cup_F M_2$ be a Heegaard splitting as in Lemma 17.2 with $M_1 = F' \times [0,1]$ and $F = \partial(F' \times [0,1]) = F' \cup_k F'$. Let $\operatorname{tr}\colon \mathrm{SU}(2) \to [-2,2]$ be the trace function on the Lie group SU(2), and define the argument function $\arg\colon \mathrm{SU}(2) \to [0,\pi]$ by the formula $\arg(A) = \arccos(\operatorname{tr}(A)/2)$. Then the manifolds $\mathcal{R}_1,\ \mathcal{R}_-(F)$, and the tubular ε-neighborhood $\mathcal{N}_\varepsilon$ of $\mathcal{R}_-(F)$, for $\varepsilon > 0$ small enough, can be identified as follows:

$$\begin{aligned}
\mathcal{R}_1 &= \{(\alpha',\alpha') \mid \alpha' \in R^{\mathrm{irr}}(F')\}/\,\mathrm{SO}(3),\\
\mathcal{R}_-(F) &= \{(\alpha',\alpha'') \mid \alpha',\alpha'' \in R(F'),\ \arg\alpha'(k) = \arg\alpha''(k) = \pi\}/\,\mathrm{SO}(3), \quad \text{and}\\
\mathcal{N}_\varepsilon &= \{(\alpha',\alpha'') \mid \alpha',\alpha'' \in R(F'),\ \alpha'(k) = \alpha''(k), \pi - \varepsilon \le \arg\alpha'(k)\}/\,\mathrm{SO}(3).
\end{aligned}$$

The cycle δ is homologous to the cycle δ_0 constructed as follows. Let

$$\begin{aligned}
\delta_\varepsilon &= \bigcup_t H_t(\mathcal{R}_1 \cap \partial\mathcal{N}_\varepsilon)\\
&= \bigcup_t \{(\alpha', \alpha'(k)^{-t}\cdot\alpha'\cdot\alpha'(k)^t) \mid\ \arg\alpha'(k) = \pi - \varepsilon\}/\,\mathrm{SO}(3)\\
&= \{(\alpha', g^{-1}\cdot\alpha'\cdot g) \mid g \in \mathrm{SU}(2)^\varepsilon,\ \arg\alpha'(k) = \pi - \varepsilon\}/\,\mathrm{SO}(3),
\end{aligned}$$

where $\mathrm{SU}(2)^\varepsilon$ consists of all $g \in \mathrm{SU}(2)$ with $0 \le \arg g \le \pi - \varepsilon$ (shown schematically as the shaded portion of the sphere in Figure 17.4).

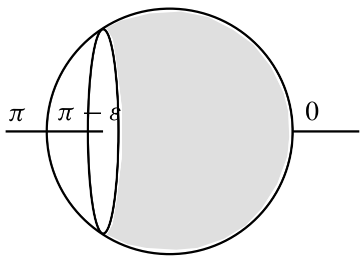


Figure 17.4

The cycle δ_0 is the limit of δ_ε as $\varepsilon \to 0$. Since the limiting value of $\mathrm{SU}(2)^\varepsilon$ as $\varepsilon \to 0$ is the entire group SU(2), we see that

$$\delta_0 = \{(\alpha', g^{-1}\cdot\alpha'\cdot g) \mid g \in \mathrm{SU}(2),\ \alpha'(k) = -1\}/\,\mathrm{SO}(3).$$

The cycle δ_0 has the following interpretation. The product action of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ on $R_-(F') \times R_-(F')$ defines the projection $\hat{p}\colon R_-(F') \times R_-(F') \to \mathcal{R}_-(F') \times \mathcal{R}_-(F')$ onto the quotient, with the fiber $\mathrm{SO}(3) \times \mathrm{SO}(3)$. The map $\hat{p}$ factors through $\mathcal{R}_-(F) = R_-(F') \times R_-(F')/\,\mathrm{SO}(3)$ (where the action of $\mathrm{SO}(3)$ is diagonal) to define the projection $p\colon \mathcal{R}_-(F) \to \mathcal{R}_-(F') \times \mathcal{R}_-(F')$ with the fiber $\mathrm{SO}(3)$. Let $\Delta \subset \mathcal{R}_-(F') \times \mathcal{R}_-(F')$ be the diagonal, then

$$\begin{aligned} p^{-1}(\Delta) = \{(\alpha', \alpha'') \mid \alpha'' = \mathrm{Ad}_g\, \alpha' &\text{ for some } g \in \mathrm{SU}(2) \\ &\text{and } \alpha'(k) = \alpha''(k) = -1\}/\,\mathrm{SO}(3). \end{aligned}$$

Since $\mathrm{SU}(2)$ is a regular double cover of $\mathrm{SO}(3)$, see Theorem 13.4, we conclude that $\delta_0 = 2 \cdot p^{-1}(\Delta)$. Thus, we have proved the following theorem.

Theorem 17.5. *Let k be a knot in a homology sphere Σ. Then*

$$\lambda'(k) = (-1)^g \,\#(p^{-1}(\Delta) \cap \mathcal{R}_2).$$

Corollary 17.6. $\lambda(\Sigma) \in \mathbb{Z}$ *for any homology sphere Σ.*

17.4 The Casson invariant for boundary links

Let $k \cup \ell$ be a boundary link in a homology sphere Σ. Choose a preferred Heegaard splitting $\Sigma = M_1 \cup_F M_2$ as in Lemma 17.3 so that $M_1 = F' \times [0,1]$ where F' is a Seifert surface of ℓ, and k is a separating curve in F'. Let τ_k and τ_ℓ be Dehn twists along k and ℓ, and h_k and h_ℓ their inverses. Note that the Dehn twists along k and ℓ commute. Then

$$\begin{aligned} &\lambda''(k,\ell) = \lambda(\Sigma + k + \ell) - \lambda(\Sigma + k) - \lambda(\Sigma + \ell) + \lambda(\Sigma) \\ &= \frac{(-1)^g}{2} \cdot \left[\#(\mathcal{R}_1 \cap \tau_k^* \tau_\ell^* \mathcal{R}_2) - \#(\mathcal{R}_1 \cap \tau_k^* \mathcal{R}_2) - \#(\mathcal{R}_1 \cap \tau_\ell^* \mathcal{R}_2) + \#(\mathcal{R}_1 \cap \mathcal{R}_2)\right] \\ &= \frac{(-1)^g}{2} \cdot \left[\#(h_k^* h_\ell^* \mathcal{R}_1 \cap \mathcal{R}_2) - \#(h_k^* \mathcal{R}_1 \cap \mathcal{R}_2) - \#(h_\ell^* \mathcal{R}_1 \cap \mathcal{R}_2) + \#(\mathcal{R}_1 \cap \mathcal{R}_2)\right] \\ &= \frac{(-1)^g}{2} \cdot \left[\#(h_k^* (h_\ell^* \mathcal{R}_1 - \mathcal{R}_1) \cap \mathcal{R}_2) - \#((h_\ell^* \mathcal{R}_1 - \mathcal{R}_1) \cap \mathcal{R}_2)\right]. \end{aligned}$$

Replace $\tau_\ell^* \mathcal{R}_1 - \mathcal{R}_1$ by the difference cycle δ_ℓ. Since $\tau_\ell^* \delta_\ell = \delta_\ell$ we have $h_\ell^* \delta_\ell = \delta_\ell$, so $\mathcal{R}_1 - h_\ell^* \mathcal{R}_1$ can be replaced by δ_ℓ. Thus

$$\lambda''(k,\ell) = -\frac{(-1)^g}{2} \cdot \#((h_k^* \delta_\ell - \delta_\ell) \cap \mathcal{R}_2).$$

We wish to prove that $h_k^* \delta_\ell - \delta_\ell = 0$ so that the property (2) of Casson invariant, which states that $\lambda''(k,\ell) = 0$ for all boundary links $k \cup \ell$, holds.

We will slightly abuse notation and denote by δ_ℓ both the cycle δ_ℓ and its homology class in $H_*(\mathcal{R}_-(F))$. As we already know, $\delta_\ell = 2 \cdot p^{-1}(\Delta)$, which implies that $\mathrm{PD}(\delta_\ell) = 2 \cdot p^* \,\mathrm{PD}(\Delta) \in H^*(\mathcal{R}_-(F))$. The map $h_k^*: \mathcal{R}_-(F') \to \mathcal{R}_-(F')$ induces homomorphisms in both the homology and cohomology of $\mathcal{R}_-(F')$; again we call them h_k^*. The following diagram is commutative:

$$\begin{array}{ccc} H^*(\mathcal{R}_-(F)) & \xrightarrow{h_k^*} & H^*(\mathcal{R}_-(F)) \\ {\scriptstyle p^*}\big\uparrow & & {\scriptstyle p^*}\big\uparrow \\ H^*(\mathcal{R}_-(F') \times \mathcal{R}_-(F')) & \xrightarrow{\tilde{h}_k^*} & H^*(\mathcal{R}_-(F') \times \mathcal{R}_-(F')) \end{array}$$

The map $\tilde{h}_k^*$ here is induced by $h_k^*: H^*(\mathcal{R}_-(F')) \to H^*(\mathcal{R}_-(F'))$, with the help of Künneth formula.

Theorem 17.7 (Newstead [124] and [125]). *Let F' be a Riemann surface with connected boundary $\partial F'$, and $h: F' \to F'$ a diffeomorphism such that* $\mathrm{id} = h_*: H_*(F') \to H_*(F')$. *Then h induces the identity map in the cohomology of $\mathcal{R}_-(F')$.*

Since k is a separating curve in F', the homomorphism $(h_k)_*: H_*(F') \to H_*(F')$ is the identity. Theorem 17.7 now implies that $\tilde{h}_k^* = \mathrm{id}$, hence $h_k^*: H^*(\mathcal{R}_-(F)) \to H^*(\mathcal{R}_-(F))$ is the identity map on the image of p^*. In particular, $h_k^*(2 \cdot p^* \,\mathrm{PD}(\Delta)) = 2 \cdot p^* \,\mathrm{PD}(\Delta)$, therefore, $h_k^*(\mathrm{PD}(\delta_\ell)) = \mathrm{PD}(\delta_\ell)$ and $\mathrm{PD}(h_k^* \delta_\ell) = \mathrm{PD}(\delta_\ell)$. Since PD is an isomorphism, $h_k^* \delta_\ell = \delta_\ell$.

17.5 The Casson invariant of a trefoil

Let k be a left-handed trefoil in S^3. We will show that $\lambda'(k) = \pm 1$. Since $S^3 - k = \Sigma(2,3,5)$ is the Poincaré homology sphere, this will imply that $\lambda(\Sigma(2,3,5)) = \pm 1$ and thus will prove property (0) of the invariant λ. Remember that λ was defined only up to a sign. As soon as we know that $\lambda(\Sigma(2,3,5)) = \pm 1$, we fix the sign of λ by requiring that $\lambda(\Sigma(2,3,5)) = -1$.

We proved in Lecture 8 that the trefoil is a fibered knot of genus 1. Let its complement in S^3 be fibered by surfaces of genus 1 with closure F', and construct a Heegaard splitting $S^3 = M_1 \cup_F M_2$ of genus 2 as in (8.2). This is a preferred Heegaard splitting in the sense of Lemma 17.2 with $F = F' \cup F'$. The corresponding representation spaces can be identified as follows:

$$\mathcal{R}(M_1) = \{(\alpha', \alpha') \mid \alpha' \in R^{\mathrm{irr}}(F')\} / \,\mathrm{SO}(3),$$
$$\mathcal{R}(M_2) = \{(\alpha', h^* \alpha') \mid \alpha' \in R^{\mathrm{irr}}(F')\} / \,\mathrm{SO}(3).$$

Lemma 17.8. *The manifold $\mathcal{R}_-(F)$ is a copy of* $\mathrm{SO}(3)$*, and the homology class $p^{-1}(\Delta) \in H_3(\mathcal{R}_-(F))$ equals (up to a sign) the fundamental class of $\mathcal{R}_-(F)$.*

Proof. The class $p^{-1}(\Delta)$ is the inverse image of the diagonal $\Delta \subset \mathcal{R}_-(F') \times \mathcal{R}_-(F')$, where p is the projection of the SO(3)-bundle $\mathcal{R}_-(F) \to \mathcal{R}_-(F') \times \mathcal{R}_-(F')$. We will show that $\mathcal{R}_-(F')$ is a point, which will imply the lemma.

Since F' is a punctured torus, its fundamental group is a free group on two generators. Therefore, $\mathcal{R}_-(F') = \{(A, B) \in \mathrm{SU}(2) \times \mathrm{SU}(2) \mid [A, B] = -1\}/\,\mathrm{SO}(3)$. After conjugation, we may assume that

$$A = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$

with $0 \le \varphi \le \pi$. Let

$$B = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}.$$

Then the condition $AB = -BA$ implies that $ae^{i\varphi} = 0$ and $be^{i\varphi} = -be^{-i\varphi}$. Therefore, $a = 0$ and $\varphi = \pi/2$. After conjugating the representation (A, B) by a matrix commuting with A, i.e. by a diagonal matrix, we can make b into a real positive number. Since $\det B = 1$ we must have $b = 1$. Therefore, $\mathcal{R}_-(F')$ contains only one point, the SO(3)-conjugacy class of the representation

$$A = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad \square$$

Recall from Theorem 17.5 that $\pm\lambda'(k) = \#(p^{-1}(\Delta) \cap \mathcal{R}(M_2))$. We know that $p^{-1}(\Delta) = \mathcal{R}_-(F)$, so we only need to compute the intersection $\mathcal{R}_-(F) \cap \mathcal{R}(M_2)$ in $\mathcal{R}(F)$. Using the description of $\mathcal{R}(M_2)$ given above, we see that

$$\mathcal{R}_-(F) \cap \mathcal{R}(M_2) = \{(\alpha', h^*\alpha') \mid \alpha' \in R_-(F')\}/\,\mathrm{SO}(3) \cong \mathcal{R}_-(F') = \{\text{point}\}.$$

The proof of the fact that $\lambda'(k) = \pm 1$ will be complete after we check the transversality condition.

Lemma 17.9. *The manifolds $\mathcal{R}_-(F)$ and $\mathcal{R}(M_2)$ intersect transversally in $\mathcal{R}(F)$.*

Proof. First, we make a dimension count. Since the Heegaard splitting has genus $g = 2$, we find that $\dim \mathcal{R}(M_2) = 3g - 3 = 3$, $\dim \mathcal{R}_-(F) = 6g - 9 = 3$, and $\dim \mathcal{R}(F) = 6g - 6 = 6$. Next, we compare the tangent spaces at $\alpha \in \mathcal{R}_-(F) \cap \mathcal{R}(M_2)$:

$$T_\alpha \mathcal{R}(M_2) = \{(\xi, dh^*(\xi)) \mid \xi \in T_{\alpha'} R^{\mathrm{irr}}(F')\}/\mathfrak{su}(2),$$
$$T_\alpha \mathcal{R}_-(F) = \{(\xi, \eta) \mid \xi, \eta \in T_{\alpha'} R_-(F')\}/\mathfrak{su}(2).$$

Their intersection consists of the pairs $(\xi, dh^*(\xi))$ such that $\xi \in T_{\alpha'} R_-(F')$ modulo the coboundaries. Therefore, $\dim (T_\alpha \mathcal{R}(M_2) \cap T_\alpha \mathcal{R}_-(F)) = 0$, which, together with the dimension count, establishes the transversality. $\square$

Lecture 18

An application of the Casson invariant

18.1 Triangulating 4-manifolds

As mentioned in the Introduction, the concepts of topological, smooth, and PL manifolds coincide in dimension 3. According to Cairns [27] and Hirsch [75], every 4-dimensional PL-manifold has a unique smooth structure, and vice versa. Topological manifolds in dimension 4 are utterly different from PL or smooth manifolds. On the one hand, there are topological 4-manifolds that admit many different smooth structures, and on the other, there are topological manifolds without any smooth structures, see Lecture 5. Recall that the latter follows from theorems of Rohlin and Freedman.

Theorem 18.1 (Rohlin). *If X is a smooth, closed, simply-connected* 4*-manifold whose intersection form is even then its signature is divisible by* 16*.*

Theorem 18.2 (Freedman). *Given a unimodular, symmetric, bilinear, integral form Q which is even, there exists exactly one simply-connected, closed, topological* 4*-manifold whose intersection form is Q, up to a homeomorphism.*

Idea of the existence proof. Freedman [51] shows that every homology 3-sphere bounds a homology 4-ball V, a compact topological 4-manifold with $H_*(V) = H_*(D^4)$. This statement fails to be true in the smooth category because homology spheres with non-trivial Rohlin invariant cannot bound smooth homology balls.

Now, given the form Q, one can find a smooth simply-connected 4-manifold X whose intersection form is Q and whose boundary is a homology sphere Σ, see Corollary 6.5. Let V be a homology ball bounding Σ, then $W = X \cup_\Sigma (-V)$ is a closed topological simply-connected 4-manifold with intersection form Q. □

Note that, according to Quinn [129], all non-compact 4-dimensional topological manifolds are smoothable, therefore, the manifold W in the proof above is smooth in the complement of a point.

Example. Let $\Sigma = \Sigma(2, 3, 5)$ be the Poincaré homology sphere. It bounds a canonical smooth simply-connected manifold X with the negative definite intersection form E_8. The intersection form of the manifold $W_8 = X \cup_\Sigma (-V)$, where V is a homology ball bounding Σ, is again E_8. The form E_8 is even and has signature -8. Therefore, W_8 is not smooth by Rohlin's theorem.

Theorem 18.3. *The manifold W_8 is not homeomorphic to a simplicial complex.*

Recall that PL manifold of dimension n is a simplicial complex that admits a combinatorial triangulation (a triangulation is combinatorial if the link of each its vertex is PL-homeomorphic to S^{n-1}). We already know that W_8 is not smooth, therefore, it is not PL. Theorem 18.3 claims more: it says that W_8 is not triangulable in the weakest possible sense – it is not homeomorphic to *any* simplicial complex, not necessarily combinatorial.

Proof. Suppose that W_8 is triangulable. After passing to a star subdivision, one may assume that the triangulation of W_8 is PL in the complement of the open star of a vertex v. This gives the link of the vertex v a PL-structure Σ so that the star of v is homeomorphic to a cone $C(\Sigma)$ over Σ. Since W_8 is a topological manifold, we may assume without loss of generality that there is an open neighborhood of v homeomorphic to an open ball D^4 that contains $C(\Sigma)$ and is contained in a bigger cone over Σ. Then we have a series of inclusions,

$$\Sigma \longrightarrow C(\Sigma) \setminus \{v\} \longrightarrow D^4 \setminus \{0\} \longrightarrow \Sigma \times I,$$

where I is an open interval. Their composition, followed by the projection $\Sigma \times I \to \Sigma$, is obviously the identity map $\Sigma \to \Sigma$. This implies that Σ is a retract of $D^4 \setminus \{0\}$ and, in particular, that Σ is a homotopy sphere.

Since the trivial group $\pi_1(\Sigma)$ does not have irreducible SU(2)-representations, the Casson invariant $\lambda(\Sigma)$ must vanish. But then $\mu(\Sigma) = \lambda(\Sigma) = 0 \bmod 2$, see Theorem 12.1. On the other hand, Σ bounds a smooth simply-connected compact 4-manifold $W_8 \setminus \operatorname{int} C(\Sigma)$ with intersection form E_8. Therefore, $\mu(\Sigma) = 1 \bmod 2$. This contradiction proves the theorem. □

A similar construction with other even unimodular forms of signature 8 modulo 16 would produce more examples of closed topological 4-manifolds that are not triangulable.

Remark. In the proof of Theorem 18.3, once we established that Σ is a homotopy sphere, we could use the Poincaré conjecture to conclude that Σ is homeomorphic to S^3. This would imply that $C(\Sigma)$ is homeomorphic to D^4 and hence the manifold W_8 is smooth, contradicting Rohlin's theorem. Our argument with the Casson and Rohlin invariants avoids using the Poincaré conjecture, which at the time of Casson's original work in 1985 was still open.

18.2 Higher-dimensional manifolds

The relations between topological, smooth, and PL manifolds in dimensions 5 and higher are more complicated than in dimension 4 – for example, the concepts of smooth and PL manifolds no longer coincide. Kirby and Siebenmann [86] showed that there exist topological manifolds in all dimensions greater than or equal to 5 that are not PL. It is still not known whether all topological manifolds in these dimensions are simplicially triangulable (i.e. homeomorphic to a simplicial complex). Surprisingly enough, this triangulation problem reduces to a problem in 3-dimensional topology, see Galewski–Stern [58] and Matumoto [108].

Theorem 18.4. *Every closed topological manifold of dimension $n \geq 5$ is simplicially triangulable if and only if there exists a homology 3-sphere Σ such that $\mu(\Sigma) = 1$ and $\Sigma \# \Sigma$ is homology cobordant to zero.*

A theorem of Fukumoto–Furuta [54] and Saveliev [141] asserts that Seifert homology spheres $\Sigma = \Sigma(a_1, \dots, a_n)$ have the following property: if $\mu(\Sigma) = 1 \bmod 2$ then no multiple $m\Sigma$ of Σ with $m \neq 0$ is homology cobordant to zero. Thus if one is to look for a homology sphere Σ as in Theorem 18.4, all Seifert fibered homology spheres, as well as all homology spheres homology cobordant to them, should be excluded from consideration.

18.3 Exercises

1. Prove that the Casson invariant of a homology sphere is not completely determined by its fundamental group. (Hint: Consider a homology sphere Σ with $\lambda(\Sigma) \neq 0$ and its two doubles, $\Sigma \# \Sigma$ and $\Sigma \# (-\Sigma)$.)
2. A knot k in S^3 has Property P if $\pi_1(S^3 + (1/n) \cdot k)$ is trivial if and only if $n = 0$. Prove that if $\Delta_k''(1) \neq 0$ then k has property P. In fact, Property P holds for all non-trivial knots in S^3, see Kronheimer and Mrowka [96].

Lecture 19
The Casson invariant of Seifert manifolds

In this lecture we give a closed form formula for the Casson invariant of Seifert homology spheres $\Sigma(a_1, \dots, a_n)$; the modulo 2 reduction of this formula will give a closed form formula for the Rohlin invariant of $\Sigma(a_1, \dots, a_n)$ promised in Lecture 11. We first deal with the manifolds $\Sigma(p, q, r)$ and describe the representation space $\mathcal{R}(\Sigma(p, q, r))$. It turns out that all the points in this representation space contribute to the Casson invariant with negative sign. As we have proved in Lecture 15, the space $\mathcal{R}(\Sigma(p, q, r))$ is non-degenerate, therefore, $\lambda(\Sigma(p, q, r))$ equals minus one-half the cardinality of $\mathcal{R}(\Sigma(p, q, r))$. The latter integer has multiple interpretations in terms of the number of integral lattice points in certain tetrahedra, Dedekind sums, and Milnor fiber signatures. The general formula for $\lambda(\Sigma(a_1, \dots, a_n))$ follows from the one for $\lambda(\Sigma(p, q, r))$ by a splicing additivity argument, and we refer the reader for that to the paper Neumann–Wahl [123].

19.1 The space $\mathcal{R}\big(\Sigma(p, q, r)\big)$

Recall from Lecture 14 that the space $\mathcal{R}(\Sigma(p, q, r))$ is finite. Moreover, the conjugacy class of a representation α in $\mathcal{R}(\Sigma(p, q, r))$ is described uniquely by a triple (ℓ_1, ℓ_2, ℓ_3) of integers ℓ_i of the parities determined by the choice of Seifert invariants and $\alpha(h) = \pm 1$, and such that $0 < \ell_1 < p$, $0 < \ell_2 < q$, $0 < \ell_3 < r$, and

$$\left|\frac{\ell_1}{p} - \frac{\ell_2}{q}\right| < \frac{\ell_3}{r} < 1 - \left|1 - \frac{\ell_1}{p} - \frac{\ell_2}{q}\right|.$$

The latter inequality can be rewritten as

$$\max\left\{\frac{\ell_1}{p} - \frac{\ell_2}{q},\, -\frac{\ell_1}{p} + \frac{\ell_2}{q}\right\} < \frac{\ell_3}{r} < \min\left\{\frac{\ell_1}{p} + \frac{\ell_2}{q},\, 2 - \frac{\ell_1}{p} - \frac{\ell_2}{q}\right\}$$

or as the system of the following four inequalities

$$\begin{aligned} -\frac{\ell_1}{p} + \frac{\ell_2}{q} + \frac{\ell_3}{r} > 0, \quad \frac{\ell_1}{p} - \frac{\ell_2}{q} + \frac{\ell_3}{r} > 0, \\ \frac{\ell_1}{p} + \frac{\ell_2}{q} - \frac{\ell_3}{r} > 0, \quad \frac{\ell_1}{p} + \frac{\ell_2}{q} + \frac{\ell_3}{r} < 2. \end{aligned} \tag{19.1}$$

If we allowed integers ℓ_1, ℓ_2, ℓ_3 of arbitrary parities, the inequalities (19.1) would describe the integral lattice points in the tetrahedron $\Delta \subset \mathbb{R}^3$ with vertices $(0, 0, 0)$,

$(0,q,r)$, $(p,0,r)$, and $(p,q,0)$. Let us denote the cardinality of $T(p,q,r) = \Delta \cap \mathbb{Z}^3 \subset \mathbb{R}^3$ by $\tau(p,q,r)$.

Lemma 19.1. *The cardinality of* $\mathcal{R}(\Sigma(p,q,r))$ *equals* $\frac{1}{4}\tau(p,q,r)$.

Proof. The group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with generators s, t acts freely on $T(p,q,r)$ by the rule $s(x_1,x_2,x_3) = (x_1, q-x_2, r-x_3)$, $t(x_1,x_2,x_3) = (p-x_1, q-x_2, x_3)$. For any given triple of integers, $(x_1,x_2,x_3) \in T(p,q,r)$, its orbit consists of the four points,

$$\begin{aligned} (x_1,x_2,x_3), &\quad (x_1, q-x_2, r-x_3), \\ (p-x_1, q-x_2, x_3), &\quad (p-x_1, x_2, r-x_3). \end{aligned} \tag{19.2}$$

The condition that a triple of integers (x_1,x_2,x_3) defines a representation in $\mathcal{R}(\Sigma(p,q,r))$ fixes the parities of x_1, x_2, and x_3; exactly one out of four triples (19.2) has the right parities. □

Remark. The cardinality of $\mathcal{R}(\Sigma(p,q,r))$ also equals one quarter the number of integral lattice points in the tetrahedron in $\mathbb{R}^3$ with vertices $(p,0,0)$, $(0,q,0)$, $(0,0,r)$, and (p,q,r). These points are identified with the points in $T(p,q,r)$ via the map $(x_1,x_2,x_3) \mapsto (p-x_1, q-x_2, r-x_3)$.

There are many curious formulas expressing $\tau(p,q,r)$ in terms of p, q, and r. Here are some of them.

Lemma 19.2. *Let* τ_k *be the number of integral lattice points* (x_1,x_2,x_3) *such that* $0 < x_1 < p$, $0 < x_2 < q$, $0 < x_3 < r$, *and*

$$k - 1 < \frac{x_1}{p} + \frac{x_2}{q} + \frac{x_3}{r} < k.$$

Then $\tau(p,q,r) = -\tau_1 + \tau_2 - \tau_3$.

Proof. Let $\pi = (p-1)(q-1)(r-1)$ be the cardinality of the set $P = [(0,p) \times (0,q) \times (0,r)] \cap \mathbb{Z}^3 \subset \mathbb{R}^3$, and denote

$$T_k = \left\{ (x_1,x_2,x_3) \in P \;\middle|\; k - 1 < \frac{x_1}{p} + \frac{x_2}{q} + \frac{x_3}{r} < k \right\},$$

so that τ_k is the cardinality of T_k. Since we always have that

$$0 < \frac{x_1}{p} + \frac{x_2}{q} + \frac{x_3}{r} < 3,$$

the set P splits into the disjoint union $P = T_1 \cup T_2 \cup T_3$, see Figure 19.1. It should be observed at this point that

$$\frac{x_1}{p} + \frac{x_2}{q} + \frac{x_3}{r} = 1 \text{ or } 2$$

is impossible, since these equalities would imply

$$x_1 qr + x_2 pr + x_3 pq = 0 \bmod (pqr)$$

and then $x_1 = 0 \bmod p$, $x_2 = 0 \bmod q$, $x_3 = 0 \bmod r$, which is prohibited by $0 < x_1 < p$, $0 < x_2 < q$, and $0 < x_3 < r$.

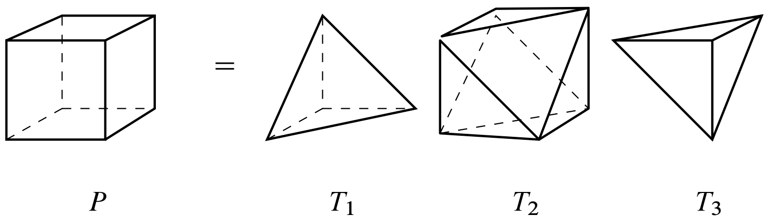


Figure 19.1

Thus $\pi = \tau_1 + \tau_2 + \tau_3$, and in addition the cardinality of $P \setminus T(p,q,r)$ is $\pi - \tau(p,q,r)$. The set $P \setminus T(p,q,r)$ can be split as shown in Figure 19.2, and then the obvious symmetries will imply that

$$\tau_1 = \tau_3 = \frac{1}{4}(\pi - \tau),$$

so that $-\tau_1 + \tau_2 - \tau_3 = \pi - 4\tau_1 = \tau(p,q,r)$. $\square$

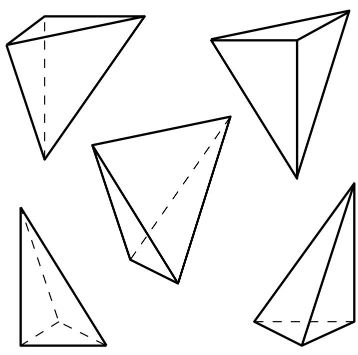


Figure 19.2

Another formula for $\tau(p,q,r)$ involves the so-called Dedekind sums. For any real number x let

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

where $[x]$ stands for the greatest integer less than or equal to x. For any pair of relatively prime integers a, b such that $a > 0$ the *Dedekind sum* $s(b, a)$ is given by

$$s(b,a) = \sum_{k=1}^{a-1} \left(\left(\frac{k}{a}\right)\right) \cdot \left(\left(\frac{kb}{a}\right)\right).$$

The next result follows from the fact that

$$\tau(p,q,r) = (p-1)(q-1)(r-1) - 4\tau_1,$$

see the proof of Lemma 19.2, and the calculation of τ_1 in Mordell [117], see also Rademacher–Grosswald [130], Theorem 5 of Section 3.E:

$$\begin{aligned}\tau(p,q,r) = 1 - \frac{1}{3pqr}&(1 - p^2q^2r^2 + p^2q^2 + q^2r^2 + p^2r^2)\\ &+ 4s(pq,r) + 4s(pr,q) + 4s(qr,p).\end{aligned}$$

It is proved in Rademacher–Grosswald [130], Section 2.C, formula (25), that

$$s(b,a) = \frac{1}{4a} \sum_{m=1}^{a-1} \frac{1+\eta^m}{1-\eta^m} \cdot \frac{1+\eta^{-bm}}{1-\eta^{-bm}},$$

where η is any primitive a-th root of unity. This implies the following formula for $\tau(p,q,r)$, which can also be found in Neumann–Wahl [123], Lemma 1.5 and Remark 1.7,

$$\begin{aligned}\tau(p,q,r) = 1 - \frac{1}{3pqr}&(1 - p^2q^2r^2 + p^2q^2 + q^2r^2 + p^2r^2)\\ &- d(r,pq) - d(q,pr) - d(p,qr).\end{aligned}$$

Here we define, for relatively prime integers a, b with $a > 0$,

$$\begin{aligned}d(a,b) &= \frac{1}{a} \sum_{\xi^a=1,\,\xi\neq 1} \frac{\xi+1}{\xi-1} \cdot \frac{\xi^b+1}{\xi^b-1}, \qquad (19.3)\\ &= -\frac{1}{a}\sum_{k=1}^{a-1} \cot\left(\frac{\pi k}{a}\right)\cot\left(\frac{\pi bk}{a}\right), \quad \text{by specifying } \xi = \exp(2\pi ik/a).\end{aligned}$$

19.2 Calculation of the Casson invariant

The next step in computing $\lambda(\Sigma(p,q,r))$ is to show that all representations in $\mathcal{R}(\Sigma(p,q,r))$ are counted with the same (negative) sign so that $\lambda(\Sigma(p,q,r))$ equals minus one-half the cardinality of $\mathcal{R}(\Sigma(p,q,r))$. We constructed a Heegaard splitting of $\Sigma(p,q,r)$ of genus 2 in Lecture 1. With the help of Kirby calculus, it can be described as follows. The manifold $\Sigma(p,q,r)$ can be thought of as the surgery on either of the links shown in Figure 19.3, where $b_1qr + b_2pr + b_3pq = 1$, compare with Figure 2.11. It is an easy exercise in Kirby calculus to show that surgeries on the links in Figure 19.3 produce homeomorphic manifolds.

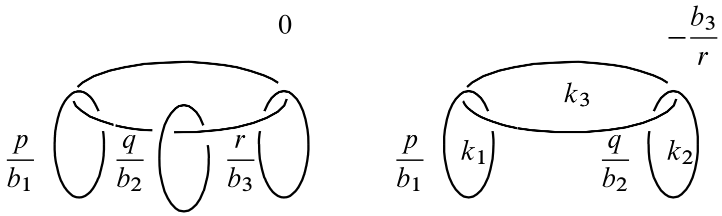


Figure 19.3

Let $N(k_1)$ be a tubular neighborhood of the knot k_1 in S^3. When we perform a (p/b_1)-surgery along k_1, we drill a tunnel along the knot k_1, which is the core of the solid torus $N(k_1)$, and fill it with a solid torus $S^1 \times D^2$ according to the (p/b_1)-rule. The so surgered solid torus $N(k_1)$ will be denoted by T_1; it is homeomorphic to a solid torus again. The solid tori T_2 and T_3 are defined similarly from surgeries on k_2 and k_3. We connect T_1 and T_2 by a solid tube as shown in Figure 19.4 to get a handlebody M_1 of genus 2 inside $\Sigma(p,q,r)$.

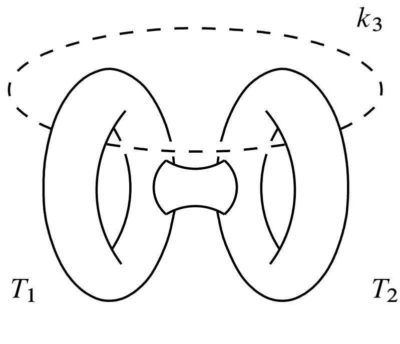


Figure 19.4

The complement of M_1 in $\Sigma(p,q,r)$ can be viewed as the solid torus T_3 with a solid tube attached as shown in Figure 19.5; the result is homeomorphic to a handlebody M_2 of genus 2.

Thus we get a Heegaard splitting $\Sigma(p,q,r) = M_1 \cup_F M_2$ of genus 2. The manifolds $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ have dimension 3 and the intersection $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ in $\mathcal{R}(F)$ is transversal, see Theorem 16.4 and the calculation of $H^1_\alpha(\pi_1\Sigma(p,q,r); \mathfrak{su}(2))$ in Lecture 15. Therefore, the intersection numbers of $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$

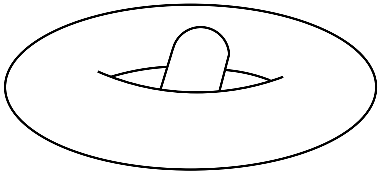


Figure 19.5

in $\mathcal{R}(F)$ can be computed directly from the Heegaard splitting description provided above. An elementary but rather technical calculation in local charts proves the following result, see Lescop [100].

Theorem 19.3. *Let $\Sigma(p,q,r) = M_1 \cup_F M_2$ be the Heegaard splitting of a Seifert homology sphere described above. Then for every $\alpha \in \mathcal{R}(M_1) \cap \mathcal{R}(M_2)$, the intersection number of $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ in $\mathcal{R}(F)$ is equal to -1.*

Another proof of this fact, which uses gauge theory, can be found in Fintushel–Stern [45]. Thus we arrive at the following formula,

$$\lambda(\Sigma(p,q,r)) = -\frac{1}{2}\,\#\mathcal{R}(\Sigma(p,q,r)),$$

where $\#\mathcal{R}(\Sigma(p,q,r))$ stands for the cardinality of the finite set $\mathcal{R}(\Sigma(p,q,r))$. Together with Lemma 19.1 and the calculation of $\tau(p,q,r)$ thereafter, this provides us with the following explicit formula for the Casson invariant of $\Sigma(p,q,r)$. The mod 2 reduction of this formula gives a closed form formula for the Rohlin invariant of $\Sigma(p,q,r)$, as promised in Section 11.2.

Theorem 19.4.

$$\begin{aligned}\lambda(\Sigma(p,q,r)) = -\frac{1}{8}\Big[1 - \frac{1}{3pqr}&(1 - p^2q^2r^2 + p^2q^2 + q^2r^2 + p^2r^2)\\ &- d(p,qr) - d(q,pr) - d(r,pq)\Big]\end{aligned}$$

where $d(a,b)$, for relatively prime integers a,b with $a > 0$, is given by the formula (19.3).

Remark. One can also use the surgery formula to compute the Casson invariant of certain Seifert homology spheres. Using the surgery description of a right-handed (p,q)-torus knot $k_{p,q}$ given in Exercise 4 of Lecture 7 one can deduce that the homology sphere $\Sigma(p,q,pq+1)$ can be obtained by (-1)-surgery on $k_{p,q}$. The surgery formula for the Casson invariant then implies that

$$\lambda(\Sigma(p,q,pq+1)) = -\frac{1}{2}\,\Delta''_{k_{p,q}}(1) = -(p^2-1)(q^2-1)/24,$$

see Exercise 1 of Lecture 9. For any positive integer m, the homology sphere $\Sigma(p,q,pqm \pm 1)$ is obtained by $(-1/m)$-surgery respectively on the right- or left-handed (p,q)-torus knot. The surgery formula implies again that $\lambda(\Sigma(p,q,pqm \pm 1)) = -m(p^2-1)(q^2-1)/24$.

Let $p,q,r \geq 2$ be integers, and $f(x,y,z) = x^p + y^q + z^r$ a polynomial in three complex variables. Let us consider the hypersurface $V(p,q,r)$ in $\mathbb{C}^3$ given by the equation $f(x,y,z) = 0$. This is a smooth non-compact manifold of real dimension 4 with the exception of the singular point $(0,0,0)$. Let $S^5 \subset \mathbb{C}^3$ be a small 5-dimensional sphere centered at $(0,0,0)$, then the intersection $V(p,q,r) \cap S^5$ is a smooth 3-manifold called the link of singularity of $V(p,q,r)$ at $(0,0,0)$. If the integers p, q, and r are pairwise relatively prime then $V(p,q,r) \cap S^5$ can be identified with the Seifert homology sphere $\Sigma(p,q,r)$, see Neumann–Raymond [122]. Note that a similar construction with the polynomial $x^p + y^q$ of two complex variables provided us with torus knots in Lecture 7. The circle action on $\Sigma(p,q,r)$ mentioned in the end of Lecture 1 is given by the formula $t \cdot (x,y,z) = (t^{qr}x, t^{pr}y, t^{pq}z)$, $t \in S^1$, with the singular fibers given by the equations $x = 0$, $y = 0$, and $z = 0$, respectively.

The formula
$$\varphi(x,y,z) = \frac{f(x,y,z)}{|f(x,y,z)|}$$
defines a map $\varphi\colon S^5 \setminus \Sigma(p,q,r) \to S^1$ onto the circle of unit complex numbers, which is a projection of a locally trivial bundle such that each fiber F is a smooth simply-connected 4-manifold, see Milnor [114]. A natural compactification of F is a smooth manifold $M(p,q,r)$ with boundary $\Sigma(p,q,r)$ called the *Milnor fiber*, compare with Lecture 8.

Theorem 19.5 (Fintushel–Stern formula, see [45]).
$$\lambda(\Sigma(p,q,r)) = \frac{1}{8}\,\operatorname{sign} M(p,q,r).$$

Proof. As we already know, $\lambda(\Sigma(p,q,r))$ equals minus one-half the cardinality of $\mathcal{R}(\Sigma(p,q,r))$, hence minus one-eighth of $-\tau_1 + \tau_2 - \tau_3$, see Lemma 19.2. The latter quantity is equal to the signature of the Milnor fiber, see Brieskorn [22]. □

Results similar to Theorem 19.4 and Theorem 19.5 also hold for Seifert homology spheres $\Sigma(a_1,\dots,a_n)$ with arbitrary n, and for more general links of singularities, see Neumann–Wahl [123].

19.3 Exercises

1. Calculate the Casson invariant of the homology spheres shown in Figure 12.9.
2. Calculate the Casson invariant of $\Sigma(3,4,5+12k)$ for all integer k.

Conclusion

Since its introduction in the 1980s, the Casson invariant has been generalized in various directions. It has turned out to be relevant to many recent developments, in particular, the application of ideas from the physics of gauge theories to the study of manifolds in 3 and 4 dimensions. In the notes below we review some of these developments without attempting an exhaustive account.

Gauge theory. Throughout the text we briefly described various applications of gauge theory. We did not advance very far in this direction because we did not want to require from the reader an extensive knowledge of differential geometry and elliptic theory. There are many books on differential geometry, for example, Kobayashi–Nomizu [90] or Warner [158]. Perhaps the book most relevant to gauge theory applications is Nicolaescu [126]. By the elliptic theory we mean the theory of (pseudo) differential elliptic operators, see e.g. first two chapters of Shubin [148], and the Index Theorem, see Atiyah–Bott–Patodi [6] for the closed manifold case and Atiyah–Patodi–Singer [7, 8, 9] for manifolds with boundary. These are the original papers which still provide an excellent treatment of the subject. Other sources include Berline–Getzler–Vergne [14] and Gilkey [59].

Gauge theory itself can be learned from Freed–Uhlenbeck [50], Donaldson–Kronheimer [39], and the Seiberg–Witten gauge theory from Morgan [118].

Instanton Floer homology. A gauge theoretic meaning of Casson's λ-invariant was discovered by Taubes [152] and Floer [47], who interpreted λ as the Euler characteristic of a homology theory now commonly known as the instanton Floer homology. More precisely, to any oriented integral homology 3-sphere Σ, Floer associated eight Abelian groups $I_n(\Sigma)$, $n = 0, \dots, 7$, so that

$$\lambda(\Sigma) = \frac{1}{2} \sum_{n=0}^{7} (-1)^n \operatorname{rank} I_n(\Sigma). \qquad (*)$$

The groups $I_*(\Sigma)$ are functorial in that, for an oriented 4-cobordism W between homology spheres Σ_0 and Σ_1 there exists a homomorphism $W_*\colon I_*(\Sigma_0) \to I_*(\Sigma_1)$, of non-zero degree, in general. The definition of $I_*(\Sigma)$ and W_* makes essential use of gauge theory on Σ and W. In short, this is an infinite-dimensional analogue of the Morse theory, see Milnor [113]. For a quick introduction to Floer homology see Braam [20]. More substantial treatment can be found in Donaldson [36].

In Floer [48] and Braam–Donaldson [21] the Floer homology was extended to include some 3-manifolds other than integral homology 3-spheres, and cobordisms between them; in particular, all 3-manifolds having the integral homology of $S^1 \times S^2$. The surgery formula for the Casson invariant was then refined in terms of the Floer exact triangle as follows.

Let $k \subset \Sigma$ be a knot in a homology sphere Σ. We introduce two other manifolds: the homology sphere Σ' obtained from Σ by (-1)-surgery on k, and the manifold $\overline{K}$, a homology $S^1 \times S^2$, obtained by 0-surgery on k. The natural cobordisms $X,\ Y$ and Z arising as the traces of the surgeries induce homomorphisms in Floer homology. These homomorphisms can be included in the following exact triangle (a long exact sequence) of total degree -1

$$\begin{tikzcd}
& I_*(\overline{K}) \arrow[dl, "Y_*"'] & \\
I_*(\Sigma) \arrow[rr, "Z_*"] & & I_*(\Sigma') \arrow[ul, "X_*"']
\end{tikzcd}$$

Here, the homomorphisms Z_* and X_* are of degree 0, while the connecting homomorphism Y_* has degree -1. Moreover, the Euler characteristic of $I_*(\overline{K})$ is expressed in terms of the Alexander polynomial as

$$\Delta''_{k\subset\Sigma}(1) = \sum_{n=0}^{7} (-1)^n \operatorname{rank} I_n(\overline{K}),$$

so Casson's surgery formula follows from the exactness of the triangle.

Floer homology is generally difficult to compute. An algorithm for computing the Floer homology $I_*(\Sigma(a_1, \dots, a_n))$ of Seifert homology spheres is described in Fintushel–Stern [45]. There also exists a closed form formula for $I_*(\Sigma(a_1, \dots, a_n))$, see Saveliev [140]. For some other computations of I_* see Fukaya [53], Kirk–Klassen–Ruberman [88], Klassen [89], Li [102], Saveliev [142], Stipsicz–Szabó [151], etc.

Another application of the Floer homology is related to the Donaldson polynomials. Recall from Lecture 5 that a Donaldson polynomial for a smooth closed 4-manifold (satisfying certain technical conditions) is a polynomial $\mathbf{D}_M\colon H^2(M, \mathbb{R}) \to \mathbb{R}$ on the second cohomology of M. For a manifold W with boundary $\partial W = \Sigma$ a homology sphere, the Donaldson polynomials take their values in $I_*(\Sigma)$,

$$\mathbf{D}_W\colon H^2(W, \mathbb{R}) \to I_*(\Sigma).$$

If a closed manifold M is split as $M = W_1 \cup_\Sigma W_2$ along a homology sphere Σ, then $\mathbf{D}_M$ is obtained by "gluing" $\mathbf{D}_{W_1}$ and $\mathbf{D}_{W_2}$ with the help of some natural pairing on $I_*(\Sigma)$, see Donaldson [36] for an accurate treatment.

Casson-type invariants in dimension four have been studied by Ruberman and Saveliev, see survey [139]. A variant of instanton Floer homology is the instanton knot Floer homology of Kronheimer and Mrowka [92]; this theory is closely related to Khovanov homology and other knot invariants.

Seiberg–Witten and Heegaard Floer homology. These are two Floer-type homology theories of 3-manifolds and links which appeared after the instanton Floer homology and whose relation to the Casson invariant is not as direct. The former theory is based on Seiberg–Witten monopoles, and its exhaustive treatment can be found in Kronheimer–Mrowka [93]. The latter theory relies on pseudo-holomorphic discs in certain symplectic manifolds, and we recommend Ozsváth–Szabó [127] for an introduction.

Casson–Lin invariant. An analogue for knots of Casson's original construction of the λ-invariant was worked out by Xiao-Song Lin in [103]. Let k be a knot in S^3, and $S^3 \setminus k$ its complement. Lin counts irreducible SU(2)-representations of $\pi_1(S^3 \setminus k)$ such that all meridians of $k \subset S^3$ are represented by trace-zero matrices in SU(2). The resulting integer $h(k)$ is an invariant of the knot k; it is usually referred to as the Casson–Lin invariant. In fact, Lin showed that $h(k)$ is not a new invariant – up to a constant, it equals the knot k signature. It should be mentioned that the Casson–Lin invariant of a knot is different from the Casson invariant of a knot described in Lecture 17, and that the the instanton knot Floer homology of Kronheimer and Mrowka [92] is closely related to representations $\pi_1(S^3 \setminus k) \to \mathrm{SU}(2)$ with trace-zero meridians.

The Casson–Lin invariant was generalized in Herald [72] and Heusener–Kroll [73] by counting representations with a fixed (but not necessarily zero) trace of the meridians. Refining the Casson–Lin invariant and its generalizations into a Floer homology theory is still ongoing.

Lin's construction was also extended to links of two components in Harper–Saveliev [69] by using the so called projective SU(2)-representations; the resulting invariant was shown to equal the linking number between the components of the link. The paper [70] constructs an instanton Floer homology for two-component links in homology spheres whose Euler characteristic is the aforementioned linking number.

Equivariant Casson invariant. The Casson–Lin invariant and its generalizations are closely related to equivariant Casson theory on manifolds with cyclic group actions. To obtain an equivariant version of the Casson invariant, one counts only irreducible representations preserved by the group action induced on the SU(2)-character variety; see Collin–Saveliev [30] and Ruberman–Saveliev [138].

Casson invariant for other Lie groups. Casson type invariants for SU(n) with $n \geq 3$ were developed in Boden–Herald [17] and Cappell–Lee–Miller [28], and for some other Lie groups in Curtis [33].

Casson invariant for general 3-manifolds. In the late 80s, the Casson invariant with all its properties was extended to homology lens spaces (3-manifolds with the integral homology of a lens space) by Boyer–Lines [19], and to all rational homology 3-spheres by Walker [156]. Later, the Casson invariant was extended to all closed oriented 3-manifolds by Lescop [99] using a combinatorial formula. The new invariant becomes simpler as the first Betti number $b_1 = \operatorname{rank} H_1(M;\mathbb{Q})$ of the manifold M increases, vanishing for all manifolds with $b_1 > 3$.

Combinatorial definition of the Casson invariant. It turns out that one can define the Casson invariant of a homology sphere Σ in purely combinatorial terms starting with a surgery presentation of Σ. It is not easy to do: since Σ has many different surgery presentations, one needs to ensure that whatever combinatorial formula one comes up with gives the same answer for all of them. Of course, any two surgery presentations of Σ are related by a sequence of Kirby moves, hence all one needs to check is that the answer does not change under these moves; the latter can be done with the help of Casson's surgery formula. The problem with this approach is that the Kirby moves can take us out of the class of integral homology 3-spheres, which is the natural domain of the Casson invariant. Walker [156] and Lescop [99] have succeeded in solving this problem by extending the Casson invariant to a larger class of 3-manifolds. Habiro [67] found a solution which does not require ever leaving the class of integral homology 3-spheres. His solution relies on Hoste's formula [77] for the Casson invariant of a homology sphere obtained by surgery on a framed link $\mathcal{L}$ as in Lemma 12.2, and on a new move called band slide which (unlike Kirby moves) preserves the linking matrix of $\mathcal{L}$.

Finite type invariants. The Casson invariant belongs to a large class of so-called finite type invariants developed by V. Vassiliev. The papers Birman [16], Bar-Natan [12] and Kontsevich [91] may serve as a good introduction to this theory; see also [97, 120, 154].

Bibliography

[1] S. Akbulut, R. Kirby, *Mazur manifolds*, Michigan Math. J. **26** (1979), 259–284.

[2] S. Akbulut, J. McCarthy, *Casson's invariant for oriented homology* 3*-spheres: an exposition*, Princeton University Press, Princeton 1990.

[3] J. W. Alexander, *On the subdivision of* 3*-space by a polyhedron*, Proc. Nat. Acad. Sci. USA **10** (1924), 6–8.

[4] J. W. Alexander, *The combinatorial theory of complexes*, Ann. Math. (2), **31** (1930), 292–320.

[5] C. Arf, *Untersuchungen über quadratische Formen in Körpern der Charakteristik* 2, J. Reine Angew. Math. **183** (1941), 148–167.

[6] M. Atiyah, R. Bott, V. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.

[7] M. Atiyah, V. Patodi, I. Singer, *Spectral asymmetry and Riemannian geometry.* I. Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[8] M. Atiyah, V. Patodi, I. Singer, *Spectral asymmetry and Riemannian geometry.* II. Math. Proc. Cambridge Philos. Soc. **78** (1975), 405–432.

[9] M. Atiyah, V. Patodi, I. Singer, *Spectral asymmetry and Riemannian geometry.* III. Math. Proc. Cambridge Philos. Soc. **79** (1976), 71–99.

[10] R. Baer, *Kurventypen auf Flächen*, J. Reine Angew. Math. **156** (1927), 231-246.

[11] R. Baer, *Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen*, J. Reine Angew. Math. **159** (1928), 101–111.

[12] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[13] S. Bauer, C. Okonek, *The algebraic geometry of representation spaces associated to Seifert fibered homology* 3*-spheres*, Math. Ann. **286** (1990), 45–76.

[14] N. Berline, E. Getzler, M. Vergne, *Heat kernels and Dirac operators*. Springer-Verlag, Berlin 2004

[15] R. H. Bing, *An alternative proof that* 3*-manifolds can be triangulated*, Ann. Math. (2) **69** (1959), 37–65.

[16] J. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. **28** (1993), 253–287.

[17] H. Boden, C. Herald, *The* SU(3) *Casson invariant for integral homology* 3*-spheres*, J. Diff. Geom. **50** (1998), 147–206.

[18] R. Bott, L. Tu, *Differential forms in algebraic topology*, Springer-Verlag, Berlin 1982.

[19] S. Boyer, D. Lines, *Surgery formulae for Casson's invariant and extensions to homology lens spaces*, J. Reine Angew. Math. **405** (1990), 181–220.

[20] P. Braam, *Floer homology groups for homology three-spheres*, Adv. Math. **88** (1991), 131–144.

[21] P. Braam, S. Donaldson, *Floer's work on instanton homology, knots and surgery*. The Floer memorial volume, Progr. Math. **133**, 195–256, Birkhäuser, Basel 1995.

[22] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14.

[23] T. Bröcker, K. Jänich, *Introduction to differential topology*, Cambridge University Press, Cambridge–New York 1982.

[24] W. Browder, *Surgery on simply-connected manifolds*, Springer-Verlag, Berlin 1972.

[25] K. Brown, *Cohomology of groups*, Springer-Verlag, Berlin 1982.

[26] G. Burde, H. Zieschang, *Knots*, de Gruyter Stud. Math. **5**, Walter de Gruyter, Berlin–New York 1985.

[27] S. Cairns, *The manifold smoothing problem*, Bull. Amer. Math. Soc. **67** (1961), 237–238.

[28] S. Cappell, R. Lee, E. Miller, *A symplectic geometry approach to generalized Casson's invariants of* 3*-manifolds*, Bull. Amer. Math. Soc. **22** (1990), 269–275.

[29] A. Casson, *Three lectures*, Spring 1985.

[30] O. Collin, N. Saveliev, *Equivariant Casson invariants via gauge theory*, J. Reine Angew. Math. **541** (2001), 143–169.

[31] P. Conner, E. Floyd, *Differentiable periodic maps*. Ergeb. Math. Grenzgeb. (N. F.) **33**, Springer-Verlag, Berlin–Göttingen–Heidelberg 1964.

[32] R. Crowell, R. Fox, *Introduction to Knot Theory*. Graduate Texts in Math. **57**, Springer-Verlag, Berlin–Heidelberg–New York 1977.

[33] C. Curtis, *Generalized Casson invariants for SO(3),U(2), Spin(4), and SO(4)*, Trans. Amer. Math. Soc. **343** (1994), 49–86.

[34] M. Dehn, *Die Gruppe der Abbildungen*, Acta Math. **69** (1938), 135–206.

[35] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.

[36] S. K. Donaldson, *Floer homology groups in Yang–Mills theory*. Cambridge University Press, Cambridge 2002.

[37] S. K. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), 257–315.

[38] S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and* 4*-manifold topology*, J. Differential Geom. **26** (1987), 397–428.

[39] S. Donaldson, P. Kronheimer, *Geometry of four-manifolds*, Oxford University Press, Oxford 1990.

[40] N. Duchon, *Involutions of plumbed manifolds*, Ph.D. Thesis, University of Maryland, College Park 1982.

[41] D. Eisenbud, W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*. Annals Math. Studies **110**, Princeton Univ. Press, Princeton 1985.

[42] D. B. A. Epstein, *Curves on* 2*-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.

[43] J. Etnyre, *Lectures on open book decompositions and contact structures*. Floer homology, gauge theory, and low-dimensional topology, 103–141, Clay Math. Proc. **5**, Amer. Math. Soc., Providence 2006.

[44] R. Fintushel, R. Stern, *A μ-invariant one homology* 3*-sphere that bounds an orientable rational ball*, Contemp. Math. **35** (1984), 265–268.

[45] R. Fintushel, R. Stern, *Instanton homology of Seifert fibred homology three spheres*, Proc. London Math. Soc. **61** (1990), 109–137.

[46] R. Fintushel, R. Stern, *Pseudofree orbifolds*, Ann. Math. **122** (1985), 335–364.

[47] A. Floer, *An instanton-invariant for* 3*-manifolds*, Comm. Math. Phys. **118** (1988), 215–240.

[48] A. Floer, *Instanton homology and Dehn surgery*. The Floer Memorial Volume, Progr. Math. **133**, 77–98, Birkhäuser, Basel 1995.

[49] A. Fomenko, S. Matveev, *Algorithmic and computer methods in three-dimensional topology* (in Russian). Moscow University Press, Moscow 1991; *Algorithmic and computer methods for three-manifolds*, Kluwer Academic Publishers, Dordrecht–Boston–London 1997.

[50] D. Freed, K. Uhlenbeck, *Instantons and four-manifolds*, Springer-Verlag, Berlin 1984.

[51] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453.

[52] M. Freedman, R. Kirby, *A geometric proof of Rochlin's theorem*, Proc. Symp. Pure Math. **32** (1978), 85–97.

[53] K. Fukaya, *Floer homology of connected sum of homology* 3*-spheres*, Topology **35** (1996), 89–136.

[54] Y. Fukumoto, M. Furuta, *Homology* 3*-spheres bounding acyclic* 4*-manifolds*, Math. Res. Lett. **7** (2000), 757–766.

[55] M. Furuta, *Homology cobordism group of homology* 3*-spheres*, Invent. Math. **100** (1990), 339–355.

[56] M. Furuta, *Monopole equation and the* $11/8$ *conjecture*, Math. Res. Lett. **8** (2001), 279–291.

[57] M. Furuta, B. Steer, *Seifert fibred homology* 3*-spheres and the Yang-Mills equations on Riemann surfaces with marked points*, Adv. Math. **96** (1992), 38–102.

[58] D. Galewski, R. Stern, *Classification of simplicial triangulations of topological manifolds*, Ann. Math. **111** (1980), 1–34.

[59] P. Gilkey, *Invariance theory, the heat equation, and the Atiyah–Singer index theorem*, Publish or Perish, Berkeley 1984.

[60] W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984), 200–225.

[61] R. Gompf, A. Stipsicz, 4-*manifolds and Kirby calculus*. American Math. Society, Providence 1999.

[62] C. Gordon, *Knots, homology spheres, and contractible* 4-*manifolds*, Topology **14** (1975), 151–172.

[63] V. Guillemin, A. Pollack, *Differential topology*, AMS Chelsea Publishing, Providence 2010.

[64] L. Guillou, A. Marin, *À la recherche de la topologie perdue*, Birkhäuser, Boston–Basel–Stuttgart 1986.

[65] L. Guillou, A. Marin, *Notes sur l'invariant de Casson des sphères d'homologie de dimension trois*, Enseign. Math. (2) **38** (1992), 233–290.

[66] N. Habegger, *Une variété de dimension* 4 *avec forme d'intersection paire et signature* -8, Comment. Math. Helv. **57** (1982), 22–24.

[67] K. Habiro, *Refined Kirby calculus for integral homology spheres*, Geom. Topol. **10** (2006), 1285–1317.

[68] J. Harer, A. Kas, R. Kirby, *Handlebody decompositions of complex surfaces*, Mem. Amer. Math. Soc. **62**, 1986.

[69] E. Harper, N. Saveliev, *A Casson-Lin type invariant for links*, Pacific J. Math. **248** (2010), 139–154.

[70] E. Harper, N. Saveliev, *Instanton Floer homology for two-component links*, J. Knot Theory Ramifications (to appear)

[71] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[72] C. Herald, *Flat connections, the Alexander invariant, and Casson's invariant*, Comm. Anal. Geom. **5** (1997), 93–120.

[73] M. Heusener, J. Kroll, *Deforming abelian* $SU(2)$*-representations of knot groups*. Comment. Math. Helv. **73** (1998), 480–498.

[74] P. Hilton, S. Wylie, *Homology theory: An introduction to algebraic topology*. Cambridge University Press, New York 1960.

[75] M. Hirsch, *Obstruction theories for smoothing manifolds and maps*, Bull. Amer. Math. Soc. **69** (1963), 352–356.

[76] F. Hirzebruch, W. Neumann, S. Koch, *Differentiable manifolds and quadratic forms*, Marcel Dekker, New York 1971.

[77] J. Hoste, *A formula for Casson's invariant*, Trans. Amer. Math. Soc. **297** (1986), 547–562.

[78] J. Igusa, *On a property of commutators in the unitary group*, Mem. Coll. Sci. Kyoto Univ., Ser A **26** (1950), 45–59.

[79] S. Kaplan, *Constructing framed* 4*-manifolds with given almost framed boundaries*, Trans. Amer. Math. Soc. **254** (1979), 237–263.

[80] L. Kauffman, *On knots*, Princeton University Press, Princeton 1987.

[81] M. Kervaire, J. Milnor, *On* 2*-spheres in* 4*-manifolds*, Proc. Nat. Acad. Sci. USA **47** (1961), 1651–1657.

[82] R. Kirby, *A calculus for framed links in* S^3, Invent. Math. **45** (1978), 36–56.

[83] R. Kirby, *Problems in low dimensional topology*, Proc. Symp. in Pure Math. **32** (1978), 273–312.

[84] R. Kirby, *The topology of* 4*-manifolds*, Lecture Notes in Math. **1374**, Springer-Verlag, Berlin 1989.

[85] R. Kirby, M. Scharlemann, *Eight faces of the Poincaré homology* 3*-sphere*, in: Geometric topology (Proc. Georgia Topology Conference, Athens, GA, 1977), 113–146, Academic Press, New York–London 1979.

[86] R. Kirby, L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton 1977.

[87] P. Kirk, E. Klassen, *Representation spaces of Seifert fibered homology spheres*, Topology **30** (1991), 77–95.

[88] P. Kirk, E. Klassen, D. Ruberman, *Splitting the spectral flow and the Alexander matrix*, Comment. Math. Helv. **69** (1994), 375–416.

[89] E. Klassen, *Representations in* SU(2) *of the fundamental groups of the Whitehead link and of doubled knots*, Forum Math. **5** (1993), 93–109.

[90] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*. John Wiley, New York, 1996.

[91] M. Kontsevich, *Vassiliev's knot invariants*, Gelfand Seminar, 137–150, Adv. Soviet Math. **16**, Amer. Math. Soc., Providence 1993.

[92] P. Kronheimer, T. Mrowka, *Instanton Floer homology and the Alexander polynomial*, Algebr. Geom. Topol. **10** (2010), 1715–1738.

[93] P. Kronheimer, T. Mrowka, *Monopoles and three-manifolds*. Cambridge University Press, Cambridge 2007.

[94] P. Kronheimer, T. Mrowka, *Recurrence relations and asymptotics for four-manifold invariants*, Bull. Amer. Math. Soc. **30** (1994), 215–221.

[95] P. Kronheimer, T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters **1** (1994), 797–808.

[96] P. Kronheimer, T. Mrowka, *Witten's conjecture and property P*, Geom. Topol. **8** (2004), 295–310.

[97] T. Le, H. Murakami, J. Murakami, T. Ohtsuki, *A three-manifold invariant derived from the universal Vassiliev–Kontsevich invariant*, Proc. Japan. Acad. Ser. A Math. Sci. **71** (1995), 125–127.

[98] K. B. Lee, F. Raymond, *Seifert fiberings*. Amer. Math. Soc., Providence 2010.

[99] C. Lescop, *Global surgery formula for the Casson–Walker invariant*, Ann. of Math. Stud. 140, Princeton Univ. Press, Princeton 1996.

[100] C. Lescop, *Un calcul élémentaire de l'invariant de Casson des sphères d'homologie entière fibrées de Seifert à trois fibres exceptionnelles*, Enseign. Math. (2) **38** (1992), 276–289.

[101] J. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

[102] W. Li, *Floer homology for connected sums of homology* 3*-spheres*, J. Diff. Geom. **40** (1994), 129–154.

[103] X.-S. Lin, *A knot invariant via representation spaces*, J. Diff. Geom. **35** (1992), 337–357.

[104] W. B. R. Lickorish, *A representation of orientable combinatorial* 3*-manifolds*. Ann. Math. (2) **76** (1962), 531–540.

[105] C. Livingston, *Knot theory*, Carus Math. Monographs **24**, Math. Assoc. America, Washington 1993.

[106] W. Massey, *Algebraic Topology: An Introduction*. Springer-Verlag, Berlin 1967.

[107] Y. Matsumoto, *An elementary proof of Rochlin's signature theorem and its extension by Guillou and Marin*, in: À la recherche de la topologie perdue, 119–139, Birkhäuser, 1986.

[108] T. Matumoto, *Triangulation of manifolds*, Proc. Symp. in Pure Math. **32** (1978), 3–6.

[109] B. Mazur, *A note on some contractible* 4*-manifolds*, Ann. Math. (2) **73** (1961), 221–228.

[110] D. McCullough, *Three–manifolds*, Notes from the graduate class at the University of Oklahoma, Spring 1994.

[111] J. Milnor, *Construction of universal bundles*. II, Ann. Math. **63** (1956), 430-436.

[112] J. Milnor, *Differentiable manifolds which are homotopy spheres* (mimeographed notes), Princeton 1959.

[113] J. Milnor, *Morse Theory*, Ann. of Math. Stud. **51**, Princeton Univ. Press, Princeton 1963.

[114] J. Milnor, *Singular points of complex hypersurfaces*, Princeton Univ. Press, Princeton 1968.

[115] J. Milnor, J. Husemoller, *Symmetric bilinear forms*, Ergeb. Math. Grenzgeb. **73**, Springer Verlag, Berlin–New York 1973.

[116] E. Moise, *Affine structures in* 3*-manifolds.* V. *The triangulation theorem and Hauptvermutung*, Ann. Math. (2) **56** (1952), 96–114.

[117] L. J. Mordell, *Lattice points in a tetrahedron and generalized Dedekind sums*, J. Indian Math. Soc. **15** (1951), 41–46.

[118] J. Morgan, *The Seiberg–Witten equations and applications to the topology of smooth four-manifolds*, Princeton University Press, Princeton 1996.

[119] J. Morgan, G. Tian, *Ricci flow and the Poincaré conjecture*. American Math. Society, Providence 2007.

[120] H. Murakami, *Quantum* SO(3)*-invariants dominate the* SU(2)*-invariant of Casson and Walker*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 237–249.

[121] W. Neumann *An invariant of plumbed homology spheres*. Topology Symposium, Siegen 1979, pp. 125–144, Lecture Notes in Math., **788**, Springer, Berlin 1980.

[122] W. Neumann, F. Raymond, *Seifert manifolds, plumbing, μ-invariant and orientation reversing maps*, Lecture Notes in Math. **664**, 163–196, Springer-Verlag, 1978.

[123] W. Neumann, J. Wahl, *Casson invariant of links of singularities*, Comment. Math. Helv. **65** (1990), 58–78.

[124] P. Newstead, *Topological properties of some spaces of stable bundles*, Topology **6** (1967), 241–262.

[125] P. Newstead, *Characteristic classes of stable bundles of rank* 2 *over an algebraic curve*, Trans. Amer. Math. Soc. **169** (1972), 337–345.

[126] L. Nicolaescu, *Lectures on the geometry of manifolds*, World Scientific Publishing, Hackensack 1996.

[127] P. Ozsváth, Z. Szabó, *An introduction to Heegaard Floer homology*. Floer homology, gauge theory, and low-dimensional topology, 3–27, Clay Math. Proc. **5**, Amer. Math. Soc., Providence 2006.

[128] F. Pham, *Formules de Picard-Lefschetz généralisées et ramification des intégrales*, Bull. Soc. Math. France **93** (1965), 333–367.

[129] F. Quinn, *Ends of maps.* III. *Dimensions* 4 *and* 5, J. Differential Geom. **17** (1982), 503–521.

[130] H. Rademacher, E. Grosswald, *Dedekind sums*, Carus Math. Monographs **16**, Math. Assoc. America, Washington 1972.

[131] F. Raymond, *Classification of the actions of the circle on* 3*-manifolds*, Trans. Amer. Math. Soc. **131** (1968), 51–78.

[132] K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 102–109.

[133] K. Reidemeister, *Zur dreidimensionalen Topologie*, Abh. Math. Sem. Univ. Hamburg **9** (1933), 189-194.

[134] R. Robertello, *An invariant of knot cobordism*, Comm. Pure Appl. Math. **18** (1965), 543–555.

[135] V. A. Rohlin, *New results in the theory of four-dimensional manifolds* (in Russian), USSR Acad. Sci. Doklady **84** (1952), 221–224.

[136] V. A. Rohlin, *Proof of Gudkov's hypothesis*, Functional Anal. Appl. **6** (1972), 136–138.

[137] D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley 1976.

[138] C. Rourke, B. Sanderson, *Introduction to piecewise-linear topology*, Springer-Verlag, Berlin–Heidelberg–New York 1972.

[139] D. Ruberman, N. Saveliev, *Casson-type invariants in dimension four*. Geometry and topology of manifolds, 281–306, Amer. Math. Soc., Providence 2005.

[140] N. Saveliev, *Floer homology of Brieskorn homology spheres*, J. Diff. Geom. **53** (1999), 15–87.

[141] N. Saveliev, *Fukumoto–Furuta invariants of plumbed homology* 3*-spheres*, Pacific J. Math. **205** (2002), 465–490.

[142] N. Saveliev, *Invariants for homology* 3*-spheres*. Springer-Verlag, Berlin 2002.

[143] N. Saveliev, *Representation spaces of Seifert fibered homology spheres*, Topology Appl. **126** (2002), 49–61.

[144] M. Schwarz, *Morse homology*, Birkhäuser Verlag, Basel 1993.

[145] J. P. Serre, *A course in Arithmetic*, Springer-Verlag, New York–Heidelberg–Berlin 1973.

[146] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, Berlin 1977.

[147] K. Shoda, *Einige Sätze über Matrizen*, Japan. J. Math. **13** (1936–37), 361–365.

[148] M. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin 1987.

[149] J. Singer, *Three dimensional manifolds and their Heegaard diagrams*, Trans. Amer. Math. Soc. **35** (1933), 88–111.

[150] E. Spanier, *Algebraic topology*, McGraw-Hill, New York 1966.

[151] A. Stipsicz, Z. Szabó, *Floer homology groups of certain algebraic links*. Low-dimensional topology, Conf. Proc. Lecture Notes Geom. Topology 3, 173–185, Internat. Press, Cambridge, 1994.

[152] C. Taubes, *Casson's invariant and gauge theory*, J. Differential Geom. **31** (1990), 547–599.

[153] C. Taubes, *Gauge theory on asymptotically periodic* 4*-manifolds*, J. Diff. Geom. **25** (1987), 363–430.

[154] V. Turaev, *Quantum invariants of knots and* 3*-manifolds*, de Gruyter Stud. Math. **18**, Walter de Gruyter, Berlin–New York 1994.

[155] C. T. C. Wall, *On simply-connected* 4*-manifolds*, J. London Math. Soc. **39** (1964), 141–149.

[156] K. Walker, *An extension of Casson's invariant*, Ann. of Math. Stud. 126, Princeton Univ. Press, Princeton 1992.

[157] A. D. Wallace, *Modifications and cobounding manifolds*. Canad. J. Math. **12** (1960), 503–528.

[158] F. Warner, *Foundations of differentiable manifolds and Lie groups*. Scott, Foresman and Co., Glenview–London 1971.

[159] J. H. C. Whitehead, *On incidence matrices, nuclei and homotopy type*, Ann. Math. **42** (1941), 1197–1239.

[160] J. H. C. Whitehead, *On simply connected* 4*-dimensional polyhedra*, Comment. Math. Helv. **22** (1949), 48–92.

[161] E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

Index