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KNOTS

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Preface to the First Edition

The phenomenon of a knot is a fundamental experience in our perception of three dimensional space. What is special about knots is that they represent a truly intrinsic and essential quality of 3-space accessible to intuitive understanding. No arbitrariness like the choice of a metric mars the nature of a knot – a trefoil knot will be universally recognizable wherever the basic geometric conditions of our world exist. (One is tempted to propose it as an emblem of our universe.)

There is no doubt that knots hold an important – if not crucial – position in the theory of 3-dimensional manifolds. As a subject for a mathematical textbook they serve a double purpose. They are excellent introductory material to geometric and algebraic topology, helping to understand problems and to recognize obstructions in this field. On the other hand they present themselves as ready and copious test material for the application of various concepts and theorems in topology.

The first nine chapters (excepting the sixth) treat standard material of classical knot theory. The remaining chapters are devoted to more or less special topics depending on the interest and taste of the authors and what they believed to be essential and alive. The subjects might, of course, have been selected quite differently from the abundant wealth of publications in knot theory during the last decades.

We have stuck throughout this book mainly to traditional topics of classical knot theory. Links have been included where they come in naturally. Higher-dimensional knot theory has been completely left out – even where it has a bearing on 3-dimensional knots such as slice knots. The theme of surgery has been rather neglected – excepting Chapter 15. Wild knots and Algebraic knots are merely mentioned.

This book may be read by students with a basic knowledge in algebraic topology – at least the first four chapters will present no serious difficulties to them. As the book proceeds certain fundamental results on 3-manifolds are used – such as the Papakyriakopoulos theorems. The theorems are stated in Appendix B and references are given where proofs may be found. There seemed to be no point in adding another presentation of these things. The reader who is not familiar with these theorems is, however, well advised to interrupt the reading to study them. At some places the theory of surfaces is needed – several results of Nielsen are applied. Proofs of these may be read in Zieschang, Vogt and Coldewey [382], but taking them for granted will not seriously impair the understanding of this book. Whenever possible we have given complete and selfcontained proofs at the most elementary level possible. To do this we occasionally refrained from applying a general theorem but gave a simpler proof for the special case in hand.

There are, of course, many pertinent and interesting facts in knot theory – especially in its recent development – that were definitely beyond the scope of such a textbook. To be complete – even in a special field such as knots – is impossible today and was not aimed at. We tried to keep up with important contributions in our bibliography.

There are not many textbooks on knots. Reidemeister's "Knotentheorie" was conceived for a different purpose and level; Neuwirth's book "Knot Groups" and Hillman's monograph "Alexander Ideals of Links" have a more specialised and algebraic interest in mind. In writing this book we had, however, to take into consideration Rolfsen's remarkable book "Knots and Links". We tried to avoid overlappings in the contents and the manner of presentation. In particular, we thought it futile to produce another set of drawings of knots and links up to ten crossings – or even more. They can – in perfect beauty – be viewed in Rolfsen's book. Knots with less than ten crossings have been added in Appendix D as a minimum of ready illustrative material. The tables of knot invariants have also been devised in a way which offers at least something new. Figures are plentiful because we think them necessary and hope them to be helpful.

Finally we wish to express our gratitude to Colin Maclachlan who read the manuscript and expurgated it from the grosser lapsus linguae (this sentence was composed without his supervision). We are indebted to U. Lüdicke and G. Wenzel who wrote the computer programs and carried out the computations of a major part of the knot invariants listed in the tables. We are grateful to U. Dederek-Breuer who wrote the program for filing and sorting the bibliography. We also want to thank Mrs. A. Huck and Mrs. M. Schwarz for patiently typing, re-typing, correcting and re-correcting abominable manuscripts.

Frankfurt (Main)/Bochum, Summer 1985

Gerhard Burde
Heiner Zieschang

Preface to the Second Edition

The text has been revised, some mistakes have been eliminated and Chapter 15 has been brought up to date, especially taking into account the Gordon–Luecke Theorem on knot complements, although we have not included a proof. Chapter 16 was added, presenting an introduction to the HOMFLY polynomial, and including a self-contained account of the fundamental facts about Hecke algebras. A proof of Markov's theorem was added in Chapter 10 on braids. We also tried to bring the bibliography up to date. In view of the vast amount of recent and pertinent contributions even approximate completeness was out of the question.

We have decided not to deal with Vassiliev invariants, quantum group invariants and hyperbolic structures on knot complements, since a thorough treatment of these topics would go far beyond the space at our disposal. Adequate introductory surveys on these topics are available elsewhere.

Since the first edition of this book in 1985, a series of books on knots and related topics have appeared. We mention especially: Kauffman [187, 188], Kawauchi [190], Murasugi [265], Turaev [358], Vassiliev [361].

Our heartfelt thanks go to Marlene Schwarz and Jörg Stümke for producing the LaTeX-file and to Richard Weidmann for proof-reading. We also thank the editors for their patience and pleasant cooperation, and Irene Zimmermann for her careful work on the final layout.

Frankfurt (Main)/Bochum, 2002

Gerhard Burde
Heiner Zieschang

Preface to the Third Edition

This text is a completely revised version of the second edition. Some sections have been rewritten and some material has been reorganized. The references have been brought up to date, taking into account recent results. Chapter 16 was added, presenting a proof of Schubert's result [319] that the bridge number minus one is additive.

We have not tried to update the nearly comprehensive bibliography of the second edition, since even approximate completeness of a bibliography would have been out of the question. Therefore, we have decided to include only references which are cited in the text. More references are now easily accessible by “MathSciNet”, “zbMATH” or through the multiple search engines on the World Wide Web. We have also reduced the amount of tables in the appendix. Only one table is included containing certain important knot invariants. Much more tables of knot and link invariants are made available by J.C. Cha and C. Livingston [65, 66]. A huge variety of knot (up to twelve crossing knots) and link (up to eleven crossing links) invariants are listed. Another very useful tool is *The Knot Atlas* [15] run by D. Bar-Natan and S. Morrison.

Since the second edition of this book in 2002, a series of books on knots and related topics have appeared. We mention especially: S. Chmutov, and S. Duzhin and J. Mostovoy [67], C. Kassel and V. Turaev [186], W. Menasco and M. Thistlethwaite [236], M. Morishita [250].

We would like to thank Leila Ben Abdelghani, Jérôme Dubois, Paul Kirk, Jennifer Schultens and Richard Weidmann for helpful conversations and some proof-reading. We also thank the editors for their professional and pleasant cooperation.

Frankfurt (Main)/Clermont-Ferrand, 2013

Gerhard Burde
Michael Heusener

Contents

Chapter 1

Knots and isotopies

The chapter contains an elementary foundation of knot theory. Sections 1.A and 1.B define and discuss knots and their equivalence classes, and Section 1.C deals with the regular projections of knots. Section 1.D contains a short review of E. Pannwitz' *quadrisecants* theorem [280] and the Fáry–Milnor theorem [96, 237] intended to further an intuitive geometric understanding for the global quality of knotting in a simple closed curve in 3-space.

1.A Knots

A knot, in the language of mathematics, is an embedding of a circle S^1 into Euclidean 3-space, $\mathbb{R}^3$, or the 3-sphere, S^3. More generally, embeddings of S^k into S^{n+k} have been studied in "higher dimensional knot theory", but this book will be strictly concerned with "classical" knots $S^1 \subset S^3$. (On occasion we digress to consider "links" or "knots of multiplicity $\mu > 1$" which are embeddings of a disjoint union of 1-spheres S_i^1, $1 \leq i \leq \mu$, into S^3 or $\mathbb{R}^3$.)

A single embedding $i\colon S^1 \to S^3$, is, of course, of little interest, and does not give rise to fruitful questions. The essential problem with a knot is whether it can be disentangled by certain moves that can be carried out in 3-space without damaging the knot. The topological object will therefore rather be a class of embeddings which are related by these moves (isotopic embeddings).

There are different notions of isotopy, and we start by investigating which one of them is best suited to our purposes.

Let X and Y be Hausdorff spaces. A mapping $f\colon X \to Y$ is called an *embedding* if $f\colon X \to f(X) \subset Y$ is a homeomorphism.

1.1 Definition (Isotopy). Two embeddings, $f_0, f_1\colon X \to Y$ are *isotopic* if there is an embedding

$$F\colon X \times I \to Y \times I$$

such that $F(x,t) = (f(x,t),t)$, $x \in X$, $t \in I = [0,1]$, with $f(x,0) = f_0(x)$, $f(x,1) = f_1(x)$.

F is called a *level-preserving isotopy* connecting f_0 and f_1.

We frequently use the notation $f_t(x) = f(x,t)$ which automatically takes care of the boundary conditions. The general notion of isotopy as defined above is not good as far as knots are concerned. Any two embeddings $S^1 \to S^3$ can be shown to be isotopic although they evidently are different with regard to their knottedness. The idea of the proof is sufficiently illustrated by the sequence of pictures in Figure 1.1. Any area where knotting occurs can be contracted continuously to a point.

1.2 Definition (Ambient isotopy). Two embeddings $f_0, f_1: X \to Y$ are *ambient isotopic* if there is a level preserving isotopy

$$H: Y \times I \to Y \times I,\ H(y,t) = (h_t(y), t),$$

with $f_1 = h_1 f_0$ and $h_0 = \mathrm{id}_Y$. The mapping H is called an *ambient isotopy*.

An ambient isotopy defines an isotopy F connecting f_0 and f_1 by $F(x,t) = (h_t f_0(x), t)$. The difference between the two definitions is the following: An isotopy moves the set $f_0(X)$ continuously over to $f_1(X)$ in Y, but takes no heed of the neighboring points of Y outside $f_1(X)$. An ambient isotopy requires Y to move continuously along with $f_t(X)$, such as a liquid filling Y will do if an object $(f_t(X))$ is transported through it.

The restriction

$$h_1|: (Y - f_0(X)) \to (Y - f_1(X))$$

of the homeomorphism $h_1: Y \to Y$ is itself a homeomorphism of the complements of $f_0(X)$ resp. $f_1(X)$ in Y, if f_0 and f_1 are ambient isotopic. This is not necessarily true in the case of mere isotopy and marks the crucial difference. We shall see in Chapter 3 that the complement of the *trefoil* knot – see the first picture of Figure 1.1 – and the complement of the unknotted circle, the *trivial knot* or *unknot*, are not homeomorphic.

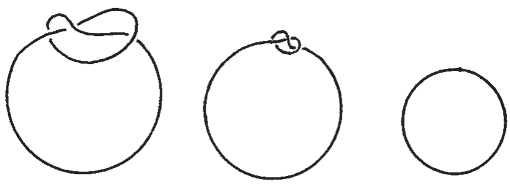


Figure 1.1. An isotopy of embeddings.

We are going to narrow further the scope of our interest. Topological embeddings $S^1 \to S^3$ may have a bizarre appearance as Figure 1.2 shows. There is an infinite sequence of similar meshes converging to a limit point L at which this knot is called *wild*. This example of a wild knot, invented by Fox, indeed has remarkable properties which show that at such a point of wildness something extraordinary may happen. E. Artin and R.H. Fox proved in [114] that the complement of the curve depicted in Figure 1.2 is different from that of a trivial knot. Nevertheless, the knot can obviously be unravelled from the right – at least finitely many stitches can.

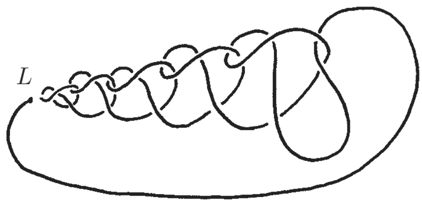


Figure 1.2. A wild knot.

1.3 Definition (Tame knots). A knot is called *tame* if it is ambient isotopic to a piecewise linear embedding in $\mathbb{R}^3$ resp. S^3. A knot is *wild* if it is not tame.

If a knot is tame, any connected proper part α of it is ambient isotopic to a straight segment and therefore the complement $S^3 - \alpha$ is simply connected. Any proper subarc of the knot of Figure 1.2 which contains the limit point L can be shown [114] to have a non-simply connected complement. From this it appears reasonable to call L a point at which the knot is wild. Wild knots are no exceptions – quite the contrary. J. Milnor proved: *"Most" knots are wild* [239]. One can even show that almost all knots are wild at every point [42]. Henceforth we shall be concerned only with tame knots. *Consequently we shall work always in the p.l.-category (p.l. = piecewise linear).* All spaces will be compact polyhedra with a finite simplicial structure, unless otherwise stated. Maps will be piecewise linear. We repeat Definitions 1.1 and 1.2 in an adjusted version:

1.4 Definition (p.l.-isotopy and p.l.-ambient isotopy). Let X, Y be polyhedra and $f_0, f_1 \colon X \to Y$ p.l.-embeddings, f_0 and f_1 are p.l.-*isotopic* if there is a level-preserving p.l.-embedding

$$F \colon X \times I \to Y \times I,\ F(x,t) = (f_t(x), t),\ 0 \le t \le 1.$$

f_0 and f_1 are p.l.-*ambient isotopic* if there is a level-preserving p.l.-isotopy

$$H \colon Y \times I \to Y \times I,\ H(y,t) = (h_t(y), t),$$

with $f_1 = h_1 f_0$ and $h_0 = \mathrm{id}_Y$.

In future we shall usually omit the prefix "p.l.".

We are now in a position to give the fundamental definition of a knot as a class of embeddings $S^1 \to S^3$ resp. $S^1 \to \mathbb{R}^3$:

1.5 Definition (Equivalence). Two (p.l.)-knots are *equivalent* if they are (p.l.)-ambient isotopic.

We use our terminology loosely in connection with this definition. A knot $\mathfrak{k}$ may be a representative of a class of equivalent knots or the class itself. If the knots $\mathfrak{k}$ and $\mathfrak{k}'$ are equivalent, we shall say they are the same, $\mathfrak{k} = \mathfrak{k}'$ and use the sign of equality.

The main topic of classical knot theory is the classification of knots with regard to equivalence.

Dropping "p.l." defines, of course, a broader field and a more general classification problem. The definition of tame knots (Definition 1.3) suggests applying the Definition 1.2 of "topological" ambient isotopy to define a topological equivalence for this class of knots. At first view one might think that the restriction to the p.l.-category will introduce equivalence classes of a different kind. We shall take up the subject in Chapter 3 to show that this is not true. In fact, two tame knots are topologically equivalent if and only if the p.l.-representatives of their topological classes are p.l.-equivalent, see Corollary 3.17.

1.B Equivalence of knots

We defined equivalence of knots by ambient isotopy in the last section. There are different notions of equivalence to be found in the literature which we propose to investigate and compare in this section.

There is a way of defining equivalence of knots which takes advantage of special properties of the embedding space, $\mathbb{R}^3$ or S^3. Fisher [103] proved that an orientation preserving homeomorphism $h\colon S^3 \to S^3$ is isotopic to the identity. (A homeomorphism with this property is called a *deformation*.) We are interested in a special case of Fisher's theorem. We shall prove it with the help of the following theorem which is well known and will not be proved here. See [242, Chapter 17] for a modern account.

1.6 Theorem (Alexander–Schoenflies [7]). *Let $i\colon S^2 \to S^3$ be a (p.l.)-embedding. Then*

$$S^3 = B_1 \cup B_2,\ i(S^2) = B_1 \cap B_2 = \partial B_1 = \partial B_2,$$

where B_i, $i = 1, 2$, is a combinatorial 3-ball (B_i is p.l.–homeomorphic to a 3-simplex).

The theorem corresponds to the Jordan curve theorem in dimension two where it holds for topological embeddings. In dimension three it is not true in this generality, Alexander gave a famous example, the *Alexander horned sphere* [6]. Brown proved in [46] that a for a locally bi-collared $(n-1)$-sphere S in $\mathbb{R}^n$ there exists an autohomeomorphism of $\mathbb{R}^n$ which maps S onto the unit sphere.

We start by proving

1.7 Proposition (Alexander–Tietze). *Any (p.l.)-homeomorphism f of a (combinatorial) n-ball B keeping the boundary fixed is isotopic to the identity by a (p.l.)-ambient isotopy keeping the boundary fixed.*

Proof. Define for $(x,t) \in \partial(B \times I)$

$$H(x,t) = \begin{cases} x & \text{for } t = 0 \\ x & \text{for } x \in \partial B \\ f(x) & \text{for } t = 1. \end{cases}$$

Every point $(x,t) \in B \times I$, $t > 0$, lies on a straight segment in $B \times I$ joining a fixed interior point P of $B \times 0$ and a variable point X on $\partial(B \times I)$. Extend $H|\partial(B \times I)$ linearly on these segments to obtain a (p.l.) level-preserving mapping $H\colon B \times I \to B \times I$, in fact, the desired ambient isotopy (*Alexander trick*, [5], Figure 1.3). □

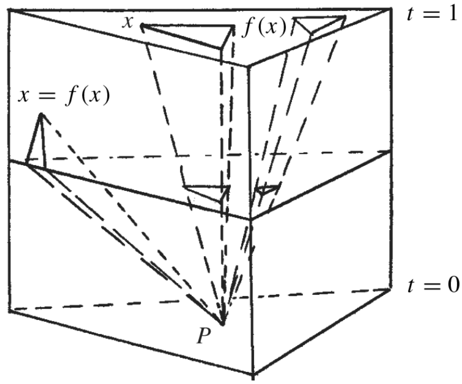


Figure 1.3. The Alexander trick.

We are now ready to prove the following proposition:

1.8 Proposition. *Let $\mathfrak{k}_0$ and $\mathfrak{k}_1$ be p.l.-knots in S^3. The following assertions are equivalent:*

(1) *There is an orientation preserving homeomorphism $f\colon S^3 \to S^3$ which carries $\mathfrak{k}_0$ onto $\mathfrak{k}_1$, $f \circ \mathfrak{k}_0 = \mathfrak{k}_1$.*

(2) *$\mathfrak{k}_0$ and $\mathfrak{k}_1$ are equivalent (ambient isotopic).*

Proof. (1) $\Longrightarrow$ (2): We begin by showing that there is an ambient isotopy $H(x,t) = (h_t(x), t)$ of S^3 such that $h_1 f$ leaves fixed a 3-simplex $[P_0, P_1, P_2, P_3]$. If $f\colon S^3 \to S^3$ has a fixed point, choose it as P_0; if not, let P_0 be any interior point of a 3-simplex $[s^3]$ of S^3. There is an ambient isotopy of S^3 which leaves $\overline{S^3 - [s^3]}$ fixed and carries P_0 over to any other interior point of $[s^3]$. If $[s^3]$ and $[s'^3]$ have a common 2-face, one can easily construct an ambient isotopy moving an interior point P_0 of $[s^3]$ to an interior point P_0' of $[s'^3]$ which is the identity outside $[s^3] \cup [s'^3]$ (Figure 1.4).

So there is an ambient isotopy H^0 with $h_1^0 f(P_0) = P_0$, since any two 3-simplices can be connected by a chain of adjoining ones. Next we choose a point $P_1 \neq P_0$ in

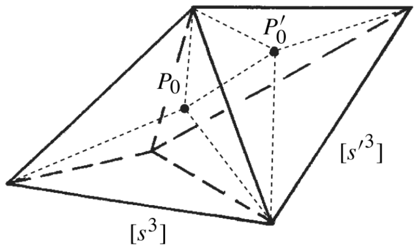


Figure 1.4

the simplex star of P_0, and by similar arguments we construct an ambient isotopy H^1 with $h_1^1 h_1^0 f$ leaving fixed the 1-simplex $[P_0, P_1]$. A further step leads to $h_1^2 h_1^1 h_1^0 f$ with a fixed 2-simplex $[P_0, P_1, P_2]$. At this juncture the assumption comes in that f is required to preserve the orientation. A point $P_3 \notin [P_0, P_1, P_2]$, but in the star of $[P_0, P_1, P_2]$ will be mapped by $h_1^2 h_1^1 h_1^0 f$ onto a point P_3' in the same half-space with regard to the plane spanned by P_0, P_1, P_2. This ensures the existence of the final ambient isotopy H^3 such that $h_1^3 h_1^2 h_1^1 h_1^0 f$ leaves fixed $[P_0, P_1, P_2, P_3]$. The assertion follows from the fact that $H = H^3 H^2 H^1 H^0$ is an ambient isotopy, $H(x,t) = (h_t(x), t)$.

By Theorem 1.6 the complement of $[P_0, P_1, P_2, P_3]$ is a combinatorial 3-ball and by Theorem 1.7 there is an ambient isotopy which connects $h_1 f$ with the identity of S^3.

(2) $\Longrightarrow$ (1) follows from the definition of an ambient isotopy. □

K. Reidemeister [294] gave an elementary introduction into knot theory stressing the combinatorial aspect, which is also the underlying concept of his book "Knotentheorie" [296, 302, 303], the first monograph written in 1932 on the subject. He considered knots as simple closed polygonal curves rather then p.l.-maps. In this context, a *polygonal knot* $\mathfrak{k}$ is a simple polygonal closed curve i.e. the image of a p.l.-map $S^1 \to \mathbb{R}^3$. Polygonal knots might be oriented or not. If S^1 is oriented and if $i \colon S^1 \to S^3$ is a knot then the image $\mathfrak{k} = i(S^1)$ inherits an orientation *(oriented knot)*. The notion of equivalence has to be adjusted: *two oriented polygonal knots are equivalent* if there is an ambient isotopy connecting them which respects the orientation of the knots. Reidemeister introduced an isotopy by moves.

1.9 Definition (Δ-move). Let u be a straight segment of a polygonal knot $\mathfrak{k}$ in $\mathbb{R}^3$ (or S^3), and D a triangle in $\mathbb{R}^3$, $\partial D = u \cup v \cup w$, u, v, w 1-faces of D. If $D \cap \mathfrak{k} = u$, then $\mathfrak{k}' = (\mathfrak{k} - u) \cup v \cup w$ defines another polygonal knot. We say $\mathfrak{k}'$ *results from* $\mathfrak{k}$ *by a* Δ*-process or a* Δ*-move.* If $\mathfrak{k}$ is oriented, $\mathfrak{k}'$ has to carry the orientation induced by that of $\mathfrak{k} - u$. The inverse process is denoted by Δ^{-1} (see Figure 1.5).

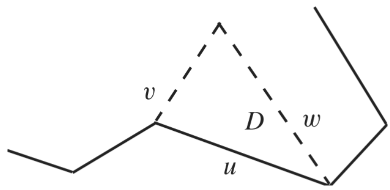


Figure 1.5. A Δ-move.

Remark: We allow Δ to degenerate as long as $\mathfrak{k}'$ remains simple. This means that Δ resp. Δ^{-1} may be a bisection resp. a reduction in dimension one.

1.10 Definition (Combinatorial equivalence). Two polygonal knots are *combinatorially equivalent* or *isotopic by moves*, if there is a finite sequence of Δ- and Δ^{-1}-moves that transforms one knot to the other.

1.11 Proposition. *Let $\mathfrak{k}_0$ and $\mathfrak{k}_1$ be two polygonal knots in $\mathbb{R}^3$. The following assertions are equivalent:*

(1) *There is an orientation preserving homeomorphism $f\colon S^3 \to S^3$ which carries $\mathfrak{k}_0$ onto $\mathfrak{k}_1$, $f(\mathfrak{k}_0) = \mathfrak{k}_1$.*

(2) *$\mathfrak{k}_0$ and $\mathfrak{k}_1$ are equivalent (ambient isotopic).*

(3) *$\mathfrak{k}_0$ and $\mathfrak{k}_1$ are combinatorially equivalent (isotopic by moves).*

Proof. The equivalence (1) $\iff$ (2) follows from the proof of Proposition 1.8.

Next we prove (1) $\implies$ (3): Let $h\colon S^3 \to S^3$ be an orientation preserving homeomorphism and $\mathfrak{k}_1 = h(\mathfrak{k}_0)$. The preceding argument (see the proof of Proposition 1.8) shows that there is another orientation preserving homeomorphism $g\colon S^3 \to S^3$, $g(\mathfrak{k}_0) = \mathfrak{k}_0$, such that hg leaves fixed some 3-simplex $[s^3]$ which will have to be chosen outside a regular neighborhood of $\mathfrak{k}_0$ and $\mathfrak{k}_1$. For an interior point P of $[s^3]$ consider $S^3 - \{P\}$ as Euclidean 3-space $\mathbb{R}^3$. There is a translation τ of $\mathbb{R}^3$, which moves $\mathfrak{k}_0$ into $[s^3] - \{P\}$. It is easy to prove that $\mathfrak{k}_0$ and $\tau(\mathfrak{k}_0)$ are isotopic by moves (see Figure 1.6). We claim that $\mathfrak{k}_1 = hg(\mathfrak{k}_0)$ and $hg\tau(\mathfrak{k}_0) = \tau(\mathfrak{k}_0)$ are isotopic by moves also, which would complete the proof. Choose a subdivision of the triangulation of S^3 such that the triangles used in the isotopy by moves between $\mathfrak{k}_0$ and $\tau(\mathfrak{k}_0)$ form a subcomplex of S^3. There is an isotopy by moves $\mathfrak{k}_0 \to \tau(\mathfrak{k}_0)$ which is defined on the triangles of the subdivision. $hg\colon S^3 \to S^3$ maps the subcomplex onto another one carrying over the isotopy by moves (see Graeub [139, 140]).

(3) $\implies$ (1). It is not difficult to construct a homeomorphism of S^3 onto itself which realizes a $\Delta^{\pm 1}$-move and leaves fixed the rest of the knot. Choose a regular neighborhood U of the 2-simplex which defines the $\Delta^{\pm 1}$-move whose boundary meets the knot at two points. By linear extension one can obtain a homeomorphism producing the $\Delta^{\pm 1}$-move in U and leaving $S^3 - U$ fixed. $\square$

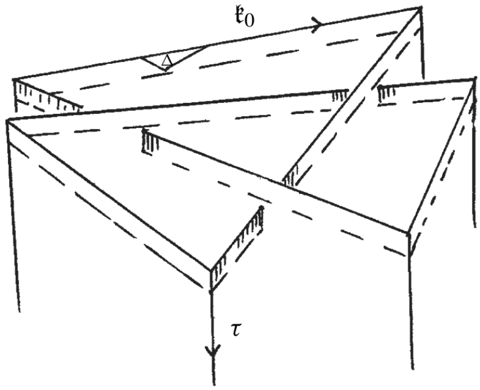


Figure 1.6. The knots $\mathfrak{k}$ and $\tau(\mathfrak{k})$ are isotopic by Δ-moves.

The image of a p.l.-embedding $i\colon S^1 \to S^3$ is a polygonal knot in S^3 and it follows from Proposition 1.11 that two p.l.-ambient isotopic p.l-embeddings give combinatorially equivalent polygonal knots. Let us investigate to what extent a polygonal knot determines a p.l.-embedding.

1.12 Lemma. *Let $i\colon S^1 \to S^3$ be a p.l.-embedding and denote by $\mathfrak{k} = i(S^1)$ the associated polygonal knot.*

If $j\colon \mathfrak{k} \to \mathfrak{k}$ is an orientation preserving p.l.-homeomorphism then there exists an orientation preserving homeomorphism $f\colon S^3 \to S^3$, supported by a regular neighborhood of $\mathfrak{k}$, such that the restriction of $f|\mathfrak{k}$ coincides with j.

Proof. Note first that every orientation preserving p.l.-homeomorphism $j\colon \mathfrak{k} \to \mathfrak{k}$ is p.l.-ambient isotopic to the identity. More precisely, j is a finite composition of homeomorphism which are supported by a closed arc of $\mathfrak{k}$. This can be proved by following the argument used in the first part of the proof of Proposition 1.8. Therefore it is sufficient to prove the lemma for a homeomorphism j supported by a closed arc $\alpha \subset \mathfrak{k}$.

Let $V = V(\alpha)$ be a regular neighborhood of $\alpha \subset S^3$ and define f to be the identity on the complement $S^3 - V$. Hence, $(V, \mathfrak{k} \cap V)$ is an unknotted ball pair and we can extend the identity on the boundary of the ball pair to a homeomorphism which agrees with j on $\mathfrak{k} \cap V$ (see Rourke-Sanderson [311, Prop. 4.4]). This extends f to V such $f|\mathfrak{k} = j$. □

1.13 Corollary. *Let S^1 be oriented and let $i_1, i_2\colon S^1 \to S^3$ be two p.l.-embeddings.*

The two oriented polygonal knots $\mathfrak{k}_1 = i_1(S^1)$ and $\mathfrak{k}_2 = i_2(S^1)$ are equivalent if and only if i_1 and i_2 are p.l.-ambient isotopic.

Proof. If there exists an orientation preserving homeomorphism $h\colon S^3 \to S^3$ such that $h|\mathfrak{k}_1\colon \mathfrak{k}_1 \to \mathfrak{k}_2$ preserves the orientation, then

$$j = i_2 \circ i_1^{-1} \circ (h|\mathfrak{k}_1)^{-1}\colon \mathfrak{k}_2 \to \mathfrak{k}_2$$

is orientation preserving. By Lemma 1.12 there exists an orientation preserving homeomorphism $f\colon S^3 \to S^3$ such that $f|\mathfrak{k}_2 = j$ and $(f \circ h) \circ i_1 = i_2$. Hence, Proposition 1.11 implies the assertion of the corollary. □

It follows that the isotopy by Δ-moves provides a means to formulate the knot problem on an elementary level. It is also useful as a method in proofs of invariance.

There will be a certain abuse of language in this book to avoid complicating the notation. A knot $\mathfrak{k}$ *will be an embedding, a class of embeddings, the image* $i(S^1) = \mathfrak{k}$ *(a simple polygonal closed curve), or a class of such curves.*

1.C Knot projections

Geometric description in 3-spaces is complicated. The data that determine a knot are usually given by a *projection of* $\mathfrak{k}$ *onto a plane* E (projection plane) in $\mathbb{R}^3$. (In this paragraph we prefer $\mathbb{R}^3$ with its Euclidean metric to S^3; a knot $\mathfrak{k}$ will always be thought of as a simple closed polygon in $\mathbb{R}^3$.) A point $P \in p(\mathfrak{k}) \subset E$ whose preimage $p^{-1}(P)$ under the projection $p\colon \mathbb{R}^3 \to E$ contains more than one point of $\mathfrak{k}$ is called a *multiple* point.

1.14 Definition (Regular projection). A projection p of a knot $\mathfrak{k}$ is called *regular* if

(1) there are only finitely many multiple points $\{P_i \mid 1 \le i \le n\}$, and all multiple points are *double points*, that is, $p^{-1}(P_i)$ contains two points of $\mathfrak{k}$,

(2) no vertex of $\mathfrak{k}$ is mapped onto a double point.

The minimal number of double points or *crossings* n in a regular projection of a knot is called the *order* of the knot. A regular projection avoids occurrences as depicted in Figure 1.7.

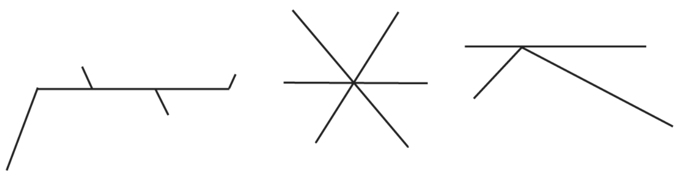


Figure 1.7. Non-regular projections.

There are sufficiently many regular projections.

1.15 Proposition. *The set of regular projections is open and dense in the space of all projections.*

Proof. Think of directed projections as points on a unit sphere $S^2 \subset \mathbb{R}^3$ with the induced topology. A standard argument (general position) shows that singular (non-regular) projections are represented on S^2 by a finite number of curves. (The reader is referred to Reidemeister [294], Crowell and Fox [80] or Burde [50] for a more detailed treatment.) □

The projection of a knot does not determine the knot, but if at every double point in a regular projection the overcrossing line is marked, the knot can be reconstructed from the projection (Figure 1.8).

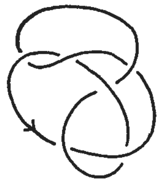


Figure 1.8

If the knot is oriented, the projection inherits the orientation. The projection of a knot with this additional information is called a *knot projection* or *knot diagram*. Two *knot diagrams* will be regarded as *equal* if they are isotopic in E as graphs, where the isotopy is required to respect overcrossing resp. undercrossing. Equivalent knots can be described by many different diagrams, but they are connected by simple operations.

1.16 Definition (Reidemeister moves). Two *knot diagrams* are called *equivalent* if they are connected by a finite sequence of Reidemeister moves Ω_i, $i = 1, 2, 3$ or their inverses Ω_i^{-1}. The moves are described in Figure 1.9.

The operations $\Omega_i^{\pm 1}$ effect local changes in the diagram. Evidently all these operations can be realized by ambient isotopies of the knot; equivalent diagrams therefore define equivalent knots. The converse is also true:

1.17 Proposition. *Two knots are equivalent if and only if all their diagrams are equivalent.*

Proof. The first step in the proof will be to verify that any two regular projections p_1, p_2 of the same simple closed polygon $\mathfrak{k}$ are connected by $\Omega_i^{\pm 1}$-moves. Let p_1, p_2 again be represented by points on S^2, and choose on S^2 a polygonal path s from p_1 to p_2 in general position with respect to the lines of singular projections on S^2. When such a line is crossed the diagram will be changed by an operation $\Omega_i^{\pm 1}$, the actual

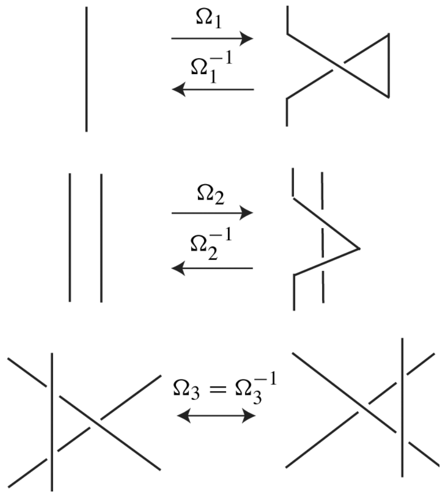


Figure 1.9. The Reidemeister moves.

type depending on the type of singularity, see Figure 1.7, corresponding to the line that is crossed.

It remains to show that for a fixed projection equivalent knots possess equivalent diagrams. According to Proposition 1.8 it suffices to show that a $\Delta^{\pm 1}$-move induces $\Omega_i^{\pm 1}$-operations on the projection. This again is easily verified (Figure 1.10). □

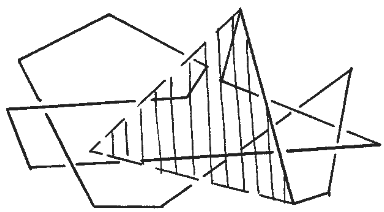


Figure 1.10

Proposition 1.17 allows an elementary approach to knot theory. It is possible to continue on this level and define invariants for diagrams with respect to equivalence. The so-called *Abbildungsmengen* are such invariants which were introduced by G. Burde in 1978 [50]. These invariants became well-known under the name *quandles* and are now systematically studied (see for example the article by R. Fenn and C. Rourke [99]).

One might be tempted to look for a finite algorithm to decide equivalence of diagrams by establishing an a priori bound for the number of crossings. Such a bound is not known, and a simple counterexample shows that it can at least not be the maximum of the crossings that occur in the diagrams to be compared. The diagram of Figure 1.11

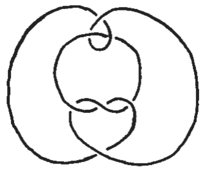


Figure 1.11

is that of a trivial knot, however, on the way to a simple closed projection, via moves $\Omega_i^{\pm 1}$ the number of crossings will increase. This follows from the fact that the diagram only allows operations Ω_1^{+1}, Ω_2^{+1} which increase the number of crossings. Additionally, Figure 1.11 demonstrates: The operations Ω_i, $i = 1, 2, 3$ are "independent" – one cannot dispense with any of them (Exercise E 1.5), see Trace [352].

1.D Global geometric properties

In this section we will discuss two theorems (without giving proofs) which connect the property of "knottedness" and "linking" with other geometric properties of the curves in $\mathbb{R}^3$. The first is the following theorem of E. Pannwitz [280]:

1.18 Theorem (E. Pannwitz). *If $\mathfrak{k}$ is a non-trivial knot in $\mathbb{R}^3$, then there is a straight line which meets $\mathfrak{k}$ in four points.*

If a link of two components $\mathfrak{k}_i$, $i = 1, 2$, is not splittable, then there is a straight line which meets $\mathfrak{k}_1$ and $\mathfrak{k}_2$ in two points A_1, B_1 resp. A_2, B_2 each with an ordering A_1, A_2, B_1, B_2 on the line. (Such a line is called a quadrisecant *of $\mathfrak{k}$).*

It is easy to see that the theorem does not hold for the trivial knot or a splittable linkage. (A link *is splittable or split* if it can be separated in $\mathbb{R}^3$ by a 2-sphere.)

What E. Pannwitz proved was actually something more general. For any knot $\mathfrak{k} \subset \mathbb{R}^3$ there is a singular disk $D \subset \mathbb{R}^3$ spanning $\mathfrak{k}$. For example, such a disk can be constructed by erecting a cone over a regular projection of $\mathfrak{k}$ (Figure 1.12). If $D \subset \mathbb{R}^3$ is immersed in general position, there will be a finite number of singular points on $\mathfrak{k}$ (boundary singularities).

1.19 Definition (Knottedness). The minimal number of boundary singularities of a disk spanning a knot $\mathfrak{k}$ is called the *knottedness k* of $\mathfrak{k}$.

1.20 Theorem (Pannwitz). *The knottedness k of a non-trivial knot is an even number. A knot of knottedness k possesses $\frac{k^2}{2}$ quadrisecants.* □

The proof of this theorem – which generalizes the first part of 1.18 – is achieved by cut-and-paste techniques as used in the proof of Dehn's Lemma. (See also J.-P. Otal [278] and G. Kuperberg [205].)

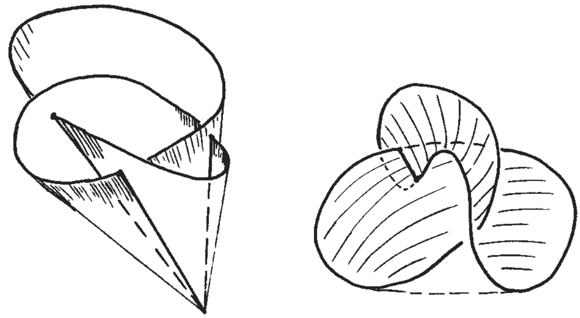


Figure 1.12

Figure 1.12 shows the trefoil spanned by a cone with 3 boundary singularities and by another disk with the minimal number of 2 boundary singularities. (The apex of the cone is not in general position, but a slight deformation will correct that.)

Another theorem about knotted curves is due to I. Fáry [96] and J. Milnor [237]. If $\mathfrak{k}$ is smooth ($\mathfrak{k} \in C^{(2)}$), the integral

$$\kappa(\mathfrak{k}) = \int_{\mathfrak{k}} |\mathfrak{x}''(s)| ds$$

is called the *total curvature* of $\mathfrak{k}$. (Here $s \mapsto \mathfrak{x}(s)$ describes $\mathfrak{k}\colon S^1 \to \mathbb{R}^3$ with s = arc length.) $\kappa(\mathfrak{k})$ is not an invariant of the knot type. Milnor generalized the notion of the total curvature so as to define it for arbitrary closed curves. In the case of a polygon this yields $\kappa(\mathfrak{k}) = \sum_{i=1}^{r} \alpha_i$, where the α_i are the angles of successive line segments (Figure 1.13).

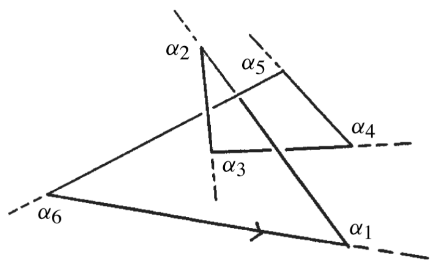


Figure 1.13

1.21 Theorem (Fáry–Milnor). *The total curvature $\kappa(\mathfrak{k})$ of a non-trivial knot $\mathfrak{k} \subset \mathbb{R}^3$ exceeds 4π.* □

We do not intend to copy Milnor's proof here. As an example, however, we give a realization of the trefoil in $\mathbb{R}^3$ with total curvature equal to $4\pi + \delta(\varepsilon)$, where $\delta(\varepsilon) > 0$ can be made arbitrarily small. This shows that the lower bound, 4π, is sharp.

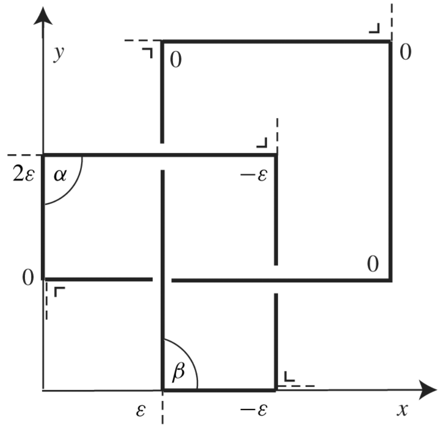


Figure 1.14

In Figure 1.14 a diagram of the trefoil is given in the (x, y)-plane, the symbol at each vertex denotes the z-coordinate of the respective point on $\mathfrak{k}$. Six of eight angles α_i are equal to $\frac{\pi}{2}$, two of them, α and β, are larger, but tend to $\frac{\pi}{2}$ as $\varepsilon \to 0$.

1.E History and sources

A systematic and scientific theory of knots developed only in the last century when combinatorial topology came under way. The first contributions M. Dehn [84, 85, 87] and J.W. Alexander [3, 8] excited quite an interest, and a remarkable amount of work in this field was done which was reflected in the first monograph on knots, K. Reidemeister's Knotentheorie [296, 302, 303]. The elementary approach to knots presented in this chapter comes from this source.

1.F Exercises

E 1.1. Let $\mathfrak{k}$ be a smooth oriented simple closed curve in $\mathbb{R}^2$, and let $-\mathfrak{k}$ denote the same curve with the opposite orientation. Show that $\mathfrak{k}$ and $-\mathfrak{k}$ are not isotopic in $\mathbb{R}^2$ whereas they are in $\mathbb{R}^3$.

E 1.2. Construct explicitly a p.l.-map of a complex K composed of two 3-simplices with a common 2-face onto itself which moves an interior point of one of the 3-simplices to an interior point of the other while keeping fixed the boundary ∂K of K (see Figure 1.4).

E 1.3. The suspension point P over a closed curve with n double points is called a *branch point of order* $n+1$. Show that there is an isotopy of $D^2 \to \mathbb{R}^3$ which transforms the suspension into a singular disk with n branch points of order two (see [281]).

E 1.4. Let $p(t)$, $0 \leq t \leq 1$, be a continuous family of projections of a fixed knot $\mathfrak{k} \subset \mathbb{R}^3$ onto $\mathbb{R}^2$, which are singular at finitely many isolated points $0 < t_1 < t_2 < \cdots < 1$. Discuss by which of the operations Ω_i the two regular projections, $p(t_k - \varepsilon)$ and $p(t_k + \varepsilon)$, $t_{k-1} < t_k - \varepsilon < t_k < t_k + \varepsilon < t_{k+1}$, are related according to the type of the singularity at t_k.

E 1.5. Prove that any projection obtained from a simple closed curve in $\mathbb{R}^3$ by using $\Omega_1^{\pm 1}$, $\Omega_2^{\pm 1}$ can also be obtained by using only Ω_1^{+1}, Ω_2^{+1}.

E 1.6. Let $p(\mathfrak{k})$ be a regular projection with n double points. By changing overcrossing arcs into undercrossing arcs at $k \leq \frac{n-1}{2}$ double points, $p(\mathfrak{k})$ can be transformed into a projection of the trivial knot.

Chapter 2

Geometric concepts

Some of the charm of knot theory arises from the fact that there is an intuitive geometric approach to it. We shall discuss in this chapter some standard constructions and presentations of knots and various geometric devices connected with them.

2.A Geometric properties of projections

Let $\mathfrak{k}$ be an oriented knot in oriented 3-space $\mathbb{R}^3$.

2.1 Definition (Symmetries). The knot obtained from $\mathfrak{k}$ by inverting its orientation is called the *inverted knot* and denoted by $-\mathfrak{k}$. The *mirror-image* of $\mathfrak{k}$ or *mirrored knot* is denoted by $\mathfrak{k}^*$, it is obtained by a reflection of $\mathfrak{k}$ in a plane.

A knot $\mathfrak{k}$ is called *invertible* if $\mathfrak{k} = -\mathfrak{k}$, and *amphicheiral* if $\mathfrak{k} = \mathfrak{k}^*$.

The trefoil is invertible; the rotation by π about the axis indicated in Figure 2.1 (a) is a symmetry which reverses the orientation of the knot. The same holds for the only knot with the minimal number 4 of crossings, the *four-knot* 4_1 (see Figure 2.1 (c)):

The trefoil was shown to be non-amphicheiral by M. Dehn in 1914 [85, 87]. The *right-handed* trefoil and its mirror-image, the *left-handed* trefoil, are displayed in Figure 2.1 (a) and Figure 2.1 (b) respectively. The four-knot is amphicheiral, this property is shown in Figure 2.2. Hence the four-knot 4_1 is both invertible and amphicheiral.

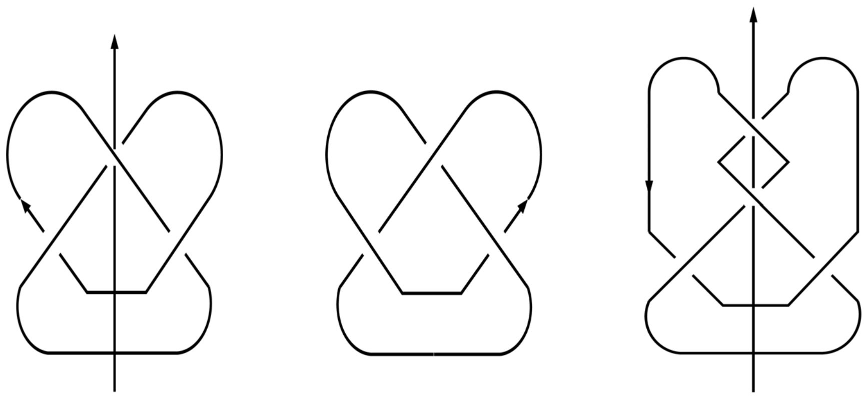


Figure 2.1 (a). The right-handed trefoil. **Figure 2.1 (b).** The left-handed trefoil. **Figure 2.1 (c).** The four-knot.

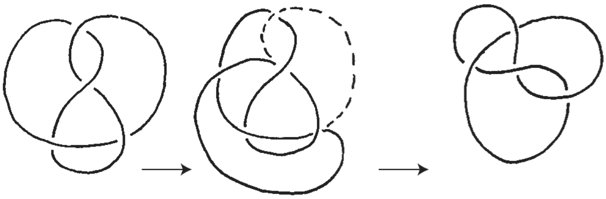


Figure 2.2. The four-knot is amphicheiral.

The existence of non-invertible knots was proved by H. Trotter [356]. That the knot 8_{17} is non-invertible was first proved by A. Kawauchi [189] and independently by F. Bonahon and L. Siebenmann in 1979 [37] using geometric methods. It is the least crossing non-invertible knot, the only one with 8 crossings. Moreover, the knot $\mathfrak{k} = 8_{17}$ is *negative amphicheiral* i.e. $\mathfrak{k} = -\mathfrak{k}^*$. By inspection, the rotation by π about the axis indicated in Figure 2.3 maps $\mathfrak{k}$ to $-\mathfrak{k}^*$.

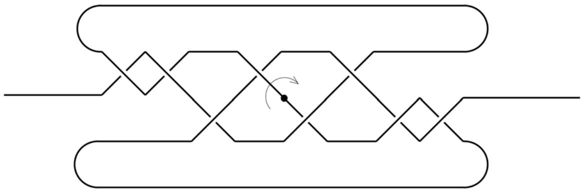


Figure 2.3. The knot 8_{17} is negative amphicheiral.

For this and more refined notions of symmetries in knot theory, see R. Hartley [150], A. Kawauchi [190, Chap. 10] and F. Bonahon and L. Siebenmann [37].

2.2 Definition (Alternating knot). A *knot projection* is called *alternating* if upper-crossings and undercrossings alternate while running along the knot. A *knot* is called *alternating* if it possesses an alternating projection; otherwise it is *non-alternating*.

The existence of non-alternating knots was first proved by C. Bankwitz [13], see Proposition 13.30.

Alternating projections are frequently printed in knot tables without marking under-crossings. It is an easy exercise to prove that any such projection can be furnished in exactly two ways with undercrossings to become alternating; the two possibilities belong to mirrored knots. Without indicating undercrossings, a closed plane curve does not hold much information about the knot whose projection it might be. Given such a curve there is always a trivial knot that projects onto it. To prove this assertion just choose a curve $\mathfrak{k}$ which ascends monotonically in $\mathbb{R}^3$ as one runs along the projection, and close it by a segment in the direction of the projection.

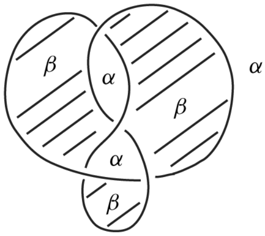


Figure 2.4. The chessboard coloring.

A finite set of closed plane curves defines a tessellation of the plane by simply connected *regions* bounded by arcs of the curves, and a single *infinite region*. (This can be avoided by substituting a 2-sphere for the plane.) The regions can be colored by two colors like a chessboard such that regions of the same color meet only at double points (Figure 2.4, E 2.2). The proof is easy. If the curve is simple, the fact is well known; if not, omit a simply closed partial curve s and color the regions by an induction hypothesis. Replace s and exchange the coloring for all points inside s.

2.3 Definition (Graph of a knot). Let a regular knot diagram be chessboard colored with colors α and β. Assign to every double point A of the projection an *index* $\theta(A) = \pm 1$ with respect to the coloring as defined by Figure 2.5. Denote by α_i, $1 \leq i \leq m$, the α-colored regions of a knot diagram. Define a graph Γ whose vertices P_i correspond to the α_i, and whose edges α_{ij}^k correspond to the double points $A^k \in \partial\alpha^i \cap \partial\alpha^j$, where α_{ij}^k joins P_i and P_j and carries the index $\theta(\alpha_{ij}^k) = \theta(A^k)$.

If β-regions are used instead of α-regions, a different graph is obtained from the regular projection. The Reidemeister moves Ω_i correspond to moves on graphs which can be used to define an equivalence of graphs (compare Definition 1.16 and Proposition 1.17). It is easy to prove (E 2.5) that the two graphs of a projection belonging to α- and β-regions are equivalent (see [376]). Another Exercise (E 2.3) shows that

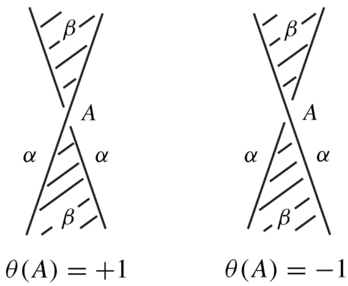


Figure 2.5. The index $\theta(A) = \pm 1$.

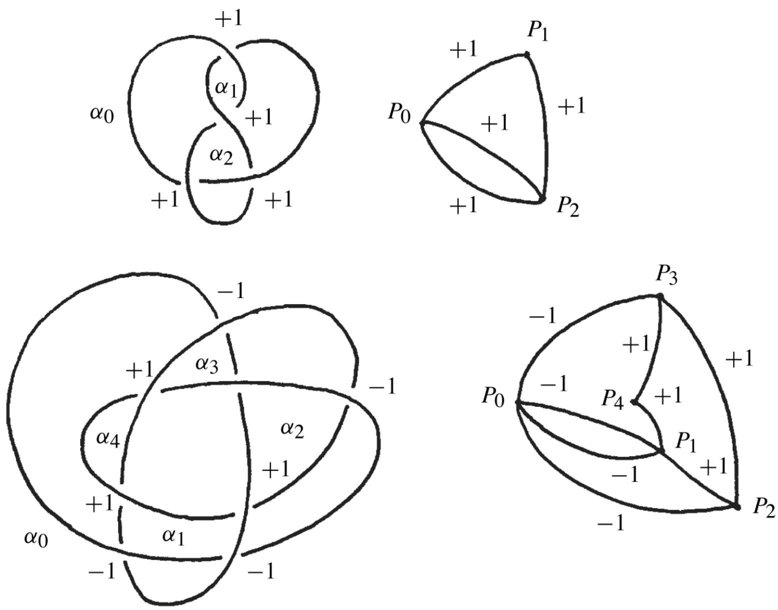


Figure 2.6

a projection is alternating if and only if the index function $\theta(A)$ on the double points is a constant (Figure 2.6).

Graphs of knots have been repeatedly employed in knot theory [12], [76], [196]. We shall take up the subject again in Chapter 13 in connection with the quadratic form of a knot.

2.B Seifert surfaces and genus

A geometric fact of some importance is the following:

2.4 Proposition (Seifert surface). *A simple closed oriented curve* $\mathfrak{k} \subset \mathbb{R}^3$ *is the boundary of an orientable surface* S*, embedded in* $\mathbb{R}^3$*. It is called a* Seifert surface of $\mathfrak{k}$.

Proof. Let $p(\mathfrak{k})$ be a regular projection of $\mathfrak{k}$ equipped with an orientation. By altering $p(\mathfrak{k})$ in the neighborhood of double points as shown in Figure 2.7, $p(\mathfrak{k})$ dissolves into a number of disjoint oriented simple closed curves which are called *Seifert cycles*. Choose an oriented 2-cell for each Seifert cycle, and embed the 2-cells in $\mathbb{R}^3$ as a disjoint union such that their boundaries are projected onto the Seifert cycles. The orientation of a Seifert cycle is to coincide with the orientation induced by the oriented

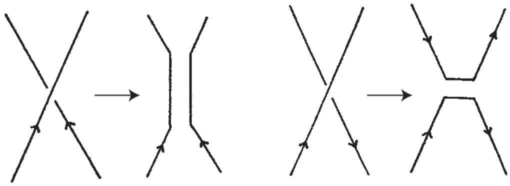


Figure 2.7

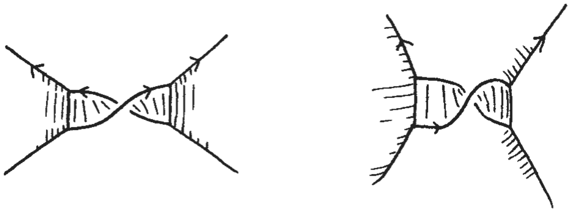


Figure 2.8

2-cell. We may place the 2-cells into planes $z = \text{const}$ parallel to the projection plane ($z = 0$), and choose planes $z = a_1, z = a_2$ for corresponding Seifert cycles c_1, c_2 with $a_1 < a_2$ if c_1 contains c_2. Now we can undo the cut-and-paste-process described in Figure 2.7 by joining the 2-cells at each double point by twisted bands such as to obtain a connected surface S with $\partial S = \mathfrak{k}$ (see Figure 2.8).

Since the oriented 2-cells (including the bands) induce the orientation of $\mathfrak{k}$, they are coherently oriented, and hence, S is orientable. □

2.5 Definition (Genus). The minimal genus $g = g(\mathfrak{k})$ of a Seifert surface spanning a knot $\mathfrak{k}$ is called the *genus of the knot* $\mathfrak{k}$.

Evidently the genus does not depend on the choice of a curve $\mathfrak{k}$ in its equivalence class: If $\mathfrak{k}$ and $\mathfrak{k}'$ are equivalent and S spans $\mathfrak{k}$, then there is a homeomorphism $h\colon S^3 \to S^3$, $h(\mathfrak{k}) = \mathfrak{k}'$ (Proposition 1.5), and $h(S) = S'$ spans $\mathfrak{k}'$. So the genus $g(\mathfrak{k})$ is a knot invariant, $g(\mathfrak{k}) = 0$ characterizes the trivial knot, because if $\mathfrak{k}$ bounds a disk D which is embedded in $\mathbb{R}^3$ (or S^3), one can use Δ-moves over 2-simplices of D and reduce $\mathfrak{k}$ to the boundary of a single 2-simplex.

The notion of the genus was first introduced by H. Seifert in [328], it holds a central position in knot theory.

The method to construct a Seifert surface by Seifert cycles assigns a surface S' of genus g' to a given regular projection of a knot. We call g' the *canonical genus associated with the projection*. It is remarkable that in many cases the canonical genus coincides with the (minimal) genus g of the knot. This is always true for alternating projections (Theorem 13.26 (a)). In our table of knot projections up to nine crossings only the projections $8_{20}, 8_{21}, 9_{42}, 9_{44}$ and 9_{45} fail to yield $g' = g$: in these cases $g' = g + 1$.

This was already observed by H. Seifert; the fact that he lists 9_{46} instead of 9_{44} in [328] is due to the choice of different projections in D. Rolfsen's [309] (and our table) and K. Reidemeister's [296, 302, 303].

There is a general algorithm to determine the genus of a knot due to H. Schubert [321], but its application is complicated. More recently, P. Ozsváth and Z. Szabó [279] showed that *knot Floer homology* detects the genus of a knot. For other more elementary methods see E 4.11.

2.6 Definition and simple properties (Meridian and longitude). A tubular neighborhood $V(\mathfrak{k})$ of a knot $\mathfrak{k} \subset S^3$ is homeomorphic to a solid torus. There is a simple closed curve m on $\partial V(\mathfrak{k})$ which is null-homologous in $V(\mathfrak{k})$ but not on $\partial V(\mathfrak{k})$; we call m *meridian* of $\mathfrak{k}$. It is easy to see that any two meridians (if suitably oriented) in $\partial V(\mathfrak{k})$ are isotopic. A Seifert surface S will meet $\partial V(\mathfrak{k})$ in a simple closed curve ℓ, if $V(\mathfrak{k})$ is suitably chosen: ℓ is called a *longitude* of $\mathfrak{k}$. We shall see later on (Proposition 3.1) that ℓ, too, is unique up to isotopy on $\partial V(\mathfrak{k})$. If $\mathfrak{k}$ and S^3 are oriented, we may assign orientations to m and ℓ: The longitude ℓ is isotopic to $\mathfrak{k}$ in $V(\mathfrak{k})$ and will be oriented as $\mathfrak{k}$. The meridian will be oriented in such a way that its linking number $\operatorname{lk}(m, \mathfrak{k})$ with $\mathfrak{k}$ in S^3 is $+1$ or equivalently, its intersection number $\operatorname{int}(m, \ell)$ with ℓ is $+1$. From this it follows that ℓ is not null-homologous on $\partial V(\mathfrak{k})$.

2.C Companion knots and product knots

Another important idea was added by H. Schubert in 1949 [317]: the product of knots.

2.7 Definition (Product of knots). Let an oriented knot $\mathfrak{k} \subset \mathbb{R}^3$ meet a plane E in two points P and Q. The arc of $\mathfrak{k}$ from P to Q is closed by an arc in E to obtain a knot $\mathfrak{k}_1$; the other arc (from Q to P) is closed in the same way and so defines a knot $\mathfrak{k}_2$. The knot $\mathfrak{k}$ is called the *product* or *composition* of $\mathfrak{k}_1$ and $\mathfrak{k}_2$, and it is denoted by $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$; see Figure 2.9. The knot $\mathfrak{k}$ is also called a *composite knot* when both knots $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are non-trivial. Moreover, $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are called *factors* of $\mathfrak{k}$.

It is easy to see that for any given knots $\mathfrak{k}_1$, $\mathfrak{k}_2$ the product $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ can be constructed; the product will not depend on the choice of representatives or on the plane E. A thorough treatment of the subject will be given in Chapter 7.

There are other procedures to construct more complicated knots from simpler ones.

2.8 Definition (Companion knot, satellite knot). Let $\widetilde{\mathfrak{k}}$ be a knot in a 3-sphere $\widetilde{S}^3$ and $\widetilde{V}$ an unknotted solid torus in $\widetilde{S}^3$ with $\widetilde{\mathfrak{k}} \subset \widetilde{V} \subset \widetilde{S}^3$. Assume that $\widetilde{\mathfrak{k}}$ is not contained in a 3-ball of $\widetilde{V}$. A homeomorphism $h\colon \widetilde{V} \to \widehat{V} \subset S^3$ onto a tubular neighborhood $\widehat{V}$ of a non-trivial knot $\widehat{\mathfrak{k}} \subset S^3$ which maps a meridian of $\widetilde{S}^3 - \widetilde{V}$ onto a longitude

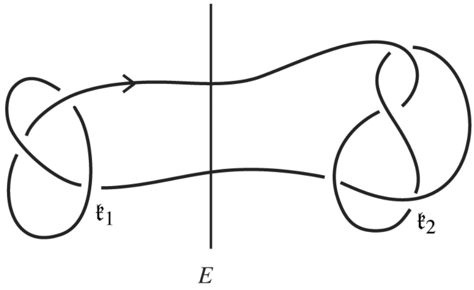


Figure 2.9. A product knot.

of $\widehat{\mathfrak{k}}$ maps $\widetilde{\mathfrak{k}}$ onto a knot $\mathfrak{k} = h(\widetilde{\mathfrak{k}}) \subset S^3$. The knot $\mathfrak{k}$ is called a *satellite* of $\widehat{\mathfrak{k}}$, and $\widehat{\mathfrak{k}}$ is its *companion* (Begleitknoten). The pair $(\widetilde{V}, \widetilde{\mathfrak{k}})$ is the *pattern* of $\mathfrak{k}$.

The companion is the simpler knot, it forgets some of the tangles of its satellite.

2.9 Examples. (1) Each factor $\mathfrak{k}_i$ of a product $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$, for instance, is a companion of $\mathfrak{k}$ (see Figure 7.2 and Proposition 16.7). There are some special cases of companion knots: If $\widetilde{\mathfrak{k}}$ is ambient isotopic in $\widetilde{V}$ to a simple closed curve on $\partial\widetilde{V}$, then $\mathfrak{k} = h(\widetilde{\mathfrak{k}})$ is called a *cable knot* on $\widehat{\mathfrak{k}}$.

(2) As another example consider $\widetilde{\mathfrak{k}} \subset \widetilde{V}$ as in Figure 2.10. For given $k \in \mathbb{Z}$ let $h_k\colon \widetilde{V} \to \widehat{V}$ be the homeomorphism which maps $\widetilde{\ell}$ to $h_k(\widetilde{\ell}) = \widehat{\ell} + k\ \widehat{m}$. The knot $\mathfrak{k} = h_k(\widetilde{\mathfrak{k}})$ is called a *doubled knot* of $\widehat{\mathfrak{k}}$ or more precisely a *k-twisted double of* $\widehat{\mathfrak{k}}$ which is also denoted by $\widehat{\mathfrak{k}}_k$. The case $k = 0$ is of particular interest and we will call $\widehat{\mathfrak{k}}_0$ an *untwisted* double of $\widehat{\mathfrak{k}}$. In Figure 2.10 the companion $\widehat{\mathfrak{k}}$ is a trefoil, the satellite $\mathfrak{k}$ is the 3-twisted double of $\widehat{\mathfrak{k}}$. Doubled knots were introduced

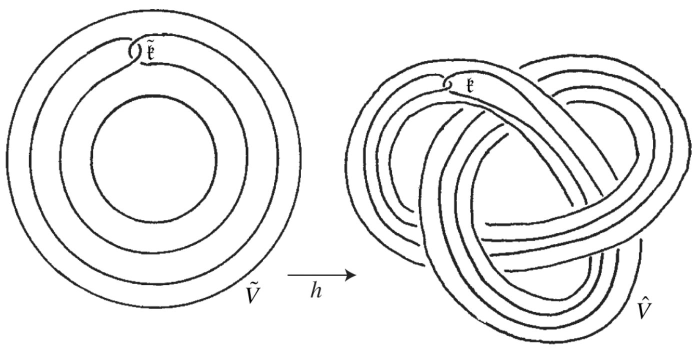


Figure 2.10. The 3-twisted double of the trefoil knot.

by J. H. C. Whitehead in [370] and form an interesting class of knots with respect to certain algebraic invariants.

For the relation between the genus of a satellite and the genus of its companion see Section 16.A.

2.D Braids, bridges, plats

There is a second topic to our main theme of knots, which has developed some weight of its own: *the theory of braids.* E. Artin introduced braids explicitly in [10], and at the same time solved the problem of their classification. (The proof there is somewhat intuitive, Artin revised it to meet rigorous standards in a later paper [11]). We shall treat braids in Chapter 10 but will introduce here the geometric idea of a braid, because it offers another possibility of representing knots (or links).

2.10 Braids. Place on opposite sides of a rectangle R in 3-space equidistant points P_i, Q_i, $1 \le i \le n$ (Figure 2.11). Let f_i, $1 \le i \le n$, be n simple disjoint polygonal arcs in $\mathbb{R}^3$, f_i starting in P_i and ending in $Q_{\pi(i)}$, where $i \mapsto \pi(i)$ is a permutation on $\{1, 2, \ldots, n\}$. The f_i are required to run "strictly downwards", that is, each f_i meets any plane perpendicular to the lateral edges of the rectangle at most once. The strings f_i constitute a *braid* $\mathfrak{z}$ (sometimes called an n-braid). The rectangle is called the *frame* of $\mathfrak{z}$, and $i \mapsto \pi(i)$ the *permutation of the braid*. In $\mathbb{R}^3$, *equivalent* or *isotopic braids* will be defined by "level-preserving" isotopies of $\mathbb{R}^3$ relative to the endpoints $\{P_i\}$, $\{Q_i\}$, which will be kept fixed, but we defer a treatment of these questions to Chapter 10.

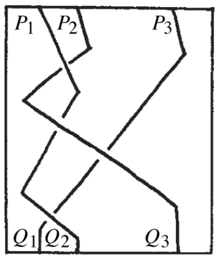


Figure 2.11

A *braid* can be *closed* with respect to an *axis* h (Figure 2.12). In this way every braid $\mathfrak{z}$ defines a *closed* braid $\widehat{\mathfrak{z}}$ which represents a link of μ components, where μ is the number of cycles of the permutation of $\mathfrak{z}$. We shall prove that every link can be presented as a closed braid. This mode of presentation is connected with another notion introduced by Schubert: the bridge number of a knot (resp. link):

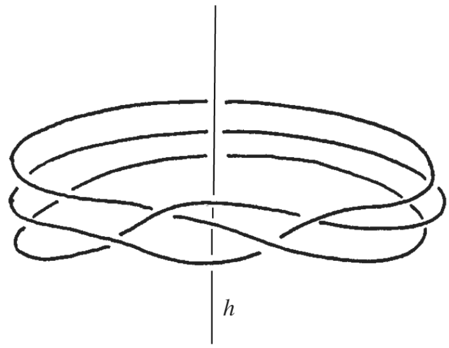


Figure 2.12

2.11 Definition (Bridge number). Let $\mathfrak{k}$ be a knot (or link) in $\mathbb{R}^3$ which meets a plane $E \subset \mathbb{R}^3$ at $2m$ points such that the arcs of $\mathfrak{k}$ contained in each half-space relative to E possess orthogonal projections onto E which are simple and disjoint. $(\mathfrak{k}, E)$ is called an *m-bridge presentation* of $\mathfrak{k}$; the minimal number m possible for a knot $\mathfrak{k}$ is called its *bridge number*.

A regular projection $p(\mathfrak{k})$ of order n (see Definition 1.14) admits an n-bridge presentation relative to the plane of projection (Figure 2.13 (a)). (If $p(\mathfrak{k})$ is not alternating, the number of bridges will even be smaller than n). The trivial knot is the only 1-bridge knot. The 2-bridge knots are an important class of knots which were classified by H. Schubert [320]. Even 3-bridge knots cannot be classified until today.

2.12 Proposition (J. W. Alexander [4]). *A link $\mathfrak{k}$ can be represented by a closed braid.*

Proof. Choose $2m$ points P_i in a regular projection $p(\mathfrak{k})$, one on each arc between undercrossing and overcrossing (or vice versa). This defines an m-bridge presentation, $m \leq n$, with arcs s_i, $1 \leq i \leq m$, between P_{2i-1} and P_{2i} in the upper half-space and arcs t_i, $1 \leq i \leq m$, joining P_{2i} and $P_{2i+1}(P_{2m+1} = P_1)$ in the lower half-space of the projection plane (see Figure 2.13 (a)).

By an ambient isotopy of $\mathfrak{k}$ we arrange the $p(t_i)$ to form m parallel straight segments bisected by a common perpendicular line h such that all P_i with odd index are on one side of h (Figure 2.13 (b)). The arc $p(s_i)$ meets h at an odd number of points $P_{i1}, P_{i2}, \ldots$.

In the neighborhood of a point P_{i2} we introduce a new bridge – we push the arc s_i in this neighborhood from the upper half-space into the lower one. Thus we obtain a bridge presentation where every arc $p(s_i), p(t_i)$ meets h at exactly one point. Now choose s_i monotonically ascending over $p(s_i)$ from P_{2i-1} until h is reached, then descending to P_{2i}. Equivalently, the t_i begin by descending and ascend afterwards. The result is a closed braid with axis h. □

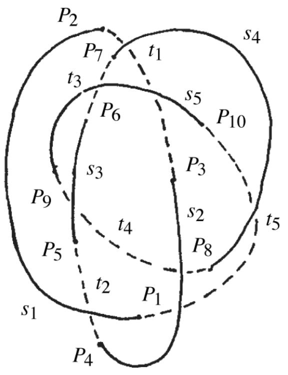


Figure 2.13 (a)

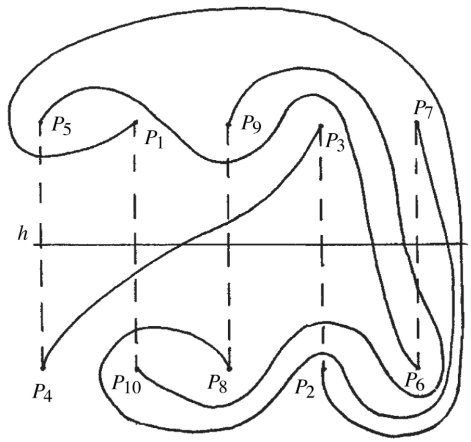


Figure 2.13 (b)

P. Vogel presents in [362] a simple algorithm that does not increase the number of Seifert circles and which limits the increase in the number of crossings.

A $2m$-braid completed by $2m$ simple arcs to make a link as depicted in Figure 2.14 is called a *plat* or a $2m$-*plat*. A closed m-braid obviously is a special $2m$-plat, hence every link can be represented as a plat. The construction used in the proof of Proposition 2.12 can be modified to show that an m-bridge representation of a knot $\mathfrak{k}$ can be used to construct a $2m$-plat representing it. In Corollary 10.4 we prove the converse: *Every $2m$-plat allows an m-bridge presentation. The 2-bridge knots (and links) hence are the 4-plats (Viergeflechte).*

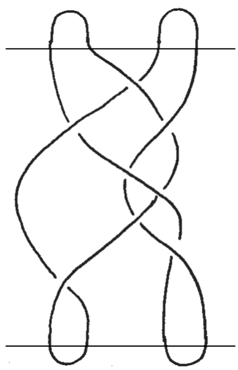


Figure 2.14. A 4-plat.

2.E Slice knots and algebraic knots

R. H. Fox and J. Milnor introduced the notion of a slice knot. It arises from the study of embeddings $S^2 \subset S^4$ (see [112]).

2.13 Definition (Slice knot). A knot $\mathfrak{k} \subset \mathbb{R}^3$ is called a *slice knot* if it can be obtained as a cross section of a locally flat 2-sphere S^2 in $\mathbb{R}^4$ by a hyperplane $\mathbb{R}^3$. ($S^2 \subset \mathbb{R}^4$ is *embedded locally flat*, if it is locally a Cartesian factor.) The local flatness is essential: Any knot $\mathfrak{k} \subset \mathbb{R}^3 \subset \mathbb{R}^4$ is a cross section of a 2-sphere S^2 embedded in $\mathbb{R}^4$. Choose the double suspension of $\mathfrak{k}$ with suspension points P_+ and P_- respectively in the half-space $\mathbb{R}^4_+$ and $\mathbb{R}^4_-$ defined by $\mathbb{R}^3$. The suspension S^2 is not locally flat at P_+ and P_- (Figure 2.15).

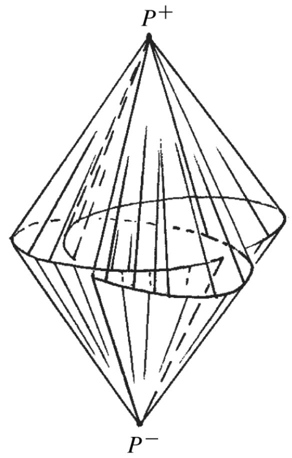


Figure 2.15

There is a disk $D^2 = S^2 \cap \mathbb{R}^4_+$ spanning the knot $\mathfrak{k} = \partial D^2$ which will be locally flat if and only if S^2 can be chosen locally flat. This leads to an equivalent definition of slice knots:

2.14 Definition. A knot $\mathfrak{k}$ in the boundary of a 4-cell, $\mathfrak{k} \subset S^3 = \partial D^4$, is a *slice knot* if there is a locally flat 2-disk $D^2 \subset D^4$, $\partial D^2 = \mathfrak{k}$, whose tubular neighborhood intersects S^3 in a tubular neighborhood of $\mathfrak{k}$.

The last condition ensures that the intersection of $\mathbb{R}^3$ and D^2 resp. S^2 is transversal. We shall give some examples of knots that are *slice* and of others that are not.

Let $f\colon D^2 \to S^3$ be an immersion, and $\partial(f(D)) = \mathfrak{k}$ a knot. If the singularities of $f(D)$ are all double lines σ, $f^{-1}(\sigma) = \sigma_1 \cup \sigma_2$, such that at least one of the preimages σ_i, $1 \leq i \leq 2$, is contained in $\mathring{D}$, then $\mathfrak{k}$ is called a *ribbon knot*.

2.15 Proposition. *Ribbon knots are slice knots.*

Proof. Double lines with boundary singularities come in two types: The type required in a ribbon knot is shown in Figure 2.16 while the second type is depicted in Figure 1.12. In the case of a ribbon knot the hatched regions of $f(D)$ can be pushed into the fourth dimension without changing the knot $\mathfrak{k}$. □

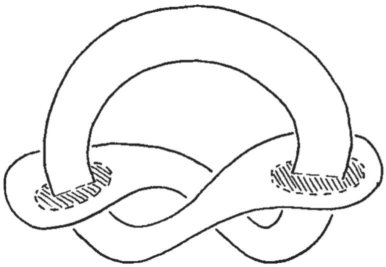


Figure 2.16. A ribbon knot.

It is not known whether all slice knots are ribbon knots. There are several criteria which allow to decide that a certain knot cannot be a slice knot [115], [261]. The trefoil, for instance, is not a slice knot. In fact, of all knots of order ≤ 7 the knot 6_1 of Figure 2.16 is the only one which is a slice knot. Slice knots can be used to define an equivalence relation on the set of knots in S^3. Two knots $\mathfrak{k}$ and $\mathfrak{k}'$ are equivalent if $\mathfrak{k} \# -\mathfrak{k}'$ is slice. With this equivalence the set of knots becomes a group, the *concordance group* of knots. Much progress has been made in studying slice knots and the concordance group (see the survey of C. Livingston [215] for more details).

Knots turn up in connection with another higher-dimensional setting: a polynomial equation $f(z_1, z_2) = 0$ in two complex variables defines a complex curve C in $\mathbb{C}^2$. At a singular point $z_0 = (z_1, z_2)$, where $\frac{\partial f}{\partial z_j}(z_0) = 0$, $i = 1, 2$, consider a small 3-sphere S^3_ε with center z_0. Then $\mathfrak{k} = C \cap S^3_\varepsilon$ may be a knot or link. (If z_0 is a regular point of C, the knot $\mathfrak{k}$ is always trivial.)

2.16 Proposition. *The algebraic surface* $f(z_1, z_2) = z_1^a + z_2^b = 0$ *with* $a, b, \in \mathbb{Z}, a, b \geq 2$, *intersects the boundary* S^3_ε *of a spherical neighborhood of* $(0,0)$ *in a torus knot (or link)* $\mathfrak{t}(a,b)$, *see Definition 3.36.*

Proof. The equations

$$r_1^a e^{ia\varphi_1} = r_2^b e^{ib\varphi_2 + i\pi}, \quad r_1^2 + r_2^2 = \varepsilon^2, \quad z_j = r_j e^{i\varphi_j},$$

define the intersection $S^3_\varepsilon \cap C$. Since $r_1^2 + r_1^{\frac{2a}{b}}$ is monotone, there are unique solutions $r_i = \rho_i > 0$, $i = 1, 2$. Thus the points of the intersection lie on $\{(z_1, z_2) \mid |z_1| = \rho_1, |z_2| = \rho_2\}$, which is an unknotted torus in S^3_ε. Furthermore $a\varphi_1 \equiv b\varphi_2 + \pi \mod 2\pi$ so that $S^3_\varepsilon \cap C = \{(\rho_1 e^{ib\varphi}, \rho_2 e^{ia\varphi + \frac{\pi i}{b}}) \mid 0 \leq \varphi \leq 2\pi\} = \mathfrak{t}(a,b)$. □

Knots that arise in this way at isolated singular points of algebraic curves are called *algebraic knots*. They are known to be *iterated torus knots*, that is, knots or links that are obtained by a repeated cabling process starting from the trivial knot. See J. Milnor [241], D. Hacon [144] and D. Eisenbud and W. Neumann [91].

2.F History and sources

To regard and treat a knot as an object of elementary geometry in 3-space was a natural attitude in the beginning, but proved to be very limited in its success. Nevertheless, direct geometric approaches occasionally were quite fruitful and inspiring, H. Brunn in 1897 [47] prepared a link in a way that practically resulted in J. W. Alexander's theorem [4] that every link can be deformed into a closed braid. The braids themselves were introduced by E. Artin in [10, 11]. H. Seifert then brought into knot theory the fundamental concept of the genus of a knot [328]. Nevertheless, the fact that a simple closed curve $\mathbb{R}^3$ bounds an orientable surface embedded in $\mathbb{R}^3$ was first proved by F. Frankl and L. Pontrjagin in 1930 [117]. Another simple geometric idea led to the product of knots introduced by H. Schubert [317], and afterwards he introduced and studied the theory of companions [318], and the notion of the bridge number of a knot [319]. Finally R. H. Fox and J. Milnor suggested looking at a knot from a 4-dimensional point of view which led to the slice knot [112].

During the last decades geometric methods have gained importance in knot theory – but they are, as a rule, no longer elementary.

2.G Exercises

E 2.1. Show that the trefoil is symmetric, and that the four-knot is both symmetric and amphicheiral.

E 2.2. Let $p(\mathfrak{k}) \subset E^2$ be a regular projection of a link $\mathfrak{k}$. The plane E^2 can be colored with two colors in such a way that regions with a common arc of $p(\mathfrak{k})$ in their boundary obtain different colors (chessboard coloring).

E 2.3. A knot projection is alternating if and only if $\theta(A)$ (see Definition 2.3) is constant.

E 2.4. Describe the operations on graphs associated to knot projections which correspond to the Reidemeister operations Ω_i, $i = 1, 2, 3$.

E 2.5. Show that the two graphs associated to the regular projection of a knot by distinguishing either α-regions or β-regions are equivalent (see Definition 2.3 and E 2.4).

E 2.6. A regular projection $p(\mathfrak{k})$ (onto S^2), of a knot $\mathfrak{k}$ defines two surfaces F_1, $F_2 \subset S^3$ spanning $\mathfrak{k} = \partial F_1 = \partial F_2$ where $p(F_1)$ and $p(F_2)$ respectively cover the regions colored by the same color of a chessboard coloring of $p(\mathfrak{k})$ (see E 2.2). Prove that at least one of the surfaces F_1, F_2 is non-orientable.

E 2.7. Construct an orientable surface of genus one spanning the four-knot 4_1.

E 2.8. Give a presentation of the knot 6_3 as a 3-braid.

E 2.9. In Definition 2.7 the following condition was imposed on the knot $\widetilde{\mathfrak{k}}$ embedded in the solid torus $\widetilde{V}$:

(1) There exists a ball $\widetilde{B}$ such that $\widetilde{\mathfrak{k}} \subset \widetilde{B} \subset \widetilde{V}$.

Show that (1) is equivalent to each of the following two conditions:

(2) There exists a meridian disk $\delta \subset \widetilde{V}$, $\partial\delta = \delta \cap \partial\widetilde{V}$, $\partial\delta$ not contractible in $\partial\widetilde{V}$ such that $\widetilde{\mathfrak{k}} \cap \delta = \emptyset$.

(3) $\pi_1(\partial\widetilde{V}) \to \pi_1(\widetilde{V} - \widetilde{\mathfrak{k}})$, induced by the inclusion, is not injective.

(Hint: in order to prove "(3) $\Rightarrow$ (1)", use the loop theorem B.5.)

Chapter 3

Knot groups

The investigation of the complement of a knot in $\mathbb{R}^3$ or S^3 has been of special interest since the beginnings of knot theory. H. Tietze [350] was the first to prove the existence of non-trivial knots by computing the fundamental group of the complement of the trefoil. He conjectured that two knot types are equal if and only if their complements are homeomorphic. In 1989 C. M. Gordon and J. Luecke [137] finally proved this conjecture – this proof is beyond the scope of this book. In the attempt to classify knot complements, homological methods are not very helpful. The fundamental group, however, is very effective and we will develop methods to present and study it. In particular, we will use it to show that there are non-trivial knots.

3.A Homology

$V = V(\mathfrak{k})$ denotes a tubular neighborhood of the knot $\mathfrak{k}$ and $C = \overline{S^3 - V}$ is called the *complement* of the knot. H_j will denote the (singular) homology with coefficients in $\mathbb{Z}$.

3.1 Theorem (Homological properties).

(a) $H_0(C) \cong H_1(C) \cong \mathbb{Z}$, $H_n(C) = 0$ *for* $n \geq 2$.

(b) *There are two simple closed curves* m *and* ℓ *on* ∂V *with the following properties*
  (1) m *and* ℓ *intersect in one point,*
  (2) $m \sim 0$, $\ell \sim \mathfrak{k}$ *in* $V(\mathfrak{k})$,
  (3) $\ell \sim 0$ *in* $C = \overline{S^3 - V(\mathfrak{k})}$,
  (4) $\mathrm{lk}(m, \mathfrak{k}) = 1$ *and* $\mathrm{lk}(\ell, \mathfrak{k}) = 0$ *in* S^3.

These properties determine m *and* ℓ *up to isotopy on* $\partial V(\mathfrak{k})$. *We call* m *a* meridian *and* ℓ *a* longitude *of the knot* $\mathfrak{k}$. *The knot* $\mathfrak{k}$ *and the longitude* ℓ *bound an annulus* $A \subset V$.

Proof. For (a) there are several proofs. Here we present one based on homological methods. We use the following well-known results:

$$H_n(S^3) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_n(\partial V) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_n(V) = H_n(S^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, \\ 0 & \text{otherwise;} \end{cases}$$

they can be found in standard books on algebraic topology, see Spanier [341], Stöcker-Zieschang [346], Hatcher [157].

Since C is connected, $H_0(C) = \mathbb{Z}$. For further calculations we use the Mayer–Vietoris sequence of the pair (V, C) where $V \cup C = S^3$, $V \cap C = \partial V$:

$$\begin{array}{ccccccccccc}
H_3(\partial V) & \to & H_3(V) & \oplus & H_3(C) & \to & H_3(S^3) & \to & H_2(\partial V) & \to \\
\| & & \| & & & & \|\wr & & \|\wr & \\
0 & & 0 & & & & \mathbb{Z} & & \mathbb{Z} & \\
\\
 & \to & H_2(V) & \oplus & H_2(C) & \to & H_2(S^3) & \to & H_1(\partial V) & \to \\
 & & \| & & & & \| & & \|\wr & \\
 & & 0 & & & & 0 & & \mathbb{Z} \oplus \mathbb{Z} & \\
\\
 & \to & H_1(V) & \oplus & H_1(C) & \to & H_1(S^3) & & & \\
 & & \|\wr & & & & \| & & & \\
 & & \mathbb{Z} & & & & 0 & . & &
\end{array}$$

It follows that $H_1(C) = \mathbb{Z}$. Since ∂V is the boundary of the orientable compact 3-manifold C, the group $H_2(\partial V)$ is mapped by the inclusion $\partial V \hookrightarrow C$ to $0 \in H_2(C)$. This implies that $H_2(C) = 0$ and that $H_3(S^3) \to H_2(\partial V)$ is surjective; hence, $H_3(C) = 0$.

Since C is a 3-manifold it follows that $H_n(C) = 0$ for $n > 3$; this is also a consequence of the Mayer–Vietoris sequence.

Consider the isomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \cong H_1(\partial V) \to H_1(V) \oplus H_1(C)$$

in the Mayer–Vietoris sequence. The generators of $H_1(V) \cong \mathbb{Z}$ and $H_1(C) \cong \mathbb{Z}$ are determined up to their inverses. Choose the homology class of $\mathfrak{k}$ as a generator of $H_1(V)$ and represent it by a simple closed curve ℓ on ∂V which is homologous to 0 in $H_1(C)$. These conditions determine the homology class of ℓ in ∂V; hence, ℓ is unique up to isotopy on ∂V. A generator of $H_1(C)$ can be represented by a curve m on ∂V that is homologous to 0 in V. The curves ℓ and m determine a system of generators of $H_1(\partial V) \cong \mathbb{Z} \oplus \mathbb{Z}$. By a well-known result, we may assume that m is simple and intersects ℓ in one point, see e.g. Stillwell [345, 6.4.3], Zieschang, Vogt and Coldewey [382, E 3.22]. As m is homologous to 0 in V it is null-homotopic in V, bounds a disk, and is a meridian of the solid torus V. The linking number of m

and $\mathfrak{k}$ is 1 or -1. If necessary we reverse the direction of m to get (4). These properties determine m up to an isotopy of ∂V. A consequence is that ℓ and $\mathfrak{k}$ bound an annulus $A \subset V$.

Since $\ell \sim 0$ in C, ℓ bounds a surface, possibly with singularities, in C. (As we already know, see Proposition 2.4, ℓ even spans a surface without singularities: a Seifert surface.) □

Theorem 3.1 can be generalized to links (E 3.2). The negative aspect of the theorem is that complements of knots cannot be distinguished by their homological properties.

3.2 On the characterization of longitudes and meridians by the complement of a knot. With respect to the complement C of a knot, the longitude ℓ and the meridian m have quite different properties: The longitude ℓ is determined up to isotopy and orientation by C; this follows from the fact that ℓ is a simple closed curve on ∂C which is not homologous to 0 on ∂C but homologous to 0 in C. The meridian m is a simple closed curve on ∂C that intersects ℓ at one point; hence, ℓ and m represent generators of $H_1(\partial C) \cong \mathbb{Z}^2$. The meridian is not determined by C because simple closed curves on ∂C which are homologous to $m^{\pm 1}\ell^r$, $r \in \mathbb{Z}$, have the same properties (see E 3.3).

3.B Wirtinger presentation

The most important and effective invariant of a knot $\mathfrak{k}$ (or link) is its *group*: the fundamental group of its complement $\mathfrak{G} = \pi_1(S^3 - \mathfrak{k})$. Frequently $S^3 - \mathfrak{k}$ is replaced by $\mathbb{R}^3 - \mathfrak{k}$ or by $\overline{S^3 - V(\mathfrak{k})}$ or $\overline{\mathbb{R}^3 - V(\mathfrak{k})}$, respectively. The fundamental groups of these various spaces are obviously isomorphic, the isomorphisms being induced by inclusion. There is a simple procedure, due to Wirtinger, to obtain a presentation of a knot group.

3.3. Embed the knot $\mathfrak{k}$ into $\mathbb{R}^3$ such that its projection onto the plane $\mathbb{R}^2$ is regular. The projecting cylinder Z has self-intersections in n projecting rays a_i corresponding to the n double points of the regular projection. The a_i decompose Z into n 2-cells Z_i (see Figure 3.1) where Z_i is bounded by a_{i-1}, a_i and the overcrossing arc σ_i of $\mathfrak{k}$.

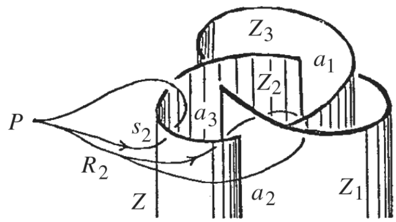


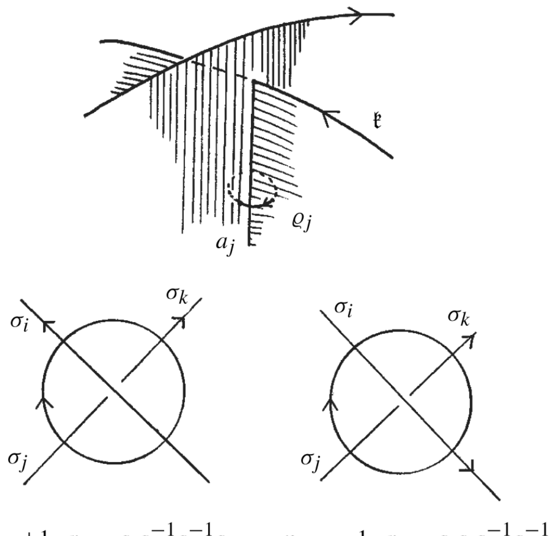
Figure 3.1

Choose the orientation of Z_i to induce on σ_i the direction of $\mathfrak{k}$. The complement of Z can be retracted parallel to the rays onto a half-space above the knot; thus it is contractible.

To compute $\pi_1 C$ for some basepoint $P \in C$ observe that there is (up to a homotopy fixing P) exactly one polygonal closed path in general position relative to Z which intersects a given Z_i with intersection number ε_i and which does not intersect the other Z_j. Paths of this type, taken for $i = 1, 2, \ldots, n$ and $\varepsilon_i = 1$, represent generators $s_i \in \pi_1 C$. To see this, let a path in general position with respect to Z represent an arbitrary element of $\pi_1 C$. Move its intersection points with Z_i into the intersection of the curves s_i. Now the assertion follows since the complement of Z is contractible. Running through an arbitrary closed polygonal path ω yields the homotopy class as a word $w(s_i) = s_{i_1}^{\varepsilon_1} \ldots s_{i_r}^{\varepsilon_r}$ if in turn each intersection with Z_{i_j} and intersection number ε_j is put down by writing $s_{i_j}^{\varepsilon_j}$.

To obtain relations, consider a small path ϱ_j in C encircling a_j and join it with P by an arc λ_j. Then $\lambda_j \varrho_j \lambda_j^{-1}$ is contractible and the corresponding word $l_j r_j l_j^{-1}$ in the generators s_i is a relation. The word $r_j(s_j)$ can easily be read off from the knot projection. According to the characteristic $\eta \in \{1, -1\}$ of a double point, see Figure 3.2, we get the relation

$$r_j = s_j s_i^{-\eta_j} s_k^{-1} s_i^{\eta_j}.$$



$\eta_j = +1,\ r_j = s_j s_i^{-1} s_k^{-1} s_i$ $\qquad$ $\eta_j = -1,\ r_j = s_j s_i s_k^{-1} s_i^{-1}$

Figure 3.2. The Wirtinger relations.

3.4 Theorem (Wirtinger presentation). *Let σ_i, $i = 1, 2, \ldots, n$, be the overcrossing arcs of a regular projection of a knot (or link) $\mathfrak{k}$. Then the knot group admits the following so-called* Wirtinger presentation:

$$\mathfrak{G} = \pi_1(\overline{S^3 - V(\mathfrak{k})}) = \langle s_1, \ldots, s_n \mid r_1, \ldots, r_n \rangle.$$

The arc σ_i corresponds to the generator s_i; a crossing of characteristic η_j as in Figure 3.2 gives rise to the defining relations

$$r_j = s_j s_i^{-\eta_j} s_k^{-1} s_i^{\eta_j}.$$

Proof. It remains to check that $r_1, \ldots, r_n$ are defining relations. Consider $\mathbb{R}^3$ as a simplicial complex Σ containing Z as a subcomplex, and denote by Σ^* the dual complex. Let ω be a contractible curve in C, starting at a vertex P of Σ^*. By simplicial approximation ω can be replaced by a path in the 1-skeleton of Σ^* and the contractible homotopy by a series of homotopy moves which replace arcs on the boundary of 2-cells σ^2 of Σ^* by the inverse of the rest. If $\sigma^2 \cap Z = \emptyset$ the deformation over σ^2 has no effect on the words $\omega(s_i)$. If σ^2 meets Z in an arc then the deformation over σ^2 either cancels or inserts a word $s_i^{\varepsilon} s_i^{-\varepsilon}, \varepsilon \in \{1, -1\}$, in $\omega(s_i)$; hence, it does not affect the element of $\pi_1 C$ represented by ω. If σ^2 intersects a double line a_j then the deformation over σ^2 omits or inserts a relation: a conjugate of r_j or r_j^{-1} for some j. □

In the case of a link $\mathfrak{k}$ of μ components the relations ensure that generators s_i and s_j are conjugate if the corresponding arcs σ_i and σ_j belong to the same component. By abelianizing $\mathfrak{G} = \pi_1(S^3 - \mathfrak{k})$ we obtain from Theorem 3.4, see also E 3.2:

3.5 Proposition. $H_1(S^3 - \mathfrak{k}) \cong \mathbb{Z}^\mu$ *where μ is the number of components of $\mathfrak{k}$.* □

Using Proposition 3.5 and duality theorems for homology and cohomology one can calculate the other homology groups of $S^3 - \mathfrak{k}$, see E 3.2:

3.6 Corollary. *Let $\mathfrak{k}$ be a knot or link and $\langle s_1, \ldots, s_n \mid r_1, \ldots, r_n \rangle$ a Wirtinger presentation of $\mathfrak{G}$. Then each defining relation r_j is a consequence of the other defining relations $r_i, i \neq j$.*

Proof. Choose the curves $\lambda_j \varrho_j \lambda_j^{-1}$ (see the paragraph before Theorem 3.4) in a plane E parallel to the projection plane and "far down" such that E intersects all a_i. Let δ be a disk in E such that $\mathfrak{k}$ is projected into δ, and let γ be the boundary of δ. We assume that P is on γ and that the λ_j have only the basepoint P in common. Then, see Figure 3.3,

$$\gamma \simeq \prod_{j=1}^{n} \lambda_j \varrho_j \lambda_j^{-1} \quad \text{in } E - \Big(\bigcup_j a_j \cap E\Big).$$

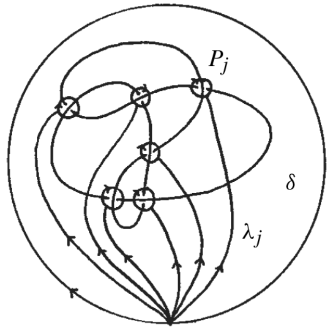


Figure 3.3

This implies the equation

$$1 \equiv \prod_{j=1}^{n} l_j r_j l_j^{-1}$$

in the free group generated by the s_i, where l_j is the word which corresponds to λ_j. Thus each relation r_j is a consequence of the other relations. □

3.7 Example (Trefoil knot = clover leaf knot). From Figure 3.4 we obtain Wirtinger generators s_1, s_2, s_3 and defining relations

$$\begin{array}{ll} s_1 s_2 s_3^{-1} s_2^{-1} & \text{at the vertex } A, \\ s_2 s_3 s_1^{-1} s_3^{-1} & \text{at the vertex } B, \\ s_3 s_1 s_2^{-1} s_1^{-1} & \text{at the vertex } C. \end{array}$$

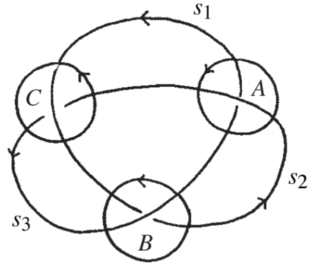


Figure 3.4. The Wirtinger relations for the trefoil knot.

Since by Corollary 3.6 one relation is a consequence of the other two, the knot group has the presentation

$$\begin{aligned}\langle\, s_1, s_2, s_3 \mid s_1 s_2 s_3^{-1} s_2^{-1}, s_3 s_1 s_2^{-1} s_1^{-1} \,\rangle &= \langle\, s_1, s_2 \mid s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1} \,\rangle \\ &= \langle\, x, y \mid x^3 = y^2 \,\rangle\end{aligned}$$

where $y = s_2 s_1 s_2$ and $x = s_1 s_2$. This group is not isomorphic to $\mathbb{Z}$, since the last presentation shows that it is a free product with amalgamated subgroup $\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$ where $\mathfrak{A}_i \cong \mathbb{Z}$ and $\mathfrak{B} = \langle\, x^3 \,\rangle = \langle\, y \,\rangle$ with $\mathfrak{B} \subsetneq \mathfrak{A}_i$. Hence, it is not commutative. This can also be shown directly using the representation $\rho\colon \mathfrak{G} \to \mathrm{SL}_2(\mathbb{Z})$ given by

$$\rho\colon x \mapsto A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \rho\colon y \mapsto B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

since

$$A \cdot B = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \neq \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = B \cdot A\,.$$

The reader should note that here for the first time in this book the existence of non-trivial knots has been proved, since the group of the trivial knot is cyclic.

We can approach the analysis of the group of the trefoil knot in a different manner by calculating its commutator subgroup using the Reidemeister–Schreier method. It turns out that $\mathfrak{G}'$ is a free group of rank 2, see E 4.2. We will use this method in the next example.

3.8 Example (Four-knot or figure-eight knot, Figure 3.5).

$$\begin{aligned}\mathfrak{G} &= \langle\, s_1, s_2, s_3, s_4 \mid s_3 s_4^{-1} s_3^{-1} s_1,\ s_1 s_2^{-1} s_1^{-1} s_3,\ s_4 s_2^{-1} s_3^{-1} s_2 \,\rangle \\ &= \langle\, s_1, s_3 \mid s_3^{-1} s_1 s_3 s_1^{-1} s_3 s_1^{-1} s_3 s_1^{-1} \,\rangle \\ &= \langle\, s, u \mid u^{-1} s u s^{-1} u^{-2} s^{-1} u s \,\rangle, \\ &\quad \text{where } s = s_1 \text{ and } u = s_1^{-1} s_3.\end{aligned}$$

The abelianizing homomorphism $\mathfrak{G} \to \mathbb{Z}$ maps s_i onto 1 which is a generator of $\mathbb{Z}$ and u onto 0. Hence, $\{s^n \mid n \in \mathbb{Z}\}$ is a system of coset representatives and $\{x_n = s^n u s^{-n} \mid n \in \mathbb{Z}\}$ the corresponding system of Schreier generators for the commutator subgroup $\mathfrak{G}'$ (see [382, 2.2]). The defining relations are

$$r_n = s^n (u^{-1} s u s^{-1} u^{-2} s^{-1} u s) s^{-n} = x_n^{-1} x_{n+1} x_n^{-2} x_{n-1},\ n \in \mathbb{Z}.$$

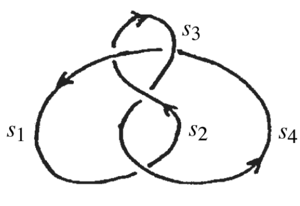


Figure 3.5. The figure-eight knot.

Using r_1, we obtain

$$x_2 = x_1 x_0^{-1} x_1^{+2};$$

hence, we may drop the generator x_2 and the relation r_1. Next we consider r_2 and obtain

$$x_3 = x_2 x_1^{-1} x_2^{+2}$$

and replace x_2 by the word in x_0, x_1 from above. Now we drop x_3 and r_2. By induction, we get rid of the relations $r_1, r_2, r_3, \ldots$ and the generators $x_2, x_3, x_4, \ldots$. Now, using the relation r_0 we obtain

$$x_{-1} = x_0^{+2} x_1^{-1} x_0;$$

thus we may drop the generator x_{-1} and the relation r_0. By induction we eliminate $x_{-1}, x_{-2}, x_{-3}, \ldots$ and the relation $r_0, r_{-1}, r_{-2}, \ldots$. Finally we are left with the generators x_0, x_1 and no relation, i.e. $\mathfrak{G}' = \langle\, x_0, x_1 | \,\rangle$ is a free group of rank 2. This proves that the figure-eight knot is non-trivial. □

The fact that the commutator subgroup is finitely generated has a strong geometric consequence, namely that the complement can be fibered locally trivial over S^1 and the fiber is an orientable surface with one boundary component. In the case of the trefoil knot and the figure-eight knot, the fiber is a punctured torus. It turns out that these are the only knots that have a fibered complement with a torus as fiber, see Proposition 5.15. We will develop the theory of fibered knots in Chapter 5.

3.9 Example (2-bridge knot $\mathfrak{b}(7,3)$). From Figure 3.6 we determine generators for $\mathfrak{G}$ as before. It suffices to use the Wirtinger generators v, w which correspond to the bridges. One obtains the presentation

$$\begin{aligned} \mathfrak{G} &= \langle\, v, w \mid vwvw^{-1}v^{-1}wvw^{-1}v^{-1}w^{-1}vwv^{-1}w^{-1} \,\rangle \\ &= \langle\, s, u \mid susu^{-1}s^{-1}usu^{-1}s^{-2}u^{-1}sus^{-1}u^{-1} \,\rangle \end{aligned}$$

where $s = v$ and $u = wv^{-1}$. A system of coset representatives is $\{s^n \mid n \in \mathbb{Z}\}$ and they lead to the generators $x_n = s^n u s^{-n}, n \in \mathbb{Z}$, of $\mathfrak{G}'$ and the defining relations

$$x_{n+1} x_{n+2}^{-1} x_{n+1} x_{n+2}^{-1} x_n^{-1} x_{n+1} x_n^{-1},\ n \in \mathbb{Z}.$$

By abelianizing we obtain the relations $-2x_n + 3x_{n+1} - 2x_{n+2} = 0$, and now it follows that this group is not finitely generated (E 3.5 (a)).

From the above relations it follows that

$$\mathfrak{G}' = \cdots *_{\mathfrak{B}_{-2}} \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{\mathfrak{B}_1 \ldots},$$

where $\mathfrak{A}_n = \langle\, x_n, y_n \mid - \,\rangle$ and $\mathfrak{B}_n = \langle a_n, b_n \mid - \rangle$ are free groups of rank 2. The injections $\varphi_n \colon \mathfrak{B}_n \to \mathfrak{A}_n$ and $\psi_n \colon \mathfrak{B}_n \to \mathfrak{A}_{n+1}$ are given by

$$\varphi_n(a_n) = x_n y_n^{-1},\ \varphi_n(b_n) = y_n^2 \text{ and } \psi_n(a_n) = x_{n+1},\ \psi_n(b_n) = y_{n+1}^2 x_{n+1}^{-1}$$

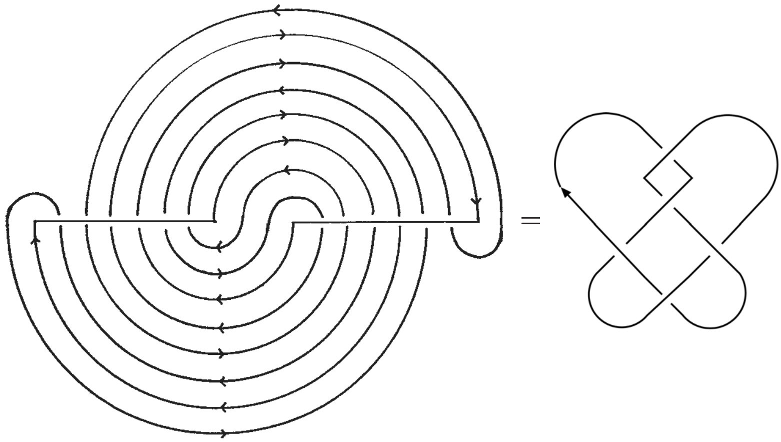


Figure 3.6. $\mathfrak{b}(7,3) = 5_2$.

respectively. It follows that $\mathfrak{A}_n \neq \varphi(\mathfrak{B}_n)$ and $\psi(\mathfrak{B}_n) \neq \mathfrak{A}_{n+1}$. Proof as E 3.5 (b). A consequence is that the complement of this knot cannot be fibered over S^1 with a surface as fiber, see Theorem 5.1. This knot also has genus one, i.e. it bounds a torus with one hole. □

The background to the calculations in Examples 3.8 and 3.9 is discussed in Chapter 4. The next proposition shows that the fundamental group of the complement characterizes the trivial knot:

3.10 Proposition. *If $\mathfrak{k}$ is a non-trivial knot the inclusion $i\colon \partial V \to C = \overline{S^3 - V}$ induces an injective homomorphism $i_\#\colon \pi_1 \partial V \to \pi_1 C$. In particular, if $\pi_1 C \cong \mathbb{Z}$ is cyclic then the knot $\mathfrak{k}$ is trivial.*

Proof. Suppose $i_\#$ is not injective. Then the Loop Theorem of Papakyriakopoulos [281], see Appendix B.5 [159, 4.2], guarantees the existence of a simple closed curve κ on ∂V and a disk δ in C such that

$$\kappa = \partial\delta; \quad (\text{hence } \kappa \simeq 0 \text{ in } C), \delta \cap V = \kappa \text{ and } \kappa \not\simeq 0 \text{ in } \partial V.$$

Since κ is simple and $\kappa \sim 0$ in C it is a longitude, see 3.2. So there is an annulus $A \subset V$ such that $A \cap \partial V = \kappa$, $\partial A = \kappa \cup \mathfrak{k}$, as has been shown in Theorem 3.1. This proves that $\mathfrak{k}$ bounds a disk in S^3 and, hence, is the trivial knot. □

3.11 Groups of satellites and companions. *Recall the notation of Definition 2.8: $\widetilde{V}$ is an unknotted solid torus in a 3-sphere $\widetilde{S}^3$ and $\widetilde{\mathfrak{k}} \subset \widetilde{V}$ a knot such that a meridian of $\widetilde{V}$ is not contractible in $\widetilde{V} - \widetilde{\mathfrak{k}}$. As, by definition, a companion $\widehat{\mathfrak{k}}$ is non-trivial the* *homomorphisms* $i_\#\colon \pi_1\partial\,\widehat{V} \to \pi_1(\,\widehat{V} - \widehat{\mathfrak{k}}\,)$, $j_\#\colon \pi_1\partial V \to \pi_1(\overline{S^3 - \widehat{V}}\,)$ *are injective, see Proposition 3.10 and E 2.9. By the Seifert–van Kampen Theorem we get:*

3.12 Proposition. *With the above notation:*

$$\mathfrak{G} = \pi_1(S^3 - \mathfrak{k}) = \pi_1(\widetilde{V} - \widetilde{\mathfrak{k}}) *_{\pi_1\partial\widehat{V}} \pi_1(S^3 - \widehat{\mathfrak{k}}) = \mathfrak{H} *_{\langle\widehat{t},\widehat{\lambda}\rangle} \widehat{\mathfrak{G}},$$

is a free product with amalgamation. Here $\widehat{\mathfrak{G}}$ *is the knot group of the companion knot* $\widehat{\mathfrak{k}}$, $\widehat{t}$ *and* $\widehat{\lambda}$ *represent meridian and longitude of* $\widehat{\mathfrak{k}}$ *and* $\mathfrak{H} = \pi_1(\widetilde{V} - \widetilde{\mathfrak{k}})$. □

Remark. A satellite is never trivial.

3.13 Proposition (Longitude). *The longitude* ℓ *of a knot* $\mathfrak{k}$ *represents an element of the second commutator group of the knot group* $\mathfrak{G}$*:*

$$\ell \in \mathfrak{G}^{(2)} = \mathfrak{G}''.$$

Proof. Consider a Seifert surface S spanning the knot $\mathfrak{k}$ such that for some regular neighborhood V of $\mathfrak{k}$ the intersection $S \cap V$ is an annulus A with $\partial A = \mathfrak{k} \cup \ell$. Thus $\ell = \partial(S - A)$ implies that $\ell \sim 0$ in $C = \overline{S^3 - V}$. A 1-cycle z of C and S have intersection number r if $z \sim r \cdot m$ in C where m is a meridian of $\mathfrak{k}$. Hence, a curve ζ represents an element of the commutator subgroup $\mathfrak{G}'$ if and only if its intersection number with S vanishes. Since S is two-sided, each curve on S can be pushed into $C - S$, and thus has intersection number 0 with S and consequently represents an element of $\mathfrak{G}'$. If $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ is a canonical system of curves on S then

$$\ell \simeq \prod_{n=1}^{g} [\alpha_n, \beta_n],$$

hence, $\ell \in \mathfrak{G}^{(2)}$. □

Remark. It follows from the proof of Proposition 3.13 that the canonical homomorphism $\mathfrak{G} \to \mathbb{Z}$ which maps the meridian to 1 is given by $\gamma \mapsto \operatorname{lk}(\gamma, \mathfrak{k})$. Hence each Wirtinger generator is mapped to 1.

In what follows we will fix the orientation of S^3 such that the

lk() = 1.

3.14 Remark. The longitude ℓ of a knot $\mathfrak{k}$ can be read off a regular projection as a word in the Wirtinger generators as follows: run through the knot projection starting on the arc assigned to the generator s_k. Write down s_i (or s_i^{-1}) when undercrossing the arc from right to left (or from left to right) corresponding to s_i. Add s_k^{α} such that the sum of all exponents equals 0. See Figure 3.7, $\mathfrak{k} = 5_2$, $k = 1$, $\alpha = 5$.

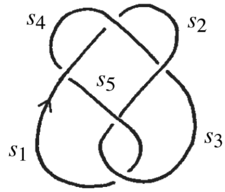


Figure 3.7. $\ell = s_4^{-1}s_5^{-1}s_2^{-1}s_1^{-1}s_3^{-1} \cdot s_1^5$.

3.C Peripheral system

In Paragraph 3.2 we assigned meridian and longitude to a given knot $\mathfrak{k}$. They define homotopy classes in the knot group. These elements are, however, not uniquely determined, but only up to a common conjugating factor. Meridian and longitude can be chosen as free Abelian generators of $\pi_1 \partial V$. (In this section $C = C(\mathfrak{k})$ always stands for the compact manifold $C = \overline{S^3 - V}$.)

3.15 Definition and Proposition (Peripheral system). *The peripheral system of a knot $\mathfrak{k}$ is a triple $(\mathfrak{G}, m, \ell)$ consisting of the knot group $\mathfrak{G}$ and the homotopy classes m, ℓ of a meridian and a longitude. These elements commute: $m \cdot \ell = \ell \cdot m$. The pair (m, ℓ) is uniquely determined up to a common conjugating element of $\mathfrak{G}$.*

The peripheral group system $(\mathfrak{G}, \mathfrak{P})$ consists of $\mathfrak{G}$ and the subgroup $\mathfrak{P}$ generated by m and ℓ, $\mathfrak{P} = \pi_1 \partial V$. As before, the inclusion $\partial V \subset C$ only defines a class of conjugate subgroups $\mathfrak{P}$ of $\mathfrak{G}$. □

Theorem 3.16 shows the strength of the peripheral system; unfortunately, its proof depends on a fundamental theorem of F. Waldhausen [367] on 3-manifolds which we cannot prove here.

3.16 Theorem (Waldhausen [367]). *Two knots $\mathfrak{k}_1$, $\mathfrak{k}_2$ in S^3 with the peripheral systems $(\mathfrak{G}_i, m_i, \ell_i)$, $i = 1, 2$, are equal if there is an isomorphism $\varphi\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ with the property that $\varphi(m_1) = m_2$ and $\varphi(\ell_1) = \ell_2$.*

Proof. Clearly, if $h\colon S^3 \to S^3$ is an orientation preserving homeomorphism such that $h(\mathfrak{k}_1) = \mathfrak{k}_2$, then the restriction of h to $S^3 - \mathfrak{k}_1$ induces an isomorphism $\varphi\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ satisfying $\varphi(m_1) = m_2$ and $\varphi(\ell_1) = \ell_2$.

Conversely, assume that $\varphi\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ is an isomorphism with the property that $\varphi(m_1) = m_2$ and $\varphi(\ell_1) = \ell_2$. By the theorem of Waldhausen on sufficiently large irreducible 3-manifolds, see Appendix B.7 [367, Cor. 6.5], [159, 13.6] the isomorphism φ is induced by a homeomorphism $h'\colon C_1 \to C_2$ mapping representative curves μ_1, λ_1 of m_1, ℓ_1 onto representatives μ_2, λ_2 of m_2, ℓ_2. The representatives can be taken on the boundaries ∂C_i. Note that Waldhausen's Theorem B.7 can be applied in our situation, see Remark B.8.

As h' maps the meridian of V_1 onto a meridian of V_2 it can be extended to a homeomorphism $h'': V_1 \to V_2$ mapping the 'core' $\mathfrak{k}_1$ onto $\mathfrak{k}_2$, see E 3.14. Together h' and h'' define the required homeomorphism $h: S^3 \to S^3$ which maps the (directed) knot $\mathfrak{k}_1$ onto the (directed) $\mathfrak{k}_2$. The orientation on S^3 defines orientations on V_1 and V_2, hence on the boundaries ∂V_1 and ∂V_2. Since $h(\mu_1) = \mu_2$ and $h(\lambda_1) = \lambda_2$ it follows that $h|\partial V_1: \partial V_1 \to \partial V_2$ is an orientation preserving mapping. This implies that $h|V_1: V_1 \to V_2$ is also orientation preserving; hence $h: S^3 \to S^3$ is orientation preserving. Thus $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are the 'same' knots. □

As direct corollary we obtain the following:

3.17 Corollary. *If two tame knots are topologically equivalent then they are p.l.-equivalent.* □

An immediate consequence of the proof of Theorem 3.16 is the following statement:

3.18 Proposition (Invertible or amphicheiral knots). *Let $(\mathfrak{G}, m, \ell)$ be the peripheral system of the knot $\mathfrak{k}$ in the oriented sphere S^3.*

(a) *$\mathfrak{k}$ is invertible if and only if there is an automorphism $\varphi: \mathfrak{G} \to \mathfrak{G}$ such that $\varphi(m) = m^{-1}$ and $\varphi(\ell) = \ell^{-1}$.*

(b) *$\mathfrak{k}$ is amphicheiral if and only if there is an automorphism $\psi: \mathfrak{G} \to \mathfrak{G}$ such that $\psi(m) = m^{-1}$ and $\psi(\ell) = \ell$.*

Proof. A knot $\mathfrak{k}$ is invertible if and only if there exists an orientation preserving homeomorphism $h: S^3 \to S^3$ such that $h(\mathfrak{k}) = -\mathfrak{k}$. Obviously, $(\mathfrak{G}, m^{-1}, \ell^{-1})$ is the peripheral system of $-\mathfrak{k}$ and assertion (a) is a direct consequence of Theorem 3.16.

In order to prove (b) note that $\mathfrak{k}$ is amphicheiral if and only if there is an orientation reserving homeomorphism $h: S^3 \to S^3$ such that $h(\mathfrak{k}) = \mathfrak{k}$. Now the peripheral system of $\mathfrak{k}$ in the reversed oriented sphere $-S^3$ is $(\mathfrak{G}, m^{-1}, \ell)$. (Note that $h|\partial V$ is orientation reversing.) □

The two trefoil knots can be distinguished by using the peripheral system. We will give a proof of this fact in a more general context in Theorem 3.39, but we suggest carrying out the calculations for the trefoil as in exercise E 3.9.

The peripheral group system $(\mathfrak{G}, \langle m, \ell \rangle)$ has not – at first glance – the same strength as the peripheral system $(\mathfrak{G}, m, \ell)$ since it classifies only the complement of the knot, see Waldhausen [367]. The question whether different knots may have homeomorphic complements was first raised by Tietze in 1908 [350, p. 83]. Gordon and Luecke proved in [137] that a knot complement determines the knot.

3.19 Theorem (Gordon–Luecke [137]). *Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be two unoriented knots in S^3. If the complements $S^3 - \mathfrak{k}_1$ and $S^3 - \mathfrak{k}_2$ are homeomorphic then there exists a homeomorphism $h: S^3 \to S^3$ such that $h(\mathfrak{k}_1) = \mathfrak{k}_2$.*

3.20 Remarks. (1) The analog of Theorem 3.19 holds for orientation preserving homeomorphisms too: if two unoriented knots have complements which are homeomorphic by an orientation preserving homeomorphism, then they are ambient isotopic.

(2) By a result of Edwards [90, Thm. 3], see also [131, Prop. 10.1], we have that $S^3 - \mathfrak{k}_1$ and $S^3 - \mathfrak{k}_2$ are homeomorphic if and only if the compact manifolds $C(\mathfrak{k}_1)$ and $C(\mathfrak{k}_2)$ are homeomorphic.

3.21 Definition (Dehn surgery). For integers $r, n \in \mathbb{Z}$, $\gcd(r,n) = 1$, let M denote the closed 3-manifold $C(\mathfrak{k}) \cup_f V'$ where V' is a solid torus with meridian m' and f an identifying homeomorphism $f\colon \partial V' \to \partial C(\mathfrak{k})$, $f(m') \sim rm + n\ell$ on $\partial C(\mathfrak{k})$. We say that M *is obtained from* S^3 *by Dehn surgery on* $\mathfrak{k}$ and write $M = S^3_{r/n}(\mathfrak{k})$. The Dehn surgery is called *trivial* if $|r| = 1$ and $n = 0$.

3.22 Remarks. (1) Note that trivial Dehn surgery on any knot in S^3 yields S^3.

(2) If $\mathfrak{k}$ is the trivial knot then $S^3_{1/n}(\mathfrak{k})$, $n \in \mathbb{Z}$, is homeomorphic to S^3.

(3) We have $H_1(S^3_{r/n}(\mathfrak{k})) \cong \mathbb{Z}_{|r|}$, see E 3.3. Therefore $M = S^3_{r/n}(\mathfrak{k})$ is a homology sphere, that is $H_*(M) \cong H_*(S^3)$, if and only if $|r| = 1$.

(4) It was proved by W. B. R. Lickorish [213] and A. H. Wallace [369] that every closed, orientable, connected 3-manifold may be obtained by Dehn surgery on a link in S^3 (see also Rolfsen [309, Chap. 9] and Saveliev [314, Sec. 2.2]).

Theorem 3.19 follows from the following result:

3.23 Theorem (Gordon–Luecke [137, Thm. 2]). *Non-trivial Dehn surgery on a non-trivial knot in* S^3 *never yields* S^3. □

We cannot give the proof here. For an outline see the announcement of C. M. Gordon and J. Luecke [138] or the lecture of A. Gramain from the *Séminaire Bourbaki* [141].

The following proposition shows that Theorem 3.23 implies Theorem 3.19.

3.24 Proposition. *Let* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$ *be two knots and* $h'\colon C(\mathfrak{k}_1) \to C(\mathfrak{k}_2)$ *a homeomorphism. If non-trivial surgery on* $\mathfrak{k}_2$ *never yields* S^3 *then there exists a homeomorphism* $h\colon S^3 \to S^3$ *such that* $h(\mathfrak{k}_1) = \mathfrak{k}_2$.

Proof. Let (m_1, ℓ_1) and (m_2, ℓ_2) be a meridian-longitude pair for $\mathfrak{k}_1$ and $\mathfrak{k}_2$ respectively. For homological reasons we have $h'_\#(m_1) = m_2^{\pm 1}\ell_2^n$ for some $n \in \mathbb{Z}$, see Paragraph 3.2. We can extend h' to a homeomorphism $h\colon S^3 \to S^3_{\pm 1/n}(\mathfrak{k})$. Since non-trivial surgery on $\mathfrak{k}_2$ never yields S^3 it follows that $n = 0$. Hence, $h\colon S^3 \to S^3$ is a homeomorphism with $h(\mathfrak{k}_1) = \mathfrak{k}_2$. □

3.25 Property P conjecture. Previous to the recent proof of the Poincaré conjecture (see Appendix B.10) it was a natural question whether one might be able to produce a counterexample to the Poincaré conjecture by a single Dehn surgery. This motivated R. Bing and J. Martin [22] and F. Gonzalez-Acuña [131] to study the following property:

3.26 Definition (Property P). A knot $\mathfrak{k}$ in S^3 has *Property P* if non-trivial Dehn surgery on $\mathfrak{k}$ never yields a simply connected 3-manifold. The *Property P conjecture* states that all non-trivial knots have Property P (see Kirby's problem list [197, Problem 1.15]).

Note that the trivial knot does not have Property P, see Remark 3.22. It is clear that a proof of the Property P conjecture implies Theorem 3.23. On the other hand, the Gordon–Luecke Theorem 3.23 and the Poincaré conjecture together imply the Property P conjecture.

The final step in the proof of the Property P conjecture was published in 2004 by P. B. Kronheimer and T. S. Mrowka [204] as the combined result of efforts of mathematicians working in several different fields. According to the 1987 article by M. Culler, C. M. Gordon, J. Luecke and P. B. Shalen [81], there are at most two possibilities to obtain a simply connected 3-manifold by Dehn surgery on a non-trivial knot:

3.27 Theorem (Culler et al. [81, Corollary 2]). *If $\mathfrak{k}$ is a non-trivial knot and $p/q \in \mathbb{Q}$ is not equal to 1 or -1, then $S^3_{p/q}(\mathfrak{k})$ is not simply connected. Moreover, $S^3_1(\mathfrak{k})$ and $S^3_{-1}(\mathfrak{k})$ cannot both be simply connected.* □

The proof of this theorem is beyond the scope of this book.

Using tools from gauge theory and symplectic topology, P. B. Kronheimer and T. S. Mrowka showed in 2004 the following result which implies the Property P conjecture. We cannot give a proof here (see the article by H. Geiges [127] for a sketch of the proof).

3.28 Theorem. *Let $\mathfrak{k}$ be a non-trivial knot in S^3. Then there exists a non-trivial homomorphism $\rho\colon \pi_1 S^3_1(\mathfrak{k}) \to \mathrm{SO}(3)$. Therefore, every non-trivial knot has Property P.* □

3.29 Remarks. (a) Observe that $\pi_1 S^3_{p/q}(\mathfrak{k})$ and $\pi_1 S^3_{-p/q}(\mathfrak{k}^*)$ are isomorphic, see E 3.4. Hence, Theorems 3.27 and 3.28 together imply the Property P conjecture.

(b) It follows from work of A. Casson (see [2, 143, 314]) that $\pi_1 S^3_1(\mathfrak{k})$ admits a non-trivial homomorphism to $\mathrm{SO}(3)$ if the symmetrized Alexander polynomial $\Delta_S(t)$ of $\mathfrak{k}$ satisfies $\Delta''_S(1) \neq 0$ (see Remark 9.26 and Exercises E 9.10). Such knots therefore have Property P. The argument used by A. Casson is closely related to the work of P. B. Kronheimer and T. S. Mrowka. The quantity $\Delta''_S(1)$ is equal to the Euler characteristic of a certain *Floer homology* group associated to $\mathfrak{k}$.

Kronheimer and Mrowka proved that this Floer homology group itself is always non-trivial if $\mathfrak{k}$ is not the unknot, even though the Euler characteristic $\Delta''_S(1)$ may vanish.

(c) It can be proved, with elementary means, that $\pi_1 S^3_{1/n}(\mathfrak{k})$, $n \neq 0$, admits a non-trivial homomorphism to SO(3) if $\mathfrak{k}$ is a non-trivial torus knot or a 2-bridge knot (see E. Klassen [198] and G. Burde [58]).

3.D Knots on handlebodies

The Wirtinger presentation of a knot group is easily obtained and is most frequently applied in the study of examples. It depends, however, strongly on the knot projection and, in general, it does not reflect geometric symmetries of the knot nor does it afford much insight into the structure of the knot group as we have seen in the preceding Sections 3.B and 3.C. In this section, we describe another method. In the simplest case, for solid tori, a detailed treatment will be given in Section 3.E.

3.30 Definition (Handlebody, Heegaard splitting).

(a) A *handlebody* V of *genus* g is obtained from a 3-ball $\mathbb{B}^3$ by attaching g handles $\mathbb{D}^2 \times I$ such that the boundary ∂V is an orientable closed surface of genus g, see Figure 3.8:

$$V = \mathbb{B}^3 \cup H_1 \cup \ldots \cup H_g, \quad H_i \cap H_j = \emptyset \quad (i \neq j),$$
$$H_i \cap \mathbb{B}^3 = D_{i1} \cup D_{i2}, \quad D_{i1} \cap D_{i2} = \emptyset, \quad D_{ij} \cong \mathbb{D}^2,$$

and $(\overline{\partial \mathbb{B}^3 - \bigcup_{i,j} D_{ij}}) \cup \bigcup_i (\overline{\partial H_i - (D_{i1} \cup D_{i2})})$ is a closed orientable surface of genus g.

Another often-used picture of a handlebody is shown in Figure 3.9.

(b) The decomposition of a closed orientable 3-manifold M^3 into two handlebodies V, W: $M^3 = V \cup W, V \cap W = \partial V = \partial W$, is called a *Heegaard splitting* or *decomposition* of M^3 of *genus* g.

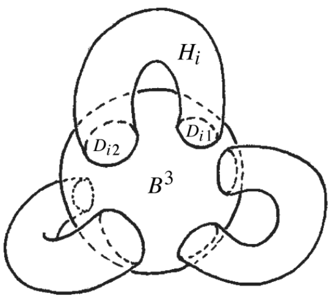


Figure 3.8

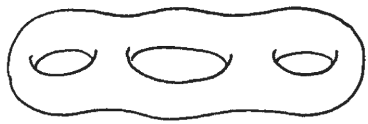


Figure 3.9

A convenient characterization of handlebodies is:

3.31 Proposition. *Let W be an orientable 3-manifold. If W contains a system $D_1, \dots, D_g$ of mutually disjoint disks such that $\partial W \cap D_i = \partial D_i$ and $\overline{W - \cup_i U(D_i)}$ is a closed 3-ball then W is a handlebody of genus g. (By $U(D_i)$ we denote closed regular neighborhoods of D_i with $U(D_i) \cap U(D_j) = \emptyset$ for $i \neq j$.)*

Proof as Exercise E 3.10. □

Each orientable closed 3-manifold M^3 admits Heegaard splittings; one of them can be constructed as follows: Consider the 1-skeleton of a triangulation of M^3, define V as a regular neighborhood of it and put $W = \overline{M^3 - V}$. Then V and W are handlebodies and form a Heegaard decomposition of M. (Proof as Exercise E 3.11; that V is a handlebody is obvious, that W is also can be proved using Proposition 3.31.) The classification problem of 3-manifolds can be reformulated as a problem on Heegaard decompositions, see K. Reidemeister [297] and J. Singer [339]. F. Waldhausen has shown in [366] that Heegaard splittings of S^3 are unique. We quote his theorem without proof.

3.32 Theorem (Heegaard splittings of S^3). *Any two Heegaard decompositions of S^3 of genus g are homeomorphic; more precisely: If (V, W) and (V', W') are Heegaard splittings of this kind then there exists an orientation preserving homeomorphism $h: S^3 \to S^3$ such that $h(V) = V'$ and $h(W) = W'$.* □

Next a direct application to knot theory.

3.33 Proposition. *Every knot in S^3 can be embedded in the boundary of the handlebodies of a Heegaard splitting of S^3.*

Proof. A (tame) knot $\mathfrak{k}$ can be represented by a regular projection onto S^2 which does not contain loops (see Figure 3.10). Let Γ be a graph of $\mathfrak{k}$ with vertices in the α-colored regions of the projection (cf. Definition 2.3), and let W be a regular neighborhood of Γ. Obviously the knot $\mathfrak{k}$ can be realized by a curve on ∂W, see Figure 3.11. $\mathfrak{k}$ can serve as a canonical curve on ∂W – if necessary add a handle to ensure $\mathfrak{k} \not\sim 0$ on ∂W.

W is a handlebody. To see this choose a tree T in Γ that contains all the vertices of Γ. It follows by induction on the number of edges of T that a regular neighborhood

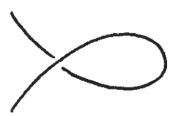


Figure 3.10. A loop in a regular projection.

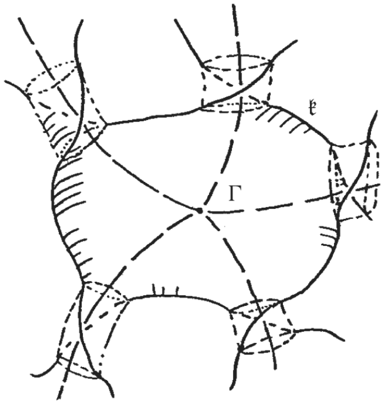


Figure 3.11

of T is a 3-ball B. A regular neighborhood W of Γ is obtained from B by attaching handles; for each of the segments of $\Gamma - T$ attach one handle.

$\overline{S^3 - W}$ is also a handlebody: The finite β-regions represent disks D_i such that $D_i \cap W = \partial D_i$. If one dissects $\overline{S^3 - W}$ along the disks D_i one obtains a ball, see Figure 3.12. □

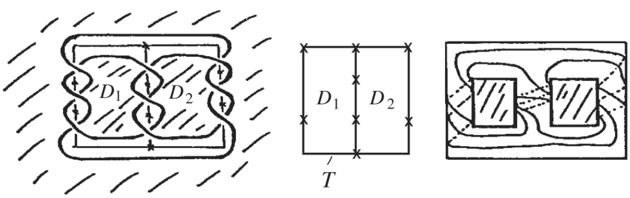


Figure 3.12

We can now obtain a new presentation of the group of the knot $\mathfrak{k}$:

3.34 Proposition. *Let W, W' be a Heegaard splitting of S^3 of genus g. Assume that the knot $\mathfrak{k}$ is represented by a curve on the surface $F = \partial W = \partial W'$. Choose free generators $s_i, s_i', 1 \le i \le g$, $\pi_1 W = \langle s_1, \dots, s_g \mid - \rangle$, $\pi_1 W' = \langle s_i', \dots, s_g' \mid - \rangle$, and a canonical system of curves $\kappa_l, 1 \le l \le 2g$, on $F = W \cap W'$ with a common basepoint P, such that $\kappa_2 = \mathfrak{k}$. If κ_i is represented by a word $w_i(s_j) \in \pi_1 W$ and by $w_i'(s_j') \in \pi_1 W'$, then*

(a) $\mathfrak{G} = \pi_1(S^3 - V(\mathfrak{k})) = \langle s_1, \dots, s_g, s_1', \dots, s_g' \mid w_i(w_i')^{-1},\ 2 \le i \le 2g \rangle$

(b) *$w_1(s_j)(w_1'(s_j'))^{-1}$ can be represented by a meridian m or m^{-1}, and, for some (well-defined) integer r, $w_2(s_j)(w_1(w_1')^{-1})^r$ can be represented by a longitude, if the basepoint is suitably chosen.*

Proof. Assertion (a) is an immediate consequence of van Kampen's theorem.

For the proof of (b) let D be a disk in the tubular neighborhood $V(\mathfrak{k})$, spanning a meridian m of $\mathfrak{k}$, and let D meet κ_1 in a subarc κ_1' which contains the basepoint P, see Figure 3.13. We choose the endpoint of σ as a basepoint for the fundamental group $\pi_1(S^3 - V(\mathfrak{k}))$. The boundary ∂D is composed of two arcs $\nu = \partial D \cap W$, $\nu' = \partial D \cap W'$, $\partial D = \nu^{-1}\nu'$, such that $\nu^{-1}\nu'$ is a meridian m or m^{-1}. For $\kappa_1'' = \kappa_1 - \kappa_1'$, the paths $\nu^{-1}\kappa_1''$ resp. $\nu'^{-1}\kappa_1''$ represent $w_1(s_j)$ resp. $w_1'(s_j)$; hence $w_1(w_1')^{-1} = \nu^{-1}\nu'$. A longitude ℓ is represented by a simple closed curve λ on ∂V, $\lambda \sim \kappa_2$ in V, which is null-homologous in $C = \overline{S^3 - V}$. Hence λ represents $w_2(s_j) \cdot (w_1(w_1')^{-1})^r$ for some (uniquely determined) integer r (see Remark 3.14). □

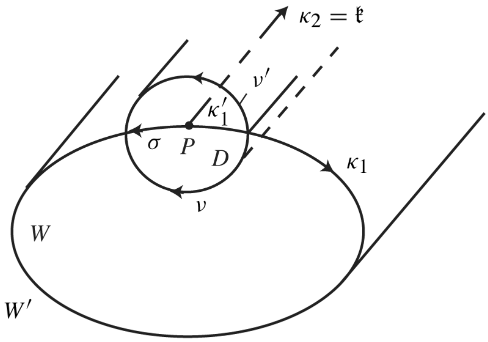


Figure 3.13

In what follows we shall denote by $\mathfrak{F}_n$ the free group of rank n.

3.35 Corollary. *Assume $S^3 = W \cup W', W \cap W' = F \supset \mathfrak{k}$ as in Proposition 3.34. If the inclusions $i\colon \overline{F - V} \to W$, $i'\colon \overline{F - V} \to W'$ induce injective homomorphisms of the corresponding fundamental groups, then*

$$\mathfrak{G} = \pi_1(\overline{S^3 - V(\mathfrak{k})}) = \pi_1 W *_{\pi_1(F-V)} \pi_1 W' = \mathfrak{F}_g *_{\mathfrak{F}_{2g-1}} \mathfrak{F}_g.$$

There is a finite algorithm by which one can decide whether the assumption of the corollary is valid. In this case the knot group $\mathfrak{G}$ has a non-trivial center if and only if $g = 1$.

Proof. Since $F - V$ is connected, it is an orientable surface of genus $g - 1$ with two boundary components. $\pi_1(\overline{F - V})$ is a free group of rank $2(g-1) + 1$.

There is an algorithm due to Nielsen [272], see [382, 1.7], by which the rank of the finitely generated subgroup $i_{\#}\pi_1(\overline{F - V})$ in the free group $\pi_1 W = \mathfrak{F}_g$ can be determined. Moreover, $i_{\#}$ is injective if and only the rank of $i_{\#}\pi_1(\overline{F - V})$ is $2g - 1$. The remark about the center follows from the fact that the center of a proper product with amalgamation is contained in the amalgamating subgroup. □

We propose to study the case $g = 1$, the torus knots, in the following section. They form the simplest class of knots and can be classified. For an intrinsic characterization of torus knots see Theorem 6.1.

3.E Torus knots

Let $S^3 = \mathbb{R}^3 \cup \{\infty\} = W \cup W'$ be a 'standard' Heegaard splitting of genus 1 of the oriented 3-sphere S^3. We may assume W to be an unknotted solid torus in $\mathbb{R}^3$ and $F = W \cap W'$ a torus carrying the orientation induced by that of W. There are meridians μ and ν of W and W' on F which intersect in the basepoint P on F, see Figure 3.14. We choose the orientation of μ and ν such they form a standard meridian and longitude of the trivial knot represented by the core of W.

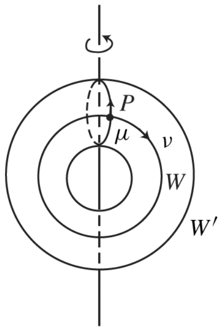


Figure 3.14

Any closed curve κ on F is homotopic to a curve $\mu^a \cdot \nu^b, a, b \in \mathbb{Z}$. Its homotopy class on F contains a (non-trivial) simple closed curve if and only if a and b are relatively prime. Such a simple curve intersects μ resp. ν exactly b resp. a times with intersection number $+1$ or -1 according to the signs of a and b. Two simple closed curves $\kappa = \mu^a \nu^b$, $\lambda = \mu^c \nu^d$ on F intersect, eventually after an isotopy, in a single point if and only if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1$, where the exact value of the determinant is the intersection number of κ with λ.

3.36 Definition (Torus knots). Let (W, W') be the Heegaard splitting of genus 1 of S^3 described above. If $\mathfrak{k}$ is a simple closed curve on F represented by the curve $\mu^a \nu^b$ and if $|a|, |b| \geq 2$ then $\mathfrak{k}$ is called a torus knot, more precisely, *the torus knot* $\mathfrak{t}(a, b)$.

3.37 Proposition.

(a) $\mathfrak{t}(-a, -b) = -\mathfrak{t}(a, b)$, $\mathfrak{t}(a, -b) = \mathfrak{t}^*(a, b)$ *(see Definition 2.1).*

(b) $\mathfrak{t}(a, b) = \mathfrak{t}(-a, -b) = \mathfrak{t}(b, a)$*: torus knots are invertible.*

Proof. The first assertion of (a) is obvious. A reflection in a plane shows $\mathfrak{t}(a,-b) = \mathfrak{t}^*(a,b)$. A rotation through π illustrates the equation $\mathfrak{t}(a,b) = \mathfrak{t}(-a,-b)$, see Figure 3.14. Exchanging the role of W and W' gives $\mathfrak{t}(a,b) = \mathfrak{t}(b,a)$. □

3.38 Proposition. (a) *The group $\mathfrak{G}$ of the torus knot $\mathfrak{t}(a,b)$ can be presented as follows:*

$$\mathfrak{G} = \langle\, u, v \mid u^a v^{-b} \,\rangle = \langle\, u|-\rangle *_{\langle\, u^a = v^b \,\rangle} \langle\, v|-\rangle,$$

where μ represents the generator u of $\pi_1 W'$ and ν represents the generator v of $\pi_1 W$. The amalgamating subgroup $\langle\, u^a \,\rangle$ is an infinite cyclic group; it represents the center $\mathfrak{C} = \langle\, u^a \,\rangle \cong \mathbb{Z}$ of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{C} \cong \mathbb{Z}_{|a|} * \mathbb{Z}_{|b|}$.

(b) *The elements $m = v^d u^{-c}$, $\ell = u^a m^{-ab}$, where $ad - bc = 1$, describe meridian and longitude of $\mathfrak{t}(a,b)$ for a suitable chosen basepoint.*

(c) *$\mathfrak{t}(a,b)$ and $\mathfrak{t}(a',b')$ have isomorphic groups if and only if $|a| = |a'|$ and $|b| = |b'|$ or $|a| = |b'|$ and $|b| = |a'|$.*

Proof. The curve $\mathfrak{t}(a,b) = \mu^a \nu^b$ belongs to the homotopy class u^a of W' and to v^b of W. This implies the first assertion of (a) by Proposition 3.34 (a). It is clear that u^a belongs to the center of the knot group $\mathfrak{G}$. If we introduce the relation $u^a = 1$ we obtain the free product

$$\langle\, u, v \mid u^a, v^b \,\rangle = \langle\, u \mid u^a \,\rangle * \langle\, v \mid v^b \,\rangle \cong \mathbb{Z}_{|a|} * \mathbb{Z}_{|b|}.$$

Since this group has a trivial center, see [382, 2.3.9] it follows that u^a generates the center. Moreover, $\mathfrak{G} = \langle\, u|-\rangle *_{\langle u^a = v^b \rangle} \langle\, v|-\rangle$ implies that each of the factor subgroups is free.

In order to prove (b) note first that the curve $\kappa_1 = \mu^c \nu^d$ with $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = 1$ is a simple closed curve on F intersecting $\kappa_2 = \mathfrak{t}(a,b) = \mu^a \nu^b$ in a unique point. From Proposition 3.34 (b) it follows that the meridian $m^{\pm 1}$ of $\mathfrak{t}(a,b)$ belongs to the homotopy class $v^d u^{-c}$. Moreover, $\operatorname{lk}(u,v) = 1$ and $\operatorname{lk}(\mathfrak{t}(a,b), m) = 1$ imply that $m = v^d u^{-c}$ is a meridian of $\mathfrak{t}(a,b)$. Since the classes $u^a = v^b$ and $v^d u^{-c}$ can be represented by two simple closed curves on ∂V intersecting in one point the class $u^a\,(v^d u^{-c})^{-ab}$ can be represented by a simple closed curve on ∂V. Since it becomes trivial by abelianizing it is the class of a longitude. This implies (b).

Assertion (c) is a consequence of the fact that u and v generate non-conjugate maximal finite cyclic subgroups in the free product $\mathbb{Z}_{|a|} * \mathbb{Z}_{|b|}$, cf. [382, 2.3.10]. □

3.39 Theorem (Classification of torus knots). (a) $\mathfrak{t}(a,b) = \mathfrak{t}(a',b')$ *if and only if (a',b') is equal to one of the following pairs:* (a,b), (b,a), $(-a,-b)$, $(-b,-a)$.

(b) *Torus knots are invertible, but not amphicheiral.*

Proof. Sufficiency follows from Proposition 3.37. Suppose $\mathfrak{t}(a,b) = \mathfrak{t}(a',b')$. It follows from Proposition 3.38 (c) that $\mathfrak{t}(a,b) = \mathfrak{t}(a',b')$ implies that $|a| = |a'|$, $|b| = |b'|$ or $|a| = |b'|, |b| = |a'|$.

By Proposition 3.37 (b) it remains to prove that torus knots are not amphicheiral i.e. $\mathfrak{t}(a,b) \neq \mathfrak{t}(a,-b)$. Let us assume $a,b > 0$ and $\mathfrak{t}(a,b) = \mathfrak{t}(a,-b)$. By Proposition 3.15 there is an isomorphism

$$\varphi\colon \mathfrak{G} = \langle\, u, v \mid u^a v^{-b} \,\rangle \to \langle\, u', v' \mid (u')^a (v')^b \,\rangle = \mathfrak{G}^*$$

mapping the peripheral system $(\mathfrak{G}, m, \ell)$ onto $(\mathfrak{G}^*, m', \ell')$:

$$m' = \varphi(v^d u^{-c}) = (v')^{d'} (u')^{-c'},$$
$$\ell' = \varphi(u^a (v^d u^{-c})^{-ab}) = (u')^a \big((v')^{d'} (u')^{-c'}\big)^{+ab}$$

with $ad - bc = ad' + bc' = 1$.

It follows that $d' = d + jb$ and $-c' = c + ja$ for some $j \in \mathbb{Z}$ and hence

$$m' = (v')^{d'} (u')^{-c'} = (v')^d \big((v')^b (u')^a\big)^j (u')^c = (v')^d (u')^c.$$

The isomorphism φ maps the center $\mathfrak{C}$ of $\mathfrak{G}$ onto the center $\mathfrak{C}^*$ of $\mathfrak{G}^*$. This implies that $\varphi(u^a) = (u')^{\varepsilon a}$ for $\varepsilon \in \{1, -1\}$. Now,

$$\ell' = (u')^a \big((v')^d (u')^c\big)^{ab} = \varphi(u^a (v^d u^{-c})^{-ab}) = \varphi(u^a) \varphi(v^d u^{-c})^{-ab}$$
$$= (u')^{\varepsilon a} \big((v')^d (u')^c\big)^{-ab};$$

hence $(u')^{(1-\varepsilon)a} = \big((v')^d (u')^c\big)^{-2ab}$. This equation is impossible: the homomorphism $\mathfrak{G}^* \to \mathfrak{G}^*/\mathfrak{C}^* \cong \mathbb{Z}_a * \mathbb{Z}_b$ maps the term on the left onto unity, whereas the term on the right represents a non-trivial element of $\mathbb{Z}_a * \mathbb{Z}_b$ because $a \nmid c$ and $b \nmid d$. This follows from the solution of the word problem in free products, see [382, 2.3.3]. □

3.F Asphericity of the knot complement

In this section we use some notions and deeper results from algebraic topology, in particular, the notion of a $K(\pi, 1)$-space, π a group: X is called a $K(\pi, 1)$-*space* if $\pi_1 X = \pi$ and $\pi_n X = 0$ for $n \neq 1$. X is also called *aspherical*. Note that a CW-complex X is aspherical if and only if its universal covering space $\widetilde{X}$ is contractible. For more details see [157, 341].

3.40 Theorem. *Let $\mathfrak{k} \subset S^3$ be a knot, C the complement of an open regular neighborhood V of $\mathfrak{k}$. Then*

(a) $\pi_n C = 0$ *for* $n \neq 1$*; in other words,* C *is a* $K(\pi_1 C, 1)$*-space.*

(b) $\pi_1 C$ *is torsion free.*

Proof. $\pi_0 C = 0$ since C is connected. Assume that $\pi_2 C \neq 0$. By the Sphere Theorem [281], see Appendix B.6 [159, 4.3] there is an embedded p.l.-2-sphere $S \subset C$ which is not null-homotopic. By the Alexander–Schoenflies Theorem 1.6 (see [242, p. 122]), S divides S^3 into two 3-balls B_1 and B_2. Since $\mathfrak{k}$ is connected it follows that one of the balls, say B_2, contains V and $B_1 \subset C$. Therefore S is null-homotopic, contradicting the assumption. This proves $\pi_2 C = 0$.

To calculate $\pi_3 C$ we consider the universal covering $\widetilde{C}$ of C. As $\pi_1 C$ is infinite, $\widetilde{C}$ is not compact, and this implies $H_3(\widetilde{C}) = 0$. As $\pi_1 \widetilde{C} = 0$ and $\pi_2 \widetilde{C} = \pi_2 C = 0$ it follows from the Hurewicz Theorem (see [341, 7.5.2], [346, 16.8.4], [157, Sec. 4.2]) that $\pi_3 \widetilde{C} = \pi_3 C = 0$. By the same argument $\pi_n C = \pi_n \widetilde{C} = H_n(\widetilde{C}) = 0$ for $n \geq 4$.

This proves (a). To prove (b) assume that $\pi_1 C$ contains a non-trivial element x of finite order $m > 1$. The cyclic group generated by x defines a covering $p\colon \overline{C} \to C$ with $\pi_1 \overline{C} = \mathbb{Z}_m$. As $\pi_n \overline{C} = 0$ for $n > 1$ it follows that $\overline{C}$ is a $K(\mathbb{Z}_m, 1)$-space hence, $H_n(\overline{C}) = \mathbb{Z}_m$ for n odd, see [221, IV Theorem 7.1]. This contradicts the fact that $\overline{C}$ is a 3-manifold. See also Hatcher [157, Prop. 2.45]. □

3.41 Remark. Theorem 3.40 holds also if C is replaced by an orientable irreducible 3-manifold M with infinite fundamental group. Here a 3-manifold M is *irreducible* if each embedded 2-sphere bounds a ball in M. The proof is identical.

3.G History and sources

The knot groups became an important tool in knot theory very early on. The method presenting groups by generators and defining relations was developed by W. Dyck [89], pursuing a suggestion of A. Cayley [64]. The best known knot group presentations were introduced by W. Wirtinger; however, in the literature only the title "Über die Verzweigung bei Funktionen von zwei Veränderlichen" of this talk at the Jahresversammlung der Deutschen Mathematiker Vereinigung in Meran 1905 in Jahresber. DMV 14, 517 (1905) is mentioned. His student K. Brauner later used the Wirtinger presentations again in the study of singularities of algebraic surfaces in $\mathbb{R}^4$ and mentioned that these presentations were introduced by Wirtinger, see [41]. M. Dehn [84, 87] introduced the notion of a knot group and implicitly used the peripheral system to show that the two trefoils are non-equivalent in [85, 87]. (He used a different presentation for the knot group, see E 3.15.) O. Schreier [316, 87] classified the groups $\langle A, B \mid A^a B^b = 1 \rangle$ and determined their automorphism groups; this permitted to classify the torus knots. R. H. Fox [116] introduced the peripheral system and showed its importance by distinguishing the square knot and the granny knot. These knots have isomorphic groups: there is, however, no isomorphism preserving the peripheral system.

Dehn's Lemma, the Loop and the Sphere Theorem, proved by C. D. Papakyriakopoulos in [282, 281] opened new ways to knot theory, in particular, C. D. Papakyriakopoulos showed that knot complements are aspherical. F. Waldhausen [367] found the full strength of the peripheral system, showing that it determines the knot complement and, hence, the knot type (see the Gordon–Luecke Theorem 3.19 [137]). New tools for the study and use of knot groups have been made available by R. Riley and W. P. Thurston discovering a hyperbolic structure in many knot complements. See for example W. P. Thurston [349], A. Marden [228] and F. Bonahon and L. Siebenmann [37] and the references therein.

3.H Exercises

E 3.1. Compute the relative homology $H_i(S^3, \mathfrak{k})$ for a knot $\mathfrak{k}$ and give a geometric interpretation of the generator of $H_2(S^3, \mathfrak{k}) \cong \mathbb{Z}$.

E 3.2. Calculate the homology $H_i(S^3 - \mathfrak{k})$ of the complement of a link $\mathfrak{k}$ with μ components.

E 3.3. Let $\mathfrak{k}$ be a knot with meridian m and longitude ℓ. Show that attaching a solid torus with meridian m' to the complement of $\mathfrak{k}$ defines a homology sphere if and only if m' is mapped to $m^{\pm 1}\ell^r$. Prove that $H_1(S^3_{r/n}(\mathfrak{k})) \cong \mathbb{Z}_{|r|}$.

E 3.4. Prove that there is an orientation reversing homeomorphism between $S^3_{-r/n}(\mathfrak{k})$ and $S^3_{r/n}(\mathfrak{k}^*)$.

E 3.5. Let $\mathfrak{G}' = \langle \{x_n, n \in \mathbb{Z}\} \mid \{x_{n+1}x_{n+2}^{-1}x_{n+1}x_{n+2}^{-1}x_n^{-1}x_{n+1}x_n^{-1}, n \in \mathbb{Z}\} \rangle$. Prove:

(a) $\mathfrak{G}'$ is not finitely generated.

(b) The subgroups $\mathfrak{A}_n = \langle x_n, x_{n+1} \rangle$, $\mathfrak{B}_n = \langle x_{n+1}, x_n x_{n+1}^{-1} x_n = (x_{n+1}x_{n+2}^{-1})^2 \rangle$ of $\mathfrak{G}'$ are free groups of rank 2, and

$$\mathfrak{G}' = \cdots *_{\mathfrak{B}_{-2}} \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{\mathfrak{B}_1} \cdots .$$

(For this exercise compare Example 3.9 and Theorem 4.7.)

E 3.6. Calculate the groups and peripheral systems of the knots in Figure 3.15. (See Lemma 15.2.)

E 3.7. Express the peripheral system of a product knot in terms of those of the factor knots.

Figure 3.15

E 3.8. Let $\mathfrak{G}$ be a knot group and $\varphi\colon \mathfrak{G} \to \mathbb{Z}$ a non-trivial homomorphism. Then $\ker\varphi = \mathfrak{G}'$.

E 3.9. Show that the two trefoil knots can be distinguished by their peripheral systems. *Hint: follow the steps:*

- *Show that the epimorphism* $\rho\colon \mathfrak{G} \to \mathrm{SL}_2(\mathbb{Z})$ *defined in Example 3.7 maps the center of* $\mathfrak{G}$ *to* $\{\pm I_2\}$ *(here we denote by* $I_2 \in \mathrm{SL}_2(\mathbb{Z})$ *the identity matrix).*
- *Use Figure 3.4 and the procedure given in Remark 3.14 to read off a longitude* $\ell = m^6 z^{-1}$ *where* $m = s_2 = yx^{-1}$ *and* $z = x^3 = y^2$ *is in the center of* $\mathfrak{G}$.
- *Suppose that there exists an isomorphism* $\varphi\colon \mathfrak{G} \to \mathfrak{G}$ *such that* $\varphi(m) = m$ *and* $\varphi(\ell) = \ell^{-1}$. *Calculate* $\varphi(z)$ *and observe that* $\rho\varphi(z)$ *is not in the center of* $\mathrm{SL}_2(\mathbb{Z})$.

E 3.10. Prove Proposition 3.31.

E 3.11. Show that a regular neighborhood V of the 1-skeleton of a triangulation of S^3 (or any closed orientable 3-manifold M) and $\overline{S^3 - V}$ ($\overline{M - V}$, respectively) form a Heegaard splitting of S^3 (or M).

E 3.12. Prove that $\mathfrak{F}_g *_{\mathfrak{F}_{2g-1}} \mathfrak{F}_g$ has a trivial center for $g > 1$. (Here $\mathfrak{F}_g$ is the free group of rank g.)

E 3.13. Let $h\colon S^3 \to S^3$ be an orientation preserving homeomorphism with $h(\mathfrak{k}) = \mathfrak{k}$ for a knot $\mathfrak{k} \subset S^3$. Show that h induces an automorphism $h_*\colon \mathfrak{G}'/\mathfrak{G}'' \to \mathfrak{G}'/\mathfrak{G}''$ which commutes with $\alpha\colon \mathfrak{G}'/\mathfrak{G}'' \to \mathfrak{G}'/\mathfrak{G}''$, $x \mapsto t^{-1}xt$, where t represents a meridian of $\mathfrak{k}$.

E 3.14. Let V_1, V_2 be solid tori with meridians m_1, m_2. A homeomorphism $h\colon \partial V_1 \to \partial V_2$ can be extended to a homeomorphism $H\colon V_1 \to V_2$ if and only if $h(m_1) \sim m_2^{\pm 1}$ on ∂V_2.

E 3.15. (*Dehn presentation*) Derive from a regular knot projection a presentation of the knot group of the following kind: Assign a generator to each of the finite regions of the projection, and a defining relation to each double point.

Chapter 4

Commutator subgroup of a knot group

There is no practicable procedure to decide whether two knot groups, given, say by Wirtinger presentations, are isomorphic. It has proved successful to investigate instead certain homomorphic images of a knot group $\mathfrak{G}$ or distinguished subgroups. The abelianized group $\mathfrak{G}/\mathfrak{G}' \cong H_1(C)$, though, is not helpful, since it is infinite cyclic for all knots, see Theorem 3.1. However, the commutator subgroup $\mathfrak{G}'$ together with the action of the infinite cyclic group $\mathfrak{Z} = \mathfrak{G}/\mathfrak{G}'$ is a strong invariant which nicely corresponds to geometric properties of the knot complement; this is studied in Chapter 4. Another fruitful invariant is the metabelian group $\mathfrak{G}/\mathfrak{G}''$ which is investigated in Chapters 8–9. All these groups are closely related to cyclic coverings of the complement.

4.A Construction of cyclic coverings

For the group $\mathfrak{G}$ of a knot $\mathfrak{k}$ the property $\mathfrak{G}/\mathfrak{G}' \cong \mathfrak{Z}$ implies that there are epimorphisms $\mathfrak{G} \to \mathfrak{Z}$ and $\mathfrak{G} \to \mathfrak{Z}_n = \mathfrak{Z}/n\mathfrak{Z}$, $n \geq 2$, such that their kernels $\mathfrak{G}'$ and $\mathfrak{G}_n$ are characteristic subgroups of $\mathfrak{G}$, hence, invariants of $\mathfrak{k}$. Moreover, $\mathfrak{G}$ and $\mathfrak{G}_n$ are semidirect products of $\mathfrak{Z}$ and $\mathfrak{G}'$:

$$\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}' \quad \text{and} \quad \mathfrak{G}_n = n\mathfrak{Z} \ltimes \mathfrak{G}',$$

where $n\mathfrak{Z}$ denotes the subgroup of index n in $\mathfrak{Z}$ and the operation of $n\mathfrak{Z}$ on $\mathfrak{G}'$ is the induced one.

The following Proposition 4.1 is a consequence of the general theory of coverings. However, in 4.4 we give an explicit construction and reprove most of Proposition 4.1.

4.1 Proposition and Definition (Cyclic coverings). *Let C denote the complement of a knot $\mathfrak{k}$ in S^3. Then there are regular coverings*

$$p_n\colon C_n \to C,\ 2 \leq n \leq \infty,$$

such that $p_{n\#}(\pi_1 C_n) = \mathfrak{G}_n$ and $p_{\infty\#}(\pi_1 C_\infty) = \mathfrak{G}'$. The n-fold covering is uniquely determined.

The group of covering transformations is $\mathfrak{Z}$ for $p_\infty\colon C_\infty \to C$ and $\mathfrak{Z}_n$ for $p_n\colon C_n \to C$, $2 \leq n < \infty$.?

The covering $p_\infty\colon C_\infty \to C$ is called the infinite cyclic covering*, the coverings $p_n\colon C_n \to C$, $2 \leq n < \infty$, are called the* finite cyclic coverings of the knot complement *(or, inexactly, of the knot $\mathfrak{k}$).* □

The main tool for the announced construction is the *cutting of the complement along a surface*; this process is inverse to pasting parts together.

4.2 Cutting along a surface. Let M be a 3-manifold and S a two-sided surface in M with $\partial S = S \cap \partial M$. Let U be a regular neighborhood of S; then $U - S = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $U_i \cong S \times (0,1]$. Let M_0', U_1', U_2' be homeomorphic copies of $\overline{M-U}, \overline{U}_1, \overline{U}_2$, respectively, and let $f_0\colon \overline{M-U} \to M_0'$, $f_i\colon \overline{U}_i \to U_i'$ be homeomorphisms. Let M' be obtained from the disjoint union $U_1' \cup M_0' \cup U_2'$ by identifying $f_0(x)$ and $f_i(x)$ when $x \in \overline{M-U} \cap \overline{U}_i = \partial(\overline{M-U}) \cap \partial U_i$, $i \in \{1,2\}$. *The result M' is a 3-manifold* and we say that M' is *obtained by cutting M along S*. There is a natural mapping $j\colon M' \to M$.

Cutting along a one-sided surface can be described in a slightly more complicated way (Exercise E 4.1). The same construction can be done in other dimensions; in fact, the classification of surfaces is usually based on cuts of surfaces along curves, see Figure 4.1. A direct consequence of the definition is the following proposition.

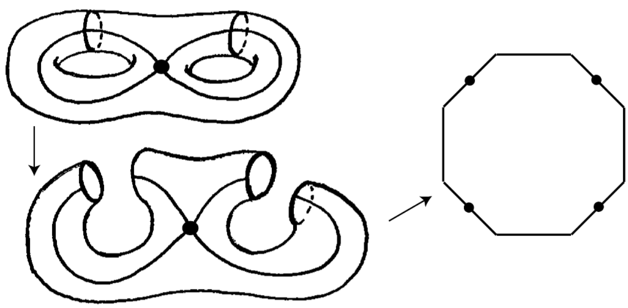


Figure 4.1. Cutting a surface along curves.

4.3 Proposition. (a) *M' is a 3-manifold homeomorphic to $\overline{M-U} = M_0'$.*

(b) *There is an identification map $j\colon M' \to M$ which induces a homeomorphism $M' - j^{-1}(S) \to M - S$.*

(c) *The restriction $j\colon j^{-1}(S) \to S$ is a two-fold covering. When S is two-sided $j^{-1}(S)$ consists of two copies of S; when S is one-sided $j^{-1}(S)$ is connected.*

(d) *When S is two-sided an orientation of M' induces orientations on both components of $j^{-1}(S)$. They are projected by j onto opposite orientations of S, if M' is connected.* □

4.4 Construction of the cyclic coverings. The notion? of cutting now permits a convenient description of the cyclic coverings $p_n\colon C_n \to C$: Let V be a regular neighborhood of the knot $\mathfrak{k}$ and S' a Seifert surface. Assume that $V \cap S'$ is an annulus and that $\lambda = \partial V \cap S'$ is a simple closed curve, that is a longitude of $\mathfrak{k}$. Define $C = \overline{S^3 - V}$ and $S = S' \cap C$. Cutting C along S defines a 3-manifold C^*. The boundary of C^*

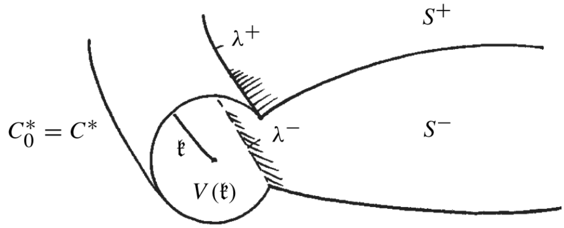


Figure 4.2. Cutting the complement C along a Seifert surface S.

is a connected surface and consists of two disjoint parts S^+ and S^-, both homeomorphic to S, and an annulus R which is obtained from the torus $\partial V = \partial C$ by cutting along λ:

$$\partial C^* = S^+ \cup R \cup S^-, S^+ \cap R = \lambda^+, S^- \cap R = \lambda^-, \partial R = \lambda^+ \cup \lambda^-,$$

see Figure 4.2. (C^* is homeomorphic to the complement of a regular neighborhood of the Seifert surface S.) Let $r\colon S^+ \to S^-$ be the homeomorphism mapping a point from S^+ to the point of S^- which corresponds to the same point of S. Let $i^+\colon S^+ \to C^*$ and $i^-\colon S^- \to C^*$ denote the inclusions.

Take homeomorphic copies C_j^* of C^* $(j \in \mathbb{Z})$ with homeomorphisms $h_j\colon C^* \to C_j^*$. The topological space C_∞ is obtained from the disjoint union $\bigcup_{j=-\infty}^{\infty} C_j^*$ by identifying $h_j(x)$ and $h_{j+1}(r(x))$ when $x \in S^+$, $j \in \mathbb{Z}$; see Figure 4.3. The space C_n is defined by starting with $\bigcup_{j=0}^{n-1} C_j^*$ and identifying $h_j(x)$ with $h_{j+1}(r(x))$ and $h_n(x)$ with $h_1(r(x))$ when $x \in S^+$, $1 \le j \le n-1$. For $2 \le n \le \infty$ define $p_n(x) = \iota(h_i^{-1}(x))$ if $x \in C_i^*$; here ι denotes the identification mapping $C^* \to C$, see Proposition 4.3 (b). It easily follows that $p_n\colon C_n \to C$ is an n-fold covering.

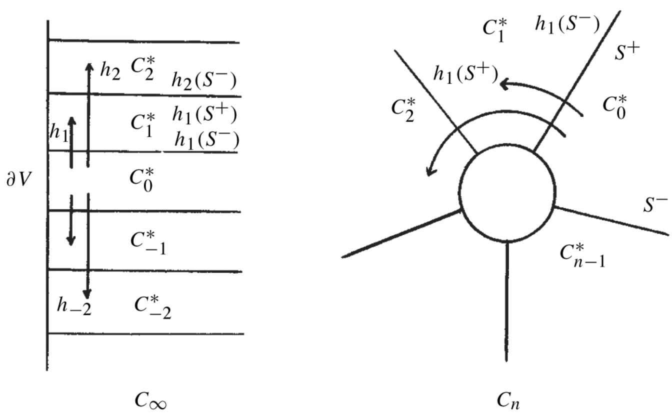


Figure 4.3. The cyclic coverings.

By $t|C_j^* = h_{j+1}h_j^{-1}, j \in \mathbb{Z}$, a covering transformation $t\colon C_\infty \to C_\infty$ of the covering $p_\infty\colon C_\infty \to C$ is defined. For any two points $x_1, x_2 \in C_\infty$ with the same p_∞-image in C there is an exponent m such that $t^m(x_1) = x_2$. Thus the covering $p_\infty\colon C_\infty \to C$ is regular, the group of covering transformations is infinite cyclic and t generates it. Hence, $p_\infty\colon C_\infty \to C$ is *the* infinite cyclic covering of Proposition 4.1. In the same way it follows that $p_n\colon C_n \to C$ $(2 \le n < \infty)$ is *the* n-fold cyclic covering. The generating covering transformation t_n is defined by

$$\begin{aligned} t_n|C_j &= h_{j+1}h_j^{-1} \qquad \text{for } 1 \le j \le n-1,\\ t_n|C_n &= h_1h_n^{-1}. \end{aligned}$$

4.B Structure of the commutator subgroup

Using the Seifert–van Kampen Theorem the groups $\mathfrak{G}' = \pi_1 C_\infty$ and $\mathfrak{G}_n = \pi_1 C_n$ can be calculated from $\pi_1(C^*)$ and the homomorphisms $i_\#^\pm\colon \pi_1 S^\pm \to \pi_1 C^*$.

4.5 Lemma (Neuwirth). *When S is a Seifert surface of minimal genus spanning the knot $\mathfrak{k}$ the inclusions $i^\pm\colon S^\pm \to C^*$ induce monomorphisms $i_\#^\pm\colon \pi_1 S^\pm \to \pi_1 C^*$.*

Proof. Note first that a simple application of the Seifert–van Kampen theorem gives that the inclusion $S^\pm \to \partial C^*$ induces an injective homomorphism $\pi_1 S^\pm \to \pi_1 \partial C^*$ since $g(S^\pm) > 0$.

If, e.g. $i_\#^+$ is not injective, then, by the Loop Theorem (see Appendix B.5) there is a simple closed curve ω on S^+, $\omega \not\simeq 0$ in ∂C^*, and a disk $\delta \subset C^*$ such that $\partial\delta = \omega = \delta \cap \partial C^* = \delta \cap S^+$. Replace S^+ by $S_1^+ = (S^+ - U(\delta)) \cup \delta_1 \cup \delta_{-1}$ where $U(\delta) = [-1, +1] \times \delta$ is a regular neighborhood of δ in C with $\delta_i = i \times \delta, 0 \times \delta = \delta$. Then $g(S_1^+) + 1 = g(S^+)$, g the genus, contradicting the minimality of $g(S)$, if S_1^+ is connected. If not, the component of S_1^+ containing ∂S^+ has smaller genus than S^+, since $\omega \not\simeq 0$ in S^+; again this leads to a contradiction to the assumption on S. Compare with Figure 4.4. □

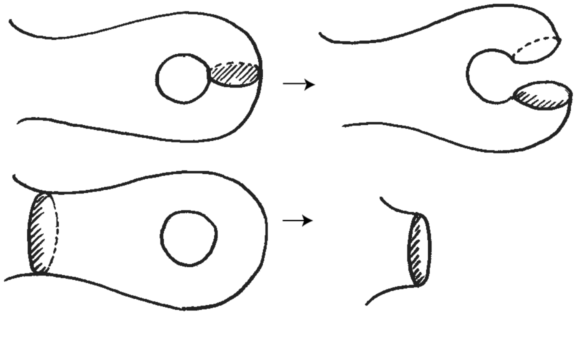


Figure 4.4

4.6 Corollary. *When S is a Seifert surface of minimal genus spanning the knot $\mathfrak{k}$ the inclusions $i\colon S \to C$ induce monomorphisms $i_\#\colon \pi_1 S \to \pi_1 C$.*

Proof. By 4.5, the inclusions $i^\pm\colon S^\pm \to C^*$ induce monomorphisms $i^\pm_\#\colon \pi_1 S^\pm \to \pi_1 C^*$. By the Seifert–van Kampen Theorem (Appendix B.3), we obtain that

$$\pi_1 C = \pi_1 C^* *_{\pi_1 S}$$

is an HNN–extension and hence $i_\#\colon \pi_1 S \to \pi_1 C$ is injective (see Scott and Wall [324, Thm. 1.7]). □

Next we prove the main theorem of this chapter:

4.7 Theorem (Structure of the commutator subgroup).

(a) *If the commutator subgroup $\mathfrak{G}'$ of a knot group $\mathfrak{G}$ is finitely generated, then $\mathfrak{G}'$ is a free group of rank $2g$ where g is the genus of the knot. In fact, $\mathfrak{G}' = \pi_1 S$, S a Seifert surface of genus g.*

(b) *If $\mathfrak{G}'$ cannot be finitely generated then*

$$\mathfrak{G}' = \dots \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{\mathfrak{B}_1} \mathfrak{A}_2 \dots$$

and the generator t of the group of covering transformations of $p_\infty\colon C_\infty \to C$ induces an automorphism τ of $\mathfrak{G}'$ such that $\tau(\mathfrak{A}_j) = \mathfrak{A}_{j+1}$, $\tau(\mathfrak{B}_j) = \mathfrak{B}_{j+1}$. Here $\mathfrak{A}_j \cong \pi_1 C^*$, $\mathfrak{B}_j \cong \pi_1 S \cong \mathfrak{F}_{2g}$ and $\mathfrak{B}_j$ is a proper subgroup of $\mathfrak{A}_j$ and $\mathfrak{A}_{j+1}$. (The subgroups $\mathfrak{B}_j$ and $\mathfrak{B}_{j+1}$ do not coincide).

Proof. We apply the construction of 4.4, for a Seifert surface of minimal genus. By Lemma 4.5, the inclusions $h_j i^\pm\colon S^\pm \to C_j^*$ induce monomorphisms $(h_j i^\pm)_\#\colon \pi_1 S^\pm \to \pi_1 C_j^*$. By the Seifert–van Kampen Theorem (see Appendix B.3), $\mathfrak{G}' = \pi_1 C_\infty$ is the direct $\lim_{n\to\infty} \mathfrak{P}_n$ of the following free products with amalgamation:

$$\mathfrak{P}_n = \mathfrak{A}_{-n} *_{\mathfrak{B}_{-n}} \mathfrak{A}_{-n+1} *_{\mathfrak{B}_{-n+1}} \dots *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{\mathfrak{B}_1} \mathfrak{A}_2 \dots *_{\mathfrak{B}_{n-1}} \mathfrak{A}_n;$$

here $\mathfrak{A}_j$ corresponds to the sheet C_j^* and $\mathfrak{B}_j$ to $h_j i^+(S^+)$ if considered as a subgroup of $\mathfrak{A}_j$ and to $h_{j+1} i^-(S^-)$ as a subgroup of $\mathfrak{A}_{j+1}$. Thus for different j the pairs $(\mathfrak{A}_j, \mathfrak{B}_j)$ are isomorphic and the same is true for the pairs $(\mathfrak{A}_{j+1}, \mathfrak{B}_j)$.

When $\mathfrak{G}'$ is finitely generated there is an n such that the generators of $\mathfrak{G}'$ are in $\mathfrak{P}_n$. This implies that $\mathfrak{B}_n = \mathfrak{A}_{n+1}$ and $\mathfrak{B}_{-n-1} = \mathfrak{A}_{-n-1}$; hence,

$$i_\#^+(\pi_1 S^+) = \pi_1 C^* = i_\#^-(\pi_1 S^-) \cong \mathfrak{F}_{2g}$$

where g is the genus of S (and $\mathfrak{k}$). Now it follows that $\pi_1 C_\infty \cong \pi_1 C^* \cong \pi_1 S \cong \mathfrak{F}_{2g}$.

There remain the cases where $i^+_\#(\pi_1 S^+) \neq \pi_1 C^*$ or $i^-_\#(\pi_1 S^-) \neq \pi_1 C^*$. Then $\mathfrak{G}'$ cannot be generated by a finite system of generators. Lemma 4.8, due to Brown and Crowell [44], shows that these two inequalities are equivalent; hence, $i^+_\#(\pi_1 S^+) \neq \pi_1 C^* \neq i^-_\#(\pi_1 S^-)$, and now the situation is as described in (b). (That $\mathfrak{B}_j$ and $\mathfrak{B}_{j+1}$ do not coincide can be deduced using facts from the proof of Theorem 5.1, see Remark 5.4). □

Section 4.C is devoted to the proof of the Lemma 4.8 of Brown and Crowell [44] and can be neglected at first reading.

4.C A lemma of Brown and Crowell

The following lemma is a special case of a result of E. M. Brown and R. H. Crowell.

4.8 Lemma (Brown–Crowell [44]). *Let M be a connected, compact, orientable 3-manifold where ∂M consists of two connected surfaces S^+ and S^- of genus g with common boundary*

$$\partial S^+ = \partial S^- = S^+ \cap S^- = \bigcup_{i=1}^{r} \kappa_i \neq \emptyset,\ \kappa_i \cap \kappa_j = \emptyset \quad \text{for } i \neq j.$$

If the inclusion $i^+\colon S^+ \to M$ induces an isomorphism $i^+_\#\colon \pi_1 S^+ \to \pi_1 M$ so does $i^-\colon S^- \to M$.

Proof by induction on the Euler characteristic of the surface S^+. As $\partial S^+ \neq \emptyset$ the Euler characteristic $\chi(S^+)$ is maximal for $r = 1$ and $g = 0$; in this case $\chi(S^+) = 1$ and S^+ and S^- are disks, $\pi_1 S^-$ and $\pi_1 S^+$ are trivial; hence, $\pi_1 M$ is trivial too, and nothing has to be proved.

If $\chi(S^+) = \chi(S^-) < 1$ there is a simple arc α on S^- with $\partial\alpha = \{A, B\} = \alpha \cap \partial S^-$ which does not separate S^-, see Figure 4.5. We want to prove that there is an arc β on S^+ with the same properties such that $\alpha^{-1}\beta$ bounds a disk δ in M.

$i^+_\#\colon \pi_1 S^+ \to \pi_1 M$ is an isomorphism by assumption, thus there is an arc β' in S^+ connecting A and B such that $(\alpha, A, B) \simeq (\beta', A, B)$ in M. In general, the arc β' is

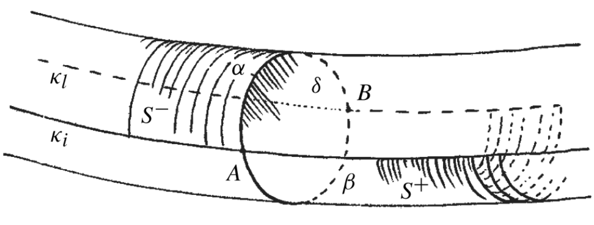


Figure 4.5

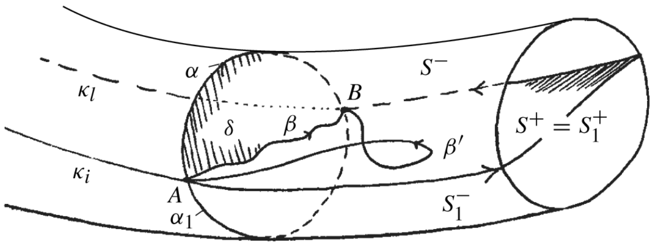


Figure 4.6

not simple. The existence of a simple arc is proved using the following *doubling trick*: Let M_1 be a homeomorphic copy of M with $\partial M_1 = S_1^+ \cup S_1^-$. Let M' be obtained from the disjoint union $M \cup M_1$ by identifying S^+ and S_1^+ and let $\alpha_1 \subset M_1$ be the arc corresponding to α. In M', $\alpha\alpha_1^{-1} \simeq \beta'\beta'^{-1} \simeq 1$. By Dehn's Lemma (Appendix B.4), there is a disk δ' in M' with boundary $\alpha\alpha_1^{-1}$. We may assume that δ' is in general position with respect to $S^+ = S_1^+$ and that $\delta' \cap \partial M' = \partial\delta' = \alpha\alpha_1^{-1}$. The disk δ' intersects S^+ in a simple arc β connecting A and B and, perhaps, in a number of closed curves. The simple closed curve $\alpha\beta^{-1}$ is null-homotopic in δ', hence in M'. By the Seifert–van Kampen Theorem,

$$\pi_1 M' = \pi_1 M *_{\pi_1 S^+} \pi_1 M_1 \cong \pi_1 M;$$

thus the inclusion $M \hookrightarrow M'$ induces an isomorphism $\pi_1 M \to \pi_1 M'$. Since $\alpha\beta^{-1}$ is contained in M it follows that $\alpha\beta^{-1} \simeq 0$ in M. By Dehn's Lemma, there is a disk $\delta \subset M$ with $\delta \cap \partial M = \partial\delta = \alpha \cap \beta$, see Figure 4.6.

The arc β does not separate S^+. To prove this let C and D be points of ∂S^+ close to β on different sides. There is an arc λ in M connecting C and D without intersecting δ; this is a consequence of the assumption that α does not separate S^-. Now deform λ into S^+ by a homotopy that leaves fixed C and D. The resulting path $\lambda' \subset S^+$ again connects C and D and has intersection number 0 with δ, the intersection number calculated in M; hence, also 0 with β when the calculation is done in S^+. This proves that β does not separate S^+.

Cut M along δ, see Figure 4.7. The result is a 3-manifold M_*. We prove that the boundary of M_* fulfills the assumptions of the lemma and that $\chi(\partial M_*) > \chi(\partial M)$. Then induction can be applied.

Assume that $A \in \kappa_i$, $B \in \kappa_\ell$. Let γ be a simple arc in δ such that $\gamma \cap \partial\delta = \partial\gamma = \{A, B\}$. By cutting M along δ, γ is cut into two arcs γ', γ'' which join the points A', B' and A'', B'' corresponding to A and B. The curves κ_i and κ_ℓ of ∂S^+ are replaced by one new curve κ_i' if $i \neq \ell$ or by two new curves $\kappa_{i,1}, \kappa_{i,2}$ if $i = \ell$. These new curves together with those κ_m that do not intersect δ decompose ∂M_* into two homeomorphic surfaces. They contain homeomorphic subsets S_*^+, S_*^- which result from removing the two copies of δ in ∂M_*. The surfaces S_*^+, S_*^- are obtained from

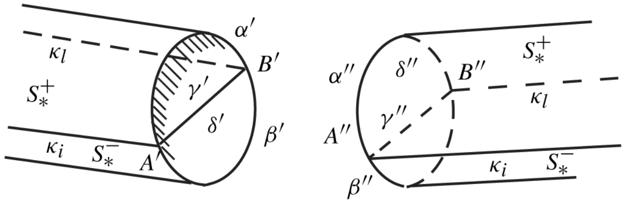


Figure 4.7

S^+ and S^- by cutting along $\partial\delta$. It follows that

$$\chi(S_*^+) = \chi(S^+) + 1,$$

since for $i = \ell$ the number r of boundary components increases by 1 and the genus decreases by 1: $r_* = r + 1$, $g_* = g - 1$, and for $i \neq \ell$ one has $r_* = r - 1$ and $g_* = g$.

The inclusions and identification mappings form the following commutative diagrams:

$$\begin{tikzcd} S_*^+ \arrow[r, "j^+"] \arrow[d, hook, "i_*^+"'] & S^+ \arrow[d, hook, "i^+"] \\ M_* \arrow[r, "j"] & M \end{tikzcd} \qquad \begin{tikzcd} S_*^- \arrow[r, "j^-"] \arrow[d, hook, "i_*^-"'] & S^- \arrow[d, hook, "i^-"] \\ M_* \arrow[r, "j"] & M \end{tikzcd}$$

From the second version of the Seifert–van Kampen Theorem, see Appendix B.3 (b), [382, 2.8.3], [346, 5.3.11], it follows that

$$\begin{aligned} \pi_1 M &= j_\#(\pi_1 M_*) * \mathfrak{Z}, \\ \pi_1 S^+ &= j_\#^+(\pi_1 S_*^+) * \mathfrak{Z}, \quad \pi_1 S^- = j_\#^-(\pi_1 S_*^-) * \mathfrak{Z}, \end{aligned}$$

where $\mathfrak{Z}$ is the infinite cyclic group generated by κ_i. By assumption the inclusion $i^+\colon S^+ \to M$ induces an isomorphism $i_\#^+$ which maps $j_\#^+(\pi_1 S_*^+)$ to $j_\# i_{*\#}^+(\pi_1 S^+) \subset j_\#(\pi_1 M_*)$ and $\mathfrak{Z}$ onto $\mathfrak{Z}$. From the solution of the word problem in free products, see [382, 2.3], it follows that $i_\#^+$ bijectively maps $j_\#^+(\pi_1 S_*^+)$ onto $j_\#(\pi_1 M_*)$; hence, $i_{*\#}^+$ is an isomorphism.

As induction hypothesis we may assume that $i_{*\#}^-$ is an isomorphism. By arguments, similar to those above, it follows that $i_\#^-$ can be described by the following commutative diagram:

$$\begin{tikzcd} j_\#^-(\pi_1 S_*^-) * \mathfrak{Z} \arrow[r, "="] \arrow[d, "i_{*\#}^- * (i_\#^- | \mathfrak{Z})"', "\cong"] & \pi_1 S^- \arrow[d, "i_\#^-"] \\ j_\#(\pi_1 M_*) * \mathfrak{Z} \arrow[r, "="] & \pi_1 M . \end{tikzcd}$$

Since the mapping on the left side is bijective, $i_\#^-$ is an isomorphism. □

4.D Examples and applications

Theorem 4.7 now throws some light on the results in Examples 3.7–3.9: the trefoil (E 4.2) and the figure-eight knot (Figure 3.5) have finitely generated commutator subgroups. The 2-bridge knot $\mathfrak{b}(7,3)$ has a commutator subgroup of infinite rank; in Example 3.9 we have already calculated $\mathfrak{G}'$ in the form of Theorem 4.7 (b) using the Reidemeister–Schreier method.

We will prove that all torus knots have finitely generated commutator subgroups. Let us begin with some consequences of Theorem 4.7.

4.9 Corollary. *Let the knot $\mathfrak{k}$ have a finitely generated commutator subgroup and let S be an orientable surface spanning $\mathfrak{k}$. If S is incompressible in the knot complement (this means that the inclusion $i\colon S \hookrightarrow C$ induces a monomorphism $i_{\#}\colon \pi_1 S \to \pi_1 C = \mathfrak{G}$) then S and $\mathfrak{k}$ have the same genus.* □

In the following $\mathfrak{G}$ always denotes a knot group.

4.10 Corollary. *The center of $\mathfrak{G}'$ is trivial.*

Proof. If $\mathfrak{G}'$ cannot be finitely generated, by Theorem 4.7 there are groups $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{G}' = \mathfrak{A} *_{\mathfrak{F}_{2g}} \mathfrak{B}$ where g is the genus of $\mathfrak{k}$, $\mathfrak{F}_{2g}$ is a free group of rank $2g$ and $\mathfrak{A} \neq \mathfrak{F}_{2g} \neq \mathfrak{B}$. From the solution of the word problem it follows that the center is contained in the amalgamated subgroup and is central in both factors, see [382, 2.3.9]. But $\mathfrak{F}_{2g}$ has trivial center ([382, E1.5]. The last argument also applies to finitely generated $\mathfrak{G}'$ because they are free groups. □

4.11 Proposition.

(a) *If the center $\mathfrak{C}$ of $\mathfrak{G}$ is non-trivial then $\mathfrak{G}'$ is finitely generated.*

(b) *The center $\mathfrak{C}$ of $\mathfrak{G}$ is trivial or infinite cyclic. When $\mathfrak{C} \neq 1$, $\mathfrak{C}$ is generated by an element $t^n u$, $n > 1$, $u \in \mathfrak{G}'$. (The coset $t\mathfrak{G}'$ generates the first homology group $\mathfrak{G}/\mathfrak{G}' \cong \mathfrak{Z}$.)*

Proof. (a) Assume that $\mathfrak{G}'$ cannot be generated by finitely many elements. Then, by Theorem 4.7, $\mathfrak{G}' = \ldots * \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 * \ldots$ where $\mathfrak{A}_j \supsetneqq \mathfrak{B}_j \subsetneqq \mathfrak{A}_{j+1}$. Denote by $\mathfrak{H}_r$ the subgroup of $\mathfrak{G}'$ which is generated by $\{\mathfrak{A}_j \mid j \leq r\}$. Then $\mathfrak{H}_{r+1} = \mathfrak{H}_r *_{\mathfrak{B}_r} \mathfrak{A}_{r+1}$ and $\mathfrak{H}_r \supsetneqq \mathfrak{B}_r \subsetneqq \mathfrak{A}_{r+1}$; hence $\mathfrak{H}_r \subsetneqq \mathfrak{H}_{r+1}$ and

$$\mathfrak{H}_r \subsetneqq \mathfrak{H}_s \qquad \text{if } r < s. \tag{4.1}$$

Let $t \in \mathfrak{G}$ be an element which is mapped onto a generator of $\mathfrak{G}/\mathfrak{G}' = \mathfrak{Z}$. Assume that $t^{-1}\mathfrak{A}_r t = \mathfrak{A}_{r+1}$; hence, $t^{-1}\mathfrak{H}_r t = \mathfrak{H}_{r+1}$.

Consider $z \in \mathfrak{C}$, $1 \neq z$. Then $z = ut^m$ where $u \in \mathfrak{G}'$. By 4.10, $m \neq 0$; without loss of generality: $m > 0$. Choose s such that $u \in \mathfrak{H}_s$. Then

$$\mathfrak{H}_s = z^{-1}\mathfrak{H}_s z$$

since $z \in \mathfrak{C}$, and

$$z^{-1}\mathfrak{H}_s z = t^{-m}u^{-1}\mathfrak{H}_s u t^m = t^{-m}\mathfrak{H}_s t^m = \mathfrak{H}_{s+m}.$$

This implies $\mathfrak{H}_s = \mathfrak{H}_{s+m}$, contradicting equation (4.1).

(b) By (a) a non-trivial center $\mathfrak{C}$ contains an element $t^n u$, $n > 0, u \in \mathfrak{G}'$ and n minimal. It follows that $\mathfrak{G}_n = \mathfrak{C}\mathfrak{G}'$ and by Corollary 4.10,

$$n\mathfrak{Z} = \mathfrak{G}_n/\mathfrak{G}' = \mathfrak{C}\mathfrak{G}'/\mathfrak{G}' \cong \mathfrak{C}/\mathfrak{C} \cap \mathfrak{G}' \cong \mathfrak{C}.$$

Hence $\mathfrak{C}$ is an infinite cyclic group.

If $n = 1$ then $\mathfrak{G} = \mathfrak{C}\mathfrak{G}' \cong \mathfrak{C} \times \mathfrak{G}'$ which contradicts the fact that $\mathfrak{G}$ collapses if the relation $t = 1$ is introduced. □

4.12 Corollary (Genus of torus knots).

(a) *The group* $\mathfrak{G}_{a,b} = \langle u, v \mid u^a v^{-b} \rangle$ *of the torus knot* $\mathfrak{t}(a, b)$, $a, b \in \mathbb{N}$, $(a, b) = 1$ *has a finitely generated commutator subgroup. It is, following Theorem 4.7 (a), a free group of rank* $2g$ *where* g *is the genus of* $\mathfrak{t}(a, b)$.

(b) $g = \frac{(a-1)\cdot(b-1)}{2}$.

Proof. Since the group of a torus knot has non-trivial center Proposition 4.11 and Theorem 4.7 (a) imply the first statement of the corollary. It remains to prove (b). Consider the natural projection

$$\lambda\colon \mathfrak{G}_{a,b} \to \mathfrak{G}_{a,b}/\mathfrak{C} \cong \mathfrak{Z}_a * \mathfrak{Z}_b.$$

The center $\mathfrak{C}$ of $\mathfrak{G}_{a,b}$ is generated by $u^a = v^b$, and it is $\mathfrak{C} = \ker \lambda$. Now the restriction of $\lambda|_{\mathfrak{G}'_{a,b}}$ maps the commutator subgroup $\mathfrak{G}'_{a,b}$ onto the commutator subgroup $(\mathfrak{Z}_a * \mathfrak{Z}_b)'$ of $\mathfrak{Z}_a * \mathfrak{Z}_b$. Moreover, $\ker(\lambda|_{\mathfrak{G}'_{a,b}}) = \mathfrak{C} \cap \mathfrak{G}'_{a,b}$ is trivial by Corollary 4.10 and hence

$$\lambda|_{\mathfrak{G}'_{a,b}}\colon \mathfrak{G}'_{a,b} \xrightarrow{\cong} (\mathfrak{Z}_a * \mathfrak{Z}_b)'$$

is an isomorphism.

The fact that $(\mathfrak{Z}_a * \mathfrak{Z}_b)'$ is isomorphic to a free group of rank $(a-1)(b-1)$ can be found in the literature (see Serre [331, 1.3 Prop. 4] or Magnus, Karrass and Solitar [224, p.196, ex 24]). Theorem 4.7 implies that the genus of $\mathfrak{t}(a, b)$ is $(a-1)(b-1)/2$.

Nevertheless, we give a geometric proof: consider the 2-complex C^2 consisting of one vertex, two edges ξ, η and two disks δ_1, δ_2 with the boundaries ξ^a and η^b, respectively. Then $\pi_1 C^2 \cong \mathfrak{Z}_a * \mathfrak{Z}_b$. Let $\widetilde{C}^2$ be the covering space of C^2 with fundamental group the commutator subgroup of $\mathfrak{Z}_a * \mathfrak{Z}_b$. The group of covering transformations is isomorphic to $\mathfrak{Z}_a \times \mathfrak{Z}_b$. Each edge of $\widetilde{C}^2$ over η (or ξ) belongs to the boundaries of exactly b (resp. a) disks of $\widetilde{C}^2$ which have the same boundary. It suffices to choose one to get a system of defining relations of $\pi_1 \widetilde{C}^2 \cong (\mathfrak{Z}_a * \mathfrak{Z}_b)'$. Then there are ab/b disks over δ_2 and ab/a disks over δ_1. The new complex $\widehat{C}^2$ contains

$$ab \text{ vertices}, \quad 2ab \text{ edges}, \quad a + b \text{ disks},$$

and each edge is in the boundary of exactly one disk of $\widehat{C}^2$. Thus $\pi_1 \widehat{C}^2$ is a free group of rank

$$2ab - (ab - 1) - (a + b) = (a - 1)(b - 1). \qquad \square$$

The isomorphism $(\mathfrak{Z}_a * \mathfrak{Z}_b)' \cong \mathfrak{F}_{(a-1)(b-1)}$ can also be proved using the (modified) Reidemeister–Schreier method, see Zieschang, Vogt and Caldewey [382, 2.2.8]. In the proof above, the geometric background of the algebraic method has been used directly.

4.E Commutator subgroups of satellites

In what follows we let $C = \overline{S^3 - V(\mathfrak{k})}$ denote the complement of a satellite $\mathfrak{k}$, $\widehat{C}$ the complement of its companion $\widehat{\mathfrak{k}}$ and $\widetilde{C} := \overline{\widehat{V} - V(\mathfrak{k})}$ where $\widehat{V}$ is a tubular neighborhood of $\widehat{\mathfrak{k}}$ which contains $\mathfrak{k}$. Note that $\widetilde{C}$ is a compact manifold with two boundary components $\partial \widehat{V} \amalg \partial V$ and that $\widetilde{C}$ is homeomorphic to the complement of the pattern $\widetilde{\mathfrak{k}}$ in the unknotted torus $\widetilde{V} \supset \widetilde{\mathfrak{k}}$ (see Definition 2.8). The complement C is a union

$$C = \widehat{C} \cup_{\widehat{T}} \widetilde{C} \text{ where } \widehat{T} := \partial \widehat{V} = \widehat{C} \cap \widetilde{C}.$$

Let $x_0 \in \widehat{T}$ be a base point for the fundamental groups. We call a subspace $X \subset C$, $x_0 \in X$, π_1*-injective* if the homomorphism $\pi_1(X, x_0) \to \pi_1(C, x_0)$ induced by the injection $(X, x_0) \hookrightarrow (C, x_0)$ is injective. In this case we will identify $\pi_1(X, x_0)$ with the corresponding subgroup of $\pi_1(C, x_0)$.

According to Proposition 3.12, the subspaces $\widetilde{C}$, $\widehat{C}$ and $\widehat{T}$ of C are π_1-injective and the groups of a satellite $\mathfrak{k}$, its companion $\widehat{\mathfrak{k}}$ and the pattern $\widetilde{\mathfrak{k}} \subset \widetilde{V}$ are related by

$$\mathfrak{G} = \widehat{\mathfrak{G}} *_{\mathfrak{A}} \pi_1(\widehat{V} - \mathfrak{k}, x_0) \cong \widehat{\mathfrak{G}} *_{\mathfrak{A}} \pi_1(\widetilde{C}, x_0) = \widehat{\mathfrak{G}} *_{\mathfrak{A}} \mathfrak{H},$$

where $\mathfrak{A} = \pi_1(\widehat{T}, x_0) = \langle \widehat{m}, \widehat{\ell} \rangle \cong \mathbb{Z}^2$ and $\mathfrak{H} = \pi_1(\widetilde{C}, x_0)$. Here $\widehat{m}$ and $\widehat{\ell}$ are meridian and longitude respectively of the companion $\widehat{\mathfrak{k}}$.

4.13 The infinite cyclic covering of the complement of a satellite knot. We let $p_\infty \colon C_\infty \to C$ denote the infinite cyclic covering of C and $x_\infty \in p_\infty^{-1}(x_0)$ a base-point such that $\mathfrak{G}' = (p_\infty)_\# \pi_1(C_\infty, x_\infty) \subset \pi_1(C, x_0) = \mathfrak{G}$. The canonical homomorphism $\varphi\colon \mathfrak{G} \to \mathbb{Z}$ which maps the meridian m of $\mathfrak{k}$ to 1 is given by $\varphi(g) =$

$\mathrm{lk}(\gamma, \mathfrak{k})$ where $\gamma\colon (I, \partial I) \to (C, x_0)$ is a loop representing $g \in \mathfrak{G}$. In the same way the canonical homomorphism $\widehat{\varphi}\colon \widehat{\mathfrak{G}} \to \mathbb{Z}$ which maps the meridian $\widehat{m}$ of $\widehat{\mathfrak{k}}$ to 1 is given by $\varphi(\hat{g}) = \mathrm{lk}(\hat{\gamma}, \widehat{\mathfrak{k}})$ where $\hat{\gamma}\colon (I, \partial I) \to (\widehat{C}, x_0)$ is a loop representing $\hat{g} \in \widehat{\mathfrak{G}}$. Note that

$$\varphi(\hat{g}) = \mathrm{lk}(\widehat{m}, \mathfrak{k}) \cdot \widehat{\varphi}(\hat{g}) \text{ for all } \hat{g} \in \widehat{\mathfrak{G}} \subset \mathfrak{G} \tag{4.2}$$

where we have identified $\widehat{\mathfrak{G}}$ with the corresponding subgroup of $\mathfrak{G}$. Recall that the commutator subgroup $\mathfrak{G}'$ is exactly the kernel of φ i.e $\mathfrak{G}' = \ker \varphi$ and let $n = |\mathrm{lk}(\widehat{m}, \mathfrak{k})|$ denote the *winding number* of $\mathfrak{k}$ in $\widehat{V}$.

4.14 Lemma. *If $n = 0$ then the commutator subgroup $\mathfrak{G}'$ is not finitely generated.*

Proof. It follows from equation (4.2) that $n = 0$ implies that $\mathfrak{G}' = \mathrm{Ker}\,\varphi$ contains $\widehat{\mathfrak{G}}$ and hence a free Abelian subgroup of rank two. The Structure Theorem 4.7 shows that $\mathfrak{G}'$ cannot be finitely generated since it is not free. □

In what follows we shall assume that the winding number $n > 0$ is nonzero. By changing the orientation of $\mathfrak{k}$ if necessary, we may also suppose that $n = \mathrm{lk}(\widehat{m}, \mathfrak{k})$. Let us recall that for a π_1-injective, path-connected subspace $X \subset C$, $x_0 \in X$, the connected components of $p_\infty^{-1}(X)$ are in one-to-one correspondence with the double cosets $\mathfrak{G}' \backslash \mathfrak{G} / \pi_1(X, x_0)$ (see [324, Lemma 3.12]). It follows that the space $\widetilde{C}_\infty := p_\infty^{-1}(\widetilde{C})$ is connected since

$$\varphi|_{\mathfrak{H}}\colon \mathfrak{H} \to \mathbb{Z}$$

is surjective and hence $\mathfrak{G} = \mathfrak{G}'\mathfrak{H}$. Let $\mathfrak{K}$ denote the kernel $\mathrm{Ker}(\varphi|_{\mathfrak{H}})$ i.e.

$$\mathfrak{G}' \cap \mathfrak{H} = \mathfrak{K} = (p_\infty)_\# \pi_1(\widetilde{C}_\infty, x_\infty).$$

We have $\varphi(\widehat{m}) = n$ and hence $\varphi(\widehat{\mathfrak{G}}) = \varphi(\mathfrak{A}) = n\mathbb{Z} \subset \mathbb{Z}$ is the subgroup of index n. Therefore $p_\infty^{-1}(\widehat{C})$ and $p_\infty^{-1}(\widehat{T})$ have exactly n components. Let $\widehat{C}_\infty$ and $\widehat{T}_\infty$ be the component of $p_\infty^{-1}(\widehat{C})$ and $p_\infty^{-1}(\widehat{T})$ respectively which contains x_∞. It follows that

$$(p_\infty)_\# \pi_1(\widehat{C}_\infty, x_\infty) = \mathrm{Ker}(\varphi|_{\widehat{\mathfrak{G}}}) = \widehat{\mathfrak{G}} \cap \mathfrak{G}' = \widehat{\mathfrak{G}}' \tag{4.3}$$

and

$$(p_\infty)_\# \pi_1(\widehat{T}_\infty, x_\infty) = \mathrm{Ker}(\varphi|_{\mathfrak{A}}) = \langle \widehat{\ell} \rangle \cong \mathbb{Z}.$$

We obtain from (4.3) that the restriction of p_∞ to $\widehat{C}_\infty$, $\hat{p}_\infty\colon \widehat{C}_\infty \to \widehat{C}$, is the infinite cyclic covering of $\widehat{C}$.

The covering transformation $t\colon C_\infty \to C_\infty$ permutes the components of $p_\infty^{-1}(\widehat{C})$ and again equation (4.2) implies that the transformation t^n preserves the components of $p_\infty^{-1}(\widehat{C})$ and hence $\hat{t} = t^n|_{\widehat{C}_\infty}\colon \widehat{C}_\infty \to \widehat{C}_\infty$ is the covering transformation of $\hat{p}_\infty$. It follows that $p_\infty^{-1}(\widehat{C})$ decomposes into path components:

$$p_\infty^{-1}(\widehat{C}) = \coprod_{k=0}^{n-1} t^k \widehat{C}_\infty .$$

4.15 Lemma. *Let $m \in \mathfrak{G}$ be a meridian of the satellite $\mathfrak{k}$. We suppose that $n > 0$. If we define $\mathfrak{K}_0 := \mathfrak{K}$ and*

$$\mathfrak{K}_{k+1} := \mathfrak{K}_k *_{\langle m^k \widehat{\ell}\, m^{-k}\rangle} m^k \widehat{\mathfrak{G}}' m^{-k},\ k = 0, \ldots, n-1,$$

then $\mathfrak{K}_0 = \pi_1(\widetilde{C}_\infty, x_\infty)$ and $\mathfrak{K}_n = \mathfrak{G}' = \pi_1(C_\infty, x_\infty)$.

Proof. Let $m \in \mathfrak{G}$ be a Wirtinger generator of $\mathfrak{k}$ represented by a loop $\mu\colon (I, \partial I) \to (\widetilde{C}, x_0)$. Let μ_∞ be the lift of μ which goes from x_∞ to tx_∞. Then

$$(p_\infty)_\#\big(\pi_1(t^k \widehat{C}_\infty, x_\infty)\big) = \mathfrak{G}' \cap m^k \widehat{\mathfrak{G}}' m^{-k} = m^k \widehat{\mathfrak{G}}' m^{-k} \tag{4.4}$$

where we define $\pi_1(t^k \widehat{C}_\infty, x_\infty)$ by using the lift μ_∞^k of the loop μ^k connecting x_∞ to $t^k x_\infty$ (for more details see [324, Lemma 3.13]). By the same method we obtain that

$$(p_\infty)_\#\big(\pi_1(t^k \widehat{T}_\infty, x_\infty)\big) = \langle m^k \hat{\ell}\, m^{-k}\rangle \cong \mathbb{Z} . \tag{4.5}$$

A repeated application of the Seifert–van Kampen theorem gives that $\pi_1(C_\infty, x_\infty)$ is the free product of the groups $\mathfrak{K} = \pi_1(\widetilde{C}_\infty, x_\infty)$ and $m^k \widehat{\mathfrak{G}}' m^{-k} = \pi_1(t^k \widehat{C}_\infty, x_\infty)$, $0 \le k \le n-1$, with amalgamated subgroups $\langle m^k \widehat{\ell}\, m^{-k}\rangle = \pi_1(t^k \widehat{T}_\infty, x_\infty)$, $0 \le k \le n-1$. More precisely, if we define $\mathfrak{K}_0 := \mathfrak{K}$ and

$$\mathfrak{K}_{k+1} := \mathfrak{K}_k *_{\langle m^k \widehat{\ell}\, m^{-k}\rangle} m^k \widehat{\mathfrak{G}}' m^{-k},\ k = 0, \ldots, n-1$$

then $\mathfrak{K}_0 = \pi_1(\widetilde{C}_\infty, x_\infty)$ and $\mathfrak{K}_n = \mathfrak{G}' = \pi_1(C_\infty, x_\infty)$. □

Remark. The description of $\mathfrak{G}'$ given by Lemma 4.15 can also be obtained by using Bass–Serre theory [331, 324, 68]: the free product with amalgamation $\mathfrak{G} = \widehat{\mathfrak{G}} *_{\mathfrak{U}} \mathfrak{H}$ acts on a tree T with fundamental domain a segment. The description of T is explicit (see [331, I.§4, Thm. 7]). Each subgroup of $\mathfrak{G}$ is the *fundamental group of a graph of groups* [331, I.§4.4].

More precisely, the subgroup theorem [324, 3.14], [68, 8.5, Thm. 27] shows that $\mathfrak{G}'$ is the *fundamental group of a tree of groups* (see Figure 4.8) with vertex groups $G_{v_0} = \mathfrak{K}$, $G_{w_k} = m^k \widehat{\mathfrak{G}}' m^{-k}$, $k = 0, \ldots, n-1$, and edge groups $G_{e_k} = \langle m^k \widehat{\ell}\, m^{-k}\rangle \cong \mathbb{Z}$.

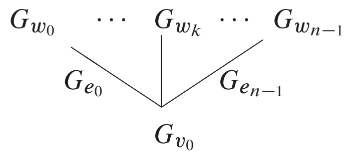


Figure 4.8. $\mathfrak{G}'$ is the fundamental group of this tree of groups.

4.16 Lemma. *Let $\mathfrak{G} = \mathfrak{G}_1 *_{\mathfrak{S}} \mathfrak{G}_2$ where $\mathfrak{S}$ is finitely generated. Then $\mathfrak{G}$ is finitely generated if and only if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are finitely generated.*

Proof (R. Bieri). When $\mathfrak{G}$ is finitely generated there are finite subsets $X_i \subset \mathfrak{G}_i$ $(i = 1, 2)$ with $\langle X_1, X_2 \rangle = \mathfrak{G}$. Since $\mathfrak{S}$ is finitely generated we may assume that both X_1 and X_2 contain generators for $\mathfrak{G}$. Let $\mathfrak{H}_i = \langle X_i \rangle \subset \mathfrak{G}_i$. Then $\mathfrak{S} = \mathfrak{G}_1 \cap \mathfrak{G}_2 \supset \mathfrak{H}_1 \cap \mathfrak{H}_2$, but on the other hand $\mathfrak{H}_1 \cap \mathfrak{H}_2 \supset \mathfrak{S}$, so that $\mathfrak{S} = \mathfrak{G}_1 \cap \mathfrak{G}_2 = \mathfrak{H}_1 \cap \mathfrak{H}_2$. It follows that the map $\mathfrak{H}_1 *_{\mathfrak{S}} \mathfrak{H}_2 \to \mathfrak{G}_1 *_{\mathfrak{S}} \mathfrak{G}_2$ induced by the embeddings $\mathfrak{H}_i \to \mathfrak{G}_i$ is an isomorphism. Now the solution of the word problem implies that $\mathfrak{H}_i = \mathfrak{G}_i$. □

Let us consider the pattern $\tilde{\mathfrak{k}}$ in the unknotted torus $\tilde{V} \subset \tilde{S}^3$ and let $\tilde{\ell}$ denote a longitude of $\tilde{V}$ i.e. a meridian of $\tilde{S}^3 - \tilde{V}$ (see Definition 2.8). We let $\tilde{\mathfrak{G}}$ denote the group of the knot $\tilde{\mathfrak{k}} \subset \tilde{S}^3$ i.e. $\tilde{\mathfrak{G}} \cong \mathfrak{H}/\mathfrak{N}$ where $\mathfrak{N} = \mathfrak{N}(\tilde{\ell})$ denotes the normal closure of $\tilde{\ell}$ in $\mathfrak{H}$.

4.17 Corollary. *With the above notations we have:*

(1) *For $n \neq 0$, $\mathfrak{G}'$ is finitely generated if and only if $\mathfrak{K}$ and $\widehat{\mathfrak{G}}'$ are finitely generated.*

(2) *If $\mathfrak{G}'$ is finitely generated, then $n \neq 0$ and $\tilde{\mathfrak{G}}'$ and $\widehat{\mathfrak{G}}'$ are finitely generated.*

Proof. The first part follows directly from Lemma 4.15 and Lemma 4.16.

If $\mathfrak{G}'$ is finitely generated, then $n \neq 0$ by Lemma 4.14 and the first part of the corollary implies that $\mathfrak{K}$ and $\widehat{\mathfrak{G}}'$ are finitely generated. Consider the canonical epimorphism $\psi\colon \mathfrak{H} \twoheadrightarrow \tilde{\mathfrak{G}}$ with $\operatorname{Ker}\psi = \mathfrak{N}$ the normal closure of $\tilde{\ell}$ in $\mathfrak{H}$. One has $\mathfrak{N} \subset \mathfrak{K}$, and

$$1 \to \mathfrak{N} \to \mathfrak{K} \to \tilde{\mathfrak{G}}' \to 1 \tag{4.6}$$

is exact. Hence $\tilde{\mathfrak{G}}'$ is finitely generated, since $\mathfrak{K}$ is. □

Remark. Note that the sequence (4.6) is always exact and that $\mathfrak{K}$ is finitely generated, if $\tilde{\mathfrak{G}}'$ and $\operatorname{Ker}\psi$ are. In the first edition of this book it was wrongly assumed that $\operatorname{Ker}\psi$ was always finitely generated. D. S. Silver pointed out the mistake and he supplied the following counterexample: No satellite with pattern $\tilde{\mathfrak{k}}$ (Figure 4.9) has a finitely generated commutator subgroup $\mathfrak{G}'$ since $\mathfrak{K}$ is not finitely generated even

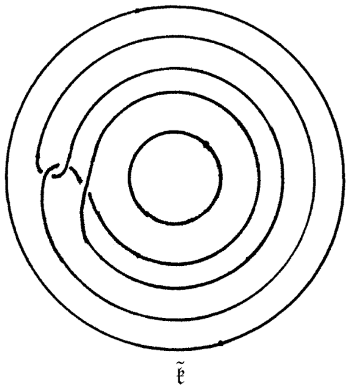


Figure 4.9. No satellite with pattern $\tilde{\mathfrak{k}}$ has a finitely generated commutator subgroup.

if the commutator groups $\widetilde{\mathfrak{G}}'$ and $\widehat{\mathfrak{G}}'$ are. More details can be found in an article by M. Hirasawa, K. Murasugi and D. S. Silver, see [173]. We will come back to this question in a more geometric context in Section 5.C.

4.F History and sources

The study of the commutator subgroup $\mathfrak{G}'$ concentrated on $\mathfrak{G}'/\mathfrak{G}''$ in the early years of knot theory. This will be the object of Chapters 8 and 9. In Reidemeister's *Knotentheorie* [296, III.§ 6], a group presentation of $\mathfrak{G}'$ is given. But the structure of $\mathfrak{G}'$ eluded the purely algebraic approach.

L. Neuwirth made the first important step by investigating the infinite cyclic covering space C_∞, $\pi_1 C_\infty = \mathfrak{G}'$, using the then (relatively) new tools *Dehn's Lemma* and *Loop Theorem* [267]: Lemma 4.7. The analysis of $\mathfrak{G}'$ resulted in splitting off a special class of knots, whose commutator subgroups are finitely generated. In this case $\mathfrak{G}'$ proves to be a free group of rank $2g$, g the genus of the knot. These knots will be treated separately in the next chapter. There remained two different possible types of infinitely generated commutator groups in Neuwirth's analysis, and it took some years till one of them could be excluded by E. M. Brown and R. H. Crowell [44]: Lemma 4.8. The remaining one, an infinite free product with amalgamations does occur. This group is rather complicated and its structure surely could do with some further investigation.

4.G Exercises

E 4.1. Describe the process of cutting along a one-sided surface.

E 4.2. Prove that the commutator subgroup of the group of the trefoil is free of rank 2.

E 4.3. Prove that the commutator subgroup of the group of the knot 6_1 cannot be finitely generated.

If the bands of a Seifert surface spanning $\mathfrak{k}$ form a plat (Figure 4.10), we call $\mathfrak{k}$ a *braid-like* knot (compare Proposition 8.2).

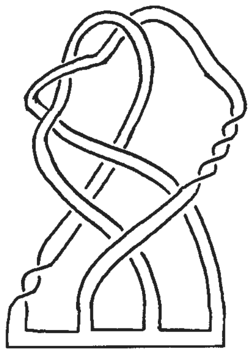


Figure 4.10. A braid-like knot.

E 4.4. Show that for a braid-like knot the group $\mathfrak{A} = \pi_1 C^*$ is always free. (For the notation see 4.4–4.7).

E 4.5. Doubled knots are not braid-like. (See Definition 2.11.)

E 4.6. If $\mathfrak{k}$ is braid-like with respect to a Seifert surface of minimal genus, then there is an algorithm by which one can decide whether $\mathfrak{G}'$ is finitely generated or not. Apply this to E 4.2.

E 4.7. Let $\mathfrak{Z}_a$ and $\mathfrak{Z}_b$ be cyclic groups of order a resp. b. Use the modified Reidemeister–Schreier method ([382, 2.2.8]) to prove that the commutator subgroup $(\mathfrak{Z}_a * \mathfrak{Z}_b)'$ of the free product is a free group of rank $(a-1)(b-1)$.

E 4.8. Let C^* be the space obtained by cutting a knot complement along a Seifert surface of minimal genus. Prove that in the case of a trefoil or 4-knot C^* is a handlebody of genus two.

E 4.9. Let $\mathfrak{l} = \mathfrak{k}_1 \cup \cdots \cup \mathfrak{k}_\mu$ be an oriented link of multiplicity μ and let $\mathfrak{G} = \pi_1(S^3 - \mathfrak{l})$ denote its group. A meridian m_i of the component $\mathfrak{k}_i$ acquires an orientation such that $\mathrm{lk}(\mathfrak{k}_i, m_i) = 1$. Let $\alpha\colon \mathfrak{G} \to \mathbb{Z}$ be the homomorphism which maps m_i to 1, $i = 1, \ldots, \mu$ i.e. α is given by $\alpha(\gamma) = \sum_{i=1}^{\mu} \mathrm{lk}(\mathfrak{k}_i, \gamma)$.

Generalize the construction of C_∞ to links by replacing $\mathfrak{G}'$ by the *augmentation subgroup* $\mathfrak{A} = \ker(\alpha)$ (see 5.17 for more details).

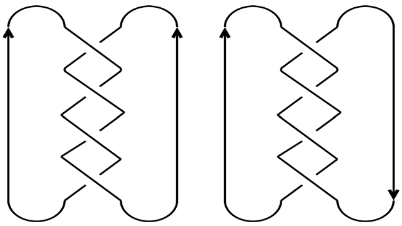


Figure 4.11. The link $\mathfrak{l}_1$ on the left-hand side and the link $\mathfrak{l}_2$ on the right-hand side.

E 4.10. Consider the two oriented links $\mathfrak{l}_1$ and $\mathfrak{l}_2$ (Figure 4.11) and define

$$\alpha_i\colon \pi_1(S^3 - \mathfrak{l}_i) \to \mathbb{Z}\,,\ i = 1, 2,$$

in the same way as in Exercise E 4.9.

Show that $\mathfrak{A}_1 = \ker(\alpha_1)$ is finitely generated, whereas $\mathfrak{A}_2 = \ker(\alpha_2)$ is not.

E 4.11. Let $\mathfrak{p}(2p+1, 2q+1, 2r+1) = \mathfrak{k}$ be a pretzel-knot, $p, q, r \in \mathbb{Z}$, Figure 8.9. Compute $i_\#^\pm\colon\ \pi_1 S \to \pi_1 C^*$ and $i_*^\pm\colon\ H_1(S) \to H_1(C^*)$ for a Seifert surface S of minimal genus spanning $\mathfrak{k}$ and decide which of these knots have a finitely generated commutator subgroup.

E 4.12. Consider the (generalized) "pretzel-knot $\mathfrak{p}(3, 1, 3, -1, -3)$", and show that it spans a Seifert surface F which is not of minimal genus such that the inclusions $i^\pm\colon\ F \to C^*$ induce injections $i_\#^\pm$. (The homomorphisms $i_*^\pm$ are necessarily not injective, compare E 8.1.)

Chapter 5

Fibered knots

By the theorem of E. M. Brown, R. H. Crowell and L. Neuwirth, knots fall into two different classes according to the structure of their commutator subgroups. The first of them comprises the knots whose commutator subgroups are finitely generated, and hence free, the second one those whose commutator subgroups cannot be finitely generated. We have seen that all torus-knots belong to the first category and we have given an example – the 2-bridge knot $\mathfrak{b}(7,3)$ – of the second variety. The aim of this chapter is to demonstrate that the algebraic distinction of the two classes reflects an essential difference in the geometric structure of the knot complements.

5.A Fibration theorem

5.1 Theorem (Stallings). *The complement* $C = \overline{S^3 - V(\mathfrak{k})}$ *of a knot* $\mathfrak{k}$ *fibers locally trivially over* S^1 *with Seifert surfaces of genus* g *as fibers if the commutator subgroup* $\mathfrak{G}'$ *of the knot group is finitely generated,* $\mathfrak{G}' \cong \mathfrak{F}_{2g}$. *Incidentally* g *is the genus of the knot.*

Theorem 5.1 is a special version of the more general Theorem 5.7 of J. Stallings [343]. The following proof of Theorem 5.1 is based on Stallings' original argument but takes advantage of the special situation, thus reducing its length and difficulty.

5.2 Fibered complement. To prepare the setting, imagine C fibered as described in Theorem 5.1. Cut along a Seifert surface S of $\mathfrak{k}$. The resulting space C^* is a fiber space with base-space the interval I, hence $C^* \cong S \times I$. The space C is re-obtained from C^* by an identification of $S \times 0$ and $S \times 1$: $(x, 0) = (h(x), 1), x \in S$, where $h\colon S \to S$ is an orientation preserving homeomorphism. We write in short:

$$C = S \times I/h.$$

Choose a basepoint P on ∂S and let $\sigma = P \times I$ denote the path leading from $(P, 1)$ to $(P, 0)$. For $w^0 = (w, 0), w^1 = (w, 1)$ and $w \in \pi_1(S, P)$ there is an equation

$$w^1 = \sigma w^0 \sigma^{-1} \quad \text{in } \pi_1(C^*, (P, 1)).$$

Let $\kappa_1, \dots, \kappa_{2g}$ be simple closed curves representing canonical generators of S. Then obviously

$$\sigma \kappa_i^0 \sigma^{-1} (\kappa_i^1)^{-1} = \rho_i' \simeq 0 \text{ in } C^*.$$

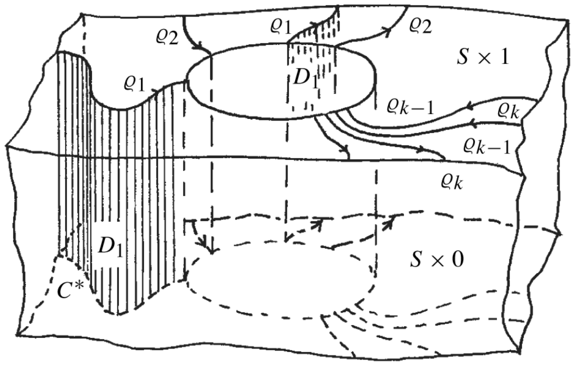


Figure 5.1

The curves $\{\varrho'_i \mid 1 \leq i \leq 2g\}$ coincide in σ; they can be replaced by a system of simple closed curves $\{\varrho_i\}$ on ∂C^* which are pairwise disjoint, where each ϱ_i is obtained from ϱ'_i by an isotopic deformation near σ, see Figure 5.1. There are disks D_i embedded in C^*, such that $\partial D_i = \varrho_i$. Cut C^* along the disks D_i to obtain a 3-ball C^{**} (Figure 5.2).

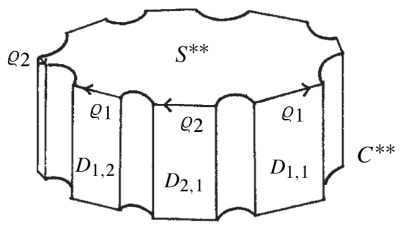


Figure 5.2

5.3 *Proof of Theorem 5.1.* We cut C along a Seifert surface S of minimal genus and get C^* with $S^{\pm} = S \times 1, 0$ in its boundary as in Chapter 4. Our aim is to produce a 3-ball C^{**} by cutting C^* along disks. The inclusions $i^+\colon S^+ \to C^*$ and $i^-\colon S^- \to C^*$ induce isomorphisms $i^{\pm}_{\#}$ of the fundamental groups. Let $m \subset \partial C$ be a meridian through the basepoint P on ∂S. Then, by the cutting process $C \to C^*$ m will become a path σ leading from $P^+ = (P, 1)$ to $P^- = (P, 0)$. Assign to $\sigma w^- \sigma^{-1}$ for $w^- \in \pi_1(S^-, P^-)$ the element $w^+ \in \pi_1(S^+, P^+)$, $w^+ = \sigma w^- \sigma^{-1}$ in $\pi_1(C^*, P^+)$. We know the map $\varphi(w^-) = w^+$ to be an isomorphism $\varphi\colon \pi_1(S^-, P^-) \to \pi_1(S^+, P^+)$ such that $\varphi(\lambda^-) = \lambda^+$. Here $\lambda^{\pm} \in \pi_1(S^{\pm}, P^{\pm})$ generates the peripheral subgroup, $\partial S^{\pm} = \lambda^{\pm}$. So by Nielsen's Theorem [273], [382, 5.7], [95, §8], there is a homeomorphism $f\colon S^- \to S^+$ inducing $f_{\#} = \varphi$. There are canonical curves κ_i^+, κ_i^- on S^+ and S^- with $f(\kappa_i^-) = \kappa_i^+$ and $\sigma \kappa_i^- \sigma^{-1} \simeq \kappa_i^+$ in C^*. Again the system $\{\sigma \kappa_i^- \sigma^{-1} (\kappa_i^+)^{-1} \mid 1 \leq i \leq 2g\}$ is replaced by a homotopic system $\{\varrho_i \mid 1 \leq i \leq 2g\}$ of disjoint simple curves, which by Dehn's Lemma [281] (see Appendix B.4) span non-singular disks D_i, $\partial D_i = \varrho_i = D_i \cap \partial C^*$ which can be chosen disjoint.

Cut C^* along the D_i. The resulting space C^{**} is a 3-ball (Figure 5.2) by Alexander's Theorem (see Theorem 1.6), because its boundary is a 2-sphere in S^3 composed of an annulus $\partial S^* \times I$ and two 2-cells $(S^+)^*$ and $(S^-)^*$ where S^*, $(S^+)^*$ and $(S^-)^*$ are, respectively, obtained from S, S^+, S^- by the cutting of C^*. So C^{**} can be fibered over I, $C^{**} = S^* \times I$, $(S^*)^+ = S^* \times 1$, $(S^*)^- = S^* \times 0$. It remains to show that the identification $C^{**} \to C^*$ inverse to the cutting process can be changed by an isotopy such as to be compatible with the fibration. Let g_i' be the identifying homeomorphisms, $g_i'(D_{i1}) = D_{i2} = D_i$, $i = 1, 2, \dots, 2g$. The fibration of C^{**} induces a fibration on D_{ij}, the fibers being parallel to $D_{ij} \cap (S^*)^{\pm}$. There are fiber-preserving homeomorphisms $g_i\colon D_{i1} \to D_{i2}$ which coincide with g_i' on the top $(S^*)^+$ and the bottom $(S^*)^-$. Since the D_{i1}, D_{i2} are 2-cells, the g_i are isotopic to the g_i'; hence, $C^* \cong S \times I$ and $C = S \times I/h$ (compare Lemma 5.8). □

5.4 Remark. Note that the isomorphism $\varphi\colon \pi_1(S^-, P^-) \to \pi_1(S^+, P^+)$ which we have used in the above proof can be defined if the subgroups $i_\#^+\pi_1(S^+, P^+)$ and $\sigma\, i_\#^- \pi_1(S^-, P^-)\,\sigma^{-1}$ coincide in $\pi_1 C^*$. This observation finishes the proof of Proposition 4.7 (b) since $\mathfrak{B}_j = \mathfrak{B}_{j+1} \subset \mathfrak{A}_{j+1}$ would imply that

$$i_\#^+\pi_1(S^+, P^+) = \sigma\, i_\#^- \pi_1(S^-, P^-)\,\sigma^{-1}.$$

Hence there would be a homeomorphism $f\colon S^- \to S^+$ inducing $f_\# = \varphi$. Now the same argument as in the Proof 5.3 shows that C fibers over the circle. This contradicts the assumption that $i_\#^+(\pi_1 S^+) \neq \pi_1 C^* \neq i_\#^-(\pi_1 S^-)$.

5.5 Corollary. *The complement C of a* fibered *knot of genus g is obtained from $S \times I$, S a compact surface of genus g with a connected nonempty boundary, by the identification*

$$(x, 0) = (h(x), 1),\ x \in S,$$

where $h\colon S \to S$ is an orientation preserving homeomorphism:

$$C = S \times I/h.$$

Now $\mathfrak{G} = \pi_1 C$ is a semidirect product $\mathfrak{G} = \mathfrak{Z} \ltimes_\alpha \mathfrak{G}'$, where $\mathfrak{G}' = \pi_1 S \simeq \mathfrak{F}_{2g}$. The automorphism $\alpha(t)\colon \mathfrak{G}' \to \mathfrak{G}'$, $a \mapsto t^{-1}at$, and $h_\#^{-1}$ belong to the same class of automorphisms, in other words, $\alpha(t) \cdot h_\#^{-1}$ or $\alpha(t) \cdot h_\#$ is an inner automorphism of $\mathfrak{G}'$.

The *proof* follows from the construction used in proving Theorem 5.1. □

Observe that σ after identification by h becomes a generator of $\mathfrak{Z}$. If t is replaced by another coset representative t^* mod $\mathfrak{G}'$, $\alpha(t^*)$ and $\alpha(t)$ will be in the same class of automorphisms. Furthermore $\alpha(t^{-1}) = \alpha^{-1}(t)$. The ambiguity $h_{\#}^{\pm 1}$ can be avoided if σ as well as t are chosen to represent a meridian of $\mathfrak{k}$. ($h_{\#}$ is called the monodromy map of C.)

There is an addendum to Theorem 5.1.

5.6 Proposition. *If the complement C of a knot $\mathfrak{k}$ of genus g fibers locally trivially over S^1 then the fiber is a compact orientable surface S of genus g with one boundary component, and $\mathfrak{G}' = \pi_1 S \cong \mathfrak{F}_{2g}$.*

Proof. Since the fibration $C \to S^1$ is locally trivial the fiber is a compact 2-manifold S. There is an induced fibration $\partial C \to S^1$ with fiber ∂S. Consider the exact fiber sequences [157, Thm. 4.41] and use that S^1 is aspherical:

$$\begin{array}{ccccccccccc} 1 & \to & \pi_1(\partial S) & \to & \pi_1(\partial C) & \to & \pi_1 S^1 & \to & \pi_0(\partial S) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \downarrow & & \\ 1 & \to & \pi_1 S & \to & \pi_1 C & \to & \pi_1 S^1 & \to & \pi_0 S & \to & 1. \end{array}$$

The diagram commutes, and $\pi_1(\partial C) \to \pi_1 S^1$ is surjective. Hence $\pi_1 C \to \pi_1 S^1$ is surjective and $\pi_0(\partial S) = \pi_0 S = 1$, that is, S and ∂S are connected. (See E 5.1.)

Now the second sequence pins down $\pi_1 S$ as $(\pi_1 C)'$. □

We conclude this paragraph by stating the general theorem of Stallings without proof:

5.7 Theorem (Stallings [343]). *Let M be a compact irreducible 3-manifold (this means that in M every 2-sphere bounds a 3-ball). Assume that $\varphi\colon \pi_1 M \to \mathbb{Z}$ is an epimorphism with a finitely generated kernel. Then:*

(a) $\ker\varphi$ *is isomorphic to the fundamental group of a compact surface S.*

(b) *M can be fibered locally trivially over S^1 with fiber S if* $\ker\varphi \not\cong \mathbb{Z}_2$. □

5.B Fibered knots

The knots of the first class whose commutator subgroups are finitely generated – in fact they are free groups of rank $2g$ – are called *fibered knots* by virtue of Theorem 5.1. The fibration of their complements affords additional mathematical tools for the treatment of these knots. In a certain way fibered knots are easier to handle: The original 3-dimensional problem can to some extent be played down to two dimensions. This is a phenomenon also known in the theory of braids (see Chapter 10) or Seifert fiber spaces.

We shall study the question: How much information on the fibered knot $\mathfrak{k}$ do we get by looking at $h\colon S \to S$ in the formula $S \times I/h = \overline{S^3 - V(\mathfrak{k})}$?

5.8 Lemma (Neuwirth). *If $h_0,\ h_1\colon\ S \to S$ are isotopic homeomorphisms then there is a fiber preserving homeomorphism*

$$H\colon\ S \times I/h_0 \to S \times I/h_1.$$

Proof. Let h_t be the isotopy connecting h_0 and h_1. Put $g_t = h_t h_0^{-1}$ and define a homeomorphism

$$H'\colon\ S \times I \to S \times I$$

by $H'(x,t) = (g_t(x), t), x \in S, t \in I$. Since $H'(x,0) = (x,0)$ and

$$H'(h_0(x), 1) = (g_1 h_0(x), 1) = (h_1(x), 1),$$

H' induces a homeomorphism H as desired. □

5.9 Lemma. *Let $f\colon\ S \to S$ be a homeomorphism. Then there is a fiber-preserving homeomorphism $F\colon\ S \times I/h \to S \times I/fhf^{-1}$. If f is orientation preserving then there is a homeomorphism F which also preserves the orientation.*

Proof. Take $F(x,t) = (f(x), t)$. □

5.10 Definition (Similarity). Homeomorphisms $h_1\colon\ S_1 \to S_1,\ h_2\colon\ S_2 \to S_2$ of homeomorphic oriented compact surfaces S_1 and S_2 are called *similar*, if there is a homeomorphism $f\colon\ S_1 \to S_2$ respecting orientations, such that fh_1f^{-1} and h_2 are isotopic.

The notion of similarity enables us to characterize homeomorphic complements C_1 and C_2 of fibered knots $\mathfrak{k}_1$ and $\mathfrak{k}_2$ of equal genus g by properties of the gluing homeomorphisms.

5.11 Proposition. *Let $\mathfrak{k}_1,\ \mathfrak{k}_2$ be two (oriented) fibered knots of genus g with (oriented) complements C_1 and C_2. There is an orientation preserving homeomorphism $H\colon\ C_1 = S_1 \times I/h_1 \to C_2 = S_2 \times I/h_2,\ \lambda_1 = \partial S_1 \simeq \mathfrak{k}_1,\ H(\partial S_1) = \partial S_2 = \lambda_2 \simeq \mathfrak{k}_2$, if and only if there is a homeomorphism $h\colon\ S_1 \to S_2$, respecting orientations, $h(\lambda_1) = \lambda_2$, such that hh_1h^{-1} and h_2 are isotopic, that is, h_1 and h_2 are similar.*

Proof. If h exists and hh_1h^{-1} and h_2 are isotopic then by Lemma 5.8 there is a homeomorphism which preserves orientation and fibration:

$$F\colon S_2 \times I/hh_1h^{-1} \to S_2 \times I/h_2.$$

Now by Lemma 5.9 $F'\colon S_1 \times I/h_1 \to S_2 \times I/hh_1h^{-1}$, $(x,t) \mapsto (h(x), t)$, gives $H = FF'$ as desired.

To show the converse, let $H\colon\ C_1 = S_1 \times I/h_1 \to S_2 \times I/h_2 = C_2$ be an orientation preserving homeomorphism, $H(\lambda_1) = \lambda_2$. There is an isomorphism

$$H_\#\colon\ \pi_1 C_1 = \mathfrak{G}_1 \to \mathfrak{G}_2 = \pi_1 C_2$$

which induces an isomorphism

$$h_\#\colon \pi_1 S_1 = \mathfrak{G}_1' \to \mathfrak{G}_2' = \pi_1 S_2$$

such that $h_\#(\lambda_1) = \lambda_2$. By Nielsen ([378, Satz V.9], [382, 5.7.2], [95, §8]), there is a homeomorphism $h\colon S_1 \to S_2$ respecting the orientations induced on ∂S_1 and ∂S_2. By Corollary 5.5 we can choose representatives m_1 and m_2 of meridians of $\mathfrak{k}_1, \mathfrak{k}_2$, such that

$$h_{i\#}\colon \pi_1 S_i \to \pi_1 S_i, x \mapsto m_i^{-1} x m_i,\ i = 1, 2.$$

Since H preserves the orientation, $H_\#(m_1) = m_2 \lambda_2^k$, $k \in \mathbb{Z}$. This gives for all $x \in \pi_1 S_1$ the following:

$$\begin{aligned} h_\# h_{1\#}(x) &= h_\#(m_1^{-1} x m_1) = H_\#(m_1^{-1}) h_\#(x)(H_\#(m_1)) \\ &= \lambda_2^{-k} m_2^{-1} h_\#(x) m_2 \lambda_2^k = \lambda_2^{-k}(h_{2\#} h_\#(x)) \lambda_2^k. \end{aligned}$$

Hence $h_{2\#}$ and $(h h_1 h^{-1})_\#$ differ by an inner automorphism of $\pi_1 S_2$. By Baer's Theorem ([378, Satz V.15], [382, 5.13,1], [95, §8]), h_2 and $h h_1 h^{-1}$ are isotopic; hence h_1 and h_2 are similar. □

Proposition 5.11 shows that the classification of fibered knot complements can be formulated in terms of the surface fiber and maps of such surfaces. The proof also shows that if fibered complements are homeomorphic then there is a fiber-preserving homeomorphism. This means: different fibrations of a complement C admit a fiber preserving autohomeomorphism. Indeed, by Waldhausen's result [367], there is even an isotopy connecting both fibrations.

In the case of fibered knots invertibility and amphicheirality can be excluded by properties of surface mappings.

5.12 Proposition. *Let $C = S \times I/h$ be the complement of a fibered knot $\mathfrak{k}$.*

(a) *$\mathfrak{k}$ is amphicheiral only if h and h^{-1} are similar.*

(b) *$\mathfrak{k}$ is invertible only if there is a homeomorphism $f\colon S \to S$, reversing orientation, such that h and $f h^{-1} f^{-1}$ are similar.*

Proof (Burde–Zieschang [61]). (a) The map $(x, t) \mapsto (x, 1-t), x \in S, t \in I$ induces a mapping

$$C = S \times I/h \to S \times I/h^{-1} = C'$$

onto the mirror image C' of C satisfying the conditions of Proposition 5.11.

(b) If $f\colon S \to S$ is any homeomorphism inverting the orientation of S, then $(x, t) \mapsto (f(x), 1-t)$ induces a homeomorphism

$$S \times I/h \to S \times I/f h^{-1} f^{-1}$$

which maps ∂S onto its inverse. Again apply Proposition 5.11. □

5.C Applications and examples

The fibration of a non-trivial knot complement is not easily visualized, even in the simplest cases. (If $\mathfrak{k}$ is trivial, C is a solid torus, hence trivially fibered by disks D^2, $C = S^1 \times D^2$.)

5.13 Fibering the complement of the trefoil. Let C be the complement of a trefoil $\mathfrak{k}$ sitting symmetrically on the boundary of an unknotted solid torus $T_1 \subset S^3$ (Figure 5.3). $T_2 = \overline{S^3 - T_1}$ is another unknotted solid torus in S^3. A Seifert surface S (hatched regions in Figure 5.3) is composed of two disks D_1 and D_2 in T_2 and three twisted 2-cells in T_1. (Figure 5.4 shows T_1 and the twisted 2-cells in a straightened position.) A rotation about the core of T_1 through φ and, at the same time, a rotation about the core of T_2 through $2\varphi/3$ combine to a mapping $f_\varphi\colon S^3 \to S^3$. Now C is fibered by $\{f_\varphi(S) \mid 0 \leq \varphi \leq \pi\}$ (see Rolfsen [309, p. 329]).

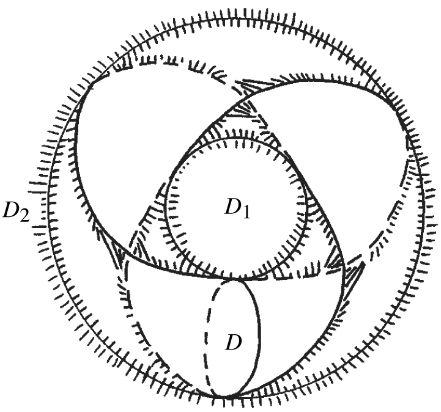


Figure 5.3

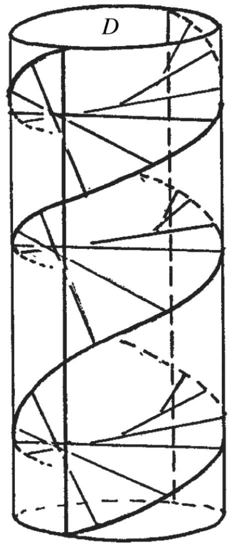


Figure 5.4

5.14 Fibring the complement of the four-knot. The above construction of a fibration takes advantage of the symmetries of the trefoil as a torus knot. It is not so easy to convince oneself of the existence of a fibration of the complement of the figure-eight knot $\mathfrak{k}$ by geometric arguments. The following sequence of Figures (5.5 (a)–(g)) tries to do it: (a) depicts a Seifert surface S spanning the four-knot in a tolerably symmetric fashion; (b) shows S thickened up to a handlebody V of genus 2. The knot $\mathfrak{k}$ is a curve on its boundary; (c) presents $V' = \overline{S^3 - V}$. In order to find $\mathfrak{k}$ on $\partial V'$ express $\mathfrak{k}$ on

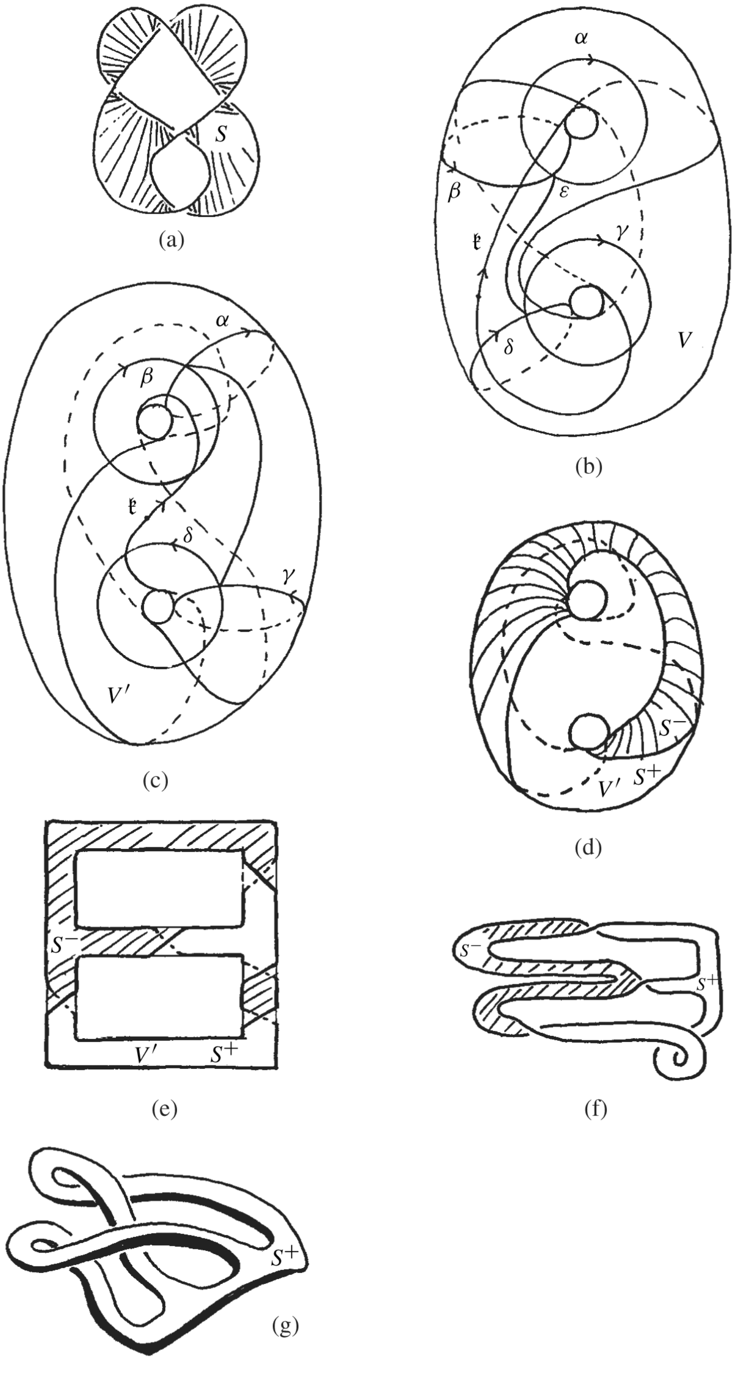


Figure 5.5. Fibring the complement of the four-knot.

∂V by canonical generators $\alpha, \beta, \gamma, \delta$ of $\pi_1(\partial V)$, $\mathfrak{k} = \beta\alpha^{-1}\gamma^{-1}\delta^{-1}\alpha\beta^{-1}\gamma\delta$. Replace every generator by its inverse to get $\mathfrak{k} = \beta^{-1}\alpha\gamma\delta\alpha^{-1}\beta\gamma^{-1}\delta^{-1}$ on $\partial V'$. The knot $\mathfrak{k}$ divides $\partial V'$ into two surfaces S^+ and S^- of genus one, Figure 5.5 (d). Figure 5.5 (e) just simplifies (d); the knot is pushed on the outline of the figure as far as possible. By way of (f) we finally reach (g), where the fibers of $V' - \mathfrak{k}$ are Seifert surfaces parallel to S^+ and S^-. The fibration extends to $V - \mathfrak{k}$ by the definition of V.

The following proposition shows that the trefoil and the four-knot are not only the two knots with the fewest crossings, but constitute a class that can be algebraically characterized.

5.15 Proposition. *The trefoil knot and the four-knot are the only fibered knots of genus one.*

At this stage we only prove a weaker result: *A fibered knot of genus one has the same complement as the trefoil or the four-knot.*

Proof (Burde–Zieschang [61]). Let $C = S \times I/h$ be the complement of a knot $\mathfrak{k}$ and assume that S is a torus with one boundary component. Then h induces automorphisms $h_\#\colon \pi_1 S \to \pi_1 S$ and $h_*\colon H_1(S) \to H_1(S) \cong \mathbb{Z}^2$. Let A denote the 2×2-matrix corresponding to h_* (after the choice of a basis).

$$\det A = 1, \tag{1}$$

since h preserves the orientation. The automorphism $h_\#$ describes the effect of the conjugation with a meridian of $\mathfrak{k}$ and it follows that $\pi_1 S$ becomes trivial by introducing the relations $h_\#(\chi) = \chi \in \pi_1 S$. This implies:

$$\det\left(A - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \pm 1. \tag{2}$$

From (1) and (2) it follows that

$$\operatorname{trace} A \in \{1, 3\}. \tag{3}$$

A matrix of trace +1 is conjugate in $\mathrm{SL}(2,\mathbb{Z})$ to $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and a matrix with trace 3 is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$, [380, 21.15]. Two automorphisms of $\mathfrak{F}_2$ which induce the same automorphism on $\mathbb{Z} \oplus \mathbb{Z}$ differ by an inner automorphism ([271], [220, I.4.5]. The Baer Theorem now implies that the gluing mappings are determined up to isotopy; hence, by Lemma 5.8, the complement of the knot is determined up to homeomorphism by the matrix above. The matrices $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ are obtained when the complements of the trefoil knots are fibered, see 5.16. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$ results in the case of the figure-eight knot as follows from the fact that in Example 3.8 the conjugation by s induces on $\mathfrak{G}'$ the mapping $x_0 \mapsto x_1$, $x_1 \mapsto x_1 x_0^{-1} x_1^2$.

Thus we have proved that the complement of a fibered knot $\mathfrak{k}$ of genus 1 is homeomorphic to the complement of a trefoil knot or the figure-eight knot. Later we shall show that $\mathfrak{k}$ is indeed a trefoil knot (Theorem 6.1) or a four-knot (Theorem 15.7). □

Remark. The proof of Proposition 5.15 gives the following result: *A fibered knot of genus one in a homology* 3*-sphere has the same complement as the trefoil or the four-knot.*

5.16 The trefoil knot. We conclude this section with an application of Proposition 5.12 and reprove the fact (see Theorem 3.39 (b)) that the trefoil knot is not amphicheiral. This was first proved by M. Dehn in 1914 [85, 87].

Figure 5.6 shows a trefoil bounding a Seifert surface S of genus one. The Wirtinger presentation of the knot group $\mathfrak{G}$ is

$$\mathfrak{G} = \langle s_1, s_2, s_3 \mid s_3 s_1 s_3^{-1} s_2^{-1}, s_1 s_2 s_1^{-1} s_3^{-1}, s_2 s_3 s_2^{-1} s_1^{-1} \rangle.$$

The curves a and b in Figure 5.6 are free generators of $\pi_1 S = \mathfrak{F}_2 = \langle a, b \rangle$. They can be expressed by the Wirtinger generators s_i (see 3.7):

$$a = s_1^{-1} s_2,\ b = s_2^{-1} s_3.$$

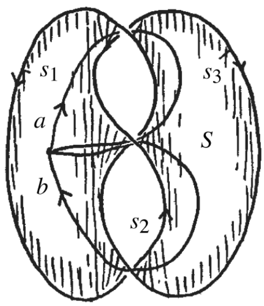


Figure 5.6. A Seifert surface of the trefoil knot.

Using the relations we get (with $t = s_1$):

$$\begin{aligned} t^{-1} a t &= s_1^{-1} s_1^{-1} s_2 s_1 = s_1^{-1} s_2 s_3^{-1} s_1 = s_1^{-1} s_2 s_3^{-1} s_2 s_2^{-1} s_1 = a b^{-1} a^{-1}, \\ t^{-1} b t &= s_1^{-1} s_2^{-1} s_3 s_1 = s_1^{-1} s_2^{-1} s_2 s_3 = s_1^{-1} s_2 \cdot s_2^{-1} s_3 = ab. \end{aligned}$$

Let $C = S \times I/h$ be the complement of the trefoil. Relative to the basis $\{a, b\}$ of $H_1(S) = \mathbb{Z} \oplus \mathbb{Z}$ the homomorphism $h_*\colon H_1(S) \to H_1(S)$ is given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ (see Corollary 5.5). If the trefoil were amphicheiral then by Proposition 5.12 there would be a unimodular matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\ 1 = \alpha\delta - \beta\gamma,$$

such that

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$
$$\begin{pmatrix}\alpha-\beta & \alpha\\ \gamma-\delta & \gamma\end{pmatrix} = \begin{pmatrix}-\gamma & -\delta\\ \alpha+\gamma & \beta+\delta\end{pmatrix}.$$

This means: $\delta = -\alpha, \gamma = \beta - \alpha$. However, $1 = \alpha\delta - \beta\gamma = -\alpha^2 - \beta(\beta-\alpha) = -(\alpha^2 - \alpha\beta + \beta^2)$ has no integral solution.

5.17 Links and satellite knots. The aim of this section is to study under which conditions a satellite knot is fibered. The main result is the *pattern fibering criterion* by M. Hirasawa, K. Murasugi and D. S. Silver [173].

Let $\mathfrak{l} = \mathfrak{l}_1 \cup \cdots \cup \mathfrak{l}_\mu$ be an oriented link of multiplicity μ. A meridian m_i of the component $\mathfrak{l}_i$ acquires an orientation such that $\operatorname{lk}(\mathfrak{l}_i, m_i) = 1$. Let $\alpha\colon \mathfrak{G} \to \mathbb{Z}$ be the homomorphism given by

$$\alpha(\gamma) = \sum_{i=1}^{\mu} \operatorname{lk}(\mathfrak{l}_i, \gamma) \text{ for } \gamma \in \pi_1(S^3 - \mathfrak{l}). \tag{5.1}$$

The kernel $\mathfrak{A} = \ker(\alpha)$ is called the *augmentation subgroup* of $\mathfrak{l}$ and was introduced by Brown and Crowell in [45] (see Exercise 4.9). The link $\mathfrak{l}$ is *fibered* with oriented connected fiber $S \subset S^3$ if $\partial S = \mathfrak{l}$ and if the complement $S^3 - \mathfrak{l}$ fibers over S^1 with fiber S. It follows from Stallings' Theorem 5.7 [343] and Neuwirth's argument (Theorem 4.7) that $\mathfrak{l}$ is fibered if and only if the kernel $\mathfrak{A}$ is finitely generated.

Remark. Exercise 4.10 shows that an orientation is required in the definition of a fibered link. An oriented link is fibered if and only if the link obtained by reversing the orientation of every component is fibered.

Let now $\mathfrak{k}$ be a satellite with companion $\widehat{\mathfrak{k}}$ and pattern $(\widetilde{V}, \widetilde{\mathfrak{k}})$. As in Section 4.E let $\varphi\colon \mathfrak{G} \to \mathbb{Z}$ be the canonical homomorphism which maps the meridian of $\mathfrak{k}$ to 1, denote by $\widetilde{C}$ the complement of the pattern $\widetilde{\mathfrak{k}} \subset \widetilde{V}$ and by $\mathfrak{K} = \ker(\varphi|_{\pi_1\widetilde{C}})$ the kernel of the restriction of φ to the subgroup $\pi_1\widetilde{C} \subset \mathfrak{G}$. We say that the pattern $(\widetilde{V}, \widetilde{\mathfrak{k}})$ is *fibered* if there exists a fibration $\widetilde{C} \to S^1$ inducing the homomorphism $\varphi|_{\pi_1\widetilde{C}}$ on the fundamental groups. Let $n = |\operatorname{lk}(\widetilde{m}, \widetilde{\mathfrak{k}})|$ denote the winding number of $\widetilde{\mathfrak{k}}$ in $\widetilde{V}$.

5.18 Theorem (Hirasawa–Murasugi–Silver [173]). *Let $\mathfrak{k}$ be a satellite with companion $\widehat{\mathfrak{k}}$ and pattern $(\widetilde{V}, \widetilde{\mathfrak{k}})$. Then the following conditions are equivalent:*

(1) *The satellite $\mathfrak{k}$ is fibered;*

(2) *The groups $\widehat{\mathfrak{G}}'$ and $\mathfrak{K}$ are finitely generated;*

(3) *The companion $\widehat{\mathfrak{k}}$ and the pattern $(\widetilde{V}, \widetilde{\mathfrak{k}})$ are fibered.*

Proof. If $\mathfrak{k}$ is fibered then $\mathfrak{G}'$ is finitely generated and by Lemma 4.14 the winding number n is nonzero. Therefore, Corollary 4.17 implies that $\mathfrak{K}$ and $\widehat{\mathfrak{G}}'$ are finitely generated.

If $\widehat{\mathfrak{G}}'$ and $\mathfrak{K}$ are finitely generated the Stallings' theorem (Theorem 5.7) implies that $\widehat{\mathfrak{k}}$ and $(\widetilde{V}, \widetilde{\mathfrak{k}})$ are fibered.

If $\widehat{\mathfrak{k}}$ and $(\widetilde{V}, \widetilde{\mathfrak{k}})$ are fibered then $\widehat{\mathfrak{G}}'$ and $\mathfrak{K}$ are isomorphic to the fundamental group of the fiber of the corresponding fibration. Hence $\widehat{\mathfrak{G}}'$ and $\mathfrak{K}$ are free and finitely generated. If the winding number $n = \varphi(\widetilde{m})$ were zero then $\mathfrak{K}$ would contain a free Abelian group $\mathbb{Z} \oplus \mathbb{Z}$ generated by the meridian $\widetilde{m}$ and longitude $\widetilde{\ell}$ of $\widetilde{V}$. This is absurd. Now again Corollary 4.17 implies that $\mathfrak{G}'$ is finitely generated and Theorem 5.1 implies that the satellite knot $\mathfrak{k}$ is fibered. $\square$

Consider a pattern $(\widetilde{V}, \widetilde{\mathfrak{k}})$ such that $\widetilde{\mathfrak{k}}$ has nonzero winding number. We associate an oriented 3-component link $\widetilde{\mathfrak{l}} := \widetilde{\mathfrak{k}} \cup \widetilde{m} \cup \widetilde{m}' \subset S^3$, where $\widetilde{m}$ and $\widetilde{m}'$ are disjoint meridians of $\widetilde{V}$ with opposite orientations (the orientation of $\widetilde{\mathfrak{k}}$ is arbitrary), see Figure 5.7.

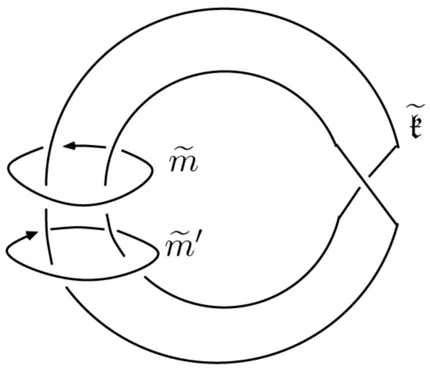


Figure 5.7. The link $\widetilde{\mathfrak{l}}$.

5.19 Theorem (Pattern fibering criterion [173]). *The pattern $(\widetilde{V}, \widetilde{\mathfrak{k}})$ is fibered if and only if the link $\widetilde{\mathfrak{l}}$ is fibered.*

Proof. According to Stallings' theorem, it is sufficient to show that $\mathfrak{K}$ is finitely generated if and only if the augmentation subgroup $\widetilde{\mathfrak{A}}$ of $\widetilde{l}$ is finitely generated. More precisely we show that $\widetilde{\mathfrak{A}}$ is isomorphic to the free product $\mathfrak{K} * \mathfrak{F}_n$ where $\mathfrak{F}_n$ is a free group of rank n. Then the Grushko–Neumann theorem [382, 2.9.2], [220, III.3.7], implies that $\widetilde{\mathfrak{A}}$ is finitely generated if and only if $\mathfrak{K}$ is finitely generated.

We have to prove that the augmentation subgroup of $\widetilde{\mathfrak{l}}$ is isomorphic to $\mathfrak{K} * \mathfrak{F}_n$. First note that the group $\pi_1(S^3 - \widetilde{\mathfrak{l}})$ is an HNN extension of $\mathfrak{H} = \pi_1 \widetilde{C}$. In order to see this, let $\langle S \mid R \rangle$ be a presentation of $\mathfrak{H}$ then

$$\pi_1(S^3 - \widetilde{\mathfrak{l}}) = \langle S, s \mid R,\ s\widetilde{m}s^{-1} = \widetilde{m} \rangle = \mathfrak{H} *_{\langle \widetilde{m} \rangle},$$

where the stable letter s is the element represented by the meridian of $\tilde{m}$. In what follows we consider $\mathfrak{H}$ as a subgroup of $\pi_1(S^3 - \tilde{\mathfrak{l}})$.

The augmentation subgroup $\tilde{\mathfrak{A}} \subset \pi_1(S^3 - \tilde{\mathfrak{l}})$ is the kernel of the homomorphism $\tilde{\alpha}\colon \pi_1(S^3 - \tilde{\mathfrak{l}}) \to \mathbb{Z}$ given by

$$\tilde{\alpha}(\gamma) = \mathrm{lk}(\tilde{\mathfrak{k}}, \gamma) + \mathrm{lk}(\tilde{m}, \gamma) + \mathrm{lk}(\tilde{m}', \gamma) \text{ for } \gamma \in \pi_1(S^3 - \tilde{\mathfrak{l}}).$$

The homomorphism $\varphi\colon \mathfrak{H} \to \mathbb{Z}$, given by $\varphi(\eta) = \mathrm{lk}(\tilde{\mathfrak{k}}, \eta)$, $\eta \in \mathfrak{H}$, and the restriction $\tilde{\alpha}|_{\mathfrak{H}}$ coincide since for every loop $\eta \subset \tilde{C}$ we have

$$\mathrm{lk}(\tilde{m}, \eta) + \mathrm{lk}(\tilde{m}', \eta) = 0$$

(here we use that $\tilde{m}$ and $\tilde{m}'$ have opposite orientations). Therefore $\ker(\tilde{\alpha}|_{\mathfrak{H}}) = \mathfrak{K} = \ker(\varphi)$.

The HNN extension $\mathfrak{H} *_{\langle \tilde{m} \rangle}$ acts on a tree T with fundamental domain a loop [331, 324, 68]. Hence $\tilde{\mathfrak{A}}$ is a fundamental group of a graph of groups [331, I.§4.4]. More precisely, the subgroup theorem [324, 3.14], [68, 8.5, Thm. 27] shows that $\tilde{\mathfrak{A}}$ is the fundamental group of a graph of groups (Figure 5.8). The vertices correspond to the double cosets $\tilde{\mathfrak{A}}\, g\, \mathfrak{H}$, $g \in \pi_1(S^3 - \tilde{\mathfrak{l}})$, with associated vertex group $\tilde{\mathfrak{A}} \cap g\mathfrak{H}g^{-1}$. The edges correspond to the double cosets $\tilde{\mathfrak{A}}\, g\, \langle \tilde{m} \rangle$, $g \in \pi_1(S^3 - \tilde{\mathfrak{l}})$, and the corresponding edge group is $\tilde{\mathfrak{A}} \cap g \langle \tilde{m} \rangle g^{-1}$. Now there is only one double coset $(\tilde{\mathfrak{A}}, \mathfrak{H})$ since $\varphi\colon \mathfrak{H} \to \mathbb{Z}$ is surjective. The corresponding vertex group is $\mathfrak{K} = \tilde{\mathfrak{A}} \cap \mathfrak{H}$. There are n double cosets $\tilde{\mathfrak{A}}\, m^k \langle \tilde{m} \rangle$, $k = 0, \dots, n-1$, since $\tilde{\alpha}(\tilde{m}) = \varphi(\tilde{m}) = n$ and $\varphi(m) = 1$ for the median m of $\tilde{\mathfrak{k}}$. The corresponding edge groups are trivial since

$$\tilde{\mathfrak{A}} \cap m \langle \tilde{m} \rangle m^{-1} = m(\tilde{\mathfrak{A}} \cap \langle \tilde{m} \rangle) m^{-1}$$

is trivial ($\varphi(\tilde{m}) = n \neq 0$). Hence $\tilde{\mathfrak{A}} \cong \mathfrak{K} * \mathfrak{F}_n$ where $\mathfrak{F}_n$ is a free group of rank n. □

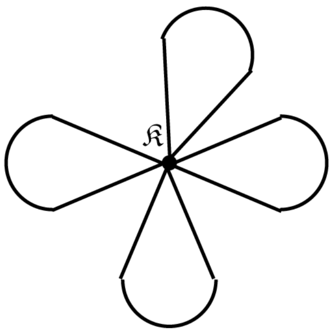


Figure 5.8. The graph of groups with trivial edge groups.

5.D History and sources

The content of this chapter is for the larger part based on J. Stallings' theorem on fibering 3-manifold [343]. The fibered complement $C = S \times I/f$ of a "fibered knot" was further investigated by L. Neuwirth in [268] and by Burde and Zieschang in [61]. In the first paper the complement was shown to be determined by the peripheral system of the knot group while in the second one C was characterized by properties of the identifying surface map f.

J. Neuwirth's result is a special case of the general theorems of F. Waldhausen [367]. In this fundamental paper manifolds with a Stallings fibration play an important role.

D. Eisenbud and W. Neumann give topological conditions for a satellite knot in a homology 3-sphere to fiber [91, Thm. 4.2]. The theorem is stated for *multilinks* in homology 3-spheres and uses the decomposition of the multilink into irreducible *splice components* arising from the *Jaco–Shalen–Johannson* splitting theorem.

5.E Exercises

E 5.1. Construct a fibration of a compact orientable 3-manifold M over S^1 such that $\pi_1 M \to \pi_1 S^1$ is not surjective. Observe that the fiber is not connected in this case.

E 5.2. Find a 2×2-matrix A representing f_*: $H_1(S) \to H_1(S)$ in the case of the complement $C = S \times I/f$ of the four-knot. Show that A and A^{-1} are conjugate.

E 5.3. Compute the powers of the automorphism $f_\#$: $\pi_1 S \to \pi_1 S$

$$\begin{aligned} f_\#(a) &= t^{-1}at = b^{-1} \\ f_\#(b) &= t^{-1}at = ba \end{aligned}$$

induced by the identifying map of the trefoil (see 5.16). Describe the manifolds $S \times I/f^i, i \in \mathbb{Z}$.

E 5.4. Show that the knot 5_2 can be spanned by a Seifert surface S of minimal genus such that the knot complement C cut along S is a handlebody C^*. Apply the method used in 5.14 to show that nevertheless 5_2 is not fibered!

E 5.5. Show that the knot 8_{20} is fibered.

Chapter 6

A characterization of torus knots

Torus knots have been repeatedly considered as examples in the preceding chapters. If knots are placed on the boundaries of handlebodies as in Chapter 3, the least possible genus of a handlebody carrying a knot defines a hierarchy for knots where the torus knots form the simplest class excepting the trivial knot. Torus knots admit a simple algebraic characterization; see Theorem 6.1.

6.A Results and sources

6.1 Theorem (Burde–Zieschang). *A non-trivial knot whose group $\mathfrak{G}$ has a non-trivial center is a torus knot.*

The theorem was first proved in [60], and had been proved for alternating knots by K. Murasugi [259] and L. Neuwirth [269]. Since torus knots have Property P (Theorem 15.5), Theorem 6.1 together with Theorem 3.39 and Proposition 3.38 show: any knot group with a non-trivial center determines its complement, and the complement in turn admits just one torus knot $\mathfrak{t}(a,b)$ and its mirror image $\mathfrak{t}(a,-b)$.

F. Waldhausen later proved a more general theorem which includes Theorem 6.1 by using Seifert's theory of fibered 3-manifolds, see [365]:

6.2 Theorem (Waldhausen). *Let M be an orientable compact irreducible 3-manifold. If either $H_1(M)$ is infinite or $\pi_1 M$ a non-trivial free product with amalgamation, and if $\pi_1 M$ has a non-trivial center, then M is homeomorphic to a Seifert fibered manifold with orientable orbit-manifold (Zerlegungsfläche).* □

Because of Theorem 3.1, Theorem 6.2 obviously applies to knot complements $C = M$. A closer inspection of the Seifert fibration of C shows that it can be extended to S^3 in such a way that the knot becomes a normal fiber. Theorem 6.1 now follows from a result of H. Seifert [326] which contains a complete description of all Seifert fibrations of S^3.

6.3 Theorem (Seifert). *A fiber of a Seifert fibration of S^3 is a torus knot or the trivial knot. Exceptional fibers are always unknotted.* □

Note that Theorem 6.2 is a special case of the following theorem:

6.4 Theorem (Seifert fiber space conjecture). *Let M be an orientable compact irreducible 3-manifold. If $\pi_1 M$ contains an infinite cyclic normal subgroup, then M is Seifert fibered.* □

Theorem 6.4 was proved for Haken manifolds by F. Waldhausen (Theorem 6.2) and C. M. Gordon & W. Heil [134]. The final proof of Theorem 6.4, which makes use of the work of several other authors, was given by D. Gabai [125] and independently by A. Casson and D. Jungreis [63].

We propose to give now a proof of Theorem 6.1 which makes use of a theorem by Nielsen [275] on mappings of surfaces. (This theorem is also used in Waldhausen's proof.) We do not presuppose Waldhausen's theory or Seifert's work on fibered manifolds, though Seifert's ideas are applied to the special case in hand. The proof is also different from that given in the original paper [60].

We now state without proof two theorems on periodic mappings of surfaces due to J. Nielsen [275, 274]. A modern exposition of Nielsen's result is contained in the book by B. Farb and D. Margalit [95, Chap. 7]. A proof of a generalization of the first theorem is due to H. Zieschang [380]; it is a deep result which requires a considerable amount of technicalities in its proof. A different approach was used by W. Fenchel [97, 98] and a combinatorial proof of his theorem was given by B. Zimmermann [384]. For a more general result see also the works of S. P. Kerckhoff [192, 193].

6.5 Theorem (Nielsen). *Let S be a compact surface different from the sphere with less than three boundary components. If $h\colon S \to S$ is a homeomorphism such that h^n is isotopic to the identity, then there is a periodic homeomorphism f of order n isotopic to h.* □

We need another theorem which provides additional geometric information on periodic surface mappings:

6.6 Theorem (Nielsen). *Let $f\colon S \to S$ be an orientation preserving periodic homeomorphism of order n, $f^n = \mathrm{id}$, of a compact orientable surface S. Let $q \in S$ be some point with $f^k(q) = q$ for some k with $0 < k < n$, and let k be minimal with this property. Then there is a neighborhood $U(q)$ of q in S, homeomorphic to an open 2-cell, such that $f^\ell(U(q)) \cap U(q) = \emptyset$ for $0 < \ell < k$. Furthermore $f^k|U(q)$ is a topological rotation of order $\frac{n}{k}$ with fixed point q.* □

For a proof of Theorem 6.6 see J. Nielsen's original paper [274], the English translation [276] or the book by B. Farb and D. Margalit [95, Chap. 7].

Points q of S for which such a k exists are called *exceptional* points.

6.7 Corollary (Nielsen). *A periodic mapping $f\colon S \to S$ as in Theorem 6.6 has at most finitely many exceptional points, none of them on $r = \partial S$.* □

Proof of Theorem 6.1. Let $\mathfrak{k}$ be a non-trivial knot whose group $\mathfrak{G}$ has a center $\mathfrak{C} \neq 1$. Then by Proposition 4.11 (a) its commutator subgroup $\mathfrak{G}'$ is finitely generated, and hence by Theorem 5.1 the complement C is a fiber space over S^1 with a Seifert surface S of minimal genus g as a fiber. Thus $C = S \times I/h$ as defined in 5.2. Let t and $r = \partial(S \times 0)$ represent a meridian and a longitude on ∂C, and choose their point of intersection P as a base point for $\pi_1(C) = \mathfrak{G}$. The homeomorphism $h\colon S \to S$ induces the automorphism:

$$h_{\#}\colon \mathfrak{G}' \to \mathfrak{G}' = \pi_1(S \times 0), x \mapsto t^{-1}xt,$$

since by Lemma 5.8 we may assume that $h(P) = P$. Again by Proposition 4.11, $\mathfrak{C} \cong \mathbb{Z}$. In the following we use the notation of Proposition 4.11.

6.8 Proposition. *Let $z = t^n u, n > 1$, be a generator of the center $\mathfrak{C}$ of $\mathfrak{G}$. Then u is a power of the longitude r, $u = r^{-m}$, $m \neq 0$, and $h_{\#}^n$ is the inner automorphism $h_{\#}^n(x) = r^m x r^{-m}$. The exponent n is the smallest one with this property. The powers of r are the only fixed elements of $h_{\#}^i$, $i \neq 0$.*

Proof. By assumption $t^{-n}xt^n = uxu^{-1}$ for all $x \in \mathfrak{G}'$. From $h_{\#}(r) = t^{-1}rt = r$ it follows that u commutes with r. The longitude r is a product of commutators of free generators of $\mathfrak{G}' \cong \mathfrak{F}_{2g}$ and it is easily verified that r is not a proper power of any other element of $\mathfrak{G}'$; hence, $u = r^{-m}, m \in \mathbb{Z}$ (see [382, E1.5] or [220, I.2.17]). We shall see $\gcd(n, m) = 1$ in equation (2) in Paragraph 6.9 and hence $m \neq 0$. Fixed elements of $h_{\#}^i, i \neq 0$, are also fixed elements of $h_{\#}^{in}$, hence they commute with r and they are therefore powers of r.

Now assume that $t^{-k}xt^k = vxv^{-1}$ for all $x \in \mathfrak{G}'$ and some $k \neq 0$ and $v \in \mathfrak{G}'$. Then $t^k v \in \mathfrak{C}$, thus $t^k v = (t^n u)^\ell = t^{n\ell}u^\ell$. This proves that n is the smallest positive exponent such that $h_{\#}^n$ is an inner automorphism of $\mathfrak{G}'$. □

It follows from Baer's Theorem [95, §8] that h^n is isotopic to the identity. Nielsen's Theorem 6.5 implies the existence of a periodic homeomorphism f of order n isotopic to h and by Lemma 5.8,

$$C = S \times I/h \cong S \times I/f.$$

The trivial fibration of $S \times I$ with fiber I defines a Seifert fibration of C. Exceptional points in S correspond to exceptional fibers by Theorem 6.6. Since a fiber $z = t^n r^{-m}$, $m \neq 0$, on ∂C is not isotopic to a meridian, the Seifert fibration of C extended to give a Seifert fibration of S^3, where $\mathfrak{k}$ is a fiber, normal or exceptional. By Theorem 6.3 normal fibers of Seifert fibrations of S^3 are torus knots or trivial knots, while exceptional fibers are always unknotted. So $\mathfrak{k}$ has to be a normal fiber i.e. a torus knot. □

6.B An elementary proof of Theorem 6.1

We shall now give a proof of Theorem 6.1 by not making use of Seifert's Theorem 6.3. In what follows let f be a periodic homeomorphism of order n isotopic to h and $C = S \times I/f$.

6.9. The orbit of an exceptional point of S relative to the cyclic group $\mathfrak{Z}_n$ generated by f consists of k_j points, $1 \leq k_j \leq n$, $k_j | n$. We denote exceptional points accordingly by $Q_{j\nu}$, $1 \leq j \leq s$, $0 \leq \nu \leq k_j - 1$, where $Q_{j,\nu+1} = f(Q_{j\nu})$, $\nu + 1 \mod k_j$. By deleting the neighborhoods $U(Q_{j\nu})$ of Theorem 6.6 we obtain $S_0 = S - \bigcup U(Q_{j\nu})$, which is a compact surface of genus g with $1 + \sum_{j=1}^{s} k_j$ boundary components, on which $\mathfrak{Z}_n = \langle f \rangle$ operates freely. So there is a regular cyclic n-fold covering $p_0\colon S_0 \to S_0^*$ with $\langle f \rangle$ as its group of covering transformations. We define a covering

$$p\colon C_0 = S_0 \times I/f \to S_0^* \times I/\mathrm{id} \cong S_0^* \times S^1 = C_0^*$$

by

$$p(x, \tau) = (p_0(x), \tau), x \in S_0, \tau \in I.$$

This covering is also cyclic of order n, and $f \times \mathrm{id}$ generates its group of covering transformations. Let $r_{j\nu}$ represent the boundary of $U(Q_{j\nu})$ in $\pi_1(S_0)$ in such a way that

$$\partial S = r = \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{s} \prod_{\nu=0}^{k_j - 1} r_{j\nu}.$$

The induced homomorphism $p_\#\colon \pi_1(C_0) \to \pi_1(C_0^*)$ then gives

$$p_\#(r) = r^{*n}, \quad p_\#(r_{j\nu}) = (r_j^*)^{m_j}, \quad m_j k_j = n, \tag{1}$$

where r^* and r_j^* represent the boundaries of S_0^* in $\pi_1(S_0^*)$ such that

$$r^* = \prod_{i=1}^{g^*} [a_i^*, b_i^*] \cdot \prod_{j=1}^{s} r_j^*.$$

Let $z^* = Q^* \times S^1$, $Q^* \in r^*$, be a simple closed curve on $r^* \times S^1$ representing a generator of $\pi_1(S^1)$. A lift of z^* is a simple arc on $r \times I/f$ connecting a point $(Q, 0)$, $Q \in p_0^{-1}(Q^*)$, to $(f^{-1}(Q), 1) \sim (f^{-2}(Q), 0)$. Since f is of order n it follows that the lift of z^{*n} is a simple closed curve on $r \times I/f$. More precisely, $p_\#^{-1}(z^{*n}) = t^n v$, $v \in \pi_1(S_0)$. The simple closed curve $t^n v$ on the torus $r \times I/f$ is central in $\pi_1(C_0)$, since z^* is central in $\pi_1(C_0^*)$. Therefore $t^n v$ is central in $\pi_1(C) \cong \mathfrak{G}$, too; hence, $p_\#^{-1}(z^{*n}) = z = t^n \cdot r^{-m}$, see Proposition 6.8. Since $t^n v = t^n r^{-m}$ represents a simple closed curve on the torus $r \times I/f$ it follows that

$$\gcd(m, n) = 1. \tag{2}$$

Furthermore, $z^{*n} = p_\#(z) = (p_\#(t))^n \cdot r^{*-mn}$. Putting $p_\#(t) = t^*$, we obtain

$$z^* = t^* r^{*-m}. \tag{3}$$

For $\alpha, \beta \in \mathbb{Z}$, satisfying

$$\alpha m + \beta n = 1, \tag{4}$$

$$q = t^\alpha r^\beta \quad \text{and} \quad p_\#(q) = q^* = t^{*\alpha} r^{*n\beta} \tag{5}$$

are simple closed curves on ∂C and $r^* \times S^1$, respectively. From these formulas we derive:

$$t^* = z^{*n\beta} \cdot q^{*m}, \tag{6}$$

$$r^* = z^{*-\alpha} \cdot q^*. \tag{7}$$

Since $f|r$ is a rotation of order n (see 6.6), the powers $\{r^{*\mu} \mid 0 \le \mu \le n-1\}$ are coset representatives in $\pi_1(C_0^*) \mod p_\#\pi_1(C_0)$. From (3) it follows that $\{z^{*\mu}\}$ also represent these cosets. By (7),

$$z^{*\alpha} r^* = q^* \in p_\#\pi_1(C_0). \tag{8}$$

We shall show that there are similar formulas for the boundaries r_j^*.

6.10 Lemma. *There are* $\alpha_j \in \mathbb{Z}, \gcd(\alpha_j, m_j) = 1$ *such that*

$$z^{*\alpha_j k_j} r_j^* = q_j^* \in p_\#\pi_1(C_0).$$

The α_j *are determined* $\operatorname{mod} m_j$.

Proof. For some $\nu \in \mathbb{Z}$ we have $z^{*\nu} r_j^* \in p_\#\pi_1(C_0)$. Now $q_j^* = z^{*\nu} r_j^*$ and z^* generate $\pi_1(r_j^* \times S^1)$. Hence $q_j = p_\#^{-1}(q_j^*)$ and $z = p_\#^{-1}(z^{*n})$ are generators of $\pi_1(p^{-1}(r_j^* \times S^1))$. Hence,

$$r_{j0} = z^{-\alpha_j} q_j^{\beta_j}, \quad \gcd(\alpha_j, \beta_j) = 1$$

and

$$z^{*-n\alpha_j} q_j^{*\beta_j} = p_\#(r_{j0}) = (r_j^*)^{m_j} = z^{*-\nu m_j} q_j^{*m_j},$$

thus

$$k_j \alpha_j = \nu, \quad m_j = \beta_j.$$

(Remember that $m_j k_j = n$, see equation (1) in Paragraph 6.9.) $\square$

6.11 The isomorphism $\kappa^*_\#$. Now let $\widehat{C}^*_0 = \widehat{S}^*_0 \times \widehat{S}^1$ be a homeomorphic copy of $C^*_0 = S^*_0 \times S^1$ with $\hat{z}^*$ generating $\pi_1(\hat{S}^1)$ and $\hat{a}^*_i, \hat{b}^*_i, \hat{r}^*, \hat{r}^*_j$ representing canonical generators of $\pi_1(\hat{S}^*_0)$, and $\hat{r}^* = \prod_{i=1}^{g^*}[\hat{a}^*_i, \hat{b}^*_i] \cdot \prod_{j=1}^{s} \hat{r}^*_j$. Define an isomorphism

$$\kappa^*_\#\colon\ \pi_1(C^*_0) \to \pi_1(\widehat{C}^*_0)$$

by

$$\kappa^*_\#(z^*) = \hat{z}^*,\ \kappa^*_\#(r^*_j) = \hat{z}^{*-\alpha_j k_j} \cdot \hat{r}^*_j,$$
$$\kappa^*_\#(a^*_i) = \hat{z}^{*-\rho_i} \cdot \hat{a}^*_i,\ \kappa^*_\#(b^*_i) = \hat{z}^{*-\sigma_i} \cdot \hat{b}^*_i,$$

where $\rho_i, \sigma_i \in \mathbb{Z}$ are chosen in such a way that $z^{*\rho_i} a^*_i,\ z^{*\sigma_i} b^*_i \in p_\#\pi_1(C_0)$. (The ρ_i, σ_i will play no role in the following.)

6.12 Lemma. $\kappa^*_\# p_\#\pi_1(C_0) = \pi_1(\widehat{S}^*_0) \times \langle z^{*n} \rangle$.

Proof. By construction $\kappa^{*-1}_\#(\hat{a}^*_i) = z^{*\rho_i} a^*_i \in p_\#\pi_1(C_0)$, likewise $\kappa^{*-1}_\#(\hat{b}^*_i)$, $\kappa^{*-1}_\#(\hat{r}^*_j) \in p_\#\pi_1(C_0)$. Since $\kappa^*_\#$ is an isomorphism, $\kappa^*_\# p_\#\pi_1(C_0)$ is a normal subgroup of index n in $\pi_1(\hat{S}^*_0) \times \langle \hat{z}^* \rangle$, which contains $\pi_1(\widehat{S}^*_0)$, because it contains its generators. This proves Lemma 6.12. □

We shall now see that $\kappa^*_\#$ can be realized by a homeomorphism $\kappa^*\colon S^*_0 \times S^1 \to \widehat{S}^*_0 \times \widehat{S}^1$, and that there is a homeomorphism $\kappa\colon C_0 \to \widehat{C}_0 = \widehat{S}_0 \times \widehat{S}^1$ covering κ^* such that the following diagram is commutative

$$\begin{array}{ccc} C_0 = S_0 \times I/f & \xrightarrow{\kappa} & \widehat{S}_0 \times \widehat{S}^1 = \widehat{C}_0 \\ \Big\downarrow p & & \hat{p}\Big\downarrow \\ C^*_0 = S^*_0 \times S^1 & \xrightarrow{\kappa^*} & \widehat{S}^*_0 \times \widehat{S}^1. \end{array} \tag{9}$$

Here $\widehat{S}_0$ is a homeomorphic copy of $\widehat{S}^*_0$ and $\hat{p}$ is the n-fold cyclic covering defined by $\hat{p}(x,\zeta) = (x, \zeta^n)$, if the 1-sphere $\widehat{S}^1$ is described by complex numbers ζ of absolute value one.

6.13 Lemma. *There is a homeomorphism* $\kappa^*\colon\ S^*_0 \times S^1 \to \widehat{S}^*_0 \times \widehat{S}^1$ *inducing the isomorphism* $\kappa^*_\#\colon \pi_1(S^*_0 \times S^1) \to \pi_1(\widehat{S}^*_0 \times \hat{S}^1)$, *and a homeomorphism* $\kappa\colon C_0 \to \widehat{C}_0$ *covering* κ^*.

Proof. First observe that S^*_0 is not a disk because in this case the Seifert surface S would be a covering space of S^*_0 and therefore a disk. The $2g^* + s$ simple closed curves $\{a^*_i, b^*_i, r^*_j \mid 1 \le i \le g^*, 1 \le j \le s\}$ joined at the basepoint $P^* =$

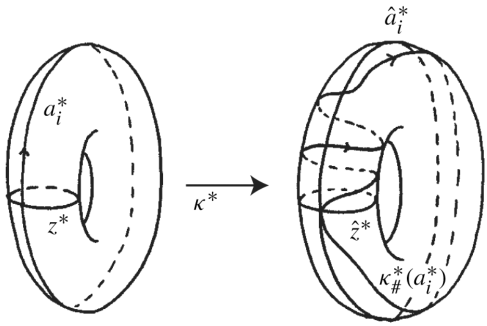


Figure 6.1

$p(P)$ represent a deformation retract R^* of S_0 as well as the respective generators $\{\hat{a}_i^*, \hat{b}_i^*, \hat{r}_j^*\} = \widehat{R}^*$ in $\widehat{S}_0^*$. It is now easy to see that there is a homeomorphism

$$\kappa^*|\colon\ R^* \times S^1 \to \widehat{R}^* \times \widehat{S}^1$$

inducing $\kappa_\#^*$ (Figure 6.1), because the homeomorphism obviously exists on each of the tori $a_i^* \times S^1, b_i^* \times S^1$ and $r_j^* \times S^1$. The extension of $\kappa^*|R^*$ to

$$\kappa^*\colon\ S_0^* \times S^1 \to \widehat{S}_0^* \times \widehat{S}^1$$

presents no difficulty. More precisely, κ^* is a Dehn twist along a family of annuli. Lemma 6.12 ensures the existence of a covering homeomorphism κ. □

We obtain by $\kappa_\#\colon \pi_1(C_0) \to \pi_1(\widehat{C}_0)$ a new presentation of $\pi_1(C_0) \cong \pi_1(\widehat{C}_0) = \langle\ \{\hat{r}_j, \hat{a}_i, \hat{b}_i \mid 1 \le i \le g^*, 1 \le j \le s\} | \ \rangle \times \langle\ \hat{z}\ \rangle$, where

$$\hat{p}_\#(\hat{r}_j) = \hat{r}_j^*,\ \hat{p}_\#(\hat{a}_i) = \hat{a}_i^*,\ \hat{p}_\#(\hat{b}_i) = \hat{b}_i^*,\ \hat{p}_\#(\hat{z}) = \hat{z}^{*n}. \tag{10}$$

From this presentation we can derive a presentation of $\mathfrak{G} \cong \pi_1(C)$ by introducing the defining relations $\kappa_\#(r_{j\nu}) = 1$. It is sufficient to choose $\nu = 0$ for all $j = 1, \ldots, s$.

We get from equations (1) in 6.9, (9) in 6.11:

$$\kappa_\#(r_{j0}) = \hat{p}_\#^{-1} \kappa_\#^*(r_j^{*m_j}) = \hat{p}_\#^{-1}(\hat{z}^{*-\alpha_j k_j m_j} \hat{r}_j^{*m_j}) = \hat{z}^{-\alpha_j} \hat{r}_j^{m_j}. \tag{11}$$

(Remember that $n = m_j k_j$, see equation (1) in 6.9.)

Furthermore (see equation (1) in 6.9 and 6.11):

$$\kappa_\#^*(r^*) = \hat{z}^{*-\sum_{j=1}^{s} k_j \alpha_j} \cdot \hat{r}^*.$$

(8) and 6.11 imply

$$\kappa_\#^*(r^*) = \hat{z}^{*-\alpha} \kappa_\#^*(q^*).$$

By (5) and (9), $\kappa_{\#}^{*}(q^{*}) \in \hat{p}_{\#}\pi_1(\widehat{C}_0)$, and by (1) and (10), $\hat{r}^{*} \in \hat{p}_{\#}\pi_1(\widehat{C}_0)$. Now the definition of $\hat{p}_{\#}$ (see (9)) yields

$$\alpha \equiv \sum_{j=1}^{s} k_j \alpha_j \mod n.$$

By Lemma 6.10 we may replace α_1 by an element of the same coset mod m_1, such that the equation

$$\alpha = \sum_{j=1}^{s} k_j \alpha_j \tag{12}$$

is satisfied. By (8), $\kappa_{\#}^{*}(q^{*}) = \hat{r}^{*}$, and, since $p_{\#}(t) = t^{*}$, it follows from (6), (9) that

$$\kappa_{\#}(t) = \hat{z}^{\beta} \cdot \hat{r}^{m}. \tag{13}$$

6.14 Lemma. $\widehat{S}_0$ *is a sphere with three boundary components:* $g^{*} = 0, s = 2$. *Moreover* $m_1 \cdot m_2 = n$, $\gcd(m_1, m_2) = 1$. *It is possible to choose* $m = 1$, $\alpha = 1$, $\beta = 0$.

There is a presentation

$$\mathfrak{G} = \langle\, \hat{z}, \hat{r}_1, \hat{r}_2 \mid \hat{z}^{-\alpha_1}\hat{r}_1^{m_1}, \hat{z}^{-\alpha_2}\hat{r}_2^{m_2}, [\hat{z}, \hat{r}_1], [\hat{z}, \hat{r}_2]\,\rangle$$

of the knot group $\mathfrak{G}$.

Proof. We have to introduce the relations $\hat{z}^{-\alpha_j}\hat{r}_j^{m_j}$ (see (11)) in

$$\pi_1(\widehat{C}_0) = \langle\, \{\hat{r}_j, \hat{a}_i, \hat{b}_i \mid 1 \le i \le g^{*}, 1 \le j \le s\} |\,\rangle \times \langle\, \hat{z}\,\rangle.$$

The additional relations $\kappa_{\#}(t) = \hat{z}^{\beta} \cdot \hat{r}^{m} = 1$ must trivialize the group. This remains true, if we put $\hat{z} = 1$.

Now $g^{*} = 0$ follows. For $s \ge 3$ the resulting groups

$$\left\langle \{\hat{r}, \hat{r}_j \mid 1 \le j \le s\} \,\middle|\, \hat{r}^{-m}, \hat{r}_j^{m_j}, \hat{r}^{-1}\prod_{j=1}^{s}\hat{r}_j \right\rangle \tag{14}$$

are known to be non-trivial (see [382, 4.16.4] or [323, §2]) since by definition $m_j > 1$. For $s = 2$ by the same argument (14) describes the trivial group only if $m = \pm 1$. The cases $s < 2$ cannot occur as $\mathfrak{k}$ was assumed to be non-trivial. By a suitable choice of the orientation of $r = \partial S$ we get $m = 1$. Thus by $\alpha = 1$, $\beta = 0$ equation (4) is satisfied. Now (12) takes the form

$$\alpha_1 k_1 + \alpha_2 k_2 = 1. \tag{15}$$

It follows that

$$\langle \hat{z}, \hat{r}_1, \hat{r}_2 \mid \hat{z}^{-\alpha_1}\hat{r}_1^{m_1}, \hat{z}^{-\alpha_2}\hat{r}_2^{m_2}, \hat{r}_1, \hat{r}_2, [\hat{z}, \hat{r}_1], [\hat{z}, \hat{r}_0] \rangle = 1$$

is a presentation of the trivialized knot group. By abelianizing this presentation yields

$$\alpha_1 m_2 + \alpha_2 m_1 = \pm 1. \tag{16}$$

The equations (15) and (16) are proportional since $m_2 k_2 - m_1 k_1 = n - n = 0$, by (1). As $m_j, k_j > 0$, they are indeed identical, $m_2 = k_1, m_1 = k_2$. □

It is a consequence of Lemma 6.14 that C_0 is obtained from a 3-sphere S^3 by removing three disjoint solid tori. Let $B := \widehat{S}_0 \cup_{\hat{r}=\partial \widehat{D}} \widehat{D}$ be an annulus with boundary $\partial B = \hat{r}_1 \cup \hat{r}_2$ obtained from $\widehat{S}_0$ by attaching a disk $\widehat{D}$ to the boundary component $\hat{r} \subset \partial \widehat{S}_0$. Equation (13) together with $m = 1, \beta = 0$ shows $\kappa_{\#}(t) = \hat{r}$. We use this equation to extend $\kappa\colon C_0 \to \widehat{C}_0$ to a homeomorphism $\hat{\kappa}$ defined on $C_0 \cup V(\mathfrak{k})$, obtained from C_0 by re-gluing the tubular neighborhood $V(\mathfrak{k})$ of $\mathfrak{k}$. We get

$$\hat{\kappa}\colon C_0 \cup V(\mathfrak{k}) \to B \times \widehat{S}^1.$$

The fundamental group $\pi_1(B \times \widehat{S}^1)$ is a free Abelian group generated by $\hat{z}$ and $\hat{r}_1 = \hat{r}_2^{-1}$. Define $\hat{q}_1$ and $\hat{q}_2$ by

$$\begin{aligned} \hat{\kappa}_{\#}(r_{10}) &= \hat{z}^{-\alpha_1}\hat{r}_1^{m_1} = \hat{q}_1^{-1}, \\ \hat{\kappa}_{\#}(r_{20}) &= \hat{z}^{-\alpha_2}\hat{r}_2^{m_2} = \hat{q}_2^{-1}, \ \alpha_1 m_2 + \alpha_2 m_1 = 1. \end{aligned} \tag{17}$$

(For the notation compare Proposition 6.8.) Now we glue two solid tori to $B \times \widehat{S}^1$ such that their meridians are identified with $\hat{q}_1, \hat{q}_2$, respectively, and obtain a closed manifold $\widehat{S}^3$. Thus $\hat{\kappa}$ can be extended to a homeomorphism $\hat{\kappa}\colon S^3 \to \widehat{S}^3$. From (17) we see that $\hat{q}_1$ and $\hat{q}_2$ are a pair of generators of $\pi_1(\hat{r}_1 \times \widehat{S}^1)$. Therefore the torus $\hat{r}_1 \times \widehat{S}^1$ defines a Heegaard splitting of $\widehat{S}^3$ which is the same as the standard Heegaard splitting of genus one of the 3-sphere. The knot $\mathfrak{k}$ is isotopic (in S^3) to $z \subset \partial C_0$. Its image $\hat{\mathfrak{k}} = \hat{\kappa}(\mathfrak{k})$ can be represented by any curve $(Q \times \widehat{S}^1) \subset \widehat{S}_0 \times \widehat{S}^1$, where Q is a point of $\widehat{S}_0$. Take $Q \in \hat{r}_1$ then $\hat{\mathfrak{k}}$ is represented by a simple closed curve on the unknotted torus $\hat{r}_1 \times \widehat{S}^1$ in $\widehat{S}^3$. This finishes the proof of Theorem 6.1. □

6.C Remarks on the proof

In Lemma 6.14 we obtained a presentation of the group of the torus knot which differs from the usual one (see Proposition 3.38). The following substitution connects both presentations:

$$\begin{aligned} u &= \hat{r}^{m_2} \cdot \hat{z}^{\alpha_2} \\ v &= \hat{r}_2^{m_1} \cdot \hat{z}^{\alpha_1}. \end{aligned}$$

First observe that $\hat{r}_1$ and $\hat{r}_2$ generate $\mathfrak{G}$:

$$\hat{r}_1^n \cdot \hat{r}_2^n = \hat{r}_1^{m_1 k_1} \cdot \hat{r}_2^{m_2 k_2} = \hat{z}^{\alpha_1 k_1 + \alpha_2 k_2} = \hat{z},$$

as follows from (16), the presentation before (16), and (15). It follows that u and v are also generators:

$$u^{\alpha_1} = \hat{r}_1^{\alpha_1 m_2} \cdot \hat{z}^{\alpha_1 \alpha_2} = \hat{r}_1 \cdot \hat{r}_1^{-\alpha_2 m_1} \cdot \hat{z}^{\alpha_1 \alpha_2} = \hat{r}_1,$$

and similarly, $v^{\alpha_2} = \hat{r}_2$. The relation $u^{m_1} = v^{m_2}$ is easily verified:

$$u^{m_1} = \hat{r}_1^{m_1 m_2} \hat{z}^{\alpha_2 m_1} = \hat{z}^{\alpha_1 m_2 + \alpha_2 m_1} = \hat{z} = v^{m_2}.$$

Starting with the presentation

$$\mathfrak{G} = \langle u, v \mid u^a = v^b \rangle, \ a = m_1, b = m_2,$$

one can re-obtain the presentation of 6.14 by introducing

$$\hat{z} = u^a = v^b \text{ and } \hat{r}_1 = u^{\alpha_1}, \hat{r}_2 = v^{\alpha_2}.$$

The argument also identifies the $\mathfrak{k}$ of Theorem 6.1 as the torus knot $\mathfrak{k}(m_1, m_2)$: for the definition of m_1, m_2 see 6.9, equation (1).

6.15. The construction used in the proof gives some additional information. The Hurwitz formula [382, 4.14.23] of the covering p_0: $S_0 \to S_0^*$ gives

$$2g + \sum_{j=1}^{s} k_j = n(2g^* + s - 1) + 1.$$

Since $g^* = 0$, $s = 2$, $k_1 = b$, $k_2 = a$, $ab = n$ it follows that $2g + a + b = ab + 1$, hence

$$g = \frac{(a-1)(b-1)}{2}$$

and, by Theorem 4.7, this reproves the genus formula from Corollary 4.12.

6.16 On cyclic coverings of torus knots. The q-fold cyclic coverings $C_{a,b}^q$ of the complement $C_{a,b}$ of the knot $\mathfrak{t}(a,b)$ obviously have a period $n = ab$:

$$C_{a,b}^q \cong C_{a,b}^{q+kn}.$$

This is a consequence of the realization of $C_{a,b} \cong S \times I/f$ by a mapping f of period n. The covering transformation of $C_{a,b}^q \to C_{a,b}$ can be interpreted geometrically as a shift along the fiber $z = t^{ab} \cdot r^{-1} \simeq t(a,b)$ such that a move from one sheet of the covering to the adjoining one shifts $\mathfrak{t}(a,b)$ through $\frac{1}{ab}$ of its "length". There is an $(ab+1)$-fold cyclic covering of $C_{a,b}$ onto itself:

$$C_{a,b} \cong C_{a,b}^{ab+1} \to C_{a,b}.$$

All its covering transformations $\neq$ id map $\mathfrak{t}(a,b)$ onto itself but no point of $\mathfrak{t}(a,b)$ is left fixed. There is no extension of the covering transformation to the $(ab+1)$-fold cyclic covering $\bar{p}$: $S^3 \to S^3$ branched along $\mathfrak{t}(a,b)$, in accordance with Smith's Theorem [340], see also Appendix B.9, [380, 36.4]. The covering transformations can indeed only be extended to a manifold $\widehat{C}_{a,b}$ which results from gluing to $C_{a,b}$ a solid torus whose meridian is tr^{-1} instead of t. The manifold $\widehat{C}_{a,b}$ is always different from S^3 as long as $\mathfrak{t}(a,b)$ is a non-trivial torus knot. In fact, one can easily compute

$$\pi_1(\widehat{C}_{a,b}) = \langle \hat{z}, \hat{r}_1, \hat{r}_2, \hat{r} \mid \hat{r}\hat{r}_1\hat{r}_2, \hat{r}^{ab+1}, \hat{r}_1^a, \hat{r}_2^b, [\hat{z},\hat{r}_1], [\hat{z},\hat{r}_2] \rangle$$

by using again the generators $\hat{r}_1, \hat{r}_2$ and $\hat{z}$. The group $\pi_1(\widehat{C}_{a,b})$ is infinite since $|a| > 1$, $|b| > 1$, $|ab+1| > 6$, see [382, 6.4.7].

In the case of the trefoil $\mathfrak{t}(3,2)$ the curves, surfaces and mappings constructed in the proof can be made visible with the help of Figure 5.3. The mapping f of order $6 = 3 \cdot 2, a = m_1 = 3, b = m_2 = 2$ is the one given by f_φ (at the end of Paragraph 5.13) for $\varphi = \pi$. Its exceptional points Q_{10}, Q_{11} are the centers of the disks D_1 and D_2 (Figure 5.3 and Figure 6.2) while Q_{20}, Q_{21}, Q_{22} are the points in which the core of T_1 meets the Seifert surface S.

Figure 6.2 shows a fundamental domain of S relative to $\mathfrak{Z}_6 = \langle f \rangle$. If its edges are identified as indicated in Figure 6.2, one obtains as orbit manifold (Zerlegungsfläche) a 2-sphere or a twice punctured 2-sphere S_0^*, if exceptional points are removed.

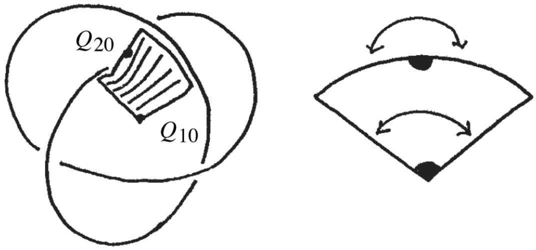


Figure 6.2

Figure 6.3 finally represents the annulus B embedded in S^3. One of its boundaries is placed on ∂T_1. The ribbon B represents the orbit manifold minus two disks. The orbit manifold itself can, of course, not be embedded in S^3, since there is no 2-sphere in S^3 which intersects a fiber z at just one point. The impossibility of such embeddings is also evident because B is twisted by 2π.

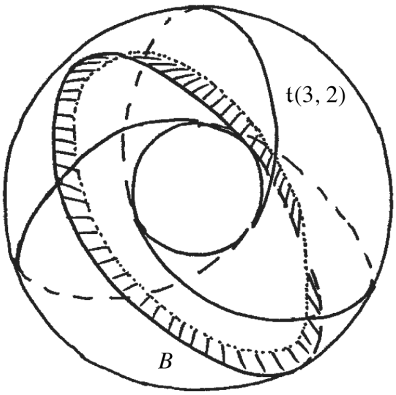


Figure 6.3

6.D History and sources

Torus knots and their groups have been studied by M. Dehn in [85, 87] and O. Schreier in [316, 87]. The question of whether torus knots are determined by their groups was treated by K. Murasugi in [259] and L. Neuwirth [269], and answered in the affirmative for alternating torus knots. This was proved in the general case by G. Burde and H. Zieschang [60], where torus knots were shown to be the only knots whose groups have a non-trivial center. A generalization of this theorem to 3-manifolds with non-trivial center is due to F. Waldhausen [365], and, as an application of it, the case of link groups with a center $\neq 1$ was investigated by G. Burde and K. Murasugi in [59]. More recently, the final step of the Seifert fiber space conjecture was proved by D. Gabai [125] and independently by A. Casson and D. Jungreis [63].

6.E Exercises

E 6.1. Let a lens space $L(p,q)$ be given by a Heegaard splitting of genus one, $L(p,q) = V_1 \cup V_2$. Define a torus knot in $L(p,q)$ by a simple closed curve on $\partial V_1 = \partial V_2$. Determine the links in the universal covering S^3 of $L(p,q)$ which cover a torus knot in $L(p,q)$. (Remark: The links that occur in this way classify the genus one Heegaard splittings of lens spaces.)

E 6.2. Show that the q-fold cyclic covering $C^q_{a,b}$ of a torus knot $\mathfrak{t}(a,b)$ is a Seifert fiber space, and that the fibration can be extended to the branched covering $C^q_{a,b}$ without adding another exceptional fiber. Compute Seifert's invariants of fiber spaces for $C^q_{a,b}$. (Remark: The 3-fold cyclic branched covering of a trefoil is a Seifert fiber space with three exceptional fibers of order two.)

Chapter 7

Factorization of knots

In Chapter 2 we defined a composition of knots. The main result of this chapter states that each tame, oriented knot is composed of finitely many in-decomposable (prime) knots and that these factors are uniquely determined.

7.A Composition of knots

In the following we often consider parts of knots, arcs, which are embedded in balls. It is useful to have the concept of knotted arcs:

7.1 Definition. Let $B \subset S^3$ be a closed ball carrying the orientation induced by the standard orientation of S^3. A simple path $\alpha\colon I \to B$ with $\alpha(\partial I) \subset \partial B$ and $\alpha(\mathring{I}) \subset \mathring{B}$ is called a *knotted arc*. Two knotted arcs $\alpha \subset B_1, \beta \subset B_2$ are called *equivalent* if there exists an orientation preserving homeomorphism $f\colon B_1 \to B_2$ such that $\beta = f\alpha$. An *arc* equivalent to a line segment is called *trivial*. Note that an arc is trivial if and only if it is isotopic to an arc in ∂B.

If α is a knotted arc in B and γ some simple curve on ∂B which connects the endpoints of α, then $\alpha\gamma$ – with the orientation induced by α – represents the *knot corresponding to* α. This knot does not depend on the choice of γ and it follows easily that equivalent knotted arcs correspond to equivalent knots.

By a slight alteration in the definition of the composition of knots we get the following two alternative versions of its description. Figures 7.1 and 7.2 show that the different definitions are equivalent.

Figure 7.1. A product knot.

7.2 Alternative descriptions. (a) Figure 7.1 describes the composition $\mathfrak{k} \# \mathfrak{l}$ of the knots $\mathfrak{k}$ and $\mathfrak{l}$ by joining representing arcs.

(b) Let $V(\mathfrak{k})$ be the tubular neighborhood of the knot $\mathfrak{k}$, and $B \subset V(\mathfrak{k})$ some ball such that $\kappa' = \mathfrak{k} \cap B$ is a trivial arc in B, $\kappa = \mathfrak{k} - \kappa'$. If κ' is replaced by a knotted arc λ defining the knot $\mathfrak{l}$, then $\kappa \cup \lambda$ represents the product $\mathfrak{k} \# \mathfrak{l} = \kappa \cup \lambda$.

The following lemma is a direct consequence of the construction in 7.2 and its proof is visualized by Figures 7.3 and 7.4.

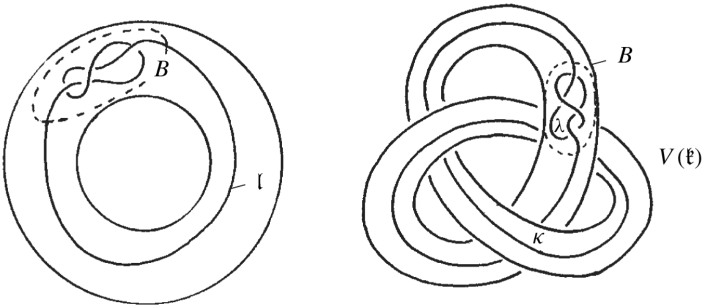


Figure 7.2. A product knot is a satellite knot.

7.3 Lemma.

(a) $\mathfrak{l} \# \mathfrak{k} = \mathfrak{k} \# \mathfrak{l}$.

(b) $\mathfrak{k}_1 \# (\mathfrak{k}_2 \# \mathfrak{k}_3) = (\mathfrak{k}_1 \# \mathfrak{k}_2) \# \mathfrak{k}_3$.

(c) *If* $\mathfrak{i}$ *denotes the trivial knot then* $\mathfrak{k} \# \mathfrak{i} = \mathfrak{k}$.

Proof. (a) Figure 7.3. (b) Figure 7.4. □

Figure 7.3. $\mathfrak{l} \# \mathfrak{k} = \mathfrak{k} \# \mathfrak{l}$.

Figure 7.4. $(\mathfrak{k}_1 \# \mathfrak{k}_2) \# \mathfrak{k}_3 = \mathfrak{k}_1 \# (\mathfrak{k}_2 \# \mathfrak{k}_3)$.

Associativity now permits us to define $\mathfrak{k}_1 \# \cdots \# \mathfrak{k}_n$ for an arbitrary $n \in \mathbb{N}$ without using brackets.

7.4 Proposition (Genus of knot compositions). *Let* $\mathfrak{k}, \mathfrak{l}$ *be knots and let* $g(\mathfrak{x})$ *denote the genus of the knot* $\mathfrak{x}$. *Then*

$$g(\mathfrak{k} \# \mathfrak{l}) = g(\mathfrak{k}) + g(\mathfrak{l}).$$

Proof. Let $B \subset S^3$ be a (p.l.-)ball. Since any two (p.l.-)balls in S^3 are ambient isotopic, see [242, Chap. 17], we can describe $\mathfrak{k} \# \mathfrak{l}$ in the following way. Let $S_{\mathfrak{k}}$ and $S_{\mathfrak{l}}$ be Seifert surfaces of minimal genus of $\mathfrak{k}$ resp. $\mathfrak{l}$ such that $S_{\mathfrak{k}}$ is contained in

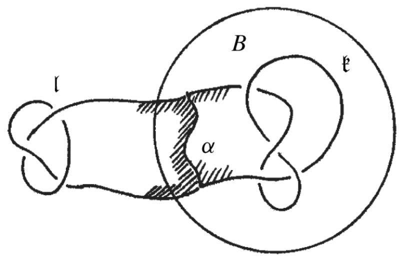


Figure 7.5

some ball $B \subset S^3$, and $S_{\mathfrak{l}}$ in $S^3 - B$. Furthermore we assume $S_{\mathfrak{k}} \cap \partial B = \alpha$ and $S_{\mathfrak{l}} \cap \partial B = -\alpha$ to be a simple arc (see Figure 7.5). Obviously $S_{\mathfrak{k}} \cup S_{\mathfrak{l}}$ is a Seifert surface spanning $\mathfrak{k} \# \mathfrak{l}$, hence:

$$g(\mathfrak{k} \# \mathfrak{l}) \leq g(\mathfrak{k}) + g(\mathfrak{l}). \tag{1}$$

Let S be a Seifert surface of minimal genus spanning $\mathfrak{k} \# \mathfrak{l}$. The 2-sphere $S^2 = \partial B$ is supposed to be in general position with respect to S. Since $\mathfrak{k} \# \mathfrak{l}$ meets ∂B in two points, $\partial B \cap S$ consists of a simple arc α joining these points, and, possibly, a set of pairwise disjoint simple closed curves. An 'innermost' curve σ on ∂B bounds a disk $\delta \subset \partial B$ such that $\delta \cap S = \sigma$. Let us assume that σ does not bound a disk on S. In the case where σ separates S replace the component not containing $\mathfrak{k} \# \mathfrak{l}$ by δ. If σ does not separate S, cut S along σ, and attach two copies of δ along their boundaries to the cuts (see proof of Lemma 4.5). In both cases we obtain a Seifert surface for $\mathfrak{k} \# \mathfrak{l}$ of a genus smaller than that of S, contradicting the assumption of minimality.

Thus σ bounds a disk on S as well as on ∂B, and there is an isotopy of S which removes σ. So we may assume $S \cap \partial B = \alpha$, which means

$$g(\mathfrak{k}) + g(\mathfrak{l}) \leq g(\mathfrak{k} \# \mathfrak{l}). \qquad \square$$

7.5 Corollary. (a) $\mathfrak{k} \# \mathfrak{l} = \mathfrak{k}$ *implies that* $\mathfrak{l}$ *is the trivial knot.*

(b) *If* $\mathfrak{k} \# \mathfrak{l}$ *is the trivial knot then* $\mathfrak{k}$ *and* $\mathfrak{l}$ *are trivial.* □

Corollary 7.5 motivates the following definition.

7.6 Definition (Prime knot). A knot $\mathfrak{k}$ which is the composition of two non-trivial knots is called *composite*; a non-trivial knot which is not composite is called a *prime knot*.

7.7 Corollary. *Genus* 1 *knots are prime.* □

7.8 Proposition. *Every* 2*-bridge knot* $\mathfrak{b}$ *is prime.*

Proof. Let δ_1 and δ_2 be disks spanning the arcs of $\mathfrak{b}$ in the upper half-space, and suppose that the other two arcs λ_i', $i \in \{1,2\}$, of $\mathfrak{b}$ are contained in the boundary E of the half space. The four "endpoints" of $E \cap \mathfrak{b}$ are joined pairwise by the simple arcs λ_i' and $\lambda_i = E \cap \delta_i$. We suppose the separating sphere S to be in general position with respect to E and δ_i. The intersections of S with $\mathfrak{b}$ may be pushed into two endpoints. Simple closed curves of $\delta_i \cap S$ and those of $E \cap S$ which do not separate endpoints can be removed by an isotopy of S. The remaining curves in $E \cap S$ must now be parallel in E, separating the arcs λ_1' and λ_2'. If there are more than one of these curves, there is a pair of neighboring curves bounding annuli on E and S which together form a torus T. The torus T intersects δ_i in simple closed curves, not null-homotopic on T, bounding disks δ in δ_i with $\delta \cap T = \partial\delta$. So T bounds a solid torus which does not intersect $\mathfrak{b}$. There is an isotopy which removes the pair of neighboring curves. We may therefore assume that $E \cap S$ consists of one simple closed curve separating λ_1' and λ_2'. The ball B bounded by S in $\mathbb{R}^3$ now intersects $\mathfrak{b}$ in, say, λ_1', and λ_1' is isotopic in $E \cap B$ to an arc of $S \cap E$. Hence this factor is trivial. □

A stronger result was proved by H. Schubert in [319, Satz 7]:

7.9 Theorem (Schubert). *The minimal bridge number* $b(\mathfrak{k})$ *minus 1 is additive with respect to the product of knots:*

$$b(\mathfrak{k}_1 \# \mathfrak{k}_2) - 1 = \big(b(\mathfrak{k}_1) - 1\big) + \big(b(\mathfrak{k}_2) - 1\big)$$

or

$$b(\mathfrak{k}_1 \# \mathfrak{k}_2) = b(\mathfrak{k}_1) + b(\mathfrak{k}_2) - 1\,.$$

We shall give a proof of this theorem in Section 16.C.

7.10 Proposition (Group of composite knots). *Let* $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ *and denote by* $\mathfrak{G}$*,* $\mathfrak{G}_2$*,* $\mathfrak{G}_2$ *the corresponding knot groups. Then* $\mathfrak{G} = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2$*, where* $\mathfrak{Z}$ *is an infinite cyclic group generated by a meridian of* $\mathfrak{k}$*, and* $\mathfrak{G}' = \mathfrak{G}_1' * \mathfrak{G}_2'$*. Here* $\mathfrak{G}_i$ *and* $\mathfrak{G}_i'$ *are – in the natural way – considered as subgroups of* $\mathfrak{G} = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2$*,* $i = 1, 2$*.*

Proof. Let S be a 2-sphere that defines the product $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$. Assume that there is a regular neighborhood V of $\mathfrak{k}$ such that $S \cap V$ consists of two disks. Then $S \cap C$ is an annulus. The complement $C = \overline{S^3 - V}$ is divided by $S \cap C$ into C_1 and C_2 with $C = C_1 \cup C_2$ and $S \cap C = C_1 \cap C_2$. Since $\pi_1(S \cap C) \cong \mathbb{Z}$ is generated by a meridian it is embedded into $\pi_1(C_i)$ and the Seifert–van Kampen Theorem implies that

$$\pi_1(C) = \pi_1(C_1) *_{\pi_1(C_1 \cap C_2)} \pi_1(C_2) = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2.$$

It is clear that $\mathfrak{G}_i' \subset \mathfrak{G}'$, $i = 1, 2$, and hence $\langle \mathfrak{G}_1', \mathfrak{G}_2' \rangle \subset \mathfrak{G}'$ where $\langle \mathfrak{G}_1', \mathfrak{G}_2' \rangle$ denotes the subgroup generated by $\mathfrak{G}_1'$ and $\mathfrak{G}_2'$. On the other hand each element $g' \in$

$\mathfrak{G}'$ can be written as a product $g' = g_1 \cdots g_k$ where $g_i \in \mathfrak{G}_{l_i}$, $1 \le i \le k$, $l_i \in \{1,2\}$. The group $\mathfrak{G}_l$, $l \in \{1,2\}$, are semidirect product $\mathfrak{G}_l = \mathfrak{Z} \ltimes \mathfrak{G}'_l$ hence $g_i = h_i g'_i$ where $h_i \in \mathfrak{Z}$ and $g'_i \in \mathfrak{G}'_{l_i}$. We can rewrite the product

$$\begin{aligned} g' &= h_1 g'_1 \cdots h_k g'_k \\ &= h_1 h_2 \cdots h_k \cdot (h_2 \cdots h_k)^{-1} g'_1 (h_2 \cdots h_k) \cdot (h_3 \cdots h_k)^{-1} g'_2 (h_3 \cdots h_k) \cdot \ldots \\ &\quad \cdot (h_{k-1} h_k)^{-1} g'_{k-2} (h_{k-1} h_k) \cdot h_k^{-1} g'_{k-1} h_k \cdot g'_k . \end{aligned}$$

Now $g' \in \mathfrak{G}'$ implies $h_1 h_2 \cdots h_k = 1$ and $g' \in \langle \mathfrak{G}'_1, \mathfrak{G}'_2 \rangle$, i.e. $\mathfrak{G}' = \langle \mathfrak{G}'_1, \mathfrak{G}'_2 \rangle$. Finally, $\mathfrak{G}'_l \cap \mathfrak{Z} = \{1\}$, $l = 1,2$, implies that every non-trivial element $g' \in \mathfrak{G}'$ can be written in an unique way as a *reduced* product $g' = g'_1 \cdots g'_k$, that is $k = 1$ or $g_i \in \mathfrak{G}'_{l_i} - \{1\}$ and $l_i \neq l_{i+1}$ for $1 \le i \le k$. Therefore, $\mathfrak{G}' = \langle \mathfrak{G}'_1, \mathfrak{G}'_2 \rangle$ is isomorphic to the free product $\mathfrak{G}'_1 * \mathfrak{G}'_2$. □

7.11 Corollary. *Torus knots are prime.*

Proof. For this fact we give a geometric and a short algebraic proof.

(1) *Geometric proof.* Let the torus knot $\mathfrak{t}(a,b)$ lie on an unknotted torus $T \subset S^3$ and let the 2-sphere S define a decomposition of $\mathfrak{t}(a,b)$. (By definition, $|a|, |b| \ge 2$.) We assume that S and T are in general position, that is, $S \cap T$ consists of finitely many disjoint simple closed curves. Such a curve either meets $\mathfrak{t}(a,b)$, is parallel to it or it bounds a disk D on T with $D \cap \mathfrak{t}(a,b) = \emptyset$. Choose γ as an innermost curve of the last kind, i.e., $D \cap S = \partial D = \gamma$. Then γ divides S into two disks D', D'' such that $S' = D \cup D'$ and $S'' = D \cup D''$ are spheres, $S' \cap S'' = D$. One of these two spheres intersects $\mathfrak{t}(a,b)$ in two points and the other sphere does not intersect $\mathfrak{t}(a,b)$. Hence, D' or D'' can be deformed into D by an isotopy of S^3 which leaves $\mathfrak{t}(a,b)$ fixed. By a further small deformation we get rid of one intersection of S with T.

Consider the curves of $T \cap S$ which intersect $\mathfrak{t}(a,b)$. There are one or two curves of this kind since $\mathfrak{t}(a,b)$ intersects S at two points only. If there is one curve it has intersection number zero with $\mathfrak{t}(a,b)$ and this implies that it is either isotopic to $\mathfrak{t}(a,b)$ or null-homotopic on T. In the first case $\mathfrak{t}(a,b)$ would be the trivial knot. In the second case it bounds a disk D_0 on T and $D_0 \cap \mathfrak{t}(a,b)$, plus an arc on S, represents one of the factor knots of $\mathfrak{t}(a,b)$; this factor would be trivial, contradicting the hypothesis.

The case remains where $S \cap T$ consists of two simple closed curves intersecting $\mathfrak{t}(a,b)$ exactly once. These curves are parallel and bound disks in one of the solid tori bounded by T. But this contradicts $|a|, |b| \ge 2$.

(2) *Algebraic proof.* Let the torus knot $\mathfrak{t}(a,b)$ the product of two knots. By Proposition 7.10,

$$\mathfrak{G} = \langle u, v \mid u^a v^{-b} \rangle = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2,$$

where $\mathfrak{Z}$ is generated by a meridian m. The center of the free product of groups with amalgamated subgroup is the intersection of the centers of the factors, see [382, 2.3.9];

hence, it is generated by a power of m. Since u^a is the generator of the center of $\mathfrak{G}$ it follows from Proposition 3.38 (b) that the longitude ℓ of $\mathfrak{t}(a,b)$ is a power of m. Since ℓ is zero homologous it follows $\ell = 1$ which contradicts the assumption that $\mathfrak{t}(a,b)$ is non-trivial. □

Now we formulate the main theorem of this chapter which was first proved by H. Schubert in 1949 [317].

7.12 Theorem (Unique prime decomposition of knots). *Each non-trivial, oriented knot $\mathfrak{k}$ is a finite product of prime knots and these factors are uniquely determined. More precisely:*

(a) $\mathfrak{k} = \mathfrak{k}_1 \# \ldots \# \mathfrak{k}_n$ *where each* $\mathfrak{k}_i$ *is a prime knot.*

(b) *If* $\mathfrak{k} = \mathfrak{k}_1 \# \ldots \# \mathfrak{k}_n = \mathfrak{k}'_1 \# \ldots \# \mathfrak{k}'_m$ *are two decompositions into prime factors* $\mathfrak{k}_i$ *or* $\mathfrak{k}'_j$*, respectively, then* $n = m$ *and* $\mathfrak{k}'_i = \mathfrak{k}_{j(i)}$ *for some permutation* $\begin{pmatrix} 1 & \ldots & n \\ j(1) & \ldots & j(n) \end{pmatrix}$.

Assertion (a) is a consequence of Proposition 7.4; part (b) will be proved in Section 7.B. The results can be summarized as follows:

7.13 Corollary (Semigroup of knots). *The tame, oriented knots in S^3 with the operation $\#$ form a commutative semigroup with a unit element such that the law of unique prime decomposition is valid.*

7.B Uniqueness of the decomposition into prime knots: proof

We will first describe a general concept for the construction of prime decompositions of a given oriented knot $\mathfrak{k}$. Then we show that any two decompositions can be connected by a chain of 'elementary processes'.

7.14 Notations and definitions. Let S_j, $1 \leq j \leq m$, be a system of disjoint 2-spheres embedded in S^3, bounding $2m$ balls B_i, $1 \leq i \leq 2m$, in S^3, and denote by B_j, B_{m+j} the two balls bounded by S_j. If B_i contains the s balls $B_{l_1}, \ldots, B_{l_s}$ as proper subsets,

$$R_i = (B_i - \bigcup_{q=1}^{s} \mathring{B}_{l_q})$$

is called the *domain determined by* B_i. Note that a domain R_i is the closure of a connected component of the complement $S^3 - \bigcup_{j=1}^{m} S_j$. It follows from Alexander duality [157, Thm. 3.44] that there are exactly $m+1$ distinct domains. See Figure 7.6 for an example with $m = 2$.

Figure 7.6. $R_1 = B_1$, $R_2 = B_2$ and $R_3 = R_4 = S^3 - (\mathring{B}_1 \cup \mathring{B}_2)$.

If each sphere S_j meets $\mathfrak{k}$ transversally at two points then the intersection $\kappa_i = \mathfrak{k} \cap R_i$, oriented as $\mathfrak{k}$, and completed by simple arcs on the boundary of R_i represents a knot $\mathfrak{k}_i \subset R_i \subset B_i$. The knot $\mathfrak{k}_i$ is called the *factor of* $\mathfrak{k}$ *determined by* B_i. It is immediately clear that $\mathfrak{k}_i$ does not depend on the choice of the arcs on ∂R_i. The sphere system gives rise to $m+1$ different factor knots.

7.15 Definition (Decomposing spheres). Let S_j, $1 \le j \le m$, be a system of disjoint 2-spheres embedded in S^3 and let $\mathfrak{k} \subset S^3$ be an oriented knot. The *spheres* S_j are said to be *decomposing* with respect to $\mathfrak{k} \subset S^3$ if the following conditions are fulfilled:

(1) Each sphere S_j meets $\mathfrak{k}$ transversally in two points.

(2) Each factor $\mathfrak{k}_i$ of $\mathfrak{k}$ determined by B_i is a prime knot.

By $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \le j \le m\}$ we denote a decomposing sphere system with respect to $\mathfrak{k}$; if $\mathfrak{k}$ itself is prime we put $\mathfrak{S} = \emptyset$.

The following lemma connects this definition with our definition of the composition of a knot.

7.16 Lemma. *If $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \le j \le m\}$ is a decomposing system of spheres, then the $m+1$ distinct domains R_{l_k}, $1 \le k \le m+1$, determining prime knots $\mathfrak{k}_{l_k}$ such that*

$$\mathfrak{k} = \mathfrak{k}_{l_1} \# \dots \# \mathfrak{k}_{l_{m+1}}.$$

Proof by induction on m. For $m = 1$ Definition 7.15 reverts to the original definition of the product of two prime knots. For $m > 1$ let B_l be a ball not containing any other ball B_i and determining the prime knot $\mathfrak{k}_l$. Note that $R_l = B_l$ is different from the regions R_i, $i \ne l$, since B_l does not contain any other ball B_i.

The boundary ∂B_l is the sphere $S_{l'}$ where $1 \le l' \le m$, $l \equiv l' \bmod m$. Replacing the knotted arc $\kappa_l = B_l \cap \mathfrak{k}$ in $\mathfrak{k}$ by a simple arc on $\partial B_l = S_{l'}$ defines a (non-trivial) knot $\mathfrak{k}' \subset S^3$. The induction hypothesis applied to $\mathfrak{S}' = \{(S_j, \mathfrak{k}') \mid 1 \le j \le m, j \ne l'\}$ gives that the m distinct regions R'_{l_j}, $1 \le j \le m$, of $\mathfrak{S}'$ determine prime knots $\mathfrak{k}'_{l_j}$ such that $\mathfrak{k}' = \mathfrak{k}'_{l_1} \# \dots \# \mathfrak{k}'_{l_m}$.

Each region R_i' determines a region $R_i - \mathring{B}_l$ and a factor $\mathfrak{k}_i'$ which coincides with $\mathfrak{k}_i$. Since the regions R_{l_j}', $1 \le j \le m$, are distinct the same holds for the regions R_{l_j}, $1 \le j \le m$. Moreover these regions are all distinct from $R_l = B_l$. Hence by setting $l_{m+1} := l$ we obtain

$$\mathfrak{k} = \mathfrak{k}' \# \mathfrak{k}_l = \mathfrak{k}_{l_1} \# \ldots \# \mathfrak{k}_{l_m} \# \mathfrak{k}_{l_{m+1}}.$$

□

Figure 7.7 illustrates Definition 7.15 and Lemma 7.16.

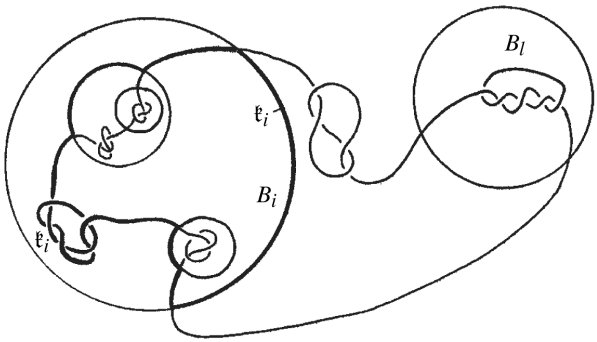


Figure 7.7. A decomposing system.

7.17 Definition. Two *decomposing systems of spheres* $\mathfrak{S} = \{(S_j, \mathfrak{k})\}$, $\mathfrak{S}' = \{(S_j', \mathfrak{k})\}$, $1 \le j \le m$, are called *equivalent* if they define the same (unordered) $m+1$ factor knots $\mathfrak{k}_{l_k}$, $1 \le k \le m+1$.

The following lemma is the crucial tool used in the proof of the Uniqueness Theorem. It describes a process by which one can pass over from a decomposing system to an equivalent one.

7.18 Lemma. *Let $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \le j \le m\}$ be a decomposing system of spheres, and let S' be another 2-sphere embedded in S^3, intersecting $\mathfrak{k}$ transversally in two points and disjoint from $\{S_j \mid 1 \le j \le m\}$. Denote by B' and B'' the balls bounded by S' in S^3. If B_l, $\partial B_l = S_l$, is a maximal ball contained in B', that is $B_l \subset B'$ but there is no B_i such that $B_l \subset B_i \subset B'$ for any $i \ne l$, and if B' determines the knot $\mathfrak{k}_l$ relative to the spheres $\{S_j \mid 1 \le j \le m, l \ne j\} \cup \{S'\}$, then these spheres define a decomposing system of spheres with respect to $\mathfrak{k}$ equivalent to $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \le j \le m\}$.*

Proof. Define $S_j' = S_j$ if $j \ne l$, $S_l' = S'$, $B_l' = B'$, $B_{m+l}' = B''$, $B_i' = B_i$ for $i \ne l, l+m$ and $\mathfrak{S}' := \{(S_j', \mathfrak{k}) \mid 1 \le j \le m\}$. Denote by $\mathfrak{k}_i$ respectively $\mathfrak{k}_i'$ the factor determined by B_i respectively B_i' relative to $\mathfrak{S}$ respectively $\mathfrak{S}'$.

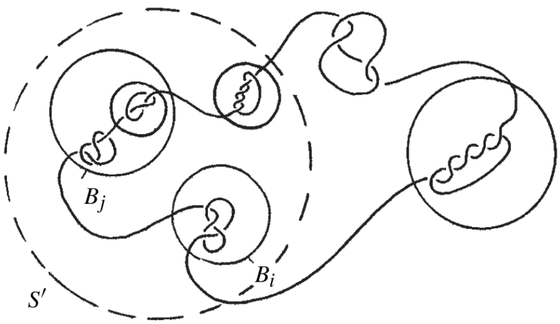


Figure 7.8. Two equivalent decomposing systems.

We have that the domains R_l and R'_l determine the same factor $\mathfrak{k}_l = \mathfrak{k}'_l$ by assumption. Note that $R'_l = (R'_l - \mathring{B}_l) \cup_{S_l} R_l$ and therefore the domain $(R'_l - \mathring{B}_l)$ determines the trivial knot (here we use that $\mathfrak{k}_l = \mathfrak{k}'_l$ is prime). The domains R_{l+m} and R'_{l+m} are related by

$$R_{l+m} = (R'_l - \mathring{B}_l) \cup_{S'_l} R'_{l+m} .$$

Hence R_{l+m} and R'_{l+m} determine the same prime factor $\mathfrak{k}_{l+m} = \mathfrak{k}'_{l+m}$.

Assume $B_i \subset B'$. For $i \neq l$, B_i and B'_i determine the same domain $R_i = R'_i$ and hence the same knot $\mathfrak{k}_i = \mathfrak{k}'_i$ since no inclusion $B_l \subset B_i \subset B'$, $i \neq l$, exists. For comparing the domains R_{i+m} and R'_{i+m} (indices modulo $2m$) there are several cases to study:

- if there exists a $k \neq i, l$ such that $B_i \subset B_k \subset B'$ then $R_{i+m} = R'_{i+m}$ and hence $\mathfrak{k}_{i+m} = \mathfrak{k}'_{m+i}$;
- if $B_i \subset B_l$ is maximal then $R'_{i+m} = (R'_l - \mathring{B}_l) \cup_{S_l} R_{i+m}$ and hence $\mathfrak{k}_{i+m} = \mathfrak{k}'_{m+i}$ since the domain $(R'_l - \mathring{B}_l)$ determines the trivial knot;
- if $B_i \subset B'$ is maximal then $R_{i+m} = R_{l+m}$, $R'_{i+m} = (R'_l - \mathring{B}_l) \cup_{S_l} R_l = R'_l$ and $\mathfrak{k}_l = \mathfrak{k}'_{m+i}$. Note that in this case $R_{l+m} = R_{i+m}$.

  So instead of $\mathfrak{k}_l$, $\mathfrak{k}_{l+m}$, $\mathfrak{k}_i$, $\mathfrak{k}_{i+m} = \mathfrak{k}_{l+m}$ determined by B_l, B_{l+m}, B_i, B_{i+m} in $\mathfrak{S}$, we get $\mathfrak{k}_l$, $\mathfrak{k}_{l+m}$, $\mathfrak{k}_i$, $\mathfrak{k}'_{i+m} = \mathfrak{k}_l$ are determined by B'_l, B'_{l+m}, B'_i, B'_{i+m} in $\mathfrak{S}'$.

The case $B_i \subset B''$ is dealt with in a similar way. □

7.19. *Proof of the Uniqueness Theorem 7.12 (b).* The proof consists in verifying the assertion that any two decomposing systems $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \leq j \leq m\}$, $\mathfrak{S}' = \{(S'_k, \mathfrak{k}) \mid 1 \leq k \leq m'\}$ with respect to the same knot $\mathfrak{k}$ are equivalent. We prove this by induction on $m + m'$. For $m + m' = 0$ nothing has to be proved. The spheres S_j and S'_k can be assumed to be in general position relative to each other.

To begin with, suppose there is a ball $B_i \cap \mathfrak{S}' = \emptyset$ not containing any other B_l or B'_k. Then by Lemma 7.18 some S'_k can be replaced by $S_i = \partial B_i$ and induction can be applied to the knot $\mathfrak{k}'$ obtained from $\mathfrak{k} \cap B_{i+m}$ by completing with a simple arc on S_i.

If there is no such B_i (or B'_i), choose an innermost curve λ' of $S'_j \cap \mathfrak{S}$ bounding a disk $\delta' \subset S'_j = \partial B'_j$ such that B'_j contains no other ball B_k of B'_l. The knot $\mathfrak{k}$ meets δ' in at most two points. The disk δ' divides B_i into two balls B_i^1 and B_i^2, and in the first two cases of Figure 7.9 one of them determines a trivial knot or does not meet $\mathfrak{k}$ at all, and the other one determines the prime knot $\mathfrak{k}_i$ with respect to $\mathfrak{S}$, because otherwise δ' would effect a decomposition of $\mathfrak{k}_i$.

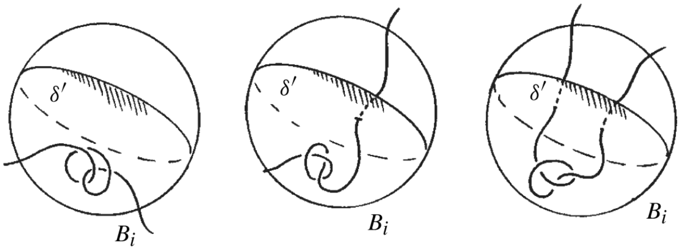


Figure 7.9

If B_i^1 determines $\mathfrak{k}_i$, replace S_i by ∂B_i^1 or rather by a sphere S' obtained from ∂B_i^1 by a small isotopy such that λ' disappears and general position is restored. The new decomposing system is equivalent to the old one by Lemma 7.18. If $\mathfrak{k}$ meets δ' at two points – the third case in Figure 7.9 – one may choose $\delta'' = S'_j - \delta'$ instead of δ' if λ' is the only intersection curve on S'_j. If not, there will be another innermost curve $\lambda'' = S'_j \cap S_k$ on S'_j bounding a disk $\delta'' \subset S'_j$. In both events the knot $\mathfrak{k}$ will not meet δ'' and we are back to case one of Figure 7.9. Thus we obtain finally an innermost ball without intersections. This proves the theorem. □

The theorem on the existence and uniqueness of decomposition carries over to the case of links without major difficulties. A proof was given by Y. Hashizume in 1958 [153].

7.C Fibered knots and decompositions

It is easily seen that the product of two fibered knots is also fibered. It is also true that factor knots of a fibered knot are fibered. We present two proofs of this assertion, an algebraic one which is quite short, and a more complicated geometric one which affords a piece of additional insight.

7.20 Proposition (Decomposition of fibered knots). *A composite knot* $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ *is a fibered knot if and only if* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$ *are fibered knots.*

Proof. Let $\mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2$ etc. denote the groups of $\mathfrak{k}$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively. By Proposition 7.10, $\mathfrak{G}_1' = \mathfrak{G}_1' * \mathfrak{G}_2'$. From the Grushko Theorem, see [382, 2.9.2], it follows that $\mathfrak{G}'$ is finitely generated if and only if $\mathfrak{G}_1'$ and $\mathfrak{G}_2'$ are finitely generated. Now the assertion 7.20 is a consequence of Theorem 5.1 □

7.21 Theorem (Decomposition of fibered knots). *Let* $\mathfrak{k}$ *be a fibered knot,* V *a regular neighborhood,* $C = \overline{S^3 - V}$ *its complement, and* $p\colon C \to S^1$ *a fibration of* C*. Let a* 2*-sphere* $S \subset S^3$ *decompose* $\mathfrak{k}$ *into two non-trivial factors. Then there is an isotopy of* S^3 *deforming* S *into a sphere* S' *with the property that* $S' \cap V$ *consists of two disks and* $S' \cap C$ *intersects each fiber* $p^{-1}(t)$, $t \in S^1$*, in a simple arc. Moreover, the isotopy leaves the points of* $\mathfrak{k}$ *fixed.*

Proof. It follows by standard arguments that there is an isotopy of S^3 that leaves the knot pointwise fixed and maps S into a sphere that intersects V in two disks. Moreover, we may assume that p maps the boundary of each of these disks bijectively onto S^1. Suppose that S already has these properties. Consider the annulus $A = C \cap S$ and the fiber $F = p^{-1}(*)$ where $* \in S^1$. We may assume that S and F are in general position and that $A \cap \partial F$ consists of two points; otherwise S can be deformed by an ambient isotopy to fulfill these conditions.

Now $A \cap F$ is composed of an arc joining the points of $A \cap \partial F$ (which are on different components of ∂A) and, perhaps, further simple closed curves. Each of them bounds a disk on A, hence also a disk on F, since $\pi_1(F) \to \pi_1(C)$ is injective. Starting with an innermost disk δ on F we find a 2-sphere $\delta \cup \delta'$ consisting of disks $\delta \subset F$ and $\delta' \subset A$ such that $\delta \cap \delta'$ is the curve $\partial\delta = \partial\delta'$ and $\delta \cap A = \partial\delta$. Now δ' can be deformed to a disk not intersecting A and the number of components of $A \cap F$ becomes smaller. Thus we may assume that $A \cap F$ consists of an arc α joining the boundary components of A, see Figure 7.10.

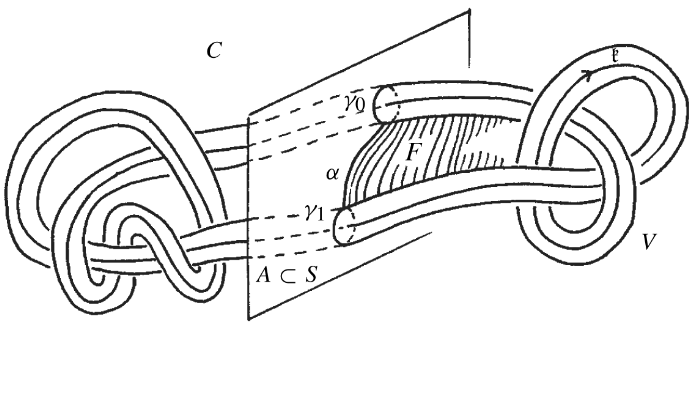


Figure 7.10

We cut C along F and obtain a space homeomorphic to $F \times I$. The cut transforms the annulus A into a disk D, $\partial D = \alpha_0\gamma_0\alpha_1^{-1}\gamma_1^{-1}$, where the $\alpha_i \subset F \times \{i\}$, $i = 0, 1$, are obtained from α and the γ_i from the meridians $\partial V \cap S$.

Let $q\colon F \times I \to F$ be the projection. The restriction $q|D$ defines a homotopy $q \circ \alpha_0 \simeq q \circ \alpha_1$. Since $q \circ \alpha_0$ and $q \circ \alpha_1$ are simple arcs with endpoints on ∂F it follows that these arcs are ambient isotopic and the isotopy leaves the endpoints fixed. (This can be proved in the same way as the refined Baer Theorem (see [382, 5.12.1]) which respects the basepoint; it can, in fact, be derived from that theorem by considering ∂F as the boundary of a 'small' disk around the basepoint of a closed surface F' containing F.) Thus there is an isotopy

$$H\colon (I \times I, (\partial I) \times I) \to (F \times I, (\partial F) \times I)$$

with

$$H(t, 0) = \alpha_0(t), H(t, 1) = \alpha_1(t)$$

which is level preserving:

$$H(x, t) = (q(H(x, t)), t) \quad \text{for } (x, t) \in I \times I.$$

Therefore $D' = H(I \times I)$ is a disk and intersects each fiber $F \times \{t\}$ in a simple arc. It is transformed by re-identifying $F \times \{0\}$ and $F \times \{1\}$ into an annulus A' which intersects each fiber $p^{-1}(t), t \in S^1$, in a simple closed curve. In addition $\partial A' = \partial A$.

It remains to prove that A' is ambient isotopic to A. An ambient isotopy takes D into general position with respect to D' while leaving its boundary ∂D fixed. Then $\mathring{D} \cap \mathring{D}'$ consists of simple closed curves. Take an innermost (relative to D') curve β. It bounds disks $\delta \subset D$ and $\delta' \subset D'$. The sphere $\delta \cup \delta' \subset F \times I \subset S^3$ bounds a 3-ball by the Alexander–Schoenflies Theorem 1.6. Thus there is an ambient isotopy of $F \times I$ which moves δ to δ' and a bit further to diminish the number of components in $D \cap D'$; during the deformation the boundary $\partial(F \times I)$ remains fixed. After a finite number of such deformations we may assume that $D \cap D' = \partial D = \partial D'$. Now $D \cup D'$ bounds a ball in $F \times I$ and D can be moved into D' by an isotopy which is the identity on $\partial(F \times I)$. Therefore the isotopy induces an isotopy of C that moves A to A' (see Figure 7.11). □

7.D History and sources

The concept and the main theorem concerning products of knots are due to H. Schubert, and they are contained in his thesis [317]. His theorem was shown to be valid for links by Y. Hashizume in [153] where a new proof was given which in some parts simplified the original one. A further simplification can be derived from J. Milnor's

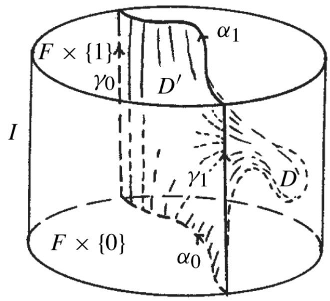


Figure 7.11

uniqueness theorem for the factorization of 3-manifolds [238]. The proof given in this chapter takes advantage of it.

Compositions of knots of a more complicated nature have been investigated by S. Kinoshita and H. Terasaka in [196] and Y. Hashizume and F. Hosokawa [154], see E 14.4 (b).

H. Schubert used Haken's theory of incompressible surfaces to give an algorithm which effects the decomposition into prime factors for a given link [321].

In the case of a fibered knot primeness can be characterized algebraically: The subgroup of fixed elements under the automorphism $\alpha(t)\colon \mathfrak{G}' \to \mathfrak{G}'$, $\alpha(x) = t^{-1}xt$, $x \in G'$, t a meridian, consists of an infinite cyclic group generated by a longitude if and only if the knot is prime. This result is due to W. Whitten [373].

For higher dimensional knots the factorization is not unique. This was proved by C. Kearton [191] and E. Bayer [17], see also E. Bayer, J. Hillman and C. Kearton [16].

7.E Exercises

E 7.1. Show that in general the product of two links $\mathfrak{l}_1 \# \mathfrak{l}_2$ (use an analogous definition) will depend on the choice of the components which are joined.

E 7.2. An m-tangle $\mathfrak{t}_m$ consists of m disjoint simple arcs α_i, $1 \le i \le m$, in a (closed) 3-ball B, $\partial B \cap \bigcup_{i=1}^{m} \alpha_i = \bigcup_{i=1}^{m} \partial\alpha_i$. An m-tangle t_m is called *m-rational*, if there are disjoint disks $\delta_i \subset B$, $\alpha_i = \overline{\mathring{B} \cap \partial\delta_i}$. Show that $\mathfrak{t}_m$ is m-rational if and only if there is an m-tangle $\mathfrak{t}_m^C$ in the complement $C = \overline{S^3 - B}$ such that $\mathfrak{t}_m \cup \mathfrak{t}_m^c$ is the trivial knot. (Observe that the complementary tangle $\mathfrak{t}_m^C$ is rational.) 2-rational tangles are called just rational.

E 7.3. Let S^3 be composed of two balls $B_1, B_2, S^3 = B_1 \cup B_2, B_1 \cap B_2 = S^2 \subset S^3$. If a knot (or link) $\mathfrak{k}$ intersects the B_i in m-rational tangles $\mathfrak{t}_i = \mathfrak{k} \cap B_i$, $i = 1, 2$, then $\mathfrak{k}$ has a bridge number $\le m$.

E 7.4. Prove Corollary 7.5 (b) using Proposition 7.10 and Proposition 3.10.

E 7.5. Show that the groups of the product knots $\mathfrak{k}_1 \# \mathfrak{k}_2$ and $\mathfrak{k}_1 \# \mathfrak{k}_2^*$ are isomorphic, where $\mathfrak{k}_2^*$ is the mirror image of $\mathfrak{k}_2$. The knots are non-equivalent if $\mathfrak{k}_2$ is not amphicheiral.

Chapter 8

Cyclic coverings and Alexander invariants

One of the most important invariants of a knot (or link) is known as the Alexander polynomial. Sections 8.A and 8.B introduce the Alexander module, which is closely related to the homomorphic image $\mathfrak{G}/\mathfrak{G}''$ of the knot group modulo its second commutator subgroup $\mathfrak{G}''$. The geometric background is the infinite cyclic covering C_∞ of the knot complement and its homology (Section 8.C). Section 8.D is devoted to the Alexander polynomials themselves. Finite cyclic coverings are investigated in Section 8.E – they provide further invariants of knots.

Let $\mathfrak{k}$ be a knot, U a regular neighborhood of $\mathfrak{k}$, $C = \overline{S^3 - U}$ the complement of the knot.

8.A Alexander module

We saw in Chapter 3 that the knot group $\mathfrak{G}$ is a powerful invariant of the knot, and the peripheral group system was even shown (compare Theorem 3.16) to characterize a knot. Torus knots could be classified by their groups (see Theorem 3.39). In general, however, knot groups are difficult to treat algebraically, and one tries to simplify matters by looking at homomorphic images of knot groups.

The knot group $\mathfrak{G}$ is a semidirect product $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, where $\mathfrak{Z} \cong \mathfrak{G}/\mathfrak{G}'$ is a free cyclic group, and we may choose $t \in \mathfrak{G}$ (representing a meridian of $\mathfrak{k}$) as a representative of a generating coset of $\mathfrak{Z}$. The knot group $\mathfrak{G}$ can be described by $\mathfrak{G}'$ and the operation of $\mathfrak{Z}$ on $\mathfrak{G}'$: $a \mapsto a^t = t^{-1}at$, $a \in \mathfrak{G}'$. In Chapter 4 we studied the group $\mathfrak{G}'$; it is a free group, if finitely generated, but if not, its structure is rather complicated. We propose to study in this chapter the abelianized commutator subgroup $\mathfrak{G}'/\mathfrak{G}''$ together with the operation of $\mathfrak{Z}$ on it. We write $\mathfrak{G}'/\mathfrak{G}''$ additively and the induced operation as a multiplication:

$$a \mapsto ta, \; a \in \mathfrak{G}'/\mathfrak{G}''.$$

(Note that the induced operation does not depend on the choice of the representative t in the coset $t\mathfrak{G}'$.) The operation $a \mapsto ta$ turns $\mathfrak{G}'/\mathfrak{G}''$ into a module over the group ring $\mathbb{Z}\mathfrak{Z} = \mathbb{Z}(t)$ of $\mathfrak{Z} \cong \langle\, t \,\rangle$ by

$$\Big(\sum_{i=-\infty}^{+\infty} n_i t^i\Big) a = \sum_{i=-\infty}^{+\infty} n_i (t^i a), \; a \in \mathfrak{G}'/\mathfrak{G}'', n_i \in \mathbb{Z}.$$

8.1 Definition (Alexander module). The $\mathfrak{Z}$-module $\mathfrak{G}'/\mathfrak{G}''$ is called the *Alexander module* $M(t)$ of the knot group where t denotes either a generator of $\mathfrak{Z} = \mathfrak{G}/\mathfrak{G}'$ or a representative of its coset in $\mathfrak{G}$. (We follow a common convention by writing merely $\mathfrak{Z}$-module instead of $\mathbb{Z}\mathfrak{Z}$-module.)

$M(t)$ is uniquely determined by $\mathfrak{G}$ except for the change from t to t^{-1}. We shall see, however, that the operations t and t^{-1} are related by a duality in $M(t)$, and that the invariants of $M(t)$ (see Appendix A.6) prove to be symmetric with respect to the substitution $t \mapsto t^{-1}$.

8.B Infinite cyclic coverings and Alexander modules

The commutator subgroup $\mathfrak{G}' \triangleleft \mathfrak{G}$ defines an infinite cyclic covering $p_\infty\colon C_\infty \to C$ of the knot complement, $\mathfrak{G}' \cong \pi_1 C_\infty$. The Alexander module $M(t)$ is the first homology group $H_1(C_\infty) \cong \mathfrak{G}'/\mathfrak{G}''$, and the group of covering transformations which is isomorphic to $\mathfrak{Z} = \mathfrak{G}/\mathfrak{G}'$ induces on $H_1(C_\infty)$ the module operation. Following Seifert [328] we investigate $M(t) \cong H_1(C_\infty)$ in a similar way as we did in the case of the fundamental group $\pi_1 C_\infty \cong \mathfrak{G}'$, see Paragraph 4.4.

Choose a Seifert surface $S \subset S^3$, $\partial S = \mathfrak{k}$ of genus h (not necessarily minimal), and cut C along S to obtain a bounded manifold C^*. Let $\{a_i \mid 1 \le i \le 2h\}$ be a canonical system of curves on S which intersect in a basepoint P. We may assume that $a_i \cap \mathfrak{k} = \emptyset$, and that $\prod_{i=1}^h [a_{2i-1}, a_{2i}] \simeq \mathfrak{k}$ on S, see Proposition 3.13. Retract S onto a regular neighborhood B of $\{a_i \mid 1 \le i \le 2h\}$ consisting of $2h$ bands that start and end in a neighborhood of P. Figures 8.1 (a) and 8.1 (b) show two examples.

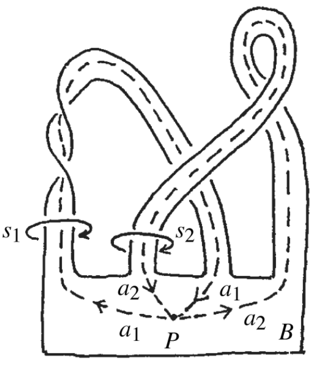


Figure 8.1 (a)

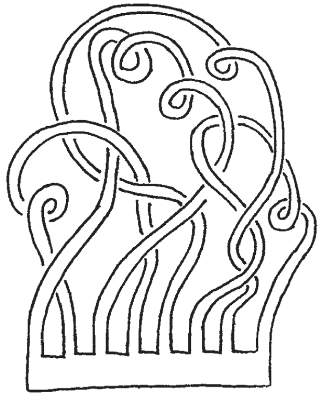


Figure 8.1 (b)

Choosing a suitable orientation we obtain $\partial B \simeq \prod_{i=1}^h [a_{2i-1}, a_{2i}]$ in B, and ∂B represents $\mathfrak{k}$ in S^3. The second assertion is proved as follows: by cutting S along $a_1, \dots, a_{2h}$ we obtain an annulus with boundaries $\mathfrak{k}$ and $\prod_{i=1}^h [a_{2i-1}, a_{2i}]$. This proves the first two parts of the following proposition:

8.2 Proposition (Band projection of a knot). *Every knot can be represented as the boundary of an orientable surface* S *embedded in* 3*-space with the following properties:*

(a) $S = D^2 \cup B_1 \cup \ldots \cup B_{2h}$ *where* D^2 *and each* B_j *is a disk.*

(b) $B_i \cap B_j = \emptyset$ *for* $i \neq j$, $\partial B_i = \alpha_i \gamma_i \beta_i \gamma_i'^{-1}$, $D^2 \cap B_i = \alpha_i \cup \beta_i$, $\partial D^2 = \alpha_1 \delta_1 \beta_2^{-1} \delta_2 \beta_1^{-1} \delta_3 \alpha_2 \delta_4 \ldots \alpha_{2h-1} \delta_{4h-3} \beta_{2h}^{-1} \delta_{4h-2} \beta_{2h-1}^{-1} \delta_{4h-1} \alpha_{2h} \delta_{4h}$.

(c) *There is a projection which is locally homeomorphic on* S *(there are no twists in the bands* B_i*.)*

A projection of this kind is called band projection of S or of $\mathfrak{k}$ *(see Figure 8.1 (b)).*

Proof. It remains to verify assertion (c). Since S is orientable every band is twisted through multiples of angle 2π (full twists). A full twist can be changed into a loop of the band (see Figure 8.2). □

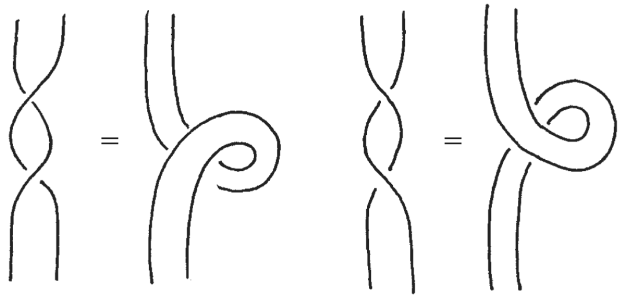


Figure 8.2

8.3 Regular neighborhood of S. There is, obviously, a handlebody W of genus $2h$ contained in a regular neighborhood of S with the following properties:

(a) $S \subset W$,

(b) $\partial W = S^+ \cup S^-$, $S^+ \cap S^- = \partial S^+ = \partial S^- = S \cap \partial W = \mathfrak{k}$, $S^+ \cong S^- \cong S$,

(c) S is a deformation retract of W.

We call S^+ the upside and S^- the downside of W. The curves $a_1, \ldots, a_{2h}$ of S are projected onto curves $a_1^+, \ldots, a_{2h}^+$ on S^+, and $a_1^-, \ldots, a_{2h}^-$ on S^-, respectively. After connecting the basepoints of S^+ and S^- with an arc, they define together a canonical system of curves on the closed orientable surface ∂W of genus $2h$; in particular, they define a basis of $H_1(\partial W) \cong \mathbb{Z}^{4h}$. Clearly

$$a_i^+ \sim a_i^- \quad \text{in } W.$$

Choose a curve s_i on the boundary of the neighborhood of the band B_i such that s_i bounds a disk in W. The orientations of the disk and of s_i are chosen such that the intersection number of the disk and a_i is $+1$ i.e. $\operatorname{lk}(a_i, s_i) = 1$ (right-hand rule), see Figure 8.3.

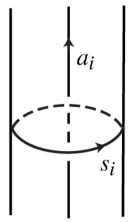


Figure 8.3

8.4 Lemma. (a) $\{a_i^+, \dots, a_{2h}^+, a_1^-, \dots, a_{2h}^-\}$ *and* $\{s_1, \dots, s_{2h}, a_1^\varepsilon, \dots, a_{2h}^\varepsilon\}$ *with* $\varepsilon = +$ *or* $\varepsilon = -$ *are bases of* $H_1(\partial W) \cong \mathbb{Z}^{4h}$.

(b) $\{a_1^\varepsilon, \dots, a_{2h}^\varepsilon\}$ *(*$\varepsilon \in \{+,-\}$*) is a basis of* $H_1(W)$*, and* $\{s_1, \dots, s_{2h}\}$ *is a basis of* $H_1(\overline{S^3 - W}) = \mathbb{Z}^{2h}$.

Proof. The first statements in (a) and (b) follow immediately from the definition of W. The second one of (a) is a consequence of the fact that either system of curves $\{s_1, \dots, s_{2h}, a_1^\varepsilon, \dots, a_{2h}^\varepsilon\}$, $\varepsilon = +$ or $-$, is canonical on ∂W, that is, cutting ∂W along these curves transforms ∂W into a disk. Finally $\{s_1, \dots, s_{2h}\}$ is a basis of $H_1(\overline{S^3 - W})$, since W can be retracted to a $2h$-bouquet of 1-spheres in S^3. The fundamental group and, hence, the first homology group of its complement can be computed in the same way as for the complement of a knot. One may also apply the Mayer–Vietoris sequence:

$$0 = H_2(S^3) \to H_1(\partial W) \xrightarrow{\varphi} H_1(W) \oplus H_1(\overline{S^3 - W}) \to H_1(S^3) = 0.$$

Here φ is given by $\varphi(s_i) = (0, s_i)$. From $H_1(\partial W) \cong \mathbb{Z}^{4h}$ and $H_1(W) \cong \mathbb{Z}^{2h}$ we get $H_1(\overline{S^3 - W}) \cong \mathbb{Z}^{2h}$. Now it follows from (a) that $\{s_1, \dots, s_{2h}\}$ is a basis of $H_1(\overline{S^3 - W})$. □

8.5 Definition (Standard Seifert matrix).

(a) Let $v_{jk} = \operatorname{lk}(a_j^-, a_k)$ be the linking number of a_j^- and a_k. The $(2h \times 2h)$-matrix $V = (v_{jk})$ is called a *standard Seifert matrix* of $\mathfrak{k}$.

(b) Define $f_{jk} = \operatorname{lk}(a_j^- - a_j^+, a_k)$ and $F = (f_{jk})$.

A standard Seifert matrix (v_{jk}) can be read off a band projection in the following way: Consider the j-th band B_j endowed with the direction of its core a_j. Denote by l_{jk} (resp. r_{jk}) the number of times when B_j overcrosses B_k from left to right (resp. from right to left), then put $v_{jk} = l_{jk} - r_{jk}$.

8.6 Lemma. (a) *Let* $i^\varepsilon \colon S^\varepsilon \to \overline{S^3 - W}$ *denote the inclusion. Then*

$$i_*^+(a_j^+) = \sum_{k=1}^{2h} v_{kj} s_k \quad \textit{and}\ i_*^-(a_j^-) = \sum_{k=1}^{2h} v_{jk} s_k .$$

(b) $$F = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}.$$

Proof. (a) Let Z_j^- be a projecting cylinder of the curve a_j^-, and close Z_j^- by a point at infinity. $Z_j^- \cap (S^3 - W)$ represents a 2-chain realizing $a_j^- \sim \sum_{k=1}^{2h} v_{jk} s_k$, Figure 8.4.

The same construction applied to a_j^+, using a projecting cylinder Z_k^+ directed upward, yields $a_j^+ \sim \sum_k v_{kj} s_k$. Note that $v_{jk} = \mathrm{lk}(a_j^-, a_k) = \mathrm{lk}(a_j, a_k^+) = \mathrm{lk}(a_k^+, a_j)$.

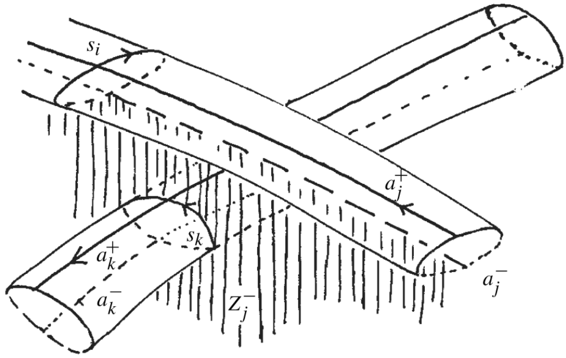


Figure 8.4

(b) There is an annulus bounded by $a_i^- - a_i^+$. It follows from the definition of the canonical system $\{a_j\}$ that

$$\begin{aligned} f_{2n-1,2n} &= \mathrm{lk}(a_{2n-1}^- - a_{2n-1}^+, a_{2n}) = \mathrm{int}(a_{2n-1}, a_{2n}) = +1, \\ f_{2n,2n-1} &= \mathrm{lk}(a_{2n}^- - a_{2n}^+, a_{2n-1}) = \mathrm{int}(a_{2n}, a_{2n-1}) = -1, \end{aligned}$$

$f_{ik} = 0$ otherwise (Figure 8.5). (A compatible convention concerning the sign of the intersection number is supposed to have been agreed on.) The matrix $F = (f_{jk})$ is the intersection matrix of the canonical curves $\{a_j\}$ (Figure 8.5). $\square$

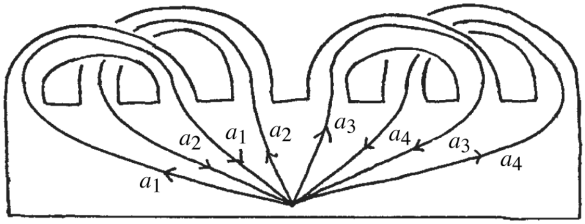


Figure 8.5

In what follows we shall write the equations

$$a_j^- \sim \sum_{k=1}^{2h} v_{jk} s_k \quad \text{and} \quad a_j^+ \sim \sum_k v_{kj} s_k$$

frequently in matrix form, $a^- = Vs$ and $a^+ = V^T s$, where a^+, a^-, s denote the $2h$-columns of the elements a_j^+, a_j^-, s_j.

Definition 8.5 and Lemma 8.6 imply that standard Seifert matrices have certain properties. The following proposition uses these properties to characterize standard Seifert matrices:

8.7 Proposition (Characterization of standard Seifert matrices). *A standard Seifert matrix V of a knot $\mathfrak{k}$ satisfies the equation $V - V^T = F$. (V^T is the transposed matrix of V and F is the intersection matrix defined in Lemma 8.6 (b)).*

Every square matrix V of even order satisfying $V - V^T = F$ is a standard Seifert matrix of a knot.

Proof. Figure 8.5 shows a realization of the matrix

$$V_0 = \begin{pmatrix} 0 & 1 & & & & & & \\ 0 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 0 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & 0 & 0 \end{pmatrix}.$$

Any $2h \times 2h$ matrix V satisfying $V - V^T = F$ is of the form $V = V_0 + Q$, $Q = Q^T$. A realization of V is easily obtained by an inductive argument on h as shown in Figure 8.6. (Here a $(2h-2) \times (2h-2)$ matrix V_1 and a 2×2 matrix V_2 are assumed to be already realized; the bands are represented just by lines.) The last two bands can be given arbitrary linking numbers with the first $2h-2$ bands. $\square$

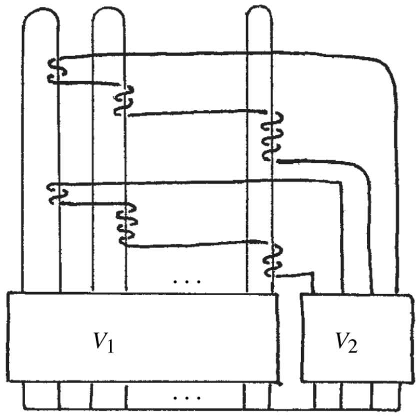


Figure 8.6

8.8 The general Seifert matrix. In this paragraph we are going to free ourselves from the geometrically defined canonical basis $\{a_i\}$ of S and introduce a more general concept of a Seifert matrix $V = (v_{ik})$ (see the notation at the beginning of this section and Definition 8.5).

8.9 Definition. Let $\{a_i \mid 1 \le i \le 2g\}$ be a basis of $H_1(S)$. A basis $\{s_i \mid 1 \le i \le 2g\}$ of $H_1(C^*)$ is called an *associated basis* with respect to $\{a_i\}$, if $\operatorname{lk}(a_i, s_k) = \delta_{ik}$. The matrix $V = (v_{ik})$ defined by the inclusion

$$i^-\colon S^- \to C^*, \quad i_*^-(a_i^-) = \sum_{i=1}^{2g} v_{ik} s_k$$

is called a *Seifert matrix*. To abbreviate notations we use vectors $s = (s_k)$, $a = (a_i)$, $a^\pm = (a_i^\pm)$ etc.

We have used standard associated bases a and s derived from a band projection. For these we deduced $i_*^+(a^+) = V^T s$ from $i_*^-(a^-) = Vs$. Moreover, in this case $V - V^T = F$ represents the intersection matrix of the canonical basis $\{a_i\}$, if a suitable convention concerning the sign of the intersection numbers is agreed upon. The following proposition shows that these assertions remain true in the general case.

8.10 Proposition. *Let a, s be associated bases of $H_1(S)$ and $H_1(C^*)$ respectively. If $i_*^-(a^-) = Vs$ then $i_*^+(a^+) = V^T s$. Moreover $\Sigma = V - V^T$ is the intersection matrix of the basis $a = (a_i)$.*

Proof. We have $i_*^-(a_i^-) = \sum_k v_{ik} s_k$ and hence $v_{ik} = \mathrm{lk}(a_i^-, a_k)$. Now observe that $v_{ik} = \mathrm{lk}(a_i^-, a_k) = \mathrm{lk}(a_i, a_k^+) = \mathrm{lk}(a_k^+, a_i)$ and hence

$$i_*^+(a_i^+) = \sum_{i=1}^{2g} w_{ik} s_k \text{ where } w_{ik} = \mathrm{lk}(a_i^+, a_k) = v_{ki}\,.$$

Therefore we have $i_*^+(a^+) = V^T s$.

Now, the same argument as in the proof of Lemma 8.6 (b) gives:

$$\sigma_{ij} = \mathrm{lk}(a_i^- - a_i^+, a_k) = \mathrm{int}(a_i, a_j).$$

Therefore $V - V^T = \Sigma = (\sigma_{ij})$ is the intersection matrix of the basis $\{a_i\}$ □

8.11 Remark. Let $\tilde{a}$, $\tilde{s}$ be the standard bases of a band projection with $i_*^-(\tilde{a}^-) = \tilde{V}\tilde{s}$ and C a unimodular $2g \times 2g$ integer matrix such that $a = C\tilde{a}$ then Σ and F are congruent i.e.

$$\Sigma = V - V^T = C\tilde{V} - \tilde{V}^T C^T = C\,F\,C^T\,. \tag{8.1}$$

8.12 Corollary. *A square matrix V of even order is a Seifert matrix of a knot if and only if* $\det(V - V^T) = 1$.

Proof. If V is a Seifert matrix of a knot then (8.1) implies that $\det(V - V^T) = 1$.

For every square matrix V of even order the matrix $Q := V - V^T$ is skew-symmetric. Hence by Theorem A.1 in Appendix A there exists a unimodular matrix L such that

$$LQL^T = \begin{pmatrix} 0 & b_1 & & & & & \\ -b_1 & 0 & & & & & \\ & & 0 & b_2 & & & \\ & & -b_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & b_h \\ & & & & & -b_h & 0 \end{pmatrix}$$

with $b_1 \mid b_2 \mid \cdots \mid b_h$. Now, $\det Q = 1$ implies that $b_i = \pm 1$. The equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

shows that we can arrange that $b_i = 1$ for all $1 \le i \le h$. Therefore, Proposition 8.7 implies that $\tilde{V} = LVL^T$ is a standard Seifert matrix of a knot. Hence V is a Seifert matrix as in Definition 8.9. □

8.C Homological properties of C_∞

Let us recall that C_∞ is the regular covering space associated to the kernel of the canonical epimorphism $\varphi\colon \pi_1 C \to \mathbb{Z}$ given by $\varphi([\gamma]) = \mathrm{lk}(\gamma, \mathfrak{k})$. As in Section 3.A the symbol H_j will denote the (singular) homology with coefficients in $\mathbb{Z}$. The group of covering transformations $\mathfrak{Z}$ is isomorphic to the integers and admits a preferred generator $\tau\colon C_\infty \to C_\infty$. The induced automorphism of $H_1(C_\infty)$ will be denoted by $t\colon H_1(C_\infty) \to H_1(C_\infty)$. Hence $H_1(C_\infty)$ turns into a $\mathbb{Z}(t)$-module.

8.13 Remark. Note that the ring $\mathbb{Z}(t)$ is Noetherian by the Hilbert basis theorem. Moreover, the chain groups $C_*(C_\infty)$ are finitely generated, free $\mathbb{Z}(t)$-modules since C is a finite complex. Hence $H_*(C_\infty)$ is a finitely generated $\mathbb{Z}(t)$-module. For details see [206, Chap. X, §1].

Let S be a Seifert surface, W its closed regular neighborhood, $C^* = \overline{S^3 - W}$, and let $C_i^* (i \in \mathbb{Z})$ be copies of C^* and $h_i\colon C^* \to C_i^*$ the corresponding homeomorphism. For the expressions a^+, a^-, s see Definition 8.9.

8.14 Theorem. *Let V be a Seifert matrix of a knot. Then $A(t) = V^T - tV$ is a presentation matrix of the Alexander module $H_1(C_\infty) = M(t)$. (We call a presentation matrix of the Alexander module an* Alexander matrix.*) More explicitly: $H_1(C_\infty)$ is generated by the elements*

$$t^i s_j, i \in \mathbb{Z}, 1 \le j \le 2h, \ \text{and}\ t^i a_j^+ = \sum_{k=1}^{2h} t^i v_{kj} s_k = \sum_{k=1}^{2h} t^{i+1} v_{jk} s_k = t^{i+1} a_j^-$$

are defining relations.

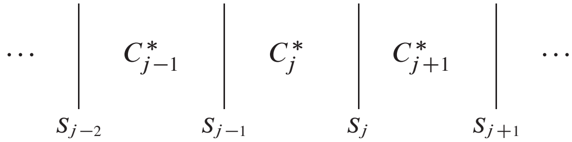
Proof (Levine [209]*, Erle* [93]*).* The Mayer–Vietoris sequence applied to the decomposition $S^3 = W \cup C^*$ gives:

$$H_3(S^3) \xrightarrow{\cong} H_2(\partial W) \xrightarrow{0} H_2(W) \oplus H_2(C^*) \to H_2(C) = 0\,.$$

We have $H_2(W) = 0$ and in Theorem 3.1 (a) we showed $H_i(C) = 0$ for $i \ge 2$. This implies $H_2(C^*) = 0$.

Recall from 4.4 that C_∞ is obtained from the infinite union $\amalg_j C_j^*$ by identifying $h_j(x)$ with $h_{j+1}(r(x))$ for $x \in S^+$, $j \in \mathbb{Z}$, where $r\colon S^+ \to S^-$ is the homeomorphism mapping a point from S^+ to the point of S^- which corresponds to the same point of the Seifert surface S. We let $j^\pm\colon S^\pm \to S$ denote the homeomorphism such that $j^+ = j^- r$. In what follows we let S_j denote the image $h_j(S^+) = h_{j+1}(S^-)$ in C_∞. The canonical homeomorphism $g_j\colon S_j \to S$ is given by $g_j(x) = j^+ h_j^{-1}(x)$

for $x \in S_j$. Note that $g_j = j^- h_{j+1}^{-1}|S_j$ since $r\, h_j^{-1}|S_j = h_{j+1}^{-1}|S_j$ and $j^+ = j^- r$.



In C_∞ we have $S_j = C_j^* \cap C_{j+1}^*$. The corresponding inclusions will be denoted by $i_j^+ \colon S_j \to C_j^*$ and $i_j^- \colon S_j \to C_{j+1}^*$ and are given by

$$i_j^+ = h_j\, i^+ h_j^{-1}|S_j \text{ and } i_j^- = h_{j+1}\, i^- h_{j+1}^{-1}|S_j\,.$$

We apply the Mayer–Vietoris sequence to the following decomposition of C_∞:

$$E_0 \cup E_1 = C_\infty,\ E_0 = \amalg_{i\in\mathbb{Z}} C_{2i}^*,\ E_1 = \amalg_{i\in\mathbb{Z}} C_{2i+1}^*,\ E_0 \cap E_1 = \amalg_{i\in\mathbb{Z}} S_i\,.$$

This gives the following exact sequence:

$$\begin{aligned} 0 = H_2(E_0) \oplus H_2(E_1) \to H_2(C_\infty) \to H_1(E_0 \cap E_1) &\xrightarrow{j_*} H_1(E_0) \oplus H_1(E_1) \\ \to H_1(C_\infty) \to H_0(E_0 \cap E_1) \to H_0(E_0) \oplus H_0(E_1) &\to H_0(C_\infty) \to 0. \end{aligned}$$

Since E_0 and E_1 consist of disjoint copies of C^*, we have $H_2(E_0) = H_2(E_1) = 0$. The homomorphism $H_0(E_0 \cap E_1) \to H_0(E_0) \oplus H_0(E_1)$ is injective, since for $i \neq j$ the surfaces S_i and S_j belong to different components of E_0 or E_1. This implies that the first line of the following sequence is exact.

$$\begin{tikzcd} 0 \to H_2(C_\infty) \arrow[r] \arrow[d, "\cong"', "\mathrm{id}"] & \bigoplus_{j\in\mathbb{Z}} H_1(S_j) \arrow[r, "\alpha"] \arrow[d, "\cong"', "\varphi"] & \bigoplus_{j\in\mathbb{Z}} H_1(C_j^*) \arrow[r] \arrow[d, "\cong"', "\psi"] & H_1(C_\infty) \to 0 \arrow[d, "\cong"', "\mathrm{id}"] \\ 0 \to H_2(C_\infty) \arrow[r] & H_1(S) \otimes_{\mathbb{Z}} \mathbb{Z}(t) \arrow[r, "\beta"] & H_1(C^*) \otimes_{\mathbb{Z}} \mathbb{Z}(t) \arrow[r, "\pi"] & H_1(C_\infty) \to 0 \end{tikzcd} \tag{8.2}$$

For $x \in H_1(S_j)$, we have by definition $\alpha(x) = (-1)^j (i_{j*}^+(x) - i_{j*}^-(x))$ and the isomorphisms φ and ψ are defined by

$$\varphi(x) = g_{j*}(x) \otimes (-1)^j t^j, \quad \psi(y) = h_{j*}^{-1}(y) \otimes t^j$$

for $x \in H_1(S_j)$ and $y \in H_1(C_j^*)$. The morphisms β and π are given by the commutativity conditions. It follows that for $z \in H_1(S)$ the equation

$$\beta(z \otimes t^i) = i_*^+\,(j_*^+)^{-1}(z) \otimes t^i - i_*^-\,(j_*^-)^{-1}(z) \otimes t^{i+1}$$

holds.

We choose now a basis $\{a_j \mid 1 \le j \le 2h\}$ of $H_1(S)$ and an associated basis $\{s_j \mid 1 \le j \le 2h\}$ of $H_1(C^*)$ with respect to $\{a_j\}$. Finally, $j_*^\pm(a_j^\pm) = a_j$ and

$$\beta(a_j \otimes 1) = i_*^+(a_j^+) \otimes 1 - i_*^-(a_j^-) \otimes t = i_*^+(a_j^+) \otimes 1 - t(i_*^-(a_j^-) \otimes 1)\,.$$

Therefore, the map β is represented by the matrix $V^T - tV$ with respect to the bases $\{a_j \otimes 1\}$, $\{s_j \otimes 1\}$ of $H_1(S) \otimes_{\mathbb{Z}} \mathbb{Z}(t)$, $H_1(C^*) \otimes_{\mathbb{Z}} \mathbb{Z}(t)$ respectively (see Lemma 8.6). □

For further use we are interested in the other homology groups of C_∞. Let R be an *integral domain* i.e. a non-trivial commutative ring with identity in which the product of any two nonzero elements is not equal to zero. We let $R(t)$ denote the group ring $R\mathfrak{Z}$ with coefficients in R i.e. $R(t) = R \otimes_{\mathbb{Z}} \mathbb{Z}(t)$.

8.15 Corollary. *Let R be an integral domain. Then there exists an exact sequence of $R(t)$-modules*

$$0 \to H_1(S;R) \otimes R(t) \xrightarrow{\beta} H_1(C^*;R) \otimes R(t) \xrightarrow{\pi} H_1(C_\infty;R) \to 0. \tag{8.3}$$

If V is the Seifert matrix of two associated basis of $H_1(S)$ and $H_1(C^*)$ then β is represented by the matrix $V^T - tV$.

Proof. The proof of Theorem 8.14 can be repeated with R coefficients. Hence sequence (8.2) is exact if $\mathbb{Z}$ is replaced by R. Now observe that β is injective since $\det(V^T - V) = 1 \neq 0$ by Corollary 8.12 (here we have used that R is an integral domain). Hence $H_2(C_\infty;R) = 0$ and the assertion follows. □

8.16 Proposition. *Let R be an integral domain.*

$$\begin{array}{lll} H_m(C_\infty;R) & = 0 & \textit{for } m > 1, \\ H_1(C_\infty,\partial C_\infty;R) & \cong H_1(C_\infty;R), & \\ H_2(C_\infty,\partial C_\infty;R) & \cong R, & \\ H_m(C_\infty,\partial C_\infty;R) & = 0 & \textit{for } m > 2. \end{array}$$

Moreover, the group of covering transformations acts trivially on $H_2(C_\infty,\partial C_\infty;R)$ and the isomorphism $H_2(C_\infty,\partial C_\infty;R) \cong R$ depends only from the orientation of the knot.

Proof. C_∞ is a 3-dimensional non-compact manifold, and ∂C_∞ is an open surface. Thus: $H_m(C_\infty;R) = H_m(C_\infty,\partial C_\infty;R) = 0$ for $m \geq 3$, and $H_2(\partial C_\infty;R) = 0$.

The vanishing of $H_2(C_\infty;R)$ follows from the argument in the proof of Corollary 8.15.

Let $S_j \subset C_\infty$. The natural inclusion $(S_j,\partial S_j) \hookrightarrow (C_\infty,\partial C_\infty)$ then yields to the following diagram (the coefficients R understood):

$$\begin{array}{ccccccccc} 0 \to H_2(S_j,\partial S_j) & \xrightarrow{\cong} & H_1(\partial S_j) & \xrightarrow{0} & H_1(S_j) & \xrightarrow{\cong} & H_1(S_j,\partial S_j) \to 0 \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow & & \downarrow \\ 0 \to H_2(C_\infty,\partial C_\infty) & \xrightarrow{\cong} & H_1(\partial C_\infty) & \xrightarrow{0} & H_1(C_\infty) & \xrightarrow{\cong} & H_1(C_\infty,\partial C_\infty) \to 0. \end{array}$$

The first line of the diagram is clear and $\partial C_\infty \cong \partial S_j \times \mathbb{R}$; this implies the 0 in the second line and the other isomorphisms. We obtain the following diagram:

$$\begin{array}{ccc} H_2(S_j, \partial S_j) & \xrightarrow[\cong]{\partial} & H_1(\partial S_j) \cong R \\ \downarrow{\cong} & & \downarrow{\cong} \\ H_2(C_\infty, \partial C_\infty) & \xrightarrow[\cong]{\partial} & H_1(\partial C_\infty) \end{array}$$

which implies the assertion of the proposition. □

8.D Alexander polynomials

The Alexander module $M(t)$ of a knot is a finitely presented $\mathbb{Z}(t)$-module. In the preceding section we have described a method of obtaining a presentation matrix $A(t)$ (an Alexander matrix) of $M(t)$. An algebraic classification of Alexander modules is not known, since the group ring $\mathbb{Z}(t)$ is not a principal ideal domain. But the theory of finitely generated modules over principal ideal domains can nevertheless be applied to obtain algebraic invariants of $M(t)$.

We call two *Alexander matrices* $A(t), A'(t)$ *equivalent*, $A(t) \sim A'(t)$, if they present isomorphic modules.

Let R be a commutative ring with a unity element 1, and A an $m \times n$-matrix over R. We define *elementary ideals* $E_k(A) \subset R$, for $k \in \mathbb{Z}$

$$E_k(A) = \begin{cases} 0, \text{ if } n-k > m \text{ or } k < 0, \\ R, \text{ if } n-k \le 0, \\ \text{ideal, generated by the } (n-k)\times(n-k) \text{ minors of } A, \text{ if } 0 < n-k \le m. \end{cases}$$

It follows from the Laplace expansion theorem that the elementary ideals form an ascending chain

$$0 = E_{-1}(A) \subset E_0(A) \subset E_1(A) \subset \cdots \subset E_n(A) = E_{n+1}(A) = \cdots = R.$$

8.17 Remark. Let M be an R-module and let A be an $m \times n$ presentation matrix of M i.e. there is an exact sequence

$$R^m \xrightarrow{A} R^n \to M \to 0.$$

The elementary ideals $E_k(A)$ are invariants of the module M (see Turaev [359, I.4], Crowell and Fox [80, Chap. VII] and Zassenhaus [377, III.§3] for more details). Elementary ideals were introduced by H. Fitting in [104] and are employed in several branches of algebra and algebraic geometry (see Lang [206, XIX.2] for details).

Given a knot $\mathfrak{k}$, its Alexander module $M(t)$ and an Alexander matrix $A(t)$ we call $E_k(t) = E_{k-1}(A(t))$ the k-th elementary ideal of $\mathfrak{k}$. The proper ideals $E_k(t)$ are invariants of $M(t)$, and hence, of $\mathfrak{k}$. Compare Appendix A.7 and Crowell and Fox [80, Chapter VII].

8.18 Definition (Alexander polynomials). The greatest common divisor $\Delta_k(t)$ of the elements of $E_k(t)$ is called the *k-th Alexander polynomial of* $M(t)$, resp. of the knot. Usually the first Alexander polynomial $\Delta_1(t)$ is simply called *the Alexander polynomial* and is denoted by $\Delta(t)$ (without an index). If there are no proper elementary ideals, we say that the Alexander polynomials are trivial, $\Delta_k(t) = 1$.

Remark. $\mathbb{Z}(t)$ is a unique factorization ring. So $\Delta_k(t)$ exists, and it is determined up to a factor $\pm t^\nu$, a unit of $\mathbb{Z}(t)$. The k-th Alexander polynomial $\Delta_k(t)$ is a generator of the smallest principal ideal containing $E_k(t)$.

8.19 Definition. Two elements $f(t), g(t) \in \mathbb{Z}(t)$ are called *associated* if there is a unit $u = \pm t^\nu$ in $\mathbb{Z}(t)$ such that $f = u\,g$. It will be convenient to introduce the following notation:

$$f(t) \doteq g(t) \text{ for } f(t), g(t) \in \mathbb{Z}(t), \text{ if } f(t) \text{ and } g(t) \text{ are associated.}$$

The *degree* of a nonzero element $f(t) = \sum_i a_i t^i \in \mathbb{Z}$ is the difference between the largest and smallest values of i for which $a_i \neq 0$. It is clear that two associated elements in $\mathbb{Z}(t)$ have the same degree.

8.20 Remark. The Alexander polynomials do not in general determine the elementary ideals (see Gordon [133] and E 8.3).

8.21 Proposition. *The (first) Alexander polynomial $\Delta(t)$ is obtained from a Seifert matrix V of a knot by*

$$|V^T - tV| = \det(V^T - tV) = \Delta(t).$$

The first elementary ideal $E_1(t)$ is a principal ideal.

Proof. By Theorem 8.14, $V^T - tV = A(t)$ is a $2h \times 2h$ presentation matrix of $M(t)$. The determinant $|A(t)|$ generates the elementary ideal $E_0(A(t)) = E_1(t)$. Since $\det(A(1)) = 1$, the ideal does not vanish, $E_1(t) \neq 0$. □

8.22 Proposition. *The Alexander matrix $A(t)$ of a knot $\mathfrak{k}$ satisfies*

(a) $A(t) \sim A^T(t^{-1})$ *(duality).*

The Alexander polynomials $\Delta_k(t)$ are polynomials of even degree with integral coefficients subject to the following conditions:

(b) *For some integer* n, Δ_k *is a unit for* $k > n$.

(c) $\Delta_k(1) = \pm 1$,

(d) $\Delta_k(t) \doteq \Delta_k(t^{-1})$ *(symmetry)*,

(e) $\Delta_k(t) \mid \Delta_{k-1}(t)$,

(f) $(\Delta_k/\Delta_{k+1}) \mid (\Delta_{k-1}/\Delta_k)$.

Remark. The symmetry (d) implies, together with $\deg \Delta_k(t) \equiv 0 \bmod 2$, that $\Delta_k(t)$ is a symmetric polynomial:

$$\Delta_k(t) \doteq \sum_{i=0}^{2r} a_i t^i, \quad a_{2r-i} = a_i, \quad a_0 = a_{2r} \neq 0.$$

Note also that $\Delta_k(t)$ is a *primitive polynomial* i.e. the integer coefficients a_i of Δ_k are coprime. This follows from $\Delta_k(1) = \pm 1$.

Proof of Proposition 8.22. Let V be a Seifert matrix of $\mathfrak{k}$. Duality follows from the fact that $A(t) = V^T - tV$ is an Alexander matrix by Theorem 8.14, $(V^T - t^{-1}V)^T = -t^{-1}(V^T - tV)$. This implies $E_k(t) = E_k(t^{-1})$ and (d). Property (e) follows from the definition. For $t = 1$ we get: $A(1) = \det(V^T - V) = 1$, see Corollary 8.12, and hence $\Delta_1(1) = \pm 1$. Then $\Delta_k \mid \Delta_{k-1}$ implies (c). The fact that $\Delta_k(t)$ is of even degree is a consequence of (c) and (d).

Property (f) follows from the fact that $A(t)$ is also a presentation matrix for the $\mathbb{Q}(t)$-module $H_1(C_\infty, \mathbb{Q})$ (see Corollary 8.15). Now, $\mathbb{Q}(t)$ is a principal ideal domain and hence $H_1(C_\infty, \mathbb{Q})$ is a direct sum of cyclic modules:

$$H_1(C_\infty, \mathbb{Q}) \cong \mathbb{Q}(t)/(\lambda_1) \oplus \cdots \oplus \mathbb{Q}(t)/(\lambda_n)$$

where $\lambda_i \in \mathbb{Q}(t)$ such that $\lambda_n \mid \lambda_{n-1} \mid \cdots \mid \lambda_1$. The λ_i are unique up to multiplication with a unit $r\,t^\nu \in \mathbb{Q}(t)$, $\nu \in \mathbb{Z}$, $r \in \mathbb{Q}$. Therefore we can chose $\lambda_i \in \mathbb{Z}(t)$ to be a primitive polynomial. The element $\epsilon_k := \lambda_n \lambda_{n-1} \cdots \lambda_k \in \mathbb{Z}(t)$ is then a generator of the ideal generated by the $(n-k+1) \times (n-k+1)$ minors of $A(t)$ in $\mathbb{Q}(t)$. Since Δ_k divides each such minor it follows that $\Delta_k \mid \epsilon_k$ in $\mathbb{Q}(t)$. Moreover, $f \in \mathbb{Q}(t)$ and $\Delta_k \cdot f = \epsilon_k$ implies that $f \in \mathbb{Z}(t)$. Otherwise the product of two primitive polynomials would be not primitive in contradiction to Gauss' Lemma (see Lang [206, IV.2]). Therefore, $\Delta_k \mid \epsilon_k$ in $\mathbb{Z}(t)$.

On the other hand, if $a \in E_k(t) \subset \mathbb{Z}(t)$ then there exists $g \in \mathbb{Q}(t)$ such that $a = g \cdot \epsilon_k$. Again, Gauss' Lemma implies $g \in \mathbb{Z}(t)$. This gives $E_k(t) \subset \epsilon_k \mathbb{Z}(t)$ and hence $\epsilon_k \mid \Delta_k$ in $\mathbb{Z}(t)$ since Δ_k is a generator of the smallest principal ideal in $\mathbb{Z}(t)$ containing $E_k(t)$. Therefore Δ_k and ϵ_k are associated elements in $\mathbb{Z}(t)$. It follows that $\lambda_k \doteq \Delta_k/\Delta_{k+1}$ and the condition $\lambda_k \mid \lambda_{k-1}$ is equivalent to (f). $\square$

8.23 Remark. The Alexander module $M(t)$ is a $\mathbb{Z}(t)$-torsion module. If $A(t)$ is a $n \times n$ presentation matrix of the Alexander module $M(t)$ and if $x \in \mathbb{Z}(t)^n$ represents an element of $M(t)$ then

$$\Delta(t) \cdot x = \det A(t) \cdot x = \left(x \cdot A^*(t)\right) A(t)$$

represents the trivial element. Here $A^*(t)$ is the the *adjoint* of $A(t)$ (see Lang [206, XIII, §4]). Now, $M(t)$ is torsion since $\Delta(t) \neq 0$.

8.24 Proposition (Crowell [78]). *The Abelian group $\mathfrak{G}'/\mathfrak{G}''$ is torsion free.*

Proof. Let $A(t)$ be an $n \times n$ presentation matrix of the Alexander module $M(t)$.

It is sufficient to prove that for $q \in \mathbb{Z}$, $x, y \in \mathbb{Z}(t)^n$ the equation $qx = yA(t)$ implies $y \equiv 0 \bmod q$ since $y = qy'$ for a certain $y' \in \mathbb{Z}(t)^n$ gives $x = y'A(t)$ and hence x represents the trivial element in $M(t)$. Now, multiplying the equation $yA(t) \equiv 0 \bmod q$ with the adjoint of $A(t)$ gives $y \cdot \Delta \equiv 0 \bmod q$ (see [206, XIII, §4]). On the other hand, $\Delta(1) = \pm 1$ implies that $\Delta(t)$ is a primitive polynomial and hence by the Gauss Lemma we obtain $y \equiv 0 \bmod q$ (see [206, IV.2]). □

8.25 Remark. Let R be a factorial ring. Then Corollary 8.15 and the proof of Proposition 8.24 show that $H_1(C_\infty; R)$ is torsion free as R-module.

The symmetry of $\Delta(t)$ suggests a transformation of variables in order to describe the function $\Delta(t)$ by a polynomial in $\mathbb{Z}(t)$ of half the degree of $\Delta(t)$. Write

$$\Delta(t) \doteq a_r + a_{r+1}(t + t^{-1}) + \ldots + a_{2r}(t^r + t^{-r}),$$

and note that $t^k + t^{-k}$ is a polynomial in $(t + t^{-1})$ with coefficients in $\mathbb{Z}$. The proof is by induction on k. For the sake of normalizing we introduce $u = t + t^{-1} - 2$ as a new variable, and obtain $\Delta(t) \doteq \sum_{i=0}^{r} c_i u^i$, $c_0 = 1, c_i \in \mathbb{Z}$.

Clearly, every polynomial $\sum_{i=0}^{r} c_i u^i$ yields a "symmetric polynomial" putting $u = t + t^{-1} - 2$.

8.26 Theorem. *The Alexander polynomial $\Delta(t) = \sum_{i=0}^{2r} a_i t^i$, $a_{2r-i} = a_i$ of a knot can be written in the form*

$$\Delta(t) \doteq \sum_{i=0}^{r} c_i u^i \quad \text{with } u = t + t^{-1} - 2,\ c_0 = 1 \text{ and } c_i \in \mathbb{Z}.$$

Conversely, given arbitrary integers $c_i \in \mathbb{Z}$, $1 \leq i \leq r$, there is a knot $\mathfrak{k}$ with Alexander polynomial

$$\Delta(t) \doteq \sum_{i=0}^{r} c_i u^i,\ c_0 = 1.$$

Consequently, every symmetric polynomial $\Delta(t) = \sum_{i=0}^{2r} a_i t^i$ *with* $\Delta(1) = \pm 1$ *is the Alexander polynomial of some knot* $\mathfrak{k} \subset S^3$.

Proof. The first assertion is clear. Suppose that integers $c_i \in \mathbb{Z}$, $1 \leq i \leq r$ are given. Let V be any $2r \times 2r$ standard Seifert matrix, see Theorem 8.7. By $V^T = V - F$, $F^T F = E$ and $\det F = 1$ we get

$$\begin{aligned} |V^T - tV| &= |V - F - tV| = |(1-t)V - F| = |(1-t)F^T V - E| \\ &= (1-t)^{2r} |F^T V - \frac{1}{1-t} E| = \lambda^{-2r} |F^T V - \lambda E| \end{aligned} \tag{8.4}$$

where $\lambda^{-1} = 1 - t$ such that $\lambda^{-2} = t\,u$. Now $(\lambda(\lambda-1))^{-1} = u$, hence with $c_0 = 1$ we obtain

$$\sum_{i=0}^{r} c_i u^i = u^r \sum_{i=0}^{r} c_i u^{i-r} = u^r \sum_{i=0}^{r} c_{r-i} (\lambda(\lambda-1))^i. \tag{8.5}$$

For any integers $d_i \in \mathbb{Z}$, $1 \leq i \leq r$, the following $(2r \times 2r)$-matrix

$$V = \left(\begin{array}{cc|cc|cc|cc|cc} d_1 & 0 & 0 & 1 & & & & & & \\ -1 & 1 & 0 & 0 & & & & & & \\ \hline 0 & 0 & d_2 & 0 & 0 & 1 & & & & \\ 1 & 0 & -1 & 0 & 0 & 0 & & & & \\ \hline & & 0 & 0 & \ddots & & & & & \\ & & 1 & 0 & & & & & & \\ \hline & & & & & & \ddots & & 0 & 1 \\ & & & & & & & & 0 & 0 \\ \hline & & & & & & 0 & 0 & d_r & 0 \\ & & & & & & 1 & 0 & -1 & 0 \end{array} \right) \tag{8.6}$$

is a Seifert matrix of a knot $\mathfrak{k}$ (see Theorem 8.7).

8.27 Claim.

$$\chi(\lambda) = |F^T V - \lambda E| = \sum_{i=0}^{r-1} d_{r-i} (-1)^{r-i-1} (\lambda(\lambda-1))^i + (\lambda(\lambda-1))^r.$$

Proof of the claim. By induction on r. We have

$$F^T V - \lambda E = \left(\begin{array}{cc|cc|cc|cc|cc}
1-\lambda & -1 & 0 & 0 & & & & & & \\
d_1 & -\lambda & 0 & 1 & & & & & & \\
\hline
-1 & 0 & 1-\lambda & 0 & 0 & 0 & & & & \\
0 & 0 & d_2 & -\lambda & 0 & 1 & & & & \\
\hline
 & & -1 & 0 & \ddots & & & & & \\
 & & 0 & 0 & & & & & & \\
\hline
 & & & & & & & & 0 & 0 \\
 & & & & & & & \ddots & 0 & 1 \\
\hline
 & & & & & & -1 & 0 & 1-\lambda & 0 \\
 & & & & & & 0 & 0 & d_r & -\lambda
\end{array}\right)$$

Denote by $D(d_1, \dots, d_r)$ the determinant of $(F^T V - \lambda E)$, and by $D'(d_1, \dots, d_r)$ the determinant of the submatrix of $(F^T V - \lambda E)$ that is formed by deleting the last row and the last but one column. We have

$$D(d_1) = \lambda(\lambda - 1) + d_1, \quad D'(d_1) = -1,$$
$$D(d_1, d_2) = -d_2 + d_1\lambda(\lambda - 1) + (\lambda(\lambda - 1))^2, \quad D'(d_1, d_2) = 1\,.$$

By expanding $D(d_1, \dots, d_r)$ by the last row, we obtain:

$$D(d_1, \dots, d_r) = -d_r D'(d_1, \dots, d_r) + \lambda(\lambda - 1) D(d_1, \dots, d_{r-1})$$

It is easy to see that

$$D'(d_1, \dots, d_r) = -D'(d_1, \dots, d_{r-1}) = (-1)^{r-2} D'(d_1, d_2) = (-1)^{r-2}\,.$$

Therefore, by induction:

$$\begin{aligned}
D(d_1, \dots, d_r) &= (-1)^{r-1} d_r + \lambda(\lambda - 1) \\
&\qquad \cdot \Big(\sum_{i=1}^{r-1} d_{r-i} (-1)^{r-i-1} (\lambda(\lambda - 1))^{i-1} + (\lambda(\lambda - 1))^{r-1} \Big) \\
&= \sum_{i=0}^{r-1} d_{r-i} (-1)^{r-i-1} (\lambda(\lambda - 1))^{i} + (\lambda(\lambda - 1))^{r}\,. \qquad \square
\end{aligned}$$

Now put $d_{r-i} = (-1)^{r-i-1} c_{r-i}$, $1 \le i \le r - 1$, and $c_0 = 1$. Then

$$|F^T V - \lambda E| = \sum_{i=0}^{r} c_{r-i} (\lambda(\lambda - 1))^i\,.$$

By equation (8.4) the Alexander polynomial of $\mathfrak{k}$ is

$$\begin{aligned}\Delta(t) = |V^T - tV| &= \lambda^{-2r}|F^T V - \lambda E| \\ &= t^r u^r \sum_{i=0}^{r} c_{r-i}\,(\lambda(\lambda-1))^i = t^r \sum_{i=0}^{r} c_i u^i\,.\end{aligned}$$

Therefore, $\Delta(t) \doteq \sum_{i=0}^{r} c_i u^i$ is the Alexander polynomial of $\mathfrak{k}$. □

Remark. The presentation of the Alexander polynomial in the concise form

$$\Delta(t) = \doteq \sum_{i=0}^{n} c_i u^i$$

was first given by Crowell and Fox in [80, Chapter IX, Exercise 4] and employed later by Burde in [51] where the coefficients c_i represented twists in a special knot projection. This connection between the algebraic invariant $\Delta(t)$ and the geometry of the knot projection has come to light very clearly through Conway's discovery [71]. The Conway polynomial is closely connected to the form $\sum_i c_i u^i$ of the Alexander polynomial. It is, however, necessary to include links in order to get a consistent theory. This will be done in Section 9.E.

8.28 Proposition. *Let $V_{\mathfrak{k}}$ and $V_{\mathfrak{l}}$ be Seifert matrices for the knots $\mathfrak{k}$ and $\mathfrak{l}$, and let $\Delta^{(\mathfrak{k})}(t)$ and $\Delta^{(\mathfrak{l})}(t)$ denote their Alexander polynomials. Then*

$$\begin{pmatrix} V_{\mathfrak{k}} & 0 \\ 0 & V_{\mathfrak{l}} \end{pmatrix} = V$$

is a Seifert matrix of the product knot $\mathfrak{k} \# \mathfrak{l}$, $M_{\mathfrak{k}\#\mathfrak{l}}(t) = M_{\mathfrak{k}}(t) \oplus M_{\mathfrak{l}}(t)$ and

$$\Delta^{(\mathfrak{k}\#\mathfrak{l})}(t) = \Delta^{(\mathfrak{k})}(t) \cdot \Delta^{(\mathfrak{l})}(t).$$

Proof. The first assertion is an immediate consequence of the construction of a Seifert surface of $\mathfrak{k} \# \mathfrak{l}$ in Proposition 7.4.

If A_i is a presentation matrix for the module M_i then $\left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right)$ is a presentation matrix for $M_1 \oplus M_2$. This implies the second assertion. The third one follows from

$$|V^T - tV| = |V^{(\mathfrak{k})^T} - tV^{(\mathfrak{k})}| \cdot |V^{(\mathfrak{l})^T} - tV^{(\mathfrak{l})}|.$$ □

8.29 Corollary. *There exists a knot $\mathfrak{k} \subset S^3$ with given arbitrary polynomials $\Delta_k(t)$ subject to the conditions (a)–(f) of Proposition* 8.22.

Proof (Gordon [133]). Let V be the Seifert matrix given in equation (8.6). Then the module represented by the matrix $A(t) = V^T - tV$ is the cyclic module $\mathbb{Z}(t)/(\Delta(t))$ (see E 8.5).

By taking direct sums it is possible to construct a knot with given arbitrary polynomials $\Delta_k(t)$ subject to the conditions (a)–(f) of Proposition 8.22. First define $\lambda_k = \Delta_k/\Delta_{k+1}$ and observe that $\lambda_k(t) \in \mathbb{Z}(t)$ is a symmetric polynomial of even degree with $\lambda_k(1) = \pm 1$. By Proposition 8.26 there exists a knot $\mathfrak{k}_k$ such that $M_{\mathfrak{k}_k}(t) \cong \mathbb{Z}(t)/(\lambda_k)$. Now, Proposition 8.28 implies that $\mathfrak{k} = \mathfrak{k}_1 \# \cdots \# \mathfrak{k}_n$ is a knot such that

$$M_{\mathfrak{k}}(t) \cong \mathbb{Z}(t)/(\lambda_1) \oplus \cdots \oplus \mathbb{Z}(t)/(\lambda_n)$$

and $\Delta_k^{(\mathfrak{k})}(t) = \lambda_n \cdots \lambda_k = \Delta_k$. (See Levine [208] for a different proof.) □

8.30 Examples. (a) The Alexander polynomials of a trivial knot are trivial: $\Delta_k(t) = 1$. (In this case $\mathfrak{G} = \mathfrak{G}/\mathfrak{G}' \cong \mathfrak{Z}, \mathfrak{G}' = 1, M(t) = (0)$.)

(b) Figure 8.7 (a) and 8.7 (b) show band projections of the trefoil 3_1 and the four-knot 4_1: the Seifert matrices are:

$$V_{3_1} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad V_{4_1} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$|V_{3_1}^T - tV_{3_1}| \doteq t^2 - t + 1, \qquad |V_{4_1}^T - tV_{4_1}| \doteq t^2 - 3t + 1.$$

(For further examples see E 8.8.)

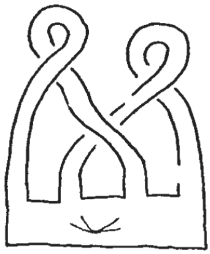


Figure 8.7 (a). The right-handed trefoil.

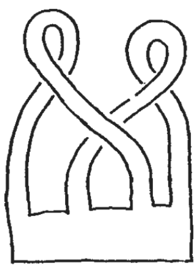


Figure 8.7 (b). The figure-eight knot.

8.31 Normalization of $\Delta(t)$. *In what follows we shall normalize $\Delta(t)$ such that $\Delta(t) = \sum_{i=0}^{2r} a_i t^i$, $a_0 = a_{2r} \neq 0$ i.e. $\Delta(0) = a_0 = a_{2r}$ is the leading coefficient of the Alexander polynomial.*

The normalized Alexander polynomial is called *monic* if $\Delta(0) = \pm 1$.

8.32 Proposition. *The Alexander polynomial of a knot $\mathfrak{k}$ is monic if and only if the Alexander module is a finitely generated as an Abelian group.*

Proof. Since $\Delta(t) = \det A(t)$ annihilates the Alexander module the assumption $\Delta(t) = \sum_{i=0}^{2g} a_i t^i$ monic implies that $H_1(C_\infty)$ is generated by finitely many elements $\{t^i s_j \mid 0 \leq i \leq 2g-1,\ 1 \leq j \leq 2h\}$ (see Theorem 8.14 for notation).

If $H_1(C_\infty)$ is finitely generated then it is a free module of finite rank (Proposition 8.24). Let $\chi(X)$ denote the characteristic polynomial of the automorphism t of $H_1(C_\infty)$. Then the Cayley–Hamilton Theorem implies that $\chi(t) \in \mathbb{Z}(t)$ annihilates the Alexander module. Moreover, (Δ_1/Δ_2) divides $\chi(t)$ (E 8.12) and hence (Δ_1/Δ_2) is monic. Then by Proposition 8.22 (f) all (Δ_i/Δ_{i+1}) are monic. This implies that

$$\Delta_1 = (\Delta_1/\Delta_2)(\Delta_2/\Delta_3)\cdots(\Delta_n/\Delta_{n+1}), \quad \Delta_{n+1} = 1,$$

is monic. □

8.33 Proposition (Alexander polynomials of fibered knots). *The Alexander polynomial*

$$\Delta(t) = \sum_{i=0}^{2g} a_i t^i$$

of a fibered knot $\mathfrak{k}$ (see Chapter 5) satisfies the conditions

(a) $\Delta(0) = a_0 = a_{2g} = \pm 1$ *i.e.* $\Delta(t)$ *is monic,*

(b) $\deg \Delta(t) = 2g$, g *the genus of* $\mathfrak{k}$.

Proof. (a) If $\mathfrak{k}$ is fibered then $H_1(C_\infty)$ is finitely generated and Proposition 8.32 implies that $\Delta(t)$ is monic.

(b) If S is a Seifert surface of minimal genus g spanning $\mathfrak{k}$, the inclusion $i^\pm\colon S^\pm \to C^*$ induces isomorphisms $i_*^\pm\colon \pi_1 S^\pm \to \pi_1 C^*$ (by Theorem 4.7). Hence, $i_*^\pm\colon H_1(S^\pm) \to H_1(C^*)$ are also isomorphisms. This means (by Lemma 8.6) that the corresponding Seifert matrix V is invertible over $\mathbb{Z}$. By Proposition 8.21: $\Delta(t) \doteq |V^T V^{-1} - tE|$, $\Delta(t)$ is the characteristic polynomial of a $2g \times 2g$ regular matrix $V^T V^{-1}$. □

Conditions (a) and (b) of Proposition 8.33 characterize Alexander polynomials of fibered knots: *There is a fibered knot with Alexander polynomial $\Delta(t)$, if $\Delta(t)$ is any polynomial satisfying (a) and (b),* (see Burde [51] and Quach [293]). Moreover, it was proved by Burde and Zieschang [61], and Bing and Martin [22] that the trefoil and the four-knot are the only fibered knots of genus one. The conjecture that fibered knots are classified by their Alexander polynomials has proved to be false in the case of genus $g > 1$ by Morton [251]. Later Morton proved in [252] that there are infinitely many different fibered knots to each Alexander polynomial of degree > 2 satisfying Proposition 8.33 (a). The methods used in Morton's paper are beyond the scope of this book; results of Johannson [179], Jaco and Shalen [177] and Thurston are employed.

It has been checked that the knots up to ten crossings are fibered if (and only if) $\Delta(0) = \pm 1$ [184]. More recently it was proved by Friedl and Vidussi [121] that a knot is fibered if and only if certain *twisted* Alexander polynomials satisfy conditions (a) and (b) of Proposition 8.33.

8.34 Proposition (Alexander modules of satellites). *Let $\mathfrak{k}$ be a satellite, $\widehat{\mathfrak{k}}$ its companion, and $\widetilde{\mathfrak{k}}$ the preimage of $\mathfrak{k}$ under the embedding $h\colon \widetilde{V} \to \widehat{V}$ as in Definition 2.8. Denote by $M(t)$, $\widehat{M}(t)$, $\widetilde{M}(t)$ and $\Delta(t)$, $\widehat{\Delta}(t)$, $\widetilde{\Delta}(t)$ the Alexander modules and Alexander polynomials of $\mathfrak{k}$, $\widehat{\mathfrak{k}}$, $\widetilde{\mathfrak{k}} \subset S^3$ respectively.*

(a) $M(t) = \widetilde{M}(t) \oplus [\mathbb{Z}(t) \otimes_{\mathbb{Z}(t^n)} \widehat{M}(t^n)]$ *where* $n = |\mathrm{lk}(\widehat{\mathfrak{m}}, \mathfrak{k})|$ *denotes the winding number of* $\mathfrak{k}$ *in* $\widehat{V}$.

(b) $\Delta(t) = \widetilde{\Delta}(t) \cdot \widehat{\Delta}(t^n)$.

Proof. We use the notation of Paragraph 4.13. Suppose first that $n > 0$. The infinite cyclic covering C_∞ is given by

$$C_\infty = \widetilde{C}_\infty \cup \left(\coprod_{k=0}^{n-1} t^k \, \widehat{C}_\infty \right) \quad \text{where} \quad \widetilde{C}_\infty \cap \left(\coprod_{k=0}^{n-1} t^k \, \widehat{C}_\infty \right) = \coprod_{k=0}^{n-1} t^k \, \widehat{T}_\infty .$$

The corresponding Mayer–Vietoris sequence gives the following short exact sequence

$$\begin{aligned} H_2(C_\infty) = 0 &\to \oplus_{k=0}^{n-1} t^k H_1(\widehat{T}_\infty) \\ &\underset{j_*}{\to} H_1(\widetilde{C}_\infty) \bigoplus \left(\oplus_{k=0}^{n-1} t^k H_1(\widehat{C}_\infty) \right) \to H_1(C_\infty) \to 0 . \end{aligned}$$

Recall that $\widehat{T}_\infty \cong (\partial \widehat{S}_\infty) \times \mathbb{R}$ where $\widehat{S}_\infty$ is a lift of a Seifert surface of $\widehat{\mathfrak{k}}$ to $\widehat{C}_\infty$. Hence a generator of $H_1(\widehat{T}_\infty) \cong \mathbb{Z}$ represented by a lift $\widehat{\lambda}_\infty = \partial \widehat{S}_\infty$ of the longitude of $\widehat{\mathfrak{k}}$ and is mapped trivially under $H_1(\widehat{T}_\infty) \to H_1(\widehat{C}_\infty)$. This implies that $\mathrm{Im}(j_*)$ is generated by the set $\{t^k \widehat{\lambda}_\infty | k = 0, \ldots, n-1\}$ and that $\mathrm{Im}(j_*) \subset H_1(\widetilde{C}_\infty)$.

The complement $C(\widetilde{\mathfrak{k}})$ of the knot $\widetilde{\mathfrak{k}} \subset \widetilde{V} \subset S^3$ is obtained from $\widetilde{C}$ by filling in the torus $S^3 - \widetilde{V}$ with meridian $h^{-1}(\widehat{\lambda})$. This implies that infinite cyclic covering $C(\widetilde{\mathfrak{k}})_\infty$ of $\widetilde{\mathfrak{k}} \subset S^3$ is obtained from $\widetilde{C}_\infty$ by gluing in n copies of $\widehat{D} \times \mathbb{R}$ to the boundary components $t^k \, \widehat{T}_\infty$, $k = 0, \ldots, n-1$, of $\widetilde{C}_\infty$ such that the generator $t^k \widehat{\lambda}_\infty$ of $H_1(t^k \, \widehat{T}_\infty)$ is killed. Therefore we obtain an isomorphism

$$\widetilde{M}(t) = H_1(C(\widetilde{\mathfrak{k}})_\infty) \cong H_1(\widetilde{C}_\infty) / \operatorname{Im} j_* .$$

The isomorphism

$$\oplus_{k=0}^{n-1} t^k H_1(\widehat{C}_\infty) \cong \mathbb{Z}(t) \otimes_{\mathbb{Z}(t^n)} \widehat{M}(t^n)$$

follows from the fact that the covering transformation $\widehat{t}$ of $\widehat{C}_\infty \subset C_\infty$ is the restriction of t^n, $\widehat{t} = t^n | \widehat{C}_\infty$. This implies (a).

Moreover, if $A(\hat{t})$ is a presentation matrix for the $\mathbb{Z}(\hat{t})$-module $\widehat{M}$, then $A(t^n)$ is a presentation matrix for the $\mathbb{Z}(t)$-module $\mathbb{Z}(t) \otimes_{\mathbb{Z}(\hat{t})} \widehat{M}$ since for all $\widehat{m} \in \widehat{M}$ the equation $1 \otimes \widehat{t} \ \widehat{m} = t^n \otimes \widehat{m}$ holds. Hence (b) follows.

If $n = 0$ then $\widehat{\mathfrak{G}} \subset \mathfrak{G}'$ by Lemma 4.14 and therefore C_∞ contains infinitely many copies of $\widehat{C}$:

$$C_\infty = \widetilde{C}_\infty \cup \big(\coprod_{k\in\mathbb{Z}} t^k \,\widehat{C}_\infty \big) \quad \text{where} \quad \widetilde{C}_\infty \cap \big(\coprod_{k\in\mathbb{Z}} t^k \,\widehat{C}_\infty \big) = \coprod_{k\in\mathbb{Z}} t^k \,\widehat{T}_\infty .$$

Here $\widehat{C}_\infty \cong \widehat{C}$ and $\widehat{T}_\infty \cong \widehat{T}$ are lifts of $\widehat{C}$ respectively $\widehat{T}$ to C_∞.The corresponding Mayer–Vietoris sequence gives the following short exact sequence

$$0 \to \underset{k\in\mathbb{Z}}{\oplus} t^k H_1(\widehat{T}_\infty) \xrightarrow{j_*} H_1(\widetilde{C}_\infty) \bigoplus \big(\underset{k\in\mathbb{Z}}{\oplus} t^k H_1(\widehat{C}_\infty)\big) \to H_1(C_\infty) \to 0 .$$

The group $H_1(\widehat{C}_\infty) \cong \mathbb{Z}$ is generated by $\widehat{m}_\infty$ and the group $H_1(\widehat{T}_\infty)$ is free Abelian generated by $\widehat{m}_\infty$ and $\widehat{\lambda}_\infty$. It follows that

$$H_1(C_\infty) \cong H_1(\widetilde{C}_\infty)/\langle t^k \,\widehat{\lambda}_\infty | k \in \mathbb{Z}\rangle$$

since the image of $\widehat{\lambda}_\infty$ is trivial under the map $H_1(\widehat{T}_\infty) \to H_1(\widehat{C}_\infty)$ and since $\widehat{m}_\infty$ maps to a generator of $H_1(\widehat{C}_\infty)$. As before we obtain an isomorphism

$$\widetilde{M}(t) = H_1(C(\widetilde{\mathfrak{k}})_\infty) \cong H_1(\widetilde{C}_\infty)/\langle t^k \,\widehat{\lambda}_\infty | k \in \mathbb{Z}\rangle$$

and therefore $H_1(C_\infty) \cong H_1(C(\widetilde{\mathfrak{k}})_\infty)$. Now (a) and (b) follow since $\widehat{M}(1)$ is trivial and $\widehat{\Delta}(1) = \pm 1$. □

8.35 Remark. The Alexander polynomial of the untwisted double of every knot has trivial Alexander module and hence trivial polynomial (see Example 2.9 and E 8.14).

8.E Finite cyclic coverings

Beyond the infinite cyclic covering C_∞ of the knot complement $C = \overline{S^3 - V(\mathfrak{k})}$ the finite cyclic coverings of C are of considerable interest in knot theory. The topological invariants of these covering spaces yield new and powerful knot invariants.

Let m be a meridian of a tubular neighborhood $V(\mathfrak{k})$ of $\mathfrak{k}$ representing the element t of the knot group $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, $\mathfrak{Z} = \langle t \rangle$. For $n \geq 0$ there are surjective homomorphisms:

$$\psi_n\colon \mathfrak{G} \to \mathfrak{Z}_n,\ (\mathfrak{Z}_0 = \mathfrak{Z}).$$

8.36 Proposition. $\ker \psi_n = n\mathfrak{Z} \ltimes \mathfrak{G}' = \mathfrak{G}_n,\ n\mathfrak{Z} = \langle t^n \rangle$.
If $\varphi_n\colon \mathfrak{G} \to \mathfrak{Z}_n \cong \mathfrak{Z}/n\mathfrak{Z}$ *is a surjective homomorphism, then* $\ker \varphi_n = \ker \psi_n$.

Proof. Since $\mathfrak{Z}_n$ is Abelian, every homomorphism $\varphi_n\colon \mathfrak{G} \to \mathfrak{Z}_n$ can be factorized, $\varphi_n = j_n \kappa$, $\ker \kappa = \mathfrak{G}'$:

$$\begin{tikzcd}
\mathfrak{G} \arrow[rr, "\varphi_n"] \arrow[dr, "\kappa"'] & & \mathfrak{Z}_n \\
& \mathfrak{G}/\mathfrak{G}' = \mathfrak{Z}. \arrow[ur, "j_n"'] &
\end{tikzcd}$$

One has $\langle \kappa(t) \rangle = \mathfrak{G}/\mathfrak{G}'$, $\ker j_n = \langle n \cdot \kappa(t) \rangle$, and

$$\ker \psi_n = \ker \varphi_n = n\mathfrak{Z} \ltimes \mathfrak{G}' = \mathfrak{G}_n. \qquad \square$$

It follows that for each $n \geq 0$ there is a (uniquely defined) regular covering space C_n, $(C_0 = C_\infty)$, with $\pi_1 C_n = \mathfrak{G}_n$, and a group of covering transformations isomorphic to $\mathfrak{Z}_n$.

8.37 Branched coverings $\widehat{C}_n$. In C_n the n-th $(n > 0)$ power m^n of the meridian is a simple closed curve on the torus ∂C_n. By attaching a solid torus T_n to C_n, $h\colon \partial T_n \to \partial C_n$, such that the meridian of T_n is mapped onto m^n, we obtain a closed manifold $\widehat{C}_n = C_n \cup_h T_n$ which is called the *n-fold branched covering of* $\mathfrak{k}$. Obviously $p_n\colon C_n \to C$ can be extended to a continuous surjective map $\widehat{p}_n\colon \widehat{C} \to S^3$ that fails to be locally homeomorphic (that is, to be a covering map) only in the points of the core $\widehat{p}^{-1}(\mathfrak{k}) = \widehat{\mathfrak{k}}$ of T_n. The restriction $p|\widehat{\mathfrak{k}}\colon \widehat{\mathfrak{k}} \to \mathfrak{k}$ is a homeomorphism. The knot $\mathfrak{k}$, $\widehat{\mathfrak{k}}$ is called the branching set of S^3, $\widehat{C}_n$ respectively, and $\widehat{\mathfrak{k}}$ is said to have *branch index* n. As $\widehat{C}_n$ is also uniquely determined by $\mathfrak{k}$, the spaces $\widehat{C}_n$ as well as C_n are knot invariants; we shall be concerned especially with their homology groups $H_1(\widehat{C}_n)$.

8.38 Proposition. (a) $\mathfrak{G}_n \cong \pi_1 C_n \cong (n\mathfrak{Z}) \ltimes \mathfrak{G}'$ *with* $n\mathfrak{Z} = \langle t^n \rangle$.

(b) $H_1(C_n) \cong (n\mathfrak{Z}) \oplus (\mathfrak{G}'/\mathfrak{G}'_n)$.

(c) $H_1(\widehat{C}_n) \cong \mathfrak{G}'/\mathfrak{G}'_n$.

(d) $H_1(C_n) \cong (n\mathfrak{Z}) \oplus H_1(\widehat{C}_n)$.

Proof. (a) by definition, (b) follows since $\mathfrak{G}'_n \lhd \mathfrak{G}'$. Assertion (c) is a consequence of the Seifert–van Kampen theorem applied to $\widehat{C}_n = C_n \cup_h T_n$. $\square$

8.39 Proposition (Homology of branched cyclic coverings $\widehat{C}_n$). *Let V be a $2h \times 2h$ standard Seifert matrix of a knot $\mathfrak{k}$ obtained from a canonical systems of cures on S and the associated band projection as in Definition* 8.5, $V - V^T = F$, $G = F^T V$, *and* $\mathfrak{Z}_n = \langle t \mid t^n \rangle$.

(a) $R_n = (G - E)^n - G^n$ *is a presentation matrix of* $H_1(\widehat{C}_n)$ *as an Abelian group. In the special case* $n = 2$ *one has* $R_2 \sim V + V^T = A(-1)$.

(b) *As a* $\mathfrak{Z}_n$*-module* $H_1(\widehat{C}_n)$ *is annihilated by* $\varrho_n(t) = 1 + t + \ldots + t^{n-1}$.

(c) $(R_n F)^T = (-1)^n (R_n F)$.

(d) $(V^T - tV)$ *is a presentation matrix of* $H_1(\widehat{C}_n)$ *as a* $\mathfrak{Z}_n$*-module.*

Proof. Denote by τ the covering transformation of the covering $p_n \colon C_n \to C$ corresponding to $\psi_n(t) \in \mathfrak{Z}_n$, see Proposition 8.36. Select a sheet C_0^* of the covering, then $\{C_i^* = \tau^i C_0^* \mid 0 \le i \le n-1\}$ are then n sheets of C_n (see Figures 4.2 and 8.8).

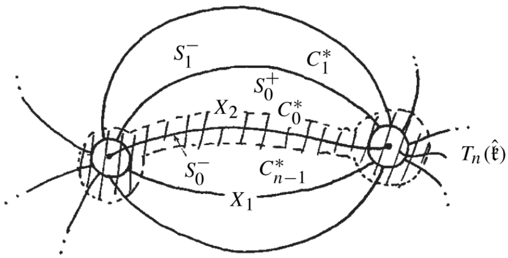


Figure 8.8

Let s_j, $a_i^{\pm}$ be defined as in 8.3. Apply the Seifert–van Kampen theorem to

$$X_1 = \mathring{C}_0^* \cup C_1^* \cup \ldots \cup C_{n-2}^* \cup \mathring{C}_{n-1}^* \quad \text{and } X_2 = U(S_0^- \cup T_n)$$

a tubular neighborhood of $S_0^- \cup T_n$. As in the proof of Theorem 4.7 one gets

$$\begin{aligned}&\pi_1 X_1 \cong \pi_1 C_0^* *_{\pi_1 S_0^+} \pi_1 C_1^* *_{\pi_1 S_1^+} \ldots *_{\pi_1 S_{n-2}^+} \pi_1 C_{n-1}^*,\\ &\pi_1 X_2 \cong \pi_1 S_0^-,\ \pi_1(X_1 \cap X_2) \cong \pi_1 S_{n-1}^+ *_{\langle \widehat{\ell} \rangle} \pi_1 S_0^-,\end{aligned}$$

where $\widehat{\ell} = \partial S_0^-$ is a longitude of $\widehat{\mathfrak{k}}$ in T_n. It follows by abelianizing $\pi_1(\widehat{C}_n) = \pi_1(X_1 \cup X_2)$ that $H_1(\widehat{C}_n) \cong \pi_1(\widehat{C}_n)/\pi_1'(\widehat{C}_n)$ is generated by $\{t^i s_j \mid 1 \le j \le 2h, 0 \le i \le n-1\}$, and its defining relations are

$$(t^i V^T - t^{i+1} V)s = 0, \quad 0 \le i \le n-1, \quad t^n = 1, \quad s^T = (s_1, s_2, \ldots, s_{2h}),$$

see Theorem 8.14. (Observe that in $H_1(\widehat{C}_n)$ the longitude $\widehat{\ell}$ is 0-homologous.) This proves (d).

Multiply the relations by F^T and introduce the abbreviation $F^T V = G$ (see Proposition 8.39). One gets:

$$G t^i s - t^i s - G t^{i+1} s = 0,\ 0 \le i \le n-1 \qquad (K_i)$$

(note that $V^T = V - F$ and $F^T F = E$). Adding these equations gives

$$(1 + t + \cdots + t^{n-1})s = 0,$$

and proves (b).

Add (K_1) to (K_0) to obtain

$$(G - E)s - ts - Gt^2 s = 0. \tag{E_1}$$

Multiply (E_1) by $G - E$ and add to (K_1): The result is

$$(G - E)^2 s - G^2 t^2 s = 0. \tag{R_2}$$

The relations (K_0), (K_1) can be replaced by the relations (E_1) and (R_2), and (E_1) can be used to eliminate ts. This procedure can be continued. Assume that after $(i-1)$ steps the generators ts, $t^2 s, \dots, t^{i-1}s$ are eliminated, and the equations (K_j), $i \le j \le n-1$ together with

$$(G - E)^i s - G^i t^i s = 0 \tag{R_i}$$

form a set of defining relations. Now multiply (K_i) by $\sum_{j=0}^{i-1} G^j$ and add to (R_i). One obtains

$$(G - E)^i s - t^i s - G \sum_{j=0}^{i-1} G^j t^{i+1} s = 0. \tag{E_i}$$

Multiply (E_i) by $(G - E)$ and add to (K_i). The result is

$$(G - E)^{i+1} s - G^{i+1} t^{i+1} s = 0. \tag{R_{i+1}}$$

The relations (R_i), (K_i) have thus been replaced by (E_i), (R_{i+1}). Eliminate $t^i s$ by (E_i) and omit (E_i).

The procedure stops when only the generators $s = (s_j)$ are left, and the defining relations

$$G^n s - (G - E)^n s = 0$$

remain. This proves (a).

Assertion (c) is easily verified using the definition of R_n and F: $G = F^T V$, $G - E = F^T(V - F) = F^T V^T$ and $F^T = -F$ imply $R_n = (F^T V^T)^n - (F^T V)^n$. Hence,

$$\begin{aligned}(R_n F)^T &= F^T((VF)^n - (V^T F)^n) = (-(FV)^n + (FV^T)^n)F \\ &= (-1)^n((F^T V^T)^n - (F^T V)^n)F = (-1)^n R_n F. \qquad \square\end{aligned}$$

8.40 Remark. It follows from Proposition 8.39 (b) for $n = 2$ that $1 + t$ is the 0-endomorphism of $H_1(\widehat{C}_2)$. This means

$$a \mapsto ta = -a \quad \text{for } a \in H_1(\widehat{C}_2).$$

8.41 Theorem. *$H_1(\widehat{C}_n)$ is finite if and only if no root of the Alexander polynomial $\Delta(t)$ of $\mathfrak{k}$ is an n-th root of unity ζ_i, $1 \le i \le n$. In this case*

$$|H_1(\widehat{C}_n)| = |\prod_{i=1}^{n} \Delta(\zeta_i)|.$$

In general, the Betti number of $H_1(\widehat{C}_n)$ is even and equals the number of roots of the Alexander polynomial which are also roots of unity; each such root is counted ν-times, if it occurs in ν different elementary divisors $\varepsilon_k(t) = \Delta_k(t)\Delta_{k+1}^{-1}(t)$, $k = 1, 2, \dots$.

Proof. Since the matrices $G - E$ and G commute,

$$R_n = (G - E)^n - G^n = \prod_{i=1}^{n} [(G - E) - \zeta_i G].$$

By $V^T = V - F$ and $F^T F = E$,

$$(G - E) - tG = F^T(V^T - tV) = F^T A(t)$$

is a presentation matrix of the Alexander module $M(t)$; thus, by Proposition 8.21,

$$\det((G - E) - tG) \doteq \Delta(t).$$

This implies that $\det R_n = \prod_{i=1}^{n} \Delta(\zeta_i)$. The order of the homology group $H_1(\widehat{C}_n)$ is $\det R_n$, if $\det R_n \neq 0$.

In the general case the Betti number of $H_1(\widehat{C}_n)$ is equal to $2h - \operatorname{rank} R_n$. To determine the rank of R_n we study the Jordan canonical form $G_0 = L^{-1} G L$ of G, where L is a non-singular matrix with coefficients in $\mathbb{C}$. Then $L^{-1} R_n L = (G_0 - E)^n - G_0^n$. The diagonal elements of G_0 are the roots $\lambda_i = (1 - t_i)^{-1}$ of the characteristic polynomial $\chi(\lambda) = \det(G - \lambda E)$, where the t_i are the roots of the Alexander polynomial, see Theorem 8.26. The nullity of $L^{-1} R_n L$ equals the number of λ_i which have the property $(\lambda_i - 1)^n - \lambda_i^n = 0 \iff t_i^n = 1$, $t_i \neq 1$, once counted in each Jordan block of G_0.

From $\Delta(1) = 1$ and the symmetry of the Alexander polynomial it follows that only non-real roots of unity may be roots of $\chi(\lambda)$ and those occur in pairs. □

The following property of $H_1(\widehat{C}_n)$ is a consequence of Proposition 8.39 (c).

8.42 Proposition (Plans [290]). $H_1(\widehat{C}_n) \cong A \oplus A$ *if* $n \equiv 1 \mod 2$.

Proof. $Q = R_n F$ is equivalent to R_n, and hence a presentation matrix of $H_1(\widehat{C}_n)$. For odd n the matrix Q is skew-symmetric, $Q = -Q^T$, see Proposition 8.39 (c). The proposition follows from the fact that Q has a canonical form

$$L^T Q L = \begin{pmatrix} 0 & a_1 & & & & & & & & & \\ -a_1 & 0 & & & & & & & & & \\ & & 0 & a_2 & & & & & & & \\ & & -a_2 & 0 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 0 & a_s & & & & \\ & & & & & -a_s & 0 & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & 0 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 0 \end{pmatrix},$$

where L is unimodular (invertible over $\mathbb{Z}$). A proof is given in Appendix A.1. □

8.F History and sources

J. W. Alexander [8] first introduced Alexander polynomials. H. Seifert [328] investigated the subject from a geometric point of view and was able to prove the characterizing properties of the Alexander polynomial (Propositions 8.21 and 8.22). The presentation of the homology of the finite cyclic coverings in Proposition 8.39 is also due to H. Seifert [328].

Further relations between Alexander invariants and the homology of finite Abelian coverings where discovered by J. P. Mayberry and K. Murasugi [231], M. Sakuma [313] and J. Porti [291].

8.G Exercises

E 8.1. Prove: deg $\Delta(t) \leq 2g$, where g is the genus of a knot, and $\Delta(t)$ its Alexander polynomial. (For knots up to ten crossings equality holds.)

E 8.2. Write $\Delta(t) = t^4 - 2t^3 + t^2 - 2t - 1$ in the reduced form $\sum_{i=0}^{2} c_i u^i$ (Proposition 8.26). Construct a knot with $\Delta(t)$ as its Alexander polynomial. Construct a fibered knot with $\Delta(t)$ as its Alexander polynomial. (Hint: use braid-like knots as defined in E 4.4.)

E 8.3. Show that the knots 6_1 and 9_{46} have the same Alexander polynomials. However, prove that for 6_1, $E_2(t) = \mathbb{Z}(t)$, whereas for 9_{46}, $E_2(t) = (t-2, 2t-1) \neq \mathbb{Z}(t)$ (see Gordon [133]).

E 8.4. Show that $H_1(C_\infty) = 0$ if and only if $\Delta(t) = 1$. Prove that $\pi_1 C_\infty$ is of finite rank, if it is free.

E 8.5. Let V be the Seifert matrix given in equation (8.6). Proof that the module represented by the matrix $A(t) = V^T - tV$ is the cyclic module $\mathbb{Z}(t)/(\Delta(t))$.

E 8.6. Prove: $H_1(\widehat{C}_n) = 0$ for $n \geq 2$ if and only if $H_1(C_\infty) = 0$.

E 8.7. Show $|H_1(\widehat{C}_2)| \equiv 1 \mod 2$; further, for a knot of genus one with $|H_1(\widehat{C}_2)| = 4a \pm 1$, show that $H_1(\widehat{C}_3) \cong \mathbb{Z}_{3a\pm1} \oplus \mathbb{Z}_{3a\pm1}, a \in \mathbb{N}$.

E 8.8. By $\mathfrak{p}(p,q,r)$, p,q,r odd integers, we denote a pretzel knot (Figure 8.9). (The sign of the integers defines the direction of the twist.) Construct a band projection of $\mathfrak{p}(p,q,r)$, and compute its Seifert matrix V and its Alexander polynomial. (Figure 8.10 shows how a band projection may be obtained.)

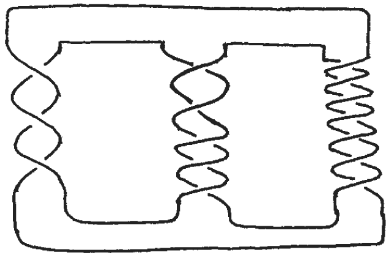


Figure 8.9. $\mathfrak{p}(3,-5,-7)$

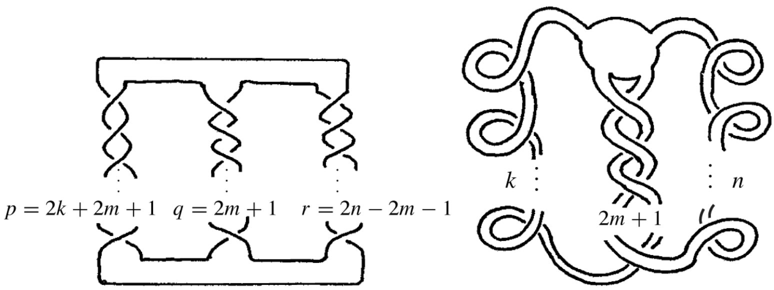


Figure 8.10

E 8.9. Let $\mathfrak{k}$ be a link of $\mu > 1$, components. Show that there is a homomorphism φ of its group $\mathfrak{G} = \pi_1(S^3 - \mathfrak{k})$ onto a free cyclic group $\mathfrak{Z} = \langle t \rangle$ which maps every Wirtinger generator of $\mathfrak{G}$ onto t. Construct an infinite cyclic covering C_∞ of the link complement using a Seifert surface S of $\mathfrak{k}$, compute its Seifert matrix and define its Alexander polynomial following the lines developed in this chapter in the case of a knot. (See also E 9.5.)

E 8.10. Let $\widehat{C}_3$ be the 3-fold cyclic branched covering of a knot. If $H_1(\widehat{C}_3) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime p, then there are generators a, b of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ such that $t\colon H_1(\widehat{C}_3) \to H_1(\widehat{C}_3)$ is given by $ta = b$, $tb = -a - b$. For all knots one has $p \neq 3$.

E 8.11. Construct a knot of genus one with the Alexander polynomials of the trefoil but not fibered – and hence different from the trefoil.

E 8.12. Show that the annihilator of the Alexander module $M(t)$ of a knot is a principal ideal which is generated by $\Delta_1(t)\Delta_2^{-1}(t)$ (see Crowell [79]).

E 8.13. Let $\mathfrak{k}$ be a fibered knot of genus g, and let $F \times I/h$ denote its complement. Describe $h_*\colon H_1(F, \mathbb{Q}) \to H_1(F, \mathbb{Q})$ by a matrix $A = \oplus A_i$ where A_i is a companion matrix determined by the Alexander polynomials of $\mathfrak{k}$. (For the notion of a companion matrix see, e.g. [360, Section 12.5].)

E 8.14. Prove that a satellite is never trivial. Show, that doubled knots (see Example 2.9) have trivial Alexander modules, and therefore trivial Alexander polynomials.

Chapter 9

Free differential calculus and Alexander matrices

In Chapter 8 we studied the homology of the cyclic coverings of the knot complement. Alexander polynomials were defined, and a general method of computing these invariants via a band projection of the knot was developed. Everyone who actually wants to carry out this task will soon find out that the calculations involved increase rapidly with the genus of the knot. There are, however, knots of arbitrary genus with groups of a relatively simple structure (for instance: torus knots). We shall present in this chapter another method of computing Alexander's knot invariants which will prove to be considerably simpler in this case – and in many other cases. The method is based on the theory of Fox derivations in the group ring of a free group. There is a geometric background to the Fox calculus with which we intend to start. It is K. Reidemeister's *theory of homotopy chains* [300], or, to use the modern terminology, *equivariant homology*.

9.A Regular coverings and homotopy chains

The one-to-one correspondence between finitely presented groups and fundamental groups of 2-complexes, and between (normal) subgroups and (regular) coverings of such complexes has been exploited in combinatorial group theory to prove group theoretical theorems (as for instance the Reidemeister–Schreier method or the Kurosh subgroup theorem [382, 2.6]) by topological methods. In the case of homology these relationships are less transparent, but some can be retained for the first homology groups.

9.1 On the homology of a covering space. Let $p\colon \tilde{X} \to X$ be a regular covering of a connected 2-complex. We assume X to be a finite CW-complex with one 0-cell P. Then a presentation

$$\mathfrak{G} = \pi_1(X, P) = \langle\, s_1, \dots s_n \mid R_1, \dots, R_m \,\rangle$$

of the fundamental group of X is obtained by assigning a generator s_i to each (oriented) 1-cell (also denoted by s_i), and a defining relation to (the boundary of) each 2-cell e_j of X. Choose a basepoint $\tilde{P} \in \tilde{X}$ over P, $p_\#(\pi_1(\tilde{X}, \tilde{P})) = \mathfrak{N} \triangleleft \mathfrak{G}$, and let $\mathfrak{D} \cong \mathfrak{G}/\mathfrak{N}$ denote the group of covering transformations.

Let $\varphi\colon \mathfrak{G} \to \mathfrak{D}$, $w \mapsto w^\varphi$ be the canonical homomorphism. The linear extension to the group ring is also denoted by $\varphi\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{D}$. Observe: $(w_1 w_2)^\varphi = w_1^\varphi w_2^\varphi$.

Our aim is to present $H_1(\tilde{X}, \tilde{X}^0)$ as a $\mathbb{Z}\mathfrak{D}$-module. (We follow a common convention by writing merely $\mathfrak{D}$-module instead of $\mathbb{Z}\mathfrak{D}$-module. $\tilde{X}^0$ denotes the 0-skeleton of $\tilde{X}$.)

The (oriented) edges s_i lift to edges $\tilde{s}_i$ with initial point $\tilde{P}$. By w we denote a closed path in the 1-skeleton X^1 of X, and, at the same time, the element it represents in the free group $\mathfrak{F} = \pi_1(X^1, P) = \langle s_1, \dots, s_n \mid - \rangle$. There is a unique lift $\tilde{w}$ of w starting at $\tilde{P}$. Clearly $\tilde{w}$ is a special element of the relative cycles $Z_1(\tilde{X}, \tilde{X}^0)$ which are called *homotopy 1-chains*. Every 1-chain can be written in the form $\sum_{j=1}^n \xi_j \tilde{s}_j$, $\xi_j \in \mathbb{Z}\mathfrak{D}$. (The expression $g\tilde{s}_j$ denotes the image of the edges $\tilde{s}_j$ under the covering transformation g.) There is a rule

$$\widetilde{w_1 w_2} = \tilde{w}_1 + w_1^\varphi \cdot \tilde{w}_2.$$

To understand it, first lift w_1 to $\tilde{w}_1$. Its endpoint is $w_1^\varphi \cdot \tilde{P}$. The covering transformation w_1^φ maps $\tilde{w}_2$ onto a chain $w_1^\varphi \tilde{w}_2$ over w_2 which starts at $w_1^\varphi \tilde{P}$. If $\tilde{w}_k = \sum_{j=1}^n \xi_{kj} \tilde{s}_j$ with $\xi_{kj} \in \mathbb{Z}\mathfrak{D}$, $k = 1, 2$, then $\widetilde{w_1 w_2} = \sum_{j=1}^n \xi_j \tilde{s}_j$ with

$$\xi_j = \xi_{1j} + w_1^\varphi \cdot \xi_{2j} \quad (1 \le j \le n).$$

(The coefficient ξ_{kj} is the algebraic intersection number of the path $\tilde{w}_k$ with the covers of s_j.) This defines mappings

$$\left(\frac{\partial}{\partial s_j}\right)^\varphi : \mathfrak{G} = \pi_1(X, P) \to \mathbb{Z}\mathfrak{D},\ w \mapsto \xi_j, \quad \text{with } \tilde{w} = \sum_{j=1}^n \xi_j \tilde{s}_j, \tag{9.1}$$

satisfying the rule

$$\left(\frac{\partial}{\partial s_j}(w_1 w_2)\right)^\varphi = \left(\frac{\partial}{\partial s_j} w_1\right)^\varphi + w_1^\varphi \cdot \left(\frac{\partial}{\partial s_j} w_2\right)^\varphi. \tag{9.2}$$

There is a linear extension to the group ring $\mathbb{Z}\mathfrak{G}$:

$$\left(\frac{\partial}{\partial s_j}(\eta + \xi)\right)^\varphi = \left(\frac{\partial}{\partial s_j}\eta\right)^\varphi + \left(\frac{\partial}{\partial s_j}\xi\right)^\varphi \quad \text{for } \eta, \xi \in \mathbb{Z}\mathfrak{G}. \tag{9.3}$$

From the definition it follows immediately that

$$\left(\frac{\partial}{\partial s_j} s_k\right)^\varphi = \delta_{jk},\ \tilde{w} = \sum \left(\frac{\partial w}{\partial s_j}\right)^\varphi \tilde{s}_j,\ \delta_{jk} = \begin{cases} 1 & j = k, \\ 0 & j \ne k. \end{cases}$$

We may now use this terminology to present $H_1(\tilde{X}, \tilde{X}^0)$ as a $\mathfrak{D}$-module: The 1-chains $\tilde{s}_i$, $1 \le i \le n$, are generators, and the lifts $\tilde{R}_j$ of the boundaries $R_j = \partial e_j$ of the 2-cells are defining relations. (The boundary of an arbitrary 2-cell of $\tilde{X}$ is of the form $\delta \tilde{R}_j$, $\delta \in \mathfrak{D}$. Hence, in a presentation of $H_1(\tilde{X}, \tilde{X}^0)$ as a $\mathfrak{D}$-module it suffices to include the $\tilde{R}_j$, $1 \le j \le m$, as defining relations.)

9.2 Proposition.

$$H_1(\tilde{X}, \tilde{X}^0) = \langle \tilde{s}_1, \dots, \tilde{s}_n \mid \tilde{R}_1, \dots, \tilde{R}_m \rangle,\ 0 = \tilde{R}_j = \sum \left(\frac{\partial R_j}{\partial s_i}\right)^\varphi \tilde{s}_i,\ 1 \le i \le m$$

is a presentation of $H_1(\tilde{X}, \tilde{X}^0)$ *as a* $\mathfrak{D}$*-module.* □

9.B Fox differential calculus

In this section we describe a purely algebraic approach to the mapping $(\frac{\partial}{\partial s_j})^\varphi$ introduced by R. H. Fox [107, 108, 109]. Let $\mathfrak{G}$ be a group and $\mathbb{Z}\mathfrak{G}$ its group ring (with integral coefficients); $\mathbb{Z}$ is identified with the multiples of the unit element 1 of $\mathfrak{G}$.

9.3 Definition (Fox derivative). (1) There is a homomorphism

$$\varepsilon\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}, \quad \tau = \sum_i n_i g_i \mapsto \sum_i n_i = \tau^\varepsilon.$$

We call ε the *augmentation homomorphism*, and its kernel $I\mathfrak{G} = \varepsilon^{-1}(0)$, the *augmentation ideal*.

(2) A mapping $\Delta\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{G}$ is called a *derivation* (of $\mathbb{Z}\mathfrak{G}$) if

$$\Delta(\xi + \eta) = \Delta(\xi) + \Delta(\eta) \qquad \text{(linearity)},$$

and

$$\Delta(\xi \cdot \eta) = \Delta(\xi) \cdot \eta^\varepsilon + \xi \cdot \Delta(\eta) \qquad \text{(product rule)},$$

for $\xi, \eta \in \mathbb{Z}\mathfrak{G}$.

From the definition it follows by simple calculations:

9.4 Lemma. (1) *The derivations of $\mathbb{Z}\mathfrak{G}$ form a (right) $\mathfrak{G}$-module under the operations defined by:*

$$(\Delta_1 + \Delta_2)(\tau) = \Delta_1(\tau) + \Delta_2(\tau),\ (\Delta\gamma)(\tau) = \Delta(\tau) \cdot \gamma.$$

(2) *Let Δ be a derivation. Then:*

$$\begin{aligned} \Delta(m) &= 0 \quad \textit{for } m \in \mathbb{Z},\\ \Delta(g^{-1}) &= -g^{-1} \cdot \Delta(g),\\ \Delta(g^n) &= (1 + g + \cdots + g^{n-1}) \cdot \Delta(g),\\ \Delta(g^{-n}) &= -(g^{-1} + g^{-2} + \cdots + g^{-n}) \cdot \Delta(g) \quad \textit{for } n \geq 1. \end{aligned}$$

□

9.5 Examples. (1) $\Delta_\varepsilon\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{G}$, $\tau \mapsto \tau - \tau^\varepsilon$, is a derivation.

(2) If $a, b \in \mathfrak{G}$ commute, $ab = ba$, then $(a-1)\Delta b = (b-1)\Delta a$. (We write Δa instead of $\Delta(a)$ when no confusion can arise.) It follows that a derivation $\Delta\colon \mathbb{Z}\mathfrak{Z}^n \to \mathbb{Z}\mathfrak{Z}^n$ of the group ring of a free Abelian group $\mathfrak{Z}^n = \langle S_1 \rangle \times \cdots \times \langle S_n \rangle$, $n \geq 2$, with $\Delta S_i \neq 0$, $1 \leq i \leq n$, is a multiple of Δ_ε in the module of derivations.

Contrary to the situation in group rings of Abelian groups the group ring of a free group admits a great many derivations.

9.6 Proposition. *Let $\mathfrak{F} = \langle \{S_i | i \in J\}\rangle$ be a free group. There is a uniquely determined derivation $\Delta\colon \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F}$, with $\Delta S_i = w_i$, for arbitrary elements $w_i \in \mathbb{Z}\mathfrak{F}$.*

Proof. $\Delta(S_i^{-1}) = -S_i^{-1}w_i$ follows from $\Delta(1) = 0$ and the product rule. Linearity and product rule imply uniqueness. Define $\Delta(S_{i_1}^{\eta_1} \cdots S_{i_k}^{\eta_k})$ using the product rule:

$$\Delta(S_{i_1}^{\eta_1} \dots S_{i_k}^{\eta_k}) = \Delta S_{i_1}^{\eta_1} + S_{i_1}^{\eta_1}\Delta S_{i_2}^{\eta_2} + \cdots + S_{i_1}^{\eta_1} \cdots S_{i_{k-1}}^{\eta_{k-1}}\Delta S_{i_k}^{\eta_k}.$$

The product rule then follows for combined words $w = uv$, $\Delta w = \Delta u + u\Delta v$. The equation

$$\begin{aligned} \Delta(uS_i^{\eta}S_i^{-\eta}v) &= \Delta u + u\Delta S_i^{\eta} + uS_i^{\eta}\Delta S_i^{-\eta} + u\Delta v = \Delta u + u\Delta v \\ &= \Delta(uv), \eta = \pm 1, \end{aligned}$$

shows that Δ is well defined on $\mathfrak{F}$. □

9.7 Definition (Partial derivations). The derivations

$$\frac{\partial}{\partial S_i}\colon \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F},\ S_j \mapsto \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j, \end{cases}$$

of the group ring of a free group $\mathfrak{F} = \langle \{S_i \mid i \in J\} \mid -\rangle$ are called *partial derivations*.

The partial derivations form a basis of the module of derivations:

9.8 Proposition. (a) $\Delta = \sum_{i\in J} \frac{\partial}{\partial S_i} \cdot \Delta(S_i)$ *for every derivation* $\Delta\colon \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F}$. *(The sum may be infinite, however, for any $\tau \in \mathbb{Z}\mathfrak{F}$ there are only finitely many $\frac{\partial\tau}{\partial S_i} \neq 0$.)*

(b) $\sum_{i\in J} \frac{\partial}{\partial S_i} \cdot \tau_i = 0 \iff \tau_i = 0,\ i \in J$.

(c) $\Delta_\varepsilon(\tau) = \tau - \tau^\varepsilon = \sum_{i\in J} \frac{\partial\tau}{\partial S_i}(S_i - 1)$ *(Fundamental formula).*

(d) $\tau - \tau^\varepsilon = \sum_{i\in J} v_i(S_i - 1) \iff v_i = \frac{\partial\tau}{\partial S_i},\ i \in J$.

Proof. $(\sum_i \frac{\partial}{\partial S_i}\Delta S_i)S_j = \sum_i \frac{\partial S_j}{\partial S_i}\Delta S_i = \Delta S_j$ proves (a) by Proposition 9.6. For $\Delta = 0$-map, and $\Delta = \Delta_\varepsilon$ one gets (b) and (c). To prove (d) apply $\frac{\partial}{\partial S_j}$ to the equation. □

The theory of derivations in $\mathbb{Z}\mathfrak{F}$ (free derivations) has been successfully used to study $\mathbb{Z}\mathfrak{F}$ and $\mathfrak{F}$ itself [379]. There are remarkable parallels to the usual derivations used in analysis. For instance, the fundamental formula resembles a Taylor expansion.

If $(S_1, \dots, S_n)$, $(S'_1, \dots, S'_n)$, $(S''_1, \dots, S''_n)$ are bases of a free group $\mathfrak{F}_n$, there is a chain rule for the Jacobian matrices:

$$\frac{\partial S''_k}{\partial S_i} = \sum_{j=1}^{n} \frac{\partial S''_k}{\partial S'_j} \cdot \frac{\partial S'_j}{\partial S_i}.$$

(Apply Proposition 9.8 (a) in the form $\Delta = \sum_{j=1}^{n} \frac{\partial}{\partial S'_j} \Delta S'_j$ for $\Delta = \frac{\partial}{\partial S_i}$ to S''_k.)

J. Birman [23] proves that $(S'_1, \dots, S'_m)$ is a basis of $\mathfrak{F} = \langle\, S_1, \dots, S_n | - \rangle$ if and only if the Jacobian $(\frac{\partial S'_j}{\partial S_i})$ is invertible over $\mathbb{Z}\mathfrak{F}$.

For further properties of derivations see E 9.7, E 9.8.

9.C Calculation of Alexander polynomials

We return to the regular covering $p\colon \tilde{X} \to X$ of 9.1. Let

$$\psi\colon \mathfrak{F} = \langle\, S_1, \dots, S_n | - \,\rangle \to \langle\, S_1, \dots, S_n \mid R_1, \dots, R_m \,\rangle = \mathfrak{G}$$

denote the canonical homomorphism of the groups and, at the same time, its extension to the group rings:

$$\psi\colon \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{G}, \ \left(\sum n_i f_i\right)^{\psi} = \sum n_i f_i^{\psi} \quad \text{for } f_i \in \mathfrak{F}, n_i \in \mathbb{Z}.$$

Combining ψ with the map $\varphi\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{D}$ of 9.1 (we use the notation $(\xi)^{\varphi\psi} = (\xi^{\psi})^{\varphi}$, $\xi \in \mathbb{Z}\mathfrak{F}$), we may state Proposition 9.2 in terms of the differential calculus.

9.9 Proposition. *$((\frac{\partial R_k}{\partial S_j})^{\varphi\psi})$, $1 \le k \le m$, $1 \le j \le n$, is a presentation matrix of $H_1(\tilde{X}, \tilde{X}^0)$ as a $\mathfrak{D}$-module. (k = row index, j = column index.)*

Proof. Comparing the linearity and the product rule of the Fox derivations (Definition 9.3) with equations (9.2) and (9.3) of Paragraph 9.1, we deduce from Proposition 9.6 that the mappings $(\frac{\partial}{\partial S_i})^{\varphi}$ in equation (9.1) coincide with those defined by $(\frac{\partial}{\partial S_i})^{\varphi\psi}$ in Definition 9.7. □

Remark. The fact that the partial derivation of equation (9.3) and Definition 9.7 are the same gives a geometric interpretation also to the fundamental formula: For $w \in \mathfrak{G}$ and $\tilde{w}$ its lift,

$$\partial\tilde{w} = (w^{\varphi\psi} - 1)\tilde{P} = \sum_i \left(\frac{\partial w}{\partial S_i}\right)^{\varphi\psi} (S_i^{\varphi\psi} - 1)\tilde{P} = \sum_i \left(\frac{\partial w}{\partial S_i}\right)^{\varphi\psi} \partial\tilde{s}_i.$$

To obtain information about $H_1(\tilde{X})$ we consider the exact homology sequence

$$\begin{array}{ccccccccc} H_1(\tilde{X}^0) & \to & H_1(\tilde{X}) & \to & H_1(\tilde{X},\tilde{X}^0) & \xrightarrow{\partial} & H_0(\tilde{X}^0) & \xrightarrow{i_*} & H_0(\tilde{X}) \to 0. \\ \| & & & & & & \|\wr & & \|\wr \\ 0 & & & & & & \mathbb{Z}\mathfrak{D} & & \mathbb{Z} \end{array} \tag{9.4}$$

$H_0(\tilde{X}^0)$ is generated by $\{w^{\varphi\psi}\cdot\tilde{P} \mid w\in\mathfrak{F}\}$ as an Abelian group. The kernel of i_* is the image $(I\mathfrak{F})^{\varphi\psi}$ of the augmentation ideal $I\mathfrak{F}\subset\mathbb{Z}\mathfrak{F}$ (see Definition 9.3 (a)). The fundamental formula shows that $\ker i_*$ is generated by $\{(S_j^{\varphi\psi}-1)\tilde{P} \mid 1\le j\le n\}$ as a $\mathfrak{D}$-module.

Thus we obtain from (9.4) a short exact sequence:

$$0\to H_1(\tilde{X})\to H_1(\tilde{X},\tilde{X}^0)\xrightarrow{\partial}\ker i_*\to 0. \tag{9.5}$$

In the case of a knot group $\mathfrak{G}$, and its infinite cyclic covering C_∞ $(\mathfrak{N}=\mathfrak{G}')$ the group of covering transformations is cyclic, $\mathfrak{D}=\mathfrak{Z}=\langle\, t\,\rangle$, and $\ker i_*$ is a free $\mathfrak{Z}$-module generated by $(t-1)\tilde{P}$. The sequence (9.5) splits, and

$$H_1(\tilde{X},\tilde{X}^0)\cong H_1(\tilde{X})\oplus\sigma(\mathbb{Z}\mathfrak{Z}\cdot(t-1)\tilde{P}), \tag{9.6}$$

where σ is a homomorphism $\sigma\colon \ker i_*\to H_1(\tilde{X},\tilde{X}^0)$, $\partial\sigma=\mathrm{id}$. This yields the following:

9.10 Theorem. *For* $\mathfrak{G}=\langle\, S_1,\dots,S_n | R_1,\dots,R_n\,\rangle$, *its* Jacobian $((\frac{\partial R_j}{\partial S_i})^{\varphi\psi})$ *and* $\varphi\colon\mathfrak{G}\to\mathfrak{G}/\mathfrak{G}'=\mathfrak{Z}=\langle\, t\,\rangle$, *a presentation matrix (Alexander matrix) of* $H_1(\tilde{X})\cong H_1(C_\infty)$ *as a* $\mathfrak{Z}$*-module is obtained from the Jacobian by omitting its i-th column, if* $S_i^{\varphi\psi}=t^{\pm1}$. *(In the case of a Jacobian derived from a Wirtinger presentation any column may be omitted.)*

Proof. It remains to show that the homomorphism $\sigma\colon\ker i_*\to H_1(\tilde{X},\tilde{X}^0)$ can be chosen in such a way that $\sigma(\ker i_*)=\mathbb{Z}\mathfrak{Z}\,\tilde{s}_i$. Put $\sigma(t-1)\tilde{P}=\pm t^\mu\tilde{s}_i$, $S_i^{\varphi\psi}=t^\nu$, $\partial\sigma=\mathrm{id}$. Then

$$(t-1)\tilde{P}=\partial\sigma(t-1)\tilde{P}=\partial(\pm t^\mu\tilde{s}_i)=\pm t^\mu(S_i^{\varphi\psi}-1)\tilde{P}=\pm t^\mu(t^\nu-1)\tilde{P},$$

that is, $(t-1)=\pm t^\mu(t^\nu-1)$. It follows $\nu=\pm1$, and in these cases σ can be chosen as desired. □

If $\mathfrak{D}$ is not free cyclic, the sequence (9.5) does not necessarily split, and $H_1(\tilde{X})$ cannot be identified as a direct summand of $H_1(\tilde{X},\tilde{X}^0)$. We shall treat the cases $\mathfrak{D}\cong\mathbb{Z}_n$ and $\mathfrak{D}\cong\mathbb{Z}^\mu$ in Section D.

There is a useful corollary to Theorem 9.10:

9.11 Corollary. *Every* $(n-1)\times(n-1)$ *minor* Δ_{ij} *of the* $n\times n$ *Jacobian of a Wirtinger presentation* $\langle\, S_i \mid R_j \,\rangle$ *of a knot group* $\mathfrak{G}$ *is a presentation matrix of* $H_1(C_\infty)$. *Furthermore,* $\det\Delta_{ij} \doteq \Delta(t)$. *The elementary ideals of the Jacobian are the elementary ideals of the knot.*

Proof. Every Wirtinger relation R_k is a consequence of the remaining ones (Corollary 3.6). Thus, by Theorem 9.10 a presentation matrix of $H_1(C_\infty) = M(t)$ is obtained from the Jacobian by leaving out an arbitrary row and an arbitrary column. □

Corollary 9.11 shows that a Jacobian of a Wirtinger presentation has nullity one. The following lemma explicitly describes the linear dependence of the rows and columns of the Jacobian of a Wirtinger presentation:

9.12 Lemma. (a) $\sum_{i=1}^{n}(\frac{\partial R_j}{\partial S_i})^{\varphi\psi} = 0$.

(b) $\sum_{j=1}^{n}\eta_j(\frac{\partial R_j}{\partial S_i})^{\varphi\psi} = 0$, $\eta_j = t^{\nu_j}$ *for suitable* $\nu_j \in \mathbb{Z}$ *for a Wirtinger presentation* $\langle\, S_1,\dots,S_n \mid R_1,\dots,R_n \,\rangle$ *of a knot group.*

Proof. The first equation follows from the fundamental formula (Proposition 9.8 (c)) applied to R_j:

$$0 = (R_j - 1)^{\varphi\psi} = \left[\sum_{i=1}^{n}\left(\frac{\partial R_j}{\partial S_i}\right)(S_i - 1)\right]^{\varphi\psi} = \sum_{i=1}^{n}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi}(t-1).$$

Since $\mathbb{Z}\mathfrak{Z}$ has no divisors of zero (E 9.1), the first equation is proved.

To prove the second equation we use the identity of Corollary 3.6 which expresses the dependence of Wirtinger relations by the equation $\prod_{j=1}^{n} L_j R_j L_j^{-1} = 1$ in the free group $\langle\, S_1,\dots,S_n | - \rangle$. Now

$$\begin{aligned}\left(\frac{\partial}{\partial S_i}L_j R_j L_j^{-1}\right)^{\varphi\psi} &= \left(\frac{\partial L_j}{\partial S_i}\right)^{\varphi\psi} + L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi} - (L_j R_j L_j^{-1})^{\varphi\psi}\left(\frac{\partial L_j}{\partial S_i}\right)^{\varphi\psi}\\ &= L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi}, \quad \text{as } (L_j R_j L_j^{-1})^{\varphi\psi} = 1.\end{aligned}$$

By the product rule:

$$\begin{aligned}0 = \frac{\partial}{\partial S_i}\left(\prod_{j=1}^{n} L_j R_j L_j^{-1}\right)^{\varphi\psi} &= \sum_{j=1}^{n}\left(\prod_{k=1}^{j-1}(L_k R_k L_k^{-1})\right)^{\varphi\psi} L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi}\\ &= \sum_{j=1}^{n} L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi},\end{aligned}$$

which proves the second equation with $L_j^{\varphi\psi} = t^{\nu_j} = \eta_j$. □

9.13 Example. A Wirtinger presentation of the group of the trefoil is

$$\langle S_1, S_2, S_3 \mid S_1S_2S_3^{-1}S_2^{-1},\ S_2S_3S_1^{-1}S_3^{-1},\ S_3S_1S_2^{-1}S_1^{-1} \rangle,$$

see Example 3.7. If $R = S_1S_2S_3^{-1}S_2^{-1}$ then

$$\frac{\partial R}{\partial S_1} = 1,\ \frac{\partial R}{\partial S_2} = S_1 - S_1S_2S_3^{-1}S_2^{-1},\ \frac{\partial R}{\partial S_3} = -S_1S_2S_3^{-1}$$

and

$$\left(\frac{\partial R}{\partial S_1}\right)^{\varphi\psi} = 1, \quad \left(\frac{\partial R}{\partial S_2}\right)^{\varphi\psi} = t - 1, \quad \left(\frac{\partial R}{\partial S_3}\right)^{\varphi\psi} = -t.$$

By similar calculations we obtain the matrix of derivatives and apply $\varphi\psi$ to get an Alexander matrix

$$\begin{pmatrix} 1 & t-1 & -t \\ -t & 1 & t-1 \\ t-1 & -t & 1 \end{pmatrix}.$$

It is easy to verify Lemma 9.12 (a) and (b). The 2×2 minor $\Delta_{11} = \begin{pmatrix} 1 & t-1 \\ -t & 1 \end{pmatrix}$, for instance, is a presentation matrix of the Alexander module. Hence, $|\Delta_{11}| = 1 - t + t^2 = \Delta(t)$, $E_1(t) = (1 - t + t^2)$. For $k > 1$: $E_k(t) = (1) = \mathbb{Z}(t)$, $\Delta_k(t) = 1$.

9.14 Proposition. *Let*

$$\langle S_1, \dots, S_n \mid R_1, \dots, R_m \rangle = \mathfrak{G} = \langle S'_1, \dots, S'_{n'} \mid R'_1, \dots, R'_{m'} \rangle$$

be two finite presentations of a knot group. The elementary ideals of the respective Jacobians $((\frac{\partial R_j}{\partial S_i})^{\varphi\psi})$ *and* $((\frac{\partial R'_j}{\partial S'_i})^{\varphi\psi})$ *coincide, and are those of the knot.*

Proof. This follows from Corollary 9.11, and from the fact (Appendix A.6) that the elementary ideals are invariant under Tietze processes. □

9.15 Example (Torus knots). $\mathfrak{G} = \langle x, y \mid x^a y^{-b} \rangle$, $a > 0, b > 0$, $\gcd(a, b) = 1$, is a presentation of the group of the knot $\mathfrak{t}(a, b)$ (see Proposition 3.38). The projection homomorphism $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}' = \mathfrak{Z} = \langle t \rangle$ is defined by: $x^\varphi = t^b, y^\varphi = t^a$ (Exercise E 9.3). The Jacobian of the presentation is:

$$\left(\frac{\partial(x^a y^{-b})}{\partial x}, \frac{\partial(x^a y^{-b})}{\partial y}\right)^{\varphi\psi} = \left(\frac{t^{ab}-1}{t^b-1}, -\frac{t^{ab}-1}{t^a-1}\right).$$

The greatest common divisor

$$\gcd\left(\frac{t^{ab}-1}{t^b-1}, \frac{t^{ab}-1}{t^a-1}\right) = \frac{(t^{ab}-1)(t-1)}{(t^a-1)(t^b-1)} = \Delta_{a,b}(t)$$

is the Alexander polynomial of $\mathfrak{t}(a,b)$, $\deg \Delta_{a,b}(t) = (a-1)(b-1)$. One may even prove something more: the Alexander module $M_{a,b}(t)$ of a torus knot $\mathfrak{t}(a,b)$ is cyclic: $M_{a,b}(t) \cong \mathbb{Z}(t)/(\Delta_{a,b}(t))$.

Proof. There are elements $\alpha(t)$, $\beta(t) \in \mathbb{Z}(t)$ such that

$$\alpha(t)(t^{a-1} + t^{a-2} + \ldots + t + 1) + \beta(t)(t^{b-1} + t^{b-2} + \ldots + t + 1) = 1. \quad (9.7)$$

This is easily verified by applying the Euclidean algorithm. It follows that

$$\alpha(t)\frac{t^{ab}-1}{t^b-1} + \beta(t)\frac{t^{ab}-1}{t^a-1} = \Delta_{a,b}(t).$$

Hence, the Jacobian can be replaced by an equivalent one:

$$\left(\frac{t^{ab}-1}{t^b-1}, -\frac{t^{ab}-1}{t^a-1}\right)\begin{pmatrix} \alpha(t) & t^{b-1}+\ldots+1 \\ -\beta(t) & t^{a-1}+\ldots+1 \end{pmatrix} = (\Delta_{a,b}(t), 0).$$

We may interpret by Proposition 9.9 the Jacobian as a presentation matrix of $H_1(\tilde{X}, \tilde{X}^0)$:

$$\left(\frac{t^{ab}-1}{t^b-1}, -\frac{t^{ab}-1}{t^a-1}\right)\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = 0,$$

where $\tilde{x}$, $\tilde{y}$ are the 1-chains that correspond to the generators x, y (see 9.1).

The transformation of the Jacobian implies a contragredient (dual) transformation of the generating 1-chains:

$$\begin{aligned} \tilde{u} &= (t^{a-1} + \cdots + 1)\tilde{x} - (t^{b-1} + \ldots + 1)\tilde{y}, \\ \tilde{v} &= \beta(t)\tilde{x} + \alpha(t)\tilde{y}. \end{aligned}$$

These 1-chains form a new basis with:

$$(\Delta_{a,b}(t), 0)\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = 0.$$

Since $\partial\tilde{x} = (t^b - 1)\tilde{P}$, $\partial\tilde{y} = (t^a - 1)\tilde{P}$, one has $\partial\tilde{u} = 0$ and

$$\partial\tilde{v} = (\beta(t)(t^b - 1) + \alpha(t)(t^a - 1))\tilde{P} = (t-1)\tilde{P}$$

by (9.7). Thus $\tilde{v}$ generates a free summand $\sigma(\ker i_*)$ (see (9.6)), and $\tilde{u}$ generates $M(t)$, subject to the relation $\Delta_{a,b}(t)\tilde{u} = 0$. □

Torus knots are fibered knots, by Proposition 4.11 and Theorem 5.1. We proved in Corollary 4.12 that the commutator subgroup $\mathfrak{G}'$ of a torus knot $\mathfrak{t}(a,b)$ is free of rank $(a-1)(b-1)$. By Theorem 4.7 the genus of $\mathfrak{t}(a,b)$ is $g = \frac{(a-1)(b-1)}{2}$, a fact which is reproved by Proposition 8.33, and $\deg \Delta_{a,b}(t) = (a-1)(b-1)$.

9.D Alexander polynomials of links

Let $\mathfrak{l} = \mathfrak{k}_1 \cup \dots \cup \mathfrak{k}_\mu$ be an oriented link of $\mu > 1$ components, and $\mathfrak{G} = \pi_1(\overline{S^3 - V(\mathfrak{l})})$ its group. $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}' = \mathfrak{Z}^\mu = \langle t_1 \rangle \times \ldots \times \langle t_\mu \rangle$ maps $\mathfrak{G}$ onto a free Abelian group of rank μ. For each component we choose a meridian t_i with $\mathrm{lk}(\mathfrak{k}_i, t_i) = +1$. We assume, as in the case of a knot, that t_i, $1 \le i \le \mu$, denotes at the same time a free generator of $\mathfrak{Z}^\mu$ or a representative in $\mathfrak{G} \mod \mathfrak{G}'$, representing a meridian of the i-th component $\mathfrak{k}_i$ of $\mathfrak{l}$ with $\varphi(t_i) = t_i$. We may consider $\mathfrak{G}'/\mathfrak{G}''$ as module over the group ring $\mathbb{Z}\mathfrak{Z}^\mu$ using the operation $a \mapsto t_i^{-1} a t_i$, $a \in \mathfrak{G}'$, to define the operation of $\mathbb{Z}\mathfrak{Z}^\mu$ on $\mathfrak{G}'/\mathfrak{G}''$. Proposition 9.2 applies to the situation with $\mathfrak{N} = \mathfrak{G}'$, $\mathfrak{D} \cong \mathfrak{Z}^\mu$. Denote by ψ the canonical homomorphism

$$\psi\colon \mathfrak{F} = \langle S_1, \ldots, S_n \mid - \rangle \to \langle S_1, \ldots, S_n \mid R_1, \ldots, R_n \rangle = \mathfrak{G}$$

onto the link group $\mathfrak{G}$, described by a Wirtinger presentation. The Jacobian $\big((\partial R_j/\partial S_i)^{\varphi\psi}\big)$, then is a presentation matrix of $H_1(\tilde{X}, \tilde{X}^0)$. The exact sequence (9.5) does not split, so that a submodule isomorphic to $H_1(\tilde{X})$ cannot easily be identified. Following Fox [108] we call $H_1(\tilde{X}, \tilde{X}^0)$ the *Alexander module* of $\mathfrak{l}$ and denote it by $M(t_1, \ldots, t_\mu)$.

9.16 Proposition. *The first elementary ideal $E_1(t_1, \ldots, t_\mu)$ of the Alexander module $M(t_1, \ldots, t_\mu)$ of a μ-component link $\mathfrak{l}$ is of the form:*

$$E_1(t_1, \ldots, t_\mu) = J_0 \cdot (\Delta(t_1, \ldots, t_\mu))$$

where J_0 is the augmentation ideal of $\mathbb{Z}\mathfrak{Z}^\mu$ (see Definition 9.3), and the second factor is a principal ideal generated by the greatest common divisor of $E_1(t_1, \ldots, t_\mu)$; it is called the Alexander polynomial $\Delta(t_1, \ldots, t_\mu)$ of $\mathfrak{l}$, and it is an invariant of $\mathfrak{l}$ – up to multiplication by a unit of $\mathbb{Z}\mathfrak{Z}^\mu$.

Proof. Corollary 3.6 is valid in the case of a link. The $(n-1) \times n$-matrix $\mathfrak{R}$ resulting from the Jacobian $\big((\partial R_j/\partial S_i)^{\varphi\psi}\big)$ by omitting its last row is, therefore, a presentation matrix of $H_1(\tilde{X}, \tilde{X}^0)$, and defines its elementary ideals. Let

$$\Delta'_i = \det(\mathfrak{a}_1, \ldots, \mathfrak{a}_{i-1}, \mathfrak{a}_{i+1}, \ldots, \mathfrak{a}_n)$$

be the determinant formed by the column-vectors $\mathfrak{a}_j, i \neq j$, of $\mathfrak{R}$. The fundamental formula $R_k - 1 = \sum_{j=1}^n \frac{\partial R_k}{\partial S_j}(S_j - 1)$ yields $\sum_{j=1}^n \mathfrak{a}_j (S_j^{\varphi\psi} - 1) = 0$. Hence,

$$\begin{aligned}
\Delta'_j(S_i^{\varphi\psi} - 1) &= \det(\mathfrak{a}_1, \ldots, \mathfrak{a}_i(S_i^{\varphi\psi} - 1), \ldots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \ldots) \\
&= \det(\mathfrak{a}_1, \ldots, -\sum_{k \neq i} \mathfrak{a}_k(S_k^{\varphi\psi} - 1), \ldots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \ldots) \\
&= \det(\mathfrak{a}_1, \ldots, -\mathfrak{a}_j(S_j^{\varphi\psi} - 1), \ldots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \ldots) \\
&= \pm \Delta'_i(S_j^{\varphi\psi} - 1);
\end{aligned}$$

thus

$$\Delta'_j(S_i^{\varphi\psi} - 1) = \pm\Delta'_i(S_j^{\varphi\psi} - 1). \tag{9.8}$$

The $S_i^{\varphi\psi}$, $1 \le i \le n$ take the value of all t_k, $1 \le k \le \mu$. Now it follows that $(S_i^{\varphi\psi}-1)|\Delta'_i$. Define Δ_i by $(S_i^{\varphi\psi}-1)\Delta_i = \Delta'_i$. Since $\mathbb{Z}\mathfrak{Z}^\mu$ is a unique factorization ring, (9.8) implies that $\Delta_i = \pm\Delta$ for $1 \le i \le n$. The first elementary ideal, therefore, is a product $J_0 \cdot (\Delta)$, where J_0 is generated by the elements $(t_k - 1)$, $1 \le k \le \mu$. It is easy to prove (E 9.1) that J_0 is the augmentation ideal $I\mathfrak{Z}^\mu$ of $\mathbb{Z}\mathfrak{Z}^\mu$.

The elementary ideal E_1 is an invariant of $\mathfrak{G}$ (Appendix A.6); hence, its greatest common divisor is an invariant of $\mathfrak{G}$, up to multiplication by a unit $\pm t_1^{r_1} \dots t_\mu^{r_\mu}$ of $\mathbb{Z}\mathfrak{Z}^\mu$. The polynomial $\Delta(t_1, \dots, t_\mu) = \Delta$, though, depends on the choice of a basis of $\mathfrak{Z}^\mu$. But it is possible to distinguish a basis of $\mathfrak{Z}^\mu \cong H_1(S^3 - \mathring{V}(\mathfrak{l}))$ geometrically by choosing a meridian for each component $\mathfrak{k}_i$ to represent t_i. □

For more information on Alexander modules of links see Crowell and Strauss [75], Hillman [167], Levine [210].

A link $\mathfrak{l}$ is called *splittable*, if it can be separated by a 2-sphere embedded in S^3.

9.17 Corollary. *The Alexander polynomial of a splittable link of multiplicity $\mu \ge 2$ vanishes, i.e. $\Delta(t_1, \dots, t_\mu) = 0$.*

Proof. A splittable link $\mathfrak{l}$ allows a Wirtinger presentation of the following form: There are two disjoint finite sets of Wirtinger generators, $\{S_i \mid i \in I\}$, $\{T_j \mid j \in J\}$, and correspondingly, two sets of relations $\{R_k(S_i)\}$, $\{N_l(T_j)\}$. For $i \in I$, $j \in J$ consider

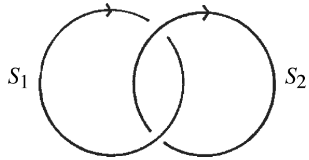
$$\Delta'_i(T_j^{\varphi\psi} - 1) = \pm\Delta'_j(S_i^{\varphi\psi} - 1).$$

The column $\mathfrak{a}_i(S_i^{\varphi\psi} - 1)$ in $\pm\Delta'_j(S_i^{\varphi\psi} - 1)$ is by $\sum_{k\in I} \mathfrak{a}_k(S_i^{\varphi\psi} - 1) = 0$ a linear combination of other columns. It follows that $\Delta'_i(T_j^{\varphi\psi} - 1) = 0$, i.e. $\Delta'_i = 0$. □

9.18 Remark. Alexander polynomials of links retain some properties of knot polynomials. G. Torres and R. H. Fox proved in [351] that they are symmetric. There are some necessary conditions, the so-called *Torres conditions*, which do not characterize Alexander polynomials of links $(\mu \ge 2)$, as J. A. Hillman [168, VII, Theorem 5], showed.

9.19 Alexander polynomial for links. There is a simplified version of the Alexander polynomial of a link. Consider the homomorphism $\chi\colon \mathfrak{Z}^\mu \to \mathfrak{Z} = \langle t \rangle$, $t_i \mapsto t$. Put $\mathfrak{N} = \ker \chi\varphi$. The sequence (9.6) now splits, and, as in the case of a knot, any $(n-1)\times(n-1)$ minor of the Jacobian $((\frac{\partial R_j}{\partial S_i})^{\chi\varphi\psi})$ is a presentation matrix of $H_1(\tilde{X}) \cong H_1(C_\infty)$, where C_∞ is the infinite cyclic covering of the complement of the link which corresponds to the normal subgroup $\mathfrak{N} = \ker \chi\varphi \triangleleft \mathfrak{G}$. The first elementary ideal is generated by $(t-1)\cdot\Delta(t,\dots,t)$ (see Proposition 9.16) where $\Delta(t_1,\dots,t_\mu)$ is the Alexander polynomial of the link. The polynomial $\Delta(t,\dots,t)$ *(the so-called reduced Alexander polynomial)* is of the form $\Delta(t,\dots,t) = (t-1)^{\mu-2}\cdot\nabla^H(t)$, and $\nabla^H(t)$ is called the *Hosokawa polynomial of the link* (E 9.5). It was shown by Hosokawa in [174] that $\nabla^H(t)$ *is of even degree and symmetric*. Furthermore, any such polynomial $f(t)\in\mathbb{Z}\mathfrak{Z}$ is the Hosokawa polynomial of a link for any $\mu > 1$.

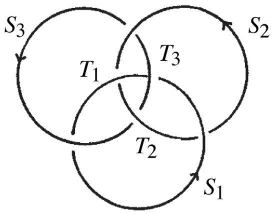
9.20 Examples. (a) For the link of Figure 9.1:
$\mathfrak{R} = ((1 - S_1S_2S_1^{-1})^{\varphi\psi}, (S_1 - S_1S_2S_1^{-1}S_2^{-1})^{\varphi\psi}) = (1-t_2, t_1-1)$ and $\Delta = 1$.



$$\mathfrak{G} = \langle S_1, S_2 \mid S_1S_2S_1^{-1}S_2^{-1}\rangle$$

Figure 9.1. The Hopf link.

(b) Borromean link (Figure 9.2).



Generators: S_1, S_2, S_3, T_1, T_2, T_3;

Relations: $T_1^{-1}S_2^{-1}T_1T_2$, $T_2^{-1}S_3T_2T_3^{-1}$, $S_2^{-1}S_3S_2T_3^{-1}$, $S_1^{-1}S_3T_1S_3^{-1}$, $S_1^{-1}S_2S_1T_2^{-1}$.

Figure 9.2. The Borromean link.

Eliminate $T_1 = S_3^{-1}S_1S_3$, $T_2 = S_1^{-1}S_2S_1$ and $T_3 = S_2^{-1}S_3S_2$, and obtain the presentation

$$\mathfrak{G} = \langle S_1, S_2, S_3 | S_3^{-1}S_1^{-1}S_3S_2^{-1}S_3^{-1}S_1S_3S_1^{-1}S_2S_1,\ S_1^{-1}S_2^{-1}S_1S_3S_1^{-1}S_2S_1S_2^{-1}S_3^{-1}S_2\rangle.$$

From this we get

$$\mathfrak{R} = \begin{pmatrix} -t_1^{-1}t_3^{-1} + t_1^{-1}t_2^{-1}t_3^{-1} - t_1^{-1}t_2^{-1} + t_1^{-1}, 0,\ -t_3^{-1} + t_1^{-1}t_3^{-1} - t_1^{-1}t_2^{-1}t_3^{-1} + t_2^{-1}t_3^{-1} \\ -t_1^{-1} + t_1^{-1}t_2^{-1} - t_1^{-1}t_2^{-1}t_3 + t_1^{-1}t_3,\ -t_1^{-1}t_2^{-1} + t_1^{-1}t_2^{-1}t_3 - t_2^{-1}t_3 + t_2^{-1},\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -t_1^{-1}t_2^{-1}t_3^{-1}(t_2-1)(t_3-1)\ ,\ 0\ ,\ -t_1^{-1}t_2^{-1}t_3^{-1}(t_2-1)(t_1-1) \\ t_1^{-1}t_2^{-1}(t_2-1)(t_3-1)\ ,\ -t_1^{-1}t_2^{-1}(t_1-1)(t_3-1)\ ,\ 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned}
\Delta_1' &= -t_1^{-2}t_2^{-2}t_3^{-1}(t_1-1)(t_2-1)(t_3-1)(t_1-1) = -\Delta\cdot(t_1-1)\\
\Delta_2' &= t_1^{-2}t_2^{-2}t_3^{-1}(t_1-1)(t_2-1)(t_3-1)(t_2-1) = \Delta\cdot(t_2-1)\\
\Delta_3' &= t_1^{-2}t_2^{-2}t_3^{-1}(t_1-1)(t_2-1)(t_3-1)(t_3-1) = \Delta\cdot(t_3-1),
\end{aligned}$$

where $\Delta = \Delta(t_1,t_2,t_3) = (t_1-1)(t_2-1)(t_3-1)$.

9.E Alexander–Conway polynomial

We have shown in Theorem 8.26 that the Alexander polynomial $\Delta(t)$ of a knot may be written as a polynomial with integral coefficients in $u = t + t^{-1} - 2$, $\Delta(t) = f(u)$. Hence, $\Delta(t^2)$ is a polynomial in $z = t - t^{-1}$. (It is even a polynomial in z^2.) J. H. Conway [71] defined a polynomial $\nabla_{\mathfrak{k}}(z)$ with integral coefficients for (oriented) links which can be inductively computed from a regular projection of a link $\mathfrak{k}$ in the following way:

9.21 Conway potential function.

(1) $\nabla_{\mathfrak{k}}(z) = 1$, if $\mathfrak{k}$ is the trivial knot.

(2) $\nabla_{\mathfrak{k}}(z) = 0$, if $\mathfrak{k}$ is a split link.

(3) $\nabla_{\mathfrak{k}_+} - \nabla_{\mathfrak{k}_-} = z\cdot\nabla_{\mathfrak{k}_0}$, if $\mathfrak{k}_+$, $\mathfrak{k}_-$, and $\mathfrak{k}_0$ differ by a local operation of the kind depicted in Figure 9.3.

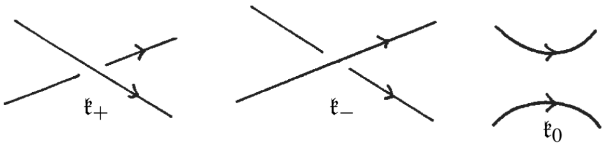


Figure 9.3. A skein relation.

Changing overcrossings into undercrossings eventually transforms any regular projection into that of a trivial knot or splittable link (compare the paragraph after Definition 2.2). Equation (3) of 9.21 may therefore be used as an algorithm (*Conway algorithm*) to compute $\nabla_{\mathfrak{k}}(z)$ with initial conditions (1) and (2). Thus, if there is a function $\nabla_{\mathfrak{k}}(z)$ satisfying conditions (1), (2), (3) which is an invariant of the link, it must be unique. The relation (3) between $\nabla_{\mathfrak{k}_+}(z)$, $\nabla_{\mathfrak{k}_-}(z)$ and $\nabla_{\mathfrak{k}_0}(z)$ is called a *skein relation*. We shall come back to this topic in a more general context in Chapter 17.

9.22 Proposition. *Let $\mathfrak{k} \subset S^3$ be an oriented knot or link.*

(a) *There is a unique integral polynomial $\nabla_{\mathfrak{k}}(z)$ satisfying* (1), (2), (3) *of* 9.21*; it is called the* Conway potential function *and is an invariant of the link.*

(b) $\nabla_{\mathfrak{k}}(t - t^{-1}) \doteq \Delta(t^2)$ *for* $\mu = 1$ *and* $\nabla_{\mathfrak{k}}(t - t^{-1}) \doteq (t^2 - 1)^{\mu-1} \bigtriangledown^H (t^2)$ *for* $\mu > 1$.

(Here μ is the number of components of $\mathfrak{k}$, $\Delta(t)$ denotes the Alexander polynomial, and $\bigtriangledown^H(t)$ the Hosokawa polynomial of $\mathfrak{k}$, see 9.19.)

Observe that the equations which relate $\nabla_{\mathfrak{k}}(t-t^{-1})$ with the Alexander polynomial and the Hosokawa polynomial suffice to show the invariance of $\nabla_{\mathfrak{k}}(z)$ in the case of knots, whereas for $\mu > 1$ there remains the ambiguity of the sign.

9.23 Proposition. *Let $\mathfrak{k} \subset S^3$ be an oriented knot and link.*

Then the function $\Omega_{\mathfrak{k}}(t) = \det(tV - t^{-1}V^T)$ is the (unique) Conway potential function for any Seifert matrix V of $\mathfrak{k}$ i.e.

- *$\Omega_{\mathfrak{k}}(t)$ is an invariant of ambient isotopy of $\mathfrak{k}$;*
- *$\Omega_{\mathfrak{k}}(t) = 1$ is $\mathfrak{k}$ is the trivial knot and $\Omega_{\mathfrak{k}}(t) = 0$ if $\mathfrak{k}$ is a split link;*
- *$\Omega_{\mathfrak{k}_+}(t) - \Omega_{\mathfrak{k}_-}(t) = (t - t^{-1}) \cdot \Omega_{\mathfrak{k}_0}(t)$, if $\mathfrak{k}_+$, $\mathfrak{k}_-$, and $\mathfrak{k}_0$ differ by a local operation of the kind as depicted in Figure 9.3.*

To prove that $\det(tV - t^{-1}V^T)$ is a link invariant, we use a result of K. Murasugi [261].

9.24 Definition (s-equivalence). Two square *integral matrices* are *s-equivalent* if they are related by a finite chain of the following operations and their inverses:

$$\Lambda_1 : V \mapsto \quad L^T V L,\ L \text{ unimodular},$$

$$\Lambda_2 : V \mapsto \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & V & \\ 0 & 0 & & & \end{array}\right), \qquad \Lambda_3 : V \mapsto \left(\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & * & & & \\ \vdots & \vdots & & V & \\ 0 & * & & & \end{array}\right).$$

9.25 Theorem (Murasugi [261]). *Two Seifert matrices obtained from two equivalent knots (or links) are s-equivalent.* □

There are several proofs of this theorem in the literature. We refer to Kawauchi's book [190, Theorem 5.2.3]. There are elementary treatments in Murasugi's book [265, Theorem 5.4.1] and in Cromwell's book [74, 5.5, 6.6]. Alternative proofs can be also found in Gordon [133], Rice [304] and in Kauffman's book [187, VII]. See also Levine [212] for a high-dimensional version.

Proof of Proposition 9.23. Let $\mathfrak{k}$ be a link of multiplicity μ, and S any Seifert surface spanning it. As in the case of a knot, one may use S to construct the infinite cyclic covering C_∞ of $\mathfrak{k}$ corresponding to the normal subgroup $\mathfrak{N} = \ker \chi\varphi$ of 9.19. There is a band projection of $\mathfrak{k}$ (see Proposition 8.2), and $H_1(C_\infty)$ – as a $\mathbb{Z}(t)$-module – is defined by a presentation matrix $(V^T - tV)$ where V is the Seifert matrix of the band projection.

It is easily checked that for s-equivalent Seifert matrices V_1 and V_2 the equation

$$\det(tV_1 - t^{-1}V_1^T) = \det(tV_2 - t^{-1}V_2^T)$$

holds. Therefore, Murasugi's Theorem 9.25 implies that $\Omega_{\mathfrak{k}}(t)$ is an invariant of ambient isotopy of $\mathfrak{k}$.

By Proposition 8.21, Paragraph 9.19 and E 9.5,

$$\begin{aligned} \Omega_{\mathfrak{k}}(t) &\doteq \Delta(t^2) \quad \text{for a knot,} \\ \Omega_{\mathfrak{k}}(t) &\doteq (t^2-1)^{\mu-1} \nabla^H (t^2) \quad \text{for a link.} \end{aligned}$$

Moreover $\Omega_{\mathfrak{k}}(1) = |V - V^T| = 1$. This proves $\Omega_{\mathfrak{k}}(t) = 1$ for the trivial knot. For a split link the reduced Alexander polynomial vanishes i.e. $0 = \Delta(t,\dots,t) = (t-1)^{\mu-2} \cdot \nabla^H(t)$, (see Corollary 9.17, Paragraph 9.19). It remains to prove the last statement. If $\mathfrak{k}_+$ is split, so is $\mathfrak{k}_-$ and $\mathfrak{k}_0$, and all functions are zero. Figure 9.4 demonstrates the position of the Seifert surfaces S_+, S_-, S_0 in the region where a change occurs. (An orientation of a Seifert surface induces the orientation of the knot). We may assume that the projection of $\mathfrak{k}_0$ is not split, because otherwise $\Omega_{\mathfrak{k}_0} = 0$, and $\mathfrak{k}_+, \mathfrak{k}_-$ are isotopic. If the projection of $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{k}_0$ are all not split, then the change from $\mathfrak{k}_0$ to $\mathfrak{k}_+$ or $\mathfrak{k}_-$ adds a free generator a to $H_1(S_0)$: $H_1(S_+) \cong \langle a \rangle \oplus H_1(S_0) \cong H_1(S_-)$. Likewise $H_1(S^3 - S_\pm) \cong H_1(S^3 - S_0) \oplus \langle s \rangle$, see Figure 9.4.

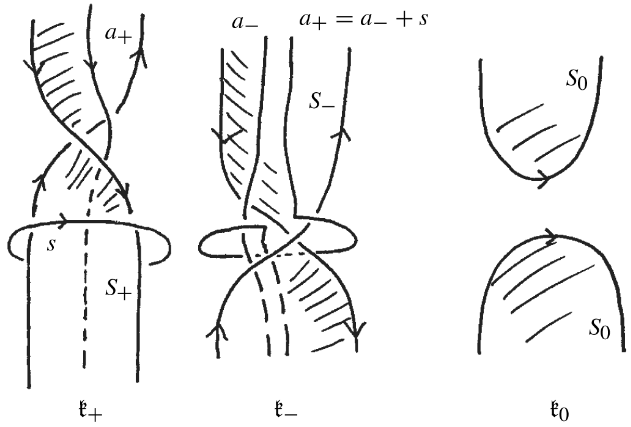


Figure 9.4

We denote by V_+, V_-, V_0 the Seifert matrices of $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{k}_0$ which correspond to the connected Seifert surfaces obtained from the projections as described in Proposition 2.4. It follows that

$$V_+ = V_- + \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right), \qquad V_- = \left(\begin{array}{c|ccc} * & * & \cdots & * \\ \hline * & & & \\ \vdots & & V_0 & \\ * & & & \end{array}\right)$$

where the first column and first row correspond to the generators s and $a_\pm$. The rest is a simple calculation:

$$\begin{aligned} \Omega_{\mathfrak{k}_+}(t) - \Omega_{\mathfrak{k}_-}(t) &= \left| tV_+ - t^{-1}V_+^T \right| - \left| tV_- - t^{-1}V_-^T \right| \\ &= \left|\begin{array}{c|ccc} t - t^{-1} & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & tV_0 - t^{-1}V_0^T & \\ 0 & & & \end{array}\right| = (t - t^{-1})\, \Omega_{\mathfrak{k}_0}(t)\,. \quad \square \end{aligned}$$

9.26 Remarks. (a) For a knot $\mathfrak{k}$ the polynomial $\Delta_S(t) = \det(t^{1/2}V - t^{-1/2}V^T)$ is called the *symmetrized* Alexander polynomial of $\mathfrak{k}$. It is characterized by the two properties $\Delta_S(t) = \Delta_S(t^{-1})$ and $\Delta_S(1) = 1$.

(b) It is possible to introduce a Conway potential function in μ variables corresponding to the Alexander polynomials of links rather than to the Hosokawa polynomial (see Hartely [149]). The function is defined as a certain normalized Alexander polynomial $\Delta(t_1^2, \dots, t_n^2) \cdot t_1^{\mu_1} \dots t_n^{\mu_n}$ where the μ_i are determined by curvature and linking numbers. Invariance is checked by considering Reidemeister moves.

9.27 More invariants. Let V be a Seifert matrix obtained from a surface spanning the knot or link $\mathfrak{k}$. By virtue of Murasugi's Theorem 9.25 it is easy to see that the absolute value of the determinant of $V + V^T$ is an invariant of $\mathfrak{k}$ (see E 9.9). It is called the *determinant* of $\mathfrak{k}$ and will be denoted by $\det(\mathfrak{k})$. The same remark applies to the signature of $V + V^T$ which is called the *signature* of $\mathfrak{k}$ and will be denoted by $\sigma(\mathfrak{k})$. We will come back to these invariants in Section 13.D.

9.F Finite cyclic coverings again

The theory of Fox derivations may also be utilized to compute the homology of finite branched cyclic coverings of knots. (For notations and results compare 8.36–8.42, 9.1.)

Let C_N, $0 < N \in \mathbb{Z}$, be the N-fold cyclic (unbranched) covering of the complement C. We know (see Proposition 8.39 (d)) that $(V^T - tV)s = 0$ are defining relations of $H_1(\widehat{C}_N)$ as a $\mathfrak{Z}_N$-module, $\mathfrak{Z}_N = \langle t \mid t^N \rangle$.

9.28 Proposition. (a) *Any Alexander matrix* $A(t)$ *(which is a presentation matrix of* $H_1(C_\infty)$ *as a* $\mathfrak{Z}$*-module,* $\mathfrak{Z} = \langle\, t\,\rangle$*) is a presentation matrix of* $H_1(\widehat{C}_N)$ *as a* $\mathfrak{Z}_N$*-module.* $\mathfrak{Z}_N = \langle\, t, \mid t^N\,\rangle$.

(b) *The matrix*

$$\begin{pmatrix} & A(t) & & \\ \varrho_N & 0 & \dots & 0 \\ 0 & \varrho_N & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \varrho_N \end{pmatrix} = B_N(t), \quad \varrho_N = 1 + t + \ldots + t^{N-1},$$

is a presentation matrix of $H_1(\widehat{C}_N)$ *as a* $\mathfrak{Z}$*-module.*

Proof. The first assertion follows from the fact that, if two presentation matrices $A(t)$ and $A'(t)$ are equivalent over $\mathbb{Z}\mathfrak{Z}$, they are equivalent over $\mathbb{Z}\mathfrak{Z}_N$. The second version is a consequence of Proposition 8.39 (b). Observe that $(t^N - 1) = \varrho_N(t)(t-1)$. □

9.29 Corollary. *The homology groups* $H_1(\widehat{C}_N)$ *of the* N*-fold cyclic branched coverings of a torus knot* $\mathfrak{t}(a,b)$ *are periodic with the period ab:*

$$H_1(\widehat{C}_{N+kab}) \cong H_1(\widehat{C}_N),\ k \in \mathbb{N}.$$

Moreover

$$H_1(\widehat{C}_N) \cong H_1(\widehat{C}_{N'}) \quad \textit{if } N' \equiv -N \mod ab.$$

Proof. By Proposition 9.28 (b), $B_N(t) = \begin{pmatrix} \Delta(t) \\ \varrho_N(t) \end{pmatrix}$ is a presentation matrix for the $\mathbb{Z}(t)$-module $H_1(\widehat{C}_N)$. Since $\Delta(t) | \varrho_{ab}(t)$ and $\varrho_{N+kab} = \varrho_N + t^N \cdot \varrho_k(t^{ab}) \cdot \varrho_{ab}$, the presentation matrices $B_N(t)$ and $B_{N+kab}(t)$ are equivalent. The second assertion is a consequence of

$$\varrho_{ab} - \varrho_N = t^N \cdot \varrho_{ab-N} \quad \text{for } 0 < N < ab.$$ □

9.30 Example. For the trefoil $\mathfrak{t}(3,2)$ the homology groups of the cyclic branched coverings are:

$$H_1(\widehat{C}_N) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } N \equiv 0 \mod 6 \\ 0 & \text{for } N \equiv \pm 1 \mod 6 \\ \mathbb{Z}_3 & \text{for } N \equiv \pm 2 \mod 6 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } N \equiv 3 \mod 6. \end{cases}$$

Proof.

$$N \equiv 0 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 0 \end{pmatrix} \sim (1-t+t^2).$$

$$N \equiv 1 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 1 \end{pmatrix} \sim (1).$$

$$N \equiv 2 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 1+t \end{pmatrix} \sim \begin{pmatrix} 3 \\ 1+t \end{pmatrix}$$

$$N \equiv 3 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 1+t+t^2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1+t+t^2 \end{pmatrix}.$$

$N \equiv 0$: $H_1(\widehat{C}_N) \cong \langle s \rangle \oplus \langle ts \rangle$ where s is the generator.
$N \equiv 1$: $H_1(\widehat{C}_N) = 0$.
$N \equiv 2$: $H_1(\widehat{C}_N) \cong \langle s \mid 3s \rangle$.
$N \equiv 3$: $H_1(\widehat{C}_N) \cong \langle s \mid 2s \rangle \oplus \langle ts \mid 2ts \rangle$.

□

9.31 Remark. In the case of a two-fold covering $\widehat{C}_2$ we get a result obtained already in Proposition 8.39 (a):

$$B_2(t) = \begin{pmatrix} & & A(t) & \\ 1+t & & & \\ & 1+t & & \\ & & \ddots & \\ & & & 1+t \end{pmatrix} \sim A(-1).$$

Proposition 8.39 gives a presentation matrix for $H_1(\widehat{C}_N)$ as an Abelian group (8.39 (a)) derived from the presentation matrix $A(t) = (V^T - tV)$ for $H_1(\widehat{C}_N)$ as a $\mathfrak{Z}_N$-module. This can also be achieved by the following trick: blow up $A(t)$ by replacing every matrix element $r_{ik}(t) = \sum_j c_{ik}^{(j)} t^j$ by an $N \times N$-matrix $R_{ik} = \sum_j c_{ik}^{(j)} \mathfrak{T}_N^j$,

$$\mathfrak{T}_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

This means introducing N generators $s_i, ts_i, \dots, t^{N-1}s_i$ for each generator s_i, observing $t(t^\nu s_i) = t^{\nu+1}s_i$, $t^N = 1$. The blown up matrix is a presentation matrix of $H_1(\widehat{C}_N)$ as an Abelian group. For practical calculations of $H_1(\widehat{C}_N)$ this procedure is not very useful, because of the high order of the matrices. It may be used, though, to give an alternative proof of Theorem 8.41, see Neuwirth [270, 5.3.1].

9.G History and sources

Homotopy chains were first introduced by K. Reidemeister [298], and they were used by K. Reidemeister and W. Franz to classify lens spaces [299, 118]. R. H. Fox gave an algebraic foundation and generalization of the theory in his free differential calculus [107, 108, 109], and introduced it to knot theory. Most of the content of this chapter is connected with the work of R. H. Fox. In connection with the Alexander polynomials of links the contribution of R. H. Crowell and D. Strauss [75] and J. Hillman [167] should be mentioned.

9.H Exercises

E 9.1. Show that

(1) the augmentation ideal $I\mathfrak{Z}^\mu$ of $\mathbb{Z}\mathfrak{Z}^\mu$ is generated as a $\mathbb{Z}\mathfrak{Z}^\mu$-module by the elements $(t_i - 1)$, $1 \le i \le \mu$,

(2) $\mathbb{Z}\mathfrak{Z}^\mu$ is a unique factorization ring with no divisors of zero,

(3) the units of $\mathbb{Z}\mathfrak{Z}^\mu$ are $\pm g$, $g \in \mathfrak{Z}^\mu$.

E 9.2. The Alexander module of a 2-bridge knot $\mathfrak{b}(a,b)$ is cyclic. Deduce from this that $\Delta_k(t) = 1$ for $k > 1$.

E 9.3. Let $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}' = \langle\, t\,\rangle$ be the abelianizing homomorphism of the group $\mathfrak{G} = \langle\, x, y | x^a y^{-b}\,\rangle$ of a torus knot $\mathfrak{t}(a,b)$. Show that $x^\varphi = t^b$, $y^\varphi = t^a$.

E 9.4. Compute the Alexander polynomial $\Delta(t_1, t_2)$ of the two component link $\mathfrak{k}_1 \cup \mathfrak{k}_2$, where $\mathfrak{k}_1$ is a torus knot, $\mathfrak{k}_1 = \mathfrak{t}(a,b)$, and $\mathfrak{k}_2$ the core of the solid torus T on whose boundary $\mathfrak{t}(a,b)$ lies. Hint: Prove that $\langle\, x, y, z \mid [x,z], x^a y^{-b} z^b\,\rangle$ is a presentation of the group of $\mathfrak{k}_1 \cup \mathfrak{k}_2$.

Result: $\Delta(t_1, t_2) = \frac{(t_1^a t_2)^b - 1}{t_1^a t_2 - 1}$.

E 9.5. Let C_∞ be the infinite cyclic covering of a link $\mathfrak{k}$ of μ components (see 9.19). Show that $H_1(C_\infty)$ has a presentation matrix of the form $(V^t - tV)$ with

$$V - V^T = F' = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

F is a $2g \times 2g$ matrix (g the genus of $\mathfrak{k}$), and the order of F' is $2g + \mu - 1$. Deduce from this that the reduced Alexander polynomial of $\mathfrak{k}$ is divisible by $(t-1)^{\mu-2}$ (compare 9.19), and from this $H_1(\widehat{C}_2; \mathbb{Z}_2) = \bigoplus_{i=1}^{\mu-1} \mathbb{Z}_2$.

Prove that $|\nabla^H(1)|$ equals the absolute value of a $(\mu-1)\times(\mu-1)$ principal minor of the linking matrix $(\mathrm{lk}(\mathfrak{k}_i, \mathfrak{k}_j))$, $1 \le i, j \le \mu$. Show that $\nabla^H(t)$ is symmetric.

E 9.6. Compute the Alexander polynomial of the twist knot $\mathfrak{d}_n$ with n half-twists (Figure 9.5). (Result: $\Delta(t) = mt^2 - (2m+1)t + m$ for $n = 2m$, $\Delta(t) = mt^2 - (2m-1)t + m$ for $n = 2m-1, m = 1, 2, \ldots$)

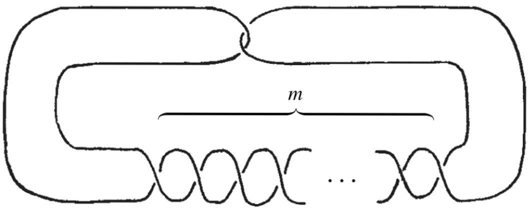


Figure 9.5. The twist knot $\mathfrak{d}_n$.

E 9.7. For $\mathfrak{F} = \langle \{s_i \mid i \in I\} | \rangle$ let l denote the usual length of words with respect to the free generators $\{s_i \mid i \in I\}$. Extend it to $\mathbb{Z}\mathfrak{F}$ by $l(n_1 x_1 + \ldots + n_k x_k) = \max\{l(x_j) \mid 1 \le j \le k, n_j \ne 0\}$; here $n_j \in \mathbb{Z}$ and $x_j \in \mathfrak{F}$ with $x_j \ne x_i$ for $i \ne j$. Introduce the following derivations:

$$\frac{\partial}{\partial s_i^{-1}} \colon \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F}, \quad s_j \mapsto \begin{cases} -s_i & \text{for } j = i, \\ 0 & \text{for } j \ne i. \end{cases}$$

Prove:

(a) $\dfrac{\partial}{\partial s_i^{-1}}(s_i^{-1}) = 1$, $\dfrac{\partial}{\partial s_i^{-1}} = -\dfrac{\partial}{\partial s_i} \cdot s_i$.

(b) $l\left(\dfrac{\partial \tau}{\partial s_i}\right) \le l(\tau)$, $l\left(\dfrac{\partial \tau}{\partial s_i^{-1}}\right) \le l(\tau)$ for all $i \in I, \tau \in \mathbb{Z}\mathfrak{F}$.

(c) $l\left(\frac{\partial}{\partial s_i^{-1}}\frac{\partial}{\partial s_i}(\tau)\right) < l(\tau)$, $l\left(\frac{\partial}{\partial s_i}\frac{\partial}{\partial s_i^{-1}}(\tau)\right) < l(\tau)$.

(d) $\frac{\partial}{\partial s_i} = \left(\frac{\partial}{\partial s_i^{-1}}\frac{\partial}{\partial s_i}\right)\cdot s_i^{-1} - \frac{\partial}{\partial s_i}\frac{\partial}{\partial s_i^{-1}}$.

E 9.8. (a) With the notation of E 9.7 prove: Let $\tau, \gamma \in \mathbb{Z}\mathfrak{F}$, $\gamma \neq 0$ and $l(\tau\gamma) \leq l(\tau)$. Then either $\gamma \in \mathbb{Z}$ or there is a s_i^δ, $i \in I$, $\delta \in \{1, -1\}$ such that $l(\tau s_i^\delta) \leq l(\tau)$. All elements $f \in \mathfrak{F}$ with $l(f) = l(\tau)$ that have a non-trivial coefficient in τ end with $s_i^{-\delta}$.

(b) If $l(\tau\gamma) < l(\tau)$ and $\gamma \neq 0$ then there is a s_i^δ, $i \in I$, $\delta \in \{1, -1\}$ such that $l(\tau s_i^\delta) < l(\tau)$.

(c) If $\tau\varrho \in \mathbb{Z}$ then either τ or ϱ is 0 or τ and ϱ have the form af with $f \in \mathfrak{F}$, $a \in \mathbb{Z}$.

E 9.9. Let V be a Seifert matrix obtained from a surface spanning the knot or link $\mathfrak{k}$. Prove that the absolute value of the determinant and the signature of $V + V^T$ are invariants of $\mathfrak{k}$ (use Theorem 9.25).

E 9.10. Let $\mathfrak{k}$ be a knot and $\Delta_S(t)$ its symmetrized Alexander polynomial. Show that $\Delta_S''(1) = a_2 \in \mathbb{Z}$ where $\nabla_{\mathfrak{k}}(z) = 1 + a_2 z^2 + a_4 z^4 + \cdots$.

Chapter 10

Braids

In this chapter we will present the basic theorems of the theory of braids including their classification or, equivalently, the solution of the word problem for braid groups, but excluding a proof of the conjugation problem. A modern account of the conjugation problem is available in the monograph *Braid Groups* by C. Kassel and V. Turaev [186].

In Section 10.C we shall consider the Fadell–Neuwirth configuration spaces which present a different aspect of the matter. Geometric reasoning will prevail, as seems appropriate in a subject of such simple beauty.

10.A The classification of braids

Braids were already defined in Chapter 2, Section 2.D. We start by defining an isotopy relation for braids, using combinatorial equivalence. We apply Δ- and Δ^{-1}-moves to the strings f_i , $1 \le i \le n$, of the braid (see Definition 1.9) assuming that each process preserves the braid properties and keeps fixed the points P_i, Q_i, $1 \le i \le n$. (See Figure 10.1.)

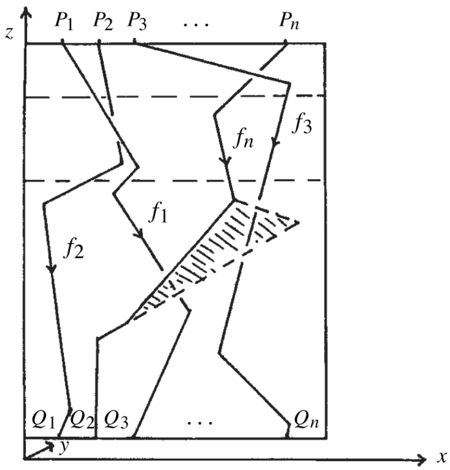


Figure 10.1. A braid in its frame.

10.1 Definition (Isotopy of braids). Two braids $\mathfrak{z}$ and $\mathfrak{z}'$ are called *isotopic* or *equivalent*, if they can be transformed into each other by a finite sequence of $\Delta^{\pm 1}$-processes.

It is obvious that a theorem similar to Proposition 1.8 can be proved. Various notions of isotopy have been introduced by Artin [11] and shown to be equivalent. As in the case of knots we shall use the term braid and the notation $\mathfrak{z}$ also for a class of equivalent braids. All braids in this section are supposed to be n-braids for some fixed $n > 1$. There is an obvious composition of two braids $\mathfrak{z}$ and $\mathfrak{z}'$ by identifying the endpoints Q_i of $\mathfrak{z}$ with the initial points P_i' of $\mathfrak{z}'$ (Figure 10.2). The composition of representatives

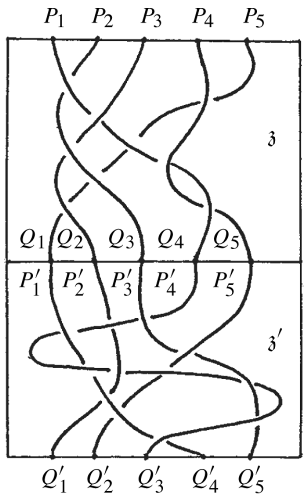


Figure 10.2. The composition of two braids.

defines a composition of equivalence classes. Since there is also a unit with respect to this composition and an inverse $\mathfrak{z}^{-1}$ obtained from $\mathfrak{z}$ by a reflection in a plane perpendicular to the braid, we obtain a group:

10.2 Proposition and Definition (Braid group $\mathfrak{B}_n$). *The isotopy classes $\mathfrak{z}$ of n-braids form a group called the braid group $\mathfrak{B}_n$.* □

We now start to look for a presentation of $\mathfrak{B}_n$. It is easy to see that $\mathfrak{B}_n$ is generated by $n-1$ generators σ_i (Figure 10.3).

For easier reference let us introduce Cartesian coordinates (x, y, z) with respect to the frames of the braids. The frames will be parallel to the plane $y = 0$ and those of their sides which carry the points P_i and Q_i will be parallel to the x-axis. Now every class of braids contains a representative such that its y-projection (onto the plane $y = 0$) has finitely many double points, all of them with different z-coordinates. Choose planes $z =$ const which bound slices of $\mathbb{R}^3$ containing parts of $\mathfrak{z}$ with just one double point in their y-projection. If the intersection points of $\mathfrak{z}$ with each of those planes $z =$ const are moved into equidistant positions on the line in which the frame meets $z = c$ (without introducing new double points in the y-projection) the braid $\mathfrak{z}$ appears as a product of the *elementary braids* σ_i, σ_i^{-1}, compare Figure 10.3.

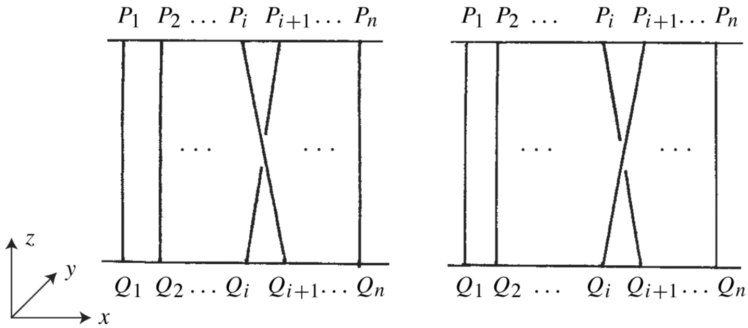


Figure 10.3. The braids σ_i and σ_i^{-1}.

To obtain defining relations for $\mathfrak{B}_n$ we proceed as we did in Chapter 3, Section 3.B, in the case of a knot group. Let $\mathfrak{z} = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$, $\varepsilon_i = \pm 1$, be a braid and consider its y-projection. We investigate how a Δ-process will effect the word $\sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$ representing $\mathfrak{z}$. We may assume that the y-projection of the generating triangle of the Δ-process contains one double point or no double points in its interior; in the second case one can assume that the projection of at most one string intersects the interior. Figure 10.4 demonstrates the possible configurations; in the first two positions it is possible to choose the triangle in a slice which contains one (Figure 10.4 (a)), or no double point (Figure 10.4 (b)) in the y-projection.

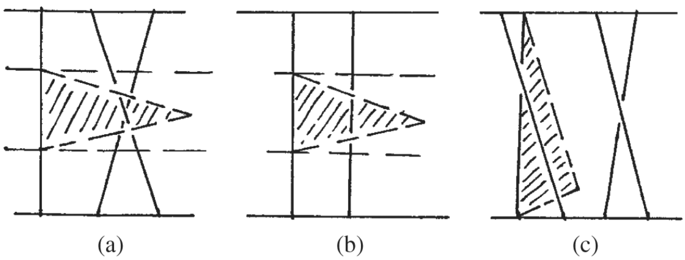


Figure 10.4. The effect of a Δ-process.

In Figure 10.4 (a), σ_{i+1} is replaced by $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1}$; (b) describes an elementary expansion, and in (c) a double point is moved along a string which may lead to a commutator relation $\sigma_i \sigma_k = \sigma_k \sigma_i$ for $|i - k| \geq 2$. It is easy to verify that any process of type (a) with differently chosen over- and undercrossings leads to the same relation $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$; (a) describes, in fact, an Ω_3-process (see Definition 1.16), and one can always think of the uppermost string as the one being moved.

10.3 Proposition (Presentation of the braid group). *The braid group* $\mathfrak{B}_n$ *can be presented as follows:*

$$\mathfrak{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_j\sigma_{j+1}\sigma_j\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+1}^{-1} \text{ for } 1 \leq j \leq n-2,$$
$$[\sigma_j, \sigma_k] \text{ for } |j-k| \geq 2 \,\rangle. \qquad \square$$

In the light of this theorem, the classification problem of braids can be understood as the word problem for $\mathfrak{B}_n$. We shall, however, solve the classification problem by a direct geometric approach and thereby reach a solution of the word problem, rather than vice versa.

As before, let (x, y, z) be the Cartesian coordinates of a point in Euclidean 3-space. We modify the geometric setting by placing the frame of the braid askew in a cuboid K. The edges of K are supposed to be parallel to the coordinate axes; the upper side of the frame which carries the points P_i coincides with an upper edge of K parallel to the x-axis, the opposite side which contains the Q_i is assumed to bisect the base-face of K (see Figure 10.5).

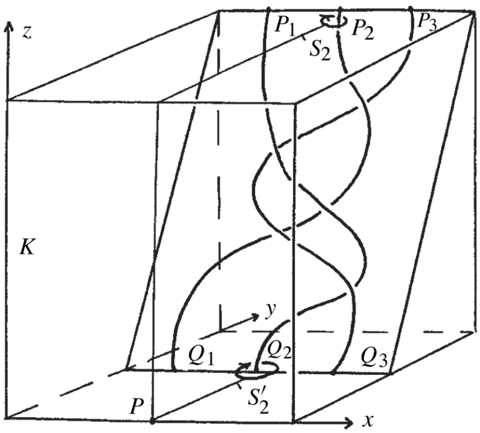


Figure 10.5. A braid in a cuboid.

10.4 Lemma. *Every class of braids contains a representative with simple* z*-projection (without double points).*

Proof. The representative $\mathfrak{z}$ of a class of braids can be chosen in such a way that its y-projection and z-projection yield the same word $\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}\ldots\sigma_{i_r}^{\varepsilon_r}$. This can be achieved by placing the strings in a neighborhood of the frame, compare 2.10.

Consider the double point in $z = 0$ corresponding to $\sigma_{i_r}^{\varepsilon_r}$, push the overcrossing along the undercrossing string over its endpoint Q_j (Figure 10.6) while preserving the z-level. Obviously this process is an isotopy of $\mathfrak{z}$ which can be carried out without disturbing the upper part of the braid which is projected onto $\sigma_{i_1}^{\varepsilon_1}\ldots\sigma_{i_{r-1}}^{\varepsilon_{r-1}}$. Proceed

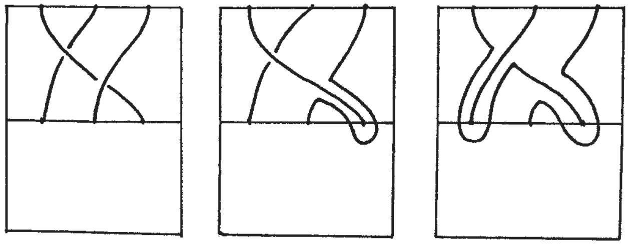


Figure 10.6

by removing the double point in $z = 0$ corresponding to $\sigma_{i_{r-1}}^{\varepsilon_{r-1}}$. The procedure eventually leads to a braid with a simple z-projection as claimed in the lemma. □

10.5 Corollary. *Every* $2m$*-plat has an m-bridge presentation.*

Proof. The corollary follows directly from Lemma 10.4 (see Figure 10.7). □

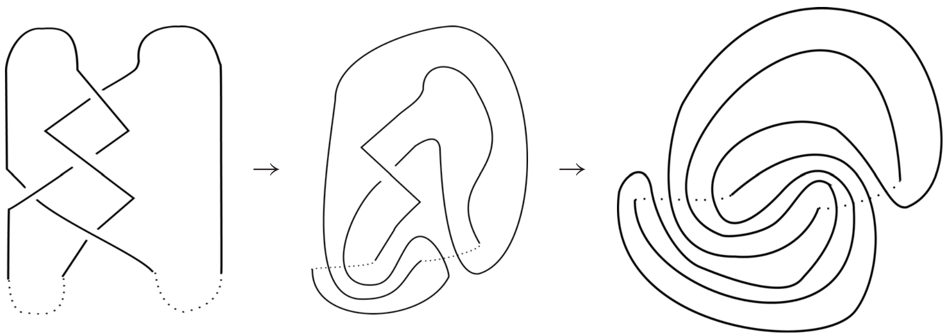


Figure 10.7

Let us denote the base-face of K by D, and the z-projections of f_i, P_i by f_i', P_i'. The simple projection of a braid then consists of a set of simple and pairwise disjoint arcs f_i' leading from P_i' to $Q_{\pi(i)}, 1 \leq i \leq n$, where π is the permutation associated with the braid $\mathfrak{z}$ (see 2.10). We call $\{f_i' \mid 1 \leq i \leq n\}$ a *normal dissection* of the punctured rectangle $D - \bigcup_{j=1}^n Q_j = D_n$. By Lemma 10.4 every braid can be represented by a set of strings which projects onto a normal dissection of D_n, and obviously every normal dissection of D_n is a z-projection of some braid in K. Two normal dissections are called isotopic if they can be transformed into each other by a sequence of $\Delta^{\pm 1}$-processes in D_n. The defining triangle of such a Δ-process intersects $\{f_i'\}$ in one or two of its sides, line segments of some f_k'. Any two braids projecting onto isotopic normal dissections evidently are isotopic. The groups $\pi_1(K - \mathfrak{z})$ as well as $\pi_1 D_n$ are free of rank n. This is clear from the fact that the projecting cylinders of a braid with a simple z-projection dissect $K - \mathfrak{z}$ into a 3-cell. Every braid $\mathfrak{z}$ in K defines two sets of free generators $\{S_i\}, \{S_i'\}, 1 \le i \le n$, of $\pi_1(K - \mathfrak{z})$: Choose a basepoint P on the x-axis and let S_i be represented by a loop on ∂K consisting of a small circle around P_i and a (shortest) arc connecting it to P. Similarly define S_i' by encircling Q_i instead of P_i (Figure 10.5).

Since every isotopy $\mathfrak{z} \mapsto \mathfrak{z}'$ can be extended to an ambient isotopy in K leaving ∂K pointwise fixed (Proposition 1.8), a class of braids defines an associated braid automorphism of $\mathfrak{F}_n \cong \pi_1(K - \mathfrak{z})$, $\zeta : \mathfrak{F}_n \to \mathfrak{F}_n$, $S_i \mapsto S_i'$. All information on ζ can be obtained by looking at the normal dissection of D_n associated to $\mathfrak{z}$. Every normal dissection defines a set of free generators of $\pi_1 D_n$. A loop in D_n which intersects $\{f_i'\}$ once positively in f_k' represents a free generator $S_k \in \pi_1 D_n$ which is mapped onto $S_k \in \pi_1(K - \mathfrak{z})$ by the isomorphism induced by the inclusion. Hence, $S_i'(S_j)$ as a word in the S_j is easily read off the normal dissection:

$$S_i' = L_i S_{\pi^{-1}(i)} L_i^{-1}. \tag{10.1}$$

To determine the word $L_i(S_j)$, run through a straight line from P to Q_i, noting down S_k or S_k^{-1} if the line is crossed by f_k' from left to right or otherwise.

The braid automorphism equation (10.1) can also be interpreted as an automorphism of $\pi_1 D_n$ with $\{S_i\}$ associated to the normal dissection $\{f_i'\}$, and $\{S_i'\}$ associated to the *standard normal dissection* consisting of the straight segments from P_i' to Q_i.

The solution of the classification problem of n-braids is contained in the following:

10.6 Proposition (Artin [10]). *Two n-braids are isotopic if and only if they define the same braid automorphism.*

Proof. Assigning a braid automorphism ζ to a braid $\mathfrak{z}$ defines a homomorphism

$$\mathfrak{B}_n \to \operatorname{Aut} \mathfrak{F}_n, \ \mathfrak{z} \mapsto \zeta.$$

To prove Proposition 10.6 we must show that this homomorphism is injective. This can be done with the help of

10.7 Lemma. *Two normal dissections define the same braid automorphism if and only if they are isotopic.*

Proof. A Δ-process does not change the $S_i' = L_i S_{\pi^{-1}(i)} L_i^{-1}$ as elements in the free group. This follows also from the fact that isotopic normal dissections are $\mathfrak{z}$-projections of isotopic braids, and the braid automorphism is assigned to the braid class. Now let $\{f_i'\}$ be some normal dissection of D_n and $S_i' = L_i S_{\pi^{-1}(i)} L_i^{-1}$ read off it as described before. If $L(S_i)$ contains a part of the form $S_j^{\varepsilon} S_j^{-\varepsilon}$, the two points on f_i' corresponding to S_j^{ε} and $S_j^{-\varepsilon}$ are connected by two simple arcs on f_i' and the loop in D_n representing S_i'. These arcs bound a 2-cell in D which contains no point Q_k, because otherwise $f_{\pi^{-1}(k)}'$ would have to meet one of the arcs which is impossible. Hence the two arcs bound a 2-cell in D_n, and there is an isotopy moving f_j' across it causing the elementary contraction in L_i which deletes $S_j^{\varepsilon} S_j^{-\varepsilon}$. Thus we can replace the normal dissection by an isotopic one such that the corresponding words $L_i(S_j)$ are reduced. Similarly we can assume $L_i S_{\pi^{-1}(i)} L_i^{-1}$ to be reduced. If the last symbol of $L_i(S_j)$ is $S^{\varepsilon}_{\pi^{-1}(i)}$, there is an isotopy of f_i' which deletes $S^{\varepsilon}_{\pi^{-1}(i)}$ in $L_i(S_j)$ (Figure 10.8).

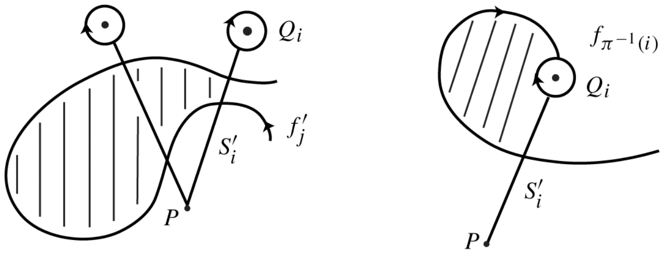


Figure 10.8

Suppose now that two normal dissections $\{f_i'\}$, $\{f_i''\}$ define the same braid automorphism $S_i \mapsto S_j' = L_i S_{\pi^{-1}(i)} L_i^{-1}$. Assume the $L_i S_{\pi^{-1}(i)} L_i^{-1}$ to be reduced, and let the points of intersection of $\{f_i'\}$ and $\{f_i''\}$ with the loops representing the S_j' coincide. It follows that two successive intersection points on some f_k' are also successive on f_k'', and, hence, the two connecting arcs on f_k' resp. f_k'' can be deformed into each other by an isotopy of $\{f_i'\}$. This is clear if $\{f_i'\}$ is the standard normal dissection and this suffices to prove Lemma 10.7. □

We return to the proof of Proposition 10.6. Let $\mathfrak{z}$ and $\mathfrak{z}'$ be n-braids inducing the same braid automorphism. By Lemma 10.4 we may assume that their z-projections are simple. Lemma 10.7 ensures that the z-projections are isotopic; hence $\mathfrak{z}$ and $\mathfrak{z}'$ are isotopic. □

Proposition 10.6 solves, of course, the word problem of the braid group $\mathfrak{B}_n$: *Two braids $\mathfrak{z}$, $\mathfrak{z}'$ are isotopic if and only if their automorphisms coincide* – a matter which can be checked easily, since $\mathfrak{F}_n$ is free.

Proposition 10.6 and Lemma 10.7 moreover imply that there is a one-to-one correspondence between braids, braid automorphisms and isotopy classes of normal dissections. These classes represent elements of *the mapping class group of D_n*; its elements are homeomorphisms of D_n which keep ∂D_n pointwise fixed, modulo deformations of D_n.

The injective image of $\mathfrak{B}_n$ in the group Aut $\mathfrak{F}_n$ of automorphisms of the free group of rank n is called the group of braid automorphisms. We shall also denote it by $\mathfrak{B}_n$. The injection $\mathfrak{B}_n \to \operatorname{Aut} \mathfrak{F}_n$ depends on a set of distinguished free generators S_i of $\mathfrak{F}_n$. It is common use to stick to these distinguished generators or rather their class

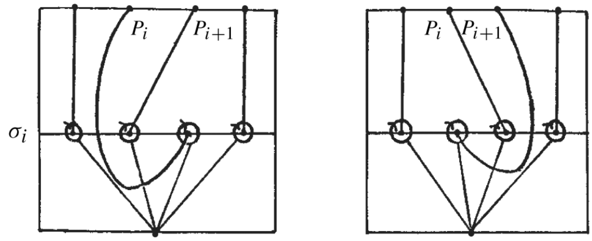


Figure 10.9. The normal dissections of σ_i and σ_i^{-1}.

modulo braid automorphisms, and braid automorphisms will always be understood in this way. We propose to study these braid automorphisms more closely.

Figure 10.9 illustrates the computations of the braid automorphisms corresponding to the elementary braids $\sigma_i^{\pm 1}$ – we denote the automorphisms by the same symbols:

$$\sigma_i(S_j) = S'_j = \begin{cases} S_i S_{i+1} S_i^{-1}, & j = i \\ S_i, & j = i+1 \\ S_j, & j \neq i, i+1 \end{cases} \tag{10.2}$$

$$\sigma_i^{-1}(S_j) = S'_j = \begin{cases} S_{i+1}, & j = i \\ S_{i+1}^{-1} S_i S_{i+1}, & j = i+1 \\ S_j, & j \neq i, i+1. \end{cases} \tag{10.3}$$

From these formulas the identity in $\mathfrak{F}_n$

$$\prod_{i=1}^{n} S'_i = \prod_{i=1}^{n} S_i \tag{10.4}$$

follows for any braid automorphism $\zeta\colon\ S_i \to S'_i$, as well as equation (10.1)

$$S'_i = L_i S_{\pi^{-1}(i)} L_i^{-1}. \tag{10.5}$$

This is also geometrically evident, since $\prod S_i$ as well as $\prod S'_i$ is represented by a loop which girds the whole braid.

At this point it seems necessary to say a few words about the correct interpretation of the symbols σ_i. If $\mathfrak{z} = \sigma_{i_1}^{\varepsilon_1} \ldots \sigma_{i_r}^{\varepsilon_r}$ is understood as a braid, the composition is defined from left to right. Denote by $\mathfrak{z}_k = \sigma_{i_1}^{\varepsilon_1} \ldots \sigma_{i_k}^{\varepsilon_k}$, $0 \leq k \leq r$, the k-th *initial section* of $\mathfrak{z}$ and by ζ_k the braid automorphism associated to $\mathfrak{z}_k$ (operating on the original generators S_i). The injective homomorphism $\mathfrak{B}_n \to \mathrm{Aut}\mathfrak{F}_n$ then maps a factor $\sigma_{i_j}^{\varepsilon_j}$ of $\mathfrak{z}$ onto an automorphism of $\mathfrak{F}_n$ defined by equation (10.2) where $\zeta_{j-1}(S_i)$ takes the place of S_i.

There is an identity in the free group generated by the $\{\sigma_i\}$:

$$\mathfrak{z} = \prod_{k=1}^{r} \sigma_{i_k}^{\varepsilon_k} = \prod_{k=1}^{r} \mathfrak{z}_{r-k} \sigma_{i_{r-k+1}}^{\varepsilon_{r-k+1}} \mathfrak{z}_{r-k}^{-1},\ \mathfrak{z}_0 = 1.$$

The automorphism $\zeta_{r-k}\sigma_{i_{r-k+1}}^{\varepsilon_{r-k+1}}\zeta_{r-k}^{-1}$ (carried out from right to left!) is the automorphism $\sigma_{i_{r-k+1}}^{\varepsilon_{r-k+1}}$ defined by equation (10.2) on the original generators S_i (from the top of the braid). We may therefore understand $\mathfrak{z} = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$ either as a product (from right to left) of automorphisms $\sigma_{i_j}^{\varepsilon_j}$ in the usual sense, or, performed from left to right, as a successive application of a rule for a substitution according to equation (10.2) with varying arguments. The last one was originally employed by Artin, and it makes the mapping $\mathfrak{B}_n \to \operatorname{Aut}\mathfrak{F}_n$ a homomorphism rather than an anti-homomorphism. The two interpretations are dual descriptions of the same automorphism.

Braid automorphisms of $\mathfrak{F}_n(S_i)$ can be characterized by equation (10.1) and equation (10.4). Artin [10] even proved a slightly stronger theorem where he does not presuppose that the given substitution is an automorphism:

10.8 Proposition. *Let $\mathfrak{F}_n(S_j)$ be a free group on a given set $\{S_j \mid 1 \leq j \leq n\}$ of free generators, and let π be a permutation on $\{1, 2, \dots, n\}$. Any set of words $S_i'(S_j), 1 \leq i \leq n$, subject to the following conditions:*

(a) $S_i' = L_i S_{\pi(i)} L_i^{-1}$,

(b) $\prod_{i=1}^n S_i' = \prod_{i=1}^n S_i$,

generates $\mathfrak{F}_n$; the homomorphism defined by $S_i \mapsto S_i'$ is a braid automorphism.

Proof. Assume S_i' to be reduced and call $\lambda(\zeta) = \sum_{i=1}^n l(L_i)$ the length of the substitution $\zeta\colon S_i \to S_i'$, where $l(L_i)$ denotes the length of L_i. If $\lambda = 0$, it follows from equation (10.4) that ζ is the identity. We proceed by induction on λ. For $\lambda > 0$ there will be reductions in

$$\prod_{i=1}^n S_i' = L_1 S_{\pi(1)} L_1^{-1} \dots L_n S_{\pi(n)} L_n^{-1}$$

such that some $S_{\pi(i)}$ is cancelled by $S_{\pi(i)}^{-1}$ contained in L_{i-1}^{-1} or L_{i+1}. (If all L_i cancel out, they have to be all equal, and hence empty, since otherwise at least one of the words S_i' would not be reduced). Suppose L_{i+1} cancels $S_{\pi(i)}$, then

$$l(L_i S_{\pi(i)} L_i^{-1} L_{i+1}) < l(L_{i+1}).$$

Apply σ_i to S_j', $\sigma_i(S_j') = S_j''$, to obtain $\lambda(\zeta\sigma_i) < \lambda(\zeta)$ while $\zeta\sigma_i$ still fulfills conditions (10.1) and (10.4). Thus, by induction, $\zeta\sigma_i$ is a braid automorphism and so is ζ. (If $S_{\pi(i)}$ is cancelled by L_{i-1}^{-1}, one has to use σ_{i-1}^{-1} instead of σ_i.) □

10.B Normal form and group structure

We have derived a presentation of the braid group $\mathfrak{B}_n$, and solved the word problem by embedding $\mathfrak{B}_n$ into the group of automorphisms of the free group of rank n. For some additional information on the group structure of $\mathfrak{B}_n$ first consider the surjective homomorphism of the braid group onto the symmetric group:

$$\mathfrak{B}_n \to \mathfrak{S}_n, \ \mathfrak{z} \mapsto \pi,$$

which assigns to each braid $\mathfrak{z}$ its permutation π. We propose to study the kernel $\mathfrak{I}_n \lhd \mathfrak{B}_n$ of this homomorphism.

10.9 Definition (Pure braids). The group $\mathfrak{I}_n \lhd \mathfrak{B}_n$ is the *pure braid group with n strings*. A braid of $\mathfrak{I}_n$ is called a *pure i-braid* if there is a representative with the strings f_j, $j \neq i$, constant (straight lines), and if its y-projection only contains double points concerning f_i and f_j, $j < i$, see Figure 10.10.

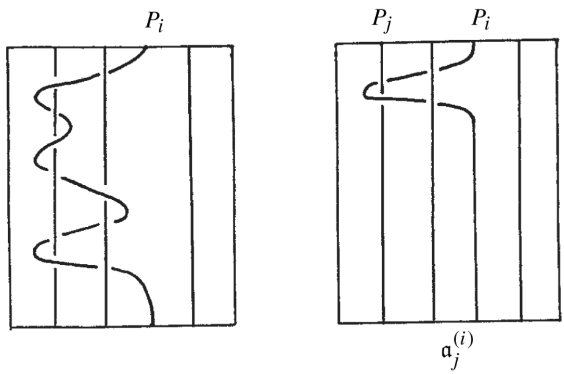


Figure 10.10. Pure i-braids.

10.10 Proposition. *The pure i-braids of $\mathfrak{I}_n$ form a free subgroup $\mathfrak{F}^{(i)}$ of $\mathfrak{I}_n$ of rank $i-1$*

Proof. It is evident that $\mathfrak{F}^{(i)}$ is a subgroup of $\mathfrak{I}_n$. Furthermore $\mathfrak{F}^{(i)}$ is obviously generated by the braids $\mathfrak{a}_j^{(i)}$, $1 \leq j < i$, as defined in Figure 10.10. Let $\mathfrak{z}^{(i)} \in \mathfrak{F}^{(i)}$ be an arbitrary pure i-braid. Note down $(\mathfrak{a}_{j_k}^{(i)})^{\varepsilon_k}$ as you traverse f_i at every double point in the y-projection where f_i overcrosses f_{j_k}, while choosing $\varepsilon_k = +1$ resp. $\varepsilon_k = -1$ according to the characteristic of the crossing. Then $\mathfrak{z}^{(i)} = \mathfrak{a}_{j_1}^{(i)\varepsilon_1}\mathfrak{a}_{j_2}^{(i)\varepsilon_2}\dots\mathfrak{a}_{j_r}^{(i)\varepsilon_r}$.

It is easy to see that the $\mathfrak{a}_j^{(i)}$ are free generators. It follows from the fact that the loops formed by the strings f_i of $\mathfrak{a}_j^{(i)}$ combined with an arc on ∂K can be considered as free generators of $\pi_1(K - \bigcup_{j=1}^{i-1} f_j) \cong \mathfrak{F}^{(i-1)}$. $\square$

10.11 Proposition. *The subgroup $\mathfrak{B}_{i-1} \subset \mathfrak{B}_n$ generated by $\{\sigma_r \mid 1 \le r \le i-2\}$ acts on $\mathfrak{F}^{(i)}$ by conjugation. For $1 \le j \le i-1$ we have*

$$\sigma_r^{-1}\mathfrak{a}_j^{(i)}\sigma_r = \begin{cases} \mathfrak{a}_j^{(i)}, & j \ne r, r+1, \\ \mathfrak{a}_r^{(i)}\mathfrak{a}_{r+1}^{(i)}(\mathfrak{a}_r^{(i)})^{-1}, & j = r, \\ \mathfrak{a}_r^{(i)}, & j = r+1. \end{cases}$$

The *proof* is given in Figure 10.11. □

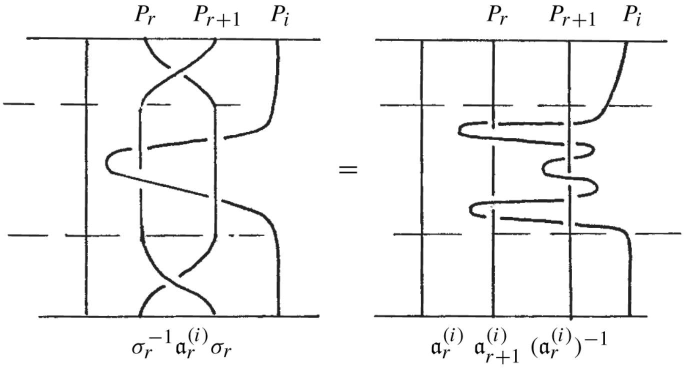


Figure 10.11

It is remarkable that σ_r induces on $\mathfrak{F}^{(i)}$ the braid automorphism σ_r with respect to the free generators $\mathfrak{a}_j^{(i)}$.

The following theorem describes the group structure of $\mathfrak{I}_n$ to a certain extent:

10.12 Proposition. *The braids $\mathfrak{z}$ of $\mathfrak{I}_n$ admit a unique decomposition:*

$$\mathfrak{z} = \mathfrak{z}_2 \cdots \mathfrak{z}_n, \ \mathfrak{z}_i \in \mathfrak{F}^{(i)}, \ \mathfrak{F}^{(1)} = 1.$$

This decomposition is called the normal form of $\mathfrak{z}$. There is a product rule for normal forms:

$$\Big(\prod_{i=2}^{n} \mathfrak{x}_i\Big)\Big(\prod_{i=2}^{n} \mathfrak{y}_i\Big) = (\mathfrak{x}_2\mathfrak{y}_2)(\mathfrak{x}_3^{\eta_2}\mathfrak{y}_3)\dots(\mathfrak{x}_n^{\eta_{n-1}\dots\eta_3\eta_2}\mathfrak{y}_n),$$

where η_i denotes the braid automorphism associated to the braid $\mathfrak{y}_i \in \mathfrak{F}^{(i)}$.

Proof. The existence of a normal form for $\mathfrak{z} \in \mathfrak{I}_n$ is an immediate consequence of Lemma 10.4. One has to realize $\mathfrak{z} \in \mathfrak{I}_n$ from a simple z-projection by letting first f_n ascend over its z-projection while representing the f_j, $j < n$, by straight lines over the endpoints Q_j. This defines the factor $\mathfrak{z}_n$. The remaining part of f_n is projected

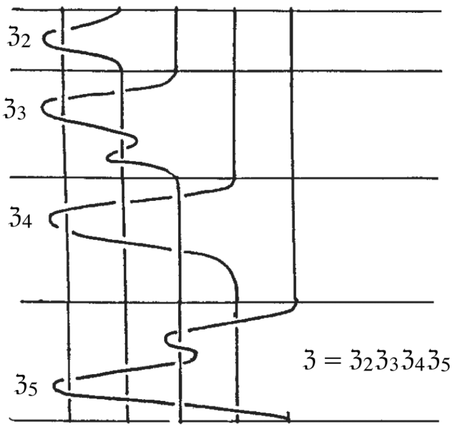


Figure 10.12. The normal form of $\mathfrak{z}$, $\mathfrak{z}_i \in \mathfrak{F}^{(i)}$.

onto P_n' and therefore has no effect on the rest of the braid. Thus the existence of the normal form follows by induction on n, Figure 10.12.

The product rule is a consequence of Proposition 10.11. Uniqueness follows from the fact that, if $\mathfrak{x}_2 \dots \mathfrak{x}_n \cdot \mathfrak{y}_n^{-1} \dots \mathfrak{y}_2^{-1} = 1$, then $(\mathfrak{x}_n \cdot \mathfrak{y}_n^{-1})^{\eta_2^{-1} \dots \eta_{n-1}^{-1}}$ is its component in $\mathfrak{F}^{(n)}$. Its string f_n is homotopic to some arc on ∂K in $K - \bigcup_{j=1}^{n-1} f_j$; hence $(\mathfrak{x}_n \cdot \mathfrak{y}_n^{-1})^{\eta_2^{-1} \dots \eta_{n-1}^{-1}} = 1$, $\mathfrak{x}_n = \mathfrak{y}_n$. The rest follows by induction. □

The normal form affords some insight into the structure of $\mathfrak{J}_n$. By definition $\mathfrak{F}^{(1)} = 1$; the group $\mathfrak{J}_n$ is a repeated semidirect product of free groups with braid automorphisms operating according to Proposition 10.10:

$$\mathfrak{J}_n = \mathfrak{F}^{(1)} \ltimes (\mathfrak{F}^{(2)} \ltimes (\dots (\mathfrak{F}^{(n-1)} \ltimes \mathfrak{F}^{(n)}) \dots)).$$

There is some more information contained in the normal form:

10.13 Proposition. *The pure braid group $\mathfrak{J}_n$ contains no elements $\neq 1$ of finite order. The center of $\mathfrak{J}_n$ and of $\mathfrak{B}_n$ is an infinite cyclic group generated by $(\sigma_1\sigma_2 \dots \sigma_{n-1})^n$ for $n > 2$.*

Proof. Let $\mathfrak{z} \in \mathfrak{J}$ and suppose for the normal forms of $\mathfrak{z}$ and $\mathfrak{z}^m$

$$(\mathfrak{x}_2 \dots \mathfrak{x}_n)^m = \mathfrak{y}_2 \dots \mathfrak{y}_n = 1, m > 1.$$

By Proposition 10.12, $\mathfrak{y}_2 = (\mathfrak{x}_2)^m = 1$. Now $\mathfrak{x}_2 = 1$ follows from Proposition 10.10. In the same way we get $\mathfrak{x}_i = 1$ successively for $i = 3, 4, \dots, n$. This proves the first assertion.

The braid $\mathfrak{z}^0 = (\sigma_1\sigma_2 \dots \sigma_{n-1})^n$ of Figure 10.13 obviously is an element of the center $Z(\mathfrak{B}_n)$. It is obtained from the trivial braid by a full twist of the lower side of

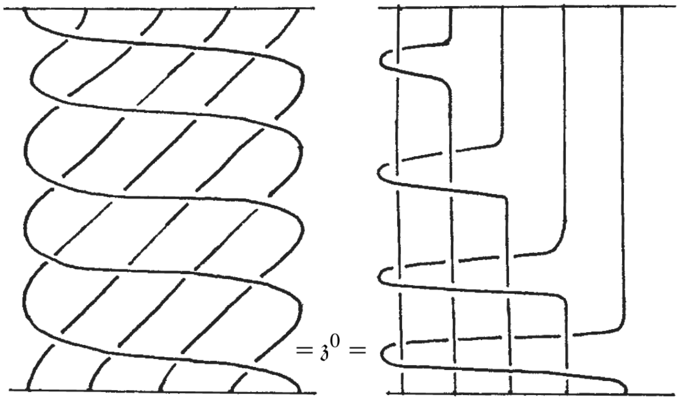


Figure 10.13. The generator of the center of $\mathfrak{B}_n$.

the frame while keeping the upper one fixed. The normal form of $\mathfrak{z}^0$ is given on the right of Figure 10.13:

$$\mathfrak{z}^0 = \mathfrak{z}_2^0 \dots \mathfrak{z}_n^0, \quad \mathfrak{z}_i^0 = \mathfrak{a}_1^{(i)} \dots \mathfrak{a}_{i-1}^{(i)}.$$

(For the definition of $\mathfrak{a}_j^{(i)}$ see Figure 10.10.) It is easily verified that $\mathfrak{z}^0$ determines the braid automorphism

$$\zeta^0\colon\ S_i \mapsto \Big(\prod_{j=1}^{n} S_j\Big) S_i \Big(\prod_{j=1}^{n} S_j\Big)^{-1},$$

and that by equation (10.4) $\mathfrak{B}_n \cap \mathfrak{T}_n = \langle \zeta^0 \rangle \cong \mathfrak{Z}$, $\mathfrak{T}_n$ the inner automorphisms of $\mathfrak{F}_n$.

Note that Proposition 10.11 yields, for $1 \le i \le r < n$

$$\begin{aligned}(\sigma_1 \dots \sigma_{r-1})^{-r} \mathfrak{a}_i^{(r+1)} (\sigma_1 \dots \sigma_{r-1})^r &= (\mathfrak{a}_i^{(r+1)})^{(\sigma_1 \dots \sigma_{r-1})^r} \\ &= (\mathfrak{a}_1^{(r+1)} \dots \mathfrak{a}_r^{(r+1)}) \mathfrak{a}_i^{(r+1)} (\mathfrak{a}_1^{(r+1)} \dots \mathfrak{a}_r^{(r+1)})^{-1}. \quad (10.6)\end{aligned}$$

For $n > 2$ the symmetric group $\mathfrak{S}_n$ has a trivial center, hence, $Z(\mathfrak{B}_n) < Z(\mathfrak{I}_n)$ for the centers $Z(\mathfrak{B}_n)$ and $Z(\mathfrak{I}_n)$ of $\mathfrak{B}_n$ and $\mathfrak{I}_n$. We may therefore write an arbitrary central element $\mathfrak{z}$ of $\mathfrak{I}_n$ or $\mathfrak{B}_n$ in normal form $\mathfrak{z} = \mathfrak{z}_2 \dots \mathfrak{z}_n$, $\mathfrak{z}_i \in \mathfrak{F}^{(i)}$. (We denote by ζ_i, ξ_i, η_i the braid automorphisms associated to the braids $\mathfrak{z}_i, \mathfrak{x}_i, \mathfrak{y}_i$.)

For every $\mathfrak{x}_3 \in \mathfrak{F}^{(3)}$:

$$\mathfrak{z}_2\, \mathfrak{x}_3^{\zeta_2}\, \mathfrak{z}_3 \dots \mathfrak{z}_n = \mathfrak{x}_3\, \mathfrak{z}_2 \dots \mathfrak{z}_n = \mathfrak{z}_2 \dots \mathfrak{z}_n\, \mathfrak{x}_3 = \mathfrak{z}_2\, \mathfrak{z}_3\, \mathfrak{x}_3\, \mathfrak{z}_4^{\xi_3} \dots \mathfrak{z}_n^{\xi_3}.$$

It follows that $\mathfrak{x}_3^{\zeta_2}\, \mathfrak{z}_3 = \mathfrak{z}_3\, \mathfrak{x}_3$, or $\mathfrak{x}_3^{\zeta_2} = \mathfrak{z}_3\, \mathfrak{x}_3\, \mathfrak{z}_3^{-1}$. Now $\mathfrak{z}_2 = (\mathfrak{a}_1^{(2)})^k = (\mathfrak{z}_2^0)^k$ for some $k \in \mathbb{Z}$. Apply equation (10.6) for $r = 2$, $\zeta_2 = \sigma_1^{2k}$: $(\mathfrak{a}_i^{(3)})^{\sigma_1^{2k}} =$

$(\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k\mathfrak{a}_i^{(3)}(\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^{-k}$. Hence, for every $\mathfrak{x}_3 \in \mathfrak{F}^{(3)}$:

$$\mathfrak{z}_3\,\mathfrak{x}_3\,\mathfrak{z}_3^{-1} = \mathfrak{x}_3^{\sigma_1^{2k}} = (\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k\mathfrak{x}_3(\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^{-k}.$$

Since $\mathfrak{F}^{(3)}$ is free with basis $\mathfrak{a}_1^{(3)}$, $\mathfrak{a}_2^{(3)}$, we get $\mathfrak{z}_3 = (\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k$.

The next step determines $\mathfrak{z}_4$ by the following property. For $\mathfrak{x}_4 \in \mathfrak{F}^{(4)}$:

$$\mathfrak{x}_4\,\mathfrak{z}_2\cdots\mathfrak{z}_n = \mathfrak{z}_2\,\mathfrak{z}_3\,\mathfrak{x}_4^{\zeta_3\zeta_2}\,\mathfrak{z}_4\cdots\mathfrak{z}_n = \mathfrak{z}_2\cdots\mathfrak{z}_n\,\mathfrak{x}_4 = \mathfrak{z}_2\,\mathfrak{z}_3\,\mathfrak{z}_4\,\mathfrak{x}_4\,\mathfrak{z}_5^{\xi_4}\cdots\mathfrak{z}_n^{\xi_4}.$$

The uniqueness of the normal form gives:

$$\mathfrak{x}_4^{\zeta_3\zeta_2} = \mathfrak{z}_4\,\mathfrak{x}_4\,\mathfrak{z}_4^{-1}.$$

The braids $\mathfrak{a}_1^{(2)}$ and $\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)}$ commute – draw a figure – and so do $\mathfrak{z}_2$ and $\mathfrak{z}_3$ and the corresponding automorphisms: $\zeta_2\,\zeta_3 = \zeta_3\,\zeta_2$.

We already know $\mathfrak{z}_2\,\mathfrak{z}_3 = (\mathfrak{a}_1^{(2)})^k(\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k = (\mathfrak{a}_1^{(2)}\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k$ hence $\zeta_2\,\zeta_3 = (\sigma_1\sigma_2)^{3k}$. By equation (10.6) we get for all $\mathfrak{x}_4 \in \mathfrak{F}^{(4)}$:

$$\begin{aligned}\mathfrak{z}_4\,\mathfrak{x}_4\,\mathfrak{z}_4^{-1} = \mathfrak{x}_4^{\zeta_3\zeta_2} &= (\sigma_1\sigma_2)^{-3k}\,\mathfrak{x}_4(\sigma_1\sigma_2)^{3k}\\ &= (\mathfrak{a}_1^{(4)}\mathfrak{a}_2^{(4)}\mathfrak{a}_3^{(4)})^k\,\mathfrak{x}_4\,(\mathfrak{a}_1^{(4)}\mathfrak{a}_2^{(4)}\mathfrak{a}_3^{(4)})^{-k}\end{aligned}$$

and hence, $\mathfrak{z}_4 = (\mathfrak{a}_1^{(4)}\mathfrak{a}_2^{(4)}\mathfrak{a}_3^{(4)})^k$. The procedure yields $\mathfrak{z}_i = (\mathfrak{z}_i^0)^k$, $\mathfrak{z} = (\mathfrak{z}^0)^k$. □

The braid group $\mathfrak{B}_n$ itself is also torsion free. This was first proved by E. Fadell and L. Neuwirth [94]. A different proof was given by K. Murasugi [264]. We discuss these proofs in Section 10.C.

10.C Configuration spaces and braid groups

In Fadell and Neuwirth [94] and Fox and Neuwirth [106] a different approach to braids was developed. We shall prove some results of it here. For details the reader is referred to the papers mentioned above.

A braid $\mathfrak{z}$ meets a plane $z = c$ in n points $(p_1, p_2, \ldots, p_n)$ if $0 \le c \le 1$, and $z = 1$ $(z = 0)$ contains the initial points P_i (endpoints Q_i) of the strings f_i. One may therefore think of $\mathfrak{z}$ as a simultaneous motion of n points in a plane E^2, $\{(p_1(t), \ldots, p_n(t)) \mid 1 \le t \le 1\}$. We shall construct a $2n$-dimensional manifold where $(p_1, \ldots, p_n)$ represents a point and $(p_1(t), \ldots, p_n(t))$ a loop such that the braid group $\mathfrak{B}_n$ becomes its fundamental group.

Every n-tuple $(p_1, \ldots, p_n)$ represents a point $P = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ in Euclidean $2n$-space E^{2n}, where (x_i, y_i) are the coordinates of $p_i \in E^2$. Let $i \prec j$ stand for the inequality $x_i < x_j$, $i \hat{=} j$ for $x_i = x_j$, $y_i < y_j$, and $i = j$ for $x_i = x_j$, $y_i = y_j$. Any distribution of these symbols in a sequence, e.g. $\pi(1) =$

$\pi(2) \hat{=} \pi(3) \prec \pi(4) \ldots \pi(n)$, $\pi \in \mathfrak{S}_n$, then describes a set of linear inequalities and, hence, a convex subset of E^{2n}. Obviously these cells form a cell division of E^{2n}. There are $n!$ cells of dimension $2n$, defined by $(\pi(1) \prec \pi(2) \prec \ldots \prec \pi(n))$.

The dimension of a cell defined by some sequence is $2n$ minus the sum of the number of occurrences of $\hat{=}$ and twice the number of occurrences of $=$. The permutations $\pi \in \mathfrak{S}_n$ under $\pi(p_1, \ldots, p_n) = (p_{\pi(1)}, \ldots p_{\pi(n)})$ form a group of cellular operations on E^{2n}. The quotient space $\hat{E}^{2n} = E^{2n}/\mathfrak{S}_n$ inherits the cell decomposition. The following example shows how we denote the projected cells:

$$(\pi(1) \prec \pi(2) \hat{=} \pi(3) \ldots = \pi(n)) \mapsto (\prec \hat{=} \ldots =).$$

(Just omit the numbers $\pi(i)$.) $\mathfrak{S}_n$ operates freely an $E^{2n} - \Lambda$, where Λ is the $(2n-2)$-dimensional subcomplex consisting of cells defined by sequences in which the sign = occurs at least once. The projection $q\colon E^{2n} \to \hat{E}^{2n}$ then maps Λ onto a $(2n-2)$-subcomplex $\hat{\Lambda}$ of $\hat{E}^{2n}$ and $q\colon E^{2n} - \Lambda \to \hat{E}^{2n} - \hat{\Lambda}$ describes a regular covering of an open $2n$-dimensional manifold with $\mathfrak{S}_n$ as its group of covering transformations. $\hat{E}^{2n}$ is called a *configuration space*.

10.14 Proposition. $\pi_1(\hat{E}^{2n} - \hat{\Lambda}) \cong \mathfrak{B}_n$, $\pi_1(E^{2n} - \Lambda) \cong \mathfrak{I}_n$.

Proof. Choose a basepoint $\hat{P}$ in the (one) $2n$-cell of $\hat{E}^{2n} - \hat{\Lambda}$ and some P, $q(P) = \hat{P}$. A braid $\mathfrak{z} \in \mathfrak{B}_n$ then defines a loop in $\hat{E}^{2n} - \hat{\Lambda}$, with basepoint $\hat{P} = q(P_1, \ldots, P_n) = q(Q_1, \ldots, Q_n)$. Two such loops $\mathfrak{z}_t = q(p_1(t), \ldots, p_n(t))$, $\mathfrak{z}'_t = q(p'_1(t), \ldots, p'_n(t))$, $0 \le t \le 1$, are homotopic relative to $\hat{P}$, if there is a continuous family $\mathfrak{z}_t(s), 0 \le s \le 1$, with $\mathfrak{z}_t(0) = \mathfrak{z}_t$, $\mathfrak{z}_t(1) = \mathfrak{z}'_t$, $\mathfrak{z}_0(s) = \mathfrak{z}_1(s) = \hat{P}$. This homotopy relation $\mathfrak{z}_t \sim \mathfrak{z}'_t$ coincides with E. Artin's definition of s-isotopy for braids $\mathfrak{z}_t, \mathfrak{z}'_t$ [11].

It can be shown by using simplicial approximation arguments that s-isotopy is equivalent to the notion of isotopy as defined in Definition 10.1, which would prove 10.14. We shall omit the proof, instead we show that $\pi_1(\hat{E}^{2n} - \hat{\Lambda})$ can be computed directly from its cell decomposition (see R. H. Fox and L. Neuwirth [106]).

We already chose the base point $\hat{P}$ in the interior of the only $2n$-cell $\hat{\lambda} = (\prec \ldots \prec)$. There are $n-1$ cells $\hat{\lambda}_i$ of dimension $2n-1$ corresponding to sequences where the sign $\hat{=}$ occurs once $(\prec \ldots \hat{=} \ldots \prec)$ at the i-th position.

Think of $\hat{P}$ as a 0-cell dual to $\hat{\lambda}$, and denote by σ_i, $1 \le i \le n-1$, the 1-cells dual to $\hat{\lambda}_i$. By a suitable choice of the orientation σ_i will represent the elementary braid. Figure 10.14 describes a loop σ_i which intersects $\hat{\lambda}_i$ at $t = \frac{1}{2}$.

It follows that the σ_i are generators of $\pi_1(\hat{E}^{2n} - \hat{\Lambda})$. Defining relations are obtained by looking at the 2-cells $\hat{r}_{ik}$ dual to the $(2n-2)$-cells $\hat{\lambda}_{ik}$ of $\hat{E}^{2n} - \hat{\Lambda}$ which are characterized by sequences in which exactly two signs $\hat{=}$ occur:

$$\hat{\lambda}_{ik} = (\prec \ldots \prec \hat{=} \prec \ldots \prec \hat{=} \prec \ldots)$$

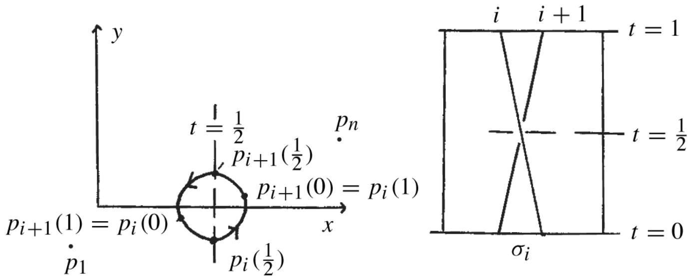


Figure 10.14. The elementary braid σ_i.

at position i and k, $1 \leq i < k \leq n-1$. The geometric situation will be quite different in the two cases $k = i + 1$ and $k > i + 1$.

Consider a plane γ transverse to $\hat{\lambda}_{i,i+1}$ in $\hat{E}^{2n} - \hat{\Lambda}$. One may describe it as the plane defined by the equations $x_i + x_{i+1} + x_{i+2} = 0$, $x_j = 0$, $j \neq i, i+1, i+2$. Figure 10.15 shows γ as an (x_i, x_{i+1})-plane with lines defined by $x_i = x_{i+1}$, $x_i = x_{i+2}$, $x_{i+1} = x_{i+2}$.

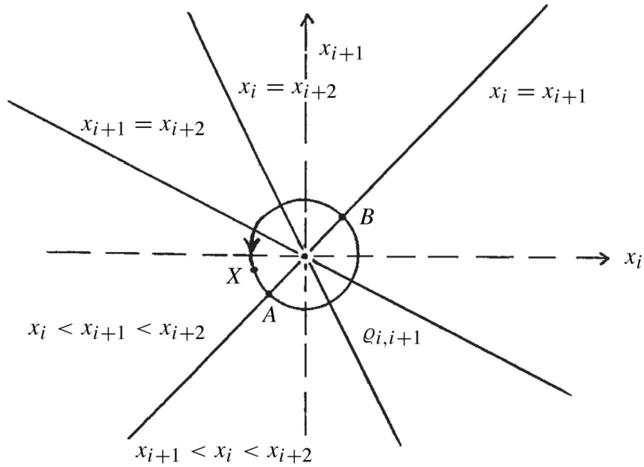


Figure 10.15. The relation which corresponds to the 2-cell $\hat{r}_{i,i+1}$.

The origin of the (x_i, x_{i+1})-plane is $\gamma \cap \hat{\lambda}_{i,i+1}$ and the half rays of the lines are $\gamma \cap \hat{\lambda}_j$, $i \leq j \leq i + 2$. We represent the points of $\gamma \cap \hat{\lambda}_{i,i+1}$ by ordered triples. We choose some point X in $x_i < x_{i+1} < x_{i+2}$ to begin with, and let it run along a simple closed curve $\varrho_{i,i+1}$ around the origin (Figure 10.15). Traversing $x_i = x_{i+1}$ corresponds to a generator $\sigma_i = (\ldots \hat{=} \prec \ldots)$, the point on $\varrho_{i,i+1}$ enters the $2n$-cell $x_{i+1} < x_i < x_{i+2}$ after that. Figure 10.16 describes the whole circuit $\varrho_{i,i+1}$.

Thus we get: $\varrho_{i,i+1} = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$. Whether to use σ_i or σ_i^{-1} can be decided in the following way. In the cross-section γ coordinates x_j, y_j different

$x_i < x_{i+1} < x_{i+2}$

$(\ldots \hat{=} \prec \ldots) \Leftrightarrow \sigma_i$

$x_{i+1} < x_i < x_{i+2}$

$(\cdots \prec \hat{=} \ldots) \Leftrightarrow \sigma_{i+1}$

$x_{i+1} < x_{i+2} < x_i$

$(\ldots \hat{=} \prec \ldots) \Leftrightarrow \sigma_i$

$x_{i+2} < x_{i+1} < x_i$

$(\cdots \prec \hat{=} \ldots) \Leftrightarrow \sigma_{i+1}^{-1}$

$x_{i+2} < x_i < x_{i+1}$

$(\ldots \hat{=} \prec \ldots) \Leftrightarrow \sigma_i^{-1}$

$x_{i+1} < x_{i+2} < x_{i+1}$

$(\cdots \prec \hat{=} \ldots) \Leftrightarrow \sigma_{i+1}^{-1}$

$x_i < x_{i+1} < x_{i+2}$

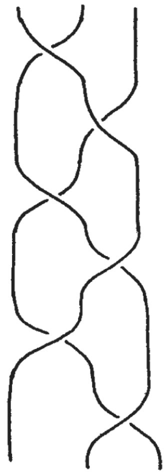


Figure 10.16. The circuit $\varrho_{i,i+1} = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$.

from x_i, x_{i+1}, x_{i+2} are kept fixed. Thus we have always $y_i < y_{i+1} < y_{i+2}$. Now Figure 10.17 shows the movement of the points $p_i, p_{i+1}, p_{i+2} \in E^2$ at the points A and B of Figure 10.15.

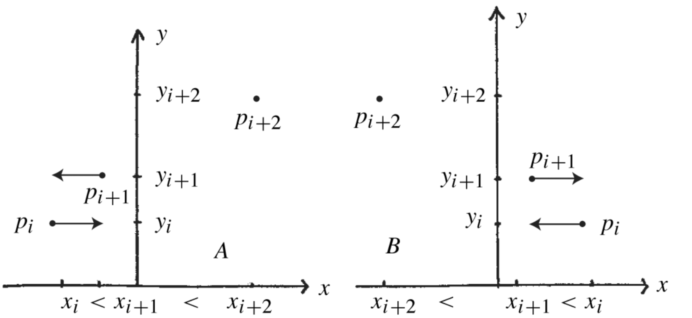


Figure 10.17

The same procedure applies to the case $k > i+1$. Here the cross-section to $\hat{\lambda}_{i,k}$ can be described by the solutions of the equations $x_i + x_{i+1} = x_k + x_{k+1} = 0$. We use an (x_i, x_k)-plane and again $\gamma \cap \hat{\lambda}_{i,k}$ is the origin and the coordinate half-rays represent $\gamma \cap \hat{\lambda}_i$, $\gamma \cap \hat{\lambda}_k$ (Figure 10.18).

It is left to the reader to verify for $i+1 < k$ that

$$\varrho_{ik} = \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}.$$

The boundaries are $\partial \hat{r}_{ik}$ homotopic to ϱ_{ik}, thus we have again obtained the standard presentation of the braid group (see Proposition 10.3), $\mathfrak{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \varrho_{ik} (1 \leq i < k \leq n-1) \rangle$. By definition $\pi_1(E^{2n} - \Lambda) \cong \mathfrak{Z}_n$. □

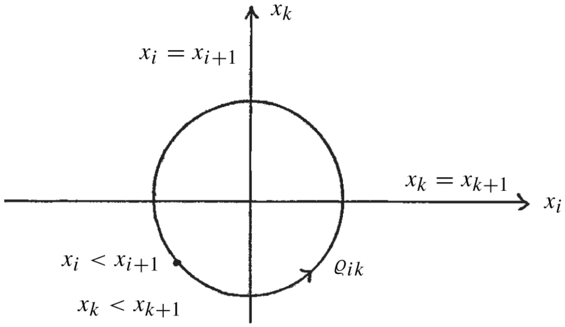


Figure 10.18. The circuit $\varrho_{ik} = \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}$ for $i + 1 < k$.

A presentation of $\mathfrak{I}_n$ might be obtained in the same way by studying the cell complex $E^{2n} - \Lambda$, but it is more easily derived from the normal form (Proposition 10.12).

Fadell and Neuwirth [94] have shown that $\hat{E}^{2n} - \hat{\Lambda}$ is aspherical; in fact, $\hat{E}^{2n} - \hat{\Lambda}$ is a $2n$-dimensional open manifold and a $K(\mathfrak{B}_n, 1)$ space. From this it follows by the argument used in Theorem 3.40 that $\mathfrak{B}_N$ has no elements $\neq 1$ of finite order.

10.15 Proposition. *The braid group $\mathfrak{B}_n$ is torsion free.*

We give a proof of this theorem using a result of F. Waldhausen [365].

Proof. Let V be a solid torus with meridian m and longitude ℓ and $\hat{\mathfrak{z}} \subset V$ a closed braid derived from an n-braid $\mathfrak{z}$ of finite order k, $\mathfrak{z}^k = 1$. The embedding $\hat{\mathfrak{z}} \subset V$ is chosen in such a way that $\hat{\mathfrak{z}}$ meets each meridional disk D of V at exactly n points. For some open tubular neighborhood $U(\hat{\mathfrak{z}})$,

$$\pi_1(V - U(\hat{\mathfrak{z}})) \cong \mathfrak{Z} \ltimes \pi_1 D_n, \quad \text{with } D_n = D \cap (V - U(\hat{\mathfrak{z}})),$$

where $\mathfrak{Z}(= \langle t \rangle)$ resp. $\pi_1 D_n (= \mathfrak{N})$ are free groups of rank 1 resp. n. The generator t can be represented by the longitude ℓ (compare Corollary 5.5). There is a k-fold cyclic covering

$$p\colon (\hat{V} - \hat{U}(\hat{\mathfrak{z}}^k)) \to (V - U(\hat{\mathfrak{z}}))$$

corresponding to the normal subgroup $\langle t^k \rangle \ltimes \mathfrak{N} \lhd \langle t \rangle \ltimes \mathfrak{N}$. Now $\langle t^k \rangle \ltimes \mathfrak{N} = \langle t^k \rangle \times \mathfrak{N}$ since $\mathfrak{z}^k$ is the trivial braid. From this it follows that $\pi_1(V - U(\hat{\mathfrak{z}}))$ has a non-trivial center containing the infinite cyclic subgroup $\langle t^k \rangle$ generated by t^k which is not contained in $\mathfrak{N} \cong \pi_1 D_n$. ($D_n$ is an incompressible surface in $V - U(\hat{\mathfrak{z}})$.)

By Waldhausen [365, Satz 4.1] $V - U(\hat{\mathfrak{z}})$ is a Seifert fiber space, $\langle t^k \rangle$ is the center of $\pi_1(V - U(\hat{\mathfrak{z}}))$ and t^k represents a fiber $\simeq \ell^k$. The fibration of $V - U(\hat{\mathfrak{z}})$ can be extended to a fibration of V [59, Theoreme 2]. This means that $\hat{\mathfrak{z}}$ is a torus link $\mathfrak{t}(a,b) = \hat{\mathfrak{z}}$. It follows that $\hat{\mathfrak{z}}^k = \mathfrak{t}(a, kb)$. Since $\mathfrak{z}^k$ is trivial, we get $kb = 0$, $b = 0$, and $\mathfrak{z} = 1$. □

The proof given above is a special version of an argument used in the proof of a more general theorem by K. Murasugi [264].

10.D Braids and links

In Chapter 2, Section 2.D we described the procedure of *closing* a braid $\mathfrak{z}$ (see Figure 2.12). The closed braid obtained from $\mathfrak{z}$ is denoted by $\hat{\mathfrak{z}}$ and its axis by h.

10.16 Definition. Two *closed braids* $\hat{\mathfrak{z}}$, $\hat{\mathfrak{z}}'$ in $\mathbb{R}^3$ are called *equivalent*, if they possess a common axis h, and if there is an orientation preserving homeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, $f(\hat{\mathfrak{z}}) = \hat{\mathfrak{z}}'$, which keeps the axis h pointwise fixed. Of course, $\mathbb{R}^3$ may again be replaced by S^3 and the axis by a trivial knot. E. Artin [10] already noticed the following:

10.17 Proposition. *Two closed braids $\hat{\mathfrak{z}}$, $\hat{\mathfrak{z}}'$ are equivalent if and only if $\mathfrak{z}$ and $\mathfrak{z}'$ are conjugate in $\mathfrak{B}_n$.*

Proof. If $\mathfrak{z}$ and $\mathfrak{z}'$ are conjugate, the equivalence of $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$ is evident. Observe that a closed braid $\hat{\mathfrak{z}}$ can be obtained from several braids which differ by a cyclic permutation of their words in the generators σ_i, and hence are conjugate.

If $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$ are equivalent, we may assume that the homeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, $f(\hat{\mathfrak{z}}) = \hat{\mathfrak{z}}'$, is constant outside a sufficiently large cube containing $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$. Since h is also kept fixed, we may choose an unknotted solid torus V containing $\hat{\mathfrak{z}}$, $\hat{\mathfrak{z}}'$ and restrict f to $f\colon V \to V$ with $f(x) = x$ for $x \in \partial V$. (We already used this construction at the end of the preceding section.) Let t again be a longitude of ∂V, and $\mathfrak{F}_n \cong \pi_1 D_n$ the free group of rank n where D_n is a disk with n holes. There is a homeomorphism $z\colon D_n \to D_n$, $z|\partial D_n = \mathrm{id}$, inducing the braid automorphism ζ of $\mathfrak{z}$, and $V - U(\hat{\mathfrak{z}}) = (D_n \times I)/z$, $\pi_1(V - U(\hat{\mathfrak{z}})) \cong \langle t \rangle \ltimes \mathfrak{F}_n$, compare 5.2, Proposition 10.6 and Lemma 10.7. For the presentation

$$\pi_1(V - U(\hat{\mathfrak{z}})) = \langle t, u_i \mid t u_i t^{-1} = \zeta(u_i) \rangle,\ 1 \le i \le n,$$

choose a basepoint on $\partial V \cap D_n$ and define the generators $\{u_i\}$ of $\pi_1 D_n$ by a normal dissection of D_n (see 10.4).

The automorphism ζ is then defined with respect to these geometrically distinguished generators up to conjugation in the group of braid automorphisms. The class of braid automorphisms conjugate to ζ is then invariant under the mapping

$$f\colon (V - U(\hat{\mathfrak{z}})) \to (V - U(\hat{\mathfrak{z}}')).$$

and, hence, the defining braids $\mathfrak{z}$, $\mathfrak{z}'$ must be conjugate. $\square$

The conjugacy problem in $\mathfrak{B}_n$ is thus equivalent to the problem of classifying closed braids. There have been many attempts since Artin's paper in 1925 to solve it. Some partial solutions had been attained by W. Fröhlich [122], until G. S. Makanin [227] and F. A. Garside [126] completely solved the problem. F. A. Garside invented an ingenious though rather complicated algorithm by which he can decide whether two braids are conjugate or not. This implies a new solution of the word problem by the use of a new normal form. We do not intend to give a proof. A modern account of the problem can be found in the book by C. Kassel and V. Turaev [186, Chapter 6].

Alexander's theorem (Proposition 2.12) can be combined with Artin's characterization of braid automorphisms (Proposition 10.8) to give a characterization of link groups in terms of special presentations.

10.18 Proposition. *A group $\mathfrak{G}$ is the fundamental group $\pi_1(S^3 - \mathfrak{l})$ for some link $\mathfrak{l}$ (a* link group*) if and only if there is a presentation of the form*

$$\mathfrak{G} = \langle\, S_1, \ldots, S_n \mid S_i^{-1} L_i S_{\pi(i)} L_i^{-1},\ 1 \le i \le n \,\rangle,$$

with π a permutation and $\prod_{i=1}^n S_i = \prod_{i=1}^n L_i S_{\pi(i)} L_i^{-1}$ in the free group generated by $\{S_i \mid 1 \le i \le n\}$. □

A group theoretical characterization of knot groups $\pi_1(S^n - S^{n-2})$ has been given by M. A. Kervaire [194] for $n \ge 5$ only. Kervaire's characterization includes $H_1(S^n - S^{n-2}) \cong \mathbb{Z}$, $H_2(\pi_1(S^n - S^{n-2})) = 0$, and that $\pi_1(S^n - S^{n-2})$ is finitely generated and the normal closure of one element. All these conditions are fulfilled in dimensions 3 and 4 too. For $n = 4$, there are stronger sufficient conditions, but theses conditions are not necessary (see J. Hillman [171, Sec. 14.6]). F. González-Acuña [132] has given a characterization for $n = 4$ in terms of group presentations in the spirit of Artin's result (see Proposition 10.18). For $n = 3$ Kervaire's conditions are definitely not sufficient. There is an example $G = \langle\, x, y \mid x^2 y x^{-1} y^{-1} \,\rangle$ given in Rolfsen [309] which satisfies all conditions, but its Jacobian (see Proposition 9.10)

$$\left(\left(\frac{\partial (x^2 y x^{-1} y^{-1})}{\partial x} \right)^{\varphi\psi}, \left(\frac{\partial (x^2 y x^{-1} y^{-1})}{\partial y} \right)^{\varphi\psi} \right) = (2 - t, 0),$$

$x^{\varphi\psi} = 1$. $y^{\varphi\psi} = t$, lacks symmetry. It seems to be a natural requirement to include a symmetry condition in a characterization of classical knot groups $\pi_1(S^3 - S^1)$. An infinite series of Wirtinger presentations satisfying Kervaire's conditions is constructed by S. Rosebrock [310]. These presentations do not belong to knot groups although they have symmetric Alexander polynomials.

We conclude this chapter by considering the relation between closed braids and the links defined by them.

Let $\mathfrak{B}_n$ be the group of braids resp. braid automorphisms ζ operating on the free group $\mathfrak{F}_n$ of rank n with free generators $\{S_i\}$, $\{S_i'\}$, $S_i' = \zeta(S_i)$, such that (a) and (b) in Proposition 10.8 are valid. There is a ring homomorphism

$$\varphi\colon \mathbb{Z}\mathfrak{F}_n \to \mathbb{Z}\mathfrak{Z},\ \mathfrak{Z} = \langle\, t \,\rangle,$$

defined by: $\varphi(S_i) = t$, mapping the group ring $\mathbb{Z}\mathfrak{F}_n$ onto the group ring $\mathbb{Z}\mathfrak{Z}$ of an infinite cyclic group $\mathfrak{Z}$ generated by t.

10.19 Proposition (Burau [49]). *The mapping* $\beta\colon \mathfrak{B}_n \to \mathrm{GL}(n, \mathbb{Z}\mathfrak{Z})$ *defined by* $\zeta \mapsto ((\frac{\partial\zeta(S_j)}{\partial S_i})^{\varphi})$ *is a homomorphism of the braid group* $\mathfrak{B}_n$ *into the group of* $(n \times n)$*-matrices over* $\mathbb{Z}\mathfrak{Z}$*. Then*

$$\beta(\sigma_i) = \begin{array}{c} \begin{array}{ccccc} & i & i+1 & & \\ \end{array} \\ \begin{pmatrix} E & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & E \end{pmatrix} \end{array} \begin{array}{l} \\ \\ i \\ i+1 \\ \\ \end{array} \quad , \quad 1 \leq i \leq n.$$

β *is called the* Burau representation.

The proof of Proposition 10.18 is a simple consequence of the chain rule for Jacobians:

$$\zeta(S_i) = S_i',\ \zeta'(S_k') = S_k'',\ \frac{\partial S_k''}{\partial S_i} = \sum_{j=1}^{n} \frac{\partial S_k''}{\partial S_j'} \frac{\partial S_j'}{\partial S_i}.$$

The calculation of $\beta(\sigma_i)$ (and $\beta(\sigma_i^{-1})$) is left to the reader. □

10.20 Proposition. $\sum_{j=1}^{n}(\frac{\partial\zeta(S_i)}{\partial S_j})^{\varphi} = 1,\ \sum_{i=1}^{n} t^{i-1} \cdot (\frac{\partial\zeta(S_i)}{\partial S_j})^{\varphi} = t^{j-1}.$

Proof. Again the proof becomes trivial by using the Fox calculus. The fundamental formula yields

$$(\zeta(S_i) - 1)^{\varphi} = t - 1 = \sum_{j=1}^{n} \left(\frac{\partial\zeta(S_i)}{\partial S_j}\right)^{\varphi} (t-1).$$

For the second equation we exploit $\prod_{i=1}^{n} \zeta(S_i) = \prod_{i=1}^{n} S_i$ in $\mathfrak{F}_n$:

$$\sum_{i=1}^{n} t^{i-1} \left(\frac{\partial\zeta(S_i)}{\partial S_j}\right)^{\varphi} = \left(\frac{\partial}{\partial S_j} \prod_{i=1}^{n} \zeta(S_i)\right)^{\varphi} = \left(\frac{\partial}{\partial S_j} \prod_{i=1}^{n} S_i\right)^{\varphi} = t^{j-1}. \qquad \square$$

The equations of Proposition 10.20 express a linear dependence between the rows and columns of the representing matrices. This makes it possible to reduce the degree

n of the representation by one. If $C(t)$ is a representing matrix, we get:

$$S^{-1}C(t)S = \begin{pmatrix} & & 0 \\ & & 0 \\ & B(t) & \\ & & 0 \\ * & * \cdots * & 1 \end{pmatrix}, \tag{5}$$

$$S = \begin{pmatrix} 1 & 1 & \dots & 1 \\ & 1 & \dots & 1 \\ & & & \\ 0 & & \dots & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & -1 & \dots & & 0 \\ \vdots & 1 & -1 & & \vdots \\ \vdots & & & & 1-1 \\ 0 & \dots & \dots & & 1 \end{pmatrix}.$$

This is easily verified and it follows that by setting $\hat{\beta}(\zeta) = B(t)$ we obtain a representation of $\mathfrak{B}_n$ in $\mathrm{GL}(n-1, \mathbb{Z}\mathfrak{Z})$ which is called the *reduced Burau representation*. Note that

$$\hat{\beta}(\sigma_1) = \left(\begin{array}{cc|c} -t & 0 & \\ 1 & 1 & \\ \hline & & E \end{array}\right)$$

$$\hat{\beta}(\sigma_i) = \begin{array}{c} \qquad\qquad i \\ \begin{pmatrix} E & & & & \\ & 1 & t & 0 & \\ & 0 & -t & 0 & \\ & 0 & 1 & 1 & \\ & & & & E \end{pmatrix} \end{array}, \quad 1 < i < n-1$$

$$\hat{\beta}(\sigma_{n-1}) = \left(\begin{array}{c|cc} E & & \\ \hline & 1 & t \\ & 0 & -t \end{array}\right)$$

($\hat{\beta}(\sigma_1) = (t)$ for $n = 2$).

In addition to the advantage of reducing the degree from n to $n-1$, the reduced representation $\hat{\beta}$ has the property of mapping the center of $\mathfrak{B}_n$ into the center of $\mathrm{GL}(n-1, \mathbb{Z}\mathfrak{Z})$; thus

$$\hat{\beta}(\sigma_1 \dots \sigma_{n-1})^n = \begin{pmatrix} t^n & & & 0 \\ & t^n & & \\ & & \ddots & \\ 0 & & & t^n \end{pmatrix}.$$

The original $\hat{\beta}$ maps the center on non-diagonal matrices.

The algebraic level of these representations is clearly that of the Alexander module (Chapter 8.A). There should be a connection.

10.21 Proposition. *For* $\mathfrak{z} \in \mathfrak{B}_n$, $\beta(\mathfrak{z}) = C(t)$, *the matrix* $(C(t) - E)$ *is a Jacobian (see* 9.19*) of the link* $\hat{\mathfrak{z}}$. *Furthermore if* $\hat{\beta}(\hat{\mathfrak{z}}) = B(t)$ *then*

$$\det(B(t) - E) \doteq (1 + t + \ldots + t^{n-1})(1 - t)^{\mu - 1} \cdot \triangledown^H(t).$$

where $\triangledown^H(t)$ *is the Hosokawa polynomial of* $\hat{\mathfrak{z}}$ *(see* 9.19*), and* μ *the multiplicity of* $\hat{\mathfrak{z}}$.

Proof. The first assertion is an immediate consequence of Proposition 10.18. The second part – first proved by W. Burau [49] – is a bit harder:

The matrix $(C(t) - E)S$, see equation (5), is a matrix with the n-th column consisting of zeroes – this is a consequence of the first identity in Proposition 10.20. If the vector $\mathfrak{a}_i$ denotes the i-th row of $(C(t) - E)$, the second identity can be expressed by $\sum_{i=1}^{n} t^{i-1}\mathfrak{a}_i = 0$. Hence

$$\sum_{i=1}^{n} t^{i-1}\mathfrak{d}_i = 0, \tag{6}$$

where $\mathfrak{d}_i$ denotes the vector composed of the first $n-1$ components of $\mathfrak{a}_i S$.

By multiplying the $\mathfrak{a}_i S$ by S^{-1} from the left we obtain

$$\det(B(t) - E) = \det(\mathfrak{d}_1 - \mathfrak{d}_2, \mathfrak{d}_2 - \mathfrak{d}_3, \ldots, \mathfrak{d}_{n-1} - \mathfrak{d}_n)$$

(compare equation (5)). From this we get that

$$\begin{aligned}
\pm\det(B(t) - E) &= \det(\mathfrak{d}_2 - \mathfrak{d}_1, \mathfrak{d}_3 - \mathfrak{d}_1, \ldots, \mathfrak{d}_n - \mathfrak{d}_1) \\
&= \det(\mathfrak{d}_2, \mathfrak{d}_3, \ldots \mathfrak{d}_n) + \sum_{i=1}^{n-1} \det(\mathfrak{d}_2, \ldots, \mathfrak{d}_i, (-\mathfrak{d}_1), \ldots, \mathfrak{d}_n) \\
&= \det(\mathfrak{d}_2, \ldots, \mathfrak{d}_n) + \sum_{i=1}^{n-1} \det(\mathfrak{d}_2, \ldots, t^i\mathfrak{d}_{i+1}, \ldots, \mathfrak{d}_n) \\
&= (1 + t + \ldots + t^{n-1}) \cdot \triangledown^H(t) \cdot (t - 1)^{\mu - 1}.
\end{aligned}$$

The last equation follows from 9.19 since $\det(\mathfrak{d}_2, \ldots, \mathfrak{d}_n)$ by (6) generates the first elementary ideal of $S^{-1}(C(t) - E)S$. □

The question of the faithfulness of the Burau presentation has received some attention: In Magnus and Peluso [226] faithfulness was proved for $n \leq 3$; only recently this was shown to be wrong for $n \geq 5$ (see Moody [246] for $n \geq 10$, Long and Paton [216] for $n \geq 6$ and Bigelow [20] for $n = 5$). At the moment of writing, the faithfulness of the Burau representation when $n = 4$ is an open problem.

It is evident that two non-equivalent closed braids may represent equivalent knots or links. For instance, $\hat{\sigma}_1$ and $\hat{\sigma}_1^{-1}$ both represent the unknot, but σ_1 and σ_1^{-1} are not conjugate in $\mathfrak{B}_2$. Figure 10.19 shows two closed n-braids which are isotopic to $\hat{\mathfrak{z}}$ as links for any $\mathfrak{z} \in \mathfrak{B}_{n-1}$.

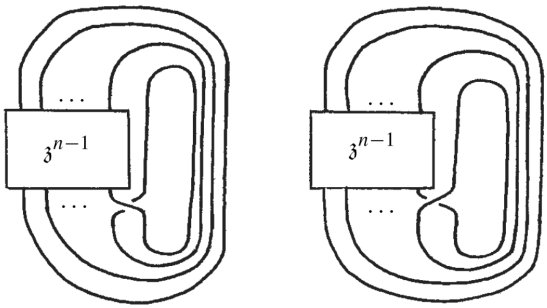


Figure 10.19. A Markov move.

A. A. Markov proved in 1936 a theorem [229] which in the case of oriented links controls the relationship between closed braids and links represented by them. The orientation in a closed braid is always defined by assuming that the strings of the braid run downward.

10.22 Definition (Markov move). The process which replaces $\mathfrak{z} \in \mathfrak{B}_{n-1}$ by $\mathfrak{z}\sigma_{n-1}^{\pm 1}$ (Figure 10.19) or vice versa is called a *Markov move*. Two braids $\mathfrak{z}$ and $\mathfrak{z}'$ are *Markov equivalent*, if they are connected by a finite chain of braids:

$$\mathfrak{z} = \mathfrak{z}_0 \to \mathfrak{z}_1 \to \mathfrak{z}_2 \to \ldots \to \mathfrak{z}_r = \mathfrak{z}',$$

where either two consecutive braids $\mathfrak{z}_i$ are conjugate or related by a Markov move.

10.23 Theorem (Markov [229]). *Two oriented links represented by the closed braids $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$ are isotopic, if and only if the braids $\mathfrak{z}$ and $\mathfrak{z}'$ are Markov equivalent.*

Proof. We revert to Alexander's theorem and its proof in Proposition 2.12. Starting with an oriented link $\mathfrak{k}$ the procedure automatically gives a closed braid with all strings oriented in the same direction, assuming that the orientation of $\mathfrak{k}$ goes along with increasing indices of the intersection points P_i, $1 \leq i \leq 2m$. We denote the oriented projections of the overcrossing arcs from P_{2i-1} to P_{2i} by s_i, and the undercrossing ones from P_{2i} to P_{2i+1} by t_i; we also give an orientation to the axis h from left to right, Figure 2.13 (b). Let S denote the set $\{P_{2i-1}\}$ of the starting point of the arcs s_i, and $F = \{P_{2i}\}$ their finishing points.

We now consider different axes for a given fixed projection $p(\mathfrak{k})$. Choose any simply closed oriented curve h' in the projection plane $\mathbb{R}^2 = p(\mathbb{R}^3)$ which separates the sets S and F and meets the projection $p(\mathfrak{k})$ transversely such that S is on the

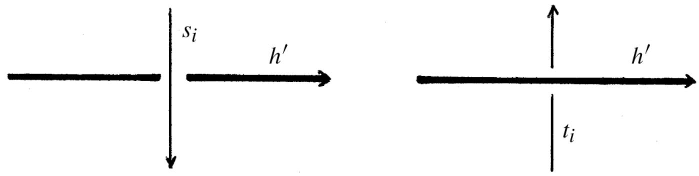


Figure 10.20

left and F on the right side of h'. We arrange that overcrossing arcs always cross h' from left to right and undercrossing ones from right to left, Figure 10.20. This can be achieved by changing $\mathfrak{k}$ while keeping $p(\mathfrak{k})$ fixed by introducing new pairs of intersection points. But the new sets $S' \supset S$, $F' \supset F$ are still separated by h' in the same way. The original axis h of Figure 2.13 (b) can be replaced by an axis in the form h' by using an arc far off the projection, and, on the other hand, any axis h' defines a closed braid $\hat{\mathfrak{z}} \cong \mathfrak{k}$ in the same way as h.

We now study the effect of changing h' while keeping $p(\mathfrak{k})$ fixed.

We first look at isotopies of h' in $\mathbb{R}^2 - (S \cup F)$ by Δ-moves, see Definition 1.10. The following cases may occur: In Figure 10.21 the bold line is always h' while the others belong to $p(\mathfrak{k})$. Intersection points are not marked.

The moves (5), (6), (7) are isotopies of the closed braid. Move (8) is a sequence of the remaining moves, Figure 10.22:

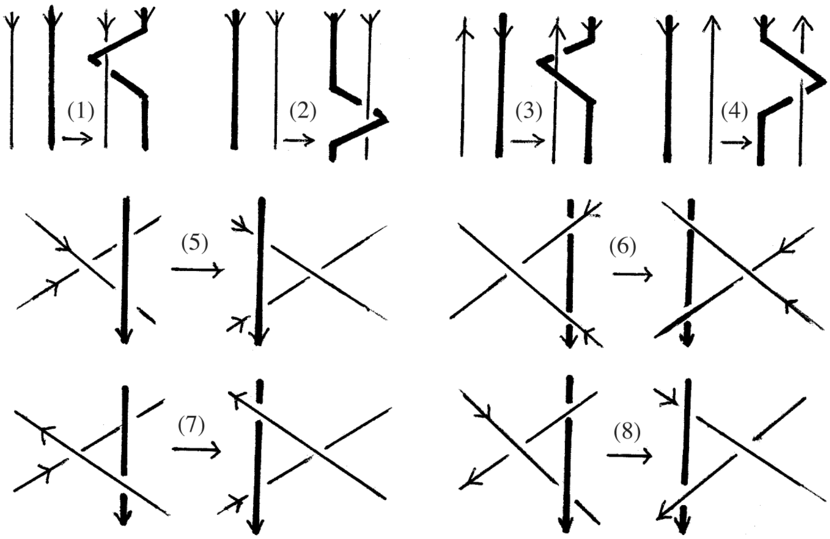


Figure 10.21

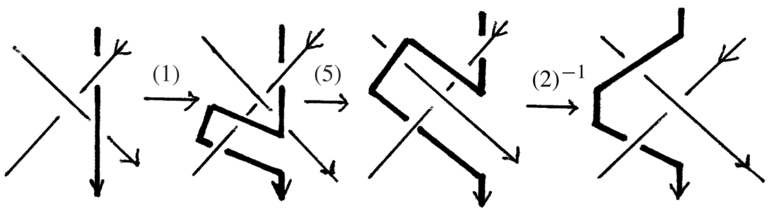


Figure 10.22. Move (8) is a sequence of the remaining moves.

Assume h' to be in the position of the x-axis of $\mathbb{R}^2$ and $S = \{P_{2i-1}\}$ in the upper half-plane H^+. We investigate the effect of the moves (1)–(4). The arcs $s_i' = H^+ \cap p(s_i)$ form a normal dissection of $H^+ - S$ and, hence, define a braid $\mathfrak{z}^+$ (the four braids corresponding to $H^\pm \cap S$, $H^\pm \cap F$ form the closed braid $\hat{\mathfrak{z}}$ determined by h').

In Figure 10.23 the move (4) is applied in a special position: we assume that on the right side of Q_m there are no intersections $h' \cap p(\mathfrak{k})$, and that the Δ-move is executed in the neighborhood of Q_m. This special position can always be produced by an isotopy of $\hat{\mathfrak{z}}$.

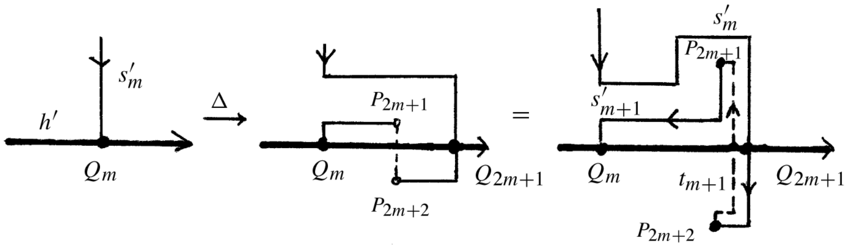


Figure 10.23

A comparison of Figure 10.23 with Figure 10.9 shows that $\mathfrak{z}^+$ is replaced by $\mathfrak{z}^+\sigma_m$ while the other three constituents of $\hat{\mathfrak{z}}$ just obtain an additional trivial string. By similar arguments we see that the moves of Figure 10.21 result in Markov moves.

In general, the isotopy which moves an axis h' into another axis h in $\mathbb{R}^2$ will not be an isotopy of $\mathbb{R}^2 - (S \cup F)$. Figure 10.24 shows the general case.

Suppose that F is contained in the interior of the closed curve h', and let the dissection lines $\{\ell_{2i+1}\}$ start in the $\{P_{2i+1}\}$ and run upwards to infinity. Consider the local process which pushes two segments of h', oppositely oriented, simultaneously over an intersection point P_{2i+1}, Figure 10.25.

This process again is an isotopy of $\hat{\mathfrak{z}}$. Applying it, we can deform h' into h'' with $h'' \cap (\bigcup \ell_{2i+1}) = \emptyset$. Then h'' is a simple closed curve containing F in its interior, S in its exterior and isotopic to h in $\mathbb{R}^2 - (S \cup F)$. So we have proved the following:

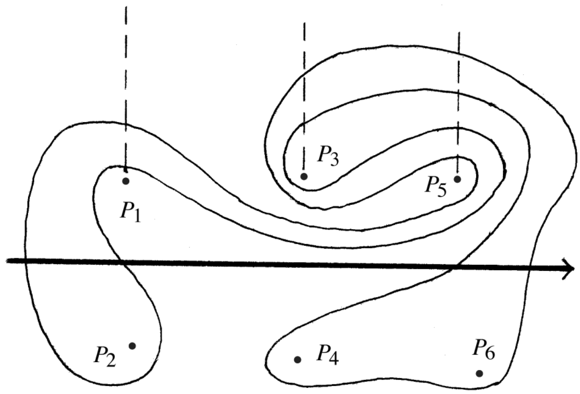


Figure 10.24

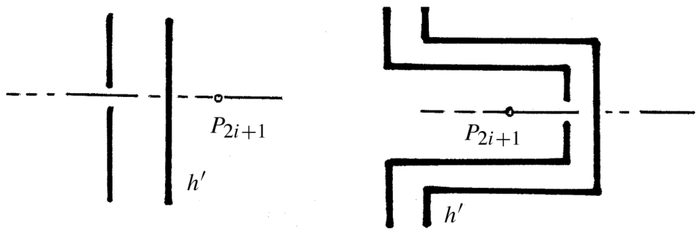


Figure 10.25

10.24 Claim. *Given a diagram $p(\mathfrak{k})$ of an oriented link with intersection points (S, F), and two axes h and h' separating S and F, then the closed braids defined by S, F, h and h' are Markov equivalent.*

10.25 Remark. Starting with $p(\mathfrak{k})$ and (S, F), a separating axis h will in general enforce additional intersection points $S^* \supset S$, $F^* \supset F$, but the separating property will be preserved.

If two closed braids for $p(\mathfrak{k})$ and axes h and h' with different intersection points (S, F) and (S', F') are given, they are still Markov equivalent, because by the claim those given by $h, (S, F)$ and h', (S', F') are, for a common refinement (S'', F'') of (S, F) and (S', F').

To complete the proof of Markov's theorem, we have to check the effect of the Reidemeister moves Ω_i, $1 \le i \le 3$, see Definition 1.16, on $p(\mathfrak{k})$. We take advantage of Claim 10.24 in choosing a suitable axis: for Ω_1 and Ω_2 the axis can be chosen away from the local region where the move is applied. In a situation where Ω_3 can be executed, Figure 10.26, the line s_i will cross the axis h'. According to the orientations, two cases arise which are shown in Figure 10.26.

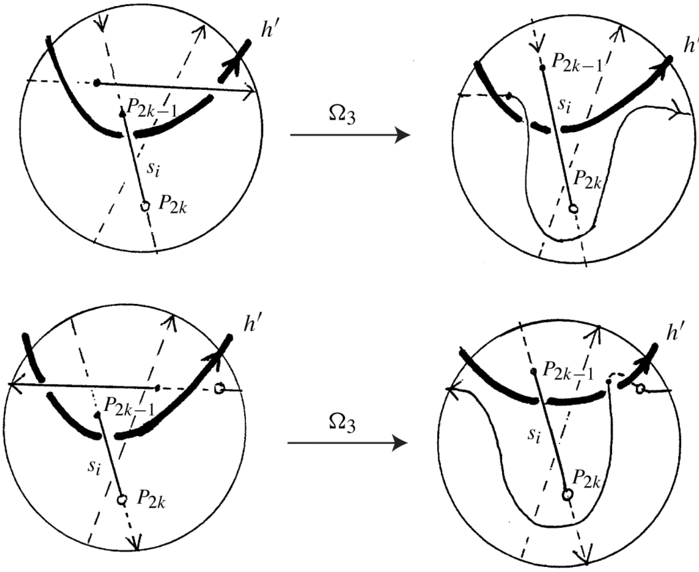


Figure 10.26

It suffices to ascertain that we can in each case place the intersection points in the region of the Ω_3-move in such a way that S and F are separated by h'. The intersection points outside the region are not changed. This completes the proof which is due to H. R Morton, [253]. □

A Markov theorem for unoriented links was proved by A. Simon [334]. In addition to conjugation and Markov moves a further move (Markov*-move) is necessary which operates not on the braid group but on the monoid of pseudobraids.

10.E History and sources

Braid groups $\mathfrak{B}_n$ and $\mathfrak{I}_n$ appeared first, in a different context and under a different name, in an article by A. Hurwitz in 1891 [175, II.§1, §2] on the theory of Riemann surfaces. Section 6.3 of M. Epple's book [92] presents this aspect of Hurwitz' work in a modern context. The connections between the paper by A. Hurwitz and later developments has been described by W. Magnus in [223].

The notation of the braid and the braid group where explicitly introduced by E. Artin in his famous paper "Theorie der Zöpfe" in 1925. (O. Schreier, who was helpful with some proofs, should, nevertheless be mentioned.) This paper already contains the fundamental isomorphism between braids and braid automorphisms by which braids are classified. The proof, though, is not satisfying. E. Artin published a new paper on braids in 1947 with rigorous definitions and proofs including the normal form of a braid. The remaining problem was the conjugacy problem. The importance of the braid group in other fields became evident in Magnus' paper on the mapping class groups of surfaces [222]. Further contributions in that direction were made by J. Birman and H. Hilden. There have been continual contributions to braid theory by several authors. For a bibliography see the books of J. Birman [24] and C. Kassel and V. Turaev [186]. The outstanding work was doubtless Makanin's and Garside's solution of the conjugacy problem [227, 126]. Braid theory from the point of view of configuration spaces was studied by E. Fadell and L. Neuwirth [94]. They assigns braid groups to manifolds – the original braid group then is the braid group of the plane $\mathbb{R}^2$. This approach has been successfully applied by V. I. Arnol'd [9] to determine the homology and cohomology groups of braid groups.

More recently some important results about the braid group have been obtained: P. Dehornoy [88] proved that Artin's braid group B_n is orderable and the linearity of $\mathfrak{B}_n$ was established by D. Krammer [202, 203] and S. J. Bigelow [21]. The book of C. Kassel and V. Turaev [186] gives a modern account of the theory of braid groups.

10.F Exercises

E 10.1. (Artin [10, Satz 2]) Prove: $\mathfrak{B}_n = \langle \sigma, \tau \mid \sigma^n = (\sigma\tau)^{n-1}, [\sigma^i \tau \sigma^{-i}, \tau] \rangle, 2 \leq i \leq \frac{n}{2}, \sigma = \sigma_1\sigma_2 \dots \sigma_{n-1}, \tau = \sigma_1$. Derive from this presentation a presentation of the symmetric groups $\mathfrak{S}_n$ [10, Satz 3].

E 10.2. $\mathfrak{B}_n/\mathfrak{B}'_n \cong \mathfrak{Z}$, $\mathfrak{I}_n/\mathfrak{I}'_n \cong \mathfrak{Z}^{\binom{n}{2}}$.

E 10.3. Let $Z(\mathfrak{B}_n)$ be the center of $\mathfrak{B}_n$. Prove that $\mathfrak{z}^m \in Z(\mathfrak{B}_n)$ and $\mathfrak{z} \in \mathfrak{I}_n$ imply $\mathfrak{z} \in Z(\mathfrak{B}_n)$.

E 10.4. Interpret $\mathfrak{I}_n$ as a group of automorphisms of $\mathfrak{F}_n$ and denote by $\mathfrak{T}_n$ the inner automorphisms of $\mathfrak{F}_n$. Show that

$$\mathfrak{T}_n\mathfrak{I}_n/\mathfrak{T}_n \cong \mathfrak{T}_{n-1}\mathfrak{I}_{n-1}, \quad \mathfrak{T}_n \cap \mathfrak{I}_n = Z(\mathfrak{I}_n) = \text{center of } \mathfrak{I}_n.$$

Derive from this that $\mathfrak{T}_n\mathfrak{I}_n$ has no elements of finite order $\neq 1$.

E 10.5. (Garside) Show that every braid $\mathfrak{z}$ can be written in the form

$$\sigma = \sigma_{i_1}^{a_1} \dots \sigma_{i_r}^{a_r} \Delta^k, a_k \geq 1, \text{ with } \Delta = (\sigma_1 \dots \sigma_{n-1})(\sigma_1 \dots \sigma_{n-2}) \dots (\sigma_1\sigma_2)\sigma_1$$

the *fundamental braid*, k an integer.

E 10.6. Show that the Burau representation β and its reduced version $\hat{\beta}$ have the same kernel i.e. $\hat{\beta}(\mathfrak{z}) = E$ if and only if $\beta(\mathfrak{z}) = E$ (see [186, Lemma 3.10]). The representations are faithful for $n \leq 3$.

E 10.7. Show that the notion of isotopy of braids as defined in 10.1 is equivalent to s-isotopy of braids as used in the proof of Proposition 10.14.

E 10.8. Find a sequence of Markov moves which relates σ_1 and σ_1^{-1}.

Chapter 11

Manifolds as branched coverings

The first section contains a treatment of Alexander's theorem [3] (Theorem 11.1). It makes use of the theory of braids and plats. The second part of this chapter is devoted to the Hilden–Montesinos theorem (Theorem 11.11) which improves Alexander's result in the case of 3-manifolds. We give a proof following H. Hilden [165], but prefer to think of the links as plats. This affords a more transparent description of the geometric relations between the branch sets and the Heegaard splittings of the covering manifolds. The Dehn–Lickorish theorem (Theorem 11.7) is used but not proved here.

11.A Alexander's theorem

11.1 Theorem (Alexander [3]). *Every orientable closed 3-manifold is a branched covering of S^3, branched along a link with branching indices ≤ 2. (Compare 8.37.)*

Proof. Let M^3 be an arbitrary closed oriented manifold with a finite simplicial structure. Define a map p on its vertices $\hat{P}_i$, $1 \leq i \leq N$, $p(\hat{P}_i) = P_i \in S^3$, such that the P_i are in general position in S^3. After choosing an orientation for S^3 we extend p to a map $p\colon M^3 \to S^3$ by the following rule. For any positively oriented 3-simplex $[\hat{P}_{i_1}, \hat{P}_{i_2}, \hat{P}_{i_3}, \hat{P}_{i_4}]$ of M^3 we define p as the affine mapping

$$p\colon [\hat{P}_{i_1}, \hat{P}_{i_2}, \hat{P}_{i_3}, \hat{P}_{i_4}] \to [P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}].$$

if $[P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}]$ is positively oriented in S^3; if not, we choose the complement $C[P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}]$ as image,

$$p\colon [\hat{P}_{i_1}, \hat{P}_{i_2}, \hat{P}_{i_3}, \hat{P}_{i_4}] \to C[P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}].$$

We will show that p is a branched covering, the 1-skeleton M^1 of M^3 being mapped by p onto the branching set $p(M^1) = T \subset S^3$. For every point $P \in S^3 - p(M^2)$, M^2 the 2-skeleton of M^3, there is a neighborhood $U \subset S^3 - p(M^2)$ containing P such that $p^{-1}(U(P))$ consists of n disjoint neighborhoods $\hat{U}_j$ of the points $p^{-1}(P)$. Suppose that $P \in p(M^2) - p(M^1)$ and $\hat{P} \in p^{-1}(P)$ is contained in the interior of the 2-simplex $[\hat{P}_1, \hat{P}_2, \hat{P}_3]$, in the boundary of $[\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3]$ and $[\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4]$. Let $[P_0, P_1, P_2, P_3]$ be positively oriented in S^3. If P_0 and P_4 are separated by the plane defined by $[P_1, P_2, P_3]$, we get that

$$p[\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3] = [P_0, P_1, P_2, P_3], \; p[\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4] = [P_1, P_2, P_3, P_4];$$

if not,

$$p[\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3] = [P_0, P_1, P_2, P_3], \; p[\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4] = C[P_1, P_2, P_3, P_4].$$

In both cases there is a neighborhood $\hat{U}$ of $\hat{P}$ which is mapped onto a neighborhood U of $P = p(\hat{P})$, see Figure 11.1.

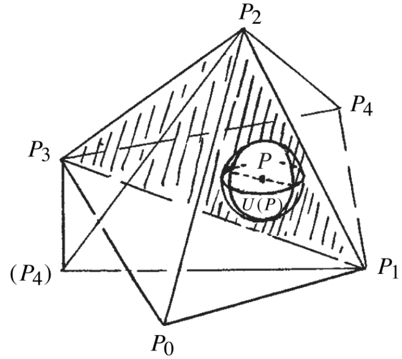


Figure 11.1

As a consequence; p: $M^3 \to S^3$ is surjective, otherwise the compact polyhedron $p(M^3) \subset S^3$ would have boundary points on $p(M^2) - p(M^1)$. It follows also that for every point $P \in S^3 - p(M^1)$ there is a neighborhood $U \subset S^3 - p(M^1)$ containing P such that $p^{-1}(U(P))$ consists of n disjoint neighborhoods $\hat{U}_j$ of the points $p^{-1}(P)$. Hence $p: M^3 - p^{-1}(p(M^1)) \to S^3 - p(M^1)$ is a covering.

It follows from the construction that the restriction $p|M^1: M^1 \to S^3$ is injective. The preimage $p^{-1}(P_i)$ of a vertex P_i consists of $\hat{P}_i$ and maybe several other points with branching index one. The same holds for the images $[P_i, P_j] = p[\hat{P}_i, \hat{P}_j]$ of edges. It remains to show that p can be modified in such a way that the branching set $T = p(M^1)$ is transformed into a link (without changing M^3).

By $U(T)$ we denote a tubular neighborhood of $T \subset S^3$, consisting of (closed) balls B_i with centers P_i and cylindrical segments Z_{ij} with axes on $[P_i, P_j]$. The intersection $Z_{ij} \cap B_k$ a disk δ_k for $k = i, j$ and empty otherwise. With $I = [0, 1]$, $Z_{ij} = I \times \delta$, and for $Y \in I$ the disk $Y \times \delta$ is covered by a collection of disjoint disks in M^3, of which at most one may contain a branching point $\hat{Y} \in M^1$ of index $r > 1$. The branched covering $p|\hat{Y}$: $\hat{Y} \times \hat{\delta} \to Y \times \delta$ (for short: p: $\hat{\delta} \to \delta$) is cyclic (Figure 11.2).

A cycle of length r may be written as a product of $r-1$ transpositions, $(12 \dots r) = (1,2)(2,3)\dots(r-1, r)$. Correspondingly there is a branched covering p': $\hat{\delta}' \to \delta$

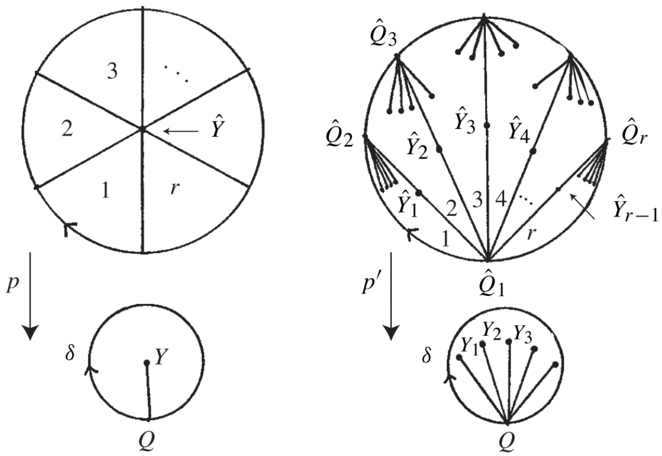


Figure 11.2. Modification of the cyclic branched covering $p\colon \hat{\delta} \to \delta$.

with $r-1$ branch points $\hat{Y}_i$, $1 \le i \le r-1$, of index two. $\hat{\delta}'$ is a disk, and $p' \mid \partial\hat{\delta}' = p|\partial\hat{\delta}$. We substitute p' for p on all cylindrical segments $\hat{Z}_{ij} \subset p^{-1}Z_{ij}$ and obtain a new branched covering

$$p'\colon \overline{M^3 - \bigcup p^{-1}(B_i)} \to \overline{S^3 - \bigcup B_i}.$$

We denote by $\hat{B}_i$ the component of $p^{-1}(B_i)$ which contains $\hat{P}_i$. The branching set consists of lines in the cylindrical segments parallel to the axis of the cylinder. $p'|\partial\hat{B}_i = \hat{S}^2 \to S^2 = \partial B_i$ is a branched covering with branching points Q_j, $1 \le j \le q$, of index two where the sphere S^2 meets the branching lines contained in the adjoining cylinders. To describe this covering we use a normal dissection of $S^2 - \bigcup_{j=1}^{q} Q_j = \Sigma_q$ joining the Q_j by simple arcs s_j to some $Q \in \Sigma_q$. (The arcs are required to be disjoint save for their common endpoint Q, Figure 11.3.)

We assign to each s_j a transposition $\tau_j \in \mathfrak{S}_n$, where n is the number of sheets of the covering $p'\colon \hat{S}^2 \to S^2$, and $\mathfrak{S}_n$ is the symmetric group of order $n!$. Crossing

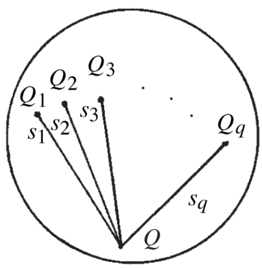


Figure 11.3. A normal dissection of Σ_q.

an arc of $p^{-1}(s_j)$ in $\hat{S}^2$ means changing from the k-th sheet to the $\tau_j(k)$-th sheet of the covering. Since Q is not a branch point, $\prod_{j=1}^{q} \tau_j = \mathrm{id}$, $q = 2m$. Computing $\chi(\hat{S}^2) = 2$ gives $(n-1) = m$. On the other hand, any set of transpositions $\{\tau_j \mid 1 \le j \le 2m\}$ which generate a transitive subgroup $\mathfrak{T}_n$ of $\mathfrak{S}_n, n = m+1$, defines a branched covering $p'\colon \hat{S}^2 \to S^2$, if $\prod_{j=1}^{2m} \tau_j = \mathrm{id}$. We may assign generators $S_j \in \pi_1(\Sigma_{2m})$ to the arcs s_j (see the text preceding Proposition 10.6), $\prod_{j=1}^{2m} S_j = 1$, and there is a homomorphism $\varphi\colon \pi_1\Sigma_{2m} \to \mathfrak{S}_n$, $\varphi(S_j) = \tau_j$. Given two normal dissections $\{s_j\}$ and $\{s'_j\}$ of Σ_{2m} there is a homeomorphism $h\colon \Sigma_{2m} \to \Sigma_{2m}$, $h(s_i) = s'_j$ which induces a braid automorphism $\zeta\colon S_j \mapsto \zeta(S_j) = S'_j = L_j S_i L_j^{-1}$, $\pi(i) = j$, where π is the permutation of a braid. The generator S'_j corresponds to the arc s'_j. The commutative diagram

$$\begin{array}{ccc} \pi_1\Sigma_{2m} & \xrightarrow{\zeta} & \pi_1\Sigma_{2m} \\ \varphi\big\downarrow & & \varphi\big\downarrow \\ \mathfrak{T}_n & \xrightarrow{\zeta^*} & \mathfrak{T}_n \end{array}$$

defines a mapping ζ^* called the *induced braid substitution in* $\mathfrak{T}_n$. This can be used to compute the transpositions $\tau'_j = \zeta^*(\tau_j)$ which have to be assigned to the arcs s'_j in order to define the covering $p'\colon \hat{S}^2 \to S^2$. It follows that the homeomorphism $h\colon \Sigma_{2m} \to \Sigma_{2m}$ can be extended and lifted to a homeomorphism $\hat{h}$:

$$\begin{array}{ccc} \hat{S}^2 & \xrightarrow{\hat{h}} & \hat{S}^2 \\ p'\big\downarrow & & \big\downarrow p' \\ S^2 & \xrightarrow{h} & S^2 \end{array}$$

We interrupt our proof to show that there are homeomorphisms h, $\hat{h}$ such that the τ_j are replaced by τ'_j with a special property.

11.2 Lemma. *If* $2m$ *transpositions* $\tau_i \in \mathfrak{S}_n$, $1 \le i \le 2m$, *satisfy* $\prod_{i=1}^{2m} \tau_i = \mathrm{id}$, *then there is a braid substitution* $\zeta^*\colon \tau_i \mapsto \tau'_i$, *such that*

$$\tau'_{2j-1} = \tau'_{2j},\ 1 \le j \le m.$$

Proof. Let $\mathfrak{T}_n$ denote the subgroup of $\mathfrak{S}_n$ generated by the transpositions τ_i. Denote by $\sigma_k^{*\pm 1}$ the braid substitutions in $\mathfrak{T}_n$ induced by the elementary braids $\sigma_k^{\pm 1}$ (Section 10.A, equation (10.2) resp. equation (10.3)). If $\tau_k = (a,b)$, $\tau_{k+1} = (c,d)$ and a,b,c,d all different, the effect of $\sigma_k^{*\pm 1}$ is to interchange the transpositions:

$$\tau'_k = \sigma_k^{*\pm 1}(\tau_k) = \tau_{k+1}, \tau'_{k+1} = \sigma_k^{*\pm 1}(\tau_{k+1}) = \tau_k.$$

If $\tau_k = (a,b)$, $\tau_{k+1} = (b,c)$ then

$$\sigma_k^*(\tau_k) = (a,c),\ \sigma_k^*(\tau_{k+1}) = (a,b) \text{ and } \sigma_k^{*-1}(\tau_k) = (b,c),\ \sigma_k^{*-1}(\tau_{k+1}) = (a,c).$$

Assume $\tau_1 = (1,2)$. Let $\tau_j = (1,a)$ be the transposition containing the figure 1, with minimal $j > 1$. (There is such a τ_j because $\prod \tau_i = \mathrm{id}$.) If $j > 2, \tau_{j-1} = (b,c)$, $b,c \neq 1$, the braid substitution $\sigma_{j-1}^{*\pm 1}$ will interchange τ_{j-1} and τ_j, if a,b,c are different. A pair $(a,b) = \tau_{j-1}$, $(1,a) = \tau_j$ is replaced by $(1,b)$, (c,b) if σ_{j-1}^* is applied, and by $(1,a)$, $(1,b)$, if σ_{j-1}^{*-1} is used.

Thus the sequence $\tau_1, \tau_2, \ldots, \tau_{2m}$ can be transformed by a braid substitution into (1,2), $(1,i_2)\ldots(1,i_\nu)$, $\tau''_{\nu+1}, \ldots, \tau''_{2m}$, where the τ''_j, $j > \nu$, do not contain the figure 1. There is an $i_j = 2$, $2 \leq j \leq \nu$. If $j = 2$, the lemma is proved by induction. Otherwise we may replace $(1,i_{j-1})$, $(1,2)$ by $(1,2)$, $(2,i_{j-1})$ using σ_{j-1}^{*-1}. □

We are now in a position to extend p': $(M^3 - \bigcup p^{-1}(\mathring{\hat{B}}_i)) \to (S^3 - \bigcup \mathring{\hat{B}}_i)$ to a covering $\tilde{p}$: $M^3 \to S^3$ and complete the proof of Theorem 11.1.

We choose a homeomorphism

$$h\colon\ \Sigma_{2m} \to \Sigma_{2m}$$

which induces a braid automorphism ζ: $\pi_1\Sigma_{2m} \to \pi_1\Sigma_{2m}$ satisfying Lemma 11.2: $\zeta^*(\tau_k) = \tau'_k$, $\tau'_{2j-1} = \tau'_{2j}$, $1 \leq j \leq m$. The homeomorphism h: $S^2 \to S^2$ is orientation preserving and hence there is an isotopy

$$H\colon\ S^2 \times I \to S^2,\ H(x,0) = x,\ H(x,1) = h(x).$$

Lift H to an isotopy

$$\hat{H}\colon\ \hat{S}^2 \times I \to \hat{S}^2,\ \hat{H}(x,0) = x,\ \hat{H}(x,1) = \hat{h}(x).$$

Now identify $S^2 \times 0$ and $\hat{S}^2 \times 0$ with ∂B_i and $\partial \hat{B}_i$, respectively. Moreover, extend p' to $\hat{S}^2 \times I$ by setting $p'(x,t) = (p'(x),t)$.

It is now easy to extend p' to a pair of balls $\hat{B}'_i$, B'_i with $\partial \hat{B}'_i = \hat{S}^2 \times 1$, $\partial B'_i = S^2 \times 1$. We replace the normal dissection $\{s'_j\}$ of $(S^2 \times 1) - U\{Q'_j\}$, $h(Q_j) = Q'_{\pi^{-1}(j)}$, by disjoint arcs t_j, $1 \leq j \leq m$, which connect Q'_{2j-1} and Q'_{2j} (Figure 11.4).

There is a branched covering p'': $\hat{B}'_i \to B'_i$ with a branching set consisting of m simple disjoint unknotted arcs t'_j, $t'_j \cap \partial B'_j = Q'_{2j-1} \cup Q'_{2j}$, and there are m disjoint disks $\delta_j \subset B'_j$ with $\partial \delta_j = t_j \cup t'_j$ (Figure 11.4).

Passing through a disk of $(p'')^{-1}(\delta_j)$ in $\hat{B}'_i$ means changing from sheet number k to sheet number $\tau'_j(k)$. Since $p''|\partial \hat{B}'_i = p'$ we may thus extend p' to a covering $\tilde{p}$: $M^3 \to S^3$. (There is no problem in extending p' to the balls of $p^{-1}(B_i)$ different from $\hat{B}_i$, since the covering is not branched in these.) □

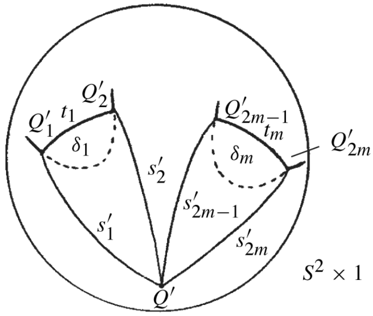


Figure 11.4. The covering $p'\colon \partial\hat{B}'_i \to \partial B'_i$ extends to $p''\colon \hat{B}'_i \to B'_i$ since $\tau'_{2j-1} = \tau'_{2j}$.

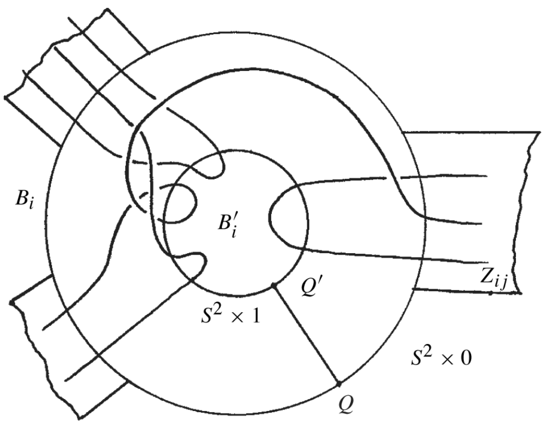


Figure 11.5. The branching set of $\tilde{p}$ in $B'_i \cup (S^2 \times I)$.

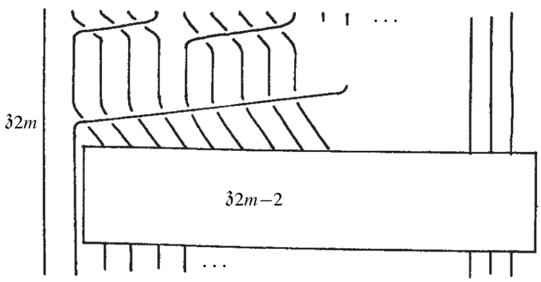


Figure 11.6

The branching set of $\tilde{p}$ in $B'_i \cup (S^2 \times I)$ is described in Figure 11.5: The orbits $\{(H(Q_i,t),t) \mid 0 \le t \le 1\} \subset S^2 \times I$ form a braid to which in B'_i the arcs $\partial\delta_i - t_i$ are added as in the case of a plat.

The braids that occur depend on the braid automorphisms required in Lemma 11.2. They can be chosen in a rather special way. It is easy to verify from the operations used in Lemma 11.2 that braids $\mathfrak{z}_{2m}$ of the type depicted in Figure 11.6 suffice. One can see that the tangle in B_i then consists of m unknotted and unlinked arcs.

11.B Branched coverings and Heegaard diagrams

By Alexander's theorem every closed oriented 3-manifold is an n-fold branched covering p: $M^3 \to S^3$ of the sphere. Suppose the branching set $\mathfrak{k}$ is a link of multiplicity μ, $\mathfrak{k} = \bigcup_{i=1}^{\mu} \mathfrak{k}_i$, and it is presented as a $2m$-plat (see Chapter 2.D), where m is the bridge number of $\mathfrak{k}$. A component $\mathfrak{k}_i$ is then presented as a $2\lambda_i$-plat, $\sum_{i=1}^{\mu} \lambda_i = m$. Think of S^3 as the union of two disjoint closed balls B_0, B_1, and $I \times S^2$, $\{j\} \times S^2 = \partial B_j = S_j^2$, $j = 0, 1$. Let the plat $\mathfrak{k}$ intersect $\mathring{B}_0$ and $\mathring{B}_1$ in m unknotted arcs spanning disjoint disks δ_i^j, $1 \le i \le m$, in B_j, $j = 0, 1$, and denote by $\mathfrak{z} = \mathfrak{k} \cap (I \times S^2)$ the braid part of $\mathfrak{k}$ (Figure 11.7). Every point of $\mathfrak{k}_i \cap (S_0^2 \cup S_1^2)$ is covered by the same number $\mu_i \le n$ of points in M^3.

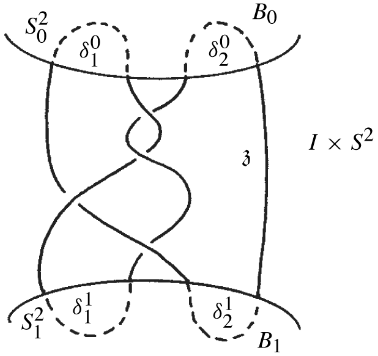


Figure 11.7

11.3 Proposition. *A manifold M^3 which is an n-fold branched covering of S^3 branched along the plat $\mathfrak{k}$ possesses a Heegaard splitting of genus*

$$g = m \cdot n - n + 1 - \sum_{i=1}^{\mu} \lambda_i \mu_i.$$

Proof. The 2-spheres S_j^2 are covered by orientable closed surfaces $\hat{F}_j = p^{-1}(S_j^2)$, $j = 0, 1$. The group $\pi_1(S^3 - \mathfrak{k})$ can be generated by m Wirtinger generators s_i, $1 \le i \le m$, encircling the arcs $\mathfrak{k} \cap B_0$. Similarly one may choose generators s_i' assigned to $\mathfrak{k} \cap B_1$; the s_i can be represented by curves in S_0^2, the s_i' by curves in S_1^2. It follows that $\hat{F}_0$ and $\hat{F}_1$ are connected. The restriction $p|\hat{F}_j$: $\hat{F}_j \to S_j^2$, $j = 0, 1$, are branched coverings with $2m$ branch points $\mathfrak{k} \cap S_j^2$ each. The genus g of $\hat{F}_0$ and $\hat{F}_1$ can easily be calculated via the Euler characteristic as follows: $p^{-1}(S_j^2 \cap \mathfrak{k}_i)$ consists of $2\lambda_i \mu_i$ points. Hence,

$$\chi(\hat{F}_0) = \chi(\hat{F}_1) = n + 2 \cdot \sum_{j=1}^{\mu} \lambda_i \mu_i - 2m \cdot n + n = 2 - 2g.$$

The balls B_j are covered by handlebodies $p^{-1}(B_j) = \hat{B}_j$ of genus g. This is easily seen by cutting the B_j along the disk δ_i^j and piecing copies of the resulting space together to obtain $\hat{B}_j$. The manifold M^3 is homeomorphic to the Heegaard splitting $\hat{B}_0 \cup_{\hat{h}} \hat{B}_1$. □

The homeomorphism $\hat{h}$: $\hat{F}_0 \to \hat{F}_1$ can be described in the following way. The braid $\mathfrak{z}$ determines a braid automorphism ζ which is induced by a homeomorphism h: $[S_0^2 - (\mathfrak{k} \cap S_0^2)] \to [S_1^2 - (\mathfrak{k} \cap S_1^2)]$. One may extend h to a homeomorphism h: $S_0^2 \to S_1^2$ and lift it to obtain $\hat{h}$:

$$\begin{tikzcd} \hat{F}_0 \arrow[r, "\hat{h}"] \arrow[d, "p"'] & \hat{F}_1 \arrow[d, "p"'] \\ S_0^2 \arrow[r, "h"] & S_1^2 \end{tikzcd}$$

Proposition 11.3 gives an upper bound for the *Heegaard genus* (g minimal) of a manifold M^3 obtained as a branched covering.

11.4 Proposition. *The Heegaard genus g^* of an n-fold branched covering of S^3 along the $2m$-plat $\mathfrak{k}$ satisfies the inequality*

$$g^* \leq m \cdot n - n + 1 - \sum_{i=1}^{\mu} \lambda_i \mu_i \leq (m-1)(n-1).$$

Proof. The second part of the inequality is obtained by putting $\mu_i = 1$. □

The 2-fold covering of knots or links with two bridges ($n = m = 2$) have Heegaard genus one – a well-known fact already observed by H. Seifert (see Chapter 12, [320, §4]). Of special interest are coverings with $g = 0$. In this case the covering space M^3 is a 3-sphere. There are many solutions of the equation $0 = mn - n + 1 - \sum_{i=1}^{\mu} \lambda_i \mu_i$; for instance, the 3-sheeted irregular coverings of 2-bridge knots, $m = 2$, $n = 3$, $\mu_i = 2$, (see Fox [110], Burde [54]). The braid $\mathfrak{z}$ of the plat then lifts to the braid $\hat{\mathfrak{z}}$ of the plat $\hat{\mathfrak{k}}$. Since $\hat{\mathfrak{z}}$ can be determined via the lifted braid automorphism $\hat{\zeta}$, $p\hat{\zeta} = \zeta p$, one can actually find $\hat{\mathfrak{k}}$. This was done for the trefoil by S. Kinoshita [195] and for the four-knot by G. Burde [54].

A simple calculation shows that our construction never yields genus zero for regular coverings – except in the trivial cases $n = 1$ or $m = 1$.

For fixed m and n the Heegaard genus of the covering space M^3 is minimized by choosing $\mu_i = n - 1$, $g = m + 1 - n$. These coverings are of the type used in our version of Alexander's Theorem 11.1. From this we get:

11.5 Proposition. *An orientable closed* 3*-manifold* M^3 *of Heegaard genus* g^* *is an* n*-fold branched covering with branching set a link* $\mathfrak{k} \subset S^3$ *with at least* $g^* + n - 1$ *bridges.* □

We propose to investigate the relation between the Heegaard splitting and the branched-covering description of a manifold M^3 in the special case of a 2-fold covering, $n = 2$. Genus and bridge number are then related by $m = g + 1$.

The covering $p|\hat{F}_0\colon\ \hat{F}_0 \to S_0^2$ is described in Figure 11.8.

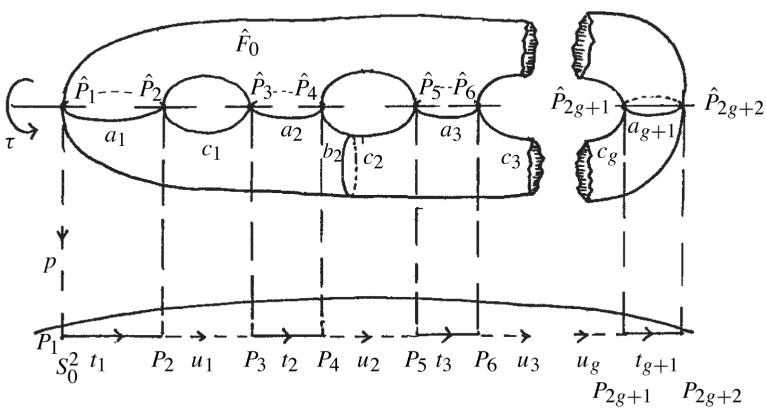


Figure 11.8. The 2-fold covering $p|\hat{F}_0\colon\ \hat{F}_0 \to S_0^2$.

Connect P_{2j} and P_{2j+1}, $1 \le j \le g$, by simple arcs u_j, such that $t_1u_1t_2u_2\dots u_gt_{g+1}$ is a simple arc, $t_i = S_0^2 \cap \delta_i^0$. A rotation through π about an axis which pierces $\hat{F}_0$ in the branch points $\hat{P}_j = p^{-1}(P_j)$, $1 \le j \le 2g+2$ is easily seen to be the covering transformation. The preimages $a_i = p^{-1}(t_i)$, $c_j = p^{-1}(u_j)$, $1 \le i \le g+1$, $1 \le j \le g$ are simple closed curves on $\hat{F}_0$. We consider homeomorphisms of the punctured sphere $S_0^2 - \bigcup_{j=1}^{2g+2} P_j$ which induce braid automorphisms, especially the homeomorphisms that induce the elementary braid automorphisms σ_k, $1 \le k \le 2g+1$. We extend them to S_0^2 and still denote them by σ_k. We are going to show that $\sigma_k\colon\ S_0^2 \to S_0^2$ lifts to a homeomorphism of $\hat{F}_0$, a so-called *Dehn twist*.

11.6 Definition (Dehn twist). Let a be a simple closed (unoriented) curve on a closed oriented surface F, and $U(a)$ a closed tubular neighborhood of a in F. A right-handed 2π-twist of $U(a)$ (Figure 11.9), extended by the identity map to F is called a *Dehn twist* α about a.

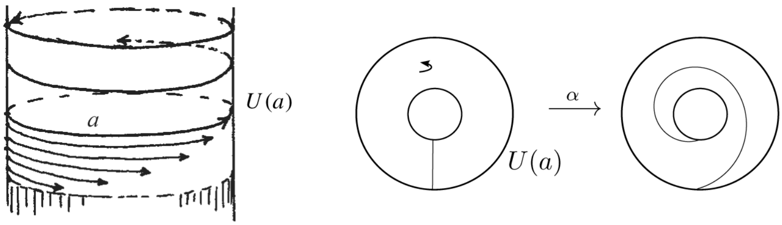


Figure 11.9. A right-handed Dehn twist along a.

Up to isotopy a Dehn twist is well defined by the simple closed curve a and a given orientation of F. Dehn twists are important because a certain finite set of Dehn twists generates the mapping class group of F – the group of autohomeomorphisms of F modulo the deformations (the homeomorphisms homotopic to the identity), see Dehn [86, 87].

11.7 Theorem (Dehn, Lickorish, Humphries). *The mapping class group of a closed orientable surface F of genus g is generated by $2g+1$ Dehn twists*

$$\alpha_1, \ldots, \alpha_g, \beta_2, \gamma_1, \ldots, \gamma_g$$

about the curves a_i, $1 \le i \le g+1$, b_2 and c_j, $1 \le j \le g$, as depicted in Figure 11.8.

For a *proof* see the book by B. Farb and D. Margalit [95, Chapter 4]. We remark that a left-handed twist about a is the inverse α^{-1} of the right-handed Dehn twist α about the same simple closed curve a. □

11.8 Lemma. *The homeomorphisms σ_{2i-1}, $1 \le i \le g+1$ lift to Dehn twists α_i about $a_i = p^{-1}(t_i)$ and the homeomorphisms σ_{2j}, $1 \le j \le g$, lift to Dehn twists γ_j about $c_j = p^{-1}(u_j)$.*

Proof. We may realize σ_{2i-1} by a half twist of a disk δ_i containing t_i (Figure 11.10), keeping the boundary $\partial\delta_i$ fixed.

The preimage $p^{-1}(\delta_i)$ consists of two annuli A_i and $\tau(A_i), A_i \cap \tau(A_i) = p^{-1}(t_i) = a_i$. The half twist of δ_i lifts to a half twist of A_i, and to a half twist of $\tau(A_i)$ in the opposite direction. Since $A_i \cap \tau(A_i) = a_i$, these two half twists add up to a full Dehn twist α_i along a_i. A similar construction shows that σ_{2j} is covered by a Dehn twist γ_j along $c_j = p^{-1}(u_j)$. □

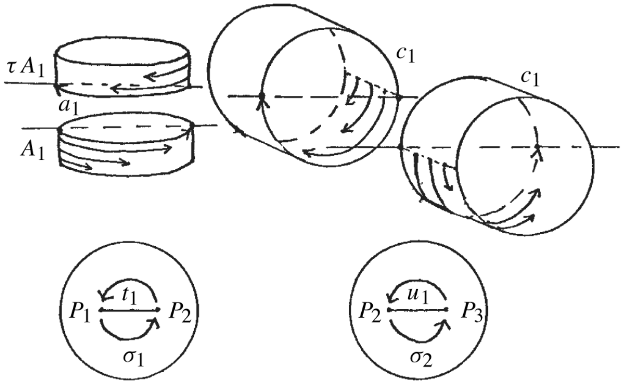


Figure 11.10. The homeomorphisms σ_i lift to Dehn twists.

There is an immediate corollary to Proposition 11.5, Theorem 11.7 and Lemma 11.8.

11.9 Corollary. *A closed oriented* 3*-manifold* M^3 *of Heegaard genus* $g \leq 2$ *is a* 2*-fold branched covering of* S^3 *with branching set a link* $\mathfrak{k} \subset S^3$ *with* $g+1$ *bridges.* □

There are, of course, closed oriented 3-manifolds which are not 2-fold coverings, if their Heegaard genus is at least three. The torus $S^1 \times S^1 \times S^1$ is a well-known example due to R. H. Fox [113].

11.10 Proposition (Fox [113]). *The manifold* $S^1 \times S^1 \times S^1$ *is not a* 2*-fold branched covering of* S^3*; its Heegaard genus is three.*

Proof. We have seen earlier that for any n-fold branched cyclic covering $\hat{C}_n$ of a knot the endomorphism $1+t+\ldots+t^{n-1}$ annihilates $H_1(\hat{C}_n)$ (Proposition 8.39 (b)). This holds equally for the second homology group, even if the branching set is merely a 1-complex. (It is even true for higher dimensions, see [113].) Let M^3 be a closed oriented manifold which is an n-fold cyclic branched covering of a homology sphere Q^3. We chose a triangulation of Q^3 and M^3 which is compatible with the branched covering map $p\colon M^3 \to Q^3$. Let $\{c_k^q\}$ denote the set of q-simplices of Q^3. For $q = 1, 2$, we choose $\hat{c}_k^q$ a simplex over c_k^q, $p(\hat{c}_k^q) = c_k^q$. Let $\langle t \rangle \cong \mathfrak{Z}_n$ denote the group covering transformations. We obtain a *transfer homomorphism* $\tau\colon C_q(Q^3) \to C_q(M^3)$ given by $\tau(c_k^q) = (1+t+\cdots+t^{n-1})\hat{c}_k^q$. The map τ is a chain homomorphism commuting with the boundary homomorphism. Notice that $p^{-1}(c_k^q) = \hat{c}_k^q \cup t\,\hat{c}_k^q \cup \cdots \cup t^{n-1}\,\hat{c}_k^q$ and that the simplices $t^i\,\hat{c}_k^q$ are not disjoint if c_k^q intersects in the branch set.

Let $\hat{c}^q = \sum_{i=0}^{n-1}\sum_k n_{ik} t^i \hat{c}_k^q$, $\partial\hat{c}^q = 0$, be a q-cycle of $C_q(M^3)$. Then $c^q := p_*(\hat{c}^q) = \sum_{i,k} n_{ik} c_k^q = \partial c^{q+1}$ is a boundary since Q^3 is a homology sphere. The following calculation shows that $(1 + t + \cdots + t^{n-1})\hat{c}^q$ is a boundary in $C_q(M^3)$:

$$\begin{aligned}\Big(\sum_{j=0}^{n-1} t^j\Big)\hat{c}^q &= \sum_{i,j,k} n_{ik} t^{i+j}\hat{c}^q_k = \sum_{i,k} n_{ik}\Big(\sum_j t^{i+j}\Big)\hat{c}^q_k \\ &= \Big(\sum_j t^j\Big)\Big(\sum_{i,k} n_{ik}\hat{c}^q_k\Big) = \tau(c^q) = \tau(\partial c^{q+1}) \\ &= \partial\,\tau(c^{q+1})\end{aligned}$$

and hence $(1 + t + \cdots + t^{n-1})\hat{c}^q \sim 0$.

Suppose $M = S^1_1 \times S^1_2 \times S^1_3$ is a 2-fold covering of S^3. One has

$$\pi_1(S^1_1 \times S^1_2 \times S^1_3) \cong H_1(S^1_1 \times S^1_2 \times S^1_3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

and t can be described by the (3×3)-matrix $-E$ with respect to the basis represented by the three factors. As $S^1_1 \times S^1_2 \times S^1_3$ is aspherical the covering transformation τ which induces t in the homology is homotopic to a map which inverts each of the 1-spheres S^1_i [341, Chap. 8, Theorem 11]. Poincaré duality assigns to each S^1_i a torus $S^1_j \times S^1_k$, i, j, k all different, which represents a free generator of $H_2(S^1_1 \times S^1_2 \times S^1_3)$. Thus t operates on $H_2(S^1_1 \times S^1_2 \times S^1_3)$ as the identity which contradicts $1 + t = 0$.

It is easy to see that $S^1 \times S^1 \times S^1$ can be presented by a Heegaard splitting of genus three – identify opposite faces of a cube K (Figure 11.11). After two pairs are identified one gets a thickened torus. Identifying its two boundary tori obviously gives $S^1 \times S^1 \times S^1$. On the other hand K_1 and $K_2 = \overline{K - K_1}$ become handlebodies of genus three under the identifying map. □

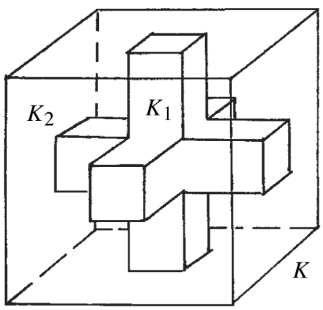


Figure 11.11

The method developed in this section can be used to study knots with two bridges by looking at their 2-fold branched covering spaces – a tool already used by H. Seifert [320, §4]. It was further developed by J. M. Montesinos who was able to classify a set of knots comprising knots with two bridges and pretzel knots by similar means. We shall take up the matter in Chapter 12.

We conclude this section by proving the following:

11.11 Theorem (Hilden–Montesinos). *Every closed orientable* 3*-manifold* M *is an irregular* 3*-fold branched covering of* S^3*. The branching set* $\mathfrak{k}$ *can be chosen in different ways, for instance as a knot or a link with unknotted components. If* g *is the Heegaard genus of* M*, it suffices to use a* $(g+2)$*-bridged branching set* $\mathfrak{k}$*.*

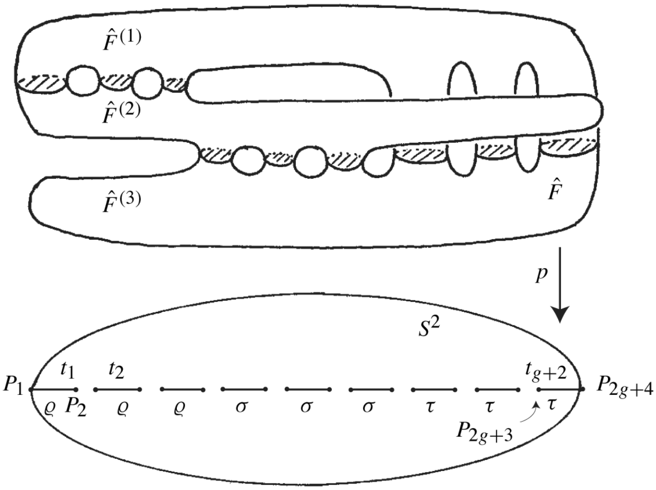


Figure 11.12. An irregular 3-fold branched coverings $p\colon \hat{F} \to S^2$ with branch indices ≤ 2.

Before starting on the actual proof in Paragraph 11.14 we study irregular 3-fold branched coverings $p\colon \hat{F} \to S^2$ of S^2 with branch indices ≤ 2. If $\hat{F}$ is an orientable closed surface of genus g, a calculation of $\chi(\hat{F})$ shows that the branching set in S^2 consists of $2(g+2)$ points P_i, $1 \leq i \leq 2(g+2)$. Let us denote by ϱ, σ, τ the transpositions $(1,2)$, $(2,3)$, $(1,3)$. Then by choosing $g+2$ disjoint simple arcs t_i, $1 \leq i \leq g+2$ in S^2, t_i connecting P_{2i-1} and P_{2i} (Figure 11.12), and assigning to each t_i one of the transpositions ϱ, σ, τ, we may construct a 3-fold branched covering $p\colon \hat{F} \to S^2$ (see Figure 11.12). The sheets $\hat{F}^{(j)}$, $1 \leq j \leq 3$, of the covering are homeomorphic to a 2-sphere with $g+2$ boundary components obtained from S^2 by cutting along the t_i, $1 \leq i \leq g+2$. Traversing an arc of $p^{-1}(t_i)$ in $\hat{F}$ means changing from $\hat{F}^{(j)}$ to $\hat{F}^{(\sigma(j))}$, if σ is assigned to t_i. (For $\hat{F}$ to be connected it is necessary and sufficient that at least two of the three transpositions are used in the construction.)

It will be convenient to use a very special version of such a covering. We assign ϱ to t_i, $1 \leq i \leq g+1$, and σ to t_{g+2} (Figure 11.13).

As in Figure 11.8 we introduce arcs u_j, $1 \leq j \leq g+1$, connecting P_{2j} and P_{2j+1}. We direct the t_i, u_j coherently (Figure 11.13) and lift these orientations. $p^{-1}(t_i)$, $1 \leq i \leq g+2$, consists of a closed curve a_i which will be regarded as unoriented,

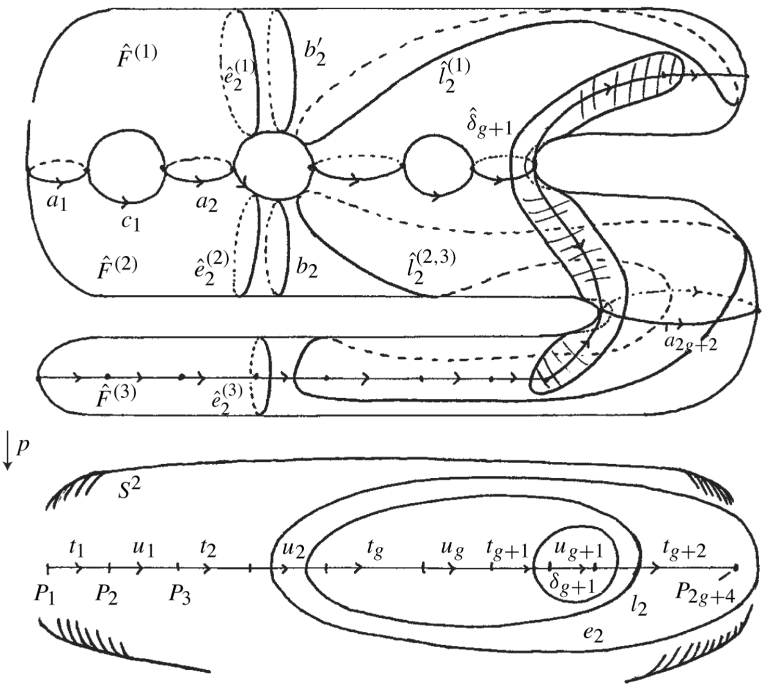


Figure 11.13

since its two parts carry opposite orientations, and an arc in $\hat{F}^{(3)}$ for $1 \le i \le g+1$, resp. in $\hat{F}^{(1)}$ for $i = g+2$. By the Dehn–Lickorish–Humphries Theorem 11.7 the mapping class group of $\hat{F}$ is generated by the Dehn twists α_i, β_2, γ_i, $1 \le i \le g$ about the curves a_i, b_2, c_i. Lemma 11.8 can be applied to the situation in hand: σ_{2i-1} in S^2 lifts to α_i, $1 \le i \le g+1$ and σ_{2j} lifts to γ_j, $1 \le j \le 2g$, because the effect of the lifting in $\hat{F}^{(3)}$ is isotopic to the identity. (Observe that σ_{2g+3} lifts to a deformation.) The only difficulty to overcome is to find homeomorphisms of $S^2 - \bigcup_{i=1}^{2g+4} P_i$ which can be lifted to homeomorphisms of $\hat{F}$ isotopic to the Dehn twist β_2. The Dehn twists β_k, $2 \le k \le g-1$ are provided by the following:

11.12 Lemma. *Let* $p\colon \hat{F} \to S^2$ *be the* 3*-fold branched covering described in Figure* 11.13.

(a) σ_{2i-1} *lifts to* α_i, $1 \le i \le g+1$; σ_{2j} *lifts to* γ_j, $1 \le j \le g$.

(b) $\omega_k = (\sigma_{2g+2}\sigma_{2g+1}\ldots\sigma_{2k+2}\sigma_{2k+1}^2\sigma_{2k+2}\ldots\sigma_{2g+2})^2$ *lifts to* β_k *for* $2 \le k \le g-1$.

(c) *The lifts of* ω_1 *resp.* ω_g *are isotopic to* α_1 *resp.* α_{g+1}.

(d) σ_{2g+2}^3 *and* σ_{2g+3} *lift to mappings isotopic to the identity.*

Proof. (a) was proved in 11.8. We proof (b): consider consider simple closed curves e_i, l_i, $1 \leq i \leq g+1$, in S^2 (Figure 11.13). The curve e_i lifts to three simple closed curves $\hat{e}_i^{(j)} \in \hat{F}^{(j)}$, $1 \leq j \leq 3$, while l_i^2 is covered by two curves $(\hat{l}_i^{(1)})^2$, $\hat{l}_i^{(2,3)}$ (Figure 11.13). This is easily checked by looking at the intersections of e_i and l_i with t_i and u_i, resp. at those of $\hat{e}_i^{(j)}$ and $\hat{l}_i^{(1)}, \hat{l}_i^{(2,3)}$ with a_i and c_i. Since $\hat{e}_k^{(1)} \simeq b'_k \simeq \hat{l}_k^{(1)}$, $\hat{e}_k^{(2)} \simeq b_k$ for $2 \leq k \leq g-1$, and $\hat{e}_i^{(3)} \simeq 1$, $1 \leq i \leq g+1$, a Dehn twist ε_k in S^2 along e_k lifts to the composition of the Dehn twists β_k and β'_k, while the square of the Dehn twist λ_k along l_k lifts to the composition of $(\beta'_k)^2$ and β_k. Thus $\varepsilon_k^2\lambda_k^{-2}$ lifts to $\beta_k^2{\beta'_k}^2(\beta'_k)^{-2}\beta_k^{-1} = \beta_k$. ε_k induces a braid automorphism resp. a $(2g+4)$-braid with strings $\{f_i | 1 \leq i \leq 2g+4\}$ represented by a full twist of the strings $f_{2k+1}, f_{2k+2}, \dots, f_{2g+4}$ (Figure 11.13). (Compare Figure 10.12). ε_k^2 is then a double twist and λ_k^{-2} a double twist in the opposite direction leaving out the last string f_{2g+4}. It follows that $\varepsilon_k^2\lambda_k^{-2}$ defines a braid $(\sigma_{2g+3}\sigma_{2g+2}\dots\sigma_{2k+1}^2\sigma_{2k+2}\dots\sigma_{2g+3})^2$ in which only the last string f_{2g+4} is not constant, encircling its neighbors $f_{2k+1}, \dots, f_{2g+3}$ to the left, twice. Since obviously σ_{2g+3} lifts to a deformation, (b) is proved.

Assertion (c) follows in the same way as (b). To prove (d) consider a disk δ_{g+1} which is a regular neighborhood of u_{g+1}. The third power $(\partial\delta_{g+1})^3$ of its boundary lifts to a simple closed curve in $\hat{F}$ bounding a disk $\hat{\delta}_{g+1} = p^{-1}(\delta_{g+1})$. The deformation σ_{2g+2}^3 in S^3 lifts to a "half-twist" of $\hat{\delta}_{g+1}$, a deformation of $\hat{F}$ which leaves the boundary $\partial\hat{\delta}_{g+1}$ pointwise fixed, and thus is isotopic to the identity. □

An easy consequence of Lemma 11.12 is the following:

11.13 Corollary. *For a given permutation $\pi \in \mathfrak{S}_{2g+4}$ there is a braid automorphism $\sigma \in \mathfrak{B}_{2g+4}$ with permutation π induced by a homeomorphism of $S^2 - \bigcup_{i=1}^{2g+4} P_i$ which lifts to a deformation of $\hat{F}$.*

Proof. Together with σ_{2g+2}^3 the conjugates

$$\sigma_i\sigma_{i+1}\dots\sigma_{2g+1}\sigma_{2g+2}^3\sigma_{2g+1}^{-1}\dots\sigma_i^{-1},\ 1 \leq i \leq 2g+1,$$

lift to deformations. Hence, the transpositions $(i, 2g+3) \in \mathfrak{S}_{2g+4}$ can be realized by deformations. Since σ_{2g+3} also lifts to a deformation, the lemma is proved. □

11.14 *Proof of Theorem* 11.11. Let $M = \hat{B}_0 \cup_{\hat{h}} \hat{B}_1$ be a Heegaard splitting of genus g, and $p_j\colon \hat{F}_j \to S_j^2$, $j \in \{0,1\}$, be 3-fold branched coverings of the type described in Figure 11.13, $\partial\hat{B}_j = \hat{F}_j$. Extend p_j to a covering $p_j\colon \hat{B}_j \to B_j$, $\partial B_j = S_j^2$, B_j a ball, in the same way as in the proof of Theorem 11.1. (Compare Figure 11.4.) The branching set of p_j consists in B_j of $g+2$ disjoint unknotted arcs, each joining a pair P_{2i-1}, P_{2i} of branch points.

By the Lemmas 11.8 and 11.12 there is a braid $\mathfrak{z}$ with given permutation π defining a homeomorphism $h\colon S_0^2 \to S_1^2$ which lifts to a homeomorphism isotopic to $\hat{h}\colon \hat{F}_0 \to \hat{F}_1$. The plat $\mathfrak{k}$ defined by $\mathfrak{z}$ is the branching set of a 3-fold irregular covering $p\colon M \to S^3$, and if π is suitably chosen, $\mathfrak{k}$ is a knot. In the case $\pi = \mathrm{id}$ the branching set $\mathfrak{k}$ consists of $g + 2$ trivial components. □

There are, of course, many plats $\mathfrak{k}$ defined by braids $\mathfrak{z} \in \mathfrak{B}_{2g+4}$ which by this construction lead to equivalent Heegaard diagrams and, hence, to homeomorphic manifolds. Replace $\mathfrak{k}$ by $\mathfrak{k}'$ with a defining braid $\mathfrak{z}' = \mathfrak{z}_1\mathfrak{z}\mathfrak{z}_0$ such that $\mathfrak{z}_i \subset B_i$, and $\mathfrak{k}' \cap B_i$ is a trivial half-plat (E 11.3). Then $\mathfrak{z}'$ lifts to a map $\hat{h}' = \hat{h}_1\hat{h}\hat{h}_0\colon \hat{F}_0 \to \hat{F}_1$, and there are homeomorphisms $\hat{H}_i\colon \hat{B}_i \to \hat{B}_i$ extending the homeomorphisms $\hat{h}_i\colon \hat{F}_i \to \hat{F}_i = \partial\hat{B}_i$, $i \in \{0,1\}$. Obviously $\hat{B}_0 \cup_{\hat{h}'} \hat{B}_1$ and $\hat{B}_0 \cup_{\hat{h}} \hat{B}_1$ are homeomorphic. The braids $\mathfrak{z}_i$ of this type form a finitely generated subgroup in $\mathfrak{B}_{2g+4}$ (Exercise E 11.3, Hilden [164] and Reidemeister [301]).

Lemmas 11.8 and 11.12 can be exploited to give some information on the mapping class group $M(g)$ of an orientable closed surface of genus g (see also Farb and Margalit [95, 3.5, 9.4]). The group $M(1)$ is well known [129], and will play an important role in Chapter 12. By Lemma 11.8 and Corollary 11.9, $M(2)$ is a homomorphic image of the braid group $\mathfrak{B}_6$. A presentation is known [24]. Since one string of the braid of $\mathfrak{B}_6$ can be kept constant, $M(2)$ is even a homomorphic image of $\mathfrak{B}_5$. For $g > 2$ the group $M(g)$ is a homomorphic image of the subgroup $\mathfrak{J}^*_{2g+3}$ of $\mathfrak{J}_{2g+3}$ generated by $\mathfrak{J}_{2g+2} \subset \mathfrak{J}_{2g+3}$ and the pure $(2g+3)$-braids ω_k, $2 \le k \le g-1$, of Lemma 11.12 (b). There is, however, a kernel $\neq 1$, which was determined in J. S. Birman and B. Wajnryb [26]. This leads to a presentation of $M(g)$, see also J. McCool [234], A. Hatcher and W. P. Thurston [155], and B. Wajnryb [363].

11.C History and sources

J. W. Alexander [3] proved that every closed oriented n-manifold M is a branched covering of the n-sphere. The branching set is an $(n-2)$-subcomplex. Alexander claims in his paper (without giving a proof) that for $n = 3$ the branching set can be assumed to be a closed submanifold – a link in S^3. J. S. Birman and H. M. Hilden [25] gave a proof, and, at the same time, obtained some information on the relations between the Heegaard genus of M, the number of sheets of the covering and the bridge number of the link. Finally, H. M. Hilden [165] and J. M. Montesinos [245] independently showed that every orientable closed 3-manifold is a 3-fold irregular covering of S^3 over a link $\mathfrak{k}$. It suffices to confine oneself to rather special types of branching sets $\mathfrak{k}$ [166].

11.D Exercises

E 11.1. Show that a Dehn twist α of an orientable surface F along a simple closed (unoriented) curve a in F is well defined (up to a deformation) by a and an orientation of F. Dehn twists α and α' represent the same element of the mapping class group ($\alpha' = \delta\alpha$, δ a deformation) if the corresponding curves are isotopic.

E 11.2. Apply the method of Lemma 11.2 to the following situation: Let p: $S^3 \to S^3$ be the cyclic 3-fold covering branched along the triangle A, B, C (Figure 11.14). Replace the branch set outside the balls around the vertices of the triangle as was done in the proof of Theorem 11.1. It follows that the 3-fold irregular covering along a trefoil is also a 3-sphere.

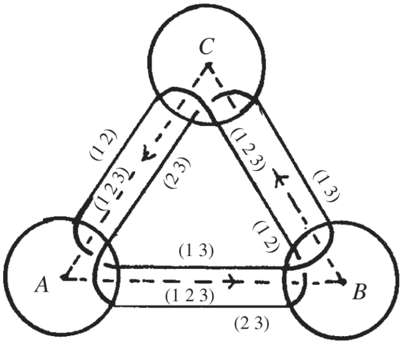


Figure 11.14. The 3-fold irregular covering along a trefoil is S^3.

E 11.3. Let $\mathfrak{k}$ be a $2m$-plat in 3-space $\mathbb{R}^3$ and (x, y, z) Cartesian coordinates of $\mathbb{R}^3$. Suppose $z = 0$ meets $\mathfrak{k}$ transversally in the $2m$ points $P_i = (i, 0, 0)$, $1 \le i \le 2m$. We call the intersection of $\mathfrak{k}$ with the upper half-space $\mathbb{R}^3_0 = \{(x, y, z) \mid z \ge 0\}$ a *half-plat* $\mathfrak{k}_0$, and denote its defining braid by $\mathfrak{z}_0 \in \mathfrak{B}_{2m}$. *The half-plat* $\mathfrak{k}_0$ *is trivial* if it is isotopic in $\mathbb{R}^3_0$ to m straight lines α_i in $x = 0$, $\partial\alpha + i = \{P_{2i-1}, P_{2i}\}$.

Show that the braids $\mathfrak{z}_0 \in \mathfrak{B}_{2m}$ defining trivial half-plats form a subgroup of $\mathfrak{B}_{2m}$ generated by the braids σ_{2i-1}, $1 \le i \le m$, $\varrho_k = \sigma_{2k}\sigma_{2k-1}\sigma_{2k+1}\sigma_{2k}$, $\tau_k = \sigma_{2k}\sigma_{2k-1}\sigma^{-1}_{2k+1}\sigma^{-1}_{2k}$, $1 \le k \le m-1$ (see Hilden [164]).

E 11.4. Construct $S^1 \times S^1 \times S^1$ as a 3-fold irregular covering of S^3 along a 5-bridged knot.

Chapter 12

Montesinos links

This chapter contains a study of a special class of knots. Section 12.A deals with the 2-bridge knots which are classified by their 2-fold branched coverings – a method due to H. Seifert. (H. Schubert published a proof of this result in [320, Satz 6], and he attributed the theorem to H. Seifert.)

Section 12.B looks at 2-bridge knots as 4-plats (Viergeflechte). This yields interesting geometric properties and a normal form due to L. Siebenmann [333]. Siebenmann's normal form is used in Section 12.C to derive some properties concerning the genus and the possibility of fibering the complement, see K. Funcke [124] and R. I. Hartley [147].

Section 12.D is devoted to the classification of the Montesinos links which generalize knots and links with two bridges with respect to the property that their 2-fold branched coverings are Seifert fiber spaces. These knots were introduced by J. M. Montesinos [243, 244], and the classification, conjectured by him, was given by F. Bonahon [38]. Here we present the proof given by H. Zieschang [381]. The last part deals with results of F. Bonahon and L. Siebenmann [37] and M. Boileau [32] from 1979 on the symmetries of Montesinos links. We prove these results following the lines of M. Boileau and B. Zimmermann [36] where a complete classification of all non-elliptic Montesinos links is given.

Montesinos knots also include the so-called pretzel knots which furnished the first examples of non-invertible knots (see Trotter [356].)

12.A Schubert's normal form of knots and links with two bridges

H. Schubert [320] classified knots and links with two bridges. His proof is a thorough and quite involved geometric analysis of the problem, his result a complete classification of these oriented knots and links. Each knot is presented in a normal form – a distinguished projection.

If one considers these knots as unoriented, their classification can be shown to rest on the classification of 3-dimensional lens spaces. This was already noticed by Seifert (see [320, Satz 6]).

12.1 Schubert's projection. We start with some geometric properties of a 2-bridge knot, using Schubert's terminology. The knot $\mathfrak{k}$ meets a projection plane $\mathbb{R}^2 \subset \mathbb{R}^3$ at four points: A, B, C, D. The plane $\mathbb{R}^2$ defines an upper and a lower half-space,

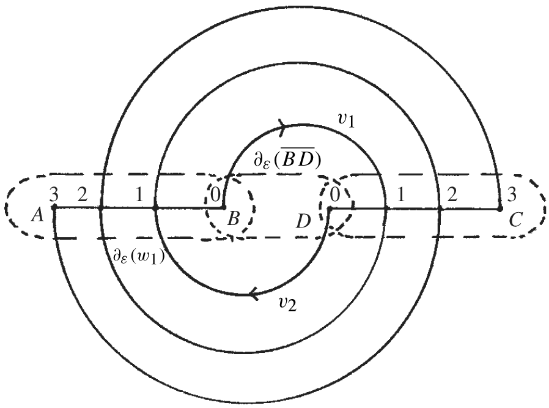


Figure 12.1. The reduced diagram of the 2-bridge knot $\mathfrak{b}(3,1)$.

and each of them intersects $\mathfrak{k}$ in two arcs. Each pair of arcs can be projected onto $\mathbb{R}^2$ without double points (see Definition 2.11). We may assume that one pair of arcs is projected onto straight segments $w_1 = AB$, $w_2 = CD$ (Figure 12.1); the other pair is projected onto two disjoint simple curves v_1 (from B to C) and v_2 (from D to A). The diagram can be reduced in the following way: v_1 first meets w_2. A first double point on w_1 can be removed by an isotopy. In the same way one can arrange for each arc v_i to meet the w_j alternately, and for each w_j to meet the v_i alternately. The number of double points, hence, is even in a reduced diagram with $\alpha - 1$ $(\alpha \in \mathbb{N}^*)$ double points on w_1 and on w_2. We attach numbers to these double points, counting against the orientation of w_1 and w_2 (Figure 12.1). Observe that for a knot α is odd; α even and $\partial v_1 = \{A, B\}$, $\partial v_2 = \{C, D\}$ yields a link.

12.2. We now add a point ∞ at infinity, $S^3 = \mathbb{R}^3 \cup \{\infty\}$, $S^2 = \mathbb{R}^2 \cup \{\infty\}$, and consider the two-fold branched covering T of S^2 with the branch set $\{A, B, C, D\}$, $\hat{p}$: $T \to S^2$, see Figure 12.2. The covering transformation τ: $T \to T$ is a rotation through angle π about an axis which pierces T at the points $\hat{A} = \hat{p}^{-1}(A)$, $\hat{B} = \hat{p}^{-1}(B)$, $\hat{C} = \hat{p}^{-1}(C)$, $\hat{D} = \hat{p}^{-1}(D)$.

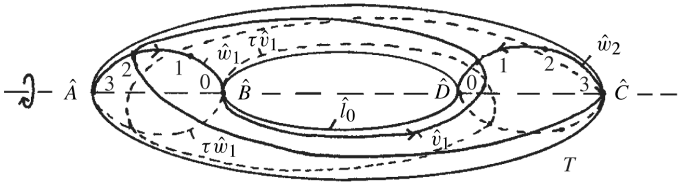


Figure 12.2

The arcs w_1 and w_2 lift to $\{\hat{w}_1, \tau\hat{w}_1\}$, $\{\hat{w}_2, \tau\hat{w}_2\}$ and in the notation of homotopy chains, see Paragraph 9.1, $(1-\tau)\hat{w}_1$ and $(1-\tau)\hat{w}_2$ are isotopic simple closed curves on T. Likewise, $(1-\tau)\hat{v}_1$, $(1-\tau)\hat{v}_2$ are two isotopic simple closed curves on T, each mapped onto its inverse by τ. They intersect with the $(1-\tau)\hat{w}_i$ alternately:

$$\operatorname{int}((1-\tau)\hat{v}_i, (1-\tau)\hat{w}_j) = \alpha.$$

Denote by $\partial_\varepsilon(c)$ the boundary of a small tubular neighborhood of an arc c in $\mathbb{R}^2$. We choose an orientation on $\mathbb{R}^2$, and let $\partial_\varepsilon(c)$ have the induced orientation. The curve $\partial_\varepsilon(w_i)$ lifts to two curves isotopic to $\pm(1-\tau)\hat{w}_i$, $1 \le i \le 2$. The preimage $p^{-1}(\partial_\varepsilon(\overline{BD}))$ consists of two curves; one of them, $\hat{\ell}_0$ together with $\hat{m}_0 = (1-\tau)\hat{w}_1$ can be chosen as canonical generators of $H_1(T)$ – we call $\hat{m}_0$ a meridian, and $\hat{\ell}_0$ a longitude. Equally $p^{-1}(\partial_\varepsilon(v_i))$ consists of two curves isotopic to $\pm(1-\tau)\hat{v}_i$.

We assume for the moment $\alpha > 1$. (This excludes the trivial knot and a splittable link with two trivial components.) Then $(1-\tau)\hat{v}_i = \beta\hat{m}_0 + \alpha\hat{\ell}_0$ where $\beta \in \mathbb{Z}$ is positive, if at the first double point of v_1 the arc w_2 crosses from left to right in the double point $|\beta|$, and negative otherwise. From the construction it follows that $|\beta| < \alpha$ and that $\gcd(\alpha, \beta) = 1$.

12.3 Proposition. *For any pair α, β of integers subject to the conditions*

$$\alpha > 0,\ -\alpha < \beta < +\alpha,\ \gcd(\alpha,\beta) = 1,\ \beta \text{ odd}, \tag{12.1}$$

there is a knot or link with two bridges $\mathfrak{k} = \mathfrak{b}(\alpha,\beta)$ with a reduced diagram with numbers α, β. We call α the torsion*, and β the* crossing number *of $\mathfrak{b}(\alpha,\beta)$. The number of components of $\mathfrak{b}(\alpha,\beta)$ is $\mu \equiv \alpha \mod 2$, $1 \le \mu \le 2$. The 2-fold covering of S^3 branched along $\mathfrak{b}(\alpha,\beta)$ is the lens space $L(\alpha,\beta)$.*

Proof. We first prove the last assertion. Suppose $\mathfrak{k} = \mathfrak{b}(\alpha,\beta)$ is a knot with two bridges whose reduced diagram determines the numbers α and β. We try to extend the covering $p\colon T \to S^2$ to a covering of S^3 branched along $\mathfrak{b}(\alpha,\beta)$. Denote by B_0, B_1 the two balls bounded by S^2 in S^3 with $\mathfrak{k} \cap B_0 = w_1 \cup w_2$. The 2-fold covering $\hat{B}_i$ of B_i branched along $B_i \cap \mathfrak{k}$ can be constructed by cutting B_i along two disjoint disks δ_1^i, δ_2^i spanning the arcs $B_i \cap \mathfrak{k}$, $i = 0, 1$.

This defines a sheet of the covering, and $\hat{B}_i$ itself is obtained by identifying corresponding cuts of two such sheets. $\hat{B}_i$, $0 \le i \le 1$, is a solid torus, and $(1-\tau)\hat{w}_1 = \hat{m}_0$ represents a meridian of $\hat{B}_0$ while $\hat{m}_1 = (1-\tau)\hat{v}_1$ represents a meridian of $\hat{B}_1$. This follows from the definition of the curves $\partial_\varepsilon(v_i)$, $\partial_\varepsilon(w_i)$. Since

$$\hat{m}_1 = (1-\tau)\hat{v}_1 \simeq \beta\hat{m}_0 + \alpha\hat{\ell}_0, \tag{12.2}$$

the covering $\hat{B}_0 \cup_T \hat{B}_1$ is the Heegaard splitting of the lens space $L(\alpha,\beta)$.

Further information is obtained by looking at the universal covering $\tilde{T} \cong \mathbb{R}^2$ of T. The curve $\hat{v}_1$ is covered by $\tilde{v}_1$ which may be drawn as a straight line through a lattice

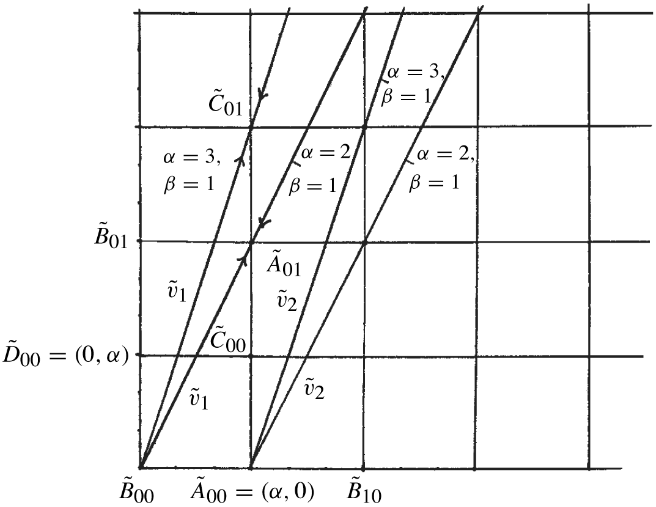


Figure 12.3

point over $\hat{B}$ and another over $\hat{C}$ (resp. $\hat{A}$) for α odd (resp. α even). If Cartesian coordinates are introduced with $\tilde{B}_{00}$ as the origin and $\tilde{D}_{00} = (0,\alpha)$, $\tilde{A}_{00} = (\alpha, 0)$, see Figure 12.3, $\tilde{v}_1$ is a straight line through $(0,0)$ and (β,α), and $\tilde{v}_2$ is a parallel through $(\alpha, 0)$ and $(\alpha+\beta,\alpha)$. The $2\alpha \times 2\alpha$ square is a fundamental domain of the covering $\tilde{p}$: $\tilde{T} \to T$. Any pair of coprime integers (α,β) defines such curves which are projected onto simple closed curves of the form $(1-\tau)\hat{v}_i$ on T, and, by $\hat{p}$: $T \to S^2$, onto a reduced diagram. □

One may choose $\alpha > 0$. If $\tilde{v}_1$ starts in $\tilde{B}_{00}$, it ends in $(\beta\alpha, \alpha^2)$. Thus $\beta \equiv 1 \mod 2$, since v_1 ends in C or A.

We attached numbers γ to the double points of the reduced projection of $\mathfrak{b}(\alpha,\beta)$ (Figure 12.1). To take into account also the characteristic of the double point we assign a residue class modulo 2α to it, represented by γ (resp. $-\gamma$) if w_i crosses v_j from left to right (resp. from right to left). Running along v_i one obtains the sequence:

$$0, \beta, 2\beta, \ldots, (\alpha-1)\beta \text{ modulo } 2\alpha. \tag{12.3}$$

This follows immediately by looking at the universal covering $\tilde{T}$ (Figure 12.3). Note that $\tilde{v}_i$ is crossed from right to left in the strips where the attached numbers run from right to left, and that $-(\alpha-\delta) \equiv \alpha+\delta$ modulo 2α.

12.4 Remark. It is common use to normalize the invariants α, β of a lens space in a different way. In this usual normalization, $L(\alpha,\beta)$ is given by $L(\alpha,\beta^*)$ where $0 < \beta^* < \alpha$, $\beta^* \equiv \beta \mod \alpha$.

12.5 Proposition. *Knots and links with two bridges are invertible.*

Proof. A rotation through angle π about the core of the solid torus $\hat{B}_0$ (or $\hat{B}_1$) commutes with the covering transformation τ. It induces therefore a homeomorphism of $S^2 = p(T)$ – a rotation through angle π about the centers of w_1 and w_2 (resp. v_1 and v_2) if the reduced diagram is placed symmetrically on S^2. This rotation can be extended to an isotopy of S^3 which carries $\mathfrak{k}$ onto $-\mathfrak{k}$. □

12.6 Theorem (H. Schubert). (a) $\mathfrak{b}(\alpha,\beta)$ *and* $\mathfrak{b}(\alpha',\beta')$ *are equivalent as oriented knots (or links), if and only if:*

$$\alpha = \alpha',\ \beta^{\pm 1} \equiv \beta' \mod 2\alpha.$$

(b) $\mathfrak{b}(\alpha,\beta)$ *and* $\mathfrak{b}(\alpha',\beta')$ *are equivalent as unoriented knots (or links), if and only if:*

$$\alpha = \alpha',\ \beta^{\pm 1} \equiv \beta' \mod \alpha.$$

Here β^{-1} *denotes the integer with the properties* $-\alpha < \beta^{-1} < \alpha$ *and* $\beta\beta^{-1} \equiv 1 \mod 2\alpha$. □

For the proof of (a) we refer to H. Schubert [320], see also Turaev [357, §3.1]. The weaker statement (b) follows from the classification of lens spaces, see Reidemeister [299], Brody [43] and Turaev [359].

12.7 Remark. In the case of knots (α odd) Theorem 12.6 (a) and (b) are equivalent – this follows also from Proposition 12.5. For links, Schubert gave examples which show that *one can obtain non-equivalent links (with linking number zero) by reversing the orientation of one component.* (A link $\mathfrak{b}(\alpha,\beta)$ is transformed into $\mathfrak{b}(\alpha,\beta')$, $\beta' \equiv \alpha + \beta \mod 2\alpha$, if one component is reoriented). The link $\mathfrak{b}(32,7)$ is an example. The sequence (12.3) can be used to compute the linking number $\mathrm{lk}(\mathfrak{b}(\alpha,\beta))$ of the link:

$$\mathrm{lk}\big(\mathfrak{b}(\alpha,\beta)\big) = \sum_{\nu=1}^{\frac{\alpha}{2}} \varepsilon_\nu, \quad \varepsilon_\nu = (-1)^{\left[\frac{(2\nu-1)\beta}{\alpha}\right]}.$$

($[a]$ denotes the integral part of a.) One obtains for $\alpha = 32$, $\beta = 7$:

$$\sum_{\nu=1}^{16} \varepsilon_\nu = 1+1-1-1-1+1+1-1-1+1+1-1-1-1+1+1 = 0.$$

12.8 Exchanging B_0 and B_1. Lastly, our construction has been unsymmetrical with respect to B_0 and B_1. If the balls are exchanged, $(\hat{m}_0, \hat{\ell}_0)$ and $(\hat{m}_1, \hat{\ell}_1)$ have to change places, where $\hat{m}_1$ is defined by (12.2) and forms a canonical basis together with $\hat{\ell}_1$:

$$\begin{matrix} \hat{m}_1 = \beta\hat{m}_0 + \alpha\hat{\ell}_0 \\ \hat{\ell}_1 = \alpha'\hat{m}_0 + \beta'\hat{\ell}_0 \end{matrix}, \quad \begin{vmatrix} \beta & \alpha \\ \alpha' & \beta' \end{vmatrix} = 1.$$

It follows $\hat{m}_0 = \beta'\hat{m}_1 - \alpha\hat{\ell}_1$. Since B_0 and B_1 induce opposite orientations on their common boundary, we may choose $(\hat{m}_1, -\hat{\ell}_1)$ as canonical curves on T. Thus $\mathfrak{b}(\alpha,\beta) = \mathfrak{b}(\alpha,\beta')$, $\beta\beta' - \alpha\alpha' = 1$, i.e. $\beta\beta' \equiv 1 \mod \alpha$.

A reflection in a plane perpendicular to the projection plane and containing the straight segments w_i transforms a normal form $\mathfrak{b}(\alpha,\beta)$ into $\mathfrak{b}(\alpha,-\beta)$. Therefore $\mathfrak{b}^*(\alpha,\beta) = \mathfrak{b}(\alpha,-\beta)$.

12.B 4-Plats (Viergeflechte)

Knots with two bridges were first studied in the form of 4-plats (see Chapter 2.D) by C. Bankwitz and G. H. Schumann [14], and certain advantages of this point of view will become apparent in the following. We return to the situation described in Section 11.B (Figure 11.7).

12.9 Euclidean algorithm. The sphere S^3 now is comprised of two balls B_0, B_1 and $I \times S^2$ in-between, containing a 4-braid $\mathfrak{z}$ which defines a 2-bridge knot $\mathfrak{b}(\alpha,\beta)$, see Figure 11.7. The 2-fold branched covering M^3 is by Proposition 12.3 the lens space $L(\alpha,\beta)$. (In this section we always choose $0 < \beta < \alpha$, β odd or even.) Lemma 11.8 shows that the braid operations σ_1, σ_2 lift to Dehn twists δ_1, δ_2 such that

$$\delta_1(\hat{m}_0) = \hat{m}_0, \qquad \delta_2(\hat{m}_0) = \hat{m}_0 - \hat{\ell}_0,$$
$$\delta_1(\hat{\ell}_0) = \hat{m}_0 + \hat{\ell}_0, \qquad \delta_2(\hat{\ell}_0) = \hat{\ell}_0.$$

Thus we may assign to σ_1, σ_2 matrices

$$\sigma_1 \mapsto A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

which describe the linear mappings induced on $H_1(\hat{F}_0)$ by δ_1, δ_2 with respect to the basis $\hat{m}_0, \hat{\ell}_0$. Note that this assignment defines a homomorphism form $\mathfrak{B}_3$ onto $\mathrm{SL}_2(\mathbb{Z})$. A braid $\mathfrak{z} = \sigma_2^{-a_1}\sigma_1^{a_2}\sigma_2^{-a_3}\dots\sigma_2^{-a_m}$ induces the transformation

$$A = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}\dots\begin{pmatrix} 1 & a_{m-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_m & 1 \end{pmatrix}. \tag{12.4}$$

Suppose the 2-fold covering M^3 of a 4-plat as in Figure 11.7 is given by a Heegaard splitting $M^3 = T_0 \cup_{\hat{h}} T_1$, $\partial T_j = \hat{F}_j$. Relative to bases $(\hat{m}_0, \hat{\ell}_0)$, $(\hat{m}_1, \hat{\ell}_1)$ of $H_1(\hat{F}_0)$, $H_1(\hat{F}_1)$, the isomorphism $\hat{h}_*\colon H_1(\hat{F}_0) \to H_1(\hat{F}_1)$ is represented by a unimodular matrix:

$$A = \begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix}; \quad \alpha, \alpha', \beta, \beta' \in \mathbb{Z};\ \beta\beta' - \alpha\alpha' = 1.$$

The integers α' and β' are determined up to a change $\alpha' \mapsto \alpha' + c\beta, \beta' \mapsto \beta' + c\alpha$ which can be achieved by

$$\begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta & \alpha' + c\beta \\ \alpha & \beta' + c\alpha \end{pmatrix}.$$

This corresponds to a substitution $\mathfrak{z} \mapsto \mathfrak{z}\sigma_1^c$ which does not alter the plat. The product (12.4) defines a sequence of equations $(r_0 = \alpha, r_1 = \beta)$:

$$\begin{aligned} r_0 &= a_1 r_1 + r_2 \\ r_1 &= a_2 r_2 + r_3 \\ &\vdots \\ r_{m-1} &= a_m r_m + 0, |r_m| = 1, \end{aligned} \tag{12.5}$$

following from the equalities

$$\begin{pmatrix} 1 & 0 \\ -a_i & 1 \end{pmatrix} \begin{pmatrix} r_i & * \\ r_{i-1} & * \end{pmatrix} = \begin{pmatrix} r_i & * \\ r_{i-1} - a_i r_i & * \end{pmatrix} = \begin{pmatrix} r_i & * \\ r_{i+1} & * \end{pmatrix},$$

$$\begin{pmatrix} 1 & -a_{i+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_i & * \\ r_{i+1} & * \end{pmatrix} = \begin{pmatrix} r_i - a_{i+1} r_{i+1} & * \\ r_{i+1} & * \end{pmatrix} = \begin{pmatrix} r_{i+2} & * \\ r_{i+1} & * \end{pmatrix}.$$

If we assume $0 \le r_i < r_{i-1}$, then equations (12.5) describe a Euclidean algorithm which is uniquely defined by $\alpha = r_0$ and $\beta = r_1$.

12.10 Definition. We call a system of equations (12.5) with $r_i, a_j \in \mathbb{Z}$, a generalized Euclidean algorithm of length m if $0 < |r_i| < |r_{i-1}|$, $1 \le i \le m$, and $r_0 \ge 0$.

Such an algorithm can also be expressed by a continued fraction:

$$\frac{\beta}{\alpha} = \frac{r_1}{r_0} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \quad \cdots + a_{m-1} + \frac{1}{a_m}}}} = [a_1, a_2, \ldots, a_m].$$

The integers a_i are called the *quotients of the continued fraction.* From $0 < |r_m| < |r_{m-1}|$ it follows that $|a_m| \ge 2$. We allow the augmentation

$$[a_1, a_2, \ldots, (a_m \pm 1), \mp 1] = [a_1, a_2, \ldots, a_m], \tag{12.6}$$

since

$$a_m \pm 1 + \frac{1}{\mp 1} = a_m.$$

Thus, by allowing $|r_{m-1}| = |r_m| = 1$, we may assume m to be odd.

12.11. To return to the 2-bridge knot $\mathfrak{b}(\alpha, \beta)$ we assume $\alpha > 0$ and $0 \leq \beta < \alpha$, $\gcd(\alpha, \beta) = 1$. For any integral solution of equations (12.5) with $r_0 = \alpha, r_1 = \beta$, one obtains a matrix equation:

$$\begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_m & 1 \end{pmatrix} \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}, m \text{ odd}, \tag{12.7}$$

$$\begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & * \end{pmatrix}, m \text{ even}. \tag{12.8}$$

The first equation (m odd) shows that a 4-plat defined by the braid

$$\mathfrak{z} = \sigma_2^{-a_1} \sigma_1^{a_2} \sigma_2^{-a_3} \ldots \sigma_2^{-a_m}$$

is the knot $\mathfrak{b}(\alpha, \beta)$, since its 2-fold branched covering is the (oriented) lens space $L(\alpha, \beta)$. The last factor on the right represents a power of σ_1 which does not change the knot, and which induces a homeomorphism of $\hat{B}_1$. In the case when m is even observe that

$$\begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

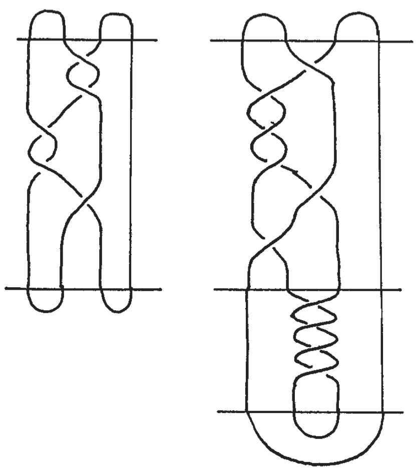


Figure 12.4

From this it follows (Figure 12.4) that $\mathfrak{b}(\alpha, \beta)$ is defined by $\mathfrak{z} = \sigma_2^{-a_1} \sigma_1^{a_2} \ldots \sigma_2^{-a_m}$ but that the plat has to be closed at the lower end in a different way, switching meridian $\hat{m}_1$ and longitude $\hat{\ell}_1$ corresponding to the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

12.12 Remark. The case $\alpha = 1$, $\beta = 0$, is described by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix}.$$

The corresponding plat (Figure 12.5) is a trivial knot. The matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix}$$

is characterized by the pair $(0,1) = (\alpha, \beta)$ (Figure 12.5). It is therefore reasonable to denote by $\mathfrak{b}(1,0)$ resp. $\mathfrak{b}(0,1)$ the unknot resp. two split unknotted components, and to put: $L(1,0) = S^3$, $L(0,1) = S^1 \times S^2$. The connection between the numbers a_i and the quotient $\beta\alpha^{-1}$ allows to invent many different normal forms of (unoriented) knots with two bridges as 4-plats. All it requires is to make the algorithm (12.5) unique and to take into account that the balls B_0 and B_1 are exchangeable.

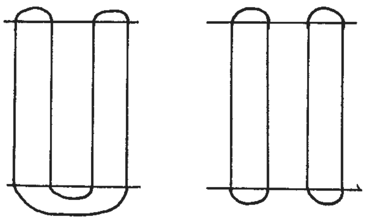


Figure 12.5

12.13 Proposition. *The (unoriented) knot (or link)* $\mathfrak{b}(\alpha, \beta)$, $0 < \beta < \alpha$, *has a presentation as a* 4*-plat with a defining braid* $\mathfrak{z} = \sigma_2^{-a_1}\sigma_1^{a_2}\dots\sigma_2^{-a_m}$, $a_i > 0$, *m odd, where the* a_i *are the quotients of the continued fraction* $[a_1, \dots, a_m] = \beta\alpha^{-1}$. *Sequences* $(a_1, \dots, a_m)$ *and* $(a'_1, \dots, a'_{m'})$ *define the same knot or link if and only if* $m = m'$, *and* $a_i = a'_i$ *or* $a_i = a'_{m+1-i}$, $1 \le i \le m$.

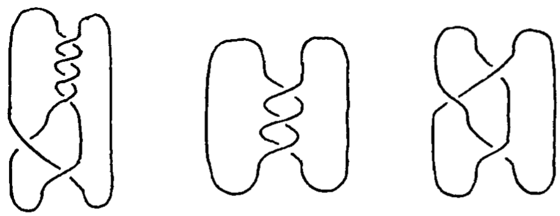
Proof. The algorithm (12.5) is unique, since $a_i > 0$ implies that $r_i > 0$ for $1 \le i \le m$. The expansion of $\beta\alpha^{-1}$ as a continued fraction of odd length m is unique, see [289]. A rotation through angle π about an axis in the projection plane containing $\overline{AB}$ and $\overline{CD}$ finally exchanges B_0 and B_1; its lift exchanges $\hat{B}_0$ and $\hat{B}_1$. □

Remark 12.14. It is an easy exercise in continued fractions (E 12.3) to prove $\beta'\alpha^{-1} = [a_m \dots, a_1]$ if $\beta\alpha^{-1} = [a_1, \dots, a_m]$, and $\beta\beta' \equiv 1 \mod \alpha$.

Note that the normal form of 4-plats described in Proposition 12.13 represents alternating plats, hence:

12.15 Proposition (Bankwitz–Schumann). *Knots and links with two bridges are alternating.* □

12.16 Examples. Consider $\mathfrak{b}(9,2) = 6_1^*$ as an example: $2/9 = [4,1,1]$. The corresponding plat is defined by $\sigma_2^{-4}\sigma_1\sigma_2^{-1}$ (Figure 12.6). (Verify: $5/9 = [1,1,4]$, $2{\cdot}5 \equiv 1$ mod 9.) Figure 12.6 also shows the normal forms of the two trefoils: $\mathfrak{z} = \sigma_2^{-3}$ corresponds to the right-handed trefoil and $\mathfrak{z}' = \sigma_2^{-1}\sigma_1\sigma_2^{-1}$ corresponds to the left-handed trefoil, according to $1/3 = [3]$, $2/3 = [1,1,1]$ and $1 \cdot 2 \equiv -1 \bmod 3$.



$\mathfrak{b}(9,2) = 6_1^*$ $\mathfrak{b}(3,1)$ $\mathfrak{b}(3,2)$

Figure 12.6

A generalized Euclidean algorithm is, of course, not unique. One may impose various conditions on it to make it so, for instance, the quotients a_i, $1 \le i \le m$ can obviously be chosen either even or odd. Combining such conditions for the quotients with $r_j > 0$ for some j gives many possibilities for normal forms of 4-plats. The following generalized Euclidean algorithm was used by L. Siebenmann [333].

12.17 Proposition. *Let α and β be integers of different parity such that $\alpha > 0$, $-\alpha < \beta < \alpha$ and* $\gcd(\alpha,\beta) = 1$. *There is a unique generalized Euclidean algorithm $(r_0 = \alpha, r_1 = \beta)$:*

$$r_{i-1} = c_i r_i + r_{i+1}$$

of length m with even coefficients c_i. In the decreasing sequence

$$r_0 > |r_1| > |r_2| \cdots > |r_m| = 1 > |r_{m+1}| = 0$$

any two adjacent numbers have opposite parities. Hence the length m of the algorithm and α have different parity.

Proof. It follows from the classical Euclidean algorithm that for $a \in \mathbb{Z}$, $b \in \mathbb{Z} - \{0\}$, a, b of different parity there are unique integers k, r such that

$$a = 2k\,b + r \text{ and } |r| < |b|.$$

It is obvious that a and r have the same parity. Hence the length m of the algorithm and α have different parity. □

12.C Alexander polynomial and genus of a knot with two bridges

In what follows we shall study the genus and the Alexander polynomial of a knot or link with two bridges. Let $\mathfrak{b}(\alpha,\beta)$ be a knot or link with two bridges, $-\alpha < \beta < \alpha$, $\gcd(\alpha,\beta) = 1$ with β even if α is odd. Then

$$\frac{\beta}{\alpha} = [-2b_1, 2b_2, \ldots, (-1)^k 2b_k]$$

describes a continued fraction expansion with even coefficients (see Proposition 12.17). One has $k \equiv \alpha + 1 \bmod 2$.

We consider the following oriented surface $S = S(b_1, b_2, \ldots, b_k)$ of genus $g = [k/2]$ (Figure 12.7). Here the integer part of x is denoted as $[x]$. The bands of the surface S form a plat and hence S spans a braid-like knot or link. It follows that the complement C^* of a regular neighborhood of S is a handlebody. A basis of $\pi_1 S$ is represented by the curves $a_1, \ldots, a_k$ as indicated in Figure 12.7.

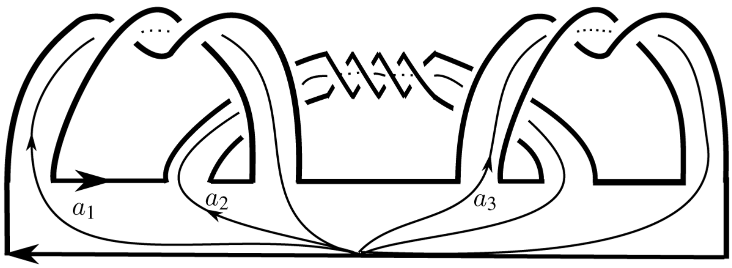


Figure 12.7. $S(-1,-2,-1)$.

The corresponding $k \times k$ Seifert matrix V is given by

$$V = \begin{pmatrix} b_1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & b_2 & -1 & 0 & & \vdots \\ 0 & 0 & b_3 & 0 & 0 & \vdots \\ 0 & 0 & 1 & b_4 & -1 & 0 \\ \vdots & & 0 & \ddots & \ddots & \eta \\ 0 & \cdots & \cdots & 0 & \epsilon & b_k \end{pmatrix} \qquad (12.9)$$

where $\epsilon = 1$, $\eta = 0$ if k is even and $\epsilon = 0$, $\eta = -1$ if k is odd. Note that the $\{a_i\}$ do not form a canonical basis of S. Hence V is a Seifert matrix in the sense of Paragraph 8.8.

The following determinant gives the Alexander polynomial of $\mathfrak{k} = \partial S(b_1, \dots, b_k)$ if $\mathfrak{k}$ is a knot (k even).

$$\Delta_{b_1,\dots,b_k}(t) = |V^T - tV| = \begin{vmatrix} (1-t)b_1 & 1 & 0 & \cdots & 0 \\ -t & (1-t)b_2 & t & 0 & \vdots \\ 0 & -1 & (1-t)b_3 & 1 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \ddots & (1-t)b_k \end{vmatrix} . \tag{12.10}$$

In the case where $\mathfrak{l} = \partial S(b_1, \dots, b_k)$ is a link (k odd), the matrix $(V^T - tV)$ is also a presentation matrix for $H_1(C_\infty)$ of the oriented link $\mathfrak{l}$. For the reduced Alexander polynomial $\Delta_{\mathfrak{l}}(t,t)$ the equation

$$(1-t) \cdot \Delta_{\mathfrak{l}}(t,t) = \Delta_{b_1,\dots,b_k}(t)$$

holds (see 9.19 and Exercise 9.5).

Note that the boundary of $S(b_1, \dots, b_k)$ is a 4-plat defined by the braid

$$\mathfrak{z} = \sigma_2^{2b_1}\sigma_1^{2b_2}\cdots\sigma_{l(k)}^{2b_k}, \text{ where } l(k) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd} \end{cases}$$

(see Figure 12.8).

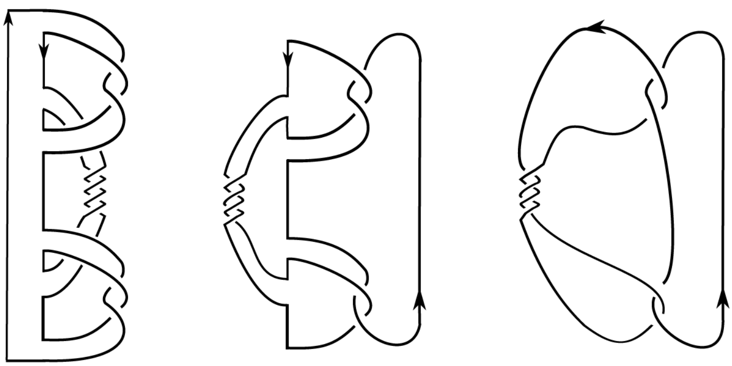


Figure 12.8. $b_1 = -1$, $b_2 = -2$ and $b_3 = -1$

In the case of a link the orientation of the components is vital – if the link is drawn as a 4-plat the outside strings must be oriented in different ways (see Figure 12.8). This implies that for $\mathfrak{b}(\alpha, \beta)$ Schubert's notation is transformed into Siebenmann's by substituting β by $\alpha + \beta$ mod 2α.

12.18 Example. The fibered 2-bridge link $\mathfrak{l}_1$ in Figure 4.11 is the link $\mathfrak{b}(4,-1)$ in Schubert's notation and the link $\mathfrak{b}(4,3)$ in Siebenmann's notation.

12.19 Lemma. *The degree of the polynomial* $\Delta_{b_1,\dots,b_k}(t)$ (*see equation* (12.10)) *is* k *and its leading coefficient is the product* $b_1 b_2 \cdots b_k$.

Proof. By expanding $\Delta_{b_1,\dots,b_k}(t)$ by the last line we obtain

$$\Delta_{b_1,\dots,b_k}(t) = (1-t)b_k\,\Delta_{b_1,\dots,b_{k-1}}(t) + t\,\Delta_{b_1,\dots,b_{k-2}}(t).$$

This together with $\Delta_{b_1}(t) = (1-t)b_1$ and $\Delta_{b_1,b_2}(t) = b_1 b_2\, t^2 + (1-2b_1 b_2)t + b_1 b_2$ proves the assertion of the lemma by induction. □

12.20 Proposition. *The genus of a 2-bridge knot or link* $\mathfrak{b}(\alpha,\beta)$ *(Siebenmann's notation) with defining braid*

$$\mathfrak{z} = \sigma_2^{2b_1}\sigma_1^{2b_2}\cdots\sigma_{l(k)}^{2b_k}$$

is $g(\alpha,\beta) = k$.

The knot or link $\mathfrak{b}(\alpha,\beta)$ *is fibered if and only if all the* b_i *in its defining braid are of absolute value one.*

(The quotients $(-1)^j 2b_j$ *of* $\beta\alpha^{-1}$ *are determined by the algorithm of Proposition 12.17.)*

Proof. The genus of the surface $S(b_1,\dots,b_k)$ is $g = [k/2]$ (integer part of x is denoted as $[x]$). If k is even, then $\mathfrak{b}(\alpha,\beta)$ is a knot and the degree of its Alexander polynomial is $2g$. Hence the genus of $\mathfrak{b}(\alpha,\beta)$ must be g since $2g$ is bigger or equal the degree of the Alexander polynomial (see Proposition 8.21). In the case of a link, k is odd and the genus is $g = (k-1)/2$. The reduced Alexander polynomial $\Delta_{\mathfrak{b}(\alpha,\beta)}(t,t)$ verifies

$$(1-t)\cdot\Delta_{\mathfrak{b}(\alpha,\beta)}(t,t) = \Delta_{b_1,\dots,b_k}(t)\,.$$

Hence the same conclusion holds by Exercise 9.5.

If the absolute value of one of the coefficients is bigger than one, then the absolute value of the leading coefficient of the Alexander polynomial is also bigger than one by Lemma 12.19. Hence by Proposition 8.33 the knot $\mathfrak{b}(\alpha,\beta)$ cannot be fibered.

Let us suppose that all the coefficients b_i, $1 \le i \le k$, have absolute value one. The complement C^* of the Seifert surface $S = S(b_1,\dots,b_k)$ in S^3 is a handlebody of genus k. We will prove that the inclusion $i_\#^-\colon \pi_1 S \to \pi_1 C^*$ is an isomorphism. Then Theorem 4.7 together with Stallings' Theorem 5.1 proves that $\mathfrak{b}(\alpha,\beta)$ is a fibered knot.

First observe that $i_\#^-\colon \pi_1 S \to \pi_1 C^*$ is injective since S is a Seifert surface of minimal genus (Lemma 4.5). A basis of the free group $\pi_1 C^*$ represented by the loops

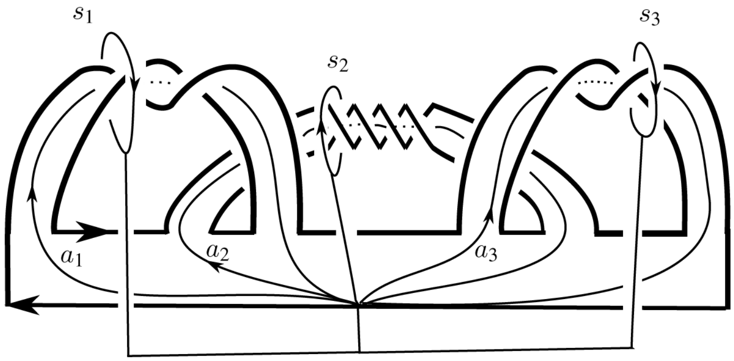


Figure 12.9

$s_1, \dots, s_k$ as indicated in Figure 12.9. It is easily seen that

$$i_{\#}^{-}(a_i) = \begin{cases} s_i^{b_i} & \text{if } i \text{ is odd,} \\ s_{i-1}^{-1}\, s_i^{b_i}\, s_{i+1} & \text{if } i \text{ is even,} \end{cases}$$

(here s_{k+1} is understood to be the trivial element). Now it follows immediately that $i_{\#}^{-}(a_i)$, $1 \le i \le k$, is a basis of $\pi_1 C^*$. Hence $i_{\#}^{-}$ is an isomorphism if $|b_i| = 1$ for all $1 \le i \le k$. □

12.21 Remark. Since 2-bridge knots and links are alternating, deg $\Delta_{\mathfrak{b}}(t) = 2g$ resp. deg $\nabla_{\mathfrak{b}}^{H}(t) = 2g$ where g is the genus of $\mathfrak{b}(\alpha, \beta)$. This was proved by Crowell [76]. Moreover, Murasugi proved that if $\mathfrak{k}$ is an alternating knot or link then $\mathfrak{k}$ is fibered if and only if $|\Delta_{\mathfrak{k}}(0)| = 1$ resp. $|\nabla_{\mathfrak{k}}^{H}(0)| = 1$ (see [258, 260]). A proof of both results is given in Theorem 13.26.

Using Proposition 12.13 we obtain:

12.22 Corollary. *There are infinitely many knots $\mathfrak{b}(a, \beta)$ of genus $g > 0$, and infinitely many fibered knots with two bridges. However, for any given genus there are only finitely many knots with two bridges which are fibered.* □

12.23 Proposition. *A knot with two bridges of genus one or its mirror image is of the form $\mathfrak{b}(\alpha, \beta)$ with:*

$$\beta = 2c,\ \alpha = 2b\beta \pm 1,\ c, b \in \mathbb{N}.$$

The trefoil and the four-knot are the only fibered 2-bridge knots of genus one.

Proof. This is a special case of Proposition 12.20 and the proof involves only straight forward computations. By Proposition 12.20, $k = 2$ and the sequence $[-2b_1, 2b_2]$

gives

$$\frac{\beta}{\alpha} = \frac{2b_2}{-4b_1b_2+1}.$$

If $b_1b_2 < 0$ then we may assume that b_2 is positive by replacing $\mathfrak{b}(\alpha,\beta)$ by $\mathfrak{b}^*(\alpha,\beta) = \mathfrak{b}(\alpha,-\beta)$ if necessary (see 12.8). Hence $\beta = 2b_2$ and $\alpha = 2\beta(-b_1)+1$.

If $b_1b_2 > 0$ we might assume that $b_2 < 0$. Then $\beta = -2b_2$ and $\alpha = 2(-b_1)\beta - 1$.

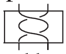
Now $\mathfrak{b}(\alpha,\beta)$ is fibered if and only if $b_i = \pm 1$, $i = 1,2$. Therefore $(b_1,b_2) \in \{(\pm 1, \pm 1)\}$. Now $2/5 = [2,2]$, $2/3 = [2,-2]$, $-2/3 = [-2,2]$ and $-2/5 = [-2,-2]$. Since the four-knot is amphicheiral the only fibered 2-bridge knots of genus one are $\mathfrak{b}(5,2)$, $\mathfrak{b}(3,2)$ and $\mathfrak{b}(3,-2) = \mathfrak{b}^*(3,2)$. □

12.24 Remark. Proposition 12.20 is a version of a theorem proved first by K. Funcke [124] and R. I. Hartley [147]. Hartley also proves in this paper a monotony property of the coefficients of the Alexander polynomial of $\mathfrak{b}(\alpha,\beta)$. See also the articles by G. Burde [56, 57].

12.D Classification of Montesinos links

The classification of knots and links with two bridges was achieved by classifying their 2-fold branched coverings – the lens spaces. It is natural to use this tool in the case of a larger class of manifolds which can be classified. J. M. Montesinos [243, 244] defined a set of links whose 2-fold branched covering spaces are Seifert fiber spaces. Their classification is a straight forward generalization of Seifert's idea in the case of 2-bridge knots.

We start with a definition of Montesinos links, and formulate the classification theorem of F. Bonahon [38]. Then we show that the 2-fold branched covering is a Seifert fiber space. Those Seifert fiber spaces are classified by their fundamental groups. By repeating the arguments for the classification of those groups we classify the Seifert fiber space together with the covering transformation. This then gives the classification of Montesinos links.

12.25 Definition (Montesinos link). A *Montesinos link* (or *knot*) has a projection as shown in Figure 12.10. The numbers e, a_i', a_i'' denote numbers of half-twists. A box $\boxed{\alpha,\beta}$ stands for a so-called *rational tangle* as illustrated in Figure 12.10 (b), and α,β are defined by the continued fraction $\frac{\beta}{\alpha} = [a_1, -a_2, a_3, \ldots, \pm a_m], a_j = a_j' + a_j''$ together with the conditions that α and β are relatively prime and $\alpha > 0$. A further assumption is that $\frac{\alpha}{\beta}$ is not an integer, that is $\boxed{\alpha,\beta}$ is not ; in this case the knot has a simpler projection. The above Montesinos link is denoted by $\mathfrak{m}(e; \alpha_1/\beta_1, \ldots, \alpha_r/\beta_r)$.

In Figure 12.10 (a): $e = 3$; in Figure 12.10 (b): $n = 5$, $a_1' = 2$, $a_1'' = 0 \Longrightarrow a_1 = 2$; $a_2' = -1$, $a_2'' = -2 \Longrightarrow a_2 = -3$; $a_3 = -1$, $a_4 = 3$, $a_5 = 5$ and $\beta/\alpha = -43/105$.

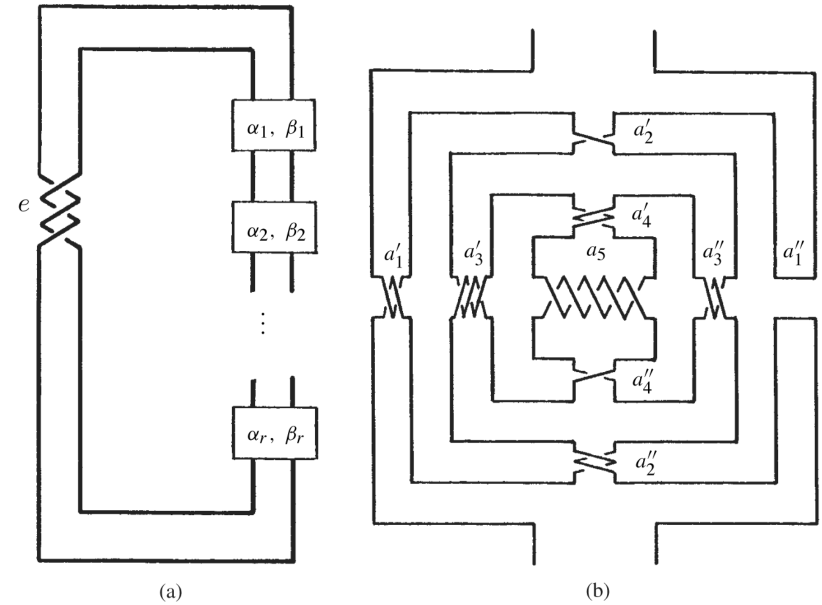


Figure 12.10. The Montesinos link $\mathfrak{m}(e;\alpha_1/\beta_1,\ldots,\alpha_r/\beta_r)$ and the rational tangle corresponding to $\beta/\alpha = -43/105$.

As before in the case of 2-bridge knots we think of $\mathfrak{m}$ as unoriented. It follows from Section 12.B that *the continued fractions* $\frac{\beta}{\alpha} = [a_1,\ldots,\pm a_m]$ *(including* $1/0 = \infty$*) classify the rational tangles up to isotopies which leave the boundary of the box pointwise fixed.*

It is easily seen that a rational tangle (α,β) is the intersection of the box with a 4-plat: there is an isotopy which reduces all twists a_j'' to 0-twists. A tangle in this position may gradually be deformed into a 4-plat working from the outside towards the inside. A rational tangle closed by two trivial bridges is a knot or link $\mathfrak{b}(\alpha,\beta)$, see the definition in 12.1 and Proposition 12.13. (Note that we excluded the trivial cases $\mathfrak{b}(0,1)$ and $\mathfrak{b}(1,0)$.)

12.26 Theorem (Classification of Montesinos links). *Montesinos links with r rational tangles, $r \geq 3$ and $\sum_{j=1}^{r}\frac{1}{\alpha_j} \leq r-2$, are classified by the ordered set of fractions* $(\frac{\beta_1}{\alpha_1} \bmod 1,\ldots,\frac{\beta_r}{\alpha_r} \bmod 1)$*, up to cyclic permutations and reversal of order, together with the rational number* $e_0 = e + \sum_{j=1}^{r}\frac{\beta_j}{\alpha_j}$.

This result was obtained by F. Bonahon [38]. Another proof was given by M. Boileau and L. Siebenmann [31]. The proof here follows the arguments of the latter, based on the method Seifert used to classify 2-bridge knots. We give a self-contained proof which is due to H. Zieschang [381] and does not use the classification of Seifert fiber spaces. We prove a special case of Zieschang's and Zimmermann's *Isomorphiesatz* from [383, Satz 3.7]. The proof of Theorem 12.26 will be finished in Paragraph 12.33.

12.27 Another construction of Montesinos links. For the following construction we use Proposition 12.3. From S^3 we remove $r+1$ disjoint balls $B_0, B_1, \dots, B_r$ and consider two disjoint disks δ_1 and δ_2 in $\overline{S^3 - \bigcup_{i=1}^r B_i} = W$ where the boundary $\partial\delta_j$ intersects B_i in an arc $\varrho_{ji} = \partial B_i \cap \delta_j = B_i \cap \delta_j$. Assume that $\partial\delta_j = \varrho_{j0}\lambda_{j0}\varrho_{j1}\lambda_{j1}\dots\varrho_{jr}\lambda_{jr}$. In B_i let κ_{1i} and κ_{2i} define a tangle of type (α_i, β_i). We assume that in B_0 there is only an e-twist, that is $\alpha_0 = 1$, $\beta_0 = e$. *Then* $\bigcup(\lambda_{ji} \cup \kappa_{ji})$ $(j = 1, 2;\ i = 0, \dots, r)$ *is the Montesinos link* $\mathfrak{m}(e; (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$, see Figure 12.11 and Proposition 12.13.

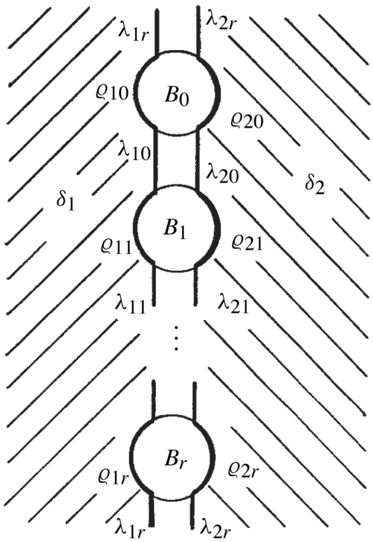


Figure 12.11

12.28 Proposition. (a) *The* 2*-fold branched covering* $\widehat{C}_2$ *of* S^3 *branched over the Montesinos link* $\mathfrak{m}(e; \alpha_1/\beta_1, \dots, \alpha_r/\beta_r)$ *is a Seifert fiber space with the fundamental group*

$$\pi_1 \widehat{C}_2 = \langle h, s_1, \dots, s_r \mid s_i^{\alpha_i} h^{\beta_i},\ [s_i, h]\ (1 \le i \le r),\ s_1 \dots s_r h^{-e} \rangle. \qquad (12.11)$$

(b) *The covering transformation* Φ *of the* 2*-fold covering induces the automorphism* $\varphi\colon \pi_1 \widehat{C}_2 \to \pi_1 \widehat{C}_2$ *given by*

$$\varphi\colon h \mapsto h^{-1}, \varphi\colon s_i \mapsto s_1 \dots s_{i-1} s_i^{-1} s_{i-1}^{-1} \dots s_1^{-1} \quad (1 \le i \le r). \qquad (12.12)$$

(c) *The covering transformations of the universal cover of* $\widehat{C}_2$ *together with a lift of* Φ *form a group* $\mathfrak{H}$ *with the following presentations:*

$$\begin{aligned}\mathfrak{H} &= \langle\, h, s_1, \dots, s_r, c \mid chc^{-1}h,\ cs_ic^{-1}\cdot(s_1\dots s_{i-1}s_is_{i-1}^{-1}\dots s_1^{-1}),\\ &\qquad [s_i,h], s_i^{\alpha_i}h^{\beta_i}\ (1\le i\le r), s_1\dots s_rh^{-e}, c^2\,\rangle\\ &= \langle\, h, c_0, \dots, c_r \mid c_i^2,\ c_ihc_i^{-1}h\ (0\le i\le r),\\ &\qquad (c_{i-1}c_i)^{\alpha_i}h^{\beta_i}\ (1\le i\le r), c_0^{-1}c_rh^{-e}\,\rangle.\end{aligned} \tag{12.13}$$

Proof. We use the notation of 12.27 and repeat the arguments of the proof 12.3. Cutting along δ_1, δ_2 turns W into the Cartesian product $(\overline{D^2 - \bigcup_{i=1}^r D_i}) \times I$ where D^2 is a 2-disk and the D_i are disjoint disks in D^2. The 2-fold covering T_r of W branched over the λ_{ji} is a solid torus with r parallel solid tori removed: $T_r = (\overline{D^2 - \bigcup_{i=1}^r D_i}) \times S^1$. The product defines an S^1-fibration of T_r. The covering transformation Φ is the rotation through angle π about the axis containing the arcs λ_{ji}, compare Figure 12.12.

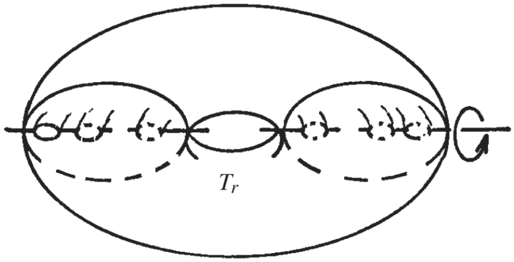


Figure 12.12

To calculate the fundamental group we choose the base point on the axis and on ∂B_0. Generators of $\pi_1 T_r$ are obtained from the curves shown in Figure 12.13, and

$$\pi_1 T_r = \langle\, h, s_0, s_1, \dots, s_r \mid [h, s_i]\ (0 \le i \le r),\ s_0s_1\dots s_r\,\rangle.$$

The covering transformation Φ maps the generators as described in Figure 12.13; hence $\Phi_*\colon \pi_1 T_r \to \pi_1 T_r$, $h \mapsto h^{-1}$, $s_0 \mapsto s_0^{-1}$, $s_1 \mapsto s_1^{-1}$, $s_2 \mapsto s_1s_2^{-1}s_1^{-1}$, ...,

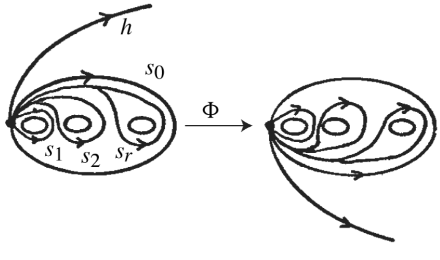


Figure 12.13

$s_r \mapsto s_1 \dots s_{r-1} s_r^{-1} s_{r-1}^{-1} \dots s_1^{-1}$. The 2-fold covering of B_i $(0 \le i \le r)$, branched over the arcs κ_{ji}, is a solid torus $\widehat{V}_i$, see Proposition 12.3. Thus the 2-fold covering $\widehat{C}_2$ of S^3 branched over $\mathfrak{m} = \mathfrak{m}(e; \alpha_1/\beta_1, \dots, \alpha_r/\beta_r)$ is $T_r \cup \bigcup_{i=0}^r \widehat{V}_i = \widehat{C}_2$ with corresponding boundaries identified. The fibration of T_r can be extended to the solid tori $\widehat{V}_i$ as we have excluded the case $(\alpha_i, \beta_i) = (0, 1)$, and $\widehat{C}_2$ obtains a Seifert fibration. Adding the solid tori $\widehat{V}_i$ introduces the relations $s_i^{\alpha_i} h^{\beta_i}$ for $1 \le i \le r$ and $s_0 h^e$. This finishes the proof of (a).

The proof of (b) follows from the effect of Φ on $\pi_1 T_r$. The first presentation of (12.13) follows from (12.11) and (12.12) by interpreting $\pi_1 \widehat{C}_2$ as the group of covering transformations of the universal covering of $\widehat{C}_2$. It remains to show $c^2 = 1$. This follows from the fact that Φ has order 2 and admits the base point as a fixed point.

Define $c_i = c s_1 \dots s_i$ $(1 \le i \le r)$ and $c_0 = c$. Then $s_i = c_{i-1}^{-1} c_i$ $(1 \le i \le r)$ and

$$\begin{aligned}
\mathfrak{H} = \langle\, h, c_0, \dots, c_r \mid\ & c_0 h c_0^{-1} h,\ c_0 (c_{i-1}^{-1} c_i) c_0^{-1} \cdot c_0^{-1} (c_i c_{i-1}^{-1}) c_0, \\
& [c_{i-1}^{-1} c_i, h],\ (c_{i-1}^{-1} c_i)^{\alpha_i} h^{\beta_i}\ (1 \le i \le r), c_0^{-1} c_r h^{-e}, c_0^2 \,\rangle \\
= \langle\, h, c_0, \dots, c_r \mid\ & c_i h c_i^{-1} h, c_i^2\ (0 \le i \le r), \\
& (c_{i-1} c_i)^{\alpha_i} h^{\beta_i}\ (1 \le i \le r),\ c_0^{-1} c_r h^{-e} \,\rangle.
\end{aligned}$$

□

12.29 Remark. For later use we note a geometric property of the 2-fold branched covering: The branch set $\widehat{\mathfrak{m}}$ in $\widehat{C}_2$ is the preimage of the Montesinos link $\mathfrak{m}$. From the construction of $\widehat{C}_2$ it follows that $\widehat{\mathfrak{m}}$ intersects each exceptional fiber exactly twice, at the "centers" of the pair of disks in Figure 12.12 belonging to one $\widehat{V}_i$.

12.30 Lemma. *For $\sum_{i=1}^r \frac{1}{\alpha_i} \le r - 2$ the element h in the presentation* (12.11) *of $\pi_1 \widehat{C}_2$ generates an infinite cyclic group $\langle h \rangle$, the center of $\pi_1 \widehat{C}_2$.*

Proof. The quotient $\pi_1 \widehat{C}_2 / \langle h \rangle$ is a discontinuous group with compact fundamental domain of motions of the Euclidean plane, if equality holds in the hypothesis, otherwise of the non-Euclidean plane, and all transformations preserve orientation; see [382, 4.5.6, 4.8.2]. In both cases the group is generated by rotations and there are r rotation centers which are pairwise non-equivalent under the action of $\pi_1 \widehat{C}_2$. A consequence is that the center of $\pi_1 \widehat{C}_2 / \langle h \rangle$ is trivial, see [382, 4.8.1 (c)]; hence, $\langle h \rangle$ is the center of $\pi_1 \widehat{C}_2$.

The proof that h has infinite order is more complicated. It is simple for $r > 3$. Then

$$\begin{aligned}
\pi_1 \widehat{C}_2 = & \langle\, h, s_1, s_2 \mid s_i^{\alpha_i} h^{\beta_i},\ [h, s_i]\ (1 \le i \le 2) \,\rangle *_{\mathbb{Z}^2} \\
& \qquad \langle\, h, s_3, \dots, s_r \mid s_i^{\alpha_i} h^{\beta_i},\ [h, s_i]\ (3 \le i \le r) \,\rangle
\end{aligned}$$

where $\mathbb{Z}^2 \cong \langle h, s_1 s_2 \rangle \cong \langle h, (s_3 \dots s_r)^{-1} \rangle$. (It easily follows by arguments on free products that the above subgroups are isomorphic to $\mathbb{Z}^2$.) In particular, $\langle h \rangle \cong \mathbb{Z}$.

To show the lemma for $r = 3$ we prove the Theorem 12.32 by repeating the arguments of the proof of Theorem 3.40. In what follows we will make use of the following result of Seifert [325]. Recall that the *rank* of a finitely generated Abelian group A is the maximal number of linearly independent elements of A. Note that the rank of A is equal to the dimension of the vector space $\mathbb{Q} \otimes A$.

12.31 Lemma (Seifert [325]). *Let M be a compact orientable 3-manifold with boundary ∂M. Then the rank of the map $H_1(\partial M) \to H_1(M)$ is equal to one half the rank of $H_1(\partial M)$.*

Proof (Hatcher [156]). The exact (co)-homology sequence of the pair $(M, \partial M)$ yields the following commutative diagram

$$\begin{array}{ccccc}
H_2(M, \partial M; \mathbb{Q}) & \xrightarrow{\partial_*} & H_1(\partial M; \mathbb{Q}) & \xrightarrow{i_*} & H_1(M; \mathbb{Q}) \\
\cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\
H^1(M; \mathbb{Q}) & \xrightarrow{i^*} & H^1(\partial M; \mathbb{Q}) & \xrightarrow{\partial^*} & H^2(M, \partial M; \mathbb{Q}).
\end{array}$$

The vertical isomorphisms are given by Poincaré–Lefschetz duality. The maps i^* and i_* are dual to each other since we have field coefficients. Therefore, we have $\operatorname{rank} i^* = \operatorname{rank} i_*$, $\operatorname{rank} \partial_* = \operatorname{rank} i^*$ and $\operatorname{Ker} i_* = \operatorname{Im} \partial_*$. Hence,

$$\dim H_1(\partial M, \mathbb{Q}) = \operatorname{rank} i_* + \dim \operatorname{Ker} i_* = \operatorname{rank} i_* + \operatorname{rank} \partial_* = 2 \operatorname{rank} i_*. \qquad \square$$

12.32 Theorem. *Let M be an orientable 3-manifold with no sphere in its boundary. If $\pi_1 M$ is infinite, non-cyclic, and not a free product then M is aspherical and $\pi_1 M$ is torsion-free.*

Proof. If $\pi_2 M \neq 0$ there is, by the Sphere Theorem [281], Appendix B.6, [159, 4.3], a 2-sphere S^2, embedded in M, which is not null-homotopic in M. If S^2 does not separate M then there is a simple closed curve λ that properly intersects S^2 in exactly one point. The regular neighborhood U of $S^2 \cup \lambda$ is bounded by a separating 2-sphere. One has

$$\pi_1 M = \pi_1 U * \pi_1(\overline{N - U}) \cong \mathbb{Z} * \pi_1(\overline{N - U})$$

contradicting the assumptions that $\pi_1 M$ is neither cyclic nor a free product. Thus S^2 separates M into two manifolds M', M''. Since $\pi_1 M$ is not a free product we may assume that $\pi_1 M' = 1$. It follows that $\partial M' = S^2$, since by assumption every other boundary component is a surface of genus ≥ 1 and, therefore, $\operatorname{Im}(H_1(\partial M') \to H_1(M')) \neq 0$, see Lemma 12.31, contradicting $\pi_1 M' = 1$. This proves that S^2 is null-homologous in M'. Since $\pi_1 M' = 1$, it follows by the Hurewicz theorem, see [341, 7.5.2], that S^2 is null-homotopic – a contradiction. This proves $\pi_2 M = 0$.

Now consider the universal cover $\widetilde{M}$ of M. Since $|\pi_1 M| = \infty$, $\widetilde{M}$ is not compact and this implies that $H_3(\widetilde{M}) = 0$. Moreover

$$1 = \pi_1 \widetilde{M}, \ H_2(\widetilde{M}) = \pi_2 \widetilde{M} = \pi_2 M = 0.$$

By the Hurewicz theorem, $\pi_3\widetilde{M} \cong H_3(\widetilde{M}) = 0$, and by induction $\pi_j\widetilde{M} \cong H_j(\widetilde{M}) = 0$ for $j \geq 3$. Since $\pi_j M \cong \pi_j\widetilde{M}$, the manifold M is aspherical and a $K(\pi_1 M, 1)$-space.

Assume that $\pi_1 M$ contains an element of finite order r. Then there is a cover M^+ of M with $\pi_1 M^+ \cong \mathbb{Z}_r$. Since $[\pi_1 M : \pi_1 M^+] = \infty$ we can apply the same argument as above to prove that M^+ is a $K(\mathbb{Z}_r, 1)$-space. This implies that $H_j(\mathbb{Z}_r) = H_j(M^+)$ for all $j \in \mathbb{N}$. Since the sequence of homology groups of a cyclic group has period 2, there are non-trivial homology groups in arbitrary high dimensions. (These results can be found in [341, 9.5].) This contradicts the fact that $H_j(M^+) = 0$ for $j \geq 3$. □

To complete the proof of Lemma 12.30 it remains to show that $\pi_1\widehat{C}_2$ is not a proper free product. Otherwise it cannot have a non-trivial center, that is, in that case $h = 1, \pi_1\widehat{C}_2 = \langle s_1, s_2 \mid s_1^{\alpha_1}, s_2^{\alpha_2}, (s_1s_2)^{\alpha_3} \rangle$. By the Grushko Theorem [382, 2.9.2, E 4.10] both factors of the free product have rank ≤ 1, and $\pi_1\widehat{C}_2$ is one of the groups $\mathbb{Z}_n * \mathbb{Z}_m$, $\mathbb{Z}_n * \mathbb{Z}$ or $\mathbb{Z} * \mathbb{Z}$. But in the group $\langle s_1, s_2 \mid s_1^{\alpha_1}, s_2^{\alpha_2}, (s_1s_2)^{\alpha_3} \rangle$ there are three non-conjugate maximal finite subgroups, namely those generated by s_1, s_2 and s_1s_2, respectively, (for a proof see [382, 4.8.1]), while there are at most 2 in the above free products of cyclic groups. This proves also that h is non-trivial; hence, by Theorem 12.32, h has infinite order. □

12.33 *Proof of the Classification Theorem 12.26.* Let $\mathfrak{H}'$ and $\mathfrak{H}$ be groups presented in the form of (12.13), and let $\psi\colon \mathfrak{H}' \to \mathfrak{H}$ be an isomorphism. By Lemma 12.30, $\psi(h') = h^{\varepsilon}$, $\varepsilon \in \{1, -1\}$, and ψ induces an isomorphism

$$\bar{\psi}\colon \mathfrak{C}' = \mathfrak{H}'/\langle h' \rangle \to \mathfrak{H}/\langle h \rangle = \mathfrak{C}.$$

The groups $\mathfrak{C}'$ and $\mathfrak{C}$ are crystallographic groups of the Euclidean or non-Euclidean plane E with compact fundamental region. Hence, $\bar{\psi}$ is induced by a homeomorphism $\chi\colon E/\mathfrak{C}' \to E/\mathfrak{C}$, see [382, 6.6.11]. Both surfaces $E/\mathfrak{C}'$ and $E/\mathfrak{C}$ are compact and have one boundary component, on which the images of the centers of the rotations $\bar{c}_1'\bar{c}_2', \bar{c}_2'\bar{c}_3', \ldots, \bar{c}_r'\bar{c}_1'$ and $\bar{c}_1\bar{c}_2, \bar{c}_2\bar{c}_3, \ldots, \bar{c}_r\bar{c}_1$, respectively, follow in this order, see [382, 4.6.3, 4]. (The induced mappings on E are denoted by a bar.) Now χ preserves or reverses this order up to a cyclic permutation, and it follows that $(\alpha_1', \ldots, \alpha_{r'}')$ differs from $(\alpha_1, \ldots, \alpha_r)$ or $(\alpha_r, \ldots, \alpha_1)$ only by a cyclic permutation. In the first case χ preserves the direction of the rotations, in the second case it reverses it, and we obtain the following equations:

$$\bar{\psi}(\bar{s}_i') = \bar{x}_i \bar{s}_{j(i)}^{\eta} \bar{x}_i^{-1} \text{ where } \eta \in \{1, -1\}, \text{ and}$$
$$\begin{pmatrix} 1 & \cdots & r \\ j(1) & \cdots & j(r) \end{pmatrix} \text{ is a permutation with } \alpha_i' = \alpha_{j(i)}.$$

Moreover,

$$\bar{x}_1 \bar{s}_{j(1)}^{\eta} \bar{x}_1^{-1} \ldots \bar{x}_r \bar{s}_{j(r)}^{\eta} \bar{x}_r^{-1} = \bar{x}(\bar{s}_1 \ldots \bar{s}_r)^{\eta} \bar{x}^{-1}$$

in the free group generated by the $\bar{s}_i$, see [382, 5.8.2]. Hence, ψ is of the following form:

$$\psi(h') = h^{\varepsilon}, \psi(s'_i) = x_i s^{\eta}_{j(i)} x_i^{-1} h^{\lambda_i},\ \lambda_i \in \mathbb{Z},$$

where the x_i are the same words in the s_i as the $\bar{x}_i$ in the $\bar{s}_i$.

The orientation of S^3 determines orientations on the 2-fold branched covering spaces $\widehat{C}'_2$ and $\widehat{C}_2$. When the links $\mathfrak{m}'$ and $\mathfrak{m}$ are isotopic then there is an orientation preserving homeomorphism from $\widehat{C}'_2$ to $\widehat{C}_2$. This implies that $\varepsilon\eta = 1$, since the orientations of $\widehat{C}'_2$ and $\widehat{C}_2$ are defined by the orientations of the fibers and the bases and $\varepsilon = -1$ corresponds to a change of the orientation in the fibers while $\eta = -1$ corresponds to a change of the bases. Therefore,

$$\begin{aligned} h^{\varepsilon\beta'_i} &= \psi(h'^{\beta'_i}) = \psi(s'^{-\alpha'_i}_i) = x_i (s^{\varepsilon}_{j(i)} h^{\lambda_i})^{-\alpha'_i} x_i^{-1} \\ &= x_i s^{-\varepsilon a_{j(i)}}_{j(i)} x_i^{-1} h^{-\alpha_{j(i)}\lambda_i} = h^{\varepsilon\beta_{j(i)} - \alpha_{j(i)}\lambda_i}, \end{aligned}$$

that is,

$$\beta'_i = \beta_{j(i)} - \varepsilon\alpha_{j(i)}\lambda_i \quad \text{for } 1 \le i \le r.$$

This proves the invariance of the β_i/α_i mod 1 and their ordering.

From the last relation we obtain:

$$\begin{aligned} h^{\varepsilon e'} &= \psi(h'^{e'}) = \psi(s'_1 \dots s'_r) = x_1 s^{\varepsilon}_{j(1)} x_1^{-1} h^{\lambda_1} \dots x_r s^{\varepsilon}_{j(r)} x_r^{-1} h^{\lambda_r} \\ &= h^{\lambda_1 + \dots + \lambda_r} x (s_1 \dots s_r)^{\varepsilon} x^{-1} = h^{\lambda_1 + \dots + \lambda_r + \varepsilon e}; \end{aligned}$$

hence,

$$e' = e + \varepsilon(\lambda_1 + \dots + \lambda_r).$$

Now,

$$e' + \sum_{i=1}^{r} \frac{\beta'_i}{\alpha'_i} = e + \varepsilon(\lambda_1 + \dots + \lambda_r) + \sum_{i=1}^{r} \frac{\beta_{j(i)} - \varepsilon\alpha_{j(i)}\lambda_i}{\alpha_{j(i)}} = e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j}. \qquad \square$$

12.34 Remark. The *orbifold* $E/\mathfrak{C}$ of fibers is a disk with r marked vertices on the boundary. A consequence of Remark 12.29 is that the image of $\widehat{\mathfrak{m}}$ consists of the edges of the boundary of $E/\mathfrak{C}$. In other words, the fundamental domain of $\mathfrak{C}$ is an r-gon, the edges of which are the images of $\widehat{\mathfrak{m}}$. Each component $\hat{\mathfrak{k}}$ of $\widehat{\mathfrak{m}}$ determines an element of $\mathfrak{C}$ which is fixed when conjugated with a suitable reflection of $\mathfrak{C}$. The reflections of $\mathfrak{C}$ are conjugate to the reflections in the (Euclidean or non-Euclidean) lines containing the edges of the fundamental domain. From geometry we know that the reflection $\bar{c}$ with axis l fixes under conjugation the following orientation preserving mappings of E:

(i) the rotations of order 2 with centers on l,

(ii) the hyperbolic transformations with axis l.

Since the image of $\hat{\mathfrak{k}}$ contains the centers of different non-conjugate rotations of $\mathfrak{C}$ it follows that $\hat{\mathfrak{k}}$ determines, up to conjugacy, a hyperbolic transformation in $\mathfrak{C}$.

Improving slightly the proof of the Classification Theorem one obtains:

12.35 Corollary. *If $\sum_{i=1}^{r} \frac{1}{\alpha_i} < r - 2$, that is, $\mathfrak{C} = \mathfrak{H}/\langle h \rangle$ is a non-Euclidean crystallographic group, each automorphism of $\mathfrak{H}$ is induced by a homeomorphism of $E \times \mathbb{R}$.* □

Proofs can be found in Conner and Raymond [69, 70], Kamishima, Lee and Raymond [183], Lee and Raymond [207], Zieschang and Zimmermann [383, 2.10].

Moreover, the outer automorphism group of $\mathfrak{H}$ can be realized by a group of homeomorphisms. This can be seen directly by looking at the corresponding extensions of $\mathfrak{H}$ and realizing them by groups of mappings of $E \times \mathbb{R}$, see the papers mentioned above.

12.E Symmetries of Montesinos links

Using the Classification Theorem 12.26 and Corollary 12.35 we can easily decide about amphicheirality and invertibility of Montesinos links.

12.36 Proposition (Amphicheiral Montesinos links). (a) *The Montesinos link*

$$\mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r), r \geq 3,$$

is amphicheiral if and only if
(1) $e_0 = 0$ *and*
(2) *there is a permutation π – an r-cycle or a reversal of the ordering – such that*

$$\beta_{\pi(i)}/\alpha_{\pi(i)} \equiv -\beta_i/\alpha_i \mod 1 \quad \textit{for } 1 \leq i \leq r.$$

(b) *For $r \geq 3$, r odd, Montesinos knots are never amphicheiral.*

Proof. The reflection in the plane maps $\mathfrak{m}$ to the Montesinos link

$$\mathfrak{m}(-e_0; -\beta_1/\alpha_1, \ldots, -\beta_r/\alpha_r);$$

hence (a) is a consequence of the Classification Theorem 12.26. Proof of (b) as Exercise E 12.5 □

A *link* $\mathfrak{l}$ is called *invertible*, see Whitten [372, 371], if there exists a homeomorphism f of S^3 which maps each component of $\mathfrak{l}$ into itself reversing the orientation. Let us use this term also for the case where f maps each component of $\mathfrak{l}$ into itself and reverses the orientation of at least one of them. In the following proof we will see that both concepts coincide for Montesinos links.

12.37 Theorem (Invertible Montesinos links). *The Montesinos link*

$$\mathfrak{m} = \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r), r \geq 3,$$

is invertible if and only if, with an appropriate enumeration,

(a) *at least one of the* α_i, $1 \leq i \leq r$, *is even, or*

(b) *all* α_i *are odd and there are three possibilities:*

$$\mathfrak{m} = \begin{cases} \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_p/\alpha_p, \beta_p/\alpha_p, \ldots, \beta_1/\alpha_1) & \text{when } r = 2p, \text{ or} \\ \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_p/\alpha_p, \beta_{p+1}/\alpha_{p+1}, \beta_p/\alpha_p, \ldots, \beta_1/\alpha_1) & \text{when } r = 2p+1; \\ \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_p/\alpha_p, \beta_{p+1}/\alpha_{p+1}, \beta_p/\alpha_p, \ldots, \beta_2/\alpha_2) & \text{when } r = 2p. \end{cases}$$

Proof. That the conditions (a) or (b) are sufficient follows easily from Corollary 12.35 (and the corresponding result for the Euclidean cases) or from Figure 12.14.

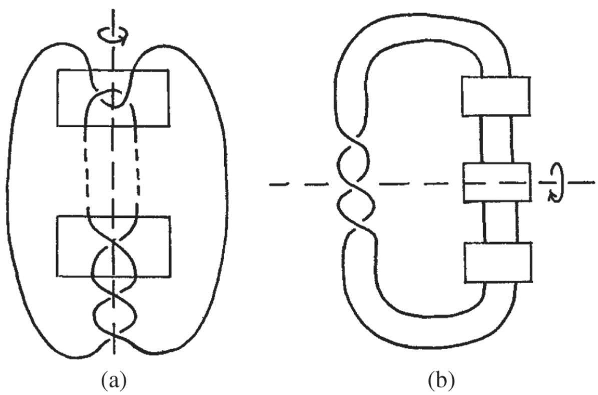


Figure 12.14

For case (a), the rotation through 180° about the dotted line maps the Montesinos link onto an equivalent one. If α_i is even, a component of $\mathfrak{m}$ enters the i-th box from above and leaves it in the same direction. The rotation inverts the components. In case (b) the rotation through 180° shown in Figure 12.14 (b) gives the required symmetry.

For the proof that the conditions are necessary we may restrict ourselves to the case where $\mathfrak{C}$ operates on the hyperbolic plane $\mathbb{H}$, since in the Euclidean cases either an exponent 2 occurs or all α_i are equal to 3 and the links are invertible. Let $f\colon S^3 \to S^3$ be an orientation preserving homeomorphism that maps $\mathfrak{m}$ onto $\mathfrak{m}$ and maps one component $\mathfrak{k}$ of $\mathfrak{m}$ onto itself, but reverses the orientation on $\mathfrak{k}$. Then, after a suitable choice of the basepoint, f induces an automorphism φ of $\mathfrak{C}$ that maps the element $k \in \mathfrak{C}$ defined by $\mathfrak{k}$ into its inverse. By Remark 12.34, k is a hyperbolic transformation.

If φ is the inner automorphism $x \mapsto g^{-1}xg$ of $\mathfrak{C}$ then g has a fixed point on the axis A of k. Hence, g is either a rotation of order 2 with center on A or g is the reflection with an axis perpendicular to A. In both cases $\mathfrak{C}$ contains an element of even order, i.e. one of the α_i is even.

If φ is not an inner automorphism then φ corresponds to a rotation or a reflection of the disk $\mathbb{H}/\mathfrak{C}$ that preserves the fractions β_i/α_i. It must reverse the orientation since the direction of one of the edges of $\mathbb{H}/\mathfrak{C}$ is reversed. Therefore φ corresponds to a reflection of the disk and this implies (b). □

Next we study the isotopy classes of symmetries of a Montesinos link $\mathfrak{m}$ with $r \geq 3$ tangles, in other words, we study the group $\mathfrak{M}(S^3, \mathfrak{m})$ of mapping classes of the pair $(S^3, \mathfrak{m})$. This group can be described as follows: using the compact-open topology on the set of homeomorphisms or diffeomorphisms of $(S^3, \mathfrak{m})$ we obtain topological spaces Homeo$(S^3, \mathfrak{m})$ and Diff$(S^3, \mathfrak{m})$, respectively. Now $\mathfrak{M}(S^3, \mathfrak{m})$ equals the set of path-components of the above spaces:

12.38. $\mathfrak{M}(S^3, \mathfrak{m}) \cong \pi_0 \operatorname{Homeo}(S^3, \mathfrak{m}) \cong \pi_0 \operatorname{Diff}(S^3, \mathfrak{m})$.

Each symmetry induces an automorphism of the knot group $\mathfrak{G}$ which maps the kernel of the homomorphism $\mathfrak{G} \to \mathbb{Z}_2$ onto itself, and maps meridians to meridians; hence, symmetries and isotopies can be lifted to the 2-fold branched covering $\widehat{C}_2$ such that the lifts commute with the covering transformation of $\widehat{C}_2 \to S^3$. Lifting a symmetry to the universal cover $\mathbb{H} \times \mathbb{R}$ of $\widehat{C}_2$ yields a homomorphism.

12.39. $\gamma\colon \mathfrak{M}(S^3, \mathfrak{m}) \to \operatorname{Out} \mathfrak{H} = \operatorname{Aut} \mathfrak{H}/\operatorname{Inn} \mathfrak{H}$: every lifted symmetry induces by conjugation an automorphism of $\mathfrak{H}$. Here, $\mathfrak{H}$ has a presentation of the form (12.13). The fundamental assertion is:

12.40 Proposition. $\gamma\colon \mathfrak{M}(S^3, \mathfrak{m}) \to \operatorname{Out} \mathfrak{H}$ *is an isomorphism.*

We cannot give a self-contained proof here, but have to use results of W. P. Thurston and others (see M. Kapovich's book [185] and the references therein). This proof shows the influence of these theorems on knot theory. An explicit and simple description of Out $\mathfrak{H}$ is given afterwards in Proposition 12.42.

Proof (Boileau–Zimmermann [36]*).* Consider first the case $\sum_{i=1}^{r} \frac{1}{\alpha_i} < r - 2$. From Corollary 12.35 it follows that γ is surjective and it remains to show that γ is injective. By Bonahon–Siebenmann [37] $\mathfrak{m}$ is a simple knot, that means, $\mathfrak{m}$ does not have a companion. By Thurston [349, 185], $S^3 - \mathfrak{m}$ has a complete hyperbolic structure with finite volume. Mostow's rigidity theorem [255] implies that $\mathfrak{M}(S^3, \mathfrak{m})$ is finite and that every element of $\mathfrak{M}(S^3, \mathfrak{m})$ can be represented by an isometry of the same order as its homotopy class. Now we represent a non-trivial element of the kernel of γ by a homeomorphism f with the above properties. Let $\bar{f}$ be the lift of f to $\mathbb{H} \times \mathbb{R}$;

then $\bar{f}^m \in \mathfrak{H}$ for a suitable $m > 0$. Since the class of f is in the kernel of γ we may assume that the conjugation by $\bar{f}$ yields the identity in $\mathfrak{H}$. As the center of $\mathfrak{H}$ is trivial it follows that $\bar{f}^m = \mathrm{id}_{\mathbb{H}\times\mathbb{R}}$ and, thus, that $\bar{f}$ is a periodic diffeomorphism commuting with the operation of $\pi_1 \widehat{C}_2$. Therefore $\bar{f}$ is a rotation of the hyperbolic 3-space and its fixed point set is a line. The elements of $\pi_1 \widehat{C}_2$ commute with f; hence, they map the axis of f onto itself and it follows from the discontinuity that $\pi_1 \widehat{C}_2$ is infinite cyclic or dihedral. This is a contradiction. Therefore γ is injective.

The Euclidean cases ($\sum_{i=1}^{r} \frac{1}{\alpha_i} = r - 2$) are left. There are four cases: (3,3,3), (2,3,6), (2,4,4) and (2,2,2,2). They can be handled using the result of Bonahon and Siebenmann [37] and Zimmermann [385]. The last paper depends strongly on Thurston's approach [349, 185] which we used above, and, furthermore, on Jaco and Shalen [177] and Johannson [179]. □

Next we determine Out $\mathfrak{H}$ for the knot $\mathfrak{m}(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$. We assume $0 < \beta_j < \alpha_j$, $1 \le j \le r$, for the sake of simplicity.

12.41 Definition. Let $\mathbb{D}_r$, denote the dihedral group of order $2r$, realized as a group of rotations and reflections of a regular polygon with vertices $(1,2,\ldots,r)$. Define

$$\tilde{\mathbb{D}}_r = \{\varrho \in \mathbb{D}_r \mid \alpha_{\varrho(i)} = \alpha_i \quad \text{for } 1 \le i \le r\}.$$

Let $\tilde{\tilde{\mathbb{D}}}_r \subset \tilde{\mathbb{D}}_r$, consist of

(i) the rotations ϱ with $\alpha_{\varrho(i)} = \alpha_i$ and $\beta_{\varrho(i)} = \beta_i$ and the reflections ϱ with $\alpha_{\varrho(i)} = \alpha_i$ and $\beta_{\varrho(i)} = \alpha_i - \beta_i$ if $e_0 \ne 0$,

(ii) the rotations ϱ with $(\alpha_{\varrho(i)},\beta_{\varrho(i)}) = (\alpha_i,\alpha_i-\beta_i)$ and the reflections ϱ with $(\alpha_{\varrho(i)},\beta_{\varrho(i)}) = (\alpha_i,\beta_i)$ if $e_0 = 0$.

12.42 Proposition. Out $\mathfrak{H}$ *is an extension of* $\mathbb{Z}_2$ *or* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *by the finite dihedral or cyclic group* $\tilde{\tilde{\mathbb{D}}}_r$.

Proposition 12.42 is a direct consequence of the following Lemmas 12.43 and 12.45.

Since $\langle h \rangle$ is the center of $\mathfrak{H}$ the projection $\mathfrak{H} \to \mathfrak{H}/\langle h \rangle = \mathfrak{C}$ is compatible with every automorphism of $\mathfrak{H}$ and we obtain a homomorphism $\chi\colon$ Out $\mathfrak{H} \to$ Out $\mathfrak{C}$. It is easy to determine the image of χ; thus the main problem is to calculate the kernel.

Consider an automorphism ψ of $\mathfrak{H}$ which induces the identity on $\mathfrak{C}$. Then $\psi(c_j) = c_j h^{m_j}$, $\psi(h) = h^{\varepsilon}$ where $\varepsilon \in \{1,-1\}$, and

$$\begin{aligned} h^{\varepsilon\beta_1} &= \psi(h^{\beta_1}) = \psi((c_0c_1)^{-\alpha_1}) = (c_0h^{m_0}c_1h^{m_1})^{-\alpha_1} \\ &= (c_0c_1)^{-\alpha_1}h^{-\alpha_1(m_1-m_0)} = h^{\beta_1-\alpha_1(m_1-m_0)}. \end{aligned} \tag{12.14}$$

(1) *Case* $\varepsilon = 1$. Since h has infinite order it follows that $m_1 = m_0$ and, by copying this argument, $m_0 = m_1 = \ldots = m_r = 2l + \eta$ with $\eta \in \{0,1\}$. Now multiply ψ by

the inner automorphism $x \mapsto h^l x h^{-l}$:

$$c_j \mapsto h^l c_j h^{-l} \mapsto h^l c_j h^{m_0} h^{-l} = c_j h^\eta;$$

hence, these automorphisms define a subgroup of $\ker \chi$ isomorphic to $\mathbb{Z}_2$.

(2) *Case* $\varepsilon = -1$. Now $2\beta_1 = -\alpha_1(m_0 - m_1)$ by (12.14). Since α_1 and β_1 are relatively prime and $0 < \beta_1 < \alpha_1$ it follows that $\alpha_1 = 2$, $\beta_1 = 1$ and $m_0 = m_1 - 1$. By induction: $\alpha_1 = \cdots = \alpha_r = 2$, $\beta_1 = \cdots = \beta_r = 1$, $m_j = m_0 + j$ for $1 \le j \le r$. Now

$$h^{-e} = \psi(h^e) = \psi(c_0^{-1} c_r) = h^{-m_0} c_0^{-1} c_r h^{m_r} = h^{e+m_r-m_0} = h^{e+r}.$$

It follows that $e = -\frac{r}{2}$ and that the Euler number e_0 vanishes:

$$e_0 = e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} = 0.$$

Thus we have proved:

12.43 Lemma. *The kernel* $\ker \chi \cong \mathbb{Z}_2$ *is generated by* $\psi_0\colon \mathfrak{H} \to \mathfrak{H}$, $c_j \mapsto c_j h$, $h \mapsto h$, *except in the case where* $(\alpha_j, \beta_j) = (2, 1)$ *for* $1 \le j \le r$ *and* $e_0 = 0$*; then* $\ker \chi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *generated by* ψ_0 *and* $\psi_1\colon \mathfrak{H} \to \mathfrak{H}$, $h \mapsto h^{-1}$, $c_j \mapsto c_j h^{r-j}$. □

Using the generalized Nielsen theorem, see [382, 5.8.3, 6.6.9], Out $\mathfrak{C}$ is easily calculated:

12.44 Lemma. (a) *An automorphism* $\varphi\colon \mathfrak{C} \to \mathfrak{C}$ *mapping each conjugacy class of elliptic subgroups* $\langle (c_j c_{j+1}) \rangle$ *onto itself is an inner automorphism of* $\mathfrak{C}$.

(b) *The canonical mapping* $\tilde{\mathbb{D}}_r \to$ Out $\mathfrak{C}$ *is an isomorphism.*

Proof. (a) By the generalized Nielsen theorem, see [382, 6.6.11], φ is induced by a homeomorphism f of $\mathbb{H}/\mathfrak{C} \cong D^2$ onto itself which fixes the rotation centers lying on ∂D^2. Now the Alexander trick [5] can be used to isotope f into the identity. This implies that φ is an inner automorphism.

(b) Each automorphism of $\mathfrak{C}$ induces a diffeomorphism of D^2 (see (a)). This diffeomorphism gives a dihedral permutation of the rotation centers and respects the α_i. By (a) it is isotopic to a standard reflection or rotation. □

12.45 Lemma. *The image of* Out $\mathfrak{H}$ *in* Out $\mathfrak{C}$ *is the subgroup* $\tilde{\tilde{\mathbb{D}}}_r$ *of* $\tilde{\mathbb{D}}_r$. *(See Definiton* 12.41.*)*

Proof. Let φ be an automorphism of $\mathfrak{H}$. By Lemma 12.44 (b), φ induces a 'dihedral' permutation π of the cyclic set $\bar{c}_1, \dots, \bar{c}_{r-1}, \bar{c}_r = \bar{c}_0$. We discuss the cases with $\varphi(h) = h$.

(1) π is a rotation. Then

$$\begin{aligned} h^{\beta_1} &= \varphi(h^{\beta_1}) = \varphi((c_0c_1)^{-\alpha_1}) = (c_{i-1}h^{m_0}c_ih^{m_1})^{-\alpha_1} \\ &= (c_{i-1}c_i)^{\alpha_i-\alpha_1}h^{\beta_i-\alpha_1(m_1-m_0)} \end{aligned}$$

and hence $\alpha_i = \alpha_1$. Since, by assumption, $0 < \beta_i < \alpha_i$, it follows that $m_1 = m_0$ and $\beta_1 = \beta_i$. Therefore φ preserves the pairs (α_j, β_j) and maps c_j to c_jh^m for a fixed m. By multiplication with an inner automorphism and, if necessary, with ψ_1 from Lemma 12.43 we obtain $m = 0$. The image of φ in Out $\mathfrak{C}$ is in $\tilde{\tilde{\mathbb{D}}}_r$, and each rotation $\pi \in \tilde{\tilde{\mathbb{D}}}_r$ is obtained from a $\varphi \in$ Out $\mathfrak{H}$.

(2) π is a reflection.Then

$$\begin{aligned} h^{\beta_1} &= \varphi(h^{\beta_1}) = \varphi((c_0c_1)^{-\alpha_1}) = (c_ih^{m_0}c_{i-1}h^{m_1})^{-\alpha_1} \\ &= (c_ic_{i-1})^{\alpha_i-\alpha_1}h^{-\beta_i-\alpha_1(m_1-m_0)} \end{aligned}$$

and hence $\alpha_1 = \alpha_i$. Therefore $m_1 - m_0 = -1$, $\beta_i + \beta_1 = \alpha_1$, and φ assigns to a pair (α_k, β_k) a pair $(\alpha_j, \beta_j) = (\alpha_k, \alpha_k - \beta_k)$. The generators c_i are mapped as follows:

$$\begin{array}{cccccccc} c_0, & c_1, & \dots, & c_i, & \dots, & c_{i+1}, & c_r, \\ \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \\ c_ih^m, & c_{i-1}h^{m-1}, & \dots, & c_0h^{m-1}, & \dots, & c_{r-1}h^{-e+m-i-1}, & c_ih^{-e+m-r}, \\ & & & \| & & & \\ & & & c_rh^{-e+m+i} & & & \end{array}$$

and

$$c_ih^{-e+m-r} = \varphi(c_r) = \varphi(c_0h^e) = c_ih^{m+e}.$$

This implies $e = -\frac{r}{2}$ and

$$e_0 = e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} = e + \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\beta_j}{\alpha_j} + \frac{\alpha_{i-j} - \beta_{i-j}}{\alpha_j}\right) = 0;$$

here $i - j$ is considered modulo r. By normalizing as before we obtain $m = 0$.

The cases for $\varphi(h) = h^{-1}$ can be handled the same way; proof as E 12.6. □

Lemmas 12.43 and 12.45 imply Proposition 12.42. As a corollary of Propositions 12.40 and 12.42 we obtain the following results of Bonahon and Siebenmann (for $r \geq 4$) and Boileau (for $r = 3$).

12.46 Corollary. *The symmetry group $\mathfrak{M}(S^3, \mathfrak{m})$ is an extension of $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, by the finite dihedral or cyclic group $\tilde{\tilde{\mathbb{D}}}_r$.* □

12.F History and sources

4-plats (Viergeflechte) were first investigated in C. Bankwitz and H. G. Schumann [14] where they were shown to be alternating and invertible. They were classified by H. Schubert [320] as knots and links with two bridges. A different proof using linking numbers of covering spaces was given by G. Burde [55]. Special properties of 2-bridge knots (genus, Alexander polynomial, fibering, group structure) were studied by K. Funcke [123, 124], R. Hartley [147] and E. Mayland [232, 233].

J. M. Montesinos then introduced a more general class of knots and links which could nevertheless be classified by essentially the same method that H. Seifert had used to classify (unoriented) knots and links with two bridges: Montesinos links are links with 2-fold branched covering spaces which are Seifert fiber spaces, see J. M. Montesinos [243, 244], M. Boileau and L. Siebenmann [31], H. Zieschang [381]. In other papers on Montesinos links their group of symmetries was determined in most cases (Bonahon and Siebenmann [37], Boileau and Zimmermann [36]).

12.G Exercises

E 12.1. Show that a reduced diagram of $\mathfrak{b}(\alpha,\beta)$ leads to the following Wirtinger presentations:

$$\mathfrak{G}(\alpha,\beta) = \langle\, S_1, S_2 \mid S_2^{-1}L_1^{-1}S_1L_1\,\rangle,$$

with

$$L_1 = S_2^{\varepsilon_1}S_1^{\varepsilon_2}\dots S_2^{\varepsilon_{\alpha-2}}S_1^{\varepsilon_{\alpha-1}}, \alpha \equiv 1 \mod 2$$

$$\mathfrak{G}(\alpha,\beta) = \langle\, S_1, S_2 \mid S_1^{-1}L_1^{-1}S_1L_1\,\rangle,$$

with

$$L_1 = S_2^{\varepsilon_1}S_1^{\varepsilon_2}\dots S_1^{\varepsilon_{\alpha-2}}S_2^{\varepsilon_{\alpha-1}}, \alpha \equiv 1 \mod 2,$$

here $\varepsilon_i = (-1)^{\left[\frac{i\beta}{\alpha}\right]}, [a] =$ integral part of α.

E 12.2. The matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

generate the mapping class group of the torus (Section B). Show that $\langle\, A_1, A_2 \mid A_1A_2A_1 = A_2A_1A_2, (A_1A_2)^6\,\rangle$ is a presentation of the group $\mathrm{SL}(2,\mathbb{Z})$ and connect it with the classical presentation

$$\mathrm{SL}(2,\mathbb{Z}) = \langle\, S, T, Z \mid S^2 = T^3 = Z,\ Z^2 = 1\,\rangle.$$

E 12.3. Let α, β, β' be positive integers, $\gcd(\alpha,\beta) = \gcd(\alpha,\beta') = 1$ and $\beta\beta' \equiv 1 \mod \alpha$. If $\beta\cdot\alpha^{-1} = [a_1,\dots,a_m]$, are the quotients of the continued fraction $\beta\cdot\alpha^{-1}$ of odd length m, then $\beta'\cdot\alpha^{-1} = [a_m,\dots,a_1]$. (Find an algebraic proof.)

E 12.4. Let $V(\alpha,\beta)$ denote a Seifert matrix for $\mathfrak{b}(\alpha,\beta)$ as described in (12.9). Prove

(a) $|\det V(\alpha,\beta)| = \prod_{i=1}^{k} b_i$,

(b) $\sigma[V(\alpha,\beta) + V^T(\alpha,\beta)] = \#\{b_i \mid b_i > 0\} - \#\{b_i \mid b_i < 0\}$.

(σ denotes their signature of a matrix, see Appendix A.2.) Deduce Lemma 12.19 from (a).

E 12.5. Prove Proposition 12.36 (b).

E 12.6. Finish the proof of Lemma 12.45 for the case $\varphi(h) = h^{-1}$.

E 12.7. Prove that Montesinos knots are prime. (Use the Smith conjecture for involutions.)

Chapter 13

Quadratic forms of a knot

In this chapter we propose to reinvestigate the infinite cyclic covering C_∞ of a knot and to extract another knot invariant from it: the quadratic form of the knot. The first section gives a cohomological definition of the quadratic form $q_{\mathfrak{k}}$ of a knot $\mathfrak{k}$. The main properties of $q_{\mathfrak{k}}$ and its signature are obtained. The second section is devoted to the description of a method of computation of $q_{\mathfrak{k}}$ from a special knot projection. In Section 12.C we will use special projections to prove results of R. Crowell [76] and K. Murasugi [258, 260] about alternating knots and links (Theorem 13.26). The last part then compares the different quadratic forms of L. Goeritz [130], H. F. Trotter [355], K. Murasugi [261], J. Milnor [240] and D. Erle [93]. Some examples are discussed.

13.A The quadratic form of a knot

H. F. Trotter [355] used algebraic methods to define a quadratic form associated to a knot $\mathfrak{k}$. Trotter worked with coefficients in a ring R which contains the integers and in which $\Delta_{\mathfrak{k}}(0)$, the leading coefficient of the Alexander polynomial, is invertible.

Let R be an integral domain with unit. In Proposition 8.16 we determined the homology groups $H_i(C_\infty; R)$, $H_i(C_\infty, \partial C_\infty; R)$ of the infinite cyclic covering C_∞ of a knot $\mathfrak{k}$. In the case of a field $R = \mathbb{F}$, these homology groups were studied by Milnor [240]. He proved that in this case the pair $(C_\infty, \partial C_\infty)$ has the homological properties of a compact 2-manifold bounded by a circle. Milnor proved also that the skew-symmetric cup product pairing

$$H^1(C_\infty, \partial C_\infty; \mathbb{F}) \otimes H^1(C_\infty, \partial C_\infty; \mathbb{F}) \xrightarrow{\cup} H^2(C_\infty, \partial C_\infty; \mathbb{F}) \cong \mathbb{F}$$

is nonsingular. A skew-symmetric pairing alone does not give much information. Some additional structure comes from the fact that $C_\infty \to C$ is an infinite cyclic covering. Milnor used the automorphism t^* of $H^1(C_\infty, \partial C_\infty, \mathbb{F})$ and the skew-symmetric cup product to construct a symmetric bilinear form

$$H^1(C_\infty, \partial C_\infty, \mathbb{F}) \otimes H^1(C_\infty, \partial C_\infty, \mathbb{F}) \to H^2(C_\infty, \partial C_\infty, \mathbb{F}) \cong \mathbb{F}$$

given by $x \otimes y \mapsto x \cup t^* y + y \cup t^* x$.

In this section we shall follow the treatment of Erle [93] which defines Milnor's quadratic form for more general coefficient rings. In a first step it will be necessary to find coefficients R such that $H^1(C_\infty, \partial C_\infty; R)$ is a finitely generated free R-module.

To this end, we will show that $(C_\infty, \partial C_\infty)$ has the cohomological properties of a compact 2-manifold bounded by a circle, provided that the coefficients are chosen in an integral domain R in which $\Delta_{\mathfrak{k}}(0)$ is invertible. The quadratic form can be defined by Milnor's formula.

In what follows we fix an integral domain R and a knot $\mathfrak{k}$. The Alexander polynomial of $\mathfrak{k}$ will be denoted by $\Delta(t)$.

13.1 Definition. An R-*unimodular matrix* P is a matrix over R with $\det P$ a unit of R. Two symmetric $n \times n$-matrices M, M' over R are called R-*equivalent* if there is an R-unimodular matrix P with $M' = PMP^T$. We use the term equivalent instead of $\mathbb{Z}$-equivalent.

13.2 Lemma (Trotter [355], Erle [93]). *Let R be an integral domain in which $\Delta(0)$ is unity. Every Seifert matrix V is R-equivalent to a matrix*

$$\begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix}$$

where W is a $2m \times 2m$ integral matrix, $|W| = \det W \neq 0$, and U is of the form

$$U = \begin{pmatrix} 0 & & & & & & \\ -1 & 0 & & & & & \\ 0 & * & 0 & & & 0 & \\ \vdots & \vdots & -1 & 0 & & & \\ 0 & * & 0 & * & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \\ 0 & * & 0 & * & \cdots & -1 & 0 \end{pmatrix}$$

(W is called a "reduced" Seifert matrix and may be empty.)

Proof. If $|V| \neq 0$, V itself is reduced and nothing has to be proved. Let us assume $|V| = 0$. There are $\mathbb{Z}$-unimodular matrices Q and R such that QVR will have a first row of zeroes . The same holds for $QVQ^T = QVRR^{-1}Q^T$. Since $F = V - V^T$ is $\mathbb{Z}$-unimodular and skew-symmetric, so is $QVQ^T - (QVQ^T)^T = QFQ^T$. Therefore its first column has a zero at the top and the remaining entries are relatively prime. But the first column of QFQ^T coincides with that of QVQ^T, because $(QVQ^T)^T$ has zero entries in its first column. So there is a $\mathbb{Z}$-unimodular P such that

$$P \cdot (QVQ^T) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & * & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & * & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}.$$

To find P look for the element of smallest absolute value in the first column of QVQ^T. Subtract its row from other rows until a smaller element turns up in the first column. Since the elements of the first column are relatively prime, one ends up with an element ± 1; the desired form is then easily reached. The operations on the rows can be realized by premultiplication by P. The matrix $(PQVQ^T) \cdot P^T$ has the same first row and column as $PQVQ^T$.

Similarly, for a suitable unimodular $\widetilde{P}$,

$$(PQVQ^T P^T) \cdot \widetilde{P}^T = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \hline 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{array}\right), \quad \widetilde{P}^T = \begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & \mathbf{0} & & 1 & * \\ & & & & 1 \end{pmatrix}$$

and $\widetilde{P} \cdot (PQVQ^T P^T \widetilde{P}^T)$ is of the same form.

By repeating this process we obtain a matrix

$$\widetilde{V} = \left(\begin{array}{ccccccc|c} 0 & & & & & & & \mathbf{0} \\ -1 & 0 & & & & & & \\ 0 & * & 0 & & & \mathbf{0} & & \\ \vdots & \vdots & -1 & 0 & & & & \vdots \\ 0 & * & 0 & * & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & & \\ 0 & * & 0 & * & \ldots & -1 & 0 & \mathbf{0} \\ \hline 0 & * & 0 & * & \ldots & 0 & * & \\ \vdots & \vdots & 0 & \vdots & & \vdots & \vdots & W \\ 0 & * & 0 & * & \ldots & 0 & * & \end{array}\right)$$

equivalent to V (over $\mathbb{Z}$) with $|W| \neq 0$. For further simplification of $\widetilde{V}$ we now make use of the assumption that $\Delta(0)$ is a unit in R. Recall that $|V^T - tV| = \Delta(t)$, $|\tilde{V}^T - t\tilde{V}|$ and $|W^T - tW|$ all represent the Alexander polynomial up to a factor $\pm t^\nu$. So $|W| = \Delta(0)$ is a unit in R. There is a unimodular P_1 over R, $|P_1| = 1$, with

$$\tilde{V}P_1 = \begin{pmatrix} * & & & \mathbf{0} \\ & \begin{array}{|cc|}\hline 0 & 0 \\ -1 & 0 \\ \hline 0 & 0 \end{array} & & \\ * & \begin{array}{|cc|} \vdots & \vdots \\ 0 & 0 \end{array} & & W \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 1 & & & \mathbf{0} & \\ & \ddots & 0 & \mathbf{0} & \\ & 0 & 1 & & \\ & & * & 1 & \mathbf{0} \\ \mathbf{0} & & \vdots & & \ddots \\ & & * & \mathbf{0} & 1 \end{pmatrix}$$

where the column adjoining W has been replaced by zeroes, because it is a linear combination of the columns of W. Now

$$P_1^T \tilde{V} P_1 = \begin{pmatrix} * & & & \mathbf{0} & \\ & \boxed{\begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix}} & \begin{matrix} 0 & \cdots & 0 \\ * & \cdots & * \end{matrix} \\ * & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & W \end{pmatrix}.$$

Since the row over W contains -1, there is a unimodular P_2, $|P_2| = 1$, with

$$(P_1^T \tilde{V} P_1) \cdot P_2 = \begin{pmatrix} * & & \mathbf{0} \\ & \boxed{\begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix}} & \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} \\ * & \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & W \end{pmatrix}$$

and $P_2^T \cdot (P_1^T \tilde{V} P_1 P_2)$ is of the same type. The process can be repeated until the desired form is reached. □

13.3 Remark. Note that we can chose the R-unimodular matrix P with

$$P\ V\ P^T = \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix}$$

such that $\det P = 1$.

For the rest of this section H_* and H^* will denote the (co)-homology with coefficients in R.

13.4 Corollary. *If R is an integral domain in which $\Delta(0)$ is a unit, then $(W^T - tW)$ is a presentation matrix of $H_1(C_\infty)$ as an $R(t)$-module, and $|W^T - tW| = \Delta(t)$. The R-module $H_1(C_\infty)$ is finitely generated and free and there is an R-basis of $H_1(C_\infty)$ such that the generating covering transformation $\tau = h_{j+1}h_j^{-1}$ (see 4.4) induces an isomorphism $t_*\colon H_1(C_\infty) \xrightarrow{t\cdot} H_1(C_\infty)$ which is represented by the matrix $W^{-1}W^T$.*

Proof. We may assume that as an $R(t)$-module $H_1(C_\infty)$ has a presentation matrix $(V^T - tV)$ where V is of the special form which can be achieved according to Lemma 13.2:

$$V^T - tV = \left(\begin{array}{c|c} U^T - tU & \mathbf{0} \\ \hline \mathbf{0} & W^T - tW \end{array}\right),$$

$$U^T - tU = \left(\begin{array}{cc|cc|cc} 0 & -1 & 0 & 0 & 0 & \cdots \\ t & 0 & 0 & * & * & \cdots \\ \hline 0 & * & 0 & -1 & 0 & 0 \\ 0 & * & t & 0 & 0 & * \\ \hline 0 & * & 0 & * & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & \end{array}\right). \tag{13.1}$$

There is an equivalent presentation matrix in whose second column all entries but the first are zero, the first remaining -1. So the first row and second column can be omitted. In the remaining matrix the first row and the first column again may be omitted. This procedure can be continued until the presentation matrix takes the form $(W^T - tW)$, or, $(W^T W^{-1} - tE)$, since $|W| = \Delta(0)$ is a unit of R. Hence we obtain an exact sequence

$$R(t)^{2m} \xrightarrow{W^T W^{-1} - tE} R(t)^{2m} \xrightarrow{p} H_1(C_\infty) \to 0\,.$$

This means that the defining relations of $H_1(C_\infty)$ as an $R(t)$-module take the form: $W^T W^{-1} s = ts$, where $s = (s_i)$ is a basis of $R(t)^n$ (see Corollary 8.15). The relation $W^T W^{-1} s = ts$ implies directly that $H_1(C_\infty)$ is generated by $p(s) = (p(s_i))$ as an R-module. We will show that $p(s)$ is an R-basis of $H_1(C_\infty)$: if $\sum_i r_i\, p(s_i) = 0$, $r_i \in R$, in $H_1(C_\infty)$ then there exists $x \in R(t)^n$ such that $W^T W^{-1} x - tx = \sum_i r_i s_i \in R^n \subset R(t)^n$. If $x \neq 0$ then we write

$$x = \sum_{j=k}^{l} x_j t^j \text{ with } x_j \in R^n \text{ and } x_k, x_l \neq 0.$$

We call the non-negative integer $l - k$ the degree of x. Note that the degree of x is zero if and only if there exists $\nu \in \mathbb{Z}$ such that $t^\nu x \in R^n$. The fact that $W^T W^{-1}$ is invertible implies that for a nonzero $x \in R(t)^n$ the degree of $W^T W^{-1} x - tx$ is strictly larger than the degree of x. Hence for all nonzero $x \in R(t)^n$ the degree of $Ax - tx$ is larger or equal to 1. Therefore, $x \in R(t)^n$ such that $Ax - tx = \sum_i r_i s_i \in R^n \subset R(t)^n$ implies $x = 0$ and hence $r_i = 0$ for all $1 \leq i \leq n$. This proves the corollary. $\square$

By Corollary 8.15, there is a distinguished generator $z \in H_2(S, \partial S) \cong R$ represented by an orientation of S which induces on ∂S the orientation of $\mathfrak{k}$. We shall now make use of cohomology to define a bilinear form. Since all homology groups $H_i(C_\infty)$, $H_i(C_\infty, \partial C_\infty)$, are torsion free, we have

$$H^i \cong \mathrm{Hom}_R(H_i, R) \cong H_i$$

for these spaces ([157, 3.1], [119, Satz 17.6], [341, 5.5.3]). For every free basis $\{b_j\}$ of a group H_i there is a dual free basis $\{b^k\}$ of H^i defined by $\langle b^k, b_j \rangle = \delta_{kj}$, where the bracket denotes the Kronecker product, that is $\langle b^k, b_j \rangle = b^k(b_j) \in R$ for $b^k \in \mathrm{Hom}_R(H_i, R)$. We use the cup product [157, 3.2], [172], [346] to define

$$\beta\colon H^1(C_\infty, \partial C_\infty) \times H^1(C_\infty, \partial C_\infty) \to R,\ (x, y) \mapsto \langle x \cup y, j_*(z) \rangle, \qquad (13.2)$$

where the inclusion $j\colon S \to C_\infty$ is the following composition:

$$S \xrightarrow{(j^-)^{-1}} S^- \xrightarrow{i^-} C^* \xrightarrow{h_0} C_0^* \subset C_\infty$$

(see the proof of Theorem 8.14).

Let $\{a_j \mid 1 \leq j \leq 2g\}$ and $\{s_i \mid 1 \leq i \leq 2g\}$ denote bases of $H_1(S)$ and $H_1(C^*)$, respectively, such that $i_*^-(j_*^-)^{-1}\colon H_1(S) \to H_1(C^*)$ according to these bases is represented by a Seifert matrix

$$V = \left(\begin{array}{c|c} W & 0 \\ \hline 0 & U \end{array}\right), \quad W = (w_{ji})$$

where the reduced Seifert matrix W is $2m \times 2m$, $m \leq g$. (See Lemma 13.2; observe that U and W are interchanged for technical reasons). From Corollary 13.4 it follows that $H_1(C_\infty)$ is already generated by $\{p(s_i) \mid 1 \leq i \leq 2m\}$. Therefore, the matrix $\binom{W}{0}$ describes the homomorphism $j_*\colon H_1(S) \to H_1(C_\infty)$ with respect to the bases $\{a_j \mid 1 \leq j \leq 2g\}$ and $\{p(s_i) \mid 1 \leq i \leq 2m\}$. The long exact sequence of the pair $(C_\infty, \partial C_\infty)$ gives a natural isomorphism $H_1(C_\infty) \xrightarrow{\cong} H_1(C_\infty, \partial C_\infty)$. The transpose $(w_{ij}) = \begin{pmatrix} W^T & 0 \end{pmatrix}$ then describes the homomorphism

$$j^*\colon H^1(C_\infty, \partial C_\infty) \to H^1(S)$$

for the dual bases $\{s^i\}$, $\{a^j\}$, and we get from (13.2)

$$B = (\beta(s^i, s^k)) = (\langle s^i \cup s^k, j_*(z) \rangle) = (\langle j^*(s^i) \cup j^*(s^k), z \rangle). \qquad (13.3)$$

We define another free basis $\{b^j \mid 1 \leq i \leq 2g\}$ of $H^1(S) \xrightarrow{\cong} H_1(S, \partial S)$ by the Lefschetz-duality-isomorphism:

$$H^1(S, R) \xrightarrow{\cap z} H_1(S, \partial S, R),\ b^j \mapsto b^j \cap z = a_j.$$

The b^j connect z with the intersection matrix

$$V - V^T = (\operatorname{int}(a_j, a_k)) = (\langle\, b^j \cup b^k, z\,\rangle) = \Sigma.$$

On the other hand

$$\langle\, a^j \cup b^k, z\,\rangle = \langle\, a^j, b^k \cap z\,\rangle = \langle\, a^j, a_k\,\rangle = \delta_{jk}.$$

(See [256, §66],[172, Theorem 4.4.13], [346, Satz 15.4.1].)

The matrix L effecting the transformation $(a^i) = L(b^i)$ is

$$(\langle\, a^i \cup a^k, z\,\rangle) = (\langle\, a^i \cup b^k, z\,\rangle) \cdot L = E \cdot L = L.$$

Now $L = L^T \Sigma L$ or $(\Sigma^T)^{-1} = L$, and, by equation (13.1),

$$\begin{pmatrix} W^T & 0 \end{pmatrix} L \begin{pmatrix} W \\ 0 \end{pmatrix} = \begin{pmatrix} W^T & 0 \end{pmatrix} (\Sigma^T)^{-1} \begin{pmatrix} W \\ 0 \end{pmatrix}.$$

From

$$\Sigma = \left(\begin{array}{c|c} W - W^T & 0 \\ \hline 0 & U - U^T \end{array}\right)$$

and equation (13.3) it follows that

$$B = -W^T (W - W^T)^{-1} W. \tag{13.4}$$

13.5 Proposition. *The bilinear form*

$$\beta\colon\ H^1(C_\infty, \partial C_\infty) \times H^1(C_\infty, \partial C_\infty) \to R, \quad (x, y) \mapsto \langle\, x \cup y, j_*(z)\,\rangle$$

can be represented by the matrix $-(W - W^T)^{-1}$*,* W *is a reduced Seifert matrix, and* β *is non-degenerate.*

Proof. It remains to show that β is non-degenerate. But, by Lemma 13.2, Remark 13.3 and equation (13.1) $|V - V^T| = 1$, and $|U - U^T| = 1$ hence $|W - W^T| = 1$. □

13.6 Remark (Milnor [240]). If $\Delta(0)$ is a unit in R, then the pair $(C_\infty, \partial C_\infty)$ has the homological properties of a compact 2-manifold bounded by a circle. The cap product with the fundamental class $j_*(z)$ gives the isomorphisms

$$H^*(C_\infty; R) \xrightarrow{\cdot \cap j_*(z)} H_*(C_\infty, \partial C_\infty; R)$$
$$H^*(C_\infty, \partial C_\infty; R) \xrightarrow{\cdot \cap j_*(z)} H_*(C_\infty; R)\,.$$

We are now in a position to define an invariant quadratic form associated to the knot $\mathfrak{k}$. Let $t\colon C_\infty \to C_\infty$ denote the generator of the group of covering transformations which corresponds to a meridian linking the knot positively in the oriented S^3.

13.7 Proposition. *The bilinear form*

$$q\colon H^1(C_\infty, \partial C_\infty) \times H^1(C_\infty, \partial C_\infty) \to R,$$

given by $q(x,y) = \langle\, x \cup t^*y + y \cup t^*x, j_*(z)\,\rangle$ *defines a quadratic form* $q(x,x)$, *which can be represented by the matrix* $W + W^T$, *where* W, *see Lemma 13.2, is a reduced Seifert matrix of* $\mathfrak{k}$. *The quadratic form is non-degenerate.* $\Delta(0)$ *is required to be a unit in* R.

Proof. Remember that t_* is represented by $W^{-1}W^T$ with respect to the basis $\{p(s_i)\}$, so t^* will be represented by $W(W^T)^{-1}$ relative to the dual basis $\{s^i\}$.

To calculate the matrix

$$Q = (q(s^i, s^k)) = (\langle\, j^*(s^i) \cup j^*t^*(s^k) + j^*(s^k) \cup j^*t^*(s^i), z\,\rangle)$$

we use $B = (\langle\, j^*(s^i) \cup j^*(s^k), z\,\rangle) = -W^T(W - W^T)^{-1}W$, see (13.3) and (13.4). We obtain

$$\begin{aligned} Q &= BW^{-1}W^T + W(W^T)^{-1}B^T \\ &= -W^T(W - W^T)^{-1}W^T + W(W - W^T)^{-1}W. \end{aligned}$$

Since $|W - W^T| = 1$, the matrices $(W - W^T)^{-1}$ and $W - W^T$ are equivalent, because there is only one skew-symmetric form over $\mathbb{Z}$ with determinant $+1$; its normal form is F (see Theorem A.1 in Appendix A). Let M be unimodular over $\mathbb{Z}$ with

$$\begin{aligned} (W - W^T)^{-1} &= M(W - W^T)M^T, \quad \text{or} \\ E &= (W - W^T)M(W - W^T)M^T. \end{aligned} \tag{13.5}$$

Now, $Q = -W^TM(W - W^T)M^TW^T + WM(W - W^T)M^TW$.

Using (13.5), we get

$$\begin{aligned} Q &= (E - WM(W - W^T)M^T)W^T + WM(W - W^T)M^TW \\ &= W^T + WM(W - W^T)M^T(W - W^T) = W^T + W. \end{aligned}$$

The quadratic form is non-degenerate, since $|W + W^T| \equiv |W - W^T| \equiv 1 \bmod 2$. □

Let us summarize the results of this section: Given a knot, we have proved that $H_1(C_\infty, \partial C_\infty; R)$ is a free R-module if $\Delta(0)$ is invertible in the integral domain R. By using the cup product, we have defined an invariant quadratic form $q_{\mathfrak{k}}$ on

$$H^1(C_\infty, \partial C_\infty, R) \cong H_1(C_\infty,\ \partial C_\infty, R),$$

associated to the knot. The form can be computed from a Seifert matrix. The form $q_{\mathfrak{k}}$ is known as Trotter's *quadratic form.*

In the course of our argument we used both an orientation of S^3 and of the knot. Nevertheless, the quadratic form is independent of the orientation of $\mathfrak{k}$. Clearly $j_*(z) = t_* j_*(z)$ in $H_1(C_\infty, \partial C_\infty)$, by the construction of C_∞ (see Corollary 8.15), so that $q(x, y) = \langle\, x \cup (t^* - t^{*-1})y,\ j_*(z)\,\rangle$ is an equivalent definition of $q(x, y)$. Replacing z by $-z$ and t by t^{-1} does not change $q(x, y)$ (see Proposition 3.18). A reflection σ of S^3 which carries $\mathfrak{k}$ into its mirror image $\mathfrak{k}^*$ induces an isomorphism $\sigma^*\colon\ H^1(C_\infty, \partial C_\infty) \to H^1(C_\infty, \partial C_\infty)$. If $q_{\mathfrak{k}}$ and $q_{\mathfrak{k}^*}$ are the quadratic forms of $\mathfrak{k}$ and $\mathfrak{k}^*$, respectively, then $q_{\mathfrak{k}^*} = -q_{\mathfrak{k}}$, because $\sigma^* t^* = t^{*-1}\sigma^*$.

13.8 Proposition. *The quadratic form of a knot is the same as that of its inverse. The quadratic forms of $\mathfrak{k}$ and its mirror image $\mathfrak{k}^*$ are related by $q_{\mathfrak{k}^*} = -q_{\mathfrak{k}}$.* □

The quadratic form is easily seen to behave naturally with respect to the composition of knots (see Definition 2.7):

13.9 Proposition. *Let us assume that in R the leading coefficients of the Alexander polynomials of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are invertible such that $q_{\mathfrak{k}_1}$ and $q_{\mathfrak{k}_2}$ are defined. Then*

$$q_{(\mathfrak{k}_1 \,\#\, \mathfrak{k}_2)} = q_{\mathfrak{k}_1} \oplus q_{\mathfrak{k}_2}.$$

Proof. Obviously the Seifert matrix of $\mathfrak{k}_1 \,\#\, \mathfrak{k}_2$ has the form

$$V = \left(\begin{array}{c|c} V_1 & 0 \\ \hline 0 & V_2 \end{array}\right)$$

with V_i Seifert matrix of $\mathfrak{k}_i$, $i = 1, 2$. The same holds for the reduced Seifert matrices. □

Invariants of the quadratic form are, of course, invariants of the knot. Moreover, every knot has a quadratic form over the rationals $R = \mathbb{Q}$.

13.10 Definition (Signature). The *signature* σ of the quadratic form of a knot $\mathfrak{k}$ over the rationals is called the signature $\sigma(\mathfrak{k})$ of $\mathfrak{k}$.

The signature of the quadratic form – the number of its positive eigenvalues minus the number of its negative eigenvalues – can be computed without much difficulty [180, Theorem 4] see Theorem A.2 in Appendix A. Obviously the signature of a quadratic form is an additive function with respect to the direct sum. Moreover the signature of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is zero.

13.11 Proposition. (1) *For any Seifert matrix V for $\mathfrak{k}$, $\sigma(\mathfrak{k}) = \sigma(V + V^T)$.*
(2) $\sigma(\mathfrak{k}_1 \# \mathfrak{k}_2) = \sigma(\mathfrak{k}_1) + \sigma(\mathfrak{k}_2)$.
(3) *If $\mathfrak{k}$ is amphicheiral, $\sigma(\mathfrak{k}) = 0$.*

Proof. We can replace V by an equivalent matrix of the form as obtained in Lemma 13.2. Then

$$V + V^T \sim \left(\begin{array}{c|c} U + U^T & \mathbf{0} \\ \hline \mathbf{0} & W + W^T \end{array}\right),$$

where

$$U + U^T = \left(\begin{array}{cc|cc|cc} 0 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & * & * & \cdots \\ \hline 0 & * & 0 & -1 & 0 & 0 \\ 0 & * & -1 & 0 & 0 & * \\ \hline 0 & * & 0 & * & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & \end{array}\right) \sim \left(\begin{array}{cc|cc|cc} 0 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & \end{array}\right).$$

□

13.B Computation of the quadratic form of a knot

The computation of the quadratic form q of a given knot $\mathfrak{k}$ was based in the last paragraph on a Seifert matrix V which in turn relied on Seifert's band projection (see 8.2). Such a projection might not be easily obtainable from some given regular knot projection. Murasugi [261] defined a knot matrix M over $\mathbb{Z}$, which can be read off any regular projection of a link. A link defines a class of s*-equivalent* matrices $\{M\}$ (see Definition 9.24), and by symmetrizing, one obtains a class of *equivalent* symmetric matrices $\{M + M^T\}$ which can be described in the following way:

13.12 Definition. Two symmetric integral matrices A and A' are called S*-equivalent* if they are related by a finite chain of the following operations and their inverses:

$$\Lambda_1 : A \mapsto L^T A\, L, \text{ where } L \text{ is } R\text{-unimodular}; \quad \Lambda_2' : A \mapsto \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & A & \\ 0 & 0 & & & \end{array}\right).$$

K. Murasugi proves in [261] that the class $\{M+M^T\}$ of S-equivalent symmetrized knot matrices is an invariant of the knot (or link). He thereby associates a class of quadratic forms to a link.

Obviously, S-equivalent matrices have the same signature (see proof of Proposition 13.11), so the signature $\sigma\{M+M^T\}$ is defined and is a knot invariant. We shall prove: *If W is a reduced Seifert matrix of $\mathfrak{k}$, then $W+W^T \in \{M+M^T\}$*. This means that the quadratic form $q_{\mathfrak{k}}$ as defined in the first section of this chapter is a member of the class of quadratic forms represented by $M+M^T$, where M is Murasugi's knot matrix. Since the rule given by Murasugi to read off M from an arbitrary regular projection is rather complicated, we shall confine ourselves to so-called special projections, which hold a position between arbitrary projections and band projections. Any projection can be converted into a special one without much difficulty. We give a simple rule in equation (13.6) to read off the matrix M from a special projection.

13.13 Definition (Special projection). Let $p(\mathfrak{k})$ be a regular projection of a knot $\mathfrak{k}$ on S^2. Choose a chessboard coloring (colors α and β) of the regions of S^2 defined by $p(\mathfrak{k})$ (see Definition 2.1). The projection $p(\mathfrak{k})$ is called a *special projection* or *special diagram*, if the union of the β-regions is the image of a Seifert surface of $\mathfrak{k}$ under the projection p.

13.14 Proposition. *Every knot $\mathfrak{k}$ possesses a special projection.*

Proof. Starting from an arbitrary regular projection of $\mathfrak{k}$ we use Seifert's procedure (see Proposition 2.4) to construct an orientable surface S spanning $\mathfrak{k}$. We obtain S as a union of several disks spanning the Seifert circuits, and a couple of bands twisted by angle π, joining the disks, which may occur in layers over each other. There is an isotopy which places the disks separately into the projection plane $\mathbb{R}^2$, so that they do not meet each other or any bands, save those which are attached to them (Figure 13.1 (a)). By giving the overcrossing section at a band crossing a half-twist (Figure 13.1 (b)) it can be arranged that only the type of crossing as shown in Figure 13.1 (b) occurs.

Now apply again Seifert's method. All Seifert circuits bound disjoint regions (β-regions) in $\mathbb{R}^2$. So they define a Seifert surface S' which – except in the neighborhood of double points – consists of β-regions. □

13.15 Remark. The surface S' is obtained from S by a 1*-handle enlargement*: the two bands of Figure 13.1 (b) are joined by a 1-handle. The corresponding Seifert matrices V and V' are s-equivalent (see Proposition 13.18, [190, 5.2], [133, Sect. 11]).

It follows that the number of edges (arcs of $p(\mathfrak{k})$ joining double points) of every α-region in a special projection must be even. This also suffices to characterize a special projection, if the boundaries of β-regions are simple closed curves, that is, if at double points always different β-regions meet. It is easy to arrange that the boundaries of β-

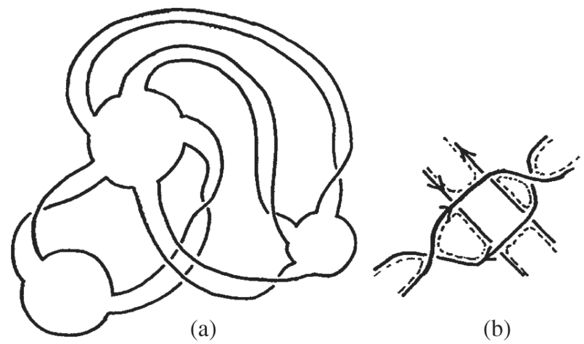


Figure 13.1. A special projection.

and α-regions are simple: in case they are not, a twist through angle π removes the double point which occurs twice in the boundary (Figure 13.2).

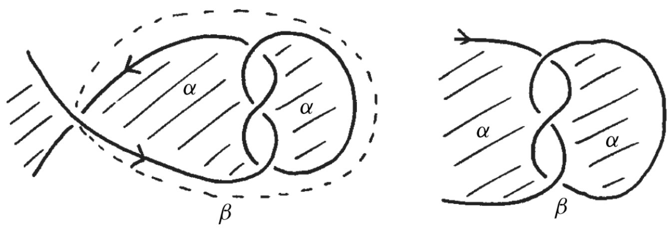


Figure 13.2

We now use a special projection to define associated bases $\{a_i\}$, $\{s_k\}$ of $H_1(S)$ and $H_1(C^*)$, respectively, and compute their Seifert matrix V. (It turns out that V is Murasugi's knot matrix M of the special diagram; see [261, 3.3].) Let S be the Seifert surface of $\mathfrak{k}$ which projects onto the β-regions $\{\beta_j\}$ of a special projection. The special projection suggests a geometric free basis of $H_1(S)$. Choose simple closed curves a_i on S whose projections are the boundaries $\partial\alpha_i$ of the finite α-regions $\{\alpha_i \mid 1 \leq i \leq 2h\}$, oriented counterclockwise in the projection plane. (See Figure 13.3.) The number of finite α-regions is $2h$, where h is the genus of S. (We denote the infinite α-region by α_0.)

Now cut the knot complement C along S to obtain C^*. There is again a geometrically defined free basis $\{s_k \mid 1 \leq k \leq 2h\}$ of $H_1(C^*)$ associated to $\{a_i\}$ by linking numbers: $\operatorname{lk}(a_i, s_k) = \delta_{ik}$. The curve representing s_k pierces the projection plane once (from below) in a point belonging to α_k and once in α_0.

The curve a_i splits up into a_i^+ and a_i^-. We move $i_*^+(a_i^+)$ by a small deformation away from S^+ and use the following convention to distinguish between $i_*^+(a_i^+)$ and $i_*^-(a_i^-)$. If in the neighborhood of a double point P the curve $i_*^+(a_i^+)$ is directed as the parallel undercrossing edge of $\partial\alpha_i$, then $i_*^+(a_i^+)$ is supposed to run above the

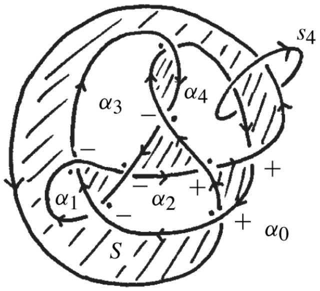


Figure 13.3

projection plane; otherwise it will run below. This arrangement is easily seen to be consistent in a special diagram.

In Definition 2.3 we defined the index $\theta(P)$ of a double point P. We need another function which takes care of the geometric situation at a double point with respect to the adjoining α-regions.

13.16 Definition (Index $\varepsilon_i(P)$). Let P be a double point in a special projection, $P \in \partial\alpha_i$. Then

$$\varepsilon_i(P) = \begin{cases} 1 & \text{if } \alpha_i \text{ is on the left of the underpassing arc at } P, \\ 0 & \text{if } \alpha_i \text{ is on the right,} \end{cases}$$

is called *ε-index of P* (see Figure 13.4).

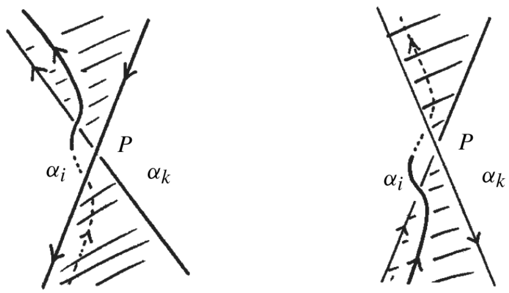


Figure 13.4

From this definition it follows that $\varepsilon_i(P) + \varepsilon_k(P) = 1$ for $P \in \partial\alpha_i \cap \partial\alpha_k$. Because of this symmetry it suffices to consider the two cases described in Figure 13.4.

We compute the Seifert matrix $V = (v_{ik})$, $i^+(a_i^+) = \Sigma v_{ik} s_k$:

$$\begin{aligned} v_{ii} &= \textstyle\sum_{P \in \partial\alpha_i} \theta(P)\varepsilon_i(P) \\ v_{ik} &= \textstyle\sum_{P \in \partial\alpha_i \cap \partial\alpha_k} \theta(P)\varepsilon_k(P). \end{aligned} \tag{13.6}$$

This can be verified from our geometric construction using Figure 13.4. The formulas (13.6) coincide with Murasugi's definition of his knot matrix M [261, Definition 3.3] in the case of a special projection. (A difference in sign is due to another choice of $\theta(P)$.)

The formulas (13.6) may be regarded as the definition of M; we do not give a definition of Murasugi knot matrix for arbitrary projections because it is rather intricate. The result of the consideration above can be formulated in the following way:

13.17 Proposition. *Let $p(\mathfrak{k})$ be a special diagram of $\mathfrak{k}$ with α-regions α_i, index functions $\theta(P)$ and $\varepsilon_i(P)$ according to Definitions* 2.3 *and* 13.16. *Then a Seifert matrix (v_{ik}) of $\mathfrak{k}$ is defined by* (13.6). *(The Seifert matrix coincides with Murasugi's knot matrix of $p(\mathfrak{k})$.)* □

13.18 Proposition. *If W is a reduced Seifert matrix, then $(W + W^T)$ is contained in the class $\{M + M^T\}$ of S-equivalent matrices. The signature $\sigma_{\mathfrak{k}}$ coincides with the signature $\sigma(M + M^T)$ of Murasugi [261].*

Proof. If S is a Seifert surface which admits a special diagram as a projection the assertion follows from Definition 13.12 and Proposition 13.17. Any Seifert surface S allows a band projection. By twists through angle π it can be arranged that the bands only cross as shown in Figure 13.1 (b). At each crossing we change S, as we did in the proof of Proposition 13.17, in order to get a spanning surface S' which projects onto the β-region of a special diagram. We then compare the band projections of S and S' and their Seifert matrices V and V'. It suffices to consider the case shown in Figure 13.5 (a). Note that S' is obtained from S by a 1-handle enlargement, see Remark 13.15.

It is not difficult to perform the local isotopy which carries Figure 13.5 (c) over to Figure 13.5 (d). The genus of the new surface is $g(S') = g(S) + 2$. Let $\{a_k\}$, $\{s_l\}$ be associated bases of $H_1(S)$, (see Definition 8.9) and $H_1(C^*)$, and let V be their Seifert matrix. Substitute $\tilde{a}, a'_j, a''_j$ for $a_j \in \{a_k\}$ and $\tilde{s}, s'_j, s''_j$ for $s_j \in \{s_l\}$ to obtain associated bases relative to S'. The corresponding Seifert matrix V' is of the form

$$\begin{array}{cc} & \begin{matrix} \tilde{s} & s'_j & s''_j & \end{matrix} \\ \begin{matrix} \tilde{a} \\ a'_j \\ a''_j \\ \end{matrix} & \begin{pmatrix} 0 & 1 & -1 & 0\dots \\ 0 & * & * & \dots \\ 0 & * & * & \dots \\ 0 & \vdots & \vdots & V \end{pmatrix} \end{array}.$$

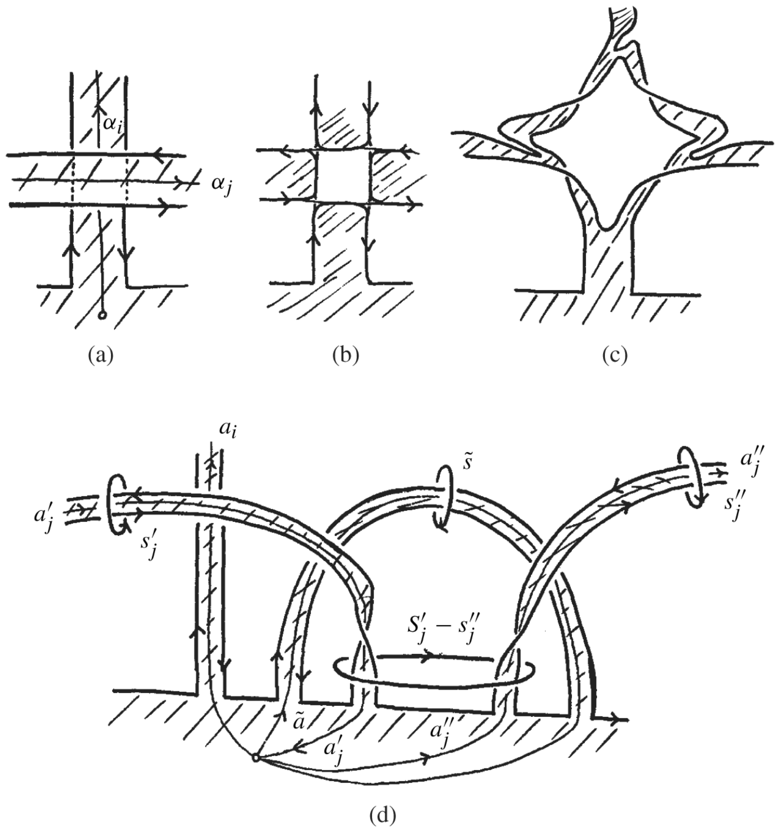


Figure 13.5. A 1-handle enlargement.

Adding the s'_j-column to the s''_j-column and then the a'_j-row to the a''_j-row we get

$$\begin{array}{rc} & \begin{array}{ccccc} \tilde{s} & s'_j - s''_j & s''_j & & \end{array} \\ \begin{array}{r} \tilde{a} \\ a'_j \\ a_j = a'_j + a''_j \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & * & * & * & \dots \\ 0 & * & & & \\ 0 & * & & V & \\ \vdots & \vdots & & & \end{pmatrix} \end{array}.$$

This follows from Figure 13.5 (d), because the overcrossings of s'_j and s''_j add up to those of s_j, and $a_j = a'_j + a''_j$. Evidently, by adding multiples of the first row to the other rows the second column can be replaced by zeroes excepting the 1 on top. After these changes the bases remain associated. We have proved: $(V' + V'^T)$ and $(V + V^T)$ are S-equivalent (see Definition 13.12). The procedure can be repeated until a Seifert surface is reached which allows a special projection. (Observe: Twists in the bands do not hamper the process.) □

13.C Alternating knots and links

The concepts which have been developed in the preceding section provide a means to obtain certain results on alternating knots and links first proved by R. H. Crowell [76] and K. Murasugi [257, 258, 260]. R. H. Crowell's paper rests on a striking application of a graph theoretical result, the Bott–Mayberry matrix tree theorem [39].

In Definition 2.3 we defined the graph of a regular projection $p(\mathfrak{k})$ of a knot (or link) by assigning a vertex P_i to each α-region α_i and an edge to each double point; we call this graph the α-graph of $p(\mathfrak{k})$ and denote it by Γ_α. Its dual Γ_β is obtained by considering β-regions instead of α-regions.

We always assume the infinite region to be the α-region α_0. The following definition endows Γ_α and Γ_β with orientations and valuations.

13.19 Definition. Let Γ_α be the α-graph of $p(\mathfrak{k})$. The edge joining $P_i \in \alpha_i$ and $P_k \in \alpha_k$ assigned to the crossing Q_λ of $p(\mathfrak{k})$ is denoted by u^λ_{ik}. The orientation of u^λ_{ik} is determined by the characteristic $\eta(Q_\lambda)$ (see Theorem 3.4): The initial point of u^λ_{ik} is P_i resp. P_k for $\eta(Q_\lambda) = +1$ resp $= -1$.

(Loops $(i = k)$ are oriented arbitrarily.) The oriented edge u^λ_{ik} obtains the *valuation* $f(u^\lambda_{ik}) = \theta(Q_\lambda)$. The edges v^μ_{jl} of the dual graph Γ_β are oriented in such a way that $\operatorname{int}(u^\lambda_{ik}, v^\mu_{jl}) = +1$ for every pair of dual edges with respect to a fixed orientation of the plane containing $p(\mathfrak{k})$. Now the valuation of Γ_β is defined by $f(v^\mu_{jl}) = -f(u^\lambda_{ik})$. Denote the graphs with orientation and valuation by Γ^*_α, Γ^*_β respectively.

13.20 The graph matrix. A Seifert matrix of a Seifert surface of $\mathfrak{k}$ which is composed of the β-regions of a special projection may now be interpreted in terms of Γ^*_α. Define a square matrix $H(\Gamma^*_\alpha) = (h_{ik})$ by

$$h_{ii} = \sum_{j,\lambda} f(u^\lambda_{ji}), \quad h_{ik} = -\sum_{\lambda} f(u^\lambda_{ik}),\ i \neq k. \tag{13.7}$$

Denote by H_{ii} the submatrix of H obtained by omitting the i-th row and column of H. From equations (13.6) and (13.7) we obtain (recall that the subscript 0 corresponds to the infinite region α_0):

13.21 Proposition. *Let $p(\mathfrak{k})$ be a special projection of a knot or link, Γ^*_α its α-graph, and H the graph matrix of Γ^*_α. Then H_{00} is a Seifert matrix of $\mathfrak{k}$ with respect to a Seifert surface which is projected onto the β-regions of $p(\mathfrak{k})$.* □

By a theorem of R. Bott and J. P. Mayberry [39], the principal minors $\det(H_{ii})$ of a graph matrix are connected with the number of *rooted trees* in a graph Γ; for definitions and the proof see Appendix A.3–A.6.

13.22 Theorem (Matrix tree theorem of Bott–Mayberry). *Let Γ^*_α be an oriented graph with vertices P_i and edges u^λ_{ik}, and a valuation $f\colon \{u^\lambda_{ik}\} \mapsto \{1,-1\}$. Then*

$$\det(H_{ii}) = \sum f(\mathrm{Tr}(i)), \tag{13.8}$$

where the sum is taken over the rooted trees $\mathrm{Tr}(i) \subset \Gamma^*_\alpha$ with root P_i, and $f(\mathrm{Tr}(i)) = \prod f(u^\lambda_{jk})$, the product taken over all $u^\lambda_{jk} \in \mathrm{Tr}(i)$. □

13.23 Proposition. *The graphs Γ^*_α and Γ^*_β of a special alternating projection have the following properties (see Figure 13.6):*

(a) *Every region of Γ^*_α can be oriented such that the induced orientation on every edge in its boundary coincides with the orientation of the edge.*

(b) *No vertex of Γ^*_β is at the same time initial point and endpoint.*

(c) *The valuation is constant (we always choose $f(u^\lambda_{ik}) = +1$).* □

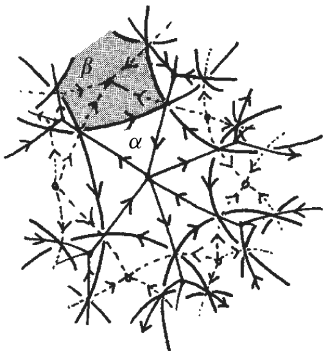


Figure 13.6

The proof of the assertion is left to the reader. It relies on geometric properties of special projections, see Figure 13.6, and the Definitions 2.3 and 13.19. Note that the edges of Γ^*_α with P_i in their boundary, cyclically ordered, have P_i alternately as initial point and endpoint, and that the edges in the boundary of a region of Γ^*_β also alternate with respect to their orientation. □

13.24 Proposition. *Let S be the Seifert surface determined by the β-regions of a special alternating projection $p(\mathfrak{k})$, and V a Seifert matrix of S. Then* $\det V \neq 0$

and S is of minimal genus. Furthermore, $\det V = \pm 1$, *if and only if* $\deg P_i = \sum_k |h_{ik}| = 2$ *for* $i \neq 0$.

Proof. It follows from Proposition 13.23 (a) that every two vertices of Γ_α^* can be joined by a path in Γ_α^*. So there is at least one rooted tree for any root P_i. Since $f(u_{ik}^\lambda) = +1$ the number of P_i-rooted trees is by Theorem 13.22 equal to $\det(H_{ii}) > 0$. If $V = H_{00}$ is an $(m \times m)$-matrix then $\deg \Delta(t) = m$ in the case of a knot, and $\deg \bigtriangledown^H(t) = m - \mu + 1$ in the case of a μ-component link. (See 9.19 for the definition of the Hosokawa polynomial $\bigtriangledown^H(t)$.) It follows that $2h = m$ where h is the genus of S. Since $\deg \Delta(t) \leq 2g$ resp. $\deg \bigtriangledown^H(t) + \mu - 1 \leq 2g$ for the genus g of $\mathfrak{k}$, we get $g = h$, see 8.21, 9.19 and E 9.5.

To prove the last assertion we characterize the graphs Γ_α^* which admit only one P_0-rooted tree. We claim that for $i \neq 0$ one must have $\deg P_i = 2$. Suppose $\deg P_k \geq 4$ for some $k \neq 0$, with $u_{ik}^\lambda \neq u_{jk}^{\lambda'}$, and u_{ik}^λ contained in a P_0-rooted tree T_0. Then $u_{jk}^{\lambda'} \notin T_0$, and there are two simple paths w_i, w_j in T_0 which intersect only in their common initial point P_l with endpoints P_i and P_j respectively, see Figure 13.7. Substitute $u_{jk}^{\lambda'}$ for u_{ik}^λ to obtain a different P_0-rooted tree.

Obviously every graph Γ_α^* with $\deg P_i = 2$ for all $i \neq 0$ has exactly one P_0-rooted tree. □

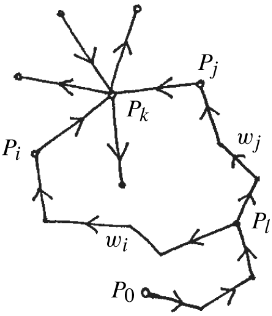


Figure 13.7

As an easy consequence one gets:

13.25 Proposition. *A knot or link $\mathfrak{k}$ with a special alternating projection is fibered, if and only if $\Delta(t)$ is monic. In this case $\mathfrak{k}$ is the product of torus knots or links $\mathfrak{k}_i = \mathfrak{t}(a_i, 2)$, $\mathfrak{k} = \mathfrak{k}_1 \# \ldots \# \mathfrak{k}_r$.*

Proof. See Figure 13.8. It follows from Proposition 13.24 that $\mathfrak{k}$ is of this form. By Corollary 4.12 and Proposition 7.20 we know that knots of this type are fibered. □

Proposition 13.25 was first proved by Murasugi [258].

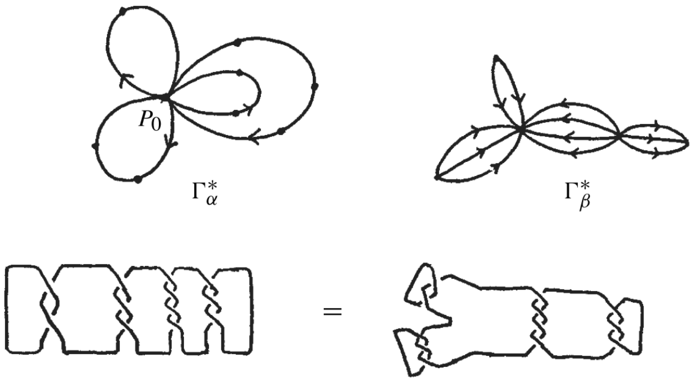


Figure 13.8. A special alternating projection.

13.26 Theorem. *Let $\mathfrak{k}$ be an alternating knot or link of multiplicity μ, and $p(\mathfrak{k})$ an alternating regular projection.*

(a) *The genus of the Seifert surface S obtained from the Seifert construction 2.4 is the genus $g(\mathfrak{k})$ of $\mathfrak{k}$ (genus and canonical genus coincide).*

(b) $\deg \Delta(t) = 2g$, *resp.* $\deg \nabla^H(t) = 2g$.

(c) $\mathfrak{k}$ *is fibered if and only if* $|\Delta(0)| = 1$ *resp.* $|\nabla^H(0)| = 1$.

Proof. Consider the Seifert cycles of the alternating projection $p(\mathfrak{k})$. If a Seifert cycle contains another Seifert cycle in the projection of the disk it spans, it is called a *cycle of the second kind*, otherwise it is *of the first kind* [258]. If there are no cycles of the second kind, the projection is special, see Proposition 2.4 and Definition 13.13. Suppose there are cycles of the second kind; choose a cycle c bounding a disk $D \subset S^3$ such that $p(D)$ contains only cycles of the first kind. Place S in $\mathbb{R}^3$ in such a way that the part of $\mathfrak{k}$ which is projected on $p(D)$ is above a plane $E \supset D$, while the rest of $\mathfrak{k}$ is in the lower half-space (Figure 13.9).

Cut S along D such that S splits into two surfaces S_1, S_2, contained in the upper resp. lower half-space defined by E such that D is replaced by two disks D_1, D_2. The knots (or links) $\mathfrak{k}_1 = \partial S_1$, $\mathfrak{k}_2 = \partial S_2$ then possess alternating projections $p(\mathfrak{k}_1)$, $p(\mathfrak{k}_2)$, and $p(\mathfrak{k}_1)$ is special. One may obtain S back from S_1 and S_2 by identifying the disks D_1 and D_2. If $\mathfrak{k}$ results in this way from the components $\mathfrak{k}_1$ and $\mathfrak{k}_2$, we write $\mathfrak{k} = \mathfrak{k}_1 * \mathfrak{k}_2$ and call it $*$-product or *Murasugi sum* [258], [190, 4.2]. (The reader is warned that the $*$-product does not depend merely on its factors $\mathfrak{k}_1$ and $\mathfrak{k}_2$.)

Let C^*, C_i^*, $1 \le i \le 2$, be obtained from the complements of $\mathfrak{k}$, $\mathfrak{k}_i$ by cutting along S, S_i, see 4.4. Choose a basepoint P on ∂D (Figure 13.9), then

$$\begin{aligned} \pi_1 C^* &\cong \pi_1 C_1^* * \pi_1 C_2^*, & \pi_1 S &\cong \pi_1 S_1 * \pi_1 S_2, \\ H_1(C^*) &\cong H_1(C_1^*) \oplus H_1(C_2^*), & H_1(S) &\cong H_1(S_1) \oplus H_1(S_2). \end{aligned} \tag{13.9}$$

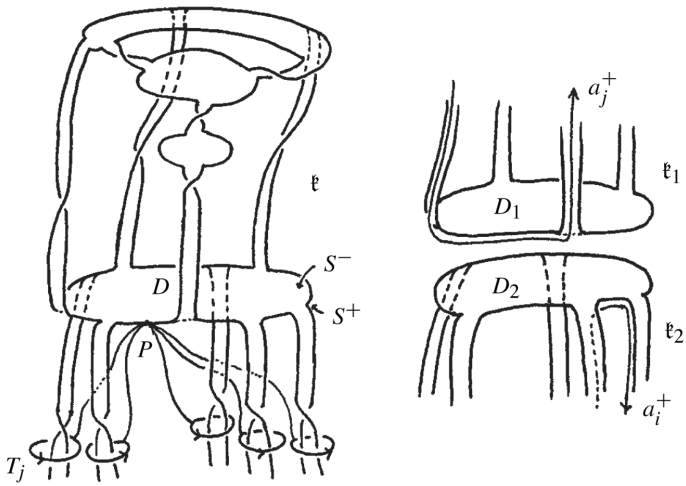


Figure 13.9. The Murasugi sum.

It is evident that every alternating projection may be obtained by forming $*$-products of special alternating projections. We shall prove Theorem 13.26 by induction on the number of $*$-products needed to build up a given alternating projection $p(\mathfrak{k})$.

Proposition 13.25 proves the assertion if $p(\mathfrak{k})$ is special alternating. Suppose $\mathfrak{k} = \mathfrak{k}_1 * \mathfrak{k}_2$ and $p(\mathfrak{k}_1)$ is special.

Let $i_1^\pm\colon S_1^\pm \to C_1^*$, $i_2^\pm\colon S_2^\pm \to C_2^*$, $i^\pm\colon S \to C^*$ denote the inclusions. If S^+ and S^- are chosen as indicated in Figure 13.9, the Seifert matrix V^+ associated with i_*^+ can be written in the form

$$V^+ = \left(\begin{array}{c|c} V_1^+ & * \\ \hline \mathbf{0} & V_2^+ \end{array}\right)$$

where V_1^+ and V_2^+ are Seifert matrices belonging to i_{1*}^+, i_{2*}^+.

Assume that $|V_2^+| \neq 0$ for $\mathfrak{k}_2$, S_2 as an induction hypothesis. By Proposition 13.24 we have $|V_1^+| \neq 0$ and $|V^+| = |V_1^+| \cdot |V_2^+|$ implies $|V^+| \neq 0$. Hence (a) and (b) follow for $\mathfrak{k}$.

To prove (c) assume that $\Delta_{\mathfrak{k}}(0) = |V^+| = \pm 1$. Then $|V^+| = |V_1^+| \cdot |V_2^+|$ implies that $|\Delta_{\mathfrak{k}_i}(0)| = 1$, $i = 1, 2$. Assume (c) for $\mathfrak{k}_2$, S_2 as an induction hypothesis. By Proposition 13.25 we know that $\mathfrak{k}_1$ is fibered. Let

$$w_1^{(1)} w_1^{(2)} w_2^{(1)} w_2^{(2)} \dots w_j^{(1)} w_j^{(2)},\ w_j^{(k)} \in \pi_1(C_k^*),\ 1 \le k \le 2,$$

be an element of $\pi_1(C_1^*) * \pi_1(C_2^*) \cong \pi_1(C^*)$. If $\mathfrak{k}_2$ is fibered, $i_{2\#}^+$ is an isomorphism. A closed curve $\omega_j^{(2)}$ in C^* representing $w_j^{(2)}$ is, therefore, homotopic rel P in C^* to a curve on S^+. Since $\mathfrak{k}_1$ is also fibered, a curve $\omega_j^{(1)}$ corresponding to a factor

$w_j^{(1)}$ is homotopic to a closed curve composed of factors a_j^+ on S^+ and $T_j^{\pm 1}$, see Figure 13.9. But the T_j can be treated as the curves $\omega_j^{(2)}$ and are homotopic to curves on S^+. Thus $i_\#^+$ is surjective; it is also injective, since S is of minimal genus (see Neuwirth's Lemma 4.5, [267]). □

This shows, together with Proposition 13.25, that a fibered alternating knot or link must be a $*$-product composed of factors

$$\mathfrak{k}_i = \mathfrak{t}(a_1, 2) \,\#\, \mathfrak{t}(a_2, 2) \,\#\, \mathfrak{t}(a_3, 2) \,\#\, \ldots \,\#\, \mathfrak{t}(a_r, 2).$$

There is a corollary:

13.27 Corollary. *The commutator subgroup of an alternating knot is either*

$$\mathfrak{G}' = \mathfrak{F}_{2g} \quad \textit{or} \quad \mathfrak{G}' = \ldots * \mathfrak{F}_{2g} *_{\mathfrak{F}_{2g}} \mathfrak{F}_{2g} * \ldots$$

where g is the genus of the knot. The 3-manifold C^* is a handlebody of genus $2g$ for a suitable Seifert surface of minimal genus.

Proof. The space C_1^* is a handlebody of genus $2g_1$, g_1 the genus of $\mathfrak{k}_1$. This follows by thickening the β-regions of $p(\mathfrak{k}_1)$. By the same inductive argument as used in the proof of Theorem 13.26 (see (13.9)) one can see that C^* is a handlebody of genus $2g$ obtained by identifying two disks D_1 and D_2 on the boundary of the handlebodies C_1^* and C_2^*. □

13.D Comparison of different concepts and examples

In the Sections 13.A and 13.B we defined the quadratic form of a knot according to Trotter and Erle, and pointed out the connection to Murasugi's class of forms [261]. Let us add now a few remarks on Goertiz's form. We shall give an example which shows that Goeritz's invariant is weaker than that of Trotter–Murasugi. Nevertheless, Goertiz's form is still of interest because it can be more easily computed than the other ones, and C. M. Gordon and R. A. Litherland [135] have shown that it can be used to compute the Trotter–Murasugi signature.

Let a regular knot diagram be chessboard colored and let $\theta(P)$ be defined as in Definition 2.3, see Figure 13.10. (Here we may assume again that at no point P the two α-regions coincide; if they do, define $\theta(P) = 0$ for such points.)

The integers

$$g_{ij} = \begin{cases} \sum\limits_{P \in \partial\alpha_i} \theta(P), & i = j \\ -\sum\limits_{P \in \partial\alpha_i \cap \partial\alpha_j} \theta(P), & i \neq j \end{cases} \tag{13.10}$$

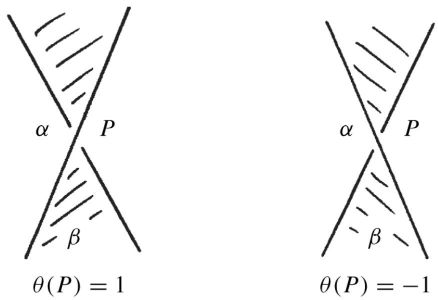


Figure 13.10

then determine a symmetric $(n \times n)$-matrix $G = (g_{ij})$, where $\{\alpha_i \mid 1 \le i \le n\}$ are the finite α-regions. The matrix G is called *Goeritz matrix* and the quadratic form, defined by G, is called *Goeritz form*. (Observe that the orientation of the arcs of the projection do not enter into the definition of the index $\theta(P)$, but that G changes its sign if $\mathfrak{k}$ is mirrored.) Transformations $G \mapsto LGL^T$ with unimodular L and the following matrix operation (and its inverse)

$$G \mapsto \left(\begin{array}{c|c} G & \mathbf{0} \\ \hline \mathbf{0} & \pm 1 \end{array}\right)$$

define a class of quadratic forms associated to the knot $\mathfrak{k}$ which Goeritz showed to be a knot invariant [130, 199]. A Goeritz matrix representing the quadratic form of a knot $\mathfrak{k}$ is denoted by $G(\mathfrak{k})$.

13.28 Proposition. *Let $p(\mathfrak{k})$ be a special diagram and V a Seifert matrix defined by* (13.6) *(see Proposition 13.17). Then $V + V^T = G(\mathfrak{k})$ is a Goeritz matrix of $p(\mathfrak{k})$.*

Proof. This is clear for elements $g_{ij}, i \neq j$, since $\varepsilon_i(P) + \varepsilon_k(P) = 1$ for $P \in \partial\alpha_i \cap \partial\alpha_k$. For $i = j$ it follows from the equality

$$v_{ii} = \sum_{P \in \partial\alpha_i} \theta(P)\varepsilon_i(P) = \sum_{P \in \partial\alpha_i} \theta(P)(1 - \varepsilon_i(P));$$

the first sum describes the linking number of $i_*^+(a_i^+)$ with $\partial\alpha_i$, the second the linking number of $i_*^-(a_i^-)$ with $\partial\alpha_i$ which are the same for geometric reasons. (There is a ribbon $S^1 \times I \subset S^3$, $S^1 \times 0 = a_i^-$, $S^1 \times 1 = a_i^+$, $S^1 \times \frac{1}{2} = \partial\alpha_i$.) □

From this it follows that each Goeritz matrix G can be interpreted as presentation matrix of $H_1(\hat{C}_2)$ (see Proposition 8.39 (a)). H. Seifert [327], M. Kneser and D. Puppe [199] have investigated this connection and were able to show that the Goeritz matrix defines the linking pairing $H_1(\hat{C}_2) \otimes H_1(\hat{C}_2) \to \mathbb{Z}$.

Figure 13.11 (a) shows a diagram of the left-handed trefoil and Figure 13.11 (b) a special diagram of it. The sign at a crossing point P denotes the sign of $\theta(P)$, a dot

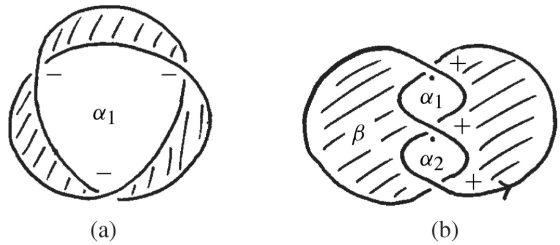


Figure 13.11. A minimal and a special diagram of the left-handed trefoil.

at P in an α-region α_i indicates $\varepsilon_i(P) = 1$ for $P \in \partial\alpha$. Thus we get $G_a = (-3)$ from Figure 13.11 (a) and

$$M + M^T = G_b = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

G_a and G_b can be transformed into each other by Goeritz moves which are described before Proposition 13.28:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \sim (-3).$$

Figures 13.12 (a) and 13.12 (b) show a minimal and a special projection of the knot 8_{19}.

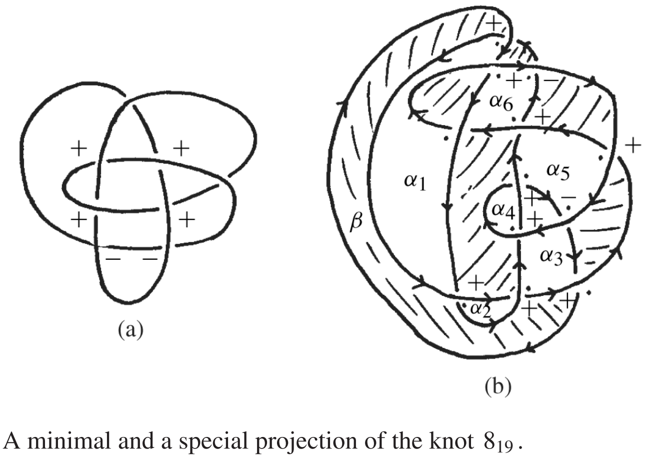


Figure 13.12. A minimal and a special projection of the knot 8_{19}.

Figure 13.12 (a) yields a Goeritz matrix

$$G = \begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \sim (3)$$

which is equivalent to that of a right-handed trefoil (the mirror image of Figure 13.10 (a)). A Seifert matrix V can be read off Figure 13.12 (b):

$$V = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Since $|V| = 1$, V is already reduced, so its quadratic form $q_{\mathfrak{k}}$ is of rank 6, different to that of a trefoil which is of rank 2.

We finally demonstrate the advantage of using a suitable integral domain R instead of $\mathbb{Z}$. Figure 13.3 shows a special diagram of 8_{20}. Its Seifert matrix is

$$V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad |V| = 1,$$

$$V + V^T = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -4 & 2 \\ 0 & -1 & 2 & 0 \end{pmatrix} \sim \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & -2 & 3 \\ & & 3 & 0 \end{array}\right).$$

Hence, $V + V^T$ is S-equivalent (see Definition 13.12) to

$$\begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix} = V' + V'^T, \ V' = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Using the construction of Proposition 8.7 one obtains a knot $\mathfrak{k}'$ with Seifert matrix V'. It follows that the Murasugi matrices $V + V^T$ and $V' + V'^T$ of 8_{20} and $\mathfrak{k}'$ respectively are S-equivalent. Over $R = \mathbb{Z}_2$ the matrix V' is not reduced and so the Trotter form of $\mathfrak{k}'$ is trivial, whereas the Trotter form of 8_{20} is represented by

$$\left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & -2 & 3 \\ & & 3 & 0 \end{array}\right).$$

Moreover both knots have zero-signature, but over $\mathbb{Z}_3$ their forms prove that they are not amphicheiral.

13.29 Corollary. *The absolute value of the determinant of the quadratic form is an invariant of the knot. It is called the* determinant of the knot. *It can be expressed in several forms:*

$$|\det(M + M^T)| = |\det(W + W^T)| = |\det G| = |H_1(\widehat{C}_2)| = |\Delta(-1)|.$$

Proof. See Propositions 8.21 and 8.39 □

In the case of alternating knots the determinant is a strong invariant; in fact, it can be used to classify alternating knots in a certain sense:

13.30 Proposition (Bankwitz [13], Crowell [77]). *The order (minimal number of crossings) with respect to regular alternating projections of a knot does not exceed its determinant.*

Proof. Let $p(\mathfrak{k})$ be a regular alternating projection of minimal order n. Consider the (unoriented) graph Γ_α of $p(\mathfrak{k})$. Since n is minimal, Γ_α does not contain any loops, and every edge of Γ_α is contained in a circuit, compare Figure 13.2. It follows from the Corollary to the Bott–Mayberry Theorem (Appendix A.5) that the determinant $\det G(\mathfrak{k})$ of $\mathfrak{k}$ is equal to the number of spanning trees of Γ_α. It remains to show that in a planar graph Γ_α with the aforesaid properties the number n of edges never exceeds the number of trees. One may reduce Γ_α by omitting points of order two and loops. If then Γ_α defines more than two regions on S^2, there exists an edge b in the boundary of two regions such that these two regions have no other edge in common. The graph $(\Gamma_\alpha - b)$ then is a connected planar graph with no loops where every edge is in a circuit. Every tree of $(\Gamma_\alpha - b)$ is a tree of Γ_α. There is at least one tree more in Γ_α which contains b. □

The inequality $n \leq |\det G(\mathfrak{k})|$ can be improved [76], see E 13.4.

Since there are only finitely many alternating knots with $|\Delta(-1)| = d$, there are only finitely many such knots with the same Alexander polynomial. If $\Delta(-1) = \pm 1$ (in particular, if $\Delta(t) = 1$), the knot is either non-alternating or any alternating projection of it can be trivialized by twists of the type of Figure 13.2. Consider as an example the knot 6_1, see Figure 13.13. The Goeritz matrix is

$$G(6_1) = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}.$$

One checks easily in Figure 13.13 that the graph has $11 = \Delta(-1) = \begin{vmatrix} 4 & -1 \\ -1 & 3 \end{vmatrix}$ maximal trees.

Proposition 13.30 of Bankwitz can also be used to show that certain knots are non-alternating, that is, do not possess any alternating projection. This is true for all non-trivial knots with trivial Alexander polynomial. Crowell was able to prove that most of the knots with less than ten crossings which are depicted in Reidemeister's table

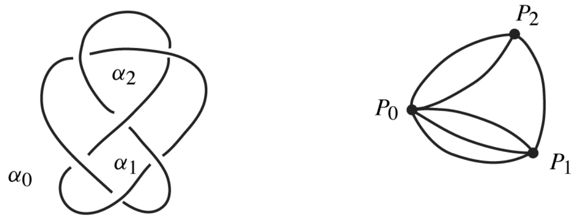


Figure 13.13. The knot 6_1.

as non-alternating, really are non-alternating. If, for instance, 8_{19} were alternating, it would have a projection of order $\Delta(-1) = 3$ or less. But 8_{19} is non-trivial and different from 3_1 by its Alexander polynomial.

We now give a description of a result of Gordon and Litherland. In a special diagram the β-regions are bounded by Seifert circuits. If in a chessboard coloring of an arbitrary projection the Seifert circuits follow the boundaries of α-regions in the neighborhood of a crossing P we call P *exceptional*, and by ν we denote the number $\nu = \sum \theta(P)$, where the sum is taken over the exceptional points of the projection. (The β-regions form an orientable Seifert surface if and only if there are no exceptional points.) Obviously the signature $\sigma(G)$ of a Goeritz matrix is not invariant in the class of equivalent Goeritz matrices. C. M. Gordon and R. A. Litherland proved in [135] the following proposition:

13.31 Proposition. $\sigma(q_{\mathfrak{k}}) = \sigma(G) - \nu$, *where* ν *is defined above.* □

The fact that $\sigma(G) - \nu$ is a knot invariant can be proved using Reidemeister moves Ω_i (Exercise E 13.3).

Since the order of G will in most cases be considerably smaller than that of $M + M^T$, Proposition 13.31 affords a useful practical method for calculating $\sigma(q_{\mathfrak{k}})$. To compute the signature of a symmetric matrix over $\mathbb{Z}$, a large spectrum of methods is available. The following proposition was used by Murasugi [261] and can be found in [180]; we give a proof in Appendix A.2.

13.32 Proposition. *Let* Q *be a symmetric matrix of rank* r *over a field. There exists a chain of principal minors* D_i, $i = 0, 1, \dots, r$ *such that* D_i *is principal minor of* D_{i+1} *and that no two consecutive determinants* D_i, D_{i+1} *vanish* $(D_0 = 1)$. *For any such sequence of minors,* $\sigma(Q) = \sum_{i=0}^{r-1} \operatorname{sign} D_i D_{i+1}$. □

As an application consider the two projections of the trefoil 3_1 in Figure 13.11. The signature of the Goeritz matrix $G_a = (-3)$ of Figure 13.11 (a) is -1 and $\nu = -3$. On the other hand, Figure 13.11 (b) is a special projection and yields $\sigma(q_{3_1}) = \sigma(G_b) = 2$ (Proposition 13.28), hence $\sigma(G_a) - \nu = \sigma(q_{3_1})$.

13.33 Remark. By symmetrizing a Seifert matrix V of $\mathfrak{k}$ to $V + V^T$, we obtain the signature. There are other signatures obtained by building Hermitian matrices: let $\omega \neq 1$ be a complex number. We consider the Hermitian matrix $H(\omega) := (1-\omega)V + (1-\bar{\omega})V^T$. The ω-signature $\sigma_{\mathfrak{k}}(\omega)$ of $\mathfrak{k}$ is defined to be the signature of $H(\omega)$ i.e $\sigma_{\mathfrak{k}}(\omega) = \text{signature}(H(\omega))$. If $\omega \in S^1 - \{1\}$ we have

$$\begin{aligned} H(\omega) &= (1-\omega)V + (1-\bar{\omega})V^T \\ &= (\omega^{-1/2} - \omega^{1/2})(\omega^{1/2}V - \omega^{-1/2}V^T). \end{aligned} \tag{13.11}$$

The Levine–Tristram [211, 353] *signature function* is the map $\sigma_{\mathfrak{k}} \colon S^1 \to \mathbb{Z}$ given by $\sigma_{\mathfrak{k}} \colon \omega \mapsto \sigma_{\mathfrak{k}}(\omega)$ if $\omega \neq 1$ and $\sigma_{\mathfrak{k}} \colon 1 \mapsto 0$. It follows from equation (13.11) that the signature function is constant away from the roots of the Alexander polynomial $\Delta(t) \doteq \det(t^{1/2}V - t^{-1/2}V^T)$. (For details see [133, Sect. 11,12].)

More recently, X.-S. Lin [214] discovered a relation between *traceless* representations of knot groups into $\mathrm{SU}(2)$ and the signature $\sigma(\mathfrak{k})$ of the knot $\mathfrak{k}$. Here traceless means that the trace of the image of a meridian is zero. This result was extended by C. Herald [162] and Heusener and Kroll [163] to the more general setting: the signatures are replaced by the Levine–Tristram signature $\sigma_{\mathfrak{k}}(e^{2\alpha i})$ and the traceless representations by representations where all the meridians have trace $2\cos\alpha$, $0 < \alpha < \pi$ and $\Delta(e^{2\alpha i}) \neq 0$. Note that $\sigma_{\mathfrak{k}}(e^{\pi i}) = \sigma(\mathfrak{k})$ and $\Delta(e^{\pi i}) = \Delta(-1) \neq 0$.

13.E History and sources

An invariant consisting of a class of quadratic forms was first defined by L. Goeritz [130]. They yielded the Minkowski units, new knot invariants introduced by K. Reidemeister [296, 303, 2.9]. Further contributions were made by H. Seifert [327], M. Kneser and D. Puppe [199], K. Murasugi [261], H. F. Trotter [355], J. Milnor [240], D. Erle [93] and others. Our exposition is based on [93] and [261], the quadratic form is that of Trotter [355].

C. M. Gordon and R. A. Litherland introduced in [135] a new quadratic form which simultaneously generalized the forms of Trotter and Goeritz. As a byproduct a simple way to compute the signature of a knot from a regular projection was obtained.

13.F Exercises

E 13.1. Compute the quadratic forms of Goeritz and Trotter and the signature of the knot 6_1, and the torus knots or links $\mathfrak{t}(2, b)$.

E 13.2. Characterize the 2×2 matrices which represent quadratic forms of knots.

E 13.3. Prove the invariance of $\sigma(G) - \nu$ (see Proposition 13.31) under Reidemeister moves.

E 13.4 (Crowell [76]). An alternating prime knot $\mathfrak{k}$ has a graph Γ_α with m vertices and k regions in S^2 such that the number of trees $\operatorname{tr}(\Gamma_\alpha)$ satisfies the inequality $\det G(\mathfrak{k}) = \operatorname{tr}(\Gamma_\alpha) \geq 1 + (m-1)(k-1)$. Show that $8_{20}, 9_{42}, 9_{43}$ and 9_{46} are non-alternating knots.

E 13.5 (Giller [128]). Suppose $\mathfrak{k}$ is a knot (but not a link). Then $\sigma(\mathfrak{k})$ can be calculated by the following procedure:

(1) If $\mathfrak{k}$ is trivial then $\sigma(\mathfrak{k}) = 0$.

(2) If $\mathfrak{k}_+$, $\mathfrak{k}_-$, and $\mathfrak{k}_0$ differ by a local operation of the kind depicted in Figure 9.3, then

$$\sigma(\mathfrak{k}_-) - 2 \leq \sigma(\mathfrak{k}_+) \leq \sigma(\mathfrak{k}_-).$$

(3) If $\Delta(t) = \det(t^{1/2}V - t^{-1/2}V^T)$ is the symmetrized Alexander polynomial (V a Seifert matrix of $\mathfrak{k}$), then $\operatorname{sign}(\Delta(-1)) = (-1)^{\sigma(\mathfrak{k})/2}$.

Prove that the knot which admits a projection with only negative crossings (see Figure 9.3) has positive signature.

Chapter 14

Representations of knot groups

Knot groups as abstract groups are rather complicated. Invariants which can be effectively calculated will, in general, be extracted from homomorphic images of knot groups.

We use the term representation in this chapter as a synonym for homomorphism, and we call two representations $\varphi, \psi\colon \mathfrak{G} \to \mathfrak{H}$ equivalent, if there is an automorphism $\alpha\colon \mathfrak{H} \to \mathfrak{H}$ with $\psi = \alpha\varphi$. There have been many contributions to the field of representations of knot groups in the past decades, and the material of this chapter comprises a selection from a special point of view – the simpler and more generally applicable types of representations.

The first section deals with metabelian (2-step metabelian) representations, the second with a class of 3-step metabelian representations which means that the third commutator group of the homomorphic image of the knot group vanishes. These representations yield an invariant derived from the peripheral system of the knot which is closely connected to linking numbers in coverings defined by the homomorphisms. These relations are studied in Section 14.C. Section 14.D contains some theorems on periodic knots, and their presence in this chapter is, perhaps, justified by the fact that a special metabelian representation in Section 14.A of a geometric type helps to prove one of the theorems and makes it clearer.

14.A Metabelian representations

14.1. Throughout this chapter we consider only knots of multiplicity $\mu = 1$. A knot group $\mathfrak{G}$ may then be written as a semidirect product $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, where $\mathfrak{Z}$ is a free cyclic group generated by a distinguished generator t represented by a meridian of the knot $\mathfrak{k}$. An Abelian homomorphic image of $\mathfrak{G}$ is always cyclic, and an *Abelian representation of* $\mathfrak{G}$ *will*, hence, *be called trivial*. A group $\mathfrak{G}$ is called k-*step metabelian*, if its k-th commutator subgroup $\mathfrak{G}^{(k)}$ vanishes. ($\mathfrak{G}^{(k)}$ is inductively defined by $\mathfrak{G}^{(k)} =$ commutator subgroup of $\mathfrak{G}^{(k-1)}$, $\mathfrak{G} = \mathfrak{G}^{(0)}$.) The 1-step metabelian groups are the Abelian groups, and 2-step metabelian groups are also called *metabelian*. It seems reasonable, therefore, to try to find metabelian representations as a first step. They turn out to be plentiful and useful.

Let $\varphi\colon \mathfrak{G} \to \mathfrak{H}$ be a surjective homomorphism onto a metabelian group $\mathfrak{H}$. Then $\varphi(\mathfrak{G}') = \mathfrak{H}'$ and $\mathfrak{H} = \varphi(\mathfrak{Z}) \cdot \mathfrak{H}'$. Moreover, it follows at once that the following diagram commutes (lines are exact and vertical maps are surjective)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{G}' & \longrightarrow & \mathfrak{Z} \ltimes \mathfrak{G}' & \longrightarrow & \mathfrak{Z} & \longrightarrow & 1 \\ & & \Big\downarrow \varphi|_{\mathfrak{G}'} & & \Big\downarrow \varphi & & \Big\downarrow \bar{\varphi} & & \\ 1 & \longrightarrow & \mathfrak{H}' & \longrightarrow & \varphi(\mathfrak{Z}) \cdot \mathfrak{H}' & \longrightarrow & \mathfrak{H}/\mathfrak{H}' \cong \mathfrak{Z}_n & \longrightarrow & 1 \end{array}$$

where $\operatorname{Ker} \varphi|_{\mathfrak{Z}} = n\mathfrak{Z} = \operatorname{Ker} \bar{\varphi}$. It turns out that $\varphi(\mathfrak{Z}) \cap \mathfrak{H}' = \{1\}$ is trivial since $\bar{\varphi}(t) = \varphi(t)\mathfrak{H}'$ generates $\mathfrak{H}/\mathfrak{H}' \cong \mathfrak{Z}_n$. Therefore, $\varphi(\mathfrak{G}) = \mathfrak{H} = \varphi(\mathfrak{Z}) \ltimes \varphi(\mathfrak{G}')$ is a semidirect product. Moreover, the Abelian group $\mathfrak{H}'$ and can be considered as a $\mathfrak{Z}$-module.

14.2 Proposition. *Let $\varphi\colon \mathfrak{G} \to \mathfrak{H}$ be any non-trivial surjective metabelian representation of a knot group $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, $\mathfrak{Z} = \langle t \rangle$, t a meridian. Then $\mathfrak{H} = \varphi(\mathfrak{Z}) \ltimes \mathfrak{H}'$, $\mathfrak{H}'$ is a finitely generated $\mathbb{Z}\mathfrak{Z}$ torsion module and $t - \mathrm{id}\colon \mathfrak{H}' \to \mathfrak{H}'$ is an isomorphism.*

Proof. Note that $\mathfrak{H}'$ is a finitely generated torsion module since the Alexander module $\mathfrak{G}'/\mathfrak{G}''$ is a finitely generated torsion module (see Remark 8.23) and the surjection $\varphi|_{\mathfrak{G}'}\colon \mathfrak{G}' \to \mathfrak{H}'$ induces a surjective $\mathfrak{Z}$-module morphism $\mathfrak{G}'/\mathfrak{G}'' \to \mathfrak{H}'$.

Since $\mathfrak{G}$ is trivialized by putting $t = 1$, the same holds for $\varphi(\mathfrak{G})$, if the φ-image of t (also denoted by t) is made a relation. For the normal closure $\overline{\langle t \rangle}$ one has $\overline{\langle t \rangle} = \mathfrak{G}$ and $\overline{\langle t \rangle} = \varphi(\mathfrak{Z}) \ltimes \mathfrak{H}'$. The relations $t\,a\,t^{-1} = a$, $a \in \mathfrak{G}'$ trivialize $\mathfrak{G}'$; hence elements of the form $(t-1)\,h$, $h \in \mathfrak{H}'$ generate $\mathfrak{H}'$. Therefore, as a $\mathfrak{Z}$-module, $\mathfrak{H}' = (t-1)\mathfrak{H}'$. Since $\mathfrak{H}'$ is a finitely generated $\mathfrak{Z}$-module and $\mathbb{Z}\mathfrak{Z}$ is Noetherian, it follows that multiplication by $(t-1)$ is also injective. □

14.3 Remark. The $\mathfrak{Z}$-module $\mathfrak{H}'$ has an annihilating polynomial of minimal degree coprime to $(t-1)$.

Since $\varphi(\mathfrak{G}') = \mathfrak{H}'$ is Abelian the homomorphism φ factors through $\mathfrak{Z} \ltimes \mathfrak{G}'/\mathfrak{G}''$. If $\varphi(\mathfrak{Z}) = \mathfrak{Z}_n$ is finite, it factors through $\mathfrak{Z}_n \ltimes \mathfrak{G}'/\mathfrak{G}_n'$, $\mathfrak{G}_n = n\mathfrak{Z} \ltimes \mathfrak{G}'$, compare Proposition 8.38. The group $\mathfrak{G}'/\mathfrak{G}''$ is the first homology group of the infinite cyclic covering C_∞ of $\mathfrak{k}$, $\mathfrak{G}'/\mathfrak{G}'' = H_1(C_\infty)$ and may be regarded as a $\mathfrak{Z}$-module (Alexander module) where the operation is defined by that of the semidirect product. Likewise $\mathfrak{G}'/\mathfrak{G}_n' = H_1(\hat{C}_n)$ is the homology group of the n-fold cyclic branched covering of $\mathfrak{k}$, see Proposition 8.38 (c). The following proposition summarizes our result:

14.4 Proposition. *A metabelian representation of a knot group*

$$\varphi\colon \mathfrak{G} \to \mathfrak{Z} \ltimes \mathfrak{A}, \textit{ respectively } \varphi_n\colon \mathfrak{G} \to \mathfrak{Z}_n \ltimes \mathfrak{A},\ \mathfrak{A} \textit{ Abelian,}$$

factors through

$$\beta\colon \mathfrak{G} \to \mathfrak{Z} \ltimes H_1(C_\infty), \textit{ respectively } \beta_n\colon \mathfrak{G} \to \mathfrak{Z}_n \ltimes H_1(\hat{C}_n),$$

mapping a meridian of $\mathfrak{k}$ *onto a generator of* $\mathfrak{Z}$ *resp.* $\mathfrak{Z}_n$. *The group* $\mathfrak{A}$ *may be considered as a* $\mathfrak{Z}$*-module resp.* $\mathfrak{Z}_n$*-module. One has* $\ker\beta = \mathfrak{G}''$, $\ker\beta_n = n\mathfrak{Z} \ltimes \mathfrak{G}'_n$. □

We give a simple example with a geometric background.

14.5 The groups of similarities. The replacing of the Alexander module $H_1(C_\infty) = \mathfrak{G}/\mathfrak{G}''$ by $H_1(C_\infty) \otimes_{\mathbb{Z}} \mathbb{C}$ suggests a metabelian representation of $\mathfrak{G}$ by linear mappings $\mathbb{C} \to \mathbb{C}$ of the complex plane. Starting from a Wirtinger presentation $\mathfrak{G} = \langle S_1, \dots, S_n \mid R_1, \dots, R_n \rangle$, a relation

$$S_k^{-1} S_i S_k S_{i+1}^{-1} = 1 \;\Leftrightarrow\; S_i S_k = S_k S_{i+1} \tag{14.1}$$

takes the form

$$u_i + t u_k = u_k + t u_{i+1} \;\Leftrightarrow\; t u_{i+1} - t u_k - u_i + u_k \tag{14.2}$$

for $u_j = \beta(S_j S_1^{-1})$, $1 \le j \le n$. ($u \mapsto tu, u \in H_1(C_\infty)$ denotes the operation of a meridian.) The equations (14.2) form a system of linear equations with coefficients in $\mathbb{Z}(t)$. We may omit one equation (Corollary 3.6) and the variable $u_1 = 0$.

The determinant of the remaining $(n-1)\times(n-1)$ linear system equals the Alexander polynomial $\Delta_1(t)$, see Definition 8.18 and Corollary 9.11. Thus, by interpreting (14.2) as linear equations over $\mathbb{C}$, one obtains non-trivial solutions if and only if t takes the value of a root α of $\Delta_1(t)$. For suitable $z_i \in \mathbb{C}$ (z a complex variable)

$$S_i \mapsto \delta_\alpha(S_i)\colon\, z \mapsto \alpha(z - z_i) + z_i \tag{14.3}$$

maps $\mathfrak{G}$ into the group $\mathfrak{C}_+$ *of orientation preserving similarities of the plane* $\mathbb{C}$, since a Wirtinger relation (14.1) results in an equation (14.2) for $t = \alpha$, $u_i = z_i$. The similarity $\delta_1(S_i)$ is the identity and for $\alpha \ne 1$ two similarities $\delta_\alpha(S_i)$ and $\delta_\alpha(S_j)$ commute if and only if $z_i = z_j$. Hence there exists a non-cyclic representation δ_α if and only if $\Delta_1(\alpha) = 0$; it is metabelian because $\mathfrak{G}'$ is mapped into the group of translations. The class K of elements in $\mathfrak{G}$ conjugate to a meridian (K = *Wirtinger class*) is mapped into the class K_α of conjugate similarities of $\mathfrak{C}^+$ characterized by α. (Note that $\alpha \ne 1$.)

14.6 Proposition. *There exists a non-cyclic representation* $\delta_\alpha\colon (\mathfrak{G}, K) \to (\mathfrak{C}^+, K_\alpha)$ *if and only if* α *is a root of the Alexander polynomial* $\Delta_1(t)$. *When* α *and* α' *are roots of an (over* $\mathbb{Z}$*) irreducible factor of* $\Delta_1(t)$ *which does not occur in* $\Delta_2(t)$*, then any two representations* $\delta_\alpha, \delta_{\alpha'}$ *are equivalent. In particular, any two such maps* δ_α, $\delta_{\alpha'}$ *differ by an inner automorphism of* $\mathfrak{C}^+$.

Proof. The first assertion has been proved above. For α satisfying $\Delta_1(\alpha) = 0$, $\Delta_2(\alpha) \ne 0$ – that means that the system of linear equations has rank $n-2$ and hence a one-dimensional space of solutions. For a nonzero complex number β let

$\gamma_\beta \in \mathfrak{C}^+$ be the similarity given by $\gamma_\beta\colon z \mapsto \beta z$. If the non-cyclic representation δ_α of the form (14.3) corresponds to the non-trivial solution $(z_2, \dots, z_n)$ then $\gamma_\beta\, \delta_\alpha\, \gamma_\beta^{-1}$ corresponds to the solution $(\beta z_2, \dots, \beta z_n)$. Hence two representations for different choices of non-trivial solutions differ by an inner automorphism of $\mathfrak{C}^+$. Finally there is a Galois automorphism $\tau\colon\ \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha')$, if α and α' are roots of an irreducible factor of $\Delta_1(t)$. Put $\delta_{\alpha'}(S_i)\colon\ z \mapsto \alpha'(z - \tau(z_i)) + \tau(z_i)$ to obtain a representation equivalent to $\delta_\alpha(S_i)\colon\ z \mapsto \alpha(z - z_i) + z_i$. (In the special case $\alpha' = \bar{\alpha}$ a reflection may be used.) □

14.7 Remark. The complex numbers α for which there are non-trivial representations

$$\delta_\alpha\colon\ (\mathfrak{G}, k) \to (\mathfrak{C}^+, K_\alpha)$$

are invariants of $\mathfrak{G}$ in their own right. The Alexander polynomial $\Delta_1(t)$, though, is a stronger invariant, because it also includes the powers of its prime factors. This is, of course, exactly what is lost when the operation of t is replaced by complex multiplication by α: $(p(\alpha))^\nu \cdot a = 0$, $a \neq 0$ implies $p(\alpha) \cdot a = 0$, but $(p(t))^\nu \cdot a = 0$ does not imply $(p(t))^{\nu-1} \cdot a = 0$. (Compare Burde [52] and de Rham [83].)

The complete structure of the $\mathbb{C}[t^{\pm 1}]$-module $H_1(C_\infty, \mathbb{C})$ is not reflected in these metabelian representations. More recently, Jebali [178] shows that this can be achieved by considering more generally metabelian representations of the knot group by upper triangular $n \times n$ matrices.

Example: Figure 14.1 shows a class of knots (compare Figure 9.5, E 9.6) with Alexander polynomials of degree two. They necessarily have trivial second Alexander polynomials. Figure 14.2 shows the configuration of the fixed points z_i of $\delta_\alpha(S_i)$ for $m = 5, k = 3$. Then $\delta_\alpha(S_i)$ are rotations through $\alpha, \cos\alpha = \frac{2k-1}{2k} = \frac{5}{6}$.

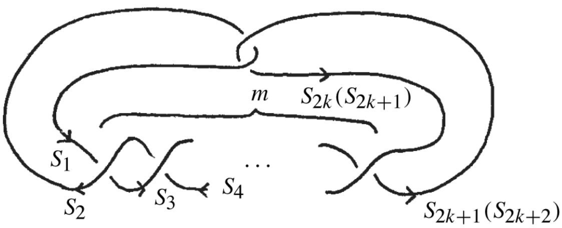


Figure 14.1

14.8 Metacyclic representations. A *representation* β^* of $\mathfrak{G}$ is called *metacyclic*, if $\beta^*(\mathfrak{G}') = \mathfrak{H}'$ is a cyclic group $\langle\, a \,\rangle \neq 1$:

$$\beta^*(\mathfrak{G}) = \langle\, t \,\rangle \ltimes \langle\, a \,\rangle.$$

The operation of t is denoted by $a \mapsto ta$. Putting

$$\beta^*(S_i) = (t, \nu_i a),\ \nu_i \in \mathbb{Z},$$

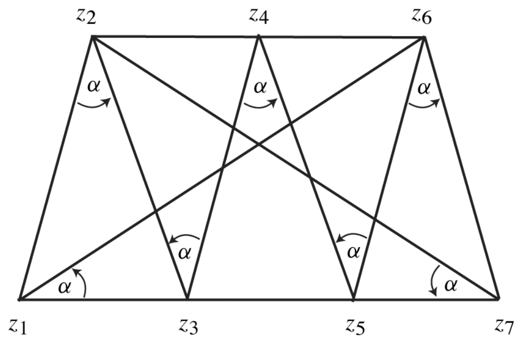


Figure 14.2

transforms a set of Wirtinger relations (14.1) into a system of n equations in n variables ν_j:

$$-\nu_{i+1} + t\nu_i + (1-t)\nu_k = 0. \qquad (14.4)$$

These equations are to be understood over $\mathbb{Z}$ if $\langle a \rangle$ is infinite, and as congruences modulo m if $\langle a \rangle \cong \mathfrak{Z}_m$.

In the first case β^* is trivial when $t = 1$. If $t = -1$, β^* must also be trivial, because the rank of (14.4) is $n-1$: Every $(n-1)\times(n-1)$ minor of its matrix is $\pm\Delta_1(-1) = \pm|H_1(\hat{C}_2)|$ which is an odd integer by Theorem 8.41 and Definition 13.19.

We may, therefore, confine ourselves to the finite case $\langle a \rangle = \mathfrak{Z}_m$.

14.9 Proposition (Fox [112]). *A non-trivial metacyclic representation of a knot group is of the form*

$$\beta_m^*\colon \mathfrak{G} \to \mathfrak{Z} \ltimes \mathfrak{Z}_m,\ m > 1,$$

mapping a meridian onto a generator t of the cyclic group $\mathfrak{Z}$. The existence of a surjective homomorphism β_m^* implies $m|\Delta_1(k)$ for $k \in \mathbb{Z}$ with $ka = ta$, $a \in \mathfrak{Z}_m$.

For a prime p, $p|\Delta_1(k)$, $\gcd(k,p) = 1$, there exists a surjective representation β_p^*. If the rank of the system (14.4) *of congruences modulo p is $n-2$, all representations β_p^* are equivalent.*

Proof. If a surjective representation β_m^* exists, the system (14.4) admits a solution with $\nu_1 = 0$, $\gcd(\nu_2,\dots,\nu_n) = 1$. Let $Ax \equiv 0 \mod m$ denote the system of congruences in matrix form obtained from (14.4) by omitting one equation and putting $\nu_1 = 0$. By multiplying Ax with the adjoint matrix A^* one obtains

$$A^*A\cdot x = (\det A)\cdot E\cdot x \equiv 0 \mod m.$$

This means $m|\Delta_1(k)$ since $\Delta_1(k) = \pm\det A$, see Corollary 9.11.

The rest of Proposition 14.9 follows from standard arguments of linear algebra, since (14.4) is a system of linear equations over a field $\mathbb{Z}_p$ if $m = p$. □

Remark: If m is not a prime, the existence of a surjective representation β_m^* does not follow from $m|\Delta_1(k)$. We shall give a counterexample in the case of a dihedral representation. By a Chinese Remainder argument, however, one can construct β_m^* for composite m, if m is square-free. One may obtain from β_m^* a homomorphism onto a finite group by mapping $\mathfrak{Z}$ onto $\mathfrak{Z}_r$, where r is a multiple of the order of the automorphism $t\colon a \mapsto ka$. As a special case we note:

14.10 Dihedral representations. *There is a surjective homomorphism*

$$\gamma_p^*\colon \mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$$

onto the dihedral group $\mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$ *if and only if the prime* p *divides the order of* $H_1(\hat{C}_2)$ *i.e.* $H_1(\hat{C}_2, \mathbb{Z}_p) \neq 0$. *If* p *does not divide the second torsion number of* $H_1(\hat{C}_2)$ *i.e.* $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$, *then all representations* γ_p^* *are equivalent. (See Appendix A.7.)* □

14.11 Remark. The prime p in 14.10 must be odd since $|\Delta(-1)| = |H_1(\hat{C}_2)|$ is an odd number.

Since any such homomorphism γ_p^* must factor through $\mathfrak{Z}_2 \ltimes H_1(\hat{C}_2)$, see Proposition 14.4, the existence of dihedral representations $\mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_m$, $m \mid |H_1(\hat{C}_2)|$, depends on the cyclic factors of $H_1(\hat{C}_2)$. If $H_1(\hat{C}_2)$ is not cyclic – for instance $H_1(\hat{C}_2) \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_3$ for 8_{18} – there is no homomorphism onto $\mathfrak{Z}_2 \ltimes \mathfrak{Z}_{45}$, though $45|\Delta_1(-1)$.

The group $\gamma_p^*(\mathfrak{G})$ can be interpreted as the symmetry group of a regular p-gon in the Euclidean plane. A meridian of the knot is mapped onto a reflection of the Euclidean plane.

14.12 Example. Consider a Wirtinger presentation of the four-knot:

$$\mathfrak{G} = \langle\, S_1, S_2, S_3, S_4 \mid S_3S_1S_3^{-1}S_2^{-1}, S_4^{-1}S_2S_4S_3^{-1}, S_1S_3S_1^{-1}S_4^{-1}, S_2^{-1}S_4S_2S_1^{-1} \,\rangle,$$

see Figure 14.3. One has $\Delta_1(-1) = 5 = p$, see Corollary 8.15 (b). The system (14.4) of congruences $\mod p$ takes the form

$$\left.\begin{array}{rrrrrrrrl} - & \nu_1 & - & \nu_2 & + & 2\nu_3 & & & \equiv 0 \\ & & - & \nu_2 & - & \nu_3 & + & 2\nu_4 & \equiv 0 \\ + & 2\nu_1 & & & - & \nu_3 & - & \nu_4 & \equiv 0 \\ - & \nu_1 & + & 2\nu_2 & & & - & \nu_4 & \equiv 0 \end{array}\right\} \mod 5.$$

Putting $\nu_1 \equiv 0$, $\nu_2 \equiv 1$, one obtains $\nu_3 \equiv 3$, $\nu_4 \equiv 2 \mod 5$. The relations of $\mathfrak{G}$ are easily verified in Figure 14.3.

Remark. More recently, irreducible metabelian representations of knot groups into $\mathrm{SL}_n(\mathbb{C})$ were studied by Boden and Friedl in a series of papers [29, 30, 27, 28].

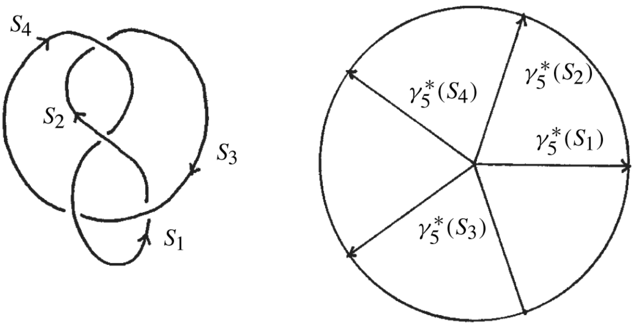


Figure 14.3

14.B Homomorphisms of $\mathfrak{G}$ into the group of motions of the Euclidean plane

We have interpreted the dihedral representations γ_p^* as homomorphisms of $\mathfrak{G}$ into the group $\mathfrak{B}$ of motions of E^2, and we studied a class of maps $\delta_\alpha\colon \mathfrak{G} \to \mathfrak{C}$ into the 2-dimensional group of similarities $\mathfrak{C}$ *of the plane* E^2. It seems rather obvious to choose any other suitable conjugacy class in one of these well-known geometric groups as a candidate to map a Wirtinger class K onto. It would be especially interesting to obtain new non-metabelian representations, because metabelian representations necessarily map a longitude, see Proposition 3.13, onto units, and are, therefore, not adequate to exploit the peripheral system of the knot. We propose to "lift" the representation γ_p^* to a homomorphism $\gamma_p\colon \mathfrak{G} \to \mathfrak{B}$ which maps the Wirtinger class K into a class of glide reflections. The representation γ_p will not be metabelian and will yield a useful tool in proving non-amphicheirality of knots. As above, p *is a prime.*

Let γ_p^* be a homomorphism of the knot group $\mathfrak{G}$ onto the dihedral group $\mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$. There is a regular covering $q\colon R_p \to C$ corresponding to the normal subgroup $\ker \gamma_p^*$. One has $2\mathfrak{Z} \ltimes \mathfrak{G}' = \mathfrak{G}_2 \supset \ker \gamma_p^* \supset \mathfrak{G}''$ and $\mathfrak{G}_2/\ker\gamma_p^* \cong \mathfrak{Z}_p$. The space R_p is a p-fold cyclic covering of the 2-fold covering C_2 of C. For a meridian m and longitude ℓ of the knot $\mathfrak{k}$ we have: $m^2 \in \ker\gamma_p^*$, $\ell \in \mathfrak{G}'' \subset \ker\gamma_p^*$. The torus ∂C is covered by p tori T_i, $0 \le i \le p-1$, in R_p. There are distinguished canonical curves $\hat{m}_i$, $\hat{\ell}_i$ on T_i with $q(\hat{m}_i) = m^2$, $q(\hat{\ell}_i) = \ell$. By a theorem of H. Seifert [325] (see Lemma 12.31) the set $\{\hat{m}_i, \hat{\ell}_i\}$ of $2p$ curves contains a subset of p (> 2) linearly independent representatives of the Betti group of $H_1(R_p)$. From this it follows that the cyclic subgroup $\mathfrak{Z}_p \lhd \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$ of covering transformations operates non-trivially on the Betti group of $H_1(R_p)$. Now abelianize $\ker\gamma_p^*$ and trivialize the torsion subgroup of $H_1(R_p) = \ker\gamma_p^*/(\ker\gamma_p^*)'$ to obtain a homomorphism of the knot group $\mathfrak{G}$ onto an extension $[\mathfrak{D}_p, \mathbb{Z}^p]$ of the Betti group $\mathbb{Z}^q$ of $H_1(R_p)$, $q \ge p$, with factor group $\mathfrak{D}_p = \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$. The operation of $\mathfrak{D}_p$ on $\mathbb{Z}^q$ is the one induced by the covering transformations. We embed $\mathbb{Z}^q$ in a vector space $\mathbb{C}^q$ over the complex numbers and use a result of the theory of representations of finite groups: The dihedral group $\mathfrak{D}_p$ admits only irreducible representations of degree 1 and degree 2 over $\mathbb{C}$.

This follows from Burnside's formula and the fact that the degree must divide the order $2p$ of $\mathfrak{D}_p$. (See [330].) Since $\mathfrak{Z}_p \triangleleft \mathfrak{D}_p$ operates non-trivially on $\mathbb{Z}^q$, the operation of $\mathfrak{D}_p$ on $\mathbb{C}^q$ contains at least one summand of degree 2. Such a representation has the form

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

with $\mathfrak{Z}_2 = \langle \tau \rangle$, $\mathfrak{Z}_p = \langle a \rangle$ and ζ a primitive p-th root of unity. (The representation is faithful, hence irreducible.)

This representation is equivalent to the following when $\mathbb{C}^2$ is replaced by $\mathbb{R}^4$:

$$\tau \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; a \mapsto \begin{pmatrix} \xi & -\eta & 0 & 0 \\ \eta & \xi & 0 & 0 \\ 0 & 0 & \xi & -\eta \\ 0 & 0 & \eta & \xi \end{pmatrix}, \; \zeta = \xi + i\eta.$$

It splits into two identical summands. Introduce again a complex structure on each of the invariant subspaces $\mathbb{R}^2$; the operation of $\mathfrak{D}_p$ on each of them may then be described by:

$$\tau(z) = \bar{z}, \quad a(z) = \zeta z. \tag{14.5}$$

By this construction the knot group $\mathfrak{G}$ is mapped onto an extension of a finitely generated (additive) subgroup $\mathfrak{T} \subset \mathbb{C}$, $\mathfrak{T} \neq 0$, with factor group $\mathfrak{D}_p$ operating on $\mathfrak{T}$ according to equation (14.5). First consider the extension $[\mathfrak{Z}_p, \mathfrak{T}]$ and denote its elements by pairs (a^ν, z).

One has

$$(a,0)\Big((a^{p-1},0)(a,0)\Big) = (a,0)(1,w) = (a,w), \quad \text{for } w = a^p \in \mathfrak{T},$$

and

$$\Big((a,0)(a^{p-1},0)\Big)(a,0) = (1,w)(a,0) = (a,\zeta w).$$

It follows that $w = \zeta w$, $\zeta \neq 1$; hence, $w = 0$, and $[\mathfrak{Z}_p, \mathfrak{T}] = \mathfrak{Z}_p \ltimes \mathfrak{T}$. Similarly one may denote the elements of $[\mathfrak{D}_p, \mathfrak{T}] = [\mathfrak{Z}_2, \mathfrak{Z}_p \ltimes \mathfrak{T}]$ by triples (τ^ν, a^μ, z). Put $(\tau, 1, 0)^2 = (1, 1, v)$, $v \in \mathbb{C}$. Then

$$(\tau, 1, 2\bar{v}) = (\tau,1,0)^2(\tau,1,0) = (\tau,1,0)(\tau,1,0)^2 = (\tau,1,2v).$$

This proves $v = \bar{v} \in \mathbb{R}$.

We obtain a homomorphism $\gamma_p \colon \mathfrak{G} \to [\mathfrak{D}_p, \mathfrak{T}] \subset \mathfrak{B}$. Put

$$\begin{aligned} &(1,a,b) &&: z \mapsto \zeta z + b,\ \zeta \text{ a primitive } p\text{-th root of unity,} \\ &(\tau,1,0) &&: z \mapsto \bar{z} + v,\ v \in \mathbb{R}. \end{aligned} \tag{14.6}$$

There are two distinct cases: $v \neq 0$ and $v = 0$. In the first case, a Wirtinger generator is mapped onto a glide reflection whereas in the second case its image is a reflection. We may in the first case choose $v = 1$ by replacing a representation by an equivalent one.

14.13 Proposition. *For any dihedral representation $\gamma_p^*\colon \mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p \subset \mathfrak{B}$ of a knot group $\mathfrak{G}$ into the group $\mathfrak{B}$ of motions of the plane there is a lifted representation $\gamma_p\colon \mathfrak{G} \to \mathfrak{B}$ such that $\gamma_p^* = \kappa\,\gamma_p$, $\kappa\colon \gamma_p(\mathfrak{G}) \to \gamma_p(\mathfrak{G})/\mathfrak{T}$, where $\mathfrak{T} \neq 0$ is the subgroup of translations in $\gamma_p(\mathfrak{G}) \subset \mathfrak{B}$ (p is a odd prime).*

An element of the Wirtinger class K is either mapped onto a glide reflection $(v = 1)$ or a reflection $(v = 0)$.

If $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$, see 14.10, the first case takes place and γ_p^*, γ_p are unique up to equivalence.

Proof. The existence of a lifted mapping γ_p has already been proved. We prove uniqueness by describing γ_p with the help of a system of linear equations which at the same time serves to carry out an effective calculation of the representation. Denote by $\mathbb{Q}(\zeta)$ the cyclotomic field over the rationals and by ζ_j a p-th root of unity. Note that p is an odd prime (Remark 14.11). Put

$$\gamma_p^*(S_j)\colon\ z \mapsto \zeta_j^2 \bar{z} \tag{14.7}$$

$$\gamma_p(S_j)\colon\ z \mapsto \zeta_j^2 \bar{z} + b_j \tag{14.8}$$

for Wirtinger generators S_j of $\mathfrak{G} = \langle\, S_1, \dots S_n \mid R_1, \dots, R_n \,\rangle$. Equation (14.8) describes a glide reflection with axis $g_j = \{\zeta_j(r + (\bar{\zeta}_j b_j - \zeta_j \bar{b}_j)/4 \mid r \in \mathbb{R}\}$ and translation

$$z \mapsto z + \frac{\zeta_j}{2}(\bar{\zeta}_j b_j + \zeta_j \bar{b}_j)\,. \tag{14.9}$$

A Wirtinger relator

$$R_j = S_j S_i^{-\eta_j} S_k^{-1} S_i^{\eta_j},\ \eta_j = \pm 1, \tag{14.10}$$

yields $(\zeta_i^2 \bar{\zeta}_j \bar{\zeta}_k)^2 = 1$ under (14.7). Since -1 is not a p-th root of unity we obtain

$$\zeta_i^2 = \zeta_j \zeta_k\,. \tag{14.11}$$

Moreover, the relation (14.10) yields

$$\begin{aligned} &\zeta_k^2 \bar{b}_i + b_k - \zeta_i^2 \bar{b}_j - b_i = 0, \text{ if } \eta_j = 1 \\ &\zeta_j^2 \bar{b}_i + b_j - \zeta_i^2 \bar{b}_k - b_i = 0, \text{ if } \eta_j = -1 \end{aligned} \tag{14.12}$$

under (14.8). Here we introduce the convention that on the right-hand side of $\gamma_p(w_1 w_2) = \gamma_p(w_1)\gamma_p(w_2)$, $w_1, w_2 \in \mathfrak{G}$, the combination is carried out from right to left, as is usual in a group of motions, whereas in the fundamental group the combination $W_1 W_2$ is understood from left to right.

We multiply the equation (14.12) with $\bar{\zeta}_k$ if $\eta_j = 1$ and with $\bar{\zeta}_j$ if $\eta_j = -1$. This gives

$$\begin{aligned} \zeta_k \bar{\zeta}_i (\zeta_i \bar{b}_i) + \bar{\zeta}_k b_k - \zeta_j \bar{b}_j - \bar{\zeta}_k \zeta_i (\bar{\zeta}_i b_i) = 0, \text{ if } \eta_j = 1 \\ \zeta_j \bar{\zeta}_i (\zeta_i \bar{b}_i) + \bar{\zeta}_j b_j - \zeta_k \bar{b}_k - \bar{\zeta}_j \zeta_i (\bar{\zeta}_i b_i) = 0, \text{ if } \eta_j = -1. \end{aligned} \tag{14.13}$$

(We have used (14.11).) Substitute

$$\nu_j + \mathbf{i}\,\omega_j = \bar{\zeta}_j b_j,\ \nu_j, \omega_j \in \mathbb{R}. \tag{14.14}$$

Note that $\nu_j = (\bar{\zeta}_j b_j + \zeta_j \bar{b}_j)/2$ is the translation length of $\gamma_p(S_j)$ in the direction ζ_j and that $\omega_j/2 = -\mathbf{i}(\bar{\zeta}_j b_j - \zeta_j \bar{b}_j)/4$ is the distance of g_j to the line $\zeta_j \mathbb{R}$ in the direction $\zeta_j \mathbf{i}$.

Instead of (14.13) we get the following system of equations

$$0 = \eta_j(\nu_k - \nu_j) + \mathbf{i}\big(-\omega_j - \omega_k + (\bar{\zeta}_j \zeta_i + \zeta_j \bar{\zeta}_i)\,\omega_i - \eta_j (\bar{\zeta}_j \zeta_i - \zeta_j \bar{\zeta}_i)\,\mathbf{i}\,\nu_i\big). \tag{14.15}$$

(Note that $(\bar{\zeta}_j \zeta_i - \zeta_j \bar{\zeta}_i)\mathbf{i}$ is a real number and that (14.10) is equivalent to $\zeta_i \bar{\zeta}_j = \bar{\zeta}_i \zeta_k$.)

The real part of equations (14.15) yields $v = \nu_j \in \mathbb{R}$ for $1 \le j \le n$. From the imaginary part of (14.15) we obtain the following system over $\mathbb{R}$ with real variables ω_j and v:

$$0 = -\omega_j - \omega_k + (\bar{\zeta}_j \zeta_i + \zeta_j \bar{\zeta}_i)\,\omega_i - \eta_j (\bar{\zeta}_j \zeta_i - \zeta_j \bar{\zeta}_i)\,\mathbf{i}\,v. \tag{14.16}$$

The lifts γ_p of γ_p^* correspond exactly to the solutions of (14.16). (The trivial solution corresponds to γ_p^* itself.) Observe that, the group of dilations

$$\mathfrak{D} = \{\delta_{r,b} \in \mathfrak{C}^+ \mid \delta_{r,b}(z) = r\,z + b,\ r \in \mathbb{R}^*, b \in \mathbb{C}\}$$

acts on the lifts of γ_p of γ_p^* by conjugation i.e. $\delta \cdot \gamma = \delta\,\gamma_p\,\delta^{-1}$ for $\delta \in \mathfrak{D}$. The the axis of $\delta \cdot \gamma_p(S_j) = \delta\,\gamma_p\,\delta^{-1}(S_j)$ is $\delta(g_j)$. If there is a proper lift γ_p – that is $\mathfrak{T} \neq 0$ – the axis $\{g_j \mid 1 \le j \le n\}$ cannot pass through one point or be parallel. Therefore, $\mathfrak{D} \cdot \gamma_p$ is a 3-dimensional subspace of lifts of γ_p^* and the dimension of the linear subspace of solutions of the homogeneous system (14.16) is at least a 3.

The rank of the system

$$0 = -\omega_j - \omega_k + (\bar{\zeta}_j \zeta_i + \zeta_j \bar{\zeta}_i)\,\omega_i \tag{14.17}$$

is a lower bound for the rank of the system (14.16). Now, the system (14.17) is a linear system of equations with coefficients in the ring of algebraic integers $\mathcal{O}(\zeta) \subset \mathbb{Q}(\zeta)$. There is a homomorphism $\psi_p\colon \mathcal{O}(\zeta) \to \mathbb{Z}_p$ given by $\psi_p\colon \zeta \mapsto [1]_p$ and $\psi_p\colon n \mapsto [n]_p$ where $[n]_p$ denote the coset of $n \in \mathbb{Z}$ modulo p. The system (14.17) transforms under ψ_p into the system of congruences modulo p:

$$-\omega_j - \omega_k + 2\,\omega_i \equiv 0 \mod p \tag{14.18}$$

which has rank $= n-1-\dim_{\mathbb{Z}_p} H_1(\hat{C}_2, \mathbb{Z}_p)$. (Compare Proposition 14.9 and (14.4).) Hence, the dimension of the linear subspace of solutions of the homogeneous system (14.16) is bounded by $2 + H_1(\hat{C}_2, \mathbb{Z}_p)$.

We prove that the case $v = 0$ cannot occur if $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$. If γ_p is a proper lift of γ_p^* and if $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ then the space of solutions of the system (14.16) is three and corresponds exactly to the orbit $\mathfrak{D} \cdot \gamma_p$. If the translation length v of γ_p were zero, we would have that the translation length vanishes for all elements in the orbit $\mathfrak{D} \cdot \gamma_p$. But then each element of $\mathfrak{D} \cdot \gamma_p$ would correspond to a solution of the system (14.17). Therefore, the rank of the system (14.17) is less or equal to $n-3$. On the other hand, the rank of the system (14.17) is greater than or equal to $n-2$, a contradiction.

Remark: The non-existence of γ_p under our assumption $v = 0$ is a property of the Euclidean plane. In a hyperbolic plane where there are no similarities such lifts γ_p may exist.

If $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$ we may assume that there is a lift γ_p of γ_p^* which maps the Wirtinger generators on glide reflections with $v = 1$. Instead of (14.16) we get the following system of inhomogeneous linear equations

$$-\omega_j - \omega_k + (\bar{\zeta}_j \zeta_i + \zeta_j \bar{\zeta}_i)\,\omega_i = \eta_j (\bar{\zeta}_j \zeta_i - \zeta_j \bar{\zeta}_i)\,\mathbf{i}. \qquad (14.19)$$

The rank of the homogeneous part of (14.19) is at least $n-2$. Since conjugation by translations gives a 2-dimensional space of solutions, the rank of (14.19) is exactly equal to $n-2$. □

For a given γ_p with $v = 1$, a primitive p-th root of unity and a suitable enumeration of the Wirtinger generators we may assume

$$\gamma_p(S_1)\colon\ z \mapsto \bar{z} + 1,\ \ \gamma_p(S_2)\colon\ z \mapsto \zeta^2 \bar{z} + \zeta.$$

This corresponds to putting $\omega_1 = \omega_2 = 0$. The fixed lines g_1 and g_2 of $\gamma_p(S_1)$ and $\gamma_p(S_2)$ meet in the origin and pass through 1 and ζ (Figure 14.4). A representation normalized in this way is completely determined up to the choice of ζ.

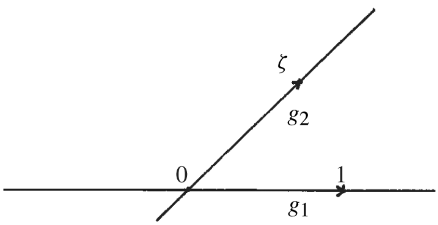


Figure 14.4

The main application of Proposition 14.13 is the exploitation of the peripheral system $(\mathfrak{G}, m, \ell)$ by a normalized representation γ_p. Let m be represented by S_1, then *γ_p maps the longitude ℓ onto a translation by $\lambda(\zeta)$:*

$$\gamma_p(\ell)\colon\ z \mapsto z + \lambda(\zeta),$$

since $\ell \in \mathfrak{G}'' \subset \ker \gamma_p^*$. The solutions ω_j of (14.19) are elements of $\mathbb{Q}(\zeta)$. From $m \cdot \ell = \ell \cdot m$ it follows that $\lambda(\zeta) \in \mathbb{Q}(\zeta) \cap \mathbb{R}$.

14.14 Definition and Proposition. *Let $\mathfrak{G}(\mathbb{Q}(\zeta) \mid \mathbb{Q})$ be the Galois group of the extension $\mathbb{Q}(\zeta) \supset \mathbb{Q}$. The set $[\lambda(\zeta)] = \{\lambda(\tau(\zeta)) \mid \tau \in \mathfrak{G}(\mathbb{Q}(\zeta) \mid \mathbb{Q})\}$ is called the* longitudinal invariant *with respect to γ_p. It is an invariant of the knot.* □

14.15 Example. We want to lift the homomorphism γ_5^* of the group of the four-knot which we computed in Example 14.12. We obtained $\zeta_1 = 1$, $\zeta_2 = \zeta$, $\zeta_3 = \zeta^3$, $\zeta_4 = \zeta^2$ for $\gamma_5^*(S_j) = \zeta_j$, and we may put $\zeta = e^{2\pi i/5}$. The equations (14.19) are then

$$\begin{aligned}
-\omega_1 - \omega_2 + (\zeta^3 + \zeta^2)\omega_3 &= -(\zeta^3 - \zeta^2)\mathbf{i},\\
-\omega_2 - \omega_3 + (\zeta + \zeta^4)\omega_4 &= (\zeta - \zeta^4)\mathbf{i},\\
-\omega_3 - \omega_4 + (\zeta^2 + \zeta^3)\omega_1 &= -(\zeta^2 - \zeta^3)\mathbf{i},\\
-\omega_4 - \omega_1 + (\zeta^4 + \zeta)\omega_2 &= (\zeta^4 - \zeta)\mathbf{i}.
\end{aligned}$$

Putting $\omega_1 = \omega_2 = 0$ yields

$$\mathbf{i}\,\omega_3 = \frac{\zeta^3 - \zeta^2}{\zeta^3 + \zeta^2} = -(\zeta^3 + \zeta)(\zeta - 1),\ \mathbf{i}\,\omega_4 = \zeta^4 - \zeta$$

and, using $b_j = \zeta_j(1 + \mathbf{i}\,\omega_j)$ (see substitution (14.14))

$$\begin{aligned}
&b_1 = 1,\ b_2 = \zeta,\ b_3 = -2(1 + \zeta^2),\ b_4 = \zeta + \zeta^2 - \zeta^3;\\
&\gamma_5(S_1)\colon\ z \mapsto \bar{z} + 1,\\
&\gamma_5(S_2)\colon\ z \mapsto \zeta^2\bar{z} + \zeta,\\
&\gamma_5(S_3)\colon\ z \mapsto \zeta\bar{z} - 2 - 2\zeta^2,\\
&\gamma_5(S_4)\colon\ z \mapsto \zeta^4\bar{z} + \zeta + \zeta^2 - \zeta^3.
\end{aligned}$$

Figure 14.5 shows the configuration of the fixed lines g_j of the glide reflections $\gamma_r(S_j)$. One may verify the Wirtinger relations by well-known geometric properties of the regular pentagon. The longitude ℓ of $(\mathfrak{G}, m, \ell)$ with $m = S_1$ may be read off the projection drawn in Figure 14.3:

$$\ell = S_3^{-1} S_4 S_1^{-1} S_2.$$

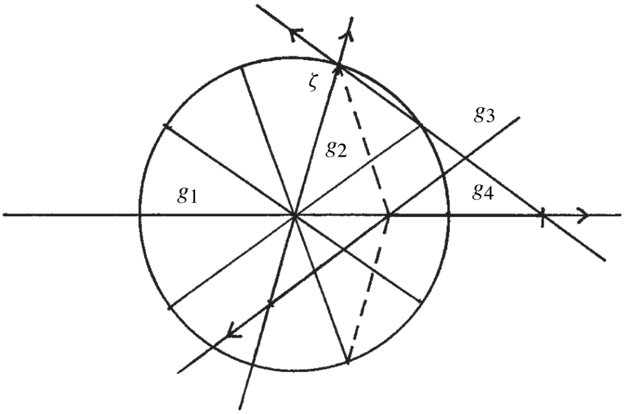


Figure 14.5

One obtains

$$\gamma_5(\ell)\colon\ z \mapsto z + \lambda(\zeta),\ \lambda(\zeta) = 2\Big(\zeta + \zeta^{-1} - (\zeta^2 + \zeta^{-2})\Big).$$

The class $[\lambda(\zeta)]$ contains only different elements, $\lambda(\zeta)$ and $-\lambda(\zeta)$ which reflects the amphicheirality of the four-knot.

14.16 Proposition. *The invariant class* $[\lambda(\zeta)]$ *of an amphicheiral knot always contains –* $\lambda(\zeta)$ *if it contains* $\lambda(\zeta)$.

Proof. A conjugation by a rotation through angle π maps $(\gamma_p(m),\ \gamma_p(\ell))$ onto $(-\gamma_p(m), -\gamma_p(\ell))$. Hence, 3.18 implies that the group of an amphicheiral knot admits normalized representations γ_p and γ'_p with $\gamma_p(\ell^{-1}) = -\gamma_p(\ell) = \gamma'_p(\ell)$. □

14.17 Remark. The argument shows at the same time that the invariant $[\lambda(\zeta)]$ is no good at detecting that a knot is non-invertible. Similarly, γ_p is not strong enough to prove that a knot has Property P : a relation $\gamma_p(\ell^a) = \gamma_p(m), a \neq 0$, would abelianize $\gamma_p(\mathfrak{G})$, and, hence, trivialize it.

Representations of the type γ_p have been defined for links by Hafer [145], Henninger [160]. In Hartley and Murasugi [151] linking numbers in covering spaces were investigated in a more general setting which yielded the invariant $[\lambda(\zeta)]$ as a special case.

14.C Linkage in coverings

The covering $q\colon R_p \to C$ of the complement C of a knot $\mathfrak{k}$ defined by $\ker\gamma_p^* \cong \pi_1 R_p$ is an invariant of $\mathfrak{k}$ as long as there is only one class of equivalent dihedral representations

$$\gamma_p^*\colon \pi_1(C) = \mathfrak{G} \to \mathfrak{D}_p = \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p.$$

The same holds for the branched covering $\hat{q}\colon \hat{R}_p \to S^3$, obtained from R_p, with branching set $\mathfrak{k}$. *In the following* p *is a prime*.

The *linking numbers* $\mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$ of the link $\hat{\mathfrak{k}} = \bigcup_{i=0}^{p-1} \hat{\mathfrak{k}}_i = \hat{q}^{-1}(\mathfrak{k})$ have been used since the beginning of knot theory to distinguish knots which could not be distinguished by their Alexander polynomials. $\ker\gamma_p^*$ is of the form $\langle t^2 \rangle \ltimes \mathfrak{R}$, t a meridian, and is contained in the subgroup $\langle t \rangle \ltimes \mathfrak{R} = \mathfrak{U} \subset \mathfrak{G}$ with $[\mathfrak{G} : \mathfrak{U}] = p$. The subgroup $\mathfrak{U}$ defines an irregular covering I_p, $\pi_1(I_p) \cong \mathfrak{U}$, and an associated branched covering $\hat{I}_p$ which was, in fact, used by Reidemeister [295, 296, 303] to study linking numbers. The regular covering $\hat{R}_p$ is a two-fold branched covering of $\hat{I}_p$, and its linking numbers $\mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$ determine those in $\hat{I}_p$ [146]. We shall, therefore, confine ourselves mainly to $\hat{R}_p$.

Linking numbers exist for pairs of disjoint closed curves in $\hat{R}_p$ which represent elements of finite order in $H_1(\hat{R}_p)$ [329] , [346, 15.6].

In the preceding section we made use of Theorem 12.31, due to Seifert [325], which guarantees that there are at least p linearly independent free elements of $H_1(R_p)$ represented in the set $\{\hat{m}_0, \dots, \hat{m}_{p-1}, \hat{\ell}_0, \dots, \hat{\ell}_{p-1}\}$. To obtain more precise information, we now have to employ a certain amount of algebraic topology (Hartley and Murasugi [151]). Consider a section of the exact homology sequence

$$\cdots \to H_2(\hat{R}_p, V; \mathbb{Q}) \xrightarrow{\partial_*} H_1(V; \mathbb{Q}) \xrightarrow{i_*} H_1(\hat{R}_p; \mathbb{Q}) \to \dots$$

of the pair $(\hat{R}_p, V)$, where V is the union $V = \bigcup_{i=0}^{p-1} V_i$, $\partial V_i = T_i$, of the tubular neighborhoods V_i of $\hat{\mathfrak{k}}_i$ in $\hat{R}_p$. As indicated, we use rational coefficients. The Lefschetz Duality Theorem [346, 14.8.5], [157, 3.43] and excision yield isomorphisms

$$H^1(R_p; \mathbb{Q}) \cong H_2(R_p, \partial R_p; \mathbb{Q}) \cong H_2(\hat{R}_p, V; \mathbb{Q}).$$

One has [346, 14.6.4(b)]

$$\begin{aligned}
&\Delta^*\colon\ H^1(R_p; \mathbb{Q}) \to H_2(\hat{R}_p, V; \mathbb{Q}),\\
&\langle z^1, z_1 \rangle = \mathrm{int}(z_2, z_1),\ z_2 = \Delta^*(z^1),\ z^1 \in H^1(R_p; \mathbb{Q})
\end{aligned}$$

where $\langle\, .\,, .\, \rangle$ denotes the Kronecker product.

We claim that the surjective homomorphism

$$\partial_*\Delta^*\colon\ H^1(R_p; \mathbb{Q}) \to \ker i_*$$

is described by

$$z^1 \mapsto \sum_{i=0}^{p-1} \langle z^1, \hat{m}_i \rangle \hat{\ell}_i. \tag{14.20}$$

To prove (14.20) put

$$\partial_* \Delta^* z^1 = \partial_* z_2 = \sum_{j=0}^{p-1} a_j \hat{\ell}_j,\ a_j \in \mathbb{Q}.$$

Let δ_i be a meridian disk in V_i bounded by $\hat{m}_i = \partial \delta_i$. Then

$$\langle z^1, \hat{m}_i \rangle = \operatorname{int}(z_2, \partial\delta_i) = \operatorname{int}(\partial_* z_2, \delta_i) = \operatorname{int}(\sum_{j=0}^{p-1} a_j \hat{\ell}_j, \delta_i) = a_i.$$

Since $j_*\colon H_1(R_p;\mathbb{Q}) \to H_1(\hat{R}_p;\mathbb{Q})$, induced by the inclusion j, is surjective, $j^*\colon H^1(\hat{R}_p;\mathbb{Q}) \to H^1(R_p;\mathbb{Q})$ is injective. The image $j^*(H^1(\hat{R}_p))$ consists exactly of the homomorphisms $\varphi\colon H_1(R_p) \to \mathbb{Q}$ which factor through $H_1(\hat{R}_p;\mathbb{Q})$. But these constitute $\ker \partial_* \Delta^*$ by (14.20). Thus, one has

$$\begin{aligned} &\dim \ker \partial_* \Delta^* = \dim H^1(\hat{R}_p) = \dim H_1(\hat{R}_p) \quad \text{and} \\ &\dim \partial_* \Delta^*(H^1(R_p;\mathbb{Q})) = \dim \ker i_*. \end{aligned}$$

14.18 Proposition (Hartley–Murasugi).

$$\dim H_1(R_p;\mathbb{Q}) = \dim H_1(\hat{R}_p;\mathbb{Q}) + \dim \ker i_*$$

where $i\colon V \to \hat{R}_p$ is the inclusion. □

It is now easy to prove that only two alternatives occur:

14.19 Proposition. *Either (case 1) all longitudes $\hat{\ell}_i$, $0 \le i \le p-1$ represent in $H_1(\hat{R}_p;\mathbb{Z})$ elements of finite order (linking numbers are defined) and the meridians $\hat{m}_i$, $0 \le i \le p-1$, generate a free Abelian group of rank p in $H_1(R_p;\mathbb{Z})$, or (case 2) the longitudes ℓ_i generate a free Abelian group of rank $p-1$ in $H_1(R_p;\mathbb{Z})$ presented by $\langle \hat{\ell}_0, \dots, \hat{\ell}_{p-1} \mid \hat{\ell}_0 + \hat{\ell}_1 + \ldots + \hat{\ell}_{p-1} \rangle$, and the meridians $\hat{m}_i$ generate a free group of rank one in $H_1(R_p;\mathbb{Z})$; more precisely, $\hat{m}_i \sim \hat{m}_j$ in $H_1(R_p;\mathbb{Q})$ for any two meridians.*

Proof. A Seifert surface S of $\mathfrak{k} = \partial S$ lifts to a surface $\hat{S}$ with $\partial \hat{S} = \sum_{i=0}^{p-1} \hat{\ell}_i \sim 0$ in R_p or $\hat{R}_p$: the construction of C_2 (see 4.4) shows that S can be lifted to S_2 in C_2 resp. $\hat{C}_2$. The inclusion $i\colon S_2 \to \hat{C}_2$ induces an epimorphism $i_*\colon H_1(S_2) \to H_1(\hat{C}_2)$. This follows from $(a^- + a^+) = Fs$ (see Lemma 8.6) and $a^+ = ta^- = -a^-$ in the case of the twofold covering where $t = -1$ (see Remark 8.40). Thus S_2 is covered in R_p by a connected surface $\hat{S}$ (see [324, 3.12]) bounded by the longitudes $\hat{\ell}_i$. If the longitudes $\hat{\ell}_i$ satisfy in $H_1(\hat{R}_p)$ only relations $c \cdot \sum \hat{\ell}_i \sim 0$, $c \in \mathbb{Z}$, which are consequences of $\sum \hat{\ell}_i \sim 0$, we have $\dim(\ker i_*) = 1$ in Proposition 14.18. Hence, by Lemma 12.31, the meridians $\hat{m}_i$ generate a 1-dimensional vector space in $H_1(R_p, \mathbb{Q})$. There is a covering transformation of $R_p \to C_2$ which maps $\hat{m}_i$ onto $\hat{m}_j \sim r\hat{m}_i$, $i \neq j$, $i \neq j$, $r \in \mathbb{Q}$. From this one gets $r^p = 1$, thus $r = 1$. This disposes of case 2. If the longitudes $\hat{\ell}_i$ are subject to a relation that is not a consequence of $\sum \hat{\ell}_i \sim 0$, then $\sum a_i \hat{\ell}_i \sim 0$ where not all a_i are equal. Furthermore, one may assume $\sum a_i \hat{\ell}_i \sim 0$, $\sum a_i \neq 0$. (If necessary, replace a_i by $a_i + 1$). Applying the cyclic group $\mathfrak{Z}_p$ of covering transformation to this relation yields a set of p relations forming a cyclic relation matrix. The determinant of such a cyclic matrix is always different from zero [266, § 19.6]. Hence, the longitudes generate a finite group. In fact, since the $\hat{\ell}_i$ are permuted by the covering transformations their orders coincide; we denote it by $|\ell|$ = order of $\hat{\ell}_i$ in $H_1(\hat{R}_p)$. It follows that $\dim\ker i_* = p$, and by Proposition 14.18 that the meridians $\hat{m}_i$ generate a free group of rank p. □

14.20 Proposition. *If $H_1(\hat{C}_2, \mathbb{Z}_p) \cong \mathbb{Z}_p$, then there is exactly one class of equivalent dihedral homomorphisms $\gamma_p^*\colon \mathfrak{G} \to \mathfrak{D}_p$ and the dihedral linking numbers $\nu_{ij} = \mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$ are defined (case 1). The invariant $[\lambda(\zeta)]$ (see 14.14) associated to the lift γ_p of γ_p^* (14.13) then takes the form*

$$\lambda_j(\zeta) = 2\sum_{i=0}^{p-1} \nu_{ij}\zeta^i \quad \textit{with } \nu_{ii} = -\sum_{j \neq i} \nu_{ij} \tag{14.21}$$

(Here we have put $[\lambda(\zeta)] = \{\lambda(\zeta^j) \mid 1 \leq j < p\}$. Case 1 and case 2 refer to Proposition 14.19.)

Proof. For each meridian $m \in \mathfrak{G}$ we have $\gamma_p^*(m^2) = 1$ and hence $\gamma_p(\hat{m}_i)$ is a translation.

The occurrence of case 2 implies $\gamma_p(\hat{m}_i) = \gamma_p(\hat{m}_j)$ for all meridians $\hat{m}_i, \hat{m}_j$. But in the case of a representation γ_p, mapping Wirtinger generators on glide reflections, $\gamma_p(\hat{m}_i)$ and $\gamma_p(\hat{m}_j)$ will be translations in different directions for some i, j. Thus the Wirtinger class is mapped onto reflections, that is, $\gamma_p(\hat{m}_i) = 0$. This contradicts Proposition 14.13.

In case 1 the longitudes $\hat{\ell}_j$ are of finite order in $H_1(\hat{R}_p; \mathbb{Z})$. Since the covering transformations permute the $\hat{\ell}_j$, they all have the same order $|\hat{\ell}_j| = |\ell|$. Consider a section of the Mayer–Vietoris sequence:

$$\cdots \to H_1(\partial V) \xrightarrow{\psi_*} H_1(R_p) \oplus H_1(V) \xrightarrow{\varphi_*} H_1(\hat{R}_p) \to \dots .$$

Since $\varphi_*(|\ell|\hat{\ell}_j, 0) = 0$, one has, for suitable $b_k, c_k \in \mathbb{Z}$,

$$(|\ell|\hat{\ell}_j, 0) = \psi_*\Big(\sum_{k=0}^{p-1} b_k \hat{m}_k + \sum_{k=0}^{p-1} c_k \hat{\ell}_k\Big) = \Big(\sum_k b_k \hat{m}_k + \sum_k c_k \hat{\ell}_k, -\sum_k c_k \hat{\ell}_k\Big).$$

This gives

$$|\ell|\,\hat{\ell}_j = \sum_{k=0}^{p-1} b_k \hat{m}_k, \quad \text{and } |\ell| \cdot \mathrm{lk}(\hat{\ell}_i, \hat{\ell}_j) = \mathrm{lk}(\hat{\ell}_i, \sum_k b_k \hat{m}_k = b_i.$$

Since $\mathrm{lk}(\hat{\ell}_i, \hat{\ell}_j) = \mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$, one has

$$\hat{\ell}_j = \sum_i \nu_{ij} \hat{m}_i. \tag{14.22}$$

The relation $\sum_{j=0}^{p-1} \hat{\ell}_j \sim 0$ yields $0 = \mathrm{lk}(\hat{\ell}_i, \sum \hat{\ell}_j) = \sum_j \nu_{ij}$. Formula (14.21) of Proposition 14.20 follows from $\gamma_p(\hat{m}_i)$: $z \mapsto z + 2\zeta^i$ for a suitable indexing after the choice of a primitive p-th root of unity ζ. □

Remark: Evidently any term $\sum_{i=0}^{p-1} a_i \zeta^i$, $a_i \in \mathbb{Q}$, can be uniquely normalized such that $\sum_i a_i = 0$ holds. But a different normalization can be chosen: $a_0 = 0$. One obtains from a sequence $\{a_1, \ldots, a_{p-1}\}$ in this normalization a set of linking number ν_{0j}, $0 < j \leq p-1$, by the formula

$$2\nu_{0j} = a_j - \frac{1}{p} \sum_{k=1}^{p-1} a_k. \tag{14.23}$$

14.21. Linking numbers associated with the dihedral representations γ_α: $\mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_\alpha$ for two-bridge knots $\mathfrak{b}(\alpha, \beta)$ have been computed explicitly. In this case a unique lift γ_α always exists even if α is not a prime. The linking matrix is

$$\begin{pmatrix} -\sum \varepsilon_j & \varepsilon_1 & \ldots & \varepsilon_{\alpha-1} \\ \varepsilon_{\alpha-1} & -\sum \varepsilon_j & \ldots & \varepsilon_{\alpha-2} \\ \vdots & \vdots & & \vdots \\ \varepsilon_1 & \varepsilon_2 & \ldots & -\sum \varepsilon_j \end{pmatrix} \tag{14.24}$$

with $\varepsilon_k = (-1)^{[\frac{k\beta}{\alpha}]}$, $[x]$ = integral part of x and $\sum = \sum_{j=1}^{\alpha-1}$, see Burde [55].

The property $|\varepsilon_k| = 1$ affords a good test for two-bridged knots. “Most” of the knots with more than two bridges (see Table C.1) can be detected by this method, compare Perko [287].

A further property of dihedral linking numbers follows from the fact that $\lambda(\zeta)$ is a real number, $\lambda(\zeta) = \overline{\lambda(\zeta)}$. This gives.

$$\nu_{i,i-j} = \nu_{ij}, i \neq j, \tag{14.25}$$

where $i - j$ is to be taken modulo p. Furthermore,

$$\nu_{ij} = \nu_{ji} = \nu_{i+k,j+k}. \tag{14.26}$$

The first equation expresses a general symmetry of linking numbers, and the second one the cyclic p-symmetry of $\hat{R}_p$.

As mentioned at the beginning of this section, $\hat{R}_p$ is a two-fold branched covering of the irregular covering space $\hat{I}_p$ with one component $\hat{\mathfrak{k}}_j$ of $\hat{\mathfrak{k}} = q^{-1}(\mathfrak{k})$ as branching set in $\hat{R}_p$. (There are, indeed, p equivalent covering spaces $\hat{I}_p$ corresponding to p conjugate subgroups $\mathfrak{U}_j = \langle t_j \rangle \ltimes \mathfrak{K}$, depending on the choice of the meridian t_j resp. the component $\hat{\mathfrak{k}}_j$.) We choose $j = 0$. Let $\hat{q}\colon \hat{R}_p \to \hat{I}_p$ be the covering map. The link $\mathfrak{k}' = \hat{q}(\hat{\mathfrak{k}})$ consists of $\frac{p+1}{2}$ components $\hat{\mathfrak{k}}'_0 = \hat{q}(\hat{\mathfrak{k}}_0)$, $\mathfrak{k}'_j = \hat{q}(\hat{\mathfrak{k}}_j) = \hat{q}(\hat{\mathfrak{k}}_{-j})$, $0 < j \leq \frac{p-1}{2}$. (Indices are read modulo p.) Going back to the geometric definition of linking numbers by intersection numbers one gets for $\mu_{ij} = \operatorname{lk}(\mathfrak{k}'_k, \mathfrak{k}'_j)$,

$$\mu_{0j} = 2\nu_{0j},\ \mu_{ij} = \nu_{ij} + \nu_{i,-j},\ i \neq j. \tag{14.27}$$

This yields by (14.25) and (14.26) Perko's identities [287]:

$$2\mu_{ij} = \mu_{0,i-j} + \mu_{0,i+j}, \quad \text{or} \quad \mu_{ij} = \nu_{0,i-j} + \nu_{0,i+j}. \tag{14.28}$$

As $\nu_{ij} = \pm 1$ for two-bridge knots, $\mu_{ij} = \pm 2$ or 0 for these.

It follows from (14.26), (14.27) and Proposition 14.20 that the linking numbers ν_{ij}, the linking numbers μ_{ij}, and the invariant $[\lambda(\zeta)]$ determine each other. All information is already contained in the ordered set $\left\{\nu_{0j} \mid 1 \leq j \leq \frac{p-1}{2}\right\}$. Equation (14.27) shows that [151, Theorem 6.3] is a consequence of Proposition 14.20.

The theory developed in this section has been generalized by Hartley [150]. Many results carry over to metacyclic homomorphisms $\beta^*_{r,p}\colon \mathfrak{G} \to \mathfrak{Z}_r \ltimes \mathfrak{Z}_p$, see Proposition 14.9 and [53]. The homomorphism $\beta^*_{r,p}$ can be lifted and the invariant $[\lambda(\zeta)]$ can be generalized to the metacyclic case. This invariant has a new quality, in that it can identify non-invertible knots which $[\lambda(\zeta)]$ cannot, as we pointed out in Remark 14.17 (see Hartley [149]).

14.22 Examples. (a) The four-knot is a two-bridge knot, $4_1 = \mathfrak{b}(5,3)$. Thus

$$\nu_{0j} = (-1)^{\left[\frac{3j}{5}\right]}, \quad (\nu_{ij}) = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix},$$

and

$$(\mu_{ij}) = \begin{pmatrix} * & 2 & -2 \\ 2 & * & 0 \\ -2 & 0 & * \end{pmatrix}.$$

The link $\mathfrak{k}' = \hat{q}^{-1}(4_1)$ in $\hat{I}_5 \cong S^3$ has been determined (Figure 14.6) in [54]. (For the definition of $\hat{I}_p$ see the beginning of Section 14.C.)

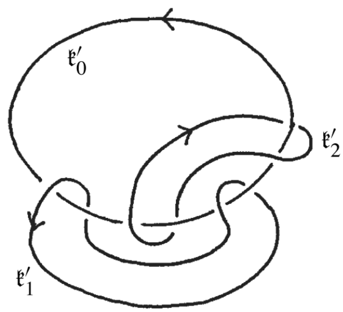


Figure 14.6

(b) As a second example consider the knot $7_4 = \mathfrak{b}(15, 11)$ and the irregular covering $\hat{I}_{15}$. Its linking matrix (μ_{ij}) is

$$(\mu_{ij}) = \begin{pmatrix} * & 2 & -2 & 2 & 2 & -2 & 2 & -2 \\ 2 & * & 2 & 0 & 0 & 2 & -2 & 0 \\ -2 & 2 & * & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & * & 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & 0 & * & 2 & -2 & 2 \\ -2 & 2 & 0 & -2 & 2 & * & 2 & 0 \\ 2 & -2 & 0 & 2 & -2 & 2 & * & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 & 0 & * \end{pmatrix}$$

by (14.28) and $\nu_{0j} = (-1)^{\left[\frac{11j}{15}\right]}$, $0 < j < 15$.

The numbers $\frac{1}{2}\sum_{j \neq i} |\mu_{ij}| = v_i$, $0 \le i \le 7$, are 7, 4, 2, 3, 4, 5, 5, 2. (Compare Reidemeister [296, III.§15, p. 69].)

In general, an effective computation of linking numbers can be carried out in various ways. One may solve equations (14.18) and (14.19) in the proof of Proposition 14.13 and thereby determine γ_p^*, γ_p and $[\lambda(\zeta)]$. A more direct way is described by Hartley and Murasugi [151] using the Reidemeister–Schreier algorithm. See also Perko [286].

14.D Periodic knots

Some knots show geometric symmetries – for instance torus knots. The term "geometric" implies "metric", a category into which topologists usually did not enter in the pre-Thurston era. Nevertheless, symmetries have been defined and considered in various ways by R. H. Fox [111]. Various other symmetries and geometric approaches can be found in Chapter 10 of A. Kawauchi's book [190] and in Bonahon and Siebenmann [37]. We shall, however, occupy ourselves with only one of the different versions of symmetry, the one most frequently investigated. It serves in this chapter as an application of the metabelian representation δ_α of the knot group introduced in 14.6 – in this section $\mathfrak{k}$ will always have one component.

A knot will be said to have period $q > 1$, if it can be represented by a curve in Euclidean 3-space E^3 which is mapped onto itself by a rotation r of E^3 of order q. The axis h must not meet the knot. The positive solution of the Smith conjecture (see Appendix B.9) allows a topological definition of periodicity.

14.23 Definition. A *knot* $\mathfrak{k} \subset S^3$ *has period* $q > 1$, if there is an orientation preserving homeomorphism $r\colon S^3 \to S^3$ of order q with a set of fixed points $h \cong S^1$ disjoint from $\mathfrak{k}$ and mapping $\mathfrak{k}$ into itself.

Remark: The orientation of $\mathfrak{k}$ is not essential in this definition. A period of an unoriented knot automatically respects an orientation of the knot (E 14.8).

Suppose a knot $\mathfrak{k}$ has period q. We assume that a regular projection of $\mathfrak{k}$ onto a plane perpendicular to the axis of the rotation has period q with respect to a rotation of the plane (Figure 14.7). Denote by $E^3_q = E^3/\mathfrak{Z}_q$ the Euclidean 3-space which

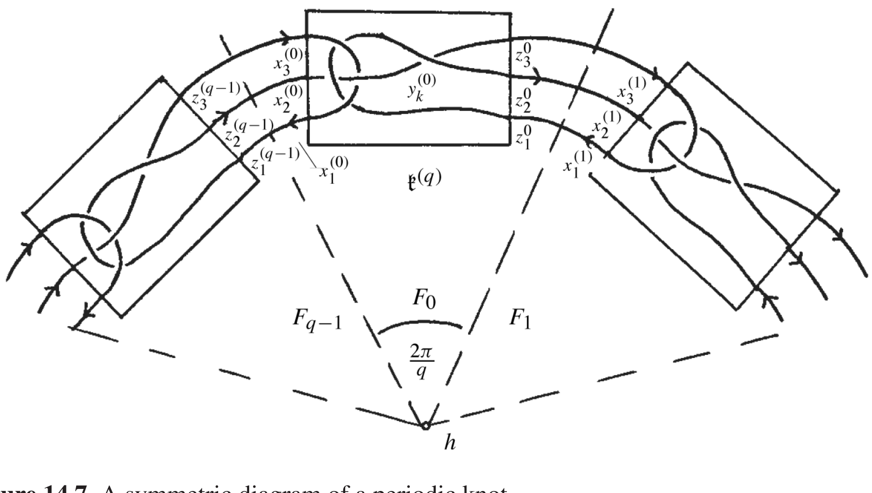


Figure 14.7. A symmetric diagram of a periodic knot.

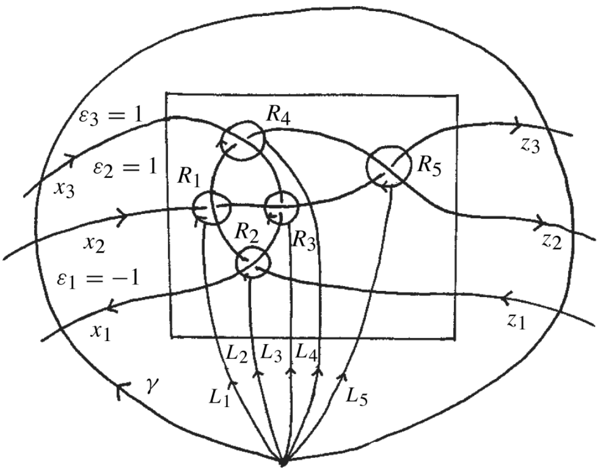


Figure 14.8

is the quotient space of E^3 under the action of $\mathfrak{Z}_q = \langle r \mid r^q \rangle$. There is a cyclic branched covering $f^{(q)}\colon E^3 \to E^3_q$ with branching set h in E^3 and $f^{(q)}(\mathfrak{k}) = \mathfrak{k}^{(q)}$ a knot in E^3_q. We call $\mathfrak{k}^{(q)}$ the *factor knot* of $\mathfrak{k}$. It is obtained from $\mathfrak{k}$ in Figure 14.8 by identifying x_i and z_i.

One has $\lambda = \mathrm{lk}(\mathfrak{k}, h) = \mathrm{lk}(\mathfrak{k}^{(q)}, h^{(q)}) \neq 0$, $h^{(q)} = f^{(q)}(h)$: the equality of the linking numbers follows by looking at the intersection of $\mathfrak{k}$ resp. $\mathfrak{k}^{(q)}$ with half-planes in E^3 resp. E^3_q spanning h resp. $h^{(q)}$. If $\lambda = \mathrm{lk}(\mathfrak{k}^{(q)}, h^{(q)}) = 0$, then $\mathfrak{k}^{(q)} \simeq 1$ in $\pi_1(E^3_q - h^{(q)})$, and $\mathfrak{k} \subset E^3$ would consist of q components. By choosing a suitable direction of h we may assume $\lambda > 0$. Moreover, we have $\gcd(\lambda, q) = 1$, (E 14.9).

The symmetric projection (Figure 14.7) yields a symmetric Wirtinger presentation of the knot group of $\mathfrak{k}$ (see Theorem 3.4):

$$\begin{aligned}\mathfrak{G} = \langle\, & x_i^{(0)}, y_k^{(0)}, z_i^{(0)}, x_i^{(1)}, y_k^{(1)}, z_i^{(1)}, \dots \mid \\ & \widetilde{R}_i^{(1)}, R_j^{(0)}, \widetilde{R}_i^{(2)}, R_j^{(1)}, \dots, \widetilde{R}_i^{(q-1)}, R_j^{(q-2)}, \widetilde{R}_i^{(0)}, R_j^{(q-1)} \,\rangle, \end{aligned} \tag{14.29}$$

where $1 \le i \le n$, $1 \le k \le m$, $1 \le j \le m+n$. The arcs entering a fundamental domain F_0 of $\mathfrak{Z}_q$, a $2\pi/q$-sector, from the left side, correspond to generators $x_i^{(0)}$ and their images under the rotation r to generators $z_i^{(0)} = r_\#(x_i^{(0)})$. The remaining arcs in F_0 give rise to generators $y_k^{(0)}$. Double points in F_0 define relations $R_j^{(0)}$. The generators $x_i^{(l)}, y_k^{(l)}, z_i^{(l)}$, $0 \le l \le q-1$, correspond to the images of the arcs of $x_i^{(0)}, y_k^{(0)}, z_i^{(0)}$ under the rotation through angle $2\pi l/q$, $\widetilde{R}_i^{(l)} = x_i^{(l)}\big(z_i^{(l-1)}\big)^{-1}$ and $R_j^{(l)} = R_j^{(0)}(x_i^{(l)}, y_k^{(l)}, z_i^{(l)})$.

The Jacobian of the Wirtinger presentation, see Proposition 9.9, is of the following form:

$$
\left(\begin{array}{ccc|ccc|ccc|c|ccc}
0_n & 0_{n\times m} & -E_n & E_n & 0_{n\times m} & 0_n & & & & & & & \\ \hline
\multicolumn{3}{c|}{\bar{A}(t)} & 0_n & 0_{n\times m} & 0_n & & & & & & & \\ \hline
0_n & 0_{n\times m} & 0_n & 0_n & 0_{n\times m} & -E_n & E_n & 0_{n\times m} & 0_n & & & & \\ \hline
0_n & 0_{n\times m} & 0_n & \multicolumn{3}{c|}{\bar{A}(t)} & 0_n & 0_{n\times m} & 0_n & & & & \\ \hline
& & & & & & & & & \ddots & & & \\ \hline
E_n & 0_{n\times m} & 0_n & & & & & & & & 0_n & 0_{n\times m} & -E_n \\ \hline
0_n & 0_{n\times m} & 0_n & & & & 0_n & 0_{n\times m} & 0_n & & \multicolumn{3}{c}{\bar{A}(t)}
\end{array}\right).
$$

Here E_n is an $n \times n$ identity matrix, 0_n, $0_{n\times m}$ are zero matrices and $\bar{A}(t)$ is a $(n+m)\times(2n+m)$ matrix over $\mathbb{Z}(t)$.

We rearrange rows and columns of $A(t)$ in such a way that the columns correspond to generators ordered in this way:

$$x_1^{(0)}, x_1^{(1)}, \dots, x_1^{(q-1)}, x_2^{(0)}, x_2^{(1)}, \dots, x_n^{(q-1)}, y_1^{(0)}, \dots, y_m^{(q-1)}, z_1^{(0)}, \dots, z_n^{(q-1)}.$$

The relations and rows have the following order:

$$x_1^{(1)}(z_1^{(0)})^{-1}, x_1^{(2)}(z_1^{(1)})^{-1}, \dots, x_1^{(0)}(z_1^{(q-1)})^{-1}, \dots, R_1^{(0)}, R_1^{(1)}, \dots .$$

This gives a matrix

$$
A^*(t) = \left(\begin{array}{cccccccc}
Z_q & & 0_q & \dots & 0_q & -E_q & & 0_q \\
& \ddots & \vdots & & \vdots & & \ddots & \\
0_q & & Z_q \; 0_q & \dots & 0_q & 0_q & & -E_q \\ \hline
\multicolumn{8}{c}{\overline{A}^*(t)}
\end{array}\right).
$$

Here $\bar{A}^*(t)$ is obtained from $\bar{A}(t)$ by replacing every element $a_{ik}(t)$ of $\bar{A}(t)$ by the $q \times q$ diagonal matrix

$$a_{ik}^{(q)}(t) = a_{ik}(t)E_q.$$

The $q \times q$-matrix

$$
Z_q = \begin{pmatrix}
0 & 1 & & & 0 \\
0 & 0 & 1 & & \\
\vdots & & \ddots & \ddots & \\
0 & & & 0 & 1 \\
1 & 0 & \dots & & 0
\end{pmatrix}
$$

is equivalent to the diagonal matrix

$$Z(\zeta) = W Z_q W^{-1} = \begin{pmatrix} 1 & & & & \mathbf{0} \\ & \zeta & & & \\ & & \zeta^2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \zeta^{q-1} \end{pmatrix}$$

over $\mathbb{Q}(\zeta)$ where ζ is a primitive q-th root of unity (Exercise E 14.10). The matrix $\widetilde{W} A^*(t) \widetilde{W}^{-1}$ with

$$\widetilde{W} = \begin{pmatrix} W & & & \mathbf{0} \\ & W & & \\ & & \ddots & \\ \mathbf{0} & & & W \end{pmatrix}$$

may be obtained from $A^*(t)$ by replacing the submatrices Z_q by $Z(\zeta)$. Returning to the original ordering of rows and columns as in $A(t)$, the matrix $\widetilde{W} A^*(t) \widetilde{W}^{-1}$ takes the form

$$A(t,\zeta) = \begin{pmatrix} A^{(q)}(t,1) & & & 0 \\ & A^{(q)}(t,\zeta) & & \\ & & \ddots & \\ 0 & & & A^{(q)}(t,\zeta^{q-1}) \end{pmatrix} \tag{14.30}$$

where

$$A^{(q)}(t,\zeta^\nu) = \left(\begin{array}{ccccccccc} \zeta^\nu & & \mathbf{0} & 0 & \cdots & 0 & -1 & & \mathbf{0} \\ & \ddots & & \vdots & & \vdots & & \ddots & \\ \mathbf{0} & & \zeta^\nu & 0 & \cdots & 0 & \mathbf{0} & & -1 \\ \hline \multicolumn{9}{c}{\overline{A}(t)} \end{array}\right)$$

$A(t,\zeta)$ is equivalent to $A(t)$ over $\mathbb{Q}(\zeta)[t^{\pm 1}]$, and $A^{(q)}(t,1)$ is a Jacobian of the factor knot $\mathfrak{k}^{(q)}$. We replace ζ^ν by a variable τ and prove:

14.24 Proposition. $\det(A^{(q)}(t,\tau)) = (\tau - 1) D(t,\tau) \in \mathbb{Z}[t^{\pm 1}, \tau]$ *with*

$$D(t,1) \doteq \rho_\lambda(t) \Delta_1^{(q)}(t), \quad \textit{where } \rho_\lambda(t) = 1 + t + \ldots + t^{\lambda - 1},\ \lambda = \mathrm{lk}(h,\mathfrak{k}).$$

Here $\Delta_1^{(q)}(t)$ *is the Alexander polynomial of the factor knot* $\mathfrak{k}^{(q)}$.
Moreover, $D(1,\tau) = \pm(1 + \tau + \ldots + \tau^{\lambda-1}) = \pm\rho_\lambda(\tau)$ *and* $|D(1,1)| = \lambda$.

Proof. Replace the first column of $A^{(q)}(t,\tau)$ by the sum of all columns and expand according to the first column:

$$\det(A^{(q)}(t,\tau)) = (\tau-1)\cdot\sum_{i=1}^{n} D_i(t,\tau)$$

where $(-1)^{i+1}D_i(t,\tau)$ denotes the minor obtained from $A^{(q)}(t,\tau)$ by omitting the first column and i-th row. Note that the sum of all columns of $\overline{A}(t)$ is zero, compare Lemma 12.9 (a). This proves the first assertion for $D(t,\tau)=\sum_{i=1}^{n} D_i(t,\tau)$.

To prove the second one we show that the rows $\mathfrak{a}_l$ of the Jacobian

$$A^{(q)}(t,1) = \left(\begin{array}{ccccccccc} 1 & & \mathbf{0} & 0 & \cdots & 0 & -1 & & \mathbf{0} \\ & \ddots & & \vdots & & \vdots & & \ddots & \\ \mathbf{0} & & 1 & 0 & \cdots & 0 & \mathbf{0} & & -1 \\ \hline \multicolumn{9}{c}{\overline{A}(t)} \end{array}\right)$$

of $\mathfrak{k}^{(q)}$ satisfy a special linear dependence

$$\sum_{l=1}^{2n+m} \alpha_l\, \mathfrak{a}_l = 0 \quad \text{with} \sum_{l=1}^{n} \alpha_l = \rho_\lambda(t)$$

(compare Lemma 9.12 (b)). Denote by $\mathfrak{F}$ the free group generated by $\{X_i, Y_k, Z_i \mid 1 \le i \le n,\ 1 \le k \le m\}$, $\psi(X_i) = x_i^{(0)}, \psi(Y_k) = y_k^{(0)}, \psi(Z_i) = z_i^{(0)}$. There is an identity

$$\Big(\prod_{i=1}^{n} X_i^{\varepsilon_i}\Big)\Big(\prod_{i=1}^{n} Z_i^{\varepsilon_i}\Big)^{-1} = \prod_{j=1}^{n+m} L_j R_j L_j^{-1} \tag{14.31}$$

for $L_j \in \mathfrak{F}$, $\varepsilon_i = \pm 1$, and $R_j = R_j^{(0)}(X_i, Z_k, Z_i)$. This follows by the argument used in the proof of 3.6: The closed path γ in Figure 14.8 can be expressed by both sides of equation (14.31). From this we define:

$$\alpha_l = \frac{\partial}{\partial X_l}\Big(\prod_{i=1}^{n} X_i^{\varepsilon_i}\Big)^{\varphi\psi} = \sum_{j=1}^{n+m} (L_j)^{\varphi\psi}\left(\frac{\partial R_j}{\partial X_l}\right)^{\varphi\psi},\ 1 \le l \le n,$$

hence,

$$-\alpha_l = \sum_{j=1}^{n+m} (L_j)^{\varphi\psi}\left(\frac{\partial R_j}{\partial Z_l}\right)^{\varphi\psi},\ 1 \le l \le n,$$

$$0 = \sum_{j=1}^{n+m} (L_j)^{\varphi\psi}\left(\frac{\partial R_j}{\partial Y_k}\right)^{\varphi\psi},\ 1 \le k \le m.$$

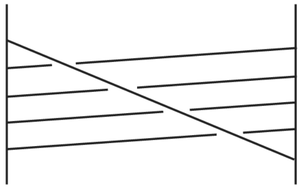


Figure 14.9

Putting $\alpha_{n+j} = -(L_j)^{\varphi\psi}$, $1 \leq j \leq n+m$, gives $\sum_{i=1}^{2n+m} \alpha_i \, \mathfrak{a}_i = 0$. The fundamental formula Lemma 9.8 (c) yields

$$(t-1)\sum_{l=1}^{n} \alpha_l = \sum_{l=1}^{n} \frac{\partial}{\partial X_l}\Big(\prod_{i=1}^{n} X_i^{\varepsilon_i}\Big)^{\varphi\psi}(t-1) = \Big(\prod_{i=1}^{n} X_i^{\varepsilon_i}\Big)^{\varphi\psi} - 1 = t^{\lambda} - 1,$$

hence $\sum\limits_{l=1}^{n} \alpha_l = \rho_\lambda(t)$. Now $D_1(t,1) \doteq \Delta_1^{(q)}(t)$, and $\alpha_i D_1(t,1) = D_i(t,1)$. The last equation is a consequence of $\sum \alpha_i \mathfrak{a}_i = 0$, compare Proposition 10.21.

To prove $D(1,\tau) = \pm(1+\tau+\ldots+\tau^{\lambda-1})$ consider $A^{(q)}(1,\tau)$. This matrix is associated to the knot projection, but it treats overcrossings in the same way as undercrossings. By a suitable choice of undercrossings and overcrossings one may replace $\mathfrak{k}$ by a closed braid of a simple type (Figure 14.9) while preserving its symmetry. The elimination of variables does not alter $|\det A^{(q)}(1,\tau)|$. Finally, $A^{(q)}(1,\tau)$ takes the form:

$$\begin{pmatrix} \tau E_\lambda & -E_\lambda \\ -E_\lambda & P_\lambda \end{pmatrix}$$

where E_λ is the $\lambda \times \lambda$-identity matrix and P_λ the representing matrix of a cyclic permutation of order λ. It follows that

$$\det(A^{(q)}(1,\tau)) = \pm\det(E_\lambda - \tau P_\lambda) = \pm(1-\tau^\lambda),$$

because the characteristic polynomial of P_λ is $\pm(1-\tau^\lambda)$. Therefore,

$$D(1,\tau) = \pm(1+\tau+\ldots+\tau^{\lambda-1}) = \pm\rho_\lambda(\tau) \quad \text{and } |D(1,1)| = \lambda. \quad \square$$

14.25 Remark. The polynomial $D(t,\tau)$ is the two variable Alexander polynomial of the link $\mathfrak{l} = h^{(q)} \cup \mathfrak{k}^{(q)}$ formed by the axis $h^{(q)}$ and the factor knot $\mathfrak{k}^{(q)}$. In particular $D(t,1) \doteq \rho_\lambda(t)\Delta_1^{(q)}(t)$ and $D(1,\tau) \doteq \rho_\lambda(\tau)$ are equivalent to the second *Torres condition* (see [190, 7.4.1] and E 14.2).

14.26 Proposition (Murasugi conditions [262]). *The Alexander polynomial $\Delta_1(t)$ of a knot $\mathfrak{k}$ with period q satisfies the equation*

$$\Delta_1(t) \doteq \Delta_1^{(q)}(t) \cdot \prod_{i=1}^{q-1} D(t,\zeta^i). \tag{14.32}$$

Here $D(t,\tau) \in \mathbb{Z}[t^{\pm 1}, \tau]$ *is a polynomial in two variables with*

$$D(t,1) \doteq \rho_\lambda(t)\Delta_1^{(q)}(t), \quad D(1,\tau) = \pm\rho_\lambda(\tau)$$

and ζ *is a primitive* q*-th root of unity.* $0 < \lambda = \mathrm{lk}(h,\mathfrak{k})$ *is the linking number of* $\mathfrak{k}$ *with the axis* h *of rotation.*

Proof. To determine the first elementary ideal of $A(t,\zeta)$ over $\mathbb{Q}(\zeta)[t^{\pm 1}]$, see (14.30), it suffices to consider the minors obtained from $A(t,\zeta)$ by omitting an i-th row and a j-th column, $1 \le i,j \le 2n+m$, because $\det(A^{(q)}(t,1)) = 0$. That the determinant of such a minor takes the form

$$\pm t^\nu \Delta_1^{(q)}(t) \cdot \prod_{i=1}^{q-1}(\zeta^i - 1)D(t,\zeta^i)$$

follows from the fact that $A^{(q)}(t,1)$ is a Jacobian of $\mathfrak{k}^{(q)}$. Now observe that $\prod_{i=1}^{q-1}(\zeta^i - 1) = q$ is a unit in $\mathbb{Q}(\zeta)[t^{\pm 1}]$ and that $\prod_{i=1}^{q-1} D(t,\zeta^i) \in \mathbb{Z}[t^{\pm 1}]$ satisfies $\prod_{i=1}^{q-1} D(1,\zeta^i) = 1$. Therefore, $\Delta_1(t)$ and $\Delta_1^{(q)}(t) \cdot \prod_{i=1}^{q-1} D(t,\zeta^i)$ are associated in $\mathbb{Z}[t^{\pm 1}]$. (See E 14.11.) □

14.27 Remarks. (1) Note that $\Delta_1^{(q)}(t)$ and $\prod_{i=1}^{q-1} D(t,\zeta^i)$ are knot polynomials i.e. they are symmetric polynomials with $|\Delta_1^{(q)}(1)| = 1 = |\prod_{i=1}^{q-1} D(1,\zeta^i)|$ (see Theorem 8.26). (See E 14.11.)

(2) If $\Delta_1(t) = q\,h(t) + 1 \equiv 1 \bmod q$, then the Murasugi conditions (14.32) are satisfied for $\Delta_1(t) = \Delta_1^{(q)}(t)$, $\lambda = 1$ and $D(t,\tau) = 1 + h(t)\,(1+\tau+\cdots+\tau^{q-1})$. Moreover, it was proved by Davis and Livingston [82, Corollary 1.2] that a knot polynomial which is congruent to 1 modulo q is the Alexander polynomial of a knot of period q.

14.28 Corollary (Murasugi's congruence [262]). *If* $\Delta_1(t)$ *is the Alexander polynomial of a knot of prime period* q *then*

$$\Delta_1(t) \doteqdot (\Delta_1^{(p^a)}(t))^{p^a} \cdot (\rho_\lambda(t))^{p^a-1} \bmod p \quad \textit{for } p^a \,|\, q,\ p \textit{ a prime.}$$

Proof. A knot $\mathfrak{k}$ with period q also has period p^a, $p^a | q$. Let $\mathcal{O}(p^a)$ denote the cyclotomic integers in $\mathbb{Q}(\zeta)$, ζ a p^a-th root of unity. There is a homomorphism

$$\psi_p \colon \mathcal{O}(p^a) \to \mathbb{Z}_p, \quad \sum_{i=1}^{p^a} n_i\zeta^i \mapsto \sum_{i=1}^{p^a}[n_i] \bmod p.$$

Extending ψ_p to the rings of polynomials over $\mathcal{O}(p^a)$ resp. $\mathbb{Z}_p$ yields the corollary. □

14.29 Proposition. *Let $\mathfrak{k}$ be a knot of period p^a and $\Delta_1(t) \not\equiv \pm 1 \bmod p$. Then $D(t,\zeta^i)$ is not a monomial for some p^a-th root of unity $\zeta^i \neq 1$. Any common root of $\Delta_1^{(p^a)}(t)$ and $D(t,\zeta^i)$ is also a root of $\Delta_2(t)$. If all roots of $D(t,\zeta^i)$ are roots of $\Delta_1^{(p^a)}(t)$, then $\lambda \equiv \pm 1 \bmod p$.*

Proof. If $D(t,\zeta^i)$ is monomial, $1 \le i \le p^a$, then equation (14.32) yields $\Delta_1(t) = \Delta_1^{(p^a)}(t)$. Apply ψ_p to this equation and use Corollary 14.28 to obtain $\Delta_1^{(p^a)} \doteqdot 1 \bmod p$ and $\lambda = 1$. From this it follows that $\Delta_1(t) \equiv 1 \bmod p$.

Suppose now that $D(t,\zeta^i)$ and $\Delta_1^{(p^r)}(t)$ have a common root η. Transform $A(t,\zeta)$ over $\mathbb{Q}(\zeta)[t^{\pm 1}]$ into a diagonal matrix by replacing each block $A^{(q)}(t,\zeta^i)$, $0 \le i \le p^a$, see (14.30), by an equivalent diagonal block. Since $\det(A^q(t,1)) = 0$, it follows that the second elementary ideal $E_2(t)$ vanishes for $t = \eta$; hence, $\Delta_2(\eta) = 0$.

If all roots of $D(t,\zeta^i)$ are roots of $\Delta_1^{(p^a)}(t)$, every prime factor $f(t)$ of $D(t,\zeta^i)$ is a prime factor of $\Delta_1^{(p^a)}(t)$ in $\mathbb{Q}(\zeta)[t^{\pm 1}]$. Since $\Delta_1^{(p^a)}(1) = \pm 1$, it follows that $\psi_p(f(1)) \equiv \pm 1 \bmod p$. But $|D(1,1)| = |\rho_\lambda(1)| = \lambda$, see Corollary 14.28. □

14.30 Proposition. *Let $\mathfrak{k}$ be a knot of period p^a, $a \ge 1$, p a prime. If $\Delta_1(t) \not\equiv 1 \bmod p$ and $\Delta_2(t) = 1$, the splitting field $\mathbb{Q}(\Delta_1)$ of $\Delta_1(t)$ over the rationals $\mathbb{Q}$ contains the p^a-th roots of unity.*

Proof. By Proposition 14.29, there is a $D(t,\zeta^i)$ which is non-constant. Hence it has a root $\alpha \in \mathbb{C}$ which is also a root of $\Delta_1(t)$. Since $\Delta_2(t) = 1$ it follows that α is not a root of $\Delta_1^{(p^a)}(t)$. Thus, there exists a uniquely determined equivalence class of representations $\delta_\alpha\colon \mathfrak{G} \to \mathfrak{C}^+$ of the knot group $\mathfrak{G}$ of $\mathfrak{k}$ into the group of similarities $\mathfrak{C}^+$ of the plane, see Proposition 14.6. If $D(\alpha,\zeta^i) = 0$, the fixed points $b_j(S_j)$ of

$$\delta_\alpha(S_j)\colon\ z \mapsto \alpha(z - b_j) + b_j$$

assigned to Wirtinger generators S_j are solutions of a linear system of equations with coefficient matrix $\bar{A}(\alpha)$, satisfying $b_j(z_j) = \zeta^i b_j(x_j)$; for the notation see (14.29). Thus the configuration of fixed points b_j associated to the symmetric projection of Figure 14.7 also shows a cyclic symmetry; its order is that of ζ^i. All representations are equivalent under similarities, and all configurations of fixed points are, therefore, similar. Since the b_j are solutions of the system of linear equations (14.2) in 14.5 for $t = \alpha$, $u_j = b_j$, they may be assumed to be elements of $\mathbb{Q}(\alpha)$. It follows that

$$b_j(z_j) b_j^{-1}(x_j) = \zeta^i \in \mathbb{Q}(\alpha).$$

We claim that there exists a representation δ_α such that the automorphism

$$r_*(\alpha)\colon\ \delta_\alpha(\mathfrak{G}) \to \delta_\alpha(\mathfrak{G})$$

induced by the rotation r has order p^a. If p^b, $b < a$, were the maximal order occurring for any δ_α, all (non-trivial) representations δ_α would induce non-trivial representations of the knot group $\mathfrak{G}^{(p^{a-b})}$ of the factor knot $\mathfrak{k}^{(p^{a-b})}$. Then α would be a root of $\Delta_2(t)$ by Proposition 14.29, contradicting $\Delta_2(t) = 1$. □

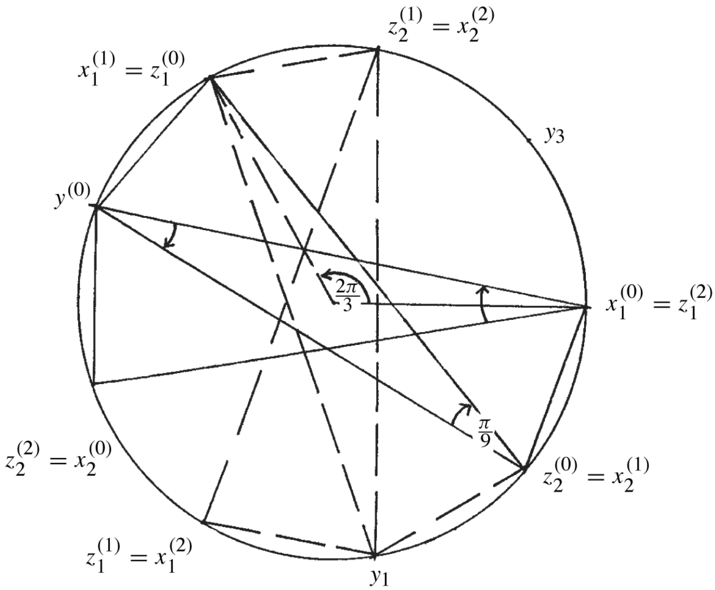


Figure 14.10. The fixed point configuration of the knot 9_1.

Figure 14.10 shows the fixed point configuration of the knot 9_1 as a knot of period three. One finds: $D(t,\tau) = t^3 + \tau$, $D(t,1) = \rho_2(t) \cdot \Delta_1^{(3)}(t)$, $\Delta_1^{(3)} = t^2 - t + 1$. For $\tau = e^{2\pi i/3}$, and $D(\alpha,\tau) = 0$, we get $\alpha = e^{-\pi i/9}$.

14.31 Corollary. *Let $\mathfrak{k}$ be a knot of period $q > 1$ with $\Delta_1(t) \neq 1$, $\Delta_2(t) = 1$. Then the splitting field of $\Delta_1(t)$ contains the q-th roots of unity or $\Delta_1(t) \equiv 1 \bmod p$ for some $p|q$.*

If $\mathfrak{k}$ is a non-trivial fibered knot of period q with $\Delta_2(t) = 1$, the splitting field of $\Delta_1(t)$ contains the q-th roots of unity [354]. □

The preceding proof contains additional information in the case of a prime period.

14.32 Corollary. *If $\mathfrak{k}$ is a knot of period p and $\Delta_1(\alpha) = 0$, $\Delta_1^{(p)}(\alpha) \neq 0$, then the p-th roots of unity are contained in $\mathbb{Q}(\alpha)$.*

Proof. There is a non-trivial representation δ_α of the knot group of $\mathfrak{k}$ with $b_j(z_j) = \zeta b_j(x_j)$, ζ a primitive p-th root of unity. □

As an application we prove:

14.33 Proposition. *The periods of a torus knot $\mathfrak{t}(a,b)$ are the divisors of a and b.*

Proof. By Example 9.15

$$\Delta_1(t) = \frac{(t^{ab}-1)(t-1)}{(t^a-1)(t^b-1)}, \quad \Delta_2(t) = 1.$$

From Corollary 14.31 we know that a period q of $\mathfrak{t}(a,b)$ must be a divisor of ab. Suppose $p_1p_2|q$, $p_1|a$, $p_2|b$ for two prime numbers p_1, p_2, then $\mathfrak{t}(a,b)$ has periods p_1, p_2, and Corollary 14.28 gives

$$(t^\lambda - 1)(t^{a'b}-1)^{p_1^c} \doteq (t^a-1)(t^b-1)[\rho_\lambda(t)\Delta_1^{(p_1^c)}(t)]^{p_1^c} \bmod p_1$$

with $a = p_1^c a'$, $\gcd(p_1, a') = 1$. Let ζ_0 be a primitive b-th root of unity. We have $\gcd(b, p_1) = 1$ and $\gcd(\lambda, p_1p_2) = 1$, hence $p_2 \nmid \lambda$. (See E 14.9.) The root ζ_0 has multiplicity s with $s \equiv 1 \mod p_1$ according to the right-hand side of the congruence, but since ζ_0 is not a λ-th root of unity, its multiplicity on the left-hand side ought to be $s \equiv 0 \mod p_1$. So there is no period q containing primes from both a and b.

It is evident that the divisors of a and b are actually periods of $\mathfrak{t}(a,b)$. □

There have been further contributions to this topic. In [218] the dihedral representations γ_p were exploited by U. Lüdicke. The periodicity of a knot is reflected in its invariant $[\lambda(\zeta)]$. In [263] K. Murasugi generalized these results, completed and formulated in terms of linking numbers of coverings. In addition, certain conditions involving the Alexander polynomial and the signature of a knot have been proved by C. M. Gordon, A. Litherland and K. Murasugi when a knot is periodic, see [136]. J. F. Davis and C. Livingston gave in [82] a more convenient formulation of the Murasugi conditions and a partial converse, showing that a polynomial which satisfies $\Delta_1(t) \equiv 1 \bmod q$ for some q is the polynomial of a knot with period q. Together all these criteria suffice to determine the periods of knots with less than ten crossings, see Table C.1. In the works of Kodama and Sakuma [201] and Henry and Weeks [161] the complete information on periods and symmetry group can be found up to 10 crossings, see also Cha and Livingston [65]. Many results on periodic knots carry over to links, see Knigge [200] and Sakuma [312].

It follows from Murasugi's congruence in 14.28 that a knot of period p^a either has Alexander polynomial $\Delta_1(t) \equiv 1 \bmod p$ or $\deg \Delta_1(t) \geq p^a - 1$. Thus a knot with $\Delta_1(t) \not\equiv 1$ can have only finitely many prime periods. No limit could be obtained for periods p^a, if $\Delta_1(t) \equiv 1 \mod p$. A fibered knot has only finitely many periods, since its Alexander polynomial is of degree $2g$ with a leading coefficient ± 1. It has been proved by E. L. Flapan [105] that only the trivial knot admits infinitely many periods. A new proof of this theorem and a generalization to links was proved by J. Hillman [170]. The generalization reads: A link with infinitely many periods consists of μ trivial components spanned by disjoint disks.

14.34 Two-bridge knots and knots with $\deg\Delta_1(t) = 2$. A simple observation shows that a 2-bridge knot $\mathfrak{b}(\alpha, \beta)$ has period two and its factor knot is trivial (see E 14.13).

Hence the Alexander polynomial $\Delta_{\alpha,\beta}(t)$ of the 2-bridge knot $\mathfrak{b}(\alpha,\beta)$ satisfies

$$\Delta_{\alpha,\beta}(t) \equiv \rho_\lambda(t) \bmod 2$$

for some odd integer λ. Moreover, every Alexander polynomial of degree two is the Alexander polynomial of a genus one 2-bridge knot (see Lemma 12.19 and E 9.6).

A quadratic polynomial $\Delta_1(t)$ cannot be the product of two non-trivial knot polynomials. Hence if $\Delta_1(t)$ is the Alexander polynomial of a periodic knot then equation (14.32) and Remark 14.27 imply that $\Delta_1(t) \doteq \Delta_1^{(q)}(t)$ or $\Delta_1^{(q)}(t) \doteq 1$. If $q = p^a$ is a prime power and if $\Delta_1(t) \doteq \Delta_1^{(q)}(t)$ then Murasugi's congruence 14.28 shows that

$$1 \doteq \prod_{i=1}^{q-1} D(t,\zeta^i) \equiv \big(\Delta_1^{(q)}(t)\big)^{p^a-1}(\rho_\lambda(t))^{p^a-1} \bmod p\,.$$

This is only possible if $\Delta_1(t) \equiv 1 \bmod p$. In the case $\Delta_1^{(q)}(t) \doteq 1$ the Murasugi's congruence 14.28 shows that

$$\Delta_1(t) \equiv (\rho_\lambda(t))^{p^a-1} \bmod p.$$

Hence, if $\Delta_1(t) \not\equiv 1 \bmod p$ then only $(\lambda, p^a) = (2,3)$ and $(\lambda, p^a) = (3,2)$ are possible. Furthermore, if the period is three then it follows that

$$\Delta_1(t) \mathrel{\dot{\equiv}} (t+1)^2 \mathrel{\dot{\equiv}} t^2 - t + 1 \quad \bmod 3.$$

Corollary 14.32 yields further information: If $\mathfrak{k}$ has period three, its Alexander polynomial has the form

$$\Delta_1(t) = nt^2 + (1-2n)t + n,\ n = 3m(m+1)+1,\ m = 0,1,\ldots,$$

see E 14.14.

There are, in fact, symmetric knots which have these Alexander polynomials, the pretzel knots $\mathfrak{p}(2m+1, 2m+1, 2m+1)$, Figure 14.11. Their factor knot $\mathfrak{p}^{(3)}$ is trivial. One obtains

$$\begin{aligned} &D(t,\tau) = (\tau + n(\tau-1))t + n(1-\tau) + 1,\\ &D(t,1) = 1+t, D(t,\tau) = 1+\tau,\ \text{hence } \lambda = 2, \end{aligned}$$

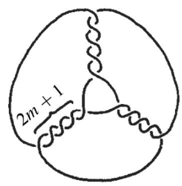


Figure 14.11. The pretzel knots $\mathfrak{p}(2m+1, 2m+1, 2m+1)$.

$$D(t,\zeta)D(t,\zeta^{-1}) = \Delta_1(t),\ \zeta \text{ a primitive third root of unity.}$$

(We omit the calculations, see exercise E 14.14.) $\mathfrak{p}(1,1,1)$ is the trefoil, $\mathfrak{p}(3,3,3) = 9_{35}$.

14.35. The different criteria or a combination of them can be applied to exclude periods of given knots. As an example consider the 2-bridge knot $\mathfrak{k} = 8_{11}$. Its polynomials are $\Delta_1(t) = (t^2-t+1)(2t^2-5t+2)$, $\Delta_2(t) = 1$. The splitting field of $\Delta_1(t)$ obviously contains the third roots of unity. The roots of the second factor are 2 and $1/2$ and hence by Corollary 14.32 the polynomial $(2t^2-5t+2)$ must be a factor of $\Delta^{(3)}(t)$. Now, Murasugi's congruence excludes this case: $\Delta_1(t) \doteq t^4+t^3+t+1 \equiv (1+t)^4 \bmod 3$, and the congruence $(1+t)^4 \doteq (\Delta^{(3)}(t))^3(\rho_\lambda(t))^2 \bmod 3$ is not possible.

Figure 14.12 shows symmetric versions of the knots of period three with less than ten crossings, 9_{35}, 9_{40}, 9_{41}, 9_{47}, 9_{49}. (The torus knots are omitted, $\mathfrak{t}(4,3) = 8_{19}$, $\mathfrak{t}(5,3) = 10_{124}$ and $\mathfrak{t}(2m+1,2)$, $1 \le m \le 4$.)

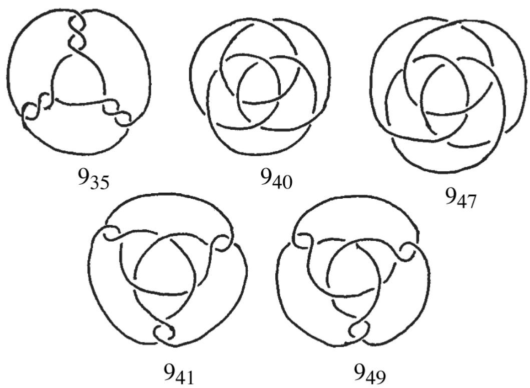


Figure 14.12. Non-torus knots of period three with less than ten crossings.

We conclude this section by showing that the condition $\Delta_2(t) = 1$ cannot be omitted in the hypotheses of Corollary 14.31. The 'rosette'-knot 8_{18} evidently has period four. The Alexander polynomials are $\Delta_1(t) = (1-t+t^2)^2(1-3t+t^2)$, $\Delta_2(t) = (1-t+t^2)$. One has $D(t,\tau) = \tau t^2 + (\tau^2-\tau+1)t + \tau$. It follows that $D(t,1) = 1+t+t^2 = \rho_3(t)$, $\Delta_1^{(4)}(t) = 1$, $D(t,-1) = 1-3t+t^2$, $D(t,\pm i) = \pm i(1-t+t^2)$. The representations δ_α, $D(\alpha,i) = \Delta_2(\alpha) = 0$, are not unique. $1-3\beta+\beta^2 = 0$ yields unique representations with period 2. In fact, the splitting fields $\mathbb{Q}(\Delta_1(t))$ do not contain i (see also [354]). Nevertheless, the condition $\Delta_2(t) = 1$ can be replaced by a more general one involving higher Alexander polynomials [169].

14.E History and sources

It seems to have been J.W. Alexander who first used homomorphic images of knot groups to obtain effectively calculable invariants [8]. The groups $\mathfrak{G}/\mathfrak{G}''$ resp. $\mathfrak{G}'/\mathfrak{G}''$, ancestral to all metabelian representations, have remained the most important source of knot invariants.

In Reidemeister's book [296, 303, III, §14] a representation of the group of alternating pretzel knots onto Fuchsian groups is used to classify these knots. This representation is not metabelian but, of course, is restricted to a rather special class of groups. It was repeatedly employed in the years to follow to produce counterexamples concerning properties which escape Alexander's invariants. Seifert employed it [328] to prove that a certain pretzel knot with the same Alexander invariants as the trivial knot is non-trivial – shattering all hopes of classifying knot types by these invariants. H. F. Trotter [356] used it to show that non-invertible knots (pretzel knots) exist. The natural class of knots to which the method developed for pretzel knots can be extended is the class of Montesinos knots (Chapter 12).

R. H. Fox drew attention to a special case of metabelian representations – the metacyclic ones. Here the image group could be chosen finite. (Compare also R. Hardley [146].) A lifting process of these representations obtained by abelianizing is kernel yielded a further class of non-metabelian representations (see G. Burde [52, 53] and R. Hardley [150]).

A class of representations of fundamental importance in the theory of 3-manifolds was introduced by R. Riley (Riley-reps), [306, 308, 307]. The image groups are discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$, and they can be understood as groups of orientation preserving motions of hyperbolic 3-space. A. Marden's book [228] gives an introduction to hyperbolic 3-manifolds.

The theory of homomorphisms onto the finite groups $\mathrm{PSL}(2,p)$ over a finite field $\mathbb{Z}_p$ has not been considered in this book, see W. Magnus and A. Peluso [225], R. Riley [305], R. Hartley and K. Murasugi [152].

14.F Exercises

E 14.1. Show that the group of symmetries of a regular a-gon is the image of a dihedral representation γ_a^* of the knot group of the torus knot $\mathfrak{t}(a,2)$. Give an example of a torus knot that does not allow a dihedral representation.

E 14.2. Prove Remark 14.25.

E 14.3. Let $\delta_a\colon \mathfrak{G} \to \mathfrak{C}^+$ be a representation into the group of similarities (14.6) of the group $\mathfrak{G}$ of a knot $\mathfrak{k}$, and $\{b_j\}$ the configuration of fixed points in $\mathbb{C}$ corresponding to Wirtinger generators S_j of a regular projection $p(\mathfrak{k})$. Show that one obtains a representation δ_a^* of $\mathfrak{k}^*$ with a fixed point configuration $\{b_j'\}$ resulting from $\{b_j\}$ by reflection in a line.

E 14.4. (a) Let $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ be a product knot and $\Delta_1^{(1)}(t) \neq 1$, $\Delta_1^{(2)}(t) \neq 1$ be the Alexander polynomials of its summands. Show that there are non-equivalent representations δ_α for $\Delta_1^{(1)}(\alpha) = \Delta_1^{(2)}(\alpha) = 0$. Derive from this that $\Delta_2(\alpha) = 0$.

(b) Consider a regular knot projection $p(\mathfrak{k})$ and a second projection $p^*(\mathfrak{k})$ in the same plane E obtained from a mirror image $\mathfrak{k}^*$ reflected in a plane perpendicular to E. Join two corresponding arcs of $p(\mathfrak{k})$ and $p^*(\mathfrak{k})$ as shown in Figure 14.13 one with an n-twist and one without a twist – the resulting projection is that of a *symmetric union* $\mathfrak{k} \cup \mathfrak{k}^*$ of $\mathfrak{k}$ [196]. Show that a representation δ_a for $\mathfrak{k}$ can always be extended to a representation δ_a for the symmetric union, hence, that every root of the Alexander polynomial of $\mathfrak{k}$ is a root of that of $\mathfrak{k} \cup \mathfrak{k}^*$. (Use E 14.3.)

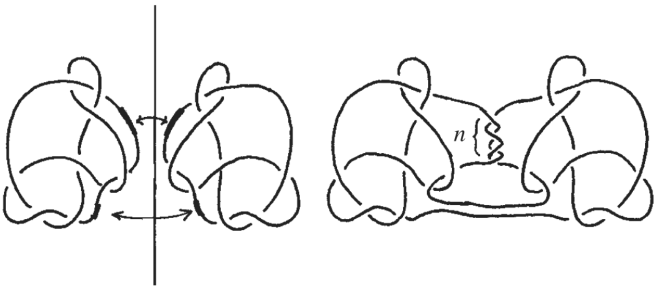


Figure 14.13. The symmetric union of $\mathfrak{k}$.

E 14.5. Compute the representations γ_a for torus knots $\mathfrak{t}(a,2)$ that lift the dihedral representations γ_a^* of E 14.1, see 14.13. Show that $[\lambda(\zeta)] = \{2a\}$. Derive from this that $\mathfrak{t}(a,2) \# \mathfrak{t}(a,2)$ and $\mathfrak{t}(a,2) \# \mathfrak{t}^*(a,2)$ have non-homeomorphic complements but isomorphic groups.

E 14.6. (Henninger) Let $\gamma_p\colon \mathfrak{G} \to \mathfrak{B}$ be a normalized representation according to Proposition 14.13, $\gamma_p(S_1)\colon z \mapsto \bar{z} + 1$, $\gamma_p(S_2)\colon z \mapsto \zeta^2\bar{z} + \zeta$, with ζ a primitive p-th root of unity. Show that $\gamma_p(\mathfrak{G}) \cong \mathfrak{D}_p \ltimes \mathbb{Z}^{p-1}$. (Hint: use a translation of the plane by $\sum_{j=0}^{\frac{p-3}{2}} \zeta^{2j+1} + \sum_{j=1}^{\frac{p-1}{2}} \zeta^{2j}$.)

E 14.7. Compute the matrix (μ_{ij}) of linking numbers (Proposition 14.13 (b)) of the irregular covering $\hat{I}_{15}$ of 9_2. Compare the invariants $\frac{1}{2}\sum_{j\neq i} |\mu_{ij}| = \nu_i$, $0 \leq i \leq 7$ with those of 7_4.

(Result: 7, 6, 5, 4, 4, 3, 2, 1, [296].)

E 14.8. If a knot has period q as an unoriented knot, it has period q as an oriented knot. Show that the axis of a rotation through angle π which maps $\mathfrak{k}$ onto $-\mathfrak{k}$ must meet $\mathfrak{k}$.

E 14.9. Let $\mathfrak{k}$ be a knot of period q and h the axis of the rotation. Prove that $\gcd(\mathrm{lk}(h, \mathfrak{k}), q) = 1$.

E 14.10. Produce a matrix W over $\mathbb{Q}(\zeta)$ such that $W Z_q W^{-1} = Z(\zeta)$,

$$Z_q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & \dots & 0 & 1 \\ 1 & 0 & & \dots & 0 \end{pmatrix}, Z(\zeta) = \begin{pmatrix} 1 & & & \mathbf{0} \\ & \zeta & & \\ & & \ddots & \\ \mathbf{0} & & & \zeta^{q-1} \end{pmatrix},$$

ζ a primitive q-th root of unity.

E 14.11. Prove that $\prod_{i=1}^{q-1} D(t, \zeta^i)$ is a knot polynomial i.e. it is a symmetric polynomial with $|\prod_{i=1}^{q-1} D(1, \zeta^i)| = 1$.

E 14.12. We call an *oriented tangle* $\mathfrak{T}_n$ *circular*, if its arcs have an even number of boundary points $X_1, \dots, X_n, Z_1, \dots, Z_n$ which can be joined pairwise (Figure 14.14) to give an oriented knot $\mathfrak{k}(\mathfrak{T}_n)$, inducing of $\mathfrak{T}_n$ the original orientation. A q-periodic knot $\mathfrak{k}$ may be obtained by joining q circular tangles $\mathfrak{T}_n$; the knot $\mathfrak{k}(\mathfrak{T}_n)$ is then the factor knot $\mathfrak{k}^{(q)} = \mathfrak{k}(\mathfrak{T}_n)$, see Figure 14.7. A circular tangle defines a polynomial $D(t, \tau)$, see Proposition 14.24.

(a) Show $D(t + \tau) = t + \tau$ for the circular tangle $\mathfrak{T}_2$ with one crossing and compute $\Delta_1(t) = \prod_{i=1}^{q-1}(t + \zeta^i)$, ζ a primitive q-th root of unity, q odd. $\Delta_1(t)$ is the Alexander polynomial of $\mathfrak{t}(q, 2)$.

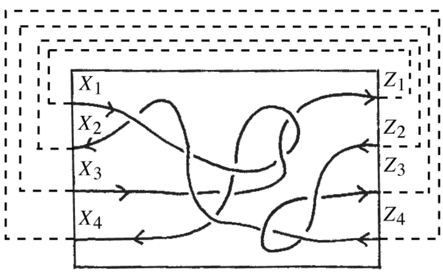


Figure 14.14. A circular tangle.

(b) Find all circular tangles with less than four crossings. Construct knots of period ≤ 4 by them.

E 14.13. Use Schubert's normal form (see Section 12.A) to prove that a 2-bridge knot $\mathfrak{b}(\alpha, \beta)$ has period two and that its factor knot is trivial.

E 14.14. If the Alexander polynomial $\Delta_1(t)$ of a periodic knot of period three is quadratic, it has the form

$$\Delta_1(t) = nt^2 + (1-2n)t + n,\ n = 3m(m+1) + 1,\ m = 0, 1, \dots.$$

Prove that the pretzel knot $\mathfrak{p}(2m+1, 2m+1, 2m+1)$ has this polynomial as $\Delta_1(t)$. Hint: Compute $D(\tau, t)$.

E 14.15. (Lüdicke [219]). Let $\mathfrak{k}$ be a knot with prime period q. Suppose there is a unique dihedral presentation $\gamma_p^*\colon \mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$ of its group, and $p \nmid \Delta_1^{(q)}(-1)$.

Then either $q = p$ or $q | p - 1$.

Chapter 15

Knots, knot manifolds, and knot groups

The long-standing problem concerning the correspondence between knots and their complements was solved by C. M. Gordon and J. Luecke [137]: “Knots are determined by their complements”. (See Theorem 3.19). The proof of the theorem is beyond the scope of this volume.

The main object of this chapter will be the relation between knot complements and their fundamental groups.

A consequence of the famous theorem of F. Waldhausen [367] (see Appendix B.7) on sufficiently large irreducible 3-manifolds is that the complements of two knots are homeomorphic if there is an isomorphism between the fundamental groups preserving the peripheral group system. We study to what extent the assumption concerning the boundary is necessary.

In Section 15.A we describe examples which show that there are links of two components which are not determined by their complements, and that there are non-homeomorphic knot complements with isomorphic groups. In Section 15.B we investigate Property P for special knots. In Section 15.C we discuss the relation between the complement and its fundamental group for prime knots and in Section 15.D for composite knots.

15.A Examples

The following example of J. H. C. Whitehead [370] shows that, in general, the complement of a link does not characterize the link.

15.1 Proposition (Whitehead). *Let* $\mathfrak{l}_n$, $n \in \mathbb{Z}$ *denote the link consisting of a trivial knot* $\mathfrak{k}$ *and the* n*-twist knot* $\mathfrak{d}_n$, *see Figure 15.1. Then:*
(a) *The links* $\mathfrak{l}_{2n}$ *and* $\mathfrak{l}_{2m}$ *are not isotopic if* $n \neq m$.
(b) $S^3 - \mathfrak{l}_{2n} \cong S^3 - \mathfrak{l}_0$ *for all* $n \in \mathbb{Z}$.

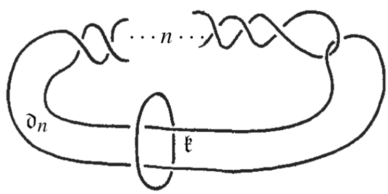


Figure 15.1

Proof. By E 9.6, the Alexander polynomial of $\mathfrak{d}_{2n}$ is $nt^2 + (1-2n)t + n$; hence, $\mathfrak{d}_{2n} = \mathfrak{d}_{2m}$ only if $n = m$.

To prove (b) take an unknotted solid torus V and the trivial doubled knot $\mathfrak{d}_0 \subset V$ "parallel" to the core of V. $W = \overline{S^3 - V}$ is a solid torus with core $\mathfrak{k}$ and $W - \mathfrak{k} \cong \partial W \times [0,1) = \partial V \times [0,1)$. Consider the following homeomorphism $V \to V$: cut V along a meridional disk, turn it $|n|$ times through 2π in the positive sense if $n > 0$, in the negative sense if $n < 0$ and glue the disks together again. This twist maps $\mathfrak{d}_0$ to $\mathfrak{d}_{2n}$. The map can be extended to $W - \mathfrak{k} \cong \partial V \times [0,1) = (\overline{S^3 - V}) - \mathfrak{k}$ to get the desired homeomorphism. □

For later use we determine from Figure 15.2 and Figure 15.3 the group and peripheral system of the twist knots $\mathfrak{d}_n$, following Bing and Martin [22]. (See E 3.6.)

15.2 Lemma. *The twist knot $\mathfrak{d}_n$ has the following group $\mathfrak{T}_n$ and peripheral system.*

(a) $\mathfrak{T}_{2m} = \langle a, b \mid b^{-1}(a^{-1}b)^m a (a^{-1}b)^{-m} a (a^{-1}b)^m a^{-1} (a^{-1}b)^{-m} \rangle$, *meridian* $m = a$, *longitude* $\ell = (a^{-1}b)^m a^{-1} (a^{-1}b)^{-2m-1} a^{-1} (a^{-1}b)^m a^2$

(b) $\mathfrak{T}_{2m-1} = \langle a, b \mid b^{-1}(a^{-1}b)^m b^{-1} (a^{-1}b)^{-m} a (a^{-1}b)^m b (a^{-1}b)^{-m} \rangle$, *meridian* $m = b$, *longitude* $(a^{-1}b)^{-m} b (a^{-1}b)^{2m-1} b (a^{-1}b)^{-m} b^{-2}$.

Proof. For $n = 2m$ the Wirtinger generators are drawn in Figure 15.2.

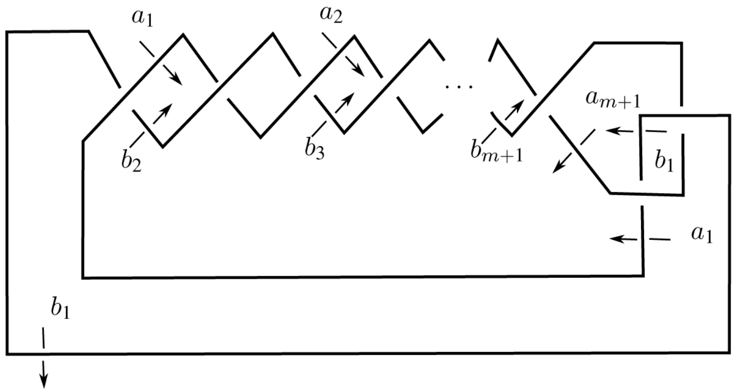


Figure 15.2. The twist knot $\mathfrak{d}_{2m}$.

We obtain the defining relations (here $a = a_1, b = b_1$):

$$\begin{aligned} b_2 &= a_1^{-1} b_1 a_1 = a^{-1}ba \\ a_2 &= b_2 a_1 b_2^{-1} = (a^{-1}b)a(a^{-1}b)^{-1} \\ b_3 &= a_2^{-1} b_2 a_2 = (a^{-1}b)^2 (a^{-1}b)^{-2} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} b_{m+1} &= a_m^{-1} b_m a_m = (a^{-1}b)^m b (a^{-1}b)^{-m} \\ a_{m+1} &= b_{m+1} a_m b_{m+1}^{-1} = (a^{-1}b)^m a (a^{-1}b)^{-m} \\ b_1 &= b = a_{m+1} a_1 a_{m+1}^{-1} = (a^{-1}b)^m a (a^{-1}b)^{-m} a (a^{-1}b)^m a^{-1} (a^{-1}b)^{-m}\,. \end{aligned}$$

For $n = 2m-1$ the last two relations from above must be replaced by one relation

$$b = b_{m+1}^{-1} a b_{m+1} = (a^{-1}b)^m b^{-1} (a^{-1}b)^{-m} a (a^{-1}b)^m b (a^{-1}b)^{-m}$$

(see Figure 15.3).

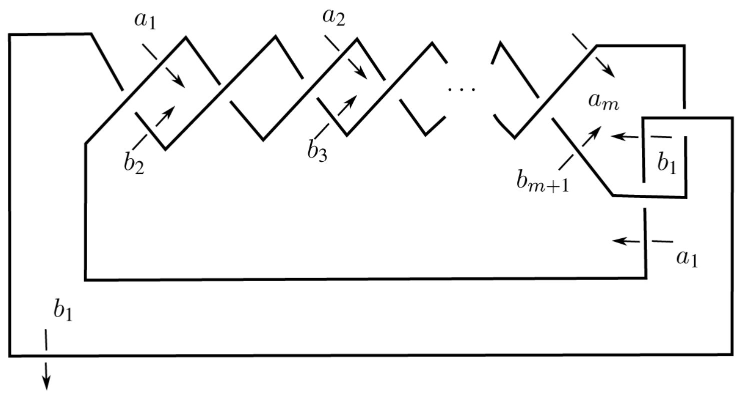


Figure 15.3. The twist knot $\mathfrak{d}_{2m-1}$.

For the calculation of the longitude we use the formulas

$$\begin{aligned} a_1 \ldots a_m &= b^m (a^{-1}b)^{-m}, \\ b_m \ldots b_2 &= (a^{-1}b)^{m-1} a^{m-1}. \end{aligned}$$

A longitude of $\mathfrak{d}_{2m}$ associated to the meridian a is given by

$$\begin{aligned} & a_{m+1}^{-1} a_1 a_2 \ldots a_m b_1^{-1} b_{m+1} \ldots b_2 a^{2-2m} \\ &= (a^{-1}b)^m a^{-1} (a^{-1}b)^{-m} b^m (a^{-1}b)^{-m} b^{-1} (a^{-1}b)^m a^m a^{2-2m} \\ &= a^m (a^{-1}b)^m a^{-1} (a^{-1}b)^{-2m-1} a^{-1} (a^{-1}b)^m a^{2-m}; \end{aligned}$$

for the last step we applied the defining relation from Lemma 15.2 (a) and replaced b^m by a conjugate of a^m and b^{-1} by $(a^{-1}b)^{-1}a^{-1}$. Since the longitude commutes with the meridian a we get the expression in Lemma 15.2 (a).

For $\mathfrak{d}_{2m-1}$ a longitude associated to the meridian b is given by

$$\begin{aligned} & a_1 a_2 \ldots a_m b_1 b_m b_{m-1} \ldots b_2 b_{m+1} b_1^{-1-2m} \\ &= b^m (a^{-1}b)^{-m} b (a^{-1}b)^{m-1} a^{m-1} (a^{-1}b)^m b (a^{-1}b)^{-m} b^{-1-2m} \\ &= b^m (a^{-1}b)^{-m} b (a^{-1}b)^{2m-1} b (a^{-1}b)^{-m} b^{-2-m}; \end{aligned}$$

here we used the relation from Lemma 15.2 (b) and replaced a^{m-1} by a conjugate of b^{m-1}. $\square$

As we have pointed out in Theorem 3.16, the results of [367] imply that the peripheral system determines the knot up to isotopy and the complement up to orientation preserving homeomorphisms. A knot and its mirror image have homeomorphic complements; however, if the knot is not amphicheiral every homeomorphism of S^3 taking the knot onto its mirror image is orientation reversing. Using this, one can construct non-homeomorphic knot complements which have isomorphic groups:

15.3 Example (Fox [116]). The knots $\mathfrak{k} \# \mathfrak{k}^*$ and $\mathfrak{k} \# \mathfrak{k}$ where $\mathfrak{k}$ is a trefoil are known as the *square* and the *granny knot* (see Figure 15.4). They are different knots by Schubert's theorem on the uniqueness of the prime decomposition of knots, see Theorem 7.12, and their complements are not homeomorphic. This is a consequence of Theorem 15.10. The first proof of this fact was given by R. H. Fox [116] who showed that the peripheral systems of the square and granny knots are different. We derive it from E 14.5: the longitudes ℓ and ℓ' are mapped by a normalized presentation γ_p, $p = 3$, onto $12 = 6 + 6$ resp. $0 = 6 - 6$, compare E 14.5 and Fox [116]. Their groups, though, are isomorphic by E 7.5.

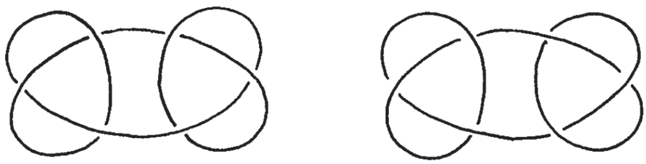


Figure 15.4. The square and the granny knot.

15.B Property P for special knots

For torus and twist knots suitable presentations of the groups provide a means to prove Property P. This method, however, reflects no geometric background. For product knots and satellite knots a nice geometric approach gives Property P. The results and methods of this section are mainly from Bing and Martin [22].

15.4 Definition (Knot type). The unoriented knots $\mathfrak{k}_1$, $\mathfrak{k}_2$ are of the *same knot type* if there is a homeomorphism $h\colon S^3 \to S^3$ with $h(\mathfrak{k}_1) = h(\mathfrak{k}_2)$.

(The homeomorphism h may be orientation reversing.)

Recall the definition of $S^3_{r/n}(\mathfrak{k})$ from Section 3.C: If $\mathfrak{k}$ is a knot with meridian m and longitude ℓ then

$$S^3_{r/n}(\mathfrak{k}) = C(\mathfrak{k}) \cup_f V'$$

where V' is a solid torus and $f\colon V' \to \partial C(\mathfrak{k})$ a homeomorphism which maps the meridian m' of V' to the curve $f(m') = m^p \ell^q$, see Definition 3.21.

Thus $H_1(S^3_{r/n}(\mathfrak{k})) = \mathbb{Z}_{|r|}$ and the knot $\mathfrak{k}$ has Property P, see Definition 3.26, if and only if $\pi_1 S^3_{r/n}(\mathfrak{k}) = 1$ implies $n = 0$.

Let $\mathfrak{k}_1$, $\mathfrak{k}_2$ be two knots with homeomorphic complements $C(\mathfrak{k}_1) \cong C(\mathfrak{k}_2)$. Now Proposition 3.24 implies that if one of the knots $\mathfrak{k}_i$, $i = 1, 2$ has Property P then $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are of the same knot type.

15.5 Proposition (Hempel [158]). *Torus knots have Property P.*

Proof. By Proposition 3.38, $\mathfrak{G}(a,b) = \langle u, v \mid u^a v^{-b} \rangle$, $m = v^d u^{-c}$, $\ell = u^a m^{-ab}$ and

$$\begin{aligned}&\pi_1 S^3_{1/n}(\mathfrak{t}(a,b)) = \langle\, u, v \mid u^a v^{-b}, v^d u^{-c} (u^a (v^d u^{-c})^{-ab})^n \,\rangle,\\ &(|a|, |b| > 1, ad + bc = 1)\end{aligned}$$

and we have to show that this group is trivial only for $n = 0$. By adding the relation u^a we obtain the factor group

$$\langle\, u, v \mid u^a, v^b, (v^d u^{-c})^{1-nab} \,\rangle = \langle\, \tilde{u}, \tilde{v} \mid \tilde{u}^a, \tilde{v}^b, (\tilde{u}\tilde{v})^{1-nab} \,\rangle$$

with $\tilde{u} = u^c$, $\tilde{v} = v^{-d}$. For $n \neq 0$ this is a non-trivial triangle group, see [382, p. 124], since $|1 - nab| > 1$. □

In the proof of Property P for twist knots we construct homeomorphisms onto the so-called *Coxeter groups*, and in the next lemma we convince ourselves that the Coxeter groups are non-trivial.

15.6 Lemma (Coxeter [73]). *The Coxeter group*

$$\mathfrak{A} = \langle\, x, y \mid x^3, y^s, (xy)^3, (x^{-1}y)^r \,\rangle$$

is not trivial when, $s, r \geq 3$.

Proof. Introducing $t = xy$ and eliminating y gives $\mathfrak{A} = \langle\, t, x \mid x^3, t^3, (x^{-1}t)^s, (xt)^r \,\rangle$. We assume that $3 \leq s \leq r$; otherwise replace x by x^{-1}.

We choose a complex number c such that

$$c\bar{c} = 4\cos^2\frac{\pi}{r} \quad \text{and } c + \bar{c} = 4\cos^2\frac{\pi}{s} - 4\cos^2\frac{\pi}{r} - 1.$$

This choice is always possible if $r \geq s \geq 3$, see Figure 15.5. Let X, T be the following 3×3 matrices:

$$X = \begin{pmatrix} 1 & c & c+1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1+\bar{c} & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}.$$

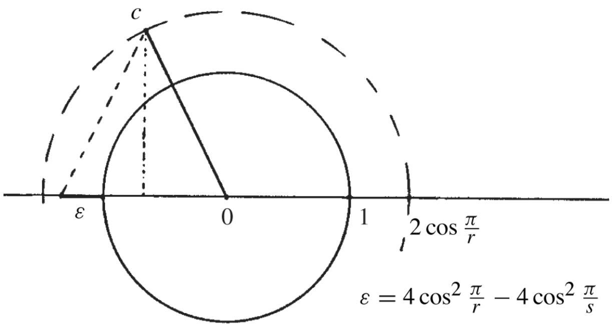


Figure 15.5

Then

$$XT = \begin{pmatrix} c\bar{c}-1 & c & 0 \\ -\bar{c} & -1 & 0 \\ 1+\bar{c} & 1 & 1 \end{pmatrix}, \quad X^{-1}T = \begin{pmatrix} c+\bar{c}+c\bar{c} & c+1 & c+1 \\ -1 & 0 & -1 \\ -\bar{c} & -1 & 0 \end{pmatrix}.$$

The characteristic polynomials are

$$\begin{aligned} P_X(\lambda) &= 1-\lambda^3, \quad P_T(\lambda) = 1-\lambda^3, \\ P_{XT}(\lambda) &= 1-(c\bar{c}-1)\lambda + (c\bar{c}-1)\lambda^2 - \lambda^3 \\ &= (\lambda-1)(\lambda^2 - 2\lambda\cos^2\frac{\pi}{r} + 1), \\ P_{X^{-1}T}(\lambda) &= 1-(c+\bar{c}+c\bar{c})\lambda + (c+\bar{c}+c\bar{c})\lambda^2 - \lambda^3 \\ &= -(\lambda-1)(\lambda^2 - 2\lambda\cos\frac{2\pi}{s} + 1). \end{aligned}$$

The roots of the last two polynomials are $1, e^{\pm 2\pi i/r}$ and $1, e^{\pm 2\pi i/s}$, respectively. This proves that $P_{XT} \mid \lambda^r - 1$ and $P_{X^{-1}T} \mid \lambda^s - 1$. By the Cayley–Hamilton Theorem the minimal polynomial of matrix divides its characteristic polynomial, hence it follows that $X^3, T^3, (XT)^r$ and $(X^{-1}T)^s$ are unit matrices. So X, T generate a non-trivial homomorphic image of $\mathfrak{A}$. □

15.7 Theorem (Bing–Martin). *The twist knot $\mathfrak{d}_n$, $n \neq 0, -1$, has Property P. In particular, the figure-eight knot $4_1 = \mathfrak{d}_2$ has Property P.*

Proof. We use the presentation 15.2 (a). Define $w = a^{-1}b$ and replace b by aw. Then

$$\mathfrak{T}_{2m} = \langle a, w \mid (aw)^{-1}w^m a w^{-m} a w^m a^{-1} w^{-m} \rangle \tag{15.1}$$

and, introducing $k = aw^{-m}$ instead of $a = kw^m$,

$$\mathfrak{T}_{2m} = \langle k, w \mid w^{-2m-1}k^{-1}w^m k^2 w^m k^{-1} \rangle. \tag{15.2}$$

The longitude is

$$\begin{aligned}\ell &= (a^{-1}b)^m a^{-1}(a^{-1}b)^{-2m-1}a^{-1}(a^{-1}b)^m a^2\\ &= w^m a^{-1} w^{-1-2m} a^{-1} w^m a^2.\end{aligned}$$

Since $m = a$ is a meridian, the $1/n$-surgery on $\mathfrak{d}_{2m}$ gives an additional relation

$$\ell^n m = (w^m a^{-1} w^{-1-2m} a^{-1} w^m a^2)^n a = 1,$$

or,

$$(k^{-1}w^{-1-3m}k^{-1}w^m k w^m k w^m)^n k w^m = 1.$$

Therefore

$$\begin{aligned}\mathfrak{H}_{2m,n} &= \pi_1 S^3_{1/n}(\mathfrak{d}_{2m})\\ &= \langle\, k, w \mid k w^{2m+1} k \cdot (w^m k^2 w^m)^{-1},\\ &\qquad (k^{-1}w^{-1-3m}k^{-1}w^m k w^m k w^m)^n k w^m \,\rangle.\end{aligned} \tag{15.3}$$

We introduce in $\mathfrak{H}_{2m,n}$ the additional relations $w^{3m+1} = 1$, $k^3 = 1$. Then the relations of (15.3) turn into $(kw^{-m})^3 = 1$, $(kw^m)^{3n+1} = 1$, and, with $v = k$ and $u = w^m$, the factor group has the presentation

$$\langle\, u, v \mid u^{3m+1}, v^3, (uv^{-1})^3, (uv)^{3n+1} \,\rangle.$$

By Lemma 15.6 this Coxeter group is not trivial if $m \neq 0$ and $|3n+1| > 2$. The latter condition is violated only if $n = 0, -1$.

For $n = -1$ the group is

$$\mathfrak{H}_{2m,-1} = \langle\, k, w \mid k w^{2m+1}(w^m k^2 w^m)^{-1},\ w^{-m}k^{-1}w^{-m}k w^{3m+1}k \,\rangle.$$

By $w \mapsto y^{-6}$, $k \mapsto yx^{-1}$, we obtain an epimorphism of $\mathfrak{H}_{2m,-1}$ to the triangle group $\langle\, x, y \mid y^{6m+1}, x^3, (xy)^2 \,\rangle$ since

$$\begin{aligned}&yx^{-1}y^{-12m-6}yx^{-1}y^{6m}xy^{-1}xy^{-1}y^{6m} = yx^{-1}y^{-3}x^{-1}xy^{-1}xy^{-1}y^{-1} =\\ &yx^{-1}y^{-2}x^2y^{-1}xy^{-2} = yx^{-1}y^{-1}x^2y^{-2} = yx^{-1}y^{-1}x^{-1}y^{-2} = yy^{-1} = 1,\end{aligned}$$

and

$$y^{6m}xy^{-1}y^{6m}yx^{-1}y^{18m-6}yx^{-1} = y^{-1}xy^{-1}x^{-1}y^{-2}x^{-1} = y^{-1}x^2y^{-1}x^{-1} = 1.$$

The triangle group is not trivial, see [382, p. 124].

Next we consider $\mathfrak{d}_{2m-1}$. To achieve a more convenient presentation we define $w = a^{-1}b$. Furthermore we substitute $k = bw^{-m}$ and eliminate b by kw^m. Then we obtain from Lemma 15.2 (b)

$$\begin{aligned}\mathfrak{T}_{2m-1} &= \langle\, b, w \mid b^{-1}w^m b^{-1} w^{-m} b w^{-1+m} b w^{-m} \,\rangle\\ &= \langle\, k, w \mid w^{-m}k^{-2}w^{-m}k w^{-1+2m}k \,\rangle.\end{aligned}$$

The longitude is

$$\ell = w^{-m} b w^{2m-1} b w^{-m} b^{-2} = w^{-m} k w^{3m-1} k w^{-m} k^{-1} w^{-m} k^{-1}.$$

Thus

$$\begin{aligned}\mathfrak{H}_{2m-1,n} =& \pi_1 S^3_{1/n}(\mathfrak{d}_{2m-1}) \\ =& \langle\, k, w \mid w^{-m} k^{-2} w^{-m} k w^{2m-1} k, \\ & \qquad (w^{-m} k w^{3m-1} k w^{-m} k^{-1} w^{-m} k^{-1})^n k w^m \,\rangle.\end{aligned}$$

Adding the relations w^{3m-1}, k^3 we obtain the group

$$\begin{aligned}&\langle\, k, w \mid k^3, w^{3m-1}, (k w^{-m})^3, (w^{-m} k^{-1})^{3n-1} \,\rangle \\ &\qquad = \langle\, x, y \mid x^{3m-1}, y^3, (x y^{-1})^3, (x y)^{3n-1} \,\rangle\end{aligned}$$

with $x = w^{-m}$, $y = k^{-1}$.

By Lemma 15.6 this group is not trivial unless $|3m-1| \le 2$ or $|3n-1| \le 2$, that is, unless $m, n = 0, 1$. For $m = 0$ we get the trivial knot and this case was excluded. In the case $m = 1$ the knot $\mathfrak{d}_1$ is the trefoil which has Property P by Proposition 15.5. So we may assume that $|3m-1| \ge 3$. For $n = 1$

$$\mathfrak{H}_{2m-1,1} = \langle\, k, w \mid w^{-m} k^{-2} w^{-m} k w^{2m-1} k,\ w^{-m} k w^{3m-1} k w^{-m} k^{-1} \,\rangle.$$

The relations are the equations

$$k w^{2m-1} k = w^m k^2 w^m,\ w^m k w^m = k w^{3m-1} k.$$

We rewrite the first as

$$(w^m k w^m) w^{-2m} (w^m k w^m) = k w^{2m-1} k$$

and substitute the second in this expression to obtain

$$k w^{-2m} k = w^{1-4m}, k w^{3m-1} k = w^m k w^m.$$

Put $k = x w^m$. Now the defining equations are

$$x w^{-m} x w^{-m} = w^{1-6m},\ w^{1-6m} = w^{-m} x w^{-m} x^{-1} w^{-m} x w^{-m}.$$

Substituting the first in the second we obtain $w^{1-6m} = (x w^{-m})^2$, $x^3 = (x w^{-m})^2$. Hence there is an epimorphism of

$$\mathfrak{H}_{2m-1,1} \cong \langle\, x, w \mid w^{1-6m} = (x w^{-m})^2,\ x^3 = (x w^{-m})^2 \,\rangle$$

onto the non-trivial triangle group $\langle\, a, b \mid a^{6m-1}, b^3, (ab)^2 \,\rangle$ given by $x \mapsto b^{-1}$ and $w \mapsto a^6$. $\square$

Next we establish Property P for product knots. It is now convenient to use a new view of the knot complement: one looks at the complement $C(\mathfrak{k})$ of a regular neighborhood of the knot $\mathfrak{k}$ from the center of a ball in the regular neighborhood. Now $C(\mathfrak{k})$ looks like a ball with a knotted hole. Following Bing and Martin [22] we say that the complement of $\mathfrak{k}$ is a *cube with a* $\mathfrak{k}$*-knotted hole* or, simply, a cube with a (knotted) hole, see Figure 15.6. A cube with an unknotted hole is a solid torus.

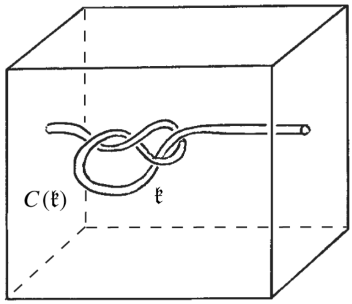


Figure 15.6. A cube with a knotted hole.

Suppose that W is a regular neighborhood of a knot $\mathfrak{h}$ and $C(\mathfrak{k})$ a knotted hole, associated to the knot $\mathfrak{k}$, such that $C(\mathfrak{k}) \subset W$ and $C(\mathfrak{k}) \cap \partial W = \partial C(\mathfrak{k}) \cap \partial W$ is an annulus, then $(\overline{S^3 - W}) \cup C(\mathfrak{k})$ is the complement of $\mathfrak{k} \# \mathfrak{h}$, if the annulus is meridional with respect to $\mathfrak{h}$ and $\mathfrak{k}$, Figure 15.7.

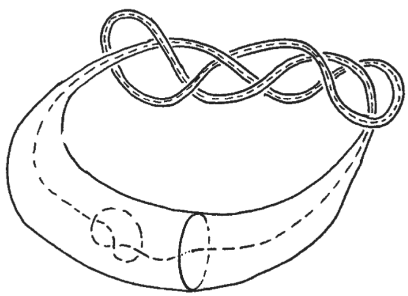


Figure 15.7

15.8 Lemma. *Let V be a homotopy solid torus, that is a 3-manifold with boundary a torus and infinite cyclic fundamental group. Suppose that K is a cube with a knotted hole in the interior of V. Then there is a homotopy 3-ball $B \subset V$ such that $K \subset B$. (B is a compact 3-manifold bounded by a sphere with trivial fundamental group).*

Proof. $\pi_1 V \cong \mathbb{Z}$ implies, as follows from the loop theorem (Appendix B.5), that there is a disk $D \subset V$ with $D \cap \partial V = \partial D$ and ∂D is not null-homologous on ∂V. By general position arguments we may assume that $D \cap \partial K$ consists of mutually disjoint simple closed curves and that, after suitable simplifications, each component of $D \cap \partial K$ is not homotopic to 0 on ∂K. Let γ be an innermost curve of the intersection on D and let D_0 be the subdisk of D bounded by γ. As K is a knotted cube, $\pi_1 \partial K \to \pi_1 K$ is injective; hence, $D_0 \subset \overline{V - K}$. By adding a regular neighborhood of D_0 to K we obtain $B \supset K$, $\partial B = S^2$. So we may assume $D \cap \partial K = \emptyset$. Let U be a regular neighborhood of D in V. Now $\overline{V - U}$ is a homotopy 3-ball containing K. □

15.9 Lemma. *Let V_1, V_2 be solid tori, $V_2 \subset \mathring{V}_1$ such that*

(a) *there is a meridional disk of V_1 whose intersection with V_2 is a meridional disk of V_2 and*
(b) *V_2 is not parallel to V_1, see Figure 15.8.*

Then the result of removing V_2 from V_1 and sewing it back differently is not a homotopy solid torus.

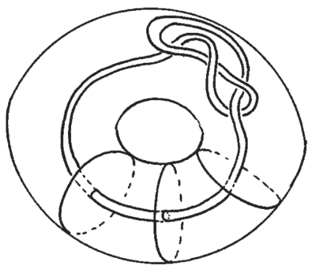


Figure 15.8

Proof. Let F be a meridional disk of V_1, that is $F \cap \partial V_1 = \partial F \not\simeq 0$ on ∂V_1, which intersects V_2 in a meridional disk of V_2. Let N be a regular neighborhood of F in V_1. Then $K_1 = \overline{V_1 - (N \cup V_2)}$ is a cube with a knotted hole since V_2 is not parallel to V_1. Now $K_1 \cap \partial V_1$ is an annulus. We push this annulus slightly into the interior of V_1 and call the resulting cube with a knotted hole $\tilde{K}_1$.

Suppose that V_2 is removed from V_1 and a solid torus V_2' is sewn back differently; denote the resulting manifold by V_1'. Assume that V_1' is a homotopy solid torus. Then there is a disk $D \subset V_1'$ such that $D \cap \partial V_1' = D \cap \partial V_1 = \partial D$ and $\partial D \not\simeq 0$ on $\partial V_1'$. Since, by Lemma 15.8, $\tilde{K}_1$ lies in a homotopy 3-ball contained in V_1' we may assume that $D \cap \tilde{K}_1 = \emptyset$ and, hence, that also $D \cap K_1 = \emptyset$. This implies that $D \cap \partial V_1 = D \cap \partial V_1'$ is parallel to $F \cap \partial V_1$. Moreover, suppose that D and $\partial V_2' = \partial V_2$ are in general position so that $D \cap \partial V_2' = D \cap \partial V_2$ is a finite collection of mutually disjoint simple closed curves, none of which is contractible on ∂V_2. Now the complement of K_1 in $\overline{V_1 - V_2}$ is the Cartesian product of an annulus and an interval, and the boundary contains an annulus on ∂V_1 and another on $\partial V_2 = \partial V_2'$. Therefore each curve of $D \cap \partial V_2'$ is homotopic on $\partial V_2'$ to the simple closed curve $F \cap \partial V_2$ which is meridional in V_2. Let γ be an innermost curve of $D \cap \partial V_2'$ and $D_0 \subset D$ the disk bounded by γ, $D_0 \cap \partial V_2' = \gamma$. Since γ is a meridian of V_2 it is not a meridian of V_2'; hence, $D_0 \subset \overline{V_1' - V_2'} = \overline{V_1 - V_2}$, in fact $D_0 \subset \overline{V_1 - (V_2 - K_1)} \cong (S^1 \times I) \times I$ which contradicts the fact that γ represents the generator of the annulus $S^1 \times I$. Consequently, $D \cap \partial V_2 = \emptyset$ and $\partial D \simeq 0$ in $\overline{V_1 - (V_2 \cup K_1)}$, contradicting the fact that ∂D also represents the generator of $\pi_1(S^1 \times I)$. This shows that V_1' is not a homotopy solid torus. □

15.10 Theorem (Bing–Martin [22], Noga [277]). *Product knots have Property P.*

Proof. Let $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ be a product knot in S^3. We use the construction shown in Figure 7.2 and Figure 15.7. Let V be a regular neighborhood of $\mathfrak{k}_2$. Replace a segment of $\mathfrak{k}_2$ by $\mathfrak{k}_1$ such that $\mathfrak{k}_1 \subset V$, see Figure 15.7. Notice that $\overline{S^3 - V}$ is a cube with a $\mathfrak{k}_2$-knotted hole and, hence, it is not a homotopy solid torus.

Now let N be a regular neighborhood of $\mathfrak{k}$, $N \subset \mathring{V}$, and let M result from S^3 by removing N and sewing it back differently. Lemma 15.9 implies that ∂V does not bound a homotopy solid torus in M. Thus $\pi_1 M$ is the free product of two groups amalgamated over $\pi_1(\partial V) \cong \mathbb{Z} \oplus \mathbb{Z}$ and therefore $\pi_1 M$ is not trivial. □

15.11 Theorem (Bing–Martin [22]). *Let $\mathfrak{k} \subset S^3$ be a satellite, $\hat{\mathfrak{k}}$ its companion and $(\tilde{V}, \tilde{\mathfrak{k}})$ its pattern. Denote by m, ℓ; $\hat{m}, \hat{\ell}$; $\tilde{m}, \tilde{\ell}$ the meridian and longitude of $\mathfrak{k}$, $\hat{\mathfrak{k}}$, $\tilde{\mathfrak{k}}$ and by m_V, ℓ_V those of $\tilde{V}$. Then $\mathfrak{k}$ has Property P if*

(a) $\tilde{\mathfrak{k}}$ *has Property P, or*

(b) $\hat{\mathfrak{k}}$ *has Property P and* $q = \operatorname{lk}(m_V, \tilde{\mathfrak{k}}) \neq 0$.

Proof of Theorem 15.11 (a). (The proof for (b) will be given in 15.14.)

There is a homeomorphism $h\colon \tilde{V} \to \hat{V}$, $h(\tilde{\mathfrak{k}}) = \mathfrak{k}$. Let $\tilde{U}$ be a regular neighborhood of $\tilde{\mathfrak{k}}$ in $\tilde{V}$. We remove $h(\tilde{U})$ from S^3 and sew it back differently to obtain a manifold M. If $\hat{\mathfrak{k}}$ is the trivial knot then h can be extended to a homeomorphism $S^3 \to S^3$ and it follows from assumption (a) that M is not simply connected.

So we may assume that $\hat{\mathfrak{k}}$ is a non-trivial knot. If the result W of a surgery on $\tilde{\mathfrak{k}}$ in $\tilde{V}$ does not yield a homotopy solid torus, then $h(\partial\tilde{V})$ divides M into two manifolds which are not homotopy solid tori. Since $\hat{\mathfrak{k}}$ is a knot, $\pi_1(h(\partial\tilde{V})) \to \pi_1(\overline{M - W}) = \pi_1(\overline{S^3 - h(\tilde{V})})$ is injective. When $\pi_1(h(\partial\tilde{V})) \to \pi_1 W$ has non-trivial kernel, there is a disk $D \subset W$, $\partial D \subset \partial W$, $\partial D \not\simeq 0$ in ∂W such that $X = \overline{W - U(D)}$ is bounded by a sphere, $U(D)$ being a regular neighborhood of D in W. Now X cannot be a homotopy ball because W is not a homotopy solid torus. Therefore $\pi_1 M \neq 1$. If $\pi_1(h(\partial\tilde{V})) \to \pi_1 W$ is injective, $\pi_1 M$ is a free product with an amalgamation over $\pi_1(h(\partial\tilde{V})) \cong \mathbb{Z}^2$, hence non-trivial.

Finally, suppose that $\hat{\mathfrak{k}}$ is non-trivial and the sewing back of $h(\tilde{U})$ in $h(\tilde{V})$ yields a homotopy solid torus W. Then a meridian of W can be presented in the form

$\overline{ph(m_V) + qh(\ell_V)}$ where p, q are relatively prime integers. From $h(\ell_V) \sim 0$ in $\overline{S^3 - h(\tilde{V})}$ it follows that $H_1(M)$ is isomorphic to $\mathbb{Z}_{|p|}$ or $\mathbb{Z}$ (for $p = 0$). To see that $|p| \neq 1$, we perform the surgery on $\tilde{\mathfrak{k}}$ in $\tilde{V}$ which transforms $\tilde{V}$ into the manifold $\tilde{V}' = h^{-1}(W)$. (The new meridian defining the surgery represents $m_V^p \ell_V^q \in \pi_1(\partial \tilde{V})$.) Now $\tilde{V}' \cup \overline{S^3 - \tilde{V}}$ is obtained from S^3 by surgery on $\tilde{\mathfrak{k}}$. Since $\ell_V \simeq 1$ in $\overline{S^3 - \tilde{V}}$ the relation $m_V^p \ell_V^q \simeq 1$ is equivalent to $m_V^p \simeq 1$, and $|p| = 1$ implies that $\tilde{V}' \cup \overline{S^3 - \tilde{V}}$ is a homotopy sphere. Thus $|p| \neq 1$ because $\tilde{\mathfrak{k}}$ has Property P. □

15.12 Remark. The knot $h(\tilde{\mathfrak{k}})$ is a satellite and $(\tilde{V}, \tilde{\mathfrak{k}})$ is the pattern of $h(\tilde{\mathfrak{k}})$. The condition $h(\ell_V) \sim 0$ in $C(\hat{\mathfrak{k}})$ ensures that the mapping h does not unknot $\tilde{\mathfrak{k}}$; this could be done, for instance, with the twist knots $\mathfrak{d}_n$, $n \neq 0, -1$ when h removes the twists. As an example, using the definition of twisted doubled knots in Example 2.9 (b) and Theorem 15.7, we obtain

15.13 Corollary. *Doubled knots with q twists, $q \neq 0, -1$ have Property P.* □

15.14. *Proof of 15.11 (b).* We consider surgery along the knot $h(\tilde{\mathfrak{k}})$ where h is a homeomorphism $h\colon \tilde{V} \to \hat{V}$, $h(\tilde{\mathfrak{k}}) = \mathfrak{k}$. Replace a tubular neighborhood $\tilde{U} \subset \tilde{V}$ on $\tilde{\mathfrak{k}}$ by another solid torus $\tilde{T}$ using a gluing map $f\colon \partial \tilde{T} \to \partial \tilde{U}$. The manifold obtained is

$$M = (\overline{S^3 - \hat{V}}) \cup_h ((\overline{\tilde{V} - \tilde{U}}) \cup_f \tilde{T}).$$

Define $\hat{C} = C(\hat{\mathfrak{k}}) = \overline{S^3 - \hat{V}}$ and $X = (\overline{\tilde{V} - \tilde{U}}) \cup_f \tilde{T}$. Since $\hat{\mathfrak{k}}$ is non-trivial the inclusion $\partial \hat{C} \to \hat{C}$ defines a monomorphism $\pi_1(\partial \hat{C}) \to \pi_1 \hat{C}$. If $\partial X \to X$ induces also a monomorphism $\pi_1(\partial X) \to \pi_1 X$, then $\pi_1 M$ is a free product with amalgamated subgroup $\pi_1(\partial \hat{C}) = \pi_1(\partial X) \cong \mathbb{Z}^2$.

Therefore, if M is a homotopy sphere, $\ker(\pi_1(\partial X) \to \pi_1 X) \neq 1$. By the loop theorem (Appendix B.5), there is a simple closed curve $\nu \subset \partial X$, ν not contractible on ∂X, which bounds a disk D in X, $\partial D \cap \partial X = \partial D = \nu$. Then $\nu \simeq \hat{m}^a \hat{\ell}^b$ of ∂X with $\gcd(a, b) = 1$; we may assume $a \geq 0$.

If W is a regular neighborhood of D in X, the boundary of $\overline{X - W}$ is a 2-sphere S^2 and

$$M = (C \cup W) \cup (\overline{X - W}),\ S^2 = (C \cup W) \cap (\overline{X - W}).$$

Therefore $\pi_1 M = \pi_1(C \cup W) * \pi_1(\overline{X - W})$. Thus $\pi_1(C \cup W) = 1$. Since by assumption 15.11 (b) $\hat{\mathfrak{k}}$ has Property P, it follows that ν must be the meridian $\hat{m}$ of $\hat{\mathfrak{k}}$ and $b = 0$ and $a = 1$; moreover, $\hat{m} = h(m_V)$ if m_V is a meridian of $\tilde{V}$.

Let $\tilde{m}$ be a meridian of the tubular neighborhood $\tilde{U}$ of $\tilde{\mathfrak{k}}$. Then, for the meridian m_V of $\tilde{V}$

$$m_V \sim q\tilde{m} \quad \text{in } \overline{\tilde{V} - \tilde{U}} \text{ with } q = \operatorname{lk}(m_V, \tilde{k}) \neq 0. \tag{15.4}$$

Moreover, there is a longitude $\tilde{\ell}$ of $\tilde{U}$ such that

$$\tilde{\ell} \sim q\ell_V \quad \text{in } \overline{\tilde{V} - \tilde{U}}. \tag{15.5}$$

$\tilde{\ell}$ can be obtained from an arbitrary longitude $\tilde{\ell}_0$ as follows. There is a 2-chain c_2 in $\overline{\tilde{V} - \tilde{U}}$ – the intersection of $\overline{\tilde{V} - \tilde{U}}$ with a projecting cylinder of $\tilde{\ell}_0$ – such that

$$\partial c_2 = \tilde{\ell}_0 + \alpha\tilde{m} + \beta m_V + \gamma\ell_V.$$

Now

$$q = \operatorname{lk}(m_V, \tilde{\mathfrak{k}}) = \operatorname{lk}(m_V, \tilde{\ell}_0) = \operatorname{lk}(m_V, -\alpha\tilde{m} - \beta m_V - \gamma\ell_V) = -\gamma,$$

and

$$\tilde{\ell} = \tilde{\ell}_0 + (\alpha + \beta q)\tilde{m} = \tilde{\ell}_0 + \beta m_V \sim q\ell_V \quad \text{in } \overline{\tilde{V} - \tilde{U}}.$$

(See E 15.1.)

For a meridian $m_{\tilde{T}}$ of $\tilde{T}$ one has

$$m_{\tilde{T}} \sim \varrho\tilde{m} + \sigma\tilde{\ell} \quad \text{on } \partial\tilde{T} = \partial\tilde{U},\ \gcd(\varrho, \sigma) = 1. \tag{15.6}$$

Here $\varrho = \pm 1$ since we assume that the surgery along $\mathfrak{k}$ gives a homotopy sphere. The disk D is bounded by m_V. We assume that D is in general position with respect to $\partial\tilde{T}$ and that $D \cap \partial\tilde{T}$ does not contain curves that are contractible on $\partial\tilde{T}$; otherwise D can be altered to get fewer components of $\partial\tilde{T} \cap D$. This implies that $\partial\tilde{T} \cap D$ is a collection of disjoint meridians of $\tilde{T}$ and that $\partial\tilde{T} \cap D$ consists of parallel meridional disks, and, thus, for a suitable p

$$m_V \sim p\, m_{\tilde{T}} \quad \text{in } \overline{\tilde{V} - \tilde{U}}. \tag{15.7}$$

ℓ_V and $\tilde{m}$ are a basis of $H_1(\overline{\tilde{V} - \tilde{U}}) \cong \mathbb{Z}^2$. The formulas (15.4)–(15.7) imply

$$q\,\tilde{m} \sim m_V \sim p\, m_{\tilde{T}} \sim p\varrho\tilde{m} + p\sigma\tilde{\ell};$$

thus

$$p\sigma = 0,\ p\varrho = q,\ \text{that is, since } p, q \neq 0, \sigma = 0, \varrho = \pm 1, p = \pm q.$$

So we may assume that $\varrho = 1$ and $p = 1$. But then $m_{\tilde{T}} = \tilde{m}$. □

15.15 Definition (Cable knots). Let W be a solid torus in S^3 with core $\mathfrak{k}$, m and ℓ meridian and longitude of W where $\ell \sim 0$ in $C(\mathfrak{k}) = \overline{S^3 - W}$. A simple closed curve $\mathrm{c} \subset \partial W$, $\mathrm{c} \sim pm + q\ell$ on ∂W, $|q| \geq 2$ is called a (p, q)*-cable knot with core* $\mathfrak{k}$. (Compare Definition 2.8.)

15.16 Proposition. (a) (p,q)*-cable knots with* $2 \le |p|,\ |q|$ *have Property P.*

(b) *Let* $\mathfrak{k}$ *be a* $(\pm 1, q)$*-cable knot about the non-trivial knot* $\hat{\mathfrak{k}}$*. If* $|q| \ge 3$ *then* $\mathfrak{k}$ *has Property P.*

Proof. The first statement is a consequence of Proposition 15.5 and Theorem 15.11 (a). For the proof of the second assertion, we consider the pattern $(\tilde{V}, \tilde{\mathfrak{k}})$. It can be constructed as follows. Let ϱ denote the rotation of the unit disk $\tilde{B}$ through the angle $2\pi/q$. Choose in $\tilde{B}$ a small disk $\tilde{D}_1$ with center $\tilde{x}_1$ such that $\tilde{D}_1$ is disjoint to all its images $\varrho^j \tilde{D}_1$, $1 \le j \le q-1$. Then the pattern consists of the solid torus $\tilde{B} \times I/\varrho$, that is, the points $(\tilde{x}, 1)$ and $(\varrho(\tilde{x}), 0)$ are identified, and the knot $\tilde{\mathfrak{k}}$ consists of the arcs $\varrho^j(\tilde{x}_1) \times I$, $0 \le j < q$. A regular neighborhood $\tilde{U}$ of $\tilde{\mathfrak{k}}$ is $\bigcup_{j=0}^{q-1}(\varrho^j(\tilde{D}_1) \times I)$, see Figure 15.9.

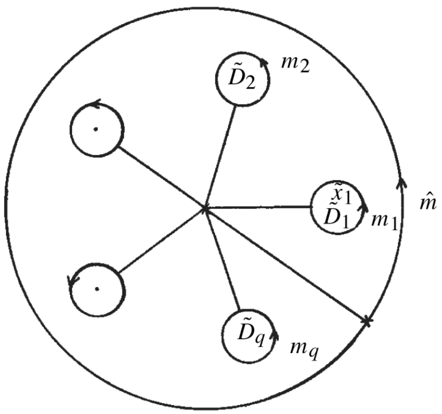


Figure 15.9

Then $C(\mathfrak{k}) = C(\hat{\mathfrak{k}}) \cup X$, $C(\hat{\mathfrak{k}}) \cap X = \partial C(\hat{\mathfrak{k}}) \subset \partial X$, where X is homeomorphic to the pattern described above. Let $\hat{m}$ be a meridian of $\hat{\mathfrak{k}}$ ($\hat{m}$ is the image of $\partial \tilde{B}$) and $m_1, \dots, m_q$ meridians of $\mathfrak{k}$ corresponding to $\partial \tilde{D}_1, \dots, \partial \tilde{D}_q$. Let $\hat{\ell}$ be the longitude of $\hat{\mathfrak{k}}$. Then

$$\begin{aligned}\pi_1 X =& \langle\, \hat{m}, m_1, \dots, m_q, \hat{\ell} \mid \hat{m}^{-1} \cdot m_1 \dots m_q, [\hat{m}, \hat{\ell}], \\ & \qquad \hat{\ell}^{-1} m_j \hat{\ell} \cdot m_{j+1}^{-1} (1 \le j < q), \hat{\ell}^{-1} m_q \hat{\ell} \cdot (\hat{m}^{-1} m_1 \hat{m})^{-1} \,\rangle \\ =& \langle\, \hat{m}, m_1, \hat{\ell} \mid \hat{m}^{-1}(m_1 \hat{\ell}^{-1})^q \hat{\ell}^q, \hat{\ell}^q m_1 \hat{\ell}^{-q} (\hat{m}^{-1} m_1 \hat{m})^{-1}, [\hat{m}, \hat{\ell}] \,\rangle \\ =& \langle\, m_1, \hat{\ell} \mid [m_1, (m_1 \hat{\ell}^{-1})^q], [(m_1 \hat{\ell}^{-1})^q, \hat{\ell}] \,\rangle.\end{aligned}$$

Note that $m_1^{-q}(m_1 \hat{\ell}^{-1})^q$ is a longitude of $\mathfrak{k}$.

Next we attach a solid torus W to $C(\mathfrak{k})$ such that the result is a homotopy sphere M. The meridian of W has the form $m_1 \cdot (m_1^{-q}(m_1 \hat{\ell}^{-1})^q)^n = m_1^{1-nq}(m_1 \hat{\ell}^{-1})^{nq}$. If we show that $n = 0$ the assertion (b) is proved. We have

$$\pi_1(X \cup W) = \langle\, m_1, \hat{\ell} \mid [m_1, (m_1\hat{\ell}^{-1})^q], [(m_1\hat{\ell}^{-1})^q, \hat{\ell}], m_1^{1-nq}(m_1\hat{\ell}^{-1})^{nq} \,\rangle$$

and $\pi_1 M = \pi_1(C(\hat{\mathfrak{k}}) \cup X \cup W)$. We obtain a quotient of $\pi_1 M$ by adding the relation $\hat{m} = (m_1\hat{\ell}^{-1})^q\hat{\ell}^q = 1$. Since $\pi_1 C(\hat{\mathfrak{k}})$ is trivialized by this relation, a presentation of $\pi_1 M/\overline{\langle\hat{m}\rangle}$ is given by

$$\langle\, m_1, \hat{\ell} \mid [m_1, (m_1\hat{\ell}^{-1})^q],\ [(m_1\hat{\ell}^{-1})^q, \hat{\ell}],\ m_1^{1-nq}(m_1\hat{\ell}^{-1})^{nq},\ (m_1\hat{\ell}^{-1})^q\hat{\ell}^q \,\rangle.$$

Here $\overline{\langle\hat{m}\rangle}$ denotes the normal closure of $\hat{m}$ in $\pi_1 M$.

Put $v = m_1\hat{\ell}^{-1}$ and replace $\hat{\ell}$ by $v^{-1}m_1$ to get

$$\pi_1 M/\overline{\langle\hat{m}\rangle} = \langle\, m_1, v \mid [m_1, v^q], [v^q, v^{-1}m_1], m_1^{1-nq}v^{nq}, v^q(v^{-1}m_1)^q \,\rangle.$$

Adding the relation $v^q = 1$ we obtain the group

$$\langle\, m_1, v \mid m_1^{1-nq}, v^q, (v^{-1}m_1)^q \,\rangle$$

which must be trivial. Since $|q| \geq 3$ this implies $1 - nq = \pm 1$, see [382, p. 122]; hence, $n = 0$. □

15.C Prime knots and their manifolds and groups

In this section we discuss to what extent the group of a prime knot determines the knot manifold. For this we need some concepts from 3-dimensional topology.

15.17 Definition. (1) A submanifold $N \subset M$ is *properly embedded* if $\partial N = N \cap \partial M$.

(2) Let A be an annulus and $a \subset A$ a non-separating properly embedded arc, a so-called *spanning arc. A mapping* $f\colon (A, \partial A) \to (M, \partial M)$, M a 3-manifold, is called *essential* if $f_\#\colon \pi_1 A \to \pi_1 M$ is injective and if the arc $f(a)$ is not homotopic rel its endpoints to an arc on ∂M. (Note that the definition of essential is independent of the choice of a.) The *annulus* $f(A)$ is also called *essential.*

(3) The properly embedded surface $F \subset M$ is *boundary parallel* if there is an embedding $g\colon F \times I \to M$ such that

$$g(F \times \{0\}) = F \text{ and } g((F \times \{1\} \cup (\partial F \times I)) \subset \partial M.$$

15.18 Remark. A properly embedded annulus $A \subset M$ is boundary parallel if and only if there is a solid torus $V \subset M$ such that $A \subset \partial V$, $\partial V - A \subset \partial M$ and the core of A is a longitude of V. (Proof as Exercise E 15.2).

To illustrate the notion of an essential annulus we give another characterizing condition and discuss two important examples.

15.19 Lemma. *Let A be a properly embedded incompressible annulus in a knot manifold C. Then A is boundary parallel if and only if the inclusion $i\colon A \to C$ is not essential.*

Proof. Clearly, if A is boundary parallel, then i is homotopic rel ∂A to a map into ∂C, thus not essential. If i is not essential then, since A is incompressible, that is $i_\#\colon \pi_1 A \to \pi_1 C$ is injective, a spanning arc a of A is homotopic to an arc $b \subset \partial C$. We may assume that b intersects ∂A transversally, intersects the two components of ∂A alternately and is simple; the last assumption is not restrictive since any arc on a torus with different endpoints can be deformed into a simple arc by a homotopy keeping the endpoints fixed. The annulus A decomposes C into two 3-manifolds C_1, C_2: $C = C_1 \cup C_2$, $A = C_1 \cap C_2$, such that $\partial C_j = (\partial C_j \cap \partial C) \cup A$ $(j = 1, 2)$ is a torus. We have

$$\pi_1 C = \pi_1 C_1 *_{\pi_1 A} \pi_1 C_2.$$

If $b \subset \partial C_j$ for some j then $b \cup a \subset \partial C_j$ is null-homotopic in C_j, thus bounds a disk in C_j. This implies that C_j is a solid torus and ∂A consists of two longitudes of C_j. By the remark above, A is boundary parallel.

If b intersects ∂A more than twice then $b = b_1 \dots b_n$ where b_j and b_{j+1} are alternately contained in C_1 and C_2. The boundary points of each b_j are on different components of ∂A. By adding segments $c_j \subset A$ we obtain

$$b \simeq (b_1 c_1)(c_1^{-1} b_2 c_2)(c_2^{-1} \dots (c_{n-1}^{-1} b_n)$$

such that ab_1c_1, $c_1^{-1}b_2c_2, \dots, c_{n-1}^{-1}b_n$ are closed and are contained in ∂C_1 or ∂C_2. If in some C_j, ab_1c_1 is contractible or homotopic to a power c^p of the core of A we replace b by $b_1c_1c^{-p}$, respectively, and argue as above. If one of the $c_{k-1}^{-1}b_kc_k$ (c_n is the trivial arc) is contractible or homotopic to a curve in A in some C_j it can be eliminated and we obtain a simpler arc, taking the role of b. Thus we may assume that none of ab_1c_1, $c_1^{-1}b_2c_2, \dots, c_{n-1}^{-1}b_n$ is homotopic to a curve in A. Then the above product determines a word in $\pi_1 C$ where consecutive factors are alternately in $\pi_1 C_1$ and $\pi_1 C_2$ and none is in the amalgamated subgroup; thus the word has length n and represents a non-trivial element of $\pi_1 C$, see [382, 2.3.3], contradicting $ab \simeq 0$ in C. □

15.20 Proposition. *Let $C(\mathfrak{k}) = C(\mathfrak{k}_1) \cup C(\mathfrak{k}_2)$ be the knot manifold of a product knot $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ with $A = C(\mathfrak{k}_1) \cap C(\mathfrak{k}_2)$ an annulus. If $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are non-trivial, then A is essential in $C(\mathfrak{k})$.*

Proof. Otherwise, by Lemma 15.19, A and one of the annuli of $\partial C(\mathfrak{k})$, defined by ∂A bounds a solid torus which must be one of the $C(\mathfrak{k}_j)$. This is impossible since a knot with complement a solid torus is trivial, see Proposition 3.10. □

15.21 Example (Cable knots). Cable knots were introduced in Definition 15.15. Let $\mathfrak{c}$ be the (p,q)-cable knot with core $\mathfrak{k}$, $|q| \geq 2$.

Another description is the following: *Let V be a solid torus with core $\mathfrak{k}$* in S^3 *and* $C(\mathfrak{k}) \cap V = (\partial C(\mathfrak{k})) \cap (\partial V) = A$ *an annulus the core of which is of type (p,q) on* $\partial C(\mathfrak{k})$. *Then $\partial(C(\mathfrak{k}) \cup V)$ is a torus and $U(\mathfrak{c}) = \overline{S^3 - (C(\mathfrak{k}) \cup V)}$ is a solid torus the core of which is (p,q)-cable knot $\mathfrak{c}$ with core $\mathfrak{k}$*, see Figure 15.10. This follows from the fact that the core of $\overline{S^3 - (C(\mathfrak{k}) \cup V))}$ is isotopic in $W = \overline{S^3 - C(\mathfrak{k})}$ to the core of A.

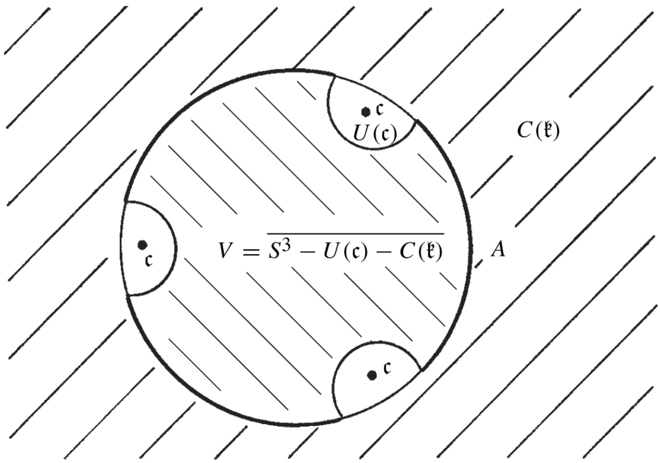


Figure 15.10. The complement of a cable knot.

We will see that the annuli of 15.20, 15.21 are the prototypes of essential annuli in knot manifolds.

15.22 Lemma (Simon [336, Lemma 2.1]). *Let C, W_0, W_1 be knot manifolds. Suppose $C = W_0 \cup (A \times [0,1]) \cup W_1$,*

$$W_0 \cap ((A \times [0,1]) \cup W_1) = A \times \{0\},\ W_1 \cap (W_0 \cup (A \times [0,1])) = A \times \{1\},$$

where A is an annulus, see Figure 15.11. Then either the components of ∂A bound disks in ∂C or the components bound meridional disks in $\overline{S^3 - C}$ and the groups $\pi_1 C$, $\pi_1 W_0$, $\pi_1 W_1$ are the normal closures of the images of $\pi_1 A$.

Proof. Since W_0 is a knot manifold, $\overline{S^3 - W_0}$ is a solid torus containing W_1. By Lemma 15.8, there is a 3-ball B such that $W_1 \subset \mathring{B} \subset B \subset S^3 - W_0$; so the 2-sphere $S^2 = \partial B$ separates W_0 and W_1 and therefore must intersect $A \times (0,1)$. We may assume that $S^2 \cap (\partial A \times (0,1))$ consists of a finite number of pairwise disjoint curves

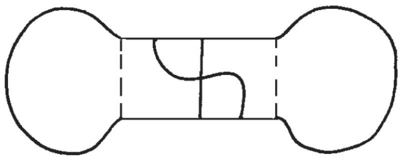


Figure 15.11

$\sigma_1, \ldots, \sigma_r$. If σ_i is innermost in S^2 then σ_i bounds a disk $D \subset S^2$ such that either $D \subset \overline{S^3 - C}$ or $D \subset A \times (0,1)$.

If σ_i also bounds a disk $E \subset \partial A \times (0,1)$ – which it necessarily does in the latter case – then the intersection line σ_i can be removed by an isotopy which replaces S^2 by a sphere S_1^2 still separating W_0 and W_1. It is impossible that all curves σ_j can be eliminated in this way, as $\partial A \times \{0\}$ and $\partial A \times \{1\}$ are separated by S^2. There exists a curve $\gamma \subset S^2 \cap (\partial A \times (0,1))$ bounding a disk in $\overline{S^3 - C}$ which is not trivial on $\partial A \times (0,1)$. So there are non-trivial curves γ_1, γ_2 on each component of $\partial A \times (0,1)$ bounding disks in $\overline{S^3 - C}$. They are isotopic on $\partial A \times [0,1]$ to the components of $\partial A \times \{0\}$, respectively, which, hence bound disks in $\overline{S^3 - C}$. □

15.23 Lemma. *Let C be a knot manifold in S^3, $C = W_0 \cup W_1$, where W_0 is a cube with a hole, W_1 is a solid torus, and $A = W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ is an annulus. Denote by $\mathfrak{k}_C$ the core of the solid torus $\overline{S^3 - C}$. Assume that $\pi_1 A \to \pi_1 W_1$ is not surjective. Then $\mathfrak{k}_C$ is a (p,q)-cable of the core $\mathfrak{k}_0$ of $\overline{S^3 - W_0}$, $|q| \geq 2$. If W_0 is a solid torus then $\mathfrak{k}_C$ is a torus knot.*

Proof. We may write $C = W_0 \cup_f W_1$ where f is an attaching map on A. This mapping f is uniquely determined up to isotopy by the choice of the core of A on ∂W_1, since $\overline{S^3 - C}$ is a solid torus. Hence, the core $\mathfrak{k}_C$ of $\overline{S^3 - C}$ is by 15.21 the (p,q)-cable of $\mathfrak{k}_0$. When $|q| = 1$ the homomorphism $\pi_1 A \to \pi_1 W_1$ is surjective. If $q = 0$, $\mathfrak{k}_C$ is trivial and C is not a knot manifold. In the special case where W_0 is a solid torus, $\mathfrak{k}_0$ is trivial and $\mathfrak{k}_C$ a torus knot. □

15.24 Lemma. *Let C be a knot manifold in S^3, and let A be an annulus in C, $\partial A \subset \partial C$, with the following properties:*

(a) *the components of ∂A do not bound disks in ∂C;*

(b) *A is not boundary parallel in C.*

Then a core $\mathfrak{k}_C$ of $\overline{S^3 - C}$ is either a product knot and each component of ∂A is a meridian of $\mathfrak{k}_C$ or $\mathfrak{k}_C$ is a cable knot isotopic to each of the components of ∂A.

Proof. By (a), the components of ∂A bound annuli in ∂C. Hence, there are submanifolds X_1 and X_2 bounded by tori such that $C = X_1 \cup X_2$, $X_1 \cap X_2 = A$, and, by Alexander's Theorem (Appendix B.2) X_i is either a knot manifold or a solid torus.

If X_1 and X_2 are both knot manifolds then, by Lemma 15.22, each component of ∂A bounds a meridional disk in $\overline{S^3 - C}$, and a core of $\overline{S^3 - C}$ is, by Definition 2.7, a product knot.

Suppose now that X_2 is a solid torus. There is an annulus $B \subset \partial C$ satisfying $A \cup B = \partial X_2$. If the homomorphism $\pi_1 A \to \pi_1 X_2$, induced by the inclusion, is not surjective, then, by Lemma 15.23, a core of $\overline{S^3 - C}$ is a cable knot. Now assume that $\pi_1 A \to \pi_1 X_2$ is surjective. Then a simple arc $\beta \subset B$ which leads from one component of ∂B to the other can be extended by a simple arc $\alpha \subset A$ to a simple closed curve $\mu \subset \partial X_2$ which is 0-homotopic in the solid torus X_2 and, hence, a meridian of X_2. Since μ intersects each component of ∂A at exactly one point it follows that A is boundary parallel, contradicting hypothesis (b). □

In order to study the relation between the complement and its fundamental group for prime knots we shall make use of the following theorem of Feustel [101, Thm. 10] and Johannson [179, Prop. 14.9], which we cannot prove here. (See also Jaco [176, X.15].)

15.25 Theorem (Feustel, Johannson). *Let M and N be compact, connected, irreducible, boundary irreducible 3-manifolds. Suppose that ∂M is a torus and that M does not admit an essential embedding of an annulus. If $\varphi\colon \pi_1 M \to \pi_1 N$ is an isomorphism then there is a homeomorphism $h\colon M \to N$ with $h_\# = \varphi$.* □

We will prove in 15.40 the following result of Simon [338], without using Theorem 15.25.

15.26 Theorem (Simon 1980). *There are at most two cable knots with the same knot group.*

A consequence of Theorems 15.25 and 15.26 is the following:

15.27 Corollary (Simon 1980). *The complements of at most two prime knot types can have the same group.*

Proof. Suppose $\mathfrak{k}_0, \mathfrak{k}_1, \mathfrak{k}_2$ are prime knots whose groups are isomorphic to $\pi_1 C(\mathfrak{k}_0)$. If $\mathfrak{k}_j$ is not a cable knot then $C(\mathfrak{k}_j)$ does not contain essential annuli, see 15.24. Now Theorem 15.25 implies that the $C(\mathfrak{k}_j)$, $j = 0, 1, 2$ are homeomorphic. So we may assume that $\mathfrak{k}_0, \mathfrak{k}_1, \mathfrak{k}_2$ are cable knots and the assertion follows from Theorem 15.26. □

In 1987, Whitten improved Simon's result:

15.28 Theorem (Whitten's Rigidity Theorem [375]). *Prime knots in S^3 with isomorphic groups have homeomorphic complements.*

We prove in 15.41 the following:

15.29 Proposition (Whitten [375]). *If there exist prime knots with isomorphic groups and non-homeomorphic complements, then there exists a non-trivial knot* $\mathfrak{k}$ *in* S^3 *and an integer* n *such that* $|n| > 2$ *and* $S^3_{1/n}(\mathfrak{k}) \cong S^3$.

Proof of Theorem 15.28. By, Proposition 15.29 the existence of two prime knots with non-homeomorphic complements implies the existence of a non-trivial knot $\mathfrak{k}$ in S^3 and an integer $n \in \mathbb{Z}$ such that $S^3_{1/n}(\mathfrak{k}) \cong S^3$ and $|n| > 2$. This contradicts Theorem 3.27. □

As a corollary of Whitten's Rigidity Theorem 15.28 and the Gordon–Luecke Theorem 3.19 we obtain:

15.30 Corollary. *If two prime knots have isomorphic groups then they are of the same knot type.* □

In what follows we shall give a proof of Theorem 15.26 and Proposition 15.29. Some parts of the this section will be applied in Section 15.D to answer the question: *How many knots have the same group?*

15.31 Lemma. *Let* $\mathfrak{k}_1$ *and* $\mathfrak{k}$ *be cable knots with complements* $C(\mathfrak{k}_1)$ *and* $C(\mathfrak{k})$. *Assume that* $\mathfrak{k}$ *is not a torus knot and that*

$$C(\mathfrak{k}) = X \cup V,\ A = X \cap V = \partial X \cap \partial V,$$

where X *is a knot manifold,* V *a solid torus, and* A *an annulus. Let* $\mathfrak{k}$ *be a* (p,q)-*curve on a torus parallel to the boundary of* $\overline{S^3 - X}$, $|q| \geq 2$.

If $\pi_1 C(\mathfrak{k}_1) \cong \pi_1 C(\mathfrak{k})$ *then there is a homotopy equivalence* $f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ *such that* $f^{-1}(A)$ *is an annulus.*

15.32 Remark. We do not use the fact that $\mathfrak{k}_1$ and $\mathfrak{k}$ are cable knots in the first part of the proof including Claim 15.34. By Theorem 6.1 we know that $\mathfrak{k}_1$ is not a torus knot. (A non-trivial knot whose group has a non-trivial center is a torus knot.)

The proof of Lemma 15.31 is quite long and of a technical nature. However, some of the intermediate steps have already been done in Chapter 5. The proof of Lemma 15.31 will be finished in 15.38.

Proof. Since $C(\mathfrak{k}_1)$ and $C(\mathfrak{k})$ are $K(\pi,1)$-spaces, see Theorem 3.40, any isomorphism $\pi_1 C(\mathfrak{k}_1) \xrightarrow{\cong} \pi_1 C(\mathfrak{k})$ is induced by a homotopy equivalence $g\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$, [157, 1.B], [341, 7.6.24], [346, p. 459]. We may assume that g has the following properties:

(1) g is transversal with respect to A, that is, there is a neighborhood $g^{-1}(A) \times [-1,1] \subset C(\mathfrak{k}_1)$ of $g^{-1}(A) = g^{-1}(A) \times \{0\}$ and a neighborhood $A \times [-1,1]$ of A such that $g(x,t) = (g(x),t)$ for $x \in g^{-1}(A)$, $t \in [-1,1]$.

(2) $g^{-1}(A)$ is a compact 2-manifold, properly embedded and two-sided in $C(\mathfrak{k}_1)$.

(3) If A' is a component of $g^{-1}(A)$ then $\ker(\pi_j A' \to \pi_j C(\mathfrak{k}_1))$ is trivial for $j = 1, 2$.

(1) (2) (3): These properties can be obtained by arguments similar to those used in the proof of Theorem 5.1; see also Stallings [343], Waldhausen [365, §1], Hempel [159, Chap. 6], Jaco [176, III.9].

Choose among all homotopy equivalences g that have the above properties one with minimal number n of components A_i of $g^{-1}(A)$.

15.33 Claim. *Each A_i is an annulus which separates $C(\mathfrak{k}_1)$ into a solid torus V_i and a knot manifold W_i, and $\pi_1 A_i \to \pi_1 V_i$ is not surjective.*

Proof of Claim 15.33. Since $\pi_2 C(\mathfrak{k}_1) = 0$ it follows from (3) that $\pi_2 A_i = 0$; moreover, since $\pi_1 A_i \to \pi_1 C(\mathfrak{k}_1)$ is injective and $g_\#\colon \pi_1 C(\mathfrak{k}_1) \to \pi_1 C(\mathfrak{k})$ is an isomorphism, $(g|A_i)_\#\colon \pi_1 A_i \to \pi_1 A$ is injective. This shows that $\pi_1 A_i$ is a subgroup of $\mathbb{Z}$, hence, trivial or isomorphic to $\mathbb{Z}$. Now A_i is an orientable compact connected surface and therefore either a disk, a sphere or an annulus. We will show that A_i is an annulus. $\pi_2 A_i = 0$ excludes spheres. If A_i is a disk then $\partial A_i \subset \partial C(\mathfrak{k}_1)$ is contractible in $C(\mathfrak{k}_1)$. If ∂A_i is not null-homotopic on $\partial C(\mathfrak{k}_1)$ then $C(\mathfrak{k}_1)$ is a solid torus and $\mathfrak{k}_1$ is the trivial knot. But then $\pi_1 C(\mathfrak{k}_1) \cong \mathbb{Z}$ and this implies that $\mathfrak{k}$ is also unknotted, contradicting the assumption that it is a (p,q)-cable knot. Therefore ∂A_i also bounds a disk $D \subset \partial C(\mathfrak{k}_1)$ and $D \cup A_i$ is a 2-sphere that bounds a ball B in $C(\mathfrak{k}_1)$. Now $Q = \overline{C(\mathfrak{k}_1) - B}$ is homeomorphic to $C(\mathfrak{k}_1)$, $g|Q\colon Q \to C(\mathfrak{k})$ satisfies the conditions (1)–(3), and $(g|Q)^{-1}(A)$ has at most $(n-1)$ components. This proves that there is also a mapping $g'\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ satisfying (1)–(3) with less components in $g'^{-1}(A)$ than in $g^{-1}(A)$, contradicting the minimality of n.

Thus we have proved that A_i is an annulus. Because of (3), ∂A_i is not null-homotopic on $\partial C(\mathfrak{k}_1)$ and decomposes $\partial C(\mathfrak{k}_1)$ into two annuli, while A_i decomposes $C(\mathfrak{k}_1)$ into two submanifolds W_i, V_i which are bounded by tori and, thus, are either knot manifolds or solid tori.

If V_i and W_i are knot manifolds then, by Lemma 15.22, $\pi_1 C(\mathfrak{k}_1)/\overline{\pi_1 A_i} = 1$, where $\overline{\pi_1 A_i}$ denotes the normal closure of $\pi_1 A_i$ in $\pi_1 C(\mathfrak{k}_1)$, and so, since g is a homotopy equivalence

$$\pi_1 C(\mathfrak{k})/\overline{\pi_1 A} = 1.$$

This implies that each 1-cycle of $C(\mathfrak{k})$ is homologous to a cycle of A, that is $H_1(A) \to H_1(C(\mathfrak{k}))$ is surjective and, hence, an isomorphism. On the other hand, if c is a core curve of A and if m is a meridian of $\mathfrak{k}$ then the map $H_1(A) \to H_1(C(\mathfrak{k}))$ is defined by $c \mapsto \pm pq\, m$, see E 15.4. It follows that $|pq| = 1$, a contradiction since $|q| \geq 2$.

So we may assume that V_i is a solid torus. If $\pi_1 A_i \to \pi_1 V_i$ is surjective, that is $|q| = 1$, then g can be modified homotopically such that A_i disappears, i.e. we can find a neighborhood U of V_i in $C(\mathfrak{k}_1)$ such that $U \cong A_i \times [-1,1]$, $A_i \times \{-1\} = V_i \cap \partial C(\mathfrak{k}_1)$, $A_i \times \{0\} = A_i$, $A_i \times [-1,0] = V_i$, $U \cap g^{-1}(A) = A_i$. Then $Q = \overline{C(\mathfrak{k}_1) - U} \cong C(\mathfrak{k}_1)$ and $g|Q\colon Q \to C(\mathfrak{k})$ is a homotopy equivalence satisfying (1)–(3) and having fewer than n components in $g^{-1}(A)$; this defines a mapping $C(\mathfrak{k}_1) \to C(\mathfrak{k})$ with the same properties, contradicting the choice of g. Therefore $\pi_1 A_i \to \pi_1 V_i$ is not surjective.

W_i is not a solid torus, since $\mathfrak{k}_1$ is not a torus knot. □

15.34 Claim. *$W_1 \subset \ldots \subset W_n$, after a suitable enumeration of the annuli A_i.*

Proof of Claim 15.34. It suffices to show that for any two components A_1, A_2 either $W_1 \subset W_2$ or $W_2 \subset W_1$. Since A_1 and A_2 are disjoint there are three configurations (see Figure 15.12):

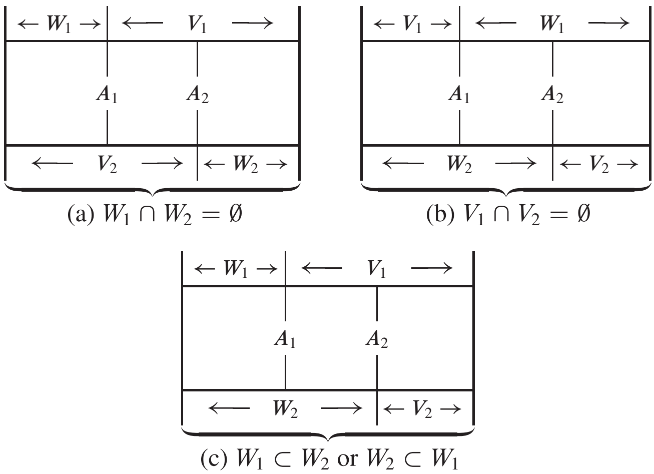


Figure 15.12. Cases (a)–(c)

Configuration (a): $W_1 \cap W_2 = \emptyset$. By Claim 15.33, W_2 is a knot manifold which can be contracted slightly in order to be contained in the interior of the solid torus V_1. By Lemma 15.8, there is a 3-ball B such that $W_2 \subset \mathring{B} \subset B \subset V_1$; hence $A_2 \subset \partial W_2$ and $\ker(\pi_1 A_2 \to \pi_1 C(\mathfrak{k}_1))$ is non-trivial, contradicting (3).

Configuration (b): $V_1 \cap V_2 = \emptyset$. Put $Y = W_1 \cap W_2$ and denote by $\mathfrak{k}_{W_i}$ the core of $\overline{S^3 - W_i}$. Since ∂Y consists of the two annuli A_1, A_2 and two parallel annuli on $\partial C(\mathfrak{k}_1)$ and since S^3 does not contain Klein bottles it follows that ∂Y is a torus. $W_2 = Y \cup V_1$, $A_1 = Y \cap V_1 = \partial Y \cap \partial V_1$ and $\pi_1 A_1 \to \pi_1 V_1$ is not surjective. When Y is a solid torus then $\mathfrak{k}_{W_2}$ is a non-trivial torus knot. When Y is a knot manifold then, by Lemma 15.23, $\mathfrak{k}_{W_2}$ is a cable about the core $\mathfrak{k}_Y$ of Y. In any case the knot $\mathfrak{k}_{W_2}$ is non-trivial and parallel to each component of ∂A_1, see Lemma 15.24. Hence, $\pi_1 A_1 \to \pi_1 \overline{S^3 - W_2}$ is an isomorphism and therefore $\pi_1 A_1 \to \pi_1 V_2$ is surjective. Now, the components of ∂A_1 and ∂A_2 are parallel on ∂W_2. This implies that $\pi_1 A_2 \to \pi_1 V_2$ is surjective contradicting Claim 15.33. Therefore, we are in the configuration (c): $W_1 \subset W_2$ or $W_2 \subset W_1$. □

15.35 Claim.

$(W_n \cap V_1, A_1, \dots, A_n)$ *is homeomorphic to* $(A_1 \times [1,n], A_1 \times \{1\}, \dots, A_1 \times \{n\})$.

Proof of Claim 15.35. $V_i \cap W_{i+1}$ is bounded by four annuli, hence by a torus. This shows that $V_i \cap W_{i+1}$ is either a knot manifold or a solid torus contained in the solid torus V_i. The first case is impossible by Lemma 15.8, since A_i is incompressible in $C(\mathfrak{k}_1)$. Now

$$V_i = (V_i \cap W_{i+1}) \cup V_{i+1},\ (V_i \cap W_{i+1}) \cap V_{i+1} = A_{i+1}$$

where V_i, V_{i+1}, $V_i \cap W_{i+1}$ are solid tori and A_{i+1} is incompressible. Therefore

$$\mathfrak{Z} \cong \pi_1 V_i = \pi_1(V_i \cap W_{i+1}) *_{\pi_1 A_{i+1}} \pi_1 V_{i+1}.$$

Since, by Claim 15.33, $\pi_1 A_{i+1}$ is a proper subgroup of $\pi_1 V_{i+1}$ it follows that $\pi_1 A_{i+1} = \pi_1(V_i \cap W_{i+1})$. Since ∂A_i is parallel to ∂A_{i+1}, which contains the generator of $\pi_1 A_{i+1}$, it follows that $\pi_1 A_i$ also generates $\pi_1(V_i \cap W_{i+1})$. Moreover, $A_i \cup A_{i+1} \subset \partial(V_i \cap W_{i+1})$ and $A_i \cap A_{i+1} = \emptyset$.

This means that

$$(V_1 \cap W_{i+1}, A_i, A_{i+1}) \cong (A_i \times [i, i+1], A_1 \times \{i\}, A_1 \times \{i+1\}). \quad \square$$

15.36 Claim. $g|A_i$ *is homotopic to a homeomorphism.*

Proof of Claim 15.36. In the following commutative diagram all groups are isomorphic to $\mathbb{Z}$.

$$\begin{array}{ccc} H_1(A_i) & \xrightarrow{j_{i*}} & H_1(C(\mathfrak{k}_1)) \\ {\scriptstyle (g|A_i)_*}\big\downarrow & & \cong\big\downarrow{\scriptstyle g_*} \\ H_1(A) & \xrightarrow[j_*]{} & H_1(C(\mathfrak{k})) \end{array}$$

where $j_i\colon A_i \hookrightarrow C(\mathfrak{k}_1)$ and $j\colon A \hookrightarrow C(\mathfrak{k})$ are the inclusions. As g is a homotopy equivalence, g_* is an isomorphism.

By Claim 15.33, A_i decomposes $C(\mathfrak{k}_1)$ into a knot manifold W_i and a solid torus V_i: $C(\mathfrak{k}_1) = W_i \cup V_i$, $A_i = W_i \cap V_i$, and by Lemma 15.24 a component b_i of ∂A_i is isotopic to $\mathfrak{k}_1$. The component b_i is, for suitable p', q', $|q'| \geq 2$, a (p', q')-curve on

$\partial(\overline{S^3 - W_i})$. For generators of the cyclic groups of the above diagram and for some $r \in \mathbb{Z}$ we obtain

$$\begin{array}{ccc} z_i & \xmapsto{j_{i*}} & t'^{\pm|p'q'|} \\ \big\downarrow & & \big\downarrow g_* \\ z^r & \longmapsto & t^{\pm r|pq|}; \end{array}$$

here we used the fact that a component of ∂A is a (p,q)-curve on $\partial(\overline{S^3 - X})$ (for the notations, see Lemma 15.31). Since g_* is an isomorphism, $g_*(t') = t^{\pm 1}$; hence, $|p'q'| = \pm r|pq|$. This implies that pq divides $p'q'$.

By a deep theorem of Schubert [318, p. 253, Satz 5], see also Gramain [142] for a modern proof, $\mathfrak{k}_1$ determines the core $\mathfrak{k}_{W_i}$ and the numbers p', q'. Hence, since g is a homotopy equivalence, we may apply the above argument with the roles of $\mathfrak{k}_1$ and $\mathfrak{k}$ interchanged and obtain that $p'q'$ divides pq; thus $|r| = 1$.

This implies that $g|A_i\colon A_i \to A$ can be deformed into a homeomorphism. Since A_i and A are two-sided, g is homotopic to a mapping g' such that $g'|A_i\colon A_i \to A$ is a homeomorphism and g' coincides with g outside a small regular neighborhood $U(A_i) \cong A_i \times [0,1]$ of A_i. □

For the following, we assume that $g|A_i$ is a homeomorphism for all A_i.

15.37 Claim. *$g^{-1}(A) \neq \emptyset$. In fact, the number of components of $g^{-1}(A)$ is odd.*

Proof of Claim 15.37. By Claim 15.35, V_n contains a core v_1 of V_1. Let δ be a path in V_1 from $x_1 \in A_1$ to v_1. Then $\pi_1 W_1$ and $\delta v_1 \delta^{-1}$ generate the group $\pi_1(C(\mathfrak{k}_1)) = \pi_1(W_1, x_1) *_{\pi_1(A_1,x_1)} \pi_1(V_1, x_1)$. Since g is transversal with respect to A, it follows that $\mathring{V}_i \cap \mathring{W}_{i+1}$ and $\mathring{V}_{i+1} \cap \mathring{W}_{i+2}$ are mapped to different sides of A; hence, if the number of components of $g^{-1}(A)$ is even, g maps W_1 and V_n – and hence v_1 – both into X or both into V. Since g is a homotopy equivalence, hence $g_\#$ an isomorphism, it follows that $\pi_1 C(\mathfrak{k})$ is isomorphic to a subgroup of $\pi_1 X$ or $\pi_1 V$, in fact, to $\pi_1 X$ or $\pi_1 V$, respectively. In the latter case $\pi_1 C(\mathfrak{k})$ is cyclic; hence, $\mathfrak{k}$ is the trivial knot, contradicting the assumption that $\mathfrak{k}$ is a cable knot. In the first case $\pi_1 C(\mathfrak{k})/\pi_1 X = 1$. This implies that $H_1(X) \to H_1(C(\mathfrak{k}))$ is surjective; hence $H_1(C(\mathfrak{k}), X) = 0$ as follows from the exact sequence

$$H_1(X) \to H_1(C(\mathfrak{k})) \xrightarrow{0} H_1(C(\mathfrak{k}), X) \xrightarrow{0} H_0(X) \xrightarrow{\cong} H_0(C(\mathfrak{k})).$$

On the other hand, by the excision theorem $H_i(C(\mathfrak{k}), X) \cong H_i(V, A)$ and the exact sequence

$$\begin{array}{ccccccccc} H_2(V,A) & \longrightarrow & H_1(A) & \xrightarrow{z \mapsto t^{\pm q}} & H_1(V) & \longrightarrow & H_1(V,A) \xrightarrow{0} H_0(A) \xrightarrow{\cong} H_0(V); \\ & & \cong\big\downarrow & & \big\downarrow\cong & & \\ & & \langle z \mid - \rangle & & \langle t \mid - \rangle & & \end{array}$$

implies $H_1(V, A) \cong \mathbb{Z}_{|q|} \neq 0$ since $|q| \geq 2$. Thus the assumption that the number of components of $g^{-1}(A)$ is even was wrong. $\square$

15.38 Claim. *The number n of components of $g^{-1}(A)$ is 1. (This finishes the proof of Lemma 15.31.)*

Proof of Claim 15.38. It will be shown that for $n > 1$ the mapping g can be homotopically deformed to reduce the number of components of $g^{-1}(A)$ by 2, contradicting the minimality of n; thus, by Claim 15.37, $n = 1$. The proof applies a variation of Stalling's technique of *binding ties* from [344] (see also Massey [230, VII.6]).

Choose $x \in A$, $x_i \in A_i$ for $1 \leq i \leq n$ such that $g(x_i) = x$. There is a path α in $C(\mathfrak{k}_1)$ from x_1 to x_n with the following properties:

(1) $g(\alpha) \simeq 0$ in $C(\mathfrak{k})$;

(2) (i) $\alpha = \alpha_1 \dots \alpha_r$ where

(ii) $\mathring{\alpha}_j \subset C(\mathfrak{k}_1) - \bigcup_{i=1}^n A_i$, $\partial\alpha_j \in \bigcup_{i=1}^n A_i$ and

(iii) α_j is either a loop with some x_i as basepoint or a path from x_i to $x_{i\pm 1}$.

A path with these properties can be obtained as follows: Let α' be a path from x_1 to x_n. Then $[g \circ \alpha'] \in \pi_1(C(\mathfrak{k}), x)$ and, since $g_\#$ is an isomorphism, there is a loop $\delta \subset C(\mathfrak{k}_1)$ in the homotopy class $g_\#^{-1}[g \circ \alpha'] \in \pi_1(C(\mathfrak{k}_1), x_n)$. Then $\alpha = \alpha'\delta^{-1}$ has property (1). We can choose α transversal to $g^{-1}(A)$ and, since each A_i is connected, intersecting an A_i only in x_i.

Assume that α is chosen such that the number r is minimal for all paths with the properties (1) and (2). In $\pi_1 C(\mathfrak{k}) = \pi_1 X *_{\pi_1 A} \pi_1 V$,

$$1 = [g \circ \alpha] = [g \circ \alpha_1] \dots [g \circ \alpha_r].$$

Since $g \circ \alpha_i$ and $g \circ \alpha_{i+1}$ are in different components X, V it follows that there is at least one α_j with $[g \circ \alpha_j] \in \pi_1 A$. A loop α_j from x_i to x_i in $V_i \cap W_{i+1}$ with $[g \circ \alpha_j] \in \pi_1 A$ can be pushed into $V_{i-1} \cap W_i$, contradicting the minimality of r. Therefore α_j connects x_i and x_{i+1}, for a suitable i.

By Claim 15.35, $V_i \cap W_{i+1} \cong A_1 \times [i, i+1]$ and $A_i = A_1 \times \{i\}$, $A_{i+1} = A_1 \times \{i+1\}$, and therefore α_j is homotopic to an arc $\beta \subset \partial(V_i \cap W_{i+1})$ connecting x_i and x_{i+1}. Let γ be an arc in $\partial(V_i \cap W_{i+1})$ such that $\beta \cup \gamma$ is a meridian of the solid torus $V_i \cap W_{i+1}$ and bounds a disk D. We may assume that $\partial D \cap A_i$ and $\partial D \cap A_{i+1}$ are arcs connecting the boundary components, see Figure 15.13.

Let B^3 be the closure of the complement of a regular neighborhood of $A_i \cup D \cup A_{i+1}$ in $V_i \cap W_{i+1}$; then B^3 is a 3-ball.

In the following we keep g fixed outside of a regular neighborhood of $V_i \cap W_{i+1}$. Since $[g \circ \beta] \in \pi_1 A$ and $g \circ \beta \simeq g \circ \gamma$, g may be deformed such that $g(\beta) \subset A$ and $g(\gamma) \subset A$. Since A is incompressible in $C(\mathfrak{k})$ and $\pi_2 X = 0 = \pi_2 V$, g can be

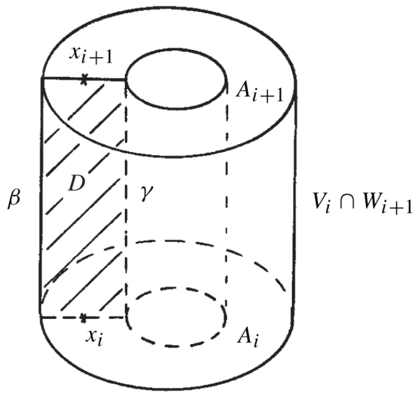


Figure 15.13

altered by an homotopy such that g maps D and also the small neighborhood into A, that is, $g(V_i \cap W_{i+1} - B^3) \subset A$. Finally, since $\pi_3 V = \pi_3 X = 0$, we can alter g by a homotopy such that $g(B^3) \subset A$; thus $g(V_i \cap W_{i+1}) \subset A$, and an additional slight adjustment eliminates both components A_i, A_{i+1} of $g^{-1}(A)$. □

This finishes the proof of Lemma 15.31. □

15.39 Lemma. *Let $\mathfrak{k}_1$ and $\mathfrak{k}$ be (p_1, q_1)- and (p, q)-cable knots about the cores $\mathfrak{h}_1$ and $\mathfrak{h}$ where $|q_1|$, $|q| \geq 2$, and let*

$$C(\mathfrak{k}) = C(\mathfrak{h}) \cup V, \quad C(\mathfrak{h}) \cap V = \partial C(\mathfrak{h}) \cap \partial V = A$$

be an annulus. If $\pi_1 C(\mathfrak{k}_1) \cong \pi_1 C(\mathfrak{k})$ then

(a) *there is a homeomorphism $F\colon C(\mathfrak{h}_1) \to C(\mathfrak{h})$ such that $A_1 = F^{-1}(A)$ defines a cable presentation of $\mathfrak{k}_1$, that is*

$$C(\mathfrak{k}_1) = \overline{C(\mathfrak{k}_1) - C(\mathfrak{h}_1)} \cup C(\mathfrak{h}_1),$$
$$\overline{C(\mathfrak{k}_1) - C(\mathfrak{h}_1)} \cap C(\mathfrak{h}_1) = \partial\overline{C(\mathfrak{k}_1) - C(\mathfrak{h}_1)} \cap \partial C(\mathfrak{h}_1) = A_1,$$

and

(b) $|p_1| = |p|$ *and* $|q_1| = |q|$.

Proof. We may assume that $\mathfrak{h}_1$ and $\mathfrak{h}$ are non-trivial, because otherwise $\mathfrak{k}_1$ and $\mathfrak{k}$ are torus knots and Lemma 15.39 follows from Proposition 15.5. We have $\pi_1 C(\mathfrak{k}) = \pi_1 C(\mathfrak{h}) *_{\pi_1 A} \pi_1 V$. Since $\pi_1 A \to \pi_1 V_1$ is not surjective (as $|q| \geq 2$), the free product with amalgamation is not trivial. By Lemma 15.31, there is a homotopy equivalence

$f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ such that $f^{-1}(A) = A_1$ is an annulus. Then A_1 decomposes $C(\mathfrak{k}_1)$ into a knot manifold X_1 and a solid torus V_1:

$$C(\mathfrak{k}_1) = X_1 \cup V_1,\ X_1 \cap V_1 = \partial X_1 \cap \partial V_1 = A_1.$$

For any basepoint $a_1 \in A_1$,

$$\pi_1(C(\mathfrak{k}_1), a_1) = \pi_1(X_1, a_1) *_{\pi_1(A_1,a_1)} \pi_1(V_1, a_1).$$

Since $f^{-1}(A) = A_1$ consists of one component only, one of the groups $f_\#(\pi_1(X_1,a_1))$ and $f_\#(\pi_1(V_1,a_1))$ is contained in $\pi_1(C(\mathfrak{h}),\ f(a_1))$ and the other in $\pi_1(V, f(a_1))$. By assumption $C(\mathfrak{h})$ and X_1 are knot manifolds, V, V_1 solid tori and $f_\#$ is an isomorphism. From the solution of the word problem in free products with amalgamated subgroups, see [382, 2.3.3] it follows that

$$f_\#(\pi_1(X_1,a_1)) = \pi_1(C(\mathfrak{h}), f(a_1)) \quad \text{and } f_\#(\pi_1(V_1,a_1)) = \pi_1(V, f(a_1)).$$

This implies

(1) $f(X_1) \subset C(\mathfrak{h})$, $f(V_1) \subset V$, and that $(f|X_1)_\#$ and $(f|V_1)_\#$ are isomorphisms and $f|X_1\colon X_1 \to C(\mathfrak{h})$ and $f|V_1\colon V_1 \to V$ are homotopy equivalences because all spaces are $K(\pi,1)$.

For the proof of (b) we note that $(f|A_1)_\#\colon \pi_1 A_1 \to \pi_1 A$ is also an isomorphism.

Assume that $f|X_1$ is homotopic to a mapping $f_0\colon X_1 \to C(\mathfrak{h})$ such that $f_0(\partial X_1) \subset \partial C(\mathfrak{h})$ and $f_0|\partial A_1 = f|\partial A_1$. Then, by Waldhausen [367, Theorem 6.1] there is a homotopy $f_t\colon (X_1, \partial X_1) \to (C(\mathfrak{h}), \partial C(\mathfrak{h}))$, $0 \le t \le 1$, such that f_1 is a homeomorphism; this proves (a).

To prove the above assumption on ∂X_1 we consider $B_1 = \partial X_1 \cap \partial C(\mathfrak{k}_1)$. Now $\partial B_1 = \partial A_1$. We have to show that $f|B_1\colon (B_1, \partial B_1) \to (C(\mathfrak{h}), \partial C(\mathfrak{h}))$ is not essential. Otherwise, by Lemma 15.19 there is a properly embedded essential annulus $A' \subset C(\mathfrak{h})$ such that $\partial A' = \partial A$. The components of ∂A are (p,q)-curves on $\partial C(\mathfrak{h})$ and $(n, \pm 1)$-curves on $\partial C(\mathfrak{k})$ for a suitable n; the last statement is a consequence of the fact that the components of ∂A are isotopic to $\mathfrak{k}$, see Lemma 15.24.

Since A' is essential, $C(\mathfrak{h})$ is either the complement of a cable knot or of a product knot, see Lemma 15.24. In the first case, the components of $\partial A'$ are isotopic to the knot $\mathfrak{h}$; hence $(n', \pm 1)$-curves on $\partial C(\mathfrak{h})$. In the latter case they are $(\pm 1, 0)$-curves. Both cases contradict the fact $\partial A = \partial A'$ and the assumption $|q| \ge 2$.

For the proof of (b), let m_1 and m be meridians on the boundaries ∂V_1, ∂V of the regular neighborhoods V_1, V of $\mathfrak{h}_1$, $\mathfrak{h}$. In the proof of (a) we saw that there is a homotopy equivalence $f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ with $f(A_1) = A$. Let s_1 be a component of ∂A_1 and $s = f(s_1)$; consider s_1 and s as oriented curves. Then s_1 represents $\pm p_1 m_1$ in $H_1(X_1)$ and s represents $\pm p\, m$ in $H_1(C(\mathfrak{h}))$. The homotopy equivalence f induces an isomorphism $f_*\colon H_1(X_1) \to H_1(C(\mathfrak{h}))$ and $f_*(p_1 m_1) = \pm p\, m$; hence, $|p_1| = |p|$.

By (a), $(f|V_1)_\#$ and $(f|A_1)_\#$ are isomorphisms, thus $f_*\colon H_1(V_1, A_1) \to H_1(V, A)$ is an isomorphism. Now $H_1(V_1, A_1) \cong \mathbb{Z}_{|q_1|}$ and $H_1(V, A) \cong \mathbb{Z}_{|q|}$ imply $|q_1| = |q|$. □

15.40 *Proof of Theorem 15.26.* Assume that $\mathfrak{k}_0$, $\mathfrak{k}_1$, $\mathfrak{k}_2$ are (p,q)-, (p_1,q_1)-, (p_2,q_2)-cables about $\mathfrak{h}_0$, $\mathfrak{h}_1$, $\mathfrak{h}_2$ with the same group. If $\mathfrak{h}_0$ is unknotted then $\mathfrak{k}_0$ is a torus knot and the equivalence of $\mathfrak{k}_0$, $\mathfrak{k}_1$, $\mathfrak{k}_2$ is a consequence of Proposition 15.5. Now we assume that $\mathfrak{h}_0$, $\mathfrak{h}_1$, $\mathfrak{h}_2$ are knotted. By Lemma 15.39, $C(\mathfrak{h}_i) \cong C(\mathfrak{h}_0)$, $|p_i| = |p|$, $|q_i| = |q|$ for $i = 1, 2$.

Let, for $i = 0, 1, 2$, an essential annulus A_i decompose $C(\mathfrak{k}_i)$ into a knot manifold $C(\mathfrak{h}_i)$ and a solid torus V_i; now the knot $\mathfrak{k}_i$ is parallel to each of the components of ∂A_i. Because of Lemma 15.39 there are homotopy equivalences

$$F_{ij}\colon C(\mathfrak{k}_i) \to C(\mathfrak{k}_j) \quad (i = 0, 1;\ j = 1, 2)$$

such that

$$\tilde{F}_{ij} = F_{ij} \mid C(\mathfrak{h}_i)\colon (C(\mathfrak{h}_i), A_i) \to (C(\mathfrak{h}_j), A_j))$$

are homeomorphisms.

It suffices to prove that $\tilde{F}_{01}$, $\tilde{F}_{12}$ or $\tilde{F}_{02} = \tilde{F}_{12} \circ \tilde{F}_{01}$ can be extended to a homeomorphism of S^3, because by a theorem of Schubert [318, p. 253] (see also Gramain [142]) cable knots are determined by their cores and winding numbers.

Let (m_i, ℓ_i) be meridian-longitude for $\mathfrak{h}_i$, $i = 0, 1, 2$; assume that they are oriented such that the components of ∂A_i are homologous to $pm_i + q\ell_i$ on $\partial C(\mathfrak{h}_i)$. There are numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \in \{1, -1\}$ and $x, y \in \mathbb{Z}$ such the $\tilde{F}_{ij}|\partial C(\mathfrak{h}_i)$ are given by the following table.

$\tilde{F}_{01}$		$\tilde{F}_{12}$		$\tilde{F}_{02}$	
m_0	$\mapsto m_1^{\alpha}\ell_1^{x}$	m_1	$\mapsto m_2^{\gamma}\ell_2^{y}$	m_0	$\mapsto m_2^{\alpha\gamma}\ell_2^{\alpha y+\delta x}$
ℓ_0	$\mapsto \ell_1^{\beta}$	ℓ_1	$\mapsto \ell_2^{\delta}$	ℓ_0	$\mapsto \ell_2^{\beta\delta}$
$m_0^p\ell_0^q$	$\mapsto (m_1^p\ell_1^q)^{\varepsilon}$	$m_1^p\ell_1^q$	$\mapsto (m_2^p\ell_2^q)^{\eta}$	$m_0^p\ell_0^q$	$\mapsto (m_2^p\ell_2^q)^{\varepsilon\eta}$;

The last row is a consequence of the fact that the $\tilde{F}_{ij}\colon A_i \to A_j$ are homeomorphisms.

If some m_i is mapped to $m_j^{\pm 1} = m_j^{\pm 1}\ell^0$ then the homeomorphism $\tilde{F}_{ij}$ can be extended to S^3 and this finishes the proof. Hence, we will show that one of the exponents x, y and $\alpha y + \delta x$ vanishes. Assume that $x \neq 0 \neq y$. Now

$$(m_1^p\ell_1^q)^{\varepsilon} = \tilde{F}_{01}(m_0^p\ell_0^q) = m_1^{\alpha p}\ell_1^{\beta q + xp}$$

implies

$$\varepsilon p = \alpha p, \quad \varepsilon q = \beta q + xp \quad \Longrightarrow \quad \varepsilon = \alpha, \quad xp = (\alpha - \beta)q.$$

Therefore, $p \neq 0 \neq x$ implies $\alpha \neq \beta$, and $|\alpha| = |\beta| = 1$ gives $\alpha = -\beta$. Therefore $xp = 2\alpha q$ and $x = \frac{2\alpha q}{p}$. The same arguments for $\tilde{F}_{12}$ imply that $\delta = -\gamma$ and $y = \frac{2\gamma q}{p}$. Finally,

$$\alpha y + \delta x = \alpha \frac{2\gamma q}{p} - \gamma \frac{2\alpha q}{p} = 0. \qquad \square$$

15.41 *Proof of Proposition 15.29.* Assume that $\mathfrak{k}_1$ and $\mathfrak{k}$ are prime knots with isomorphic groups $\pi_1 C(\mathfrak{k}_1) \cong \pi_1 C(\mathfrak{k})$ and non-homeomorphic complements $C(\mathfrak{k}_1)$ and $C(\mathfrak{k})$. Theorem 15.25 implies that both $C(\mathfrak{k}_1)$ and $C(\mathfrak{k})$ must contain an essential annulus. If either $\mathfrak{k}_1$ or $\mathfrak{k}$ is a torus knot, so is the other, see Theorem 6.1, and it follows from the classification of torus knots, see Theorem 3.39, that $\mathfrak{k}_1$ and $\mathfrak{k}$ are of the same knot type. Hence, $\mathfrak{k}_1$ and $\mathfrak{k}$ are (p_1, q_1)-cable and (p, q)-cable knots about the non-trivial cores $\mathfrak{h}_1$ and $\mathfrak{h}$ where $|q_1|, |q| \geq 2$. Let A be an annulus and let V be a solid torus such that $C(\mathfrak{k}) \cong C(\mathfrak{h}) \cup V$ and $C(\mathfrak{h}) \cap V = \partial C(\mathfrak{h}) \cup \partial V = A$.

By Lemma 15.39, there exists a homeomorphism $F: C(\mathfrak{h}_1) \to C(\mathfrak{h})$ such that $A_1 = F^{-1}(A)$ defines a cable presentation of $\mathfrak{k}_1$ and $|p_1| = |p|$ and $|q_1| = |q|$.

Let (m_1, ℓ_1) and (m, ℓ) be a meridian-longitude pair for $\mathfrak{h}_1$ and $\mathfrak{h}$ respectively. Note that a (p_1, q_1)-curve on $\partial C(\mathfrak{h}_1)$ is mapped to a $\pm(p, q)$-curve on $\partial C(\mathfrak{k})$. By changing the orientation of $\mathfrak{h}_1$ (or $\mathfrak{h}$) along with the orientation of both m_1 and ℓ_1 (or m and ℓ) we can arrange that $q_1 = q \geq 2$. Set $p_1 = \epsilon\, p$ where $\epsilon \in \{\pm 1\}$. Thus homologically we have $F_*(\ell_1) = \ell^{\pm 1}$ and $F_*(m_1) = m^{\pm 1} \ell^n$ for some $n \in \mathbb{Z}$, and $F_*(m_1^{p_1} \ell_1^{q_1}) = m^{\alpha p} \ell^{\alpha q}$, $\alpha \in \{\pm 1\}$. On the other hand,

$$F_*(m_1^{p_1} \ell_1^{q_1}) = m^{\pm p_1} \ell^{p_1 n \pm q_1},$$

and it follows $\epsilon\, p\, n \pm q = \alpha q$. This implies $n = 0$ or $p\, n = \pm 2q$. If $n = 0$ then we can extend $F: C(\mathfrak{h}_1) \to C(\mathfrak{h})$ to a homeomorphism $F: S^3 \to S^3$ with $F(\mathfrak{h}_1) = \mathfrak{h}$. Since a boundary component of A and A_1 belongs to the same knot type as $\mathfrak{k}$ and $\mathfrak{k}_1$ respectively and since $F(A_1) = A$ it follows that $\mathfrak{k}_1$ and $\mathfrak{k}$ are of the same knot type. Hence, $n \neq 0$ and $n\, p = \pm 2q$. Since p and q are coprime it follows that $1 \leq |p| \leq 2$ and

$$|n| = \begin{cases} 2q, & \text{if } |p| = 1 \\ q \text{ (odd)}, & \text{if } |p| = 2, \end{cases}$$

and so $|n| \neq 0, 1, 2$, since $q \geq 2$.

Finally, $F: C(\mathfrak{h}_1) \to C(\mathfrak{h})$ is a homeomorphism and $F_*(m_1) = m^{\pm 1} \ell^n$. Hence, either $S^3_{1/n}(\mathfrak{h}) \cong S^3$ or $S^3_{-1/n}(\mathfrak{h}) \cong S^3$. $\square$

15.D Groups of product knots

Next we consider problems for product knots similar to those in Section 15.C. The situation now is in some sense simpler, as product knots have Property P, see Theorem 15.10; hence, product knots with homeomorphic complements are of the same type. However, the groups of two product knots of different type may be isomorphic as we have shown in 15.3. We will now prove that there are no other possibilities than those described in Example 15.3.

15.42 Lemma. *Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be knots with $\pi_1 C(\mathfrak{k}_1) \cong \pi_1 C(\mathfrak{k}_2)$. Then both knots are prime or both are product knots.*

Proof. Assume that $\mathfrak{k}_2$ is a product knot. Then there is a properly embedded incompressible annulus $A \subset C(\mathfrak{k}_2)$ such that $C(\mathfrak{k}_2) = X' \cup X''$, $A = X' \cap X''$ where X' and X'' are knot manifolds. Since $\pi_n C(\mathfrak{k}_k) = 0$ for $i = 1, 2, n \geq 2$ there is a homotopy equivalence $f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k}_2)$. By Claim 15.33, see Remark 15.32, we may assume that the components of $f^{-1}(A)$ are incompressible, properly embedded annuli which are not boundary parallel in $C(\mathfrak{k}_1)$. Now $f^{-1}(A) = \emptyset$ is impossible, since, otherwise, $f_\#(\pi_1 C(\mathfrak{k}_1)) \subset \pi_1 X'$ or $f_\#(\pi_1 C(\mathfrak{k}_1)) \subset \pi_1 X''$, contradicting the assumption that $\pi_1 X'$ and $\pi_1 X''$ are proper subgroups of $\pi_1 C(\mathfrak{k}_2)$ and that $f_\#$ is an isomorphism. By Lemma 15.24, $C(\mathfrak{k}_1)$ is the complement of a product knot or a cable knot. In the first case the assertion is proved. In the latter case, $\pi_1 C(\mathfrak{k}_2) \cong \pi_1 C(\mathfrak{k}_1)$ is the group of a cable knot and, thus, applying the arguments of Claim 15.33 to $C(\mathfrak{k}_2)$ and the inverse homotopy equivalence, it follows that $C(\mathfrak{k}_2)$ is also the complement of a cable knot. Since products knots have Property P (Theorem 15.10), we conclude that $\mathfrak{k}_2$ is a cable knot, contradicting the fact that cable knots are prime, see Schubert [318, p. 250, Satz 4], Gramain [142, Cor. 2], Cromwell [74, Cor. 4.4.3]. □

15.43 Theorem (Feustel–Whitten [102]). *Let $\mathfrak{k} = \mathfrak{k}_1 \# \ldots \# \mathfrak{k}_m$ and $\mathfrak{h} = \mathfrak{h}_1 \# \ldots \# \mathfrak{h}_n$ be knots in S^3, where the $\mathfrak{k}_i$ and $\mathfrak{h}_j$ are prime and $n > 1$. If $\pi_1(S^3 - \mathfrak{k}) \cong \pi_1(S^3 - \mathfrak{h})$ then $\mathfrak{k}$ is a product knot, $m = n$ and there is a permutation σ such that $\mathfrak{k}_j$ and $\mathfrak{h}_{\sigma(j)}$ are of the same type.*

Proof. By Lemma 15.42, $\mathfrak{k}$ is also a product knot, i.e. $m > 1$. Let A be a properly embedded annulus in $C(\mathfrak{h}) = X' \cup X''$, $A = X' \cap X''$ where X' and X'' are knot manifolds. As in the proof above we conclude that there is a homotopy equivalence $f\colon C(\mathfrak{k}) \to C(\mathfrak{h})$ such that $f^{-1}(A)$ consists of disjoint incompressible, properly embedded essential annuli. Let A_1 be a component of $f^{-1}(A)$. In the following commutative diagram all groups are isomorphic to $\mathbb{Z}$.

$$\begin{array}{ccc} H_1(A_1) & \xrightarrow{j_{1*}} & H_1(C(\mathfrak{k}_1)) \\ {\scriptstyle (f|A_1)_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ H_1(A) & \xrightarrow{j_*} & H_1(C(\mathfrak{k})); \end{array}$$

where $j_1\colon A_1 \to C(\mathfrak{k}_1)$, $j\colon A \to C(\mathfrak{h})$ are the inclusions. As f is a homotopy equivalence, f_* is an isomorphism. Since $C(\mathfrak{k})$ and $C(\mathfrak{h})$ are complements of product knots the components of ∂A_1 and ∂A bound disks in $\overline{S^3 - C(\mathfrak{k})}$ and $\overline{S^3 - C(\mathfrak{h})}$, respectively, see Lemma 15.22. The boundaries of these disks are generators of $H_1(C(\mathfrak{k}))$ and $H_1(C(\mathfrak{h}))$; hence, j_{1*} and j_* are isomorphisms. This proves that $(f|A_1)_*$ is a isomorphism and, consequently, that $f|A_1\colon A_1 \to A$ is a homotopy equivalence homotopic to a homeomorphism. Since f is transversal with respect to A, see (1) in the proof of 15.31, there is a neighborhood $A \times [0,1) \subset C(\mathfrak{h})$ such that the homotopy $f|A_1$ can be extended to a homotopic deformation of f which is constant outside of $A \times [0,1)$. By the same arguments as in the proof of 15.38, one concludes that in addition f can be chosen such that $A_1 = f^{-1}(A)$ is connected. The annulus A_1 decomposes $C(\mathfrak{k})$ into two subspaces Y', Y'' of S^3 bounded by tori, which are mapped to X' and X'', respectively: $f(Y') \subset X'$, $f(Y'') \subset X''$. It follows that $(f|Y')_\#$ and $(f|Y'')_\#$ are isomorphisms. This proves that Y' and Y'' are knot manifolds. Therefore $\mathfrak{k} = \mathfrak{k}' \# \mathfrak{k}''$ and $\mathfrak{h} = \mathfrak{h}' \# \mathfrak{h}''$ where $\mathfrak{k}'$ and $\mathfrak{h}'$ have isomorphic groups. This isomorphism maps meridional elements to meridional elements, since they are realized by the components of ∂A_1 and ∂A. The same is true for $\mathfrak{k}''$ and $\mathfrak{h}''$.

Assume that $\mathfrak{h}'$ and, hence, $\mathfrak{k}'$ are prime knots. Then $\partial(C(\mathfrak{k}')) = B_1 \cup A_1$ where B_1 is an annulus. If $f(B_1)$ is essential then there is a properly embedded essential annulus in $C(\mathfrak{k}')$. One has $\partial B_1 = \partial A_1$ and $f(\partial B_1) = \partial A$. Now ∂A bounds meridional disks in $\overline{S^3 - C(\mathfrak{h})}$ and therefore also in $\overline{S^3 - C(\mathfrak{h}')}$; this contradicts the assumption that $\mathfrak{h}'$ is prime. Therefore $f(B_1)$ is not essential and thus $f|B_1$ is homotopic to a mapping with image in $\partial C(\mathfrak{h}')$ – by a homotopy constant on $\partial B_1 = \partial A_1$. This homotopy can be extended to a homotopy of f which is constant on A_1. Finally one obtains a homotopy equivalence $(Y', \partial Y') \to (X', \partial X')$ which preserves meridians. By Corollary 6.5 of [367], see Appendix B.7, $Y' \cong X'$, where the homeomorphism maps meridians to meridians and, thus, can be extended to S^3, see Theorem 3.16. This proves that $\mathfrak{k}'$ and $\mathfrak{h}'$ are of the same knot type.

Now the theorem follows from the uniqueness of the prime factor decomposition of knots. □

In fact, we have proved more than claimed in Theorem 15.43:

15.44 Proposition. *Under the assumptions of Theorem* 15.43*, there is a system of pairwise disjoint, properly embedded annuli* $A_1, \dots, A_{n-1}$ *in* $C(\mathfrak{k})$ *and a homeomorphism* $f\colon C(\mathfrak{k}) \to C(\mathfrak{h})$ *such that* $\{A_1, \dots, A_{n-1}\}$ *decomposes* $C(\mathfrak{k})$ *into the knot manifolds* $C(\mathfrak{k}_1), \dots, C(\mathfrak{k}_n)$ *and* $\{f(A_1), \dots, f(A_{n-1})\}$ *decomposes* $C(\mathfrak{h})$ *into* $C(\mathfrak{h}_{\sigma(1)}), \dots, C(\mathfrak{h}_{\sigma(n)})$.

Since product knots have the Property P, the system of homologous meridians $(m(\mathfrak{k}_1), \dots, m(\mathfrak{k}_n))$ *is mapped, for a fixed* $\varepsilon \in \{1, -1\}$*, onto the system of homologous meridians* $(m(\mathfrak{h}_{\sigma(1)})^\varepsilon, \dots, m(\mathfrak{h}_{\sigma(n)})^\varepsilon)$. □

15.45 Proposition (Simon [338]). *If* $\mathfrak{G}$ *is the group of a knot with* n *prime factors* $(n \geq 2)$*, then* $\mathfrak{G}$ *is the group of at most* 2^{n-1} *knots of mutually different knot types.*

Moreover, when the n prime factors are of mutually different knot types and when each of them is non-invertible and non-amphicheiral, then $\mathfrak{G}$ is the group of exactly 2^{n-1} knots of mutually different types and of 2^{n-1} knot manifolds.

Proof. By Theorem 3.16, an oriented knot $\mathfrak{k}$ is determined up to isotopy by the peripheral system $(\mathfrak{G}, m, \ell)$ and we use this system now to denote the knot. Clearly (proof as E 15.5, see also Proposition 3.18),

$$-\mathfrak{k} = (\mathfrak{G}, m^{-1}, \ell^{-1}),\ \mathfrak{k}^* = (\mathfrak{G}, m^{-1}, \ell),\ -\mathfrak{k}^* = (\mathfrak{G}, m, \ell^{-1}),$$

and

$$\mathfrak{k}_1 \# \mathfrak{k}_2 = (\mathfrak{G}_1 *_{m_1=m_2} \mathfrak{G}_2, m_1, \ell_1\ell_2).$$

Let $\mathfrak{k} = \mathfrak{k}_1 \# \ldots \# \mathfrak{k}_n$, $n \geq 2$. By Theorem 15.10, $\mathfrak{k}$ has Property P; hence, on $\partial C(\mathfrak{k})$ the meridian is uniquely determined up to isotopy and reversing the orientation. It is

$$(\mathfrak{G}, m, \ell) = (\mathfrak{G}_1, m_1, \ell_1) \# \ldots \# (\mathfrak{G}_n, m_n, \ell_n) = (\mathfrak{G}_1 \underset{m_1=\ldots m_n}{* \ldots *} \mathfrak{G}_n, m_1, \ell_1\ell_2 \ldots \ell_n).$$

Suppose $\mathfrak{h}$ is a knot whose group is isomorphic to $\mathfrak{G}$. Now the above remark and Proposition 15.44 imply that the peripheral system of $\mathfrak{h}$ is given by

$$(\mathfrak{G}_1, m_1^{\varepsilon}, \ell_1^{\delta_1}) \# \ldots \# (\mathfrak{G}_n, m_n^{\varepsilon}, \ell_n^{\delta_n}) = (\mathfrak{G}_1 \underset{m_1=\ldots m_n}{* \ldots *} \mathfrak{G}_n, m_1^{\varepsilon}, \ell_1^{\delta_1} \ldots \ell_n^{\delta_n}).$$

Corresponding to the choices of $\varepsilon, \delta_1, \ldots, \delta_n$ there are 2^{n+1} choices for $\mathfrak{h}$. Therefore $\mathfrak{h}$ represents one of, possible, 2^{n+1} oriented isotopy types and $\frac{1}{4}2^{n+1}$ knot types.

Clearly, this number is attained for knots with the properties mentioned in the second assertion of the proposition. □

15.46 Corollary. *Since prime knots are determined by their groups, see Whitten's Rigidity Theorem 15.28, the hypothesis $n \geq 2$ in Proposition 15.45 is unnecessary.* □

15.E History and sources

The theorem of F. Waldhausen [365] on sufficiently large irreducible 3-manifolds, see Appendix B.7, implies that the peripheral group system determines the knot complement. Then the question arises to what extent the knot group characterizes the knot type. The difficulty of this problem becomes obvious by the example of J. H. C. Whitehead [370] of different links with homeomorphic complements, see Proposition 15.1. First results were obtained by D. Noga [277] who proved Property P for product knots, and by R. H. Bing and J. M. Martin [22] who showed it for the four-knot, twist knots, product knots and again for satellites. The 2-bridge knots have Property P by Takahaski [347].

A first final answer was given by C. Gordon and J. Luecke [137] proving that the knot complement determines the knot type.

The Annulus and the Torus Theorem of C. D. Feustel [100, 101], Cannon and Feustel [62] gave strong tools to approach the problem of to what extent the group determines the complement. The results of J. Simon [335, 336, 337, 338], W. Whitten [374], Feustel and Whitten [102], K. Johannson [179], Whitten [375] and Culler, Shalen, Gordon and Luecke [81] combine to give a positive answer to the question: Is the complement of a prime knot determined by its group?

The final step in the proof of the Property P conjecture was published in 2004 by P. B. Kronheimer and T. S. Mrowka [204], as the combined result of efforts of mathematicians working in several different fields, see Paragraph 3.25.

15.F Exercises

E 15.1. Use Lemma 16.3 to prove that $h^{-1}(\ell) = \pm\tilde{\ell}$ satisfies equation (15.5) in 15.14.

E 15.2. Let M be a 3-manifold, $V \subset M$ a solid torus, $\overline{\partial V \cap \mathring{M}} = A$ an annulus such that the core of A is a longitude of V. Then A is boundary parallel.

E 15.3. Show that both descriptions in Definition 15.15 and Example 15.21 define the same knot.

E 15.4. Let $\mathfrak{k}$ be a (p,q)-cable knot and let A be an annulus, defining $\mathfrak{k}$ as cable. Then $\mathbb{Z} \cong H_1(A) \to H_1(C(\mathfrak{k})) \cong \mathbb{Z}$ is defined by $c \mapsto \pm pq\, m$, where c is a core curve of A and if m is a meridian of $\mathfrak{k}$.

E 15.5. Let $\mathfrak{k} = (\mathfrak{G}, m, \ell)$ and $\mathfrak{k}_i = (\mathfrak{G}_i, m_i, \ell_i)$. Prove that $-\mathfrak{k} = (\mathfrak{G}, m^{-1}, \ell^{-1})$, $\mathfrak{k}^* = (\mathfrak{G}, m^{-1}, \ell)$, $-\mathfrak{k}^* = (\mathfrak{G}, m, \ell^{-1})$, and $\mathfrak{k}_1 \# \mathfrak{k}_2 = (\mathfrak{G}_1 *_{m_1=m_2} \mathfrak{G}_2, m_1, \ell_1\ell_2)$.

Chapter 16

Bridge number and companionship

In what follows we consider solid tori in the 3-sphere which contain a given knot $\mathfrak{k} \subset S^3$ in its interior. Such tori were studied in a seminal paper by Schubert, the famous *Knoten und Vollringe* [318], where he introduced the *companions* of a knot. Schubert was motivated by the question of if a knot has only finitely many companions. More precisely, he asks if each companion occurs in only a finite number of *orders* and in each order it has only a finite *multiplicity*. This multiplicity is defined in a natural way that generalizes the multiplicity of a prime factor in a factorization of a knot. Schubert answered these questions in the affirmative (see [319, §8]).

In order to prove these results Schubert introduced an invariant, the so-called bridge number $b(\mathfrak{k})$, which is compatible with companionship i.e. the bridge number of a *proper companion* of $\mathfrak{k}$ is smaller than $b(\mathfrak{k})$. The aim of this chapter is to prove this result (Theorem 16.28), and that the minimal bridge number $b(\mathfrak{k})$ minus one is additive (Theorem 16.27).

16.A Seifert surfaces for satellites

In this section we will introduce some notation and we will study the Seifert surfaces for satellite knots.

Let $\widehat{V}$ be a solid torus i.e. $\widehat{V}$ is homeomorphic to $S^1 \times D^2$. A simple closed non-oriented curve α in the interior of $\widehat{V}$ is called a *core curve* if there exists a homeomorphism $f\colon \widehat{V} \to S^1 \times D^2$ which maps α to the curve $S^1 \times \{*\}$. Up to orientation, two core curves of $\widehat{V}$ are ambient isotopic in V. A solid torus $\widehat{V} \subset S^3$ is called *knotted* if the knot $\mathfrak{c}$ represented by one of its core curves is a non-trivial knot.

Let $\mathfrak{k}$ be an oriented knot in S^3 and let $\widehat{V} \subset S^3$ be a solid torus which contains $\mathfrak{k}$ in its interior. The minimal number of intersection points of $\mathfrak{k}$ with a meridian disk of $\widehat{V}$ is called the *order* of $\widehat{V}$ with respect to $\mathfrak{k}$. The *winding number* of $\mathfrak{k}$ in $\widehat{V}$ is the absolute value of the algebraic intersection number of $\mathfrak{k}$ with a meridian disk of $\widehat{V}$. Note that the winding number does not depend on the choices of the orientation of $\mathfrak{k}$ and the meridian disk. The difference of the winding number and the order is always even. The double knots (see Example 2.9) have vanishing winding number but nonzero order (see E 16.1).

In the case where the winding number $n > 0$ is nonzero we can orient the knot $\mathfrak{c}$ represented by a core curve of $\widehat{V}$ such that $\mathfrak{k}$ is homologous to $n \cdot \mathfrak{c}$ in $\widehat{V}$. In this case we will say that $\mathfrak{c}$ is *positively oriented* with respect to $\mathfrak{k}$. We will say that $\mathfrak{k}$ is contained *non-trivially* in $\widehat{V}$ if the order of $\widehat{V}$ with respect of $\mathfrak{k}$ is nonzero. Recall that the order of $\widehat{V}$ is zero if and only if $\mathfrak{k}$ is contained in a ball $B \subset \widehat{V}$ (see E 2.9). The knot $\mathfrak{c}$ represented by a core curve of $\widehat{V}$ is called a *companion* of $\mathfrak{k}$ if $\mathfrak{k}$ is non-trivially embedded in $\widehat{V}$ and if $\widehat{V}$ is knotted. We will say that $\mathfrak{c}$ is a *proper companion* if in addition $\mathfrak{k}$ is not a core curve of $\widehat{V}$. A companion $\mathfrak{c}$ is oriented if the winding number n is nonzero and non-oriented if $n = 0$. If $\mathfrak{k}$ admits a proper companion then $\mathfrak{k}$ is called a *satellite knot*.

16.1 Remark. The knot $\mathfrak{k}$ possesses a proper companion if and only if $\mathfrak{k}$ contains an *incompressible*, non-boundary parallel torus in its complement.

If $\mathfrak{c}$ is a companion of $\mathfrak{k}$, then the torus $i\colon \widehat{T} \hookrightarrow S^3 - \mathfrak{k}$ is *incompressible*, i.e. the inclusion induces an injective map $i_\#\colon \pi_1 \widehat{T} \to \pi_1(S^3 - \mathfrak{k})$, see Proposition 3.12. If moreover $\mathfrak{c}$ is a proper companion then $\widehat{T}$ is not parallel to the boundary of a regular neighborhood of $\mathfrak{k}$.

On the other hand, if $i\colon \widehat{T} \hookrightarrow C$ is an incompressible torus in the complement of the knot $\mathfrak{k}$, then $\widehat{T}$ bounds a knotted solid torus $\widehat{V} \subset S^3$ and $\widehat{V}$ contains $\mathfrak{k}$ non-trivially in its interior. Hence, the core curve $\mathfrak{c}$ of $\widehat{V}$ is a companion of $\mathfrak{k}$. More precisely, $\mathfrak{c}$ is a proper companion of $\mathfrak{k}$ if $\widehat{T}$ is not parallel to ∂C.

Let $\tilde{\mathfrak{k}}$ be a knot in a 3-sphere $\tilde{S}^3$ and $\tilde{V}$ an unknotted solid torus in $\tilde{S}^3$ with $\tilde{\mathfrak{k}} \subset \tilde{V} \subset \tilde{S}^3$. Assume that $\tilde{\mathfrak{k}}$ is not contained in a 3-ball of $\tilde{V}$. A homeomorphism $h\colon \tilde{V} \to \widehat{V} \subset S^3$ onto a tubular neighborhood $\widehat{V}$ of a non-trivial knot $\widehat{\mathfrak{k}} \subset S^3$ which maps a meridian of $\tilde{S}^3 - \tilde{V}$ onto a longitude of $\widehat{\mathfrak{k}}$ maps $\tilde{\mathfrak{k}}$ onto a knot $\mathfrak{k} = h(\tilde{\mathfrak{k}}) \subset S^3$. The knot $\mathfrak{k}$ is called a *satellite* of $\widehat{\mathfrak{k}}$, and $\widehat{\mathfrak{k}}$ is its *companion* (Begleitknoten). The pair $(\tilde{V}, \tilde{\mathfrak{k}})$ is the *pattern* of $\mathfrak{k}$.

There is a relation between the genus of a satellite knot and the genus of its companion.

16.2 Proposition (Schubert [318]). *Let $\widehat{\mathfrak{k}}$ be a companion of a satellite $\mathfrak{k}$, $(\tilde{V}, \tilde{\mathfrak{k}})$ the pattern and $h\colon \tilde{V} \to \widehat{V}$ the homeomorphism such that $\mathfrak{k} = h(\tilde{\mathfrak{k}})$ (as in Definition 2.8). Denote by $\widehat{g}$, g, $\tilde{g}$, the genera of $\widehat{\mathfrak{k}}$, $\mathfrak{k}$, $\tilde{\mathfrak{k}}$, and by $n \geq 0$ the winding number of $\mathfrak{k}$ in $\widehat{V}$. Then*

$$g \geq n\widehat{g} + \tilde{g}.$$

This result is due to H. Schubert [318]. We start by proving the following lemma:

16.3 Lemma. *There is a Seifert surface S of minimal genus g spanning the satellite $\mathfrak{k}$ such that $S \cap \partial \widehat{V}$ consists of n homologous (on $\partial \widehat{V}$) longitudes of the companion $\widehat{\mathfrak{k}}$. The intersection $S \cap (S^3 - \hat{V})$ consists of n components.*

Proof. Let S be an oriented Seifert surface of minimal genus spanning $\mathfrak{k}$. We assume that S is in general position with respect to $\partial \widehat{V}$: that is, $S \cap \partial \widehat{V}$ consists of a system of simple closed curves which are pairwise disjoint. If one of them, γ, is null-homologous on $\partial \widehat{V}$, it bounds a disk δ on $\partial \widehat{V}$. We may assume that δ does not contain another simple closed curve with this property, $\delta \cap S = \gamma$. Cut S along γ and glue two disks δ_1, δ_2 (parallel to δ) to the curves obtained from γ. Since S was of minimal genus the new surface cannot be connected. Substituting the component containing $\mathfrak{k}$ for S reduces the number of curves. So we may assume that the curves $\{\gamma_1, \ldots, \gamma_r\} = S \cap \partial\hat{V}$ are not null-homologous on the torus $\partial\hat{V}$; hence, they are parallel. Re-index so that $\gamma_1, \gamma_2, \ldots, \gamma_r$ are cyclically ordered on $\partial\hat{V}$ and give each γ_i the boundary orientation induced by $S \cap (S^3 - \hat{V})$.

If for some index $\gamma_i \sim -\gamma_{i+1}$ on $\partial \widehat{V}$ we may cut S along γ_i and γ_{i+1} and glue to the cuts two annuli parallel to one of the annuli on $\partial\hat{V}$ bounded by γ_i and γ_{i+1}. The resulting surface S' may not be connected but the Euler characteristic will remain invariant. Replace S by the component of S' that contains $\mathfrak{k}$. The genus g of S' can only be larger than that of S if the other component is a sphere. In this case γ_i spans a disk in $\overline{S^3 - \hat{V}}$, and this means that the companion $\hat{\mathfrak{k}}$ is trivial which contradicts its definition. By the cut-and-paste process the pair $\gamma_i \sim -\gamma_{i+1}$ vanishes; so we may assume $\gamma_i \sim \gamma_{i+1}$ for all i. Hence $\mathfrak{k}$ is homologous to $r\gamma_1$ in $\hat{V}$, and $r\gamma_1$ is homologous to 0 in $S^3 - \hat{V}$. We show that $S \cap (\overline{S^3 - \hat{V}})$ consists of r components. Suppose there is a component $\widehat{S}$ of $S \cap S^3 - \widehat{V}$ with $\hat{r} > 1$ boundary components. Then there are two curves $\gamma_i, \gamma_j \subset \partial\hat{S}$ such that $\gamma_k \cap \hat{S} = \emptyset$ for $i > k > j$. Connect γ_i and γ_j by a simple arc α in the annulus on $\partial\hat{V}$ bounded by γ_i and γ_j, and join its boundary points by a simple arc λ on $\widehat{S}$. A curve u parallel to $\alpha \cup \lambda$ in $S^3 - \hat{V}$ will intersect $\hat{S}$ in one point (Figure 16.1), $\operatorname{int}(u, \hat{S}) = \pm 1$. Since u does not meet $\hat{V}$, we get $\pm 1 = \operatorname{int}(u, \hat{S}) = \operatorname{lk}(u, \partial\hat{S}) = \hat{r} \cdot \operatorname{lk}(u, \mathfrak{k})$; hence $\hat{r} = 1$, a contradiction.

This implies that the γ_i are longitudes of $\hat{\mathfrak{k}}$; moreover

$$n = |\operatorname{lk}(\widehat{m}, \mathfrak{k})| = \operatorname{lk}(\widehat{m}, r\gamma_i) = r \cdot \operatorname{lk}(\widehat{m}, \gamma_i) = r. \qquad \square$$

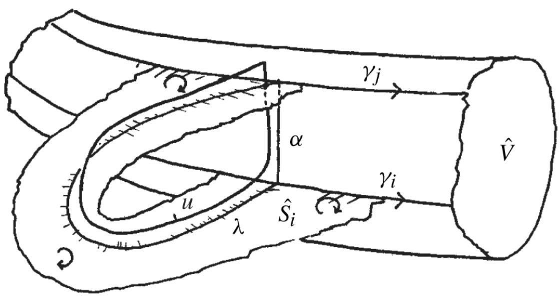


Figure 16.1

Proof of Proposition 16.2. Let S be a Seifert surface of $\mathfrak{k}$ according to Lemma 16.3. Each component $\widehat{S}_i$ of $S \cap (S^3 - \widehat{V})$ is a surface of genus $\widehat{h}$ which spans a longitude γ_i of $\widehat{\mathfrak{k}}$, hence $\mathfrak{k}$ itself. The curves $\widehat{l}_i = h^{-1}(\gamma_i)$ are longitudes of the unknotted solid torus $\widetilde{V} \subset \widetilde{S}^3$ bounding disjoint disks $\widetilde{\delta}_i \subset \widetilde{S}^3 - \widetilde{V}$. Thus $h^{-1}(S \cap \widehat{V}) \cup (\bigcup_i \widetilde{\delta}_i)$ is a Seifert surface spanning $\widetilde{\mathfrak{k}} = h^{-1}(\mathfrak{k})$. Its genus $\widetilde{h}$ is the genus of $S \cap \widehat{V}$. As $S = (S \cap \widehat{V}) \cup \bigcup_{i=1}^{n} \widehat{S}_i$ we get

$$g = n\widehat{h} + \widetilde{h} \geq n\widehat{g} + \widetilde{g}. \qquad \square$$

16.4 Remark. Note that the inequality might be strict: for an untwisted doubled knot $\mathfrak{k}$ we have $n = 0$, $\widetilde{g} = 0$ and $\widetilde{h} = g = g(\mathfrak{k}) = 1$. Therefore, the genus of a knot is an invariant which is not compatible with companionship: the genus of a proper companion $\widehat{\mathfrak{k}}$ of $\mathfrak{k}$ can be greater than the genus of $\mathfrak{k}$.

It can be proved (E 16.2) that the equality $g = n\,\widehat{g} + \widetilde{g}$ holds if there exists a Seifert surface of minimal genus $\widetilde{S}$ for $\widetilde{\mathfrak{k}}$ which intersects $\partial \widetilde{V}$ at exactly n homologous longitudes $\widetilde{\ell}$ of $\widetilde{V}$ (see Definition 2.8).

16.B Companions of order one.

Companions of order one are factors of $\mathfrak{k}$:

16.5 Proposition. *Let $\mathfrak{k}$ be an oriented knot in S^3 and let $\widehat{V} \subset S^3$ be a solid torus which contains $\mathfrak{k}$ in its interior. If the order of $\widehat{V}$ with respect of $\mathfrak{k}$ is one, then either $\mathfrak{k}$ is a core curve of $\widehat{V}$ or $\mathfrak{k}$ is a product knot $\mathfrak{k} = \mathfrak{k}' \# \mathfrak{c}$ where $\mathfrak{k}'$ is a non-trivial knot and $\mathfrak{c}$ is a positively oriented core curve of $\widehat{V}$.*

Proof. Let $\widehat{\delta}$ be a median disk of $\widehat{V}$, $\partial \widehat{V} \cap \widehat{\delta} = \partial \widehat{\delta}$, $\partial \widehat{\delta} \not\sim 0$ in $\partial \widehat{V}$, which intersects $\mathfrak{k}$ transversally in one point. The regular neighborhood B of $\widehat{\delta}$ in $\widehat{V}$ is a ball. We can choose B small enough such that $\alpha = \mathfrak{k} \cap B$ is a trivial arc. The boundary $\partial B = S = \widehat{\delta}_1 \cup \widehat{\delta}_2 \cup A \cong S^2$ of B is the union of two disjoint meridian disks $\widehat{\delta}_1$, $\widehat{\delta}_2$ and an annulus $A \subset \partial \widehat{V}$. The knot $\mathfrak{k}$ intersects $\widehat{\delta}_i$ transversally in one point P_i, say.

The two meridian disks $\widehat{\delta}_1$ and $\widehat{\delta}_2$ decompose $\widehat{V}$ into two balls B and B' such that $B \cup B' = V$, $B \cap B' = \widehat{\delta}_1 \cup \widehat{\delta}_2$. We obtain that $\mathfrak{k} = \alpha\alpha'$ where $\alpha' = \mathfrak{k} \cap B'$ is the oriented arc in B' determined by $\mathfrak{k}$. We choose the numbering in such a way that α connects P_1 to P_2 and α' connects P_2 to P_1. Note that the knot $\mathfrak{k}$ is a core curve of $\widehat{V}$ if and only if the arc $\alpha' = \mathfrak{k} \cap B'$ is trivial, too.

Let us assume that $\alpha' \subset B'$ is a non-trivial arc and choose an oriented arc γ' on $\partial B'$ connecting P_2 to P_1. The knot $\mathfrak{c} := \alpha\gamma'$ is in S^3 equivalent to a core curve of $\widehat{V}$.

The oriented knot $\mathfrak{k}' = (\gamma')^{-1}\alpha'$ is non-trivial since $\alpha' \subset B'$ is a non-trivial arc by assumption. By the very definition of the product knot, Definition 2.7, we obtain $\mathfrak{k} = \mathfrak{c} \# \mathfrak{k}' = \alpha\alpha'$. □

16.6 Corollary (Schubert [318]). *If* $\mathfrak{c}$ *is a proper companion of order one of the oriented knot* $\mathfrak{k}$*, then* $\mathfrak{k}$ *is a product knot of* $\mathfrak{c}$ *and a non-trivial knot.* □

The converse of Corollary 16.6 holds too:

16.7 Proposition. *Let* $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ *be a product knot with non-trivial factors* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$*. Then the factors* $\mathfrak{k}_i$*,* $i = 1, 2$*, are companions of order one of* $\mathfrak{k}$*.*

Proof. Let S be a 2-sphere that defines the product $\mathfrak{k} = \mathfrak{k}_0 \# \mathfrak{k}_1$ and let $S \times I$ be a small product neighborhood of S which meets $\mathfrak{k}$ in two line segments. Then $S \times I$ decomposes S^3 into two disjoint balls B_0 and B_1, $S^3 = B_0 \cup (S \times I) \cup B_1$, $B_i \cap (S \times I) = S \times \{i\}$, $i = 0, 1$, such that B_i defines the knot $\mathfrak{k}_i$ (see 7.14).

Let H_i denote a regular neighborhood of the arc $\alpha_i = \mathfrak{k} \cap B_i$ in B_i, $i = 0, 1$. We have that H_i is homeomorphic to $I \times D^2$ and that $(S \times I) \cap H_i$ is the union of two disjoint disks $\delta_i \cup \delta_i'$, $i = 0, 1$. Therefore $V_0 = B_1 \cup (S \times I) \cup H_0$ and $V_1 = B_0 \cup (S \times I) \cup H_1$ are solid tori containing $\mathfrak{k}$ in its interior. Moreover δ_i and δ_i' are meridian disks of V_i, $i = 1, 2$, intersecting $\mathfrak{k}$ transversally in a single point. The knot represented by a core curve $\mathfrak{c}_i$ of V_i is given by the union of the core α_i of H_i, $i = 0, 1$ with a trivial arc in $S \times \{i\}$. Hence $\mathfrak{c}_i = \mathfrak{k}_i$, $i = 0, 1$. This proves that $\mathfrak{k}_i$ is a companion of order one of $\mathfrak{k}$. □

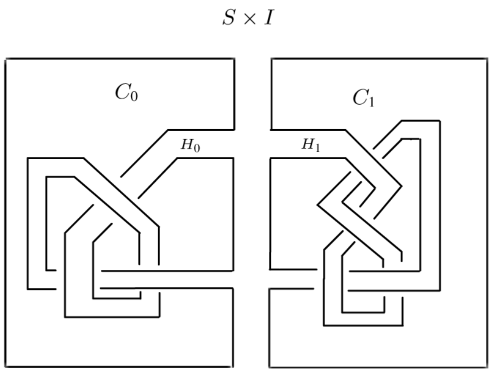


Figure 16.2. The two swallow-follow tori

16.8 Remark. The two solid tori constructed in the previous proof are sometimes called *swallow-follow* tori: the torus V_0 follows the knot $\mathfrak{k}_0$ and swallows $\mathfrak{k}_1$. The boundaries T_i of the tori V_i are disjoint and their complements $C_i := S^3 - \mathring{V}_i$ are also disjoint $C_0 \cap C_1 = \emptyset$. We obtain $C_1 \subset \mathring{V}_0$ and $C_0 \subset \mathring{V}_1$. Note that the solid tori are not disjoint $V_0 \cap V_1 = H_0 \cup (S \times I) \cup H_1$.

16.C Bridge number and height functions

Up to now we have defined a knot as the p.l.-embedding of the circle S^1 in S^3. The distinction between piecewise linear, smooth, and locally flat topological embeddings is unimportant in these dimensions. For convenience we will use in this section the language which is associated to the smooth category. The discussion in this section follows Scharlemann's [315] and Schultens' [322] expositions.

16.9 Notations and definitions. In what follows let $\mathfrak{k} \subset S^3$ be a smooth knot and let $h: S^3 \to \mathbb{R}$ be a *standard height function*, that is h is a Morse function with exactly two critical points, one maximum P_+ and one minimum P_-. The height function h induces a foliation of $S^3 - \{P_+, P_-\}$ by level surfaces which are 2-spheres: for each $P \in S^3 - \{P_+, P_-\}$ the set $S = S_P := h^{-1}(h(P))$ is a 2-sphere which decomposes S^3 into two balls $S^3 = B_+ \cup_S B_-$ where $P_\pm \in B_\pm$. We will call B_+ the *upper ball* of the level surface S_P and B_- the *lower ball* of S.

After a small isotopy of $\mathfrak{k}$ we may assume that h restricts to a Morse function $h|\mathfrak{k}$ on $\mathfrak{k}$ and $P_\pm \notin \mathfrak{k}$.

16.10 Lemma. *Let $\mathfrak{k} \subset S^3$ and suppose that the standard height function $h: S^3 \to \mathbb{R}$ restricted to $\mathfrak{k}$ is a Morse function $h|\mathfrak{k}$ on $\mathfrak{k}$ with m maxima and m minima. Then $\mathfrak{k}$ is a $2m$-plat.*

Proof. Without loss of generality we will assume that $h(P_\pm) = \pm 2$ and that $h(\mathfrak{k}) \subset (-1/2, 1/2)$. Each point on $\mathfrak{k}$ on which $h|\mathfrak{k}$ has a local maximum can be pushed higher by an isotopy of $\mathfrak{k}$ along an arc α that rises monotonically from the local maximum and misses the rest of $\mathfrak{k}$. Similarly, each point on which $h|\mathfrak{k}$ has a local minimum can be pushed lower. Therefore we can assume that all the local maxima of $h|\mathfrak{k}$ occur near the same height (say in the interval $(1 + \epsilon, 1 - \epsilon)$ and that all the local minima of $h|\mathfrak{k}$ occur near the same height (say in the interval $(-1 + \epsilon, -1 - \epsilon)$).

Now $B_+ = h^{-1}([2, 1-\epsilon])$ is a 3-ball containing all the local maxima of $h|\mathfrak{k}$, $B_- = h^{-1}([-1 + \epsilon, -2])$ is a 3-ball containing all the local minima of $h|\mathfrak{k}$ and $h^{-1}([1 - \epsilon, -1 + \epsilon]) \cong S^2 \times I$ intersects $\mathfrak{k}$ in $2m$ monotone arcs (see Figure 2.14). □

16.11 Corollary. *Let $h: S^3 \to \mathbb{R}$ be a standard height function and let $\mathfrak{k} \subset S^3$ be a knot. The bridge number $b(\mathfrak{k})$ is the minimal number of maxima that the restriction $h|\mathfrak{k}'$ of h has on a knot $\mathfrak{k}'$ which is ambient isotopic to $\mathfrak{k}$.* □

16.12 Remark. Note that a 1-bridge knot is trivial (E 16.3).

16.13 Critical points and leaves. In what follows we will consider a family $\mathfrak{T} = \{T_1, \dots, T_n\}$ of disjoint tori in S^3 such that the two points $P_\pm$ do not lie on $\mathfrak{T}$. After a small isotopy, which is supported in a regular neighborhood of $\mathfrak{T}$, we can assume that the restriction $h|T_i$, $i = 1, \dots, n$, of the height function is a Morse function. Moreover, we may assume that all the critical values are different i.e. each level surface contains at most one critical point.

A critical point $P \in T_i$ of $h|T_i$ is a maximum, minimum or a saddle point (see Figure 16.3 and Figure 16.4). Let $P \in T_i$ be a saddle point. The level surface of h passing through P intersects $\mathfrak{T}$ in a certain number of simple closed curves, which do not contain P, and in a component $\sigma \subset T_i$ which contains P. The component $\sigma = s_1^\sigma \vee s_2^\sigma$ consist of two circles s_1^σ and s_2^σ wedged at P. We will call σ the *leaf* corresponding to the saddle point P. The curves s_i^σ, $i = 1, 2$, bound disjoint disks D_i^σ, $i = 1, 2$, on the level surface $S_\sigma = h^{-1}(h(P))$ of h containing P. The complement $A^\sigma := S_\sigma - (\mathring{D}_1^\sigma \cup \mathring{D}_2^\sigma)$ is homeomorphic to a *pinched annulus*, i.e. the one point compactification of $\mathbb{R} \times I$. If there is no confusion as to which saddle we are referring, we will drop the superscripts σ.

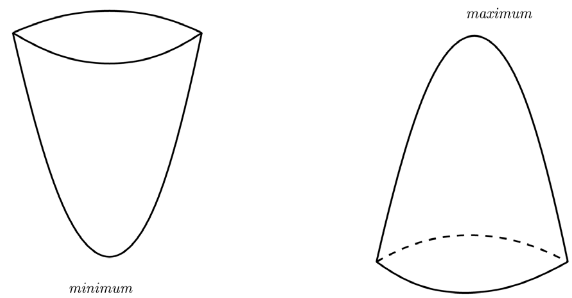


Figure 16.3. A minimum and a maximum.

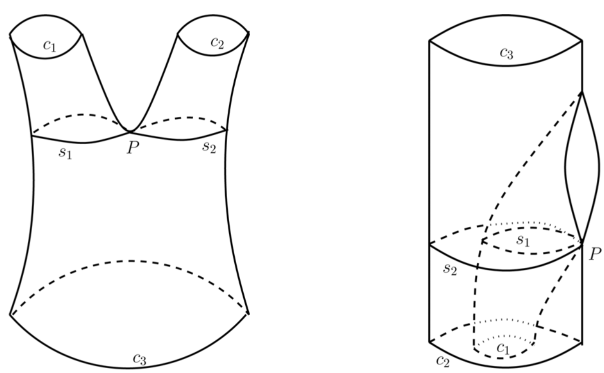


Figure 16.4. A saddle point.

If either s_1^σ or s_2^σ is null-homologous on T_i, then we call P an *inessential saddle* and σ an *inessential leaf*. Otherwise, P is called an essential saddle and σ an essential leaf.

16.14 Remark. Consider a tubular neighborhood of an essential leaf $\sigma = s_1^\sigma \vee s_2^\sigma$ in T. The boundary of this tubular neighborhood is a *pair of pants* and has three boundary components c_1^σ, c_2^σ and c_3^σ where c_i^σ is parallel to s_i^σ for $i = 1, 2$ (see Figure 16.4). Since the Euler characteristic $\chi(T) = 0$ it follows that c_3^σ bounds a disk D^σ in T.

We will see that we can simplify the torus T if $h|T$ has an inessential saddle. The easiest situation occurs if one of the curves s_i^σ, $i = 1, 2$, of a leaf σ bounds a disk in the level surface S_σ which does not intersect $\mathfrak{k}$:

16.15 Lemma. *Let $h\colon S^3 \to \mathbb{R}$ be a standard height function, $\mathfrak{k} \subset S^3$ a knot and let $\mathfrak{V} = \{V_1, \dots, V_n\} \subset S^3$ be a family of solid tori, with $\mathfrak{k} \subset \mathring{V}_i$ non-trivially for $i = 1, \dots, n$. We will assume that the boundary tori $T_i = \partial V_i$ are disjoint and that the restrictions $h|\mathfrak{k}$ and $h|T_i$ are Morse functions for $i = 1, \dots, n$.*

If there exists a saddle $P \in T_i$ with leaf $\sigma = s_1 \vee s_2$ such that one of the curves s_i bounds a disk D' in the level surface S_σ disjoint from $\mathfrak{k}$, then P is an inessential saddle and there exists an ambient isotopy of S^3 which is the identity in a neighborhood of $\mathfrak{k}$ and which transforms $\mathfrak{T} = \{T_1, \dots, T_n\}$ into a family of tori $\mathfrak{T}'$ with fewer critical points.

Proof. If the intersection $\mathring{D}' \cap \mathfrak{T}$ is nonempty we choose an innermost curve γ in the intersection. The curve γ is contained on one of the tori, say T_j, and bounds a disk $D'' \subset D'$ such that $\mathring{D}'' \cap \mathfrak{T} = \emptyset$. It follows that γ bounds a disk D on the torus T_j. Otherwise, the disk D'' would be a meridian disk of V_j with $D'' \cap \mathfrak{k} = \emptyset$. This contradicts the assumption that the order of V_j is nonzero.

The 2-sphere $S' = D' \cup D$ decomposes the 3-sphere into two balls B and B'. One of these balls, say B, does not contain $\mathfrak{k}$. It follows that B neither contains $T_j - D$ nor any other torus T_i, since otherwise B' would be a ball containing $\mathfrak{k} \subset B'$ and B' would be contained in one of the solid tori V_i contradicting the assumption.

We can now define an ambient isotopy of S^3 which is supported in a regular neighborhood of B which maps D to D' and transforms T_j into T_j'. In particular the isotopy is the identity on $\mathfrak{k}$ and on any torus T_i, $i \neq j$. After a small deformation which is supported in a regular neighborhood of D' we can arrange that $h|T_j'$ is a Morse function and that the number of connected components of $\mathring{D}' \cap T_j'$ has decreased. We replace T_j by T_j' and continue the procedure.

Eventually the intersection $\mathring{D}' \cap \mathfrak{T}$ is empty and s_i bounds a disk in one torus, say T_j. The same procedure allows us to eliminate the saddle P and we obtain a torus T_j' such that the number of critical points of $h|T_j'$ is less than the number of critical points of $h|T_j$. $\square$

In the case where one of the tori T_i contains an inessential saddle it is possible to simplify the situation.

16.16 Lemma. *Suppose we have the same assumptions as in Lemma 16.15.*

If $h|T_j$ has an inessential saddle then there exists an ambient isotopy of S^3 which is the identity in a neighborhood of $\mathfrak{k}$ and which transforms the family $\mathfrak{T}$ into $\mathfrak{T}'$ and T_j into T_j' which has an inessential saddle $P \in T_j'$ with leaf $\sigma = s_1 \vee s_2$ satisfying the following properties:

(a) *one of the circles, say s_1, bounds a disk $D \subset T_j'$ such that $h|D$ has exactly one critical point Q which is a maximum or a minimum;*

(b) *the curve s_1 bounds a disk D' in the level surface S_σ such that s_2 lies outside D' and $D \cup D'$ bounds a ball B such that*

- *$B \subset B_+$ if Q is a maximum and $P_+ \notin B$;*
- *$B \subset B_-$ if Q is a minimum and $P_- \notin B$.*

Here $B_\pm$ is the upper respectively lower ball of the level surface S_σ (see 16.9).

Moreover, the $h|\mathfrak{T}'$ has the same number of critical points as $h|\mathfrak{T}$.

Proof. First we show (a). Suppose that $P' \in T_j$ is an inessential saddle and that s_1' bounds a disk $D' \subset T_j$, $\partial D' \cap S_{P'} = s_1'$. If $h|\mathring{D}'$ has one critical point then this point must be a maximum or a minimum since $\chi(D') = 1$. We choose $P := P'$ and $s_1 := s_1'$ to satisfy (a). If $h|\mathring{D}'$ has more than one critical point then one of these must be a saddle P since $\chi(D') = 1$. The two leaves s_1 and s_2 of P are bounding disks D_1 and D_2 in T. If one of the disks D_i satisfies (a) we are done. Otherwise we replace P' by P and we continue the procedure. Eventually we arrive at the inessential saddle which is innermost in D' and which satisfies (a).

Suppose that P is an inessential saddle satisfying (a) and that the critical point Q of $h|D$ is a maximum. It follows that the disk D is contained in the upper ball B_+ of the level surface S_P and $s_1 = D \cap S_P$. The disk D decomposes B_+ into two balls B' and B'', $B_+ = B' \cup_D B''$. The curve $s_1 = \partial D$ decomposes the level surface S_P into two disks D' and D''. We chose the notation such that $\partial B' = D \cup D'$, $\partial B'' = D \cup D''$. One of the balls, say B'', does contain P_+. If the disk D'' does contain s_2' then we may take $B := B'$ and we are done. Hence suppose that $s_2 \subset D'$. In this case we chose a monotone arc α starting a the maximum Q of D' and ending at P_+. We can chose the arc α disjoint form $\mathfrak{k}$ by a general position argument. We cannot prevent α from intersecting $\mathfrak{T}$ but by looking at the intersection of the level surfaces with $\mathfrak{T}$ it is easy to see that we can chose α such that it intersects $\mathfrak{T}$ only in finitely many maxima $Q_0, \dots, Q_k$ of $h|\mathfrak{T}$. The arc α avoids $\mathfrak{T}$ as long as possible. The numbering of the Q_i is chosen such that $Q = Q_0$ and $h(Q_i) < h(Q_{i+1})$, $i = 0, \dots, k-1$.

Let β be the subarc of α between $Q_k \in T_l$ and P_+ and choose a small ball $\tilde{B}$ centered at P_+ and disjoint from $\mathfrak{T}$. Let $\tilde{U}$ be the closure of a small tubular neighborhood of β. After a small isotopy we may assume that $T_l \cap \tilde{U}$ consists of a small disk $\tilde{D} \subset T_l$ centered at Q_k. The union $U := \tilde{U} \cup \tilde{B}$ is a 3-ball which does not contain $\mathfrak{k}$ and $T_l - \tilde{D}$ or any other torus T_i. We can define an ambient isotopy of S^3 which is supported in a regular neighborhood of U and which transforms T_l into the torus $T_l' = (T_l - \tilde{D}) \cup (\partial U - \tilde{D})$ (see Figure 16.5). After a small isotopy, which is the identity outside a regular neighborhood of ∂U, we can arrange that $h|T_l'$ is a Morse function and that the maximum Q_k of $h|T_l$ has turned into a maximum of $h|T_l'$ at a higher level. The isotopy is the identity in a neighborhood of $\mathfrak{k}$ and on any torus T_i, $i \neq l$. The number of critical points of $h|\mathfrak{T}'$ is the same as that of $h|\mathfrak{T}$. We replace T_l by T_l' and continue the procedure.

Eventually the intersection of $\alpha \cap \mathfrak{T} = Q_0 \in T_j$ is the maximum of $h|D$. The same construction using α describes an ambient isotopy of S^3 which transforms the torus T_j into a torus T_j'. This isotopy augments B' to contain P_+ and shrinks B'' to avoid P_+. We choose B to be the shrunk version of B''. □

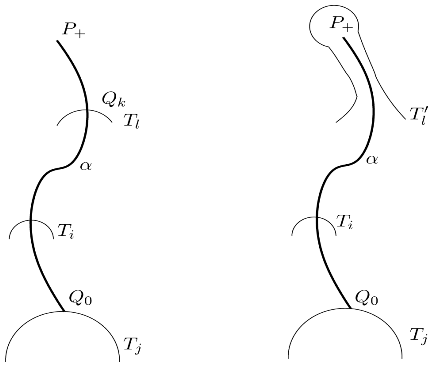


Figure 16.5. Overturning a cap (Überschlag einer Haube).

16.17 Proposition. *Suppose we have the same assumptions as in Lemma 16.15.*

If $h|T_j$ has an inessential saddle, then there exists an ambient isotopy of S^3 which transforms $\mathfrak{k}$ into $\mathfrak{k}'$ and which transforms the family $\mathfrak{T}$ into $\mathfrak{T}'$ such that $h|\mathfrak{k}'$ has the same number of maxima as $h|\mathfrak{k}$ and $h|\mathfrak{T}'$ has a smaller number of critical points than $h|\mathfrak{T}$.

Proof. Suppose that there are inessential saddles. We may suppose that there is an inessential saddle $P \in T_j$ satisfying the conclusions of Lemma 16.16. We may also suppose that Q is a maximum and that D and B lie above the level surface S_P. Let S' be a level surface below S_P such that there are no critical points of $h|\mathfrak{T}$ and $h|\mathfrak{k}$ between S_P and S'.

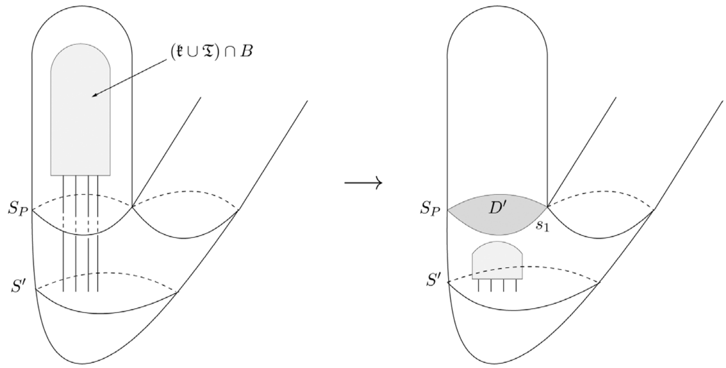


Figure 16.6

The intersection $(\mathfrak{k} \cup \mathfrak{T}) \cap \mathring{B}$ can be shrunk horizontally and lowered to lie between S_P and S' (see Figure 16.6). This can be done by an ambient isotopy of S^3 which is the identity outside a regular neighborhood of B and which transforms $\mathfrak{k}$ into a knot $\mathfrak{k}'$ and $\mathfrak{T}$ into a family $\mathfrak{T}'$ such that $h|\mathfrak{k}'$ has the same number of maxima as $h|\mathfrak{k}$ and such that the nature of critical points of $h|\mathfrak{T}$ and $h|\mathfrak{T}'$ are the same.

We obtain an inessential saddle P of $h|\mathfrak{T}'$ with a leaf s_1 which bounds a disk D' in the level surface S_P such that $D' \cap \mathfrak{k}' = \emptyset$. Hence Lemma 16.15 applies. □

16.18 Remark. If the height function $h|\mathfrak{T}$ has only essential leaves, then for each leaf σ the disk bounded by c_3^σ contains exactly one singular point, a maximum or a minimum, which will be denoted by m_σ. We call m_σ the *maximum* or the *minimum corresponding to* σ. If there are no inessential saddles then every maximum of minimum of $h|\mathfrak{T}$ corresponds to a saddle in this way.

16.19 Definition. Let $P \in T_j$ be an essential saddle of $h|T_j$ and $\sigma = s_1^\sigma \vee s_2^\sigma$ the corresponding essential leaf. The saddle is called *nested* if a collar neighborhood of $\sigma = s_1^\sigma \vee s_2^\sigma$ on the pinched annulus A^σ is contained in the solid torus V_j.

Figure 16.7 shows a pair of *adjacent* saddles, one nested and the other not nested. Here two essential saddles σ_1, σ_2 are called *adjacent saddles* if they are contained in the same torus T_j and if one component of the surface obtained by cutting T along $\sigma_1 \cup \sigma_2$ is a cylinder C which contains no critical points of $h|T$.

16.20 Black-white coloring (J. Schultens). Let us consider a single solid torus $V \subset S^3$ and $h|T$, $T = \partial V$. Let S be a level sphere such that $h(S)$ is a regular value of $h|T$. Then $S \cap T$ is a collection of simple closed curves in S which decompose S into planar regions. There is a black-white coloring of these regions: we color a region black if it is contained in $V \cap S$ and white if it is contained in $C := S^3 - \mathring{V}$.

If σ is a saddle contained in the level surface S_σ and if S' and S'' are regular level surfaces such that $h(S') < h(S_\sigma) < h(S'')$ and such that σ is the only critical point between S' and S'', then the number of black regions (respectively the white regions) of S' and S'' changes by one if σ is not nested (respectively nested). Hence, if the saddle P is not nested then the number of white regions does not change between S' and S''.

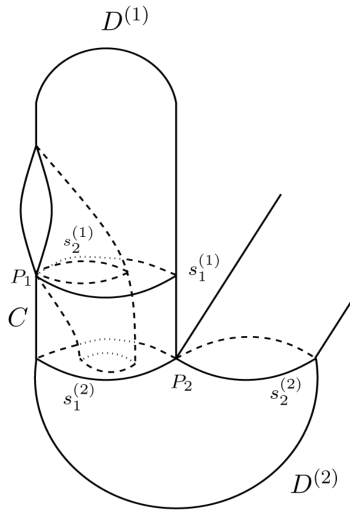


Figure 16.7. A pair of adjacent saddles, one of which is nested.

16.21 Lemma. *Suppose we have the same assumptions as in Lemma 16.15. If all saddles are essential then the highest saddle of $h|T_j$ is not nested.*

Proof. Let P be the highest saddle of $h|T_j$ and $\sigma = s_1 \vee s_2$ its leaf. Let D_1 and D_2 be the two disjoint disks in the level surface S_σ bounded by s_1 and s_2, $D_1 \cap D_2 = \{P\}$. Any curve of $T_j \cap \mathring{D}_i$ bounds a disk in T_j lying above S_σ since P is the highest saddle. Therefore, we obtain from D_i a disk D_i' such that D_i and D_i' coincide in a collar neighborhood of $\partial D_i = s_i = \partial D_i'$ and such that $D_i' \cap T_j = s_i$. On the other hand, s_i, $i = 1, 2$, is essential in T_j and since V_j is knotted, the curve s_i is a meridian, D_i' is a meridian disk and hence $D_i' \subset V$. Therefore a collar neighborhood of $\sigma = s_1 \vee s_2$ on the pinched annulus A^σ is contained in the complement of V_j i.e. σ is not nested. □

16.22 Proposition. *Suppose we have the same assumptions as in Lemma 16.15.*

If all saddles of $h|\mathfrak{T}$ are essential and if there is a nested saddle then there exists an ambient isotopy of S^3 which transforms $\mathfrak{k}$ into $\mathfrak{k}'$ and which transforms the family $\mathfrak{T}$ into $\mathfrak{T}'$ such that $h|\mathfrak{k}'$ has the same number of maxima as $h|\mathfrak{k}$ and $h|\mathfrak{T}'$ has a smaller number of critical points than $h|\mathfrak{T}$.

Proof. Suppose that there is a nested saddle on T_j. By Lemma 16.21 there is also a non-nested saddle on T_j. Hence we can find a pair of adjacent saddles $\sigma_1 = s_1^{(1)} \vee s_2^{(1)}$, $\sigma_2 = s_1^{(2)} \vee s_2^{(2)}$ with σ_1 nested and σ_2 not nested. We will chose the notations such that $s_1^{(1)}$ and $s_1^{(2)}$ meet the cylindrical component C without critical points of $h|T_j$.

Without loss of generality, we may assume that σ_1 lies above σ_2. The component of $T_j - \sigma_1$ lying above S_{σ_1} is an open disk $D^{(1)} \subset T_j$ whose boundary is σ_1, and the component of $T_j - \sigma_2$ lying below S_{σ_2} is an open disk $D^{(2)} \subset T_j$ whose boundary is σ_2. We construct a new disk D by adding the closure of $D^{(1)}$ to C and by capping off $s_2^{(1)}$ with the disk $D_2^{(1)} \subset S_{\sigma_1}$, $\partial D_2^{(1)} = s_2^1$. Note that the *horizontal* part $D_2^{(1)}$ of D will meet $\mathfrak{k}$ and T_j. More precisely, if $D_2^{(1)}$ does not meet $\mathfrak{k}$ then Lemma 16.15 applies and σ_1 would be inessential. Moreover, $\mathring{D}_2^{(1)} \cap T_j = \emptyset$ would imply that $D_2^{(1)}$ is a meridian disk of V_j contradicting the assumption that σ_1 is nested.

We now proceed as in Lemma 16.16. We have $\partial D = s_1^{(2)}$ and D decomposes the upper ball $B_+^{(2)}$ of the level surface S_{σ_2} into two balls B' and B'', $B_+^{(2)} = B' \cup_D B''$. The curve $s_1^{(2)} = \partial D$ decomposes the level surface S_{σ_2} into two disks D' and D''. We choose the notation such that $\partial B' = D \cup D'$, $\partial B'' = D \cup D''$. As in the proof of Lemma 16.16 we may assume that B'' contains P_+ and that $s_2^{(2)} \subset B''$, see Figure 16.7.

The next step is to proceed as in the proof of Proposition 16.17. We want to apply a similar isotopy to the portion $(\mathfrak{k} \cup \mathfrak{T}) \cap B$ where $B = B'$. The difference here is that the horizontal portion $D_2^{(1)}$ of D will meet $\mathfrak{k} \cup \mathfrak{T}$.

There are product neighborhoods of S_{σ_1} and S_{σ_2} which intersect $(\mathfrak{T} - D) \cup \mathfrak{k}$ only in vertical arcs and cylinders. We let $\widetilde{B}$ denote the portion of B between S_{σ_1} and S_{σ_2} and we choose a level surface S' below S_{σ_2} such that there are no critical points between S_{σ_2} and S' (Figure 16.8 shows a cross-section).

The intersection $\widetilde{B} \cap (\mathfrak{T} \cup \mathfrak{k})$ can be shrunk horizontally and lowered to lie between S_{σ_2} and S' as in the proof of Proposition 16.17. At the same time the portion of B above S_{σ_1} does not change and the product neighborhood below S_{σ_1} is extended such that $\widetilde{B}$ intersects $\mathfrak{T}' \cup \mathfrak{k}$ only in vertical lines and cylinders (see Figure 16.8).

Now observe that there is a vertical disk $\widetilde{D} \subset \widetilde{B}$ with the following properties:

- The boundary $\partial \widetilde{D}$ consists of three monotone arcs α_1, α_2 and α_3. The arc α_1 is contained in C and connects the saddle point P_1 with a point $Q_1 \in D^{(2)}$ just below S_{σ_2} and avoids P_2 (here $D^{(2)}$ is the component of $T_j - \sigma_2$ below S_{σ_2}). The arc α_2 is contained in the cylinder C' attached to $s_2^{(1)}$ it connects P_1 with a point Q_2 which is on a slightly lower level then Q_1. The last arc α_3 connects Q_1 and Q_2. (See Figure 16.9.)

- We have $\mathfrak{T} \cap \widetilde{D} = \alpha_1 \cap \alpha_2$ and $\widetilde{D} \cap \mathfrak{k} = \emptyset$.

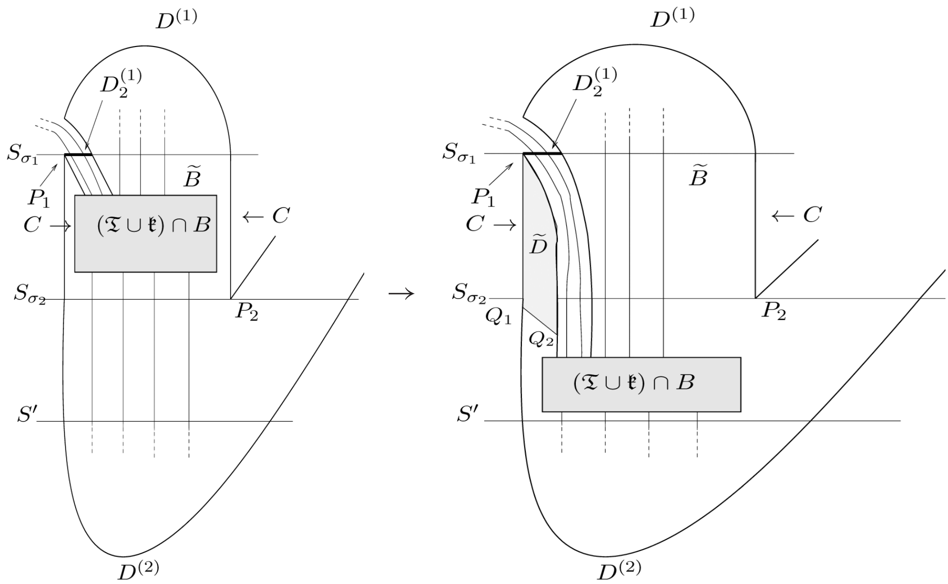


Figure 16.8. A cross section.

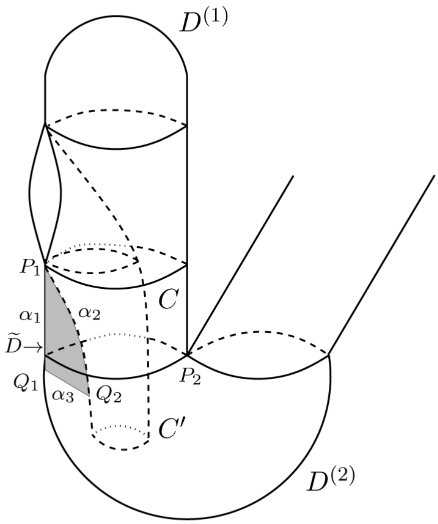


Figure 16.9

The existence of $\widetilde{D}$ follows from the fact that $\widetilde{B}$ intersects $\mathfrak{T}' \cup \mathfrak{k}$ only in vertical lines and cylinders.

In the last step we deform T_j along $\widetilde{D}$ such that the saddle P_1 is pushed into Q_1. Formally, we cut T_j along $\alpha_1 \cup \alpha_2$ and we add two parallel copies of $\widetilde{D}$ which meet along the arc α_3 to obtain a new torus T_j'. By a small isotopy, which is the identity outside a regular neighborhood of $\widetilde{D}$, we can arrange that the restriction $h|T_j'$ is a Morse function. The number of critical points of $h|T_j'$ is the same as the number of critical points of $h|T_j$. The critical point P_1 of $h|T_j$ was replaced by the critical point Q_1 of $h|T_j'$. Now, observe that P_1 as well as Q_1 became inessential saddles (Figure 16.10). Hence Proposition 16.17 applies. □

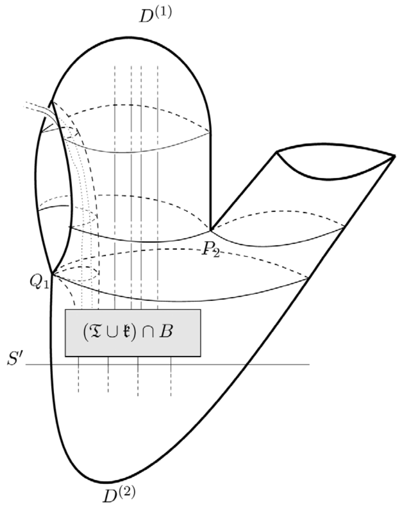


Figure 16.10

16.23 Lemma. *Assume the same conditions as in Lemma 16.15. Let r be a regular value of $h|T_j$, $1 \leq j \leq n$, and let $\sigma_1, \dots, \sigma_m$ be the saddles of $h|T_j$ and $m_i = m_{\sigma_i}$, $i = 1, \dots, n$, the corresponding maximum or minimum of $h|T_j$.*

If $h|T_j$ has no nested saddles then the level surface $S_r = h^{-1}(r)$ intersects V_j in disks. Moreover, between the level surfaces $S_{m_i} := h^{-1}(h(m_i))$ and S_{σ_i} lies a portion B_{σ_i} of V_j that is a three ball such that $B_{\sigma_i} \cap S_{\sigma_i} = D_1^{\sigma_i} \cup D_2^{\sigma_i}$. The disks $D_1^{\sigma_i} \cup D_2^{\sigma_i}$, $i = 1, \dots, n$, cut V into $B_{\sigma_1}, \dots, B_{\sigma_m}$ and vertical cylinders.

Proof. Let us consider the collection of simple closed curves $T_j \cap S_r \subset S_r$ and the corresponding black and white coloring of the planar regions (see Remark 16.20). The black regions corresponds to the intersection $V_j \cap S_r$. It follows from Alexander Duality [157, Thm. 3.44] that each black region is a disk if and only if the white region is connected.

Since there are no nested saddles the number of white regions cannot change if r varies and passes through a singular value corresponding to a saddle point of $h|T_j$. Now recall that the complement $C_j := S^3 - \mathring{V}_j$ is connected. Hence if we had two disjoint white regions then there must be a path in C_j connecting them. But this would be only possible if we had nested saddles.

Hence there is at most one white region and each black region is a disk. The second assertion follows from the first part by observing how the intersections $V_j \cap S_r$ change as r varies. □

16.24 Corollary. *Assume the same conditions as in Lemma 16.15.*

If $h|T_j$, $1 \leq j \leq n$, has no inessential and no nested saddles then a core curve of V_j represents a knot $\mathfrak{c}_j$ whose bridge number $b(\mathfrak{c}_j)$ is less than or equal to the number of maxima of $h|T_j$.

Proof. Let $\sigma_1, \dots, \sigma_m$ be the singular leaves of T_j. The meridian disks

$$D_1^{\sigma_1} \cup D_2^{\sigma_1}, \dots, D_1^{\sigma_n} \cup D_2^{\sigma_n}$$

cut V_j into 3-balls $B_{\sigma_1}, \dots, B_{\sigma_n}$ and vertical cylinders which are also 3-balls. In order to construct a core curve of V_j we chose a midpoint in each disk $D_i^{\sigma_j}$ and we connect the midpoints of $D_1^{\sigma_j}$ and $D_2^{\sigma_j}$ by an unknotted arc in B_{σ_j} on which the height function has only one extremum. The midpoints of adjacent disks are connected by a monotone and hence unknotted arc in the cylinder. These unknotted arcs fit together to give a core curve $\mathfrak{c}_j$ of V_j and the number of maxima of $h|\mathfrak{c}_j$ is exactly the number of maxima of $h|T_j$. □

16.25 Definition. Let $\mathfrak{k} \subset S^3$ be a knot and $\mathfrak{V} = \{V_1, \dots, V_n\} \subset S^3$ a family of solid tori such that each V_i is knotted and contains $\mathfrak{k} \subset \overset{\circ}{V}_i$ non-trivially, $i = 1, \dots, n$.

The family $\mathfrak{V}$ is called *adjoint* to $\mathfrak{k}$ if the complements $C_i := S^3 - \overset{\circ}{V}_i$, $i = 1, \dots, n$, are pairwise disjoint.

Example: The swallow-follow tori V_0, V_1 of a product knot $\mathfrak{k} = \mathfrak{k}_0 \# \mathfrak{k}_1$ are adjoined to $\mathfrak{k}$ (see Remark 16.8).

16.26 Theorem (Schubert [319, Satz 2]). *Let $\mathfrak{k} \subset S^3$ be a knot and suppose that $\mathfrak{V} = \{V_1, \dots, V_n\} \subset S^3$ is a family of solid tori adjoined to $\mathfrak{k}$ such that each V_i has the order α with respect to $\mathfrak{k}$.*

If $\mathfrak{c}_i$ denotes the companion of $\mathfrak{k}$ represented by a core curve in V_i then

$$\alpha \sum_{i=1}^{n} \big(b(\mathfrak{c}_i) - 1\big) + \alpha \leq b(\mathfrak{k}), \tag{16.1}$$

where b denotes the bridge number.

Proof. Note first that the boundary tori $T_i = \partial V_i$ are disjoint by Definition 16.25. We choose a standard height function $h\colon S^3 \to \mathbb{R}$ with critical points $P_\pm$. We arrange that $\mathfrak{k}$ and $\mathfrak{T} := \{T_1, \dots, T_n\}$ avoid $P_\pm$. After a small deformation which is the identity outside a regular neighborhood of $\mathfrak{T} \cup \mathfrak{k}$ we may suppose that the restrictions $h|\mathfrak{k}$ and $h|\mathfrak{T}$ are Morse functions.

By Corollary 16.11 we may also assume that the number of maxima of $h|\mathfrak{k}$ coincides with the bridge number $b(\mathfrak{k})$. Moreover, by Proposition 16.17 and Proposition 16.22 we may assume that $h|\mathfrak{T}$ has no inessential saddles and no nested saddles.

In the next step we chose a level sphere S_+ close to P_+ such that $S_+ \cap \mathfrak{T} = \emptyset$. We raise all maxima of $h|\mathfrak{T}$ above S_+ by an isotopy which is the identity in a neighborhood of $\mathfrak{k}$ and which does not change the number of critical points of $h|\mathfrak{T}$. More precisely, as in Lemma 16.16 (see also Figure 16.5) we can chose a monotone arc starting at a maximum of $h|\mathfrak{T}$ and raising above S_+. The maximum can be pushed along α by an isotopy which is similar to the isotopy constructed in the proof of Lemma 16.16 (see also Figure 16.5).

Now the intersection $\mathfrak{T} \cap S_+$ is a union of simple closed curves and each curve corresponds to a maximum of $h|\mathfrak{T}$. For each $j \in \{1, \dots, n\}$ the intersection $V_j \cap S_+$ consists of b_j disjoint disks and the intersection $F_j := C_j \cap S_+$ of the complement $C_j := S^3 - \mathring{V}_j$ with S_+ is connected by Lemma 16.23. Hence the connected planar surface F_j has b_j boundary components which is exactly the number of maxima of $h|T_j$.

We obtain again a black-white coloring of S_+: a region of $S_+ - \mathfrak{T}$ is colored black if it is contained in the intersection $\cap_{i=1}^n V_i$ and a region F_i is colored white. Note that the surfaces F_i are pairwise disjoint.

We will count the number of maxima of $h|\mathfrak{k}$ with the help of the black regions. To this end we push each point on $\mathfrak{k}$ on which $h|\mathfrak{k}$ has a local maximum above S_+ by an isotopy of $\mathfrak{k}$ along an arc that rises monotonically from the local maximum and misses the rest of $\mathfrak{k}$ and $\mathfrak{T}$. This is always possible by a general position argument and since all the local maxima of $h|\mathfrak{T}$ lie above S_+.

The number of black regions is the number of connected components of the complement of $(\cup_{i=1}^n F_i)$ in S_+. By Alexander duality, we obtain

$$\widetilde{H}_0\Big(S_+ - \Big(\coprod_{i=1}^n F_i\Big), \mathbb{Z}\Big) \cong \widetilde{H}^1\Big(\coprod_{i=1}^n F_i, \mathbb{Z}\Big) \cong \bigoplus_{i=1}^n H^1(F_i, \mathbb{Z})$$

and hence, if γ denotes the number of black regions,

$$\gamma = \sum_{i=1}^n (b_i - 1) + 1\,.$$

Corollary 16.24 implies that $b(\mathfrak{c}_i) \le b_i$ for $i = 1, \dots, n$. Therefore, we obtain equation (16.1) if we can show that $\alpha\gamma \le b(\mathfrak{k})$.

Let $G \subset S_+$ be a black region and let $t_1, \dots, t_r$ denote the boundary components of G. The curves $t_1, \dots, t_r$ are contained in different tori since the intersection $V_i \cap S_+$

consist of disjoint disks. Hence, we may suppose that $t_i \subset T_i$, $i = 1, \dots, r$. To each curve t_i corresponds a local maximum of $h|T_i$ and hence a saddle $\sigma_i \subset T_i$, $i = 1, \dots, r$. The component of $T_i - \sigma_i$ lying above S_{σ_i} is a disk D^{σ_i} whose boundary is σ_i.

We are interested in the saddles $\{\sigma_i \mid i = 1, \dots, r\}$ and we choose the numbering such that $\sigma_1 = s_1^{\sigma_1} \vee s_2^{\sigma_1} \subset T_1$ is the saddle on the highest level. The disks $D_1^{\sigma_1}, D_2^{\sigma_1} \subset S_{\sigma_1}$ are meridian disks of V_1 and $D_1^{\sigma_1} \cup D_2^{\sigma_1}$ cuts off a 3-ball B_{σ_1} from V_1. The boundary of B_{σ_1} is $D_1^{\sigma_1} \cup D_2^{\sigma_2} \cup D^{\sigma_1}$. The intersection of D^{σ_j}, $j = 2, \dots, r$, with B_{σ_1} is a disk D'_j which contains t_j and whose boundary $\partial D'_j$ lies on $\mathring{D}_1^{\sigma_1} \cup \mathring{D}_2^{\sigma_1}$. Each curve $\partial D'_j$ bounds also a disk $D''_j \subset \mathring{D}_1^{\sigma_1} \cup \mathring{D}_2^{\sigma_1}$. We construct two new meridian disks $\widetilde{D}_1^{\sigma_1}, \widetilde{D}_2^{\sigma_1}$ from $D_1^{\sigma_1}, D_2^{\sigma_1}$ by replacing D''_j by D'_j, $j = 2, \dots, r$. Since the order of V_1 with respect to $\mathfrak{k}$ is α we have that $\mathfrak{k} \cap (\widetilde{D}_1^{\sigma_1} \cup \widetilde{D}_2^{\sigma_1})$ consists of at least 2α points.

Now $\mathfrak{k}$ does not intersect $D'_j \subset T_j$ and there is no local maximum of $h|\mathfrak{k}$ below S_+. Hence $\mathfrak{k} \cap G$ also consists of 2α points and there are at least α local maxima of $h|\mathfrak{k}$ above G. Hence $h|\mathfrak{k}$ admits at least $\alpha\gamma$ local maxima and therefore $\alpha\gamma \le b(\mathfrak{k})$. □

16.27 Theorem. *If $\mathfrak{k} = \mathfrak{k}_0 \# \mathfrak{k}_1$ is a product knot then $b(\mathfrak{k}_0) + b(\mathfrak{k}_1) - 1 = b(\mathfrak{k})$.*

Proof. The two swallow-follow tori V_0 and V_1 constructed in the proof of Proposition 16.7 are of order one with respect to $\mathfrak{k}$. Moreover, V_0, V_1 are adjoined to $\mathfrak{k}$ (see Remark 16.8). Hence Theorem 16.26 implies $b(\mathfrak{k}_0) + b(\mathfrak{k}_1) - 1 \le b(\mathfrak{k})$.

The inequality $b(\mathfrak{k}_0) + b(\mathfrak{k}_1) - 1 \ge b(\mathfrak{k})$ follows from the fact that the product of a $2m$-plat and a $2n$-plat is a $2(m + n - 1)$-plat as illustrated in Figure 16.11. □

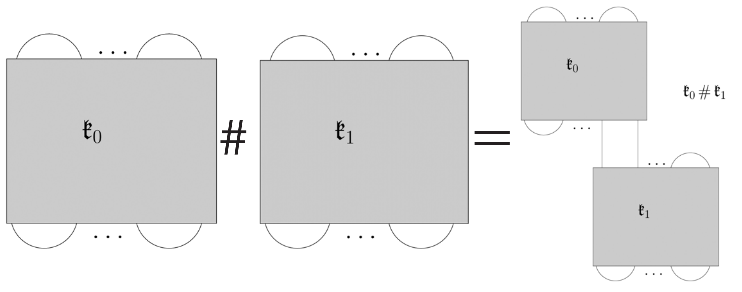


Figure 16.11. The product of a $2m$-plat and a $2n$-plat is a $2(m + n - 1)$-plat.

16.28 Theorem. *Let $\mathfrak{c}$ be a companion of order α of $\mathfrak{k}$. Then*

$$\alpha \cdot b(\mathfrak{c}) \le b(\mathfrak{k}) \text{ and } \alpha \le \frac{b(\mathfrak{k})}{2}.$$

Moreover, if $\mathfrak{c}$ is a proper companion of $\mathfrak{k}$ then $b(\mathfrak{c}) < b(\mathfrak{k})$.

Proof. The first inequality $\alpha \cdot b(\mathfrak{c}) \leq b(\mathfrak{k})$ is a special case ($n = 1$) of equation (16.1). This inequality implies $\alpha \leq b(\mathfrak{k})/2$ since a companion is always a non-trivial knot and since the bridge number of a non-trivial knot is bigger or equal to 2 by Remark 16.12.

Now suppose that $\mathfrak{c}$ is a proper companion of $\mathfrak{k}$. If $\alpha \geq 2$ then $\alpha \cdot b(\mathfrak{c}) \leq b(\mathfrak{k})$ implies $b(\mathfrak{c}) < b(\mathfrak{k})$. Hence we suppose $\alpha = 1$, i.e. $\mathfrak{c}$ is a companion of order one. Corollary 16.6 implies that $\mathfrak{k} = \mathfrak{c} \# \mathfrak{k}'$ is a product knot of two non-trivial knots $\mathfrak{c}$ and $\mathfrak{k}'$. By Theorem 16.27 we obtain $b(\mathfrak{c}) + b(\mathfrak{k}') - 1 = b(\mathfrak{k})$ with $b(\mathfrak{k}') \geq 2$ by Remark 16.12 (E 16.3) and hence $b(\mathfrak{c}) < b(\mathfrak{k})$. □

16.D History and sources

By Remark 16.1, the knot $\mathfrak{k}$ possesses a proper companion if and only if the complement of $\mathfrak{k}$ contains an incompressible, non-boundary parallel torus. Therefore, H. Schubert studied incompressible tori in knot complements. Such tori were studied in his seminal paper *Knoten und Vollringe* [318], where he introduced the *companions* of a knot. The companions generalize the factors of a knot. Schubert's article *Über eine numerische Knoteninvariante* [319] was motivated by the question if a knot has only finitely many companions. Schubert answered this question in the affirmative, he introduced an invariant, the so-called bridge number $b(\mathfrak{k})$, which is compatible with companionship, i.e. the bridge number of a *proper companion* of $\mathfrak{k}$ is smaller than $b(\mathfrak{k})$ (see Theorem 16.28). Our exposition in Section 16.C is based on J. Schultens' article [322].

The study of incompressible tori in knot complements is now part of the JSJ-theory, see Jaco and Shalen [177] and Johannson [179], applied to the complement of links in S^3. R. Budney studied in [48] the global nature of the JSJ-decomposition for knot and link complements in S^3.

16.E Exercises

E 16.1. Prove that double knots (see Example 2.9) have vanishing winding number but nonzero order.

E 16.2. Prove the statement in Remark 16.4.

E 16.3. Show that a 1-bridge knot is trivial.

E 16.4. Let $\mathfrak{k}$ be a doubled knot of $\widehat{\mathfrak{k}}$ (see Remarks 2.9). Prove that $\widehat{\mathfrak{k}}$ is a companion of order 2 of $\mathfrak{k}$ and that $b(\mathfrak{k}) = 2\,b(\widehat{\mathfrak{k}})$.

E 16.5. Let $\mathfrak{c}$ be a (p, q)-cable knot with core $\mathfrak{k}$, see Definition 15.15. Prove that $b(\mathfrak{c}) = |q|\, b(\mathfrak{k})$.

E 16.6 (S. Orevkov). Prove that a 3-bridge knot $\mathfrak{k}$ with irreducible Alexander polynomial is prime.

Chapter 17

The 2-variable skein polynomial

In Section 9.E we studied the Conway polynomial as an invariant closely connected with the Alexander polynomial. It can be computed by using the *skein relations*, Figure 9.3, and hence is called a skein invariant. Shortly after the discovery of the famous Jones polynomial several authors independently contributed to a new invariant for oriented knots and links, a Laurent polynomial $P(z, v)$ in two variables which is also a skein invariant and which comprises both the Jones and the Alexander–Conway polynomials. It has become known as the HOMFLY-PT polynomial after the contributors of [120]: Hoste, Ocneano, Millet, Floyd, Lickorish, Yetter. Nevertheless, it was independently discovered by Przytycki and Traczyk [292].

17.A Construction of a trace function on a Hecke algebra

In the following the HOMFLY-PT polynomial is established via representations of the braid groups $\mathfrak{B}_n$ into a Hecke algebra using Markov's theorem, see Theorem 10.23. We follow Jones [182] and Morton [254].

17.1 On the symmetric group. The symmetric group $\mathfrak{S}_n$ admits a presentation

$$\begin{aligned}\mathfrak{S}_n = \langle\, \tau_1, \dots, \tau_{n-1} \mid\ & \tau_i^2 = 1 \text{ for } 1 \le i \le n-1,\\ & \tau_i \tau_j = \tau_j \tau_i \text{ for } 1 \le i < j-1 \le n-2,\\ & \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \text{ for } 1 \le i \le n-2 \,\rangle,\end{aligned}$$

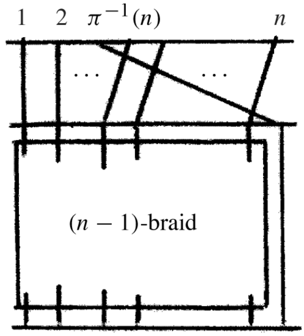
where τ_i is the transposition $(i, i+1)$. We write the group operation in $\mathfrak{S}_n$ from left to right; for example, the product of the transpositions $(1,2) \times (2,3) = (1,2)(2,3)$ is the cycle $(1,3,2)$.

We identify $\mathfrak{S}_{n-1}$ with the subgroup of $\mathfrak{S}_n$ of permutations leaving n fixed.

Every permutation $\pi \in \mathfrak{S}_n$ can be written as a word in the generators τ_i in many ways; we choose a unique representative $b_\pi(\tau_i)$ for each π in the following. If $\pi(n) = j$ we put

$$b_\pi(\tau_i) = (j, j+1)(j+1, j+2) \cdots (n-1, n) \cdot b_{\pi'}(\tau_i) \quad \text{with } \pi' \in \mathfrak{S}_{n-1},$$

see Figure 17.1. The words $\mathcal{W}_n = \{b_\pi(\tau_i) \mid \pi \in \mathfrak{S}_n\}$ satisfy the "Schreier" condition which means that, if $b_\pi(\tau_i) = w(\tau_i)\,\tau_k$, then $w(\tau_i) = b_{\pi \cdot \tau_k}(\tau_i)$, and b_{id} is the empty word. Furthermore the $b_\pi(\tau_i)$ are of minimal length, and the generator τ_{n-1} occurs at most once in each $b_\pi(\tau_i) \in \mathcal{W}_n$; both assertions are evident in Figure 17.1. Figure 17.2 shows for the cycle $(1\,3\,2\,5)$ the representative $b_\pi(\tau_i) = \tau_2\tau_3\tau_4\tau_3\tau_1\tau_2\tau_1$.



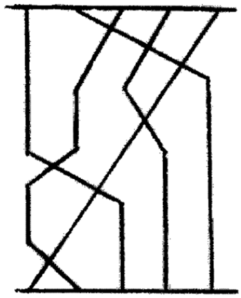


Figure 17.1 **Figure 17.2**

17.2 Definition. The following presentation

$$\begin{array}{llll} \hat{\mathfrak{S}}_n = \langle\, \hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_{n-1} \mid & \hat{\tau}_i \hat{\tau}_j = \hat{\tau}_j \hat{\tau}_i & \text{for } 1 \le i < j-1 \le n-2, \\ & \hat{\tau}_i \hat{\tau}_{i+1} \hat{\tau}_i = \hat{\tau}_{i+1} \hat{\tau}_i \hat{\tau}_{i+1} & \text{for } 1 \le i \le n-2 \,\rangle \end{array}$$

defines a semigroup $\hat{\mathfrak{S}}_n$.

The elements of $\hat{\mathfrak{S}}_n$ are the classes of words defined by the following equivalence relation $\hat{=}$: two words $w(\hat{\tau}_i)$ and $w'(\hat{\tau}_i)$ are equivalent, $w(\hat{\tau}_i) \hat{=} w'(\hat{\tau}_i)$, if and only if they are connected by a chain of substitutions

$$\hat{\tau}_i \hat{\tau}_j \mapsto \hat{\tau}_j \hat{\tau}_i, \ \hat{\tau}_i \hat{\tau}_{i+1} \hat{\tau}_i \mapsto \hat{\tau}_{i+1} \hat{\tau}_i \, \hat{\tau}_{i+1}$$

employing the relations of Definition 17.2. (The building of inverses is not permitted.)

There is a canonical homomorphism $\kappa\colon \hat{\mathfrak{S}}_n \to \mathfrak{S}_n$, $\kappa(\hat{\tau}_i) = \tau_i$; we write $\hat{b}_\pi = b_\pi(\hat{\tau}_i)$ and $\hat{\mathcal{W}}_n = \{\hat{b}_\pi \mid \pi \in \mathfrak{S}_n\}$.

Two cases occur in forming a product $\hat{b}_\pi \cdot \hat{\tau}_k$: either $\hat{b}_\pi \hat{\tau}_k \hat{=} \hat{b}_\varrho$, the class of $\hat{b}_\pi \hat{\tau}_k$ contains a representative $\hat{b}_\varrho \in \hat{\mathcal{W}}_n$ (case α), or not (case β). Case α occurs when the strings crossing at τ_k do not cross in b_π (Figure 17.3), $\varrho = \pi\tau_k$. In case β they do, and Figure 17.4 shows

$$\hat{b}_\pi \, \hat{\tau}_k = \hat{b}_\varrho \, \hat{\tau}_k^2, \ \varrho\, \tau_k = \pi.$$

We note down the result in:

17.3 Lemma.

$$\hat{b}_\pi \cdot \hat{\tau}_k \hat{=} \begin{cases} \hat{b}_\varrho, & \varrho = \pi\tau_k, \ \textit{case } \alpha \\ \hat{b}_\varrho \hat{\tau}_k^2 & \varrho\, \tau_k = \pi, \ \textit{case } \beta. \quad \square \end{cases} \tag{17.1}$$

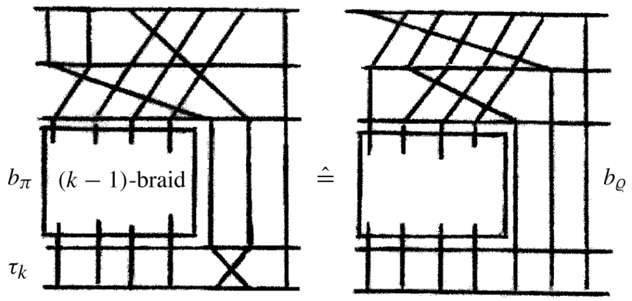


Figure 17.3

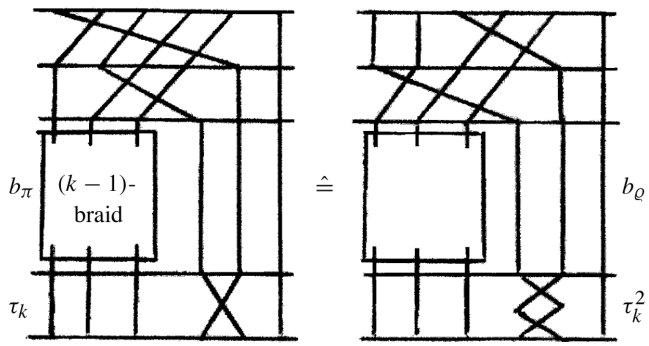


Figure 17.4

17.4 Construction of a Hecke algebra. Next we construct a special algebra, a so-called *Hecke algebra*. We define a free module M_n of rank $n!$ over a unitary commutative ring $R \ni 1$ using the $n!$ words of $\mathcal{W}_n$. We replace the generators $\hat{\tau}_i$ by c_i, $1 \le i \le n-1$ and write $w(c_i) = w'(c_i)$ if and only if $\hat{w}(\hat{\tau}_i) \mathrel{\hat{=}} \hat{w}'(\hat{\tau}_i)$. Let M_n be the free R-module with basis $\mathcal{W}_n(c_i) = \{b_\pi(c_i) \mid \pi \in \mathfrak{S}_n\}$. Note that $\mathcal{W}_n(c_i) \ni c_j = b_{\tau_j}(c_i)$, $1 \le j \le n-1$. We introduce an associative product in M_n which transforms M_n into an R-algebra $H_n(z)$ of rank $n!$.

17.5 Definition. We put $c_k^2 = zc_k + 1$, $1 \le k \le n-1$ for some fixed element $z \in R$. Then (17.1) takes the form

$$b_\pi(c_i) \cdot c_k = \begin{cases} b_{\pi\tau_k}(c_i), & \text{case } \alpha, \\ z\, b_\pi(c_i) + b_\varrho(c_i),\ \varrho\,\tau_k = \pi, & \text{case } \beta. \end{cases} \tag{17.2}$$

By iteration (17.2) defines a product for the elements of the basis $\mathcal{W}_n(c_i)$ and, thus, a product on M_n by distributivity. It remains to prove associativity for the product on $\mathcal{W}_n(c_i)$.

17.6 Lemma. *The product defined in Definition 17.5 is associative on $\mathcal{W}_n(c_i)$.*

Proof. Given a word $w(c_i)$ we apply the rule (17.2) from left to right (product algorithm) to obtain an element

$$\sum_j \gamma_j\, b_{\pi_j}(c_i) = \overline{w(c_i)} \in M_n,\ \gamma_i \in R.$$

One has $\overline{b_\pi(c_i)} = b_\pi(c_i)$ by the Schreier property. We prove

$$(b_1 b_2) b_3 = b_1(b_2 b_3), \quad b_j \in \mathcal{W}_n(c_i),$$

by induction on $|b_1|+|b_2|+|b_3|$ where $|b_i|$ denotes the length of b_i. We may assume $|b_i| \geq 1$. Applying the product algorithm on the left side lets case β occur for the first time for some c_k in b_2. (It cannot happen in b_1 since $\overline{b}_1 = b_1$.) We have

$$b_2 = b_2' b_2'' \quad \text{and} \quad \overline{b_1 b_2'} = \sum \gamma_j\, b_{\pi_j}(c_i),\ |b_{\pi_j}| < |b_1| + |b_2'|. \tag{17.3}$$

We stop the product algorithm at this point and get:

$$\sum \gamma_j\, b_{\pi_j}(c_i) \cdot b_2'.$$

On the right side we have

$$b_1\left((b_2' b_2'') b_3\right) = b_1\left((b_2'(b_2'' b_3)\right)$$

by induction. Applying the algorithm and stopping at the same c_k we obtain:

$$\left(\sum \gamma_j\, b_{\pi_j}(c_i)\right)(b_2'' b_3).$$

Using the distributivity and the induction hypothesis, compare (17.3), we get the desired equality.

If now the case β occurs for the first time in b_3 at c_k when applying the algorithm, then we have $\overline{b_1 b_2} = b_1 b_2$. Since the strings meeting in τ_k have not met in $b_1(\tau_i) \cdot b_2(\tau_i)$, they have not met in $b_2(\tau_i)$. So case β cannot have occurred when the algorithm is applied to $b_2 b_3$ at an earlier time. Now the same argument applies as in the first case. If case β does not occur at all, equality is trivial. □

The module M_n has become an R–algebra of rank $n!$, a so-called *Hecke algebra*; we denote it by $H_n(z)$.

17.7 Theorem. *Let R be a commutative unitary ring, and $z \in R$. An algebra generated by elements $\{c_i \mid 1 \leq i \leq n-1\}$ and defined by the relations*

$$\begin{aligned} c_i c_{i+1} c_i &= c_{i+1} c_i\, c_{i+1}, && 1 \leq i \leq n-2, \\ c_i c_j &= c_j c_i, && 1 \leq i < j-1 \leq n-2, \\ c_i^2 &= z c_i + 1, && 1 \leq i \leq n-1 \end{aligned}$$

is isomorphic to the Hecke algebra $H_n(z)$.

The proof follows from the construction above. □

17.8 Remark. One has $(c_j - z)c_j = c_j^2 - zc_j = 1$; hence, $c_j^{-1} = c_j - z$.

We choose $R = \mathbb{Z}[z^{\pm 1}, v^{\pm 1}]$ to be the 2–variable ring of Laurent polynomials and denote by $H_n(z, v) = H_n$ the Hecke algebra with respect to $R = \mathbb{Z}[z^{\pm 1}, v^{\pm 1}]$. Next we define a representation of the braid group $\mathfrak{B}_n$:

$$\varrho_v \colon \mathfrak{B} \to H_n, \quad \varrho_v(\sigma_j) = vc_j, \quad 1 \leq j \leq n-1,$$

see Proposition 10.3. There are natural inclusions

$$H_{n-1} \hookrightarrow H_n, \quad \mathcal{W}_{n-1}(c_i) \hookrightarrow \mathcal{W}_n(c_i),$$

and we define

$$H = \bigcup_{n=1}^{\infty} H_n, \quad \mathcal{W}(c_i) = \bigcup_{n=1}^{\infty} \mathcal{W}_n(c_i), \quad H_1 = R.$$

For the following definition we use temporarily the ring $R = \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ adding a further variable T.

17.9 Definition. (Trace). A function tr: $H_n \to \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ is called a *trace on* H if it satisfies the following conditions for all $n \in \mathbb{N}$.

(α)	$\operatorname{tr}(\sum_{\pi \in \mathfrak{S}_n} \alpha_\pi b_\pi) = \sum_{\pi \in \mathfrak{S}_n} \alpha_\pi \operatorname{tr}(b_\pi)$	where $\alpha_\pi \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}]$	(linearity);
(β)	$\operatorname{tr}(ba) = \operatorname{tr}(ab)$	for $a, b \in H_n$;	
(γ)	$\operatorname{tr}(1) = 1$;		
(δ)	$\operatorname{tr}(xc_{n-1}) = T \cdot \operatorname{tr}(x)$	for $x \in H_{n-1}$.	

17.10 Lemma. *There is a unique trace on* H.

Proof. It suffices to show that a trace on H_n can be uniquely extended to a trace on H_{n+1}. From (β) and (δ) we get

$$\operatorname{tr}(xc_n y) = \operatorname{tr}(yxc_n) = T \cdot \operatorname{tr}(yx) = T \cdot \operatorname{tr}(xy) \quad \text{for } x, y \in H_n.$$

The basic elements of H_{n+1} which do not belong to H_n are of the form $xc_n y$ with $x, y \in H_n$; this follows from the remark in Paragraph 17.1 that τ_n appears only once in b_π. So we must define the extension of the trace by

$$\operatorname{tr}(xc_n y) = T \cdot \operatorname{tr}(xy) \quad \text{for } xc_n y \in \mathcal{W}_{n+1}(c_i) \setminus \mathcal{W}_n(c_i).$$

We have to show that the linear extension of this definition to H_{n+1} is in fact a trace. Condition (α) is the linearity which is valid by definition. We first prove $\operatorname{tr}(xc_n y) = T \cdot \operatorname{tr}(xy)$ for arbitrary $x, y \in H_n$. An element $\xi \in \mathcal{W}_n$ has the form $\xi = c_j c_{j+1} \cdots c_{n-1} \cdot \xi'$, $\xi' \in \mathcal{W}_{n-1}(c_i)$. Now

$$\xi\, c_n\, y = c_j c_{j+1} \ldots c_{n-1} \xi' c_n y = c_j c_{j+1} \ldots c_{n-1} c_n \xi' y$$

by the braid relation $\xi' c_n = c_n \xi'$. Put $\xi' y = \sum \beta_j \eta_j$; by the linearity (α),

$$\operatorname{tr}(\xi c_n y) = \sum \beta_j \operatorname{tr}(c_1 \dots c_n \eta_j) = \sum \beta_j \cdot T \cdot \operatorname{tr}(c_1 \dots c_{n-1} \eta_j)$$

since $c_1 \dots c_n \eta_j$ is in the basis of H_{n+1}. It follows

$$\operatorname{tr}(\xi c_n y) = T \cdot \operatorname{tr}(c_1 \dots c_{n-1} \xi' y) = T \cdot \operatorname{tr}(\xi y).$$

Since x is a linear combination of elements like ξ from above, we obtain by (α)

$$\operatorname{tr}(x c_n y) = T \cdot \operatorname{tr}(x y) \quad \text{for } x, y \in H_n.$$

This implies (δ).

It remains to prove (β). A basis element $b_{n+1} \in \mathcal{W}_{n+1}(c_i)$ is of the form $b_{n+1} = x\, c_n\, y$, $x = c_j \cdots c_{n-1}$, $y \in H_n$. For $k < n$ we have

$$\operatorname{tr}(c_k \cdot x\, c_n\, y) = T \cdot \operatorname{tr}(c_k\, x\, y) = T \cdot \operatorname{tr}(x\, y\, c_k) = \operatorname{tr}(x\, c_n\, y \cdot c_k)$$

by induction. To prove $\operatorname{tr}(b_{n+1} \cdot b'_{n+1}) = \operatorname{tr}(b'_{n+1} \cdot b_{n+1})$ – which implies (β) by (α) – we now need only to prove $\operatorname{tr}(b_{n+1} \cdot c_n) = \operatorname{tr}(c_n \cdot b_{n+1})$ for $b_{n+1} = x c_n y$.

Case 1: If $b_{n+1} = x\, c_n\, y$ with $x, y \in H_{n-1}$ then $b_{n+1} c_n = c_n b_{n+1}$ since c_n commutes with $x, y \in H_{n-1}$.

Case 2: If $x = a c_{n-1} b$ with $a, b, y \in H_{n-1}$ then

$$\begin{aligned}
\operatorname{tr}(c_n \cdot a c_{n-1} b c_n y) &= \operatorname{tr}(a c_n c_{n-1} c_n b y) = \operatorname{tr}(a c_{n-1} c_n c_{n-1} b y)\\
&= T \cdot \operatorname{tr}(a c_{n-1}^2 b y) = T \cdot \operatorname{tr}\left(a(z c_{n-1} + 1) b y\right)\\
&= z \cdot T \cdot \operatorname{tr}(a c_{n-1} b y) + T \cdot \operatorname{tr}(a b y) = (z T^2 + T) \operatorname{tr}(a b y);\\
\operatorname{tr}(a c_{n-1} b c_n y \cdot c_n) &= \operatorname{tr}(a c_{n-1} b c_n^2 y) = \operatorname{tr}\left(a c_{n-1} b (z c_n + 1) y\right)\\
&= z \operatorname{tr}(a c_{n-1} b c_n y) + \operatorname{tr}(a c_{n-1} b y)\\
&= z \cdot T \cdot \operatorname{tr}(a c_{n-1} b y) + T \cdot \operatorname{tr}(a b y) = (z T^2 + T) \operatorname{tr}(a b y).
\end{aligned}$$

Case 3: The case $y = c c_{n-1} d$ with $x, c, d \in H_{n-1}$ can be dealt with analogously.

Case 4: Let $x = a c_{n-1} b$, $y = d c_{n-1} e$ with $a, b, d, e \in H_{n-1}$. Then

$$\begin{aligned}
\operatorname{tr}(c_n \cdot a c_{n-1} b \cdot c_n \cdot d c_{n-1} e) &= T \cdot \operatorname{tr}(a c_{n-1}^2 b \cdot d c_{n-1} e)\\
&= T \cdot z \cdot \operatorname{tr}(a c_{n-1} b \cdot d c_{n-1} e) + T^2 \cdot \operatorname{tr}(a b d e);\\
\operatorname{tr}(a c_{n-1} b \cdot c_n \cdot d c_{n-1} e \cdot c_n) &= T \cdot \operatorname{tr}(a c_{n-1} b d c_{n-1}^2 e)\\
&= T \cdot z \cdot \operatorname{tr}(a c_{n-1} b d c_{n-1} e) + T^2 \cdot \operatorname{tr}(a b d e).
\end{aligned}$$

We deduce from $c_n^{-1} = c_n - z$ that:

17.11 Remark. $\operatorname{tr}(x c_n^{-1}) = \operatorname{tr}(x c_n) - z \cdot \operatorname{tr}(x) = (T - z) \cdot \operatorname{tr}(x),\ \forall x \in H_n.$

17.B The HOMFLY-PT polynomial

Consider the representation

$$\varrho_v\colon \mathfrak{B}_n \to H_n, \quad \varrho_v(\sigma_i) \mapsto v\, c_i,$$

where the Hecke algebra H_n is understood over $\mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$. We put

$$P_{\mathfrak{z}_n} = k_n \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n)), \quad \mathfrak{z}_n \in \mathfrak{B}_n,$$

for some $k_n \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ which is still to be determined. Property (β) in Definition 17.9 of the trace implies that $P_{\mathfrak{z}_n} \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ is invariant under conjugation of $\mathfrak{z}_n$ in $\mathfrak{B}_n$, and is, hence, a polynomial $P_{\hat{\mathfrak{z}}_n}$ assigned to the closed braid $\hat{\mathfrak{z}}_n$. To turn $P_{\mathfrak{z}_n} \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ to an invariant of the link represented by $\hat{\mathfrak{z}}_n$, we have to check the effect of a Markov move $\mathfrak{z}_n \mapsto \mathfrak{z}_n\sigma_n^{\pm 1}$ on $\hat{\mathfrak{z}}_n$, see Proposition 10.21 and Definition 10.22. We postulate:

$$k_n \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n)) = k_{n+1}\operatorname{tr}(\varrho_v(\mathfrak{z}_n\sigma_n)).$$

It follows $k_n = k_{n+1} \cdot v \cdot T$ since

$$\operatorname{tr}(\varrho_v(\mathfrak{z}_n\sigma_n)) = v \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n) \cdot c_n) = v \cdot T \cdot \operatorname{tr}(\varrho(\mathfrak{z}_n)).$$

Another condition follows in the second case:

$$k_{n+1}\operatorname{tr}(\varrho_v(\mathfrak{z}_n\sigma_n^{-1})) = k_{n+1}v^{-1}\operatorname{tr}(\varrho_v(\mathfrak{z}_n) \cdot c_n^{-1}) = k_{n+1}v^{-1}(T - z) \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n))$$

(see Remark 17.11); hence $k_n = k_{n+1} \cdot v^{-1}(T - z)$. We solve $v^{-1}(T - z) = vT$ in the quotient field of $\mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ by $T = \frac{zv^{-1}}{v^{-1}-v}$ and define inductively

$$k_{n+1} = k_n \cdot \frac{1}{v \cdot T} = k_n \cdot z^{-1}(v^{-1} - v),\ k_1 = 1 \quad \Longrightarrow \quad k_n = \frac{(v^{-1} - v)^{n-1}}{z^{n-1}}.$$

17.12 Remark. The extension $H_n \subset H_{n+1}$ introduces the factor $T = \frac{zv^{-1}}{v^{-1}-v}$, but the denominator $v^{-1} - v$ is eliminated by the factor $k_{n+1}k_n^{-1} = z^{-1}(v^{-1} - v)$ such that $P_{\mathfrak{z}_n}(z, v)$ is indeed a Laurent polynomial in z and v.

From the above considerations we obtain the first part of the following:

17.13 Theorem and Definition. *The Laurent polynomial*

$$P_{\mathfrak{z}_n}(z, v) = \frac{(v^{-1} - v)^{n-1}}{z^{n-1}} \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n))$$

associated to a braid $\mathfrak{z}_n \in \mathfrak{Z}_n$ is an invariant of the oriented link $\mathfrak{l}$ represented by the closed braid $\hat{\mathfrak{z}}_n$.

$P_{\mathfrak{l}}(z, v) = P_{\mathfrak{z}_n}(z, v)$ *is called the 2-variable skein polynomial or* HOMFLY-PT *polynomial of the oriented link $\mathfrak{l}$.*

The trivial braid with n strings represents the trivial link with n components; its polynomial is $\frac{(v^{-1}-v)^{n-1}}{z^{n-1}}$.

To prove the last statement observe that $\varrho_v(\mathfrak{z}_n) = 1$ for the trivial braid $\mathfrak{z}_n$, the empty word in σ_i. As a special case we have $P_{\mathfrak{z}_n}(z,v) = 1$ for $\hat{\mathfrak{z}}_n$ the trivial knot. □

17.14 Definition. For an oriented link $\mathfrak{k}$ the smallest number n for which $\mathfrak{k}$ is isotopic to some $\hat{\mathfrak{z}}_n$ is called the *braid index* $\beta(\mathfrak{k}) = n$ of $\mathfrak{k}$.

The following proposition gives a lower bound for the braid index $\beta(\mathfrak{k})$ in terms of the HOMFLY-PT polynomial $P_{\mathfrak{k}}(z,v)$ of $\mathfrak{k}$. Write

$$\begin{aligned} P_{\mathfrak{k}}(z,v) &= a_n(z)v^m + \ldots + a_n(z)v^n, \quad a_j(z) \in \mathbb{Z}[z,z^{-1}], \\ & m \le n,\ m,n \in \mathbb{Z},\ \text{and}\ a_m \ne 0 \ne a_n. \end{aligned}$$

By $\mathrm{Sp}_v(P_{\mathfrak{k}}(z,v)) = n - m$ we denote the "v–span" of $P_{\mathfrak{k}}(z,v)$.

17.15 Proposition.

$$\beta(\mathfrak{k}) \ge 1 + \frac{1}{2}\mathrm{Sp}_v\left(P_{\mathfrak{k}}(z,v)\right).$$

Proof. Suppose that $\mathfrak{k}$ is isotopic to $\hat{\mathfrak{z}}_n$. From Definition 17.9 (δ) it follows by induction that the trace of an element of H_n is a polynomial in T of degree at most $n-1$. Hence, for $\mathfrak{z}_n = \prod_{j=0}^{n-1} \sigma_{i_j}^{\varepsilon_{i_j}}$ we obtain

$$\begin{aligned} \varrho_v(\mathfrak{z}_n) &= v^k \cdot \prod_{j=0}^{n-1} c_{i_j}^{\varepsilon_j} \quad \text{with } k = \sum_j \varepsilon_j \\ \Longrightarrow \quad \mathrm{tr}\left(\varrho_v(\mathfrak{z}_n)\right) &= v^k \cdot \sum_{j=0}^{n-1} a_j(z) T^j \quad \text{where } T^j = \frac{z^j v^{-j}}{(v^{-1}-v)^j} \\ \Longrightarrow \quad P_{\mathfrak{z}_n}(z,v) &= \frac{(v^{-1}-v)^{n-1}}{z^{n-1}} \cdot \mathrm{tr}\left(\varrho_v(\mathfrak{z}_n)\right) \\ &= v^k \cdot \sum_{j=0}^{n-1} a_j(z) \cdot z^{-n+j-1} v^{-j} (v^{-1}-v)^{n-j-1}. \end{aligned}$$

Therefore, $\mathrm{Sp}_v(P_{\mathfrak{z}_n}) \le 2(n-1)$. □

17.16 Example. $6_1, 7_2, 7_4$ have braid index 4.
Let $\mathfrak{k}_+$ be a diagram of an oriented link. We focus on a crossing and denote by $\mathfrak{k}_-$ resp. $\mathfrak{k}_0$ the projections which are altered in the way depicted in Figure 17.5, but are unchanged otherwise.

17.17 Proposition. *Let $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{k}_0$ be link projections related as in Figure 17.5. Then there is the skein relation*

$$v^{-1} P_{\mathfrak{k}_+} - v P_{\mathfrak{k}_-} = z P_{\mathfrak{k}_0}.$$

There exists an algorithm to calculate $P_{\mathfrak{k}}$ for an arbitrary link $\mathfrak{k}$ given by a projection.

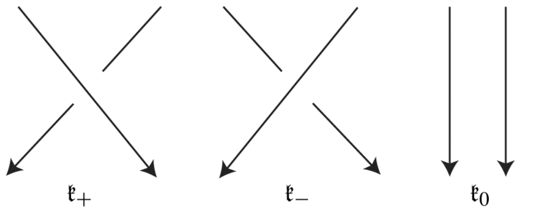


Figure 17.5. A skein relation.

Proof. The braiding process which turns an arbitrary link projection into that of a closed braid as described in Proposition 2.12 can be executed in such a way that a neighborhood of any chosen crossing point of the projection is kept fixed. Furthermore, the representing braid $\mathfrak{z}_n$ can be suitably chosen such that $\mathfrak{k}_+ = \mathfrak{z}_n \sigma_i$, $\mathfrak{k}_- = \mathfrak{z}_n \sigma_i^{-1}$ and $\mathfrak{k}_0 = \mathfrak{z}_n$. Now,

$$v^{-1} P_{\mathfrak{k}_+} - v P_{\mathfrak{k}_-} = v^{-1} k_n \operatorname{tr}(\varrho_v(\mathfrak{z}_n \sigma_i)) - v k_n \operatorname{tr}\left(\varrho_v(\mathfrak{z}_n \sigma_i^{-1})\right) = k_n z \operatorname{tr}(\varrho_v(\mathfrak{z}_n)) = z P_{\mathfrak{k}_0}$$

since

$$v^{-1}\varrho_v(\mathfrak{z}_n \sigma_i) - v\varrho_v(\mathfrak{z}_n \sigma_i^{-1}) = \varrho_v(\mathfrak{z}_n)\left(v^{-1}\varrho_v(\sigma_i) - v\varrho_v(\sigma_i^{-1})\right)$$

and

$$\varrho_v(\mathfrak{z}_n)\left(c_i - c_i^{-1}\right) = \varrho_v(\mathfrak{z}_n) \cdot z. \qquad \square$$

17.18 Remark. The skein relation permits to calculate each of the polynomials $P_{\mathfrak{k}_+}$, $P_{\mathfrak{k}_-}$, $P_{\mathfrak{k}_0}$ from the remaining two. By changing overcrossings into undercrossings or vice versa any link projection can be turned into the projection of an unlink. This implies that the skein relation supplies an algorithm for the computation of $P_{\mathfrak{k}}$. The process is illustrated in Figure 17.6: each vertex of the "skein tree" (Figure 17.6 (b)) represents a projection; the root at the top represents the projection of $\mathfrak{k}$, the terminal

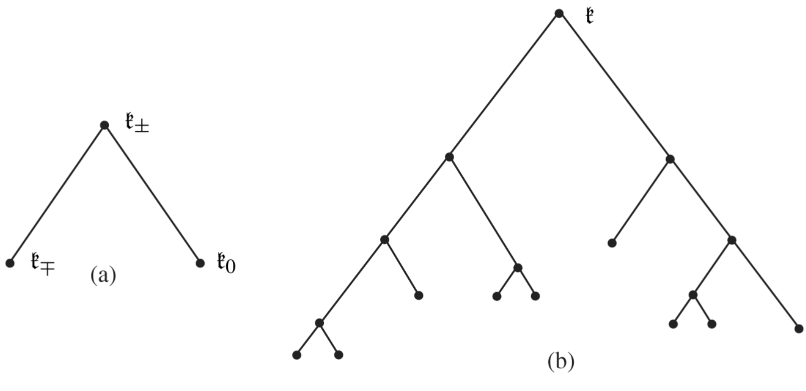


Figure 17.6. Skein trees.

points represent unlinks. Starting with the polynomials of these one can work the way upwards to compute $P_{\mathfrak{k}}$. The procedure is of exponential time complexity.

17.19 Proposition. *Let* $-\mathfrak{k}$ *and* $\mathfrak{k}^*$ *denote the inverted and mirrored knot respectively. Then:*

$$\begin{array}{lrcl} \text{(a)} & P_{\mathfrak{k}}(z,v) &=& P_{-\mathfrak{k}}(z,v); \\ \text{(b)} & P_{\mathfrak{k}}(z,v) &=& P_{\mathfrak{k}^*}(z,-v^{-1}), \\ \text{(c)} & P_{\mathfrak{k}_1 \# \mathfrak{k}_2}(z,v) &=& P_{\mathfrak{k}_1}(z,v) \cdot P_{\mathfrak{k}_2}(z,v); \\ \text{(d)} & P_{\mathfrak{k}_1 \sqcup \mathfrak{k}_2}(z,v) &=& z^{-1}(v^{-1}-v) P_{\mathfrak{k}_1}(z,v) \cdot P_{\mathfrak{k}_2}(z,v) \end{array}$$

where $\#$ *and* $\sqcup$ *denote the product and the disjoint union respectively.*

Proof. (a) Changing $\mathfrak{k}$ into $-\mathfrak{k}$ allows to use the same skein tree.

(b) If $\mathfrak{k}$ is replaced by $\mathfrak{k}^*$, we can still use the same skein tree, and at each vertex the associated projection is also replaced by its mirror image. The skein relation

$$v^{-1} P_{\mathfrak{k}_+}(z,v) - v\, P_{\mathfrak{k}_-}(z,v) = z\, P_{\mathfrak{k}_0}(z,v)$$

remains valid if v is changed into $-v^{-1}$:

$$-v\, P_{\mathfrak{k}_+}(z,-v^{-1}) + v^{-1} P_{\mathfrak{k}_-}(z,-v^{-1}) = z\, P_{\mathfrak{k}_0}(z,-v^{-1}),$$

but

$$P_{\mathfrak{k}_+}(z,-v^{-1}) = P_{\mathfrak{k}_-^*}(z,v), \quad P_{\mathfrak{k}_-}(z,-v^{-1}) = P_{\mathfrak{k}_+^*}(z,v),$$
$$P_{\mathfrak{k}_0}(z,-v^{-1}) = P_{\mathfrak{k}_0^*}(z,v),$$

and $z^{-(n-1)} \cdot (v^{-1}-v)^{n-1}$ is invariant under the substitution $v \mapsto -v^{-1}$.

The formulae (c) and (d) for the product knot and a split union easily follow by similar arguments. □

17.20 Example. We calculate the HOMFLY-PT polynomials of the trefoil and its mirror image; using this invariant they are shown to be different, a result first obtained by Dehn [85, 87]. Let us call $b(3,1) = 3_1^+$ (see Figure 12.6) the *right-handed* trefoil and $b(3,-1) = 3_1^-$ the *left-handed* one. Figure 17.7 describes the skein tree starting with $\mathfrak{k} = \mathfrak{k}_+ = 3_1^-$. The crossing where the skein relation is applied is distinguished by a circle. One has:

$$v^{-1} P_{\mathfrak{k}_+} - v\, P_{\mathfrak{k}_-} = v^{-1} P_{\mathfrak{k}} - v = z\, P_{\mathfrak{k}_0}.$$

Moreover, with $\mathfrak{k}_{0+} = \mathfrak{k}_0$ the second crossing gives:

$$v^{-1} P_{\mathfrak{k}_{0+}} - v\, P_{\mathfrak{k}_{0-}} = v^{-1} P_{\mathfrak{k}_{0+}} - v\, P_{\mathfrak{k}_{00}} = z;$$

hence

$$P_{\mathfrak{k}_{0+}} = v^2 P_{00} + vz$$

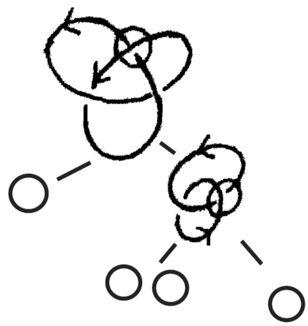


Figure 17.7

where $P_{00} = z^{-1}(v^{-1} - v)$ denotes polynomial of the 2-component trivial link (see Theorem 17.13). Therefore the first equation gives:

$$P_{3_1^-}(z, v) = -v^4 + 2v^2 + z^2v^2.$$

By Proposition 17.19 we have

$$P_{3_1^+}(z, v) = -v^{-4} + 2v^{-2} + z^2v^{-2},$$

and, hence, $3_1^+ \neq 3_1^-$. (For an exercise do a calculation of $P_{3_1^+}(z, v)$ using a skein tree.)

We give a second computation of $P_{3_1^-}(z, v)$ using the definition in Theorem 17.13:

$$P_{3_1^-}(z, v) = z^{-1}(v^{-1} - v)\mathrm{tr}\left(\varrho(\sigma_1^3)\right).$$

Here $n = 2$, and $3_1^- = b(3, -1) = \hat{\sigma}_1^3$, see Figure 12.6. We have $\varrho(\sigma_1^3) = v^3 \cdot c_1^3$. Applying $c_1^2 = vzc_1 + 1$ twice we get $c_1^3 = (z^2 + 1)c_1 + z$. By Definition 17.9 (α)

$$P_{3_1^-}(z, v) = z^{-1}(v^{-1} - v) \cdot v^3 \left((z^2 + 1)\mathrm{tr}(c_1) + z\right) = v^2z^2 + 2v^2 - v^4,$$

using Definition 17.9 (δ): $\mathrm{tr}(c_1) = T = zv^{-1}(v^{-1} - v)^{-1}$.

The HOMFLY-PT polynomial $P(z, v)$ contains as special cases the Alexander–Conway polynomial and the Jones polynomial.

17.21 Theorem. *The HOMFLY-PT polynomial specializes to the Jones polynomial and to the Alexander–Conway polynomial:*

$$P\left((t^{\frac{1}{2}} - t^{-\frac{1}{2}}), 1\right) = \Delta(t) = \text{Alexander–Conway polynomial}$$
$$P\left((t^{\frac{1}{2}} - t^{-\frac{1}{2}}), t\right) = V(t) = \text{Jones polynomial.}$$

Proof. By 9.21 the following skein relation holds for the Conway polynomial $\nabla_{\mathfrak{k}}(z)$:

$$\nabla_{\mathfrak{k}_+}(z) - \nabla_{\mathfrak{k}_-}(z) = z\,\nabla_{\mathfrak{k}_0}(z)\,.$$

The same skein relation holds for $P(z,1)$, and since both sides are equal to 1 for the trivial knot, equality must hold. Now, Proposition 9.23 gives $\Delta(t) = \nabla(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ and hence

$$P\left((t^{\frac{1}{2}} - t^{-\frac{1}{2}}), 1\right) = \Delta(t)\,.$$

In the second case we obtain the skein relation

$$t^{-1} V_{\mathfrak{k}_+}(t^{\frac{1}{2}}) - t\, V_{\mathfrak{k}_-}(t^{\frac{1}{2}}) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) V_{\mathfrak{k}_0}(t^{\frac{1}{2}})$$

of the Jones polynomial $V(t)$. □

17.22 Remark. For a one component link (knot), $V(t^{\frac{1}{2}})$ is in fact a Laurent polynomial in t.

17.C History and sources

The discovery of a new knot polynomial by V. F. R. Jones in 1985 [181, 182] which can distinguish mirror images of knots had the makings of a sensation. The immediate success in proving long-standing conjectures of Tait as an application added to its fame. Consequently many authors (Hoste, Ocneano, Millet, Floyd, Lickorish, Yetter, and Conway, Kauffman, Prytycki, Traczyk etc.) combined to study new and old (Alexander-) polynomials under the view of skein theory; as a result the 2-variable skein polynomial (HOMFLY-PT) was established which comprises both old and new knot polynomials (see [120, 292]).

17.D Exercises

E 17.1. Prove Proposition 17.19 using the Definition 17.13 of $P(z, v)$.

E 17.2. Compute the HOMFLY-PT polynomial for the Borromean link, see Example 9.20 (b) and Figure 9.2.

E 17.3. Prove that $6_1, 7_2, 7_4, 7_6, 7_7$ are the only knots with less than eight crossings whose braid index exceeds 3.

Appendix A

Algebraic theorems

A.1 Theorem. *Let Q be an $n \times n$ skew-symmetric matrix ($Q = -Q^T$) over the integers $\mathbb{Z}$. Then there is an integral unimodular matrix L such that*

$$L^T QL = \begin{pmatrix} 0 & a_1 & & & & & & & & \\ -a_1 & 0 & & & & & & & & \\ & & 0 & a_2 & & & & & & \\ & & -a_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & a_s & & & \\ & & & & & -a_s & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{pmatrix}$$

with $a_1|a_2|\dots|a_s$.

Proof. Let $\mathfrak{M}$ denote the module of $2n$-columns with integral coefficients: $\mathfrak{M} \cong \mathbb{Z}^{2n}$. Every $\mathfrak{x}_1 \in \mathfrak{M}$ defines a principal ideal

$$\{\mathfrak{x}_1^T Q\mathfrak{y} \mid \mathfrak{y} \in \mathfrak{M}\} = (a_1) \subset \mathbb{Z}.$$

We may choose $a_1 > 0$ if $Q \neq 0$. So there is a vector $\mathfrak{y}_1 \in \mathfrak{M}$ such that $\mathfrak{x}_1^T Q\mathfrak{y}_1 = a_1$, hence, $\mathfrak{y}_1^T Q\mathfrak{x}_1 = -a_1$. If follows that a_1 is contained in the ideal defined by $\mathfrak{y}_1$. Let $\mathfrak{x}_1$ be chosen in such a way that $a_1 > 0$ is minimal. Then a_1 generates the ideal defined by $\mathfrak{x}_1$ and $\mathfrak{y}_1$.

Put

$$\mathfrak{M}_1 = \{\mathfrak{u} \mid \mathfrak{x}_1^T Q\mathfrak{u} = \mathfrak{y}_1^T Q\mathfrak{u} = 0\}.$$

We prove that

$$\mathfrak{M} = \mathbb{Z}\mathfrak{x}_1 \oplus \mathbb{Z}\mathfrak{y}_1 \oplus \mathfrak{M}_1;$$

in particular, $\mathfrak{M}_1 \cong \mathbb{Z}^{2n-2}$.

Consider $\mathfrak{z} \in \mathfrak{M}$ and define $\alpha, \beta \in \mathbb{Z}$ by

$$\mathfrak{x}_1^T Q\mathfrak{z} = \beta a_1,\ \mathfrak{y}_1^T Q\mathfrak{z} = \alpha a_1.$$

Then

$$\begin{aligned}\mathfrak{x}_1^T Q(\mathfrak{z} - \beta\mathfrak{y}_1 - \alpha\mathfrak{x}_1) &= \beta a_1 - \beta a_1 - 0 = 0\\ \mathfrak{y}_1^T Q(\mathfrak{z} - \beta\mathfrak{y}_1 - \alpha\mathfrak{x}_1) &= \alpha a_1 - 0 - \alpha a_1 = 0;\end{aligned}$$

note that $Q^T = -Q$ implies that $\mathfrak{x}^T Q\mathfrak{x} = 0$. Now $\mathfrak{z} - \beta\mathfrak{y}_1 - \alpha\mathfrak{x}_1 \in \mathfrak{M}_1$ and $\mathfrak{x}_1$ and $\mathfrak{y}_1$ generate a module isomorphic to $\mathbb{Z}^2$. From

$$\mathfrak{x}_1^T Q(\xi\mathfrak{x}_1 + \eta\mathfrak{y}_1) = \eta a_1,\ \mathfrak{y}_1^T Q(\xi\mathfrak{x}_1 + \eta\mathfrak{y}_1) = -\xi a_1$$

it follows that $\xi\mathfrak{x}_1 + \eta\mathfrak{y}_1 \in \mathfrak{M}_1$ implies that $\xi = \eta = 0$. Thus $\mathfrak{M} = \mathbb{Z}\mathfrak{x}_1 \oplus \mathbb{Z}\mathfrak{h}_1 \oplus \mathfrak{M}_1$.

The skew-symmetric form Q induces on $\mathfrak{M}_1$ a skew-symmetric form Q'. As an induction hypothesis we may assume that there is a basis $\mathfrak{x}_2, \mathfrak{h}_2, \dots, \mathfrak{x}_n, \mathfrak{h}_n$ of $\mathfrak{M}_1$ such that Q' is represented by a matrix as desired.

To prove $1 \leqq a_1|a_2|\dots|a_s$, we may assume by induction $1 \leqq a_2|a_3|\dots|a_s$ already to be true. If $1 \leqq d = \gcd(a_1, a_2)$ and $d = ba_1 + ca_2$ then

$$(b\mathfrak{x}_1 + c\mathfrak{x}_2)^T Q(\mathfrak{y}_1 + \mathfrak{y}_2) = ba_1 + ca_2 = d.$$

Hence, by minimality of a_1 we obtain $d = a_1$. □

A.2 Theorem (B. W. Jones [180]). *Let $Q_n = (q_{ik})$ be a symmetric $n \times n$ matrix over $\mathbb{R}$, and $p(Q_n)$ the number of its positive, $q(Q_n)$ the number of its negative eigenvalues, then $\sigma(Q_n) = p(Q_n) - q(Q_n)$ is called the signature of Q_n. There is a sequence of principal minors $D_0 = 1,\ D_1, D_2, \dots$ such that D_i is principal minor of D_{i+1} and no two consecutive D_i, D_{i+1} are both singular for $i <$ rank Q_n. For any such (admissible) sequence*

$$\sigma(Q_n) = \sum_{i=0}^{n-1} \operatorname{sign}(D_i D_{i+1}). \tag{A.1}$$

Proof. The rank r of Q_n is the number of non-vanishing eigenvalues λ_i of Q_n; it is, at the same time, the maximal index i for which a non-singular principal minor exists – this follows from the fact that Q_n is equivalent over $\mathbb{R}$ to a diagonal matrix containing the eigenvalues λ_i in its diagonal. We may, therefore, assume $r = n$ and $D_i = \lambda_1 \dots \lambda_i$, $D_n \neq 0$.

The proof is by induction on n. Assume first that we have chosen a sequence $D_0, D_1, \dots$ with a non-singular minor D_{n-1}. (It will be admissible by induction.) We may suppose that $D_{n-1} = \det Q_{n-1}$ where Q_{n-1} is the submatrix of Q_n consisting of its first $n-1$ rows and columns. Now sign $(D_{n-1}D_n) = \operatorname{sign}\lambda_n$, and (A.1) follows by induction.

Suppose we choose a sequence with $D_{n-1} = 0$. Then $D_{n-2} \neq 0$, and, since $D_n \neq 0$, we obtain an admissible sequence for Q_n. There is a transformation

$B_n^T Q_n B_n = Q_n'$ with

$$B_n = \left(\begin{array}{c|c} & 0 \\ B_{n-1} & \vdots \\ & 0 \\ \hline 0 \quad \dots \quad 0 & 1 \end{array}\right), \quad B_{n-1} \in SO(n-1, \mathbb{R})$$

which takes Q_{n-1} into diagonal form

$$Q_{n-1}' = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_{n-2} & \\ & & & 0 \end{pmatrix}, \quad \lambda_i \neq 0.$$

By a further transformation

$$C_n^T Q_n' C_n = Q_n'', \quad C_n = \left(\begin{array}{c|c} & t_1 \\ E_{n-1} & \vdots \\ & t_{n-2} \\ & 0 \\ \hline 0 \quad \dots \quad 0 & 1 \end{array}\right), \quad t_i \in \mathbb{R},$$

one can achieve the following form

$$Q_n'' = \left(\begin{array}{ccc|cc} \lambda_1 & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & \lambda_{n-2} & 0 & 0 \\ \hline 0 & \dots & 0 & 0 & \alpha \\ 0 & \dots & 0 & \alpha & \beta \end{array}\right).$$

Since $D_n \neq 0$ it follows that $\alpha \neq 0$. Thus there exists an admissible sequence, and we can use the induction hypothesis for $n-2$. Now

$$\sigma \begin{pmatrix} 0 & \alpha \\ \alpha & \beta \end{pmatrix} = 0, \quad \text{and} \quad \sigma(Q_n) = \sigma \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-2} \end{pmatrix}.$$

The same result is obtained by (A.1) if $D_{n-1} = 0$. □

Let Γ be a finite oriented graph with vertices $\{P_i \mid 1 \leqq i \leqq n\}$ and oriented edges $\{u_{ij}^\lambda\}$, such that P_i is the initial point and P_j the terminal point of u_{ij}^λ. (For the basic terminology see Berge [18]). By a rooted tree (root P_1) we mean a subgraph of $n-1$ edges such that every point P_k is terminal point of a path with initial point P_1.

Let a_{ij} denote the number of edges with initial point P_i and terminal point P_j.

A.3 Theorem (Bott–Mayberry [39]). *Let* Γ *be a finite oriented graph without loops* $(a_{ii} = 0)$. *The principal minor* H_{ii} *of the graph matrix*

$$H(\Gamma) = \begin{pmatrix} (\sum\limits_{k \neq 1} a_{k1}) & -a_{12} & -a_{13} & \dots & -a_{1n} \\ -a_{21} & (\sum\limits_{k \neq 2} a_{k2}) & -a_{23} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ -a_{n1} & -a_{n2} & \dots & & (\sum\limits_{k \neq n} a_{kn}) \end{pmatrix}$$

is equal to the number of rooted trees with root P_i.

Proof. The principal $(n-1) \times (n-1)$-minor H_{ii} is the determinant of the submatrix obtained from $H(\Gamma)$ by omitting the i-th row and column. We need a:

A.4 Lemma. A graph C (without loops) with n vertices and $n-1$ edges is a rooted tree, root P_i, if $H_{ii}(C) = 1$; otherwise $H_{ii}(C) = 0$.

Proof of Lemma A.4. Suppose C is a rooted tree with root P_1. One has $\sum_{k \neq j} a_{kj} = 1$ for $j \neq 1$, because there is just one edge in C with terminal point P_j. If the indexing of vertices is chosen in such a way that indices increase along any path in C, then H_{11} has the form

$$H_{11} = \begin{vmatrix} 1 & * & \dots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{vmatrix} = 1.$$

To prove the converse it suffices to show that C is connected, if $H_{11} \neq 0$. Assuming this, use the fact that every point P_j, $j \neq 1$, must be a terminal point of C, otherwise the j-th column would consist of zeroes, contradicting $H_{11} \neq 0$. There is, therefore, an unoriented spanning tree in the (unoriented) graph C. The graph C coincides with this tree, since a spanning tree has $n-1$ edges. It must be a tree, rooted in P_1, because every vertex P_j, $j \neq 1$, is a terminal point.

The rest is proved by induction on n. We assume that C is not connected. Then we may arrange the indexing such that H_{11} is of the form:

$$H_{11} = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}, \ \det B' \neq 0, \ \det B'' \neq 0.$$

By the induction hypothesis we know that the subgraphs Γ' resp. Γ'' each containing P_1 and the vertices associated with the rows of B' resp. B'' – together with all edges of C joining these points – are P_1-rooted trees. This contradicts the assumption that C is not connected. □

We return to the proof of the main theorem. One may consider H_{11} as a multilinear function in the $n-1$ column vectors $\mathfrak{a}_j$, $j = 2, \ldots, n$ of the matrix (a_{ij}), $i \neq j$. This is true since the diagonal elements $\sum_{k \neq j} a_{kj}$ are themselves linear functions. Let $\mathfrak{e}_i$ denote a column vector with an i-th coordinate equal to one, and the other coordinates equal to zero. Then

$$H_{11}(\mathfrak{a}_2, \ldots, \mathfrak{a}_n) = \sum_{1 \leqq k_2, \ldots, k_n \leqq n} a_{k_2 2} \ldots a_{k_n n} H_{11}(\mathfrak{e}_{k_2}, \ldots, \mathfrak{e}_{k_n}) \tag{1}$$

with

$$\mathfrak{a}_i = \sum_{k_i = 1}^{n} a_{k_i i} \mathfrak{e}_{k_i}.$$

By the lemma $H_{11}(\mathfrak{e}_{k_2}, \ldots, \mathfrak{e}_{k_n}) = 1$ if and only if the $n-1$ edges $u_{k_2 2}$, $u_{k_2 2}, \ldots, u_{k_n n}$ form a P_1-rooted tree. Any such tree is to be counted $a_{k_2 2} \ldots a_{k_n n}$ times. □

Two corollaries follow easily.

A.5 Corollary. *Let Γ be an unoriented finite graph without loops, and let b_{ij} the number of edges joining P_i and P_j. A principal minor H_{ii} of*

$$\begin{pmatrix} \sum_{k \neq 1} b_{k1} & -b_{12} & -b_{13} & \ldots \\ -b_{21} & \sum_{k \neq 2} b_{k2} & \ldots & \\ \vdots & & & \vdots \\ \vdots & & & \sum_{k \neq n} b_{kn} \end{pmatrix}$$

gives the number of spanning trees of Γ, independent of i.

Proof. Replace every unoriented edges of Γ by a pair of edges with opposite directions, and apply Theorem A.3. □

A.6 Corollary. *Let Γ be a finite oriented loopless graph with a valuation $f\colon \{u_{ij}^{\lambda}\} \to \{1, -1\}$ on edges. Then the principal minor H_{ii} of $(f(a_{ij}))$, $f(a_{ij}) = \sum_{\lambda} f(u_{ij}^{\lambda})$, satisfies the following equation:*

$$H_{ii} = \sum f(\mathrm{Tr}(i))$$

where the sum is to be taken over all P_i*-rooted trees* $\mathrm{Tr}(i)$*, and where*

$$f(\mathrm{Tr}(i)) = \prod_{u_{kj}^{\lambda} \in \mathrm{Tr}(i)} f(u_{kj}^{\lambda}).$$

Proof. The proof of Theorem A.3 applies; it is only necessary to replace a_{ij} by $f(a_{ij})$. □

For other proofs and generalizations see Bott and Mayberry [39].

We add a well-known theorem without giving a proof. For a proof see Bourbaki [40, Chap. 7] or Zassenhaus [377, III].

A.7 Theorem. *Let* M *be a finitely generated module over a principal ideal domain. Then*

$$M \cong M_{\varepsilon_1} \oplus \ldots \oplus M_{\varepsilon_r} \oplus M_{\beta}$$

where M_β *is a free module of rank* β *and* $M_{\varepsilon_i} = \langle a \mid \varepsilon_i a \rangle$ *is a cyclic module generated by an element a and defined by* $\varepsilon_i a = 0$, $\varepsilon_i \in R$*. The* ε_i *are not units of* R*, different from zero, and form a chain of divisors* $\varepsilon_i \mid \varepsilon_{i+1}$, $1 \leqq i \leqq r$*. They are called the* elementary divisors of M*; the rank* β *of the free part of* M *is called the* Betti number of M.

The Betti numbers β *and* β' *of finitely generated modules* M *and* M' *coincide and their elementary divisors are pairwise associated,* $\varepsilon_i' = \alpha_i \varepsilon_i$, α_i *a unit of* R*, if and only if* M *and* M' *are isomorphic.* □

Remark. If M is a finitely presented module over commutative ring R with unit element, the theorem is not true. Nevertheless the elementary ideals of its presentation matrix are invariants of M. (See [359, I.4], [80, Chapter VII] and [377, III.§3] for more details.)

In the special case $R = \mathbb{Z}$ the theorem applies to finitely generated Abelian groups. The elementary divisors form a chain $T_1 | T_2 | \ldots | T_r$ of positive integers > 1, the orders of the cyclic summands. T_r is called the *first,* T_{r-1} the *second torsion numbers* etc. of the Abelian group.

Appendix B

Theorems of 3-dimensional topology

This section contains a collection of theorems in the field of 3-dimensional manifolds which have been frequently used in this book. In each case a source is given where a proof may be found.

B.1 Theorem (Alexander). *Let S^2 be a semilinearly embedded 2-sphere in S^3. There is a semilinear homeomorphism $h\colon S^3 \to S^3$ mapping S^2 onto the boundary $\partial[\sigma^3]$ of a 3-simplex σ^3.* □

Proofs can be found Alexander [6], Graeub [139] and Schubert [318]. A modern account is given the book of Moise [242, Chapter 17] and in Hatcher's notes [156].

B.2 Theorem (Alexander). *Let T be a semilinearly embedded torus in S^3. Then $S^3 - T$ consists of two components X_1 and X_2, $\overline{X}_1 \cup \overline{X}_2 = S^3$, $\overline{X}_1 \cap \overline{X}_2 = T$, and at least one of the subcomplexes $\overline{X}_1$, $\overline{X}_2$ is a torus.* □

Proofs are given by Alexander [6] and Schubert [318]. A short proof using Theorem B.1 and the loop theorem B.5 is given in Rolfsen's book [309, Chap. 4, C].

B.3 Theorem (Seifert–van Kampen). (a) *Let X be a connected polyhedron and X_1, X_2 connected subpolyhedra with $X = X_1 \cup X_2$ and $X_1 \cap X_2$ a (nonempty) connected subpolyhedron. Suppose*

$$\pi_1(X_1, P) = \langle\, S_1, \dots, S_n \mid R_1, \dots, R_m \,\rangle$$
$$\pi_1(X_2, P) = \langle\, T_1, \dots, T_k \mid N_1, \dots, N_l \,\rangle$$

with respect to a basepoint $P \in X_1 \cap X_2$. A set $\{v_j \mid 1 \leqq j \leqq r\}$ of generating loops of $\pi_1(X_2 \cap X_2, P)$ determines sets $\{V_{1j}(S_i)\}$ and $\{V_{2j}(T_j)\}$ respectively of elements in $\pi_1(X_1, P)$ or $\pi_1(X_2, P)$ respectively. Then

$$\pi_1(X, P) = \\ \langle\, S_1, \dots, S_n, T_1, \dots, T_k \mid R_1, \dots, R_m, N_1, \dots, N_l, V_{11}V_{21}^{-1}, \dots, V_{1r}V_{2r}^{-1} \,\rangle.$$

(b) *Let X_1, X_2 be disjoint connected homeomorphic subpolyhedra of a connected polyhedron X, and denote by $\bar{X} = X/h$ the polyhedron which results from identifying X_1 and X_2 via the homeomorphism $h\colon X_1 \to X_2$. For a base point $P \in X_1$ and its image $\bar{P}$ under the identification a presentation of $\pi_1(\bar{X}; \bar{P})$ is obtained from one of $\pi_1(X; P)$ by adding a generator S and the defining relations $ST_iS^{-1} = h_\#(T_i)$, $1 \leqq i \leqq r$ where $\{T_i \mid 1 \leqq i \leqq r\}$ generate $\pi_1(X_1; P)$.* □

For a proof see Zieschang, Vogt and Coldewey [382, 2.8.2]. A topological version of Theorem B.3 (a) is valid when X, X_1,, X_2, $X_1 \cap X_2$, are path connected and X_1, X_2 are open, Crowell and Fox [80], Massey [230], Hatcher [157, 1.2], Stöcker and Zieschang [346, 5.3.11]. A topological version of Theorem B.3 (b) may be obtained if X, X_1, X_2 are path connected, X_1, X_2 are closed, and if the identifying homeomorphism can be extended to a collaring.

B.4 Theorem (Generalized Dehn's lemma). *Let $h\colon S(0,r) \to M$ be a simplicial immersion of an orientable compact surface $S(0,r)$ of genus 0 with r boundary components into the 3-manifold M with no singularities on the boundary $\partial h(S(0,r)) = \{C_1, C_2, \ldots, C_r\}$, C_i a closed curve. Suppose that the normal closure $\langle \overline{C_1, \ldots, C_r} \rangle$ in $\pi_1(M)$ is contained in the subgroup $\hat{\pi}_1(M) \subset \pi_1(M)$ of orientation preserving paths. Then there is a non-singular planar surface $S(0,q)$ embedded in M with $\partial S(0,q)$ a non-vacuous subset of $\{C_1, \ldots, C_r\}$.* □

Proof are made available by Shapiro and Whitehead [332], Hempel [159], Rolfsen [309] and Jaco [176]. See also Hatcher's notes [156].

Remark. Theorem B.4 was proved by Shapiro and Whitehead. The original Lemma of Dehn $r = 1 = q$ was formulated by Dehn in 1910 [84, 87] but proved only in 1957 by Papakyriakopoulos [281].

B.5 Theorem (Generalized loop theorem). *Let M be a 3-manifold and let B be a component of its boundary. If there are elements in $\ker(\pi_1 B \to \pi_1 M)$ which are not contained in a given normal subgroup $\mathfrak{R}$ of $\pi_1(B)$ then there is a simple loop C on B such that C bounds a non-singular disk in M and $[C] \notin \mathfrak{R}$.* □

Theorem B.5 was proved by Papakyriakopoulos [281] and Stallings [342]. Proofs in books are available, see Rolfsen [309], Hempel [159] and Jaco [176]. See also Hatcher's notes [156].

Remark. The proof is given in the second reference. The original version of the loop theorem ($\mathfrak{R} = 1$) was first formulated and proved by Papakyriakopoulos. Another generalization analogous to the Shapiro–Whitehead version of Dehn's Lemma was proved by Waldhausen [364].

B.6 Theorem (Sphere theorem). *Let M be an orientable 3-manifold and $\mathfrak{R}$ a $\pi_1 M$-invariant subgroup of $\pi_2 M$, $\mathfrak{R} \subsetneq \pi_2 M$. ($\mathfrak{R}$ is $\pi_1 M$-invariant if the operation of $\pi_1 M$ on $\pi_2 M$ maps $\mathfrak{R}$ onto itself.) Then there is an embedding $g\colon S^2 \to M$ such that $[g] \notin \mathfrak{R}$.* □

Theorem B.5 was proved by Papakyriakopoulos [281]. Proofs in books are available, see Hempel [159] and Jaco [176]. See also Hatcher's notes [156].

This triad of Papakyriakopoulos theorems started a new era in 3-dimensional topology. The next impulse came from W. Haken and F. Waldhausen:

A surface F is *properly embedded* in a 3-manifold M if $\partial F = F \cap \partial M$. A 2-sphere ($F = S^2$) is called *incompressible in* M, if it does not bound a 3-ball in M, and $F \neq S^2$ is called *incompressible*, if there is no disk $D \subset M$ with $D \cap F = \partial D$, and ∂D not contractible in F. A manifold is *sufficiently large* when it contains a properly embedded 2-sided incompressible surface. A 3-manifold M is *irreducible* if each embedded 2-sphere is compressible in M.

B.7 Theorem (Waldhausen). *Let M, N be oriented, sufficiently large, irreducible 3-manifolds whose boundaries are empty or incompressible surfaces. If there is an isomorphism $f_\#: (\pi_1 M, \pi_1 \partial M) \to (\pi_1 N, \pi_1 \partial N)$ between the peripheral group systems, then there exists a homeomorphism $f: M \to N$ inducing $f_\#$.* □

Waldhausen [367, Cor. 6.5], Hempel [159] and Jaco [176].

B.8 Remarks. 1. The Waldhausen theorem states for a large class of manifolds what has long been known of surfaces.

2. Theorem B.7 applies to knot complements $C = M$. A Seifert surface of minimal genus is a properly embedded incompressible surface in C, see Corollary 4.6. By Alexander's Theorem B.1 a knot complement is irreducible, and by Proposition 3.10 C has irreducible boundary.

B.9 Theorem (Smith conjecture). *A simplicial orientation preserving map h: $S^3 \to S^3$ of period q is conjugate to a rotation.* □

A conference on the Smith conjecture was held in 1979 at Columbia University in New York. The proceedings, edited by Morgan and Bass [248], contains a proof. The case $q = 2$ is due to Waldhausen, see [368]. The Smith Conjecture is now a special case of *Thurston's Orbifold Theorem*. A general introduction to this subject can be found in Scott's survey article [323], in the books by D. Cooper, C. D. Hodgso and S. P. Kerckhoff [72] and M. Boileau, S. Maillot and J. Porti [34]. A partial result was obtained by M. Boileau and J. Porti in 2001 [35] and a complete proof of the Orbifold Theorem was published by M. Boileau, B. Leeb and J. Porti in 2005 [33].

B.10 Theorem (Poincaré conjecture). *Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.* □

G. Perelman presented a proof of the Poincaré conjecture in three papers made available in 2002 and 2003 on arXiv, see [283, 285, 284]. The proof followed R. Hamilton's program to use the Ricci flow to attack the problem, and several teams of mathematicians have verified that Perelman's proof is correct. (See J. Lott's speech at the ICM

2006 [217].) J. Morgan and G. Tian present a proof of the Poincaré conjecture in the monograph [247].

In 1982, W. P. Thurston [348] proposed a conjectural picture of 3-manifolds as built up by geometric pieces. This conjectural picture became known as *Thurston's geometrization conjecture* (it includes the Poincaré conjecture). The geometrization conjecture is an analogue for 3-manifolds of the uniformization theorem for surfaces, see Scott [323]. Perelman's work also includes a proof of Thurston's geometrization conjecture. The monographs by L. Bessières, G. Besson, S. Maillot, M. Boileau and J. Porti [19] and J. W Morgan and F. T.-H. Fong [249] offer an accessible account of the subject.

Appendix C

Table

The following table lists certain invariants of knots up to ten crossings. The identification (first column) follows Rolfsen [309] but takes into account that there is a duplication $(10_{161} = 10_{162})$ in his table which was detected by Perko. For each crossing number alternating knots are grouped in front, a star indicates the first non-alternating knot in each order.

The first column $(\Delta_1, (t), \Delta_2(t))$ contains the Alexander polynomials, factorized into irreducible polynomials. The polynomials $\Delta_k(t), k > 2$, are always trivial (see Chapter 8). Alexander polynomials of links or of knots with eleven crossings are to be found in Rolfsen [309], Conway [71] and Perko [288].

The second column (T) gives the torsion numbers of the first homology group $H_1(\hat{C}_2)$ of the two-fold branched covering of the knot. The numbers are $T_r, T_{r-1}, \ldots$ where $T_1|T_2|\ldots|T_r$ is the chain of elementary divisors of $H_1(\hat{C}_2)$ (see Chapter 9). For torsion numbers of cyclic coverings of order $n > 2$, see Mehta [235]. Torsion numbers for $n = 3$ (knots with less than ten crossings) are listed in Reidemeister [296].

The column $(|\sigma|)$ records the absolute value of the signature of the knot (see Chapter 13).

The column (q) states the periods of the knot (see Section 14.D). The sign "–" in this column indicates that the knot has no period.

The column headed α, β contains Schubert's notation of the knot as a two-bridged knot. (The first number α always coincides with T_r.) Where no entry appears, the bridge number is three (see Chapter 12).

The column (s) contains complete information about symmetries in Conway's notation (see Chapter 2).

	amphicheiral	non-amphicheiral
invertible	f	r
non-invertible	i	n

It has been checked that for knots with up to ten crossings the genus of a knot always equals half the degree of its Alexander polynomial.

***KnotInfo, LinkInfo* and *The Knot Atlas*.** Very powerful tools are the tables of knot and link invariants of J. C. Cha and C. Livingston [65, 66]. A huge variety of knot (up to twelve crossing knots) and link (up to eleven crossing links) invariants are listed.

Another very useful tool is *The Knot Atlas* [15] run by D. Bar-Natan and S. Morrison. Every knot and link up to some size has a page with much data about it: words, pictures, and values of invariants.

Acknowledgment: The Alexander polynomials, the signature and most of the periods have been computed by U. Lüdicke. Periods up to nine crossings were taken from Murasugi [263]. Symmetries and 2-bridge numbers (α, β) were copied from Conway [71] and compared with other results on amphicheirality and invertibility from Hartley [148]. Periods and symmetries have been corrected and brought up-to-date using Adams et al. [1], Kodama and Sakuma [201], Henry and Weeks [161] and Kawauchi [190]. As far as possible, the data has been compared with J. C. Cha and C. Livingston's table of knot invariants *KnotInfo* [65].

C.1 Table

$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α, β	s
3_1	$t^2 - t + 1$		3	2	2, 3	3, 1	r
4_1	$t^2 - 3t + 1$		5	0	2	5, 2	f
5_1	$t^4 - t^3 + t^2 - t + 1$		5	4	2, 5	5, 1	r
5_2	$2t^2 - 3t + 2$		7	2	2	7, 3	r
6_1	$2t^2 - 5t + 2$		9	0	2	9, 2	r
6_2	$t^4 - 3t^3 + 3t^2 - 3t + 1$		11	2	2	11, 4	r
6_3	$t^4 - 3t^3 + 5t^2 - 3t + 1$		13	0	2	13, 5	f
7_1	$t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$		7	6	2, 7	7, 1	r
7_2	$3t^2 - 5t + 3$		11	2	2	11, 5	r
7_3	$2t^4 - 3t^3 + 3t^2 - 3t + 2$		13	4	2	13, 4	r
7_4	$4t^2 - 7t + 4$		15	2	2	15, 4	r
7_5	$2t^4 - 4t^3 + 5t^2 - 4t + 2$		17	4	2	17, 7	r
7_6	$t^4 - 5t^3 + 7t^2 - 5t + 1$		19	2	2	19, 7	r
7_7	$t^4 - 5t^3 + 9t^2 - 5t + 1$		21	0	2	21, 8	r
8_1	$3t^2 - 7t + 3$		13	0	2	13, 2	r
8_2	$t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 1$		17	4	2	17, 6	r
8_3	$4t^2 - 9t + 4$		17	0	2	17, 4	f
8_4	$2t^4 - 5t^3 + 5t^2 - 5t + 2$		19	2	2	19, 5	r
8_5	$(t^2 - t + 1)(-t^4 + 2t^3 - t^2 + 2t - 1)$		21	4	2		r
8_6	$2t^4 - 6t^3 + 7t^2 - 6t + 2$		23	2	2	23, 10	r
8_7	$t^6 - 3t^5 + 5t^4 - 5t^3 + 5t^2 - 3t + 1$		23	2	2	23, 9	r
8_8	$2t^4 - 6t^3 + 9t^2 - 6t + 2$		25	0	2	25, 9	r

$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α, β	s
8_9	$t^6 - 3t^5 + 5t^4 - 7t^3 + 5t^2 - 3t + 1$		25	0	2	25, 7	f
8_{10}	$(t^2 - t + 1)^3$		27	2	–		r
8_{11}	$(2t^2 - 5t + 2)(t^2 - t + 1)$		27	2	2	27, 10	r
8_{12}	$t^4 - 7t^3 + 13t^2 - 7t + 1$		29	0	2	29, 12	f
8_{13}	$2t^4 - 7t^3 + 11t^2 - 7t + 2$		29	0	2	29, 11	r
8_{14}	$2t^4 - 8t^3 + 11t^2 - 8t + 2$		31	2	2	31, 12	r
8_{15}	$(t^2 - t + 1)(3t^2 - 5t + 3)$		33	4	2		r
8_{16}	$t^6 - 4t^5 + 8t^4 - 9t^3 + 8t^2 - 4t + 1$		35	2	–		r
8_{17}	$t^6 - 4t^5 + 8t^4 - 11t^3 + 8t^2 - 4t + 1$		37	0	–		i
8_{18}	$(t^2 - t + 1)^2(t^2 - 3t + 1)$,	$t^2 - t + 1$	15, 3	0	2, 4		f
$*8_{19}$	$(t^2 - t + 1)(t^4 - t^2 + 1)$		3	6	2, 3, 4		r
8_{20}	$(t^2 - t + 1)^2$		9	0	–		r
8_{21}	$(t^2 - t + 1)(t^2 - 3t + 1)$		15	2	2		r
9_1	$(t^2 - t + 1)(t^6 - t^3 + 1)$		9	8	2, 3, 9	9, 1	r
9_2	$4t^2 - 7t + 4$		15	2	2	15, 7	r
9_3	$2t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 2$		19	6	2	19, 6	r
9_4	$3t^4 - 5t^3 + 5t^2 - 5t + 3$		21	4	2	21, 5	r
9_5	$6t^2 - 11t + 6$		23	2	2	23, 6	r
9_6	$(t^2 - t + 1)(-2t^4 + 2t^3 - t^2 + 2t - 2)$		27	6	2	27, 5	r
9_7	$3t^4 - 7t^3 + 9t^2 - 7t + 3$		29	4	2	29, 13	r
9_8	$2t^4 - 8t^3 + 11t^2 - 8t + 2$		31	2	2	31, 11	r
9_9	$2t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 2$		31	6	2	31, 9	r
9_{10}	$4t^4 - 8t^3 + 9t^2 - 8t + 4$		33	4	2	33, 10	r
9_{11}	$t^6 - 5t^5 + 7t^4 - 7t^3 + 7t^2 - 5t + 1$		33	4	2	33, 14	r
9_{12}	$(t^2 - 3t + 1)(2t^2 - 3t + 2)$		35	2	2	35, 13	r
9_{13}	$4t^4 - 9t^3 + 11t^2 - 9t + 4$		37	4	2	37, 10	r
9_{14}	$2t^4 - 9t^3 + 15t^2 - 9t + 2$		37	0	2	37, 14	r
9_{15}	$2t^4 - 10t^3 + 15t^2 - 10t + 2$		39	2	2	39, 16	r
9_{16}	$(t^2 - t + 1)(-2t^4 + 3t^3 - 3t^2 + 3t - 2)$		39	6	2		r
9_{17}	$t^6 - 5t^5 + 9t^4 - 9t^3 + 9t^2 - 5t + 1$		39	2	2	39, 14	r
9_{18}	$4t^4 - 10t^3 + 13t^2 - 10t + 4$		41	4	2	41, 17	r
9_{19}	$2t^4 - 10t^3 + 17t^2 - 10t + 2$		41	0	2	41, 16	r
9_{20}	$t^6 - 5t^5 + 9t^4 - 11t^3 + 9t^2 - 5t + 1$		41	4	2	41, 15	r
9_{21}	$2t^4 - 11t^3 + 17t^2 - 11t + 2$		43	2	2	43, 18	r
9_{22}	$t^6 - 5t^5 + 10t^4 - 11t^3 + 10t^2 - 5t + 1$		43	2	–		r
9_{23}	$(t^2 - t + 1)(4t^2 - 7t + 4)$		45	4	2	45, 19	r

$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α, β	s
9_{24}	$(t^2-t+1)^2(t^2-3t+1)$		45	0	–		r
9_{25}	$3t^4-12t^3+17t^2-12t+3$		47	2	–		r
9_{26}	$t^6-5t^5+11t^4-13t^3+11t^2-5t+1$		47	2	2	47, 18	r
9_{27}	$t^6-5t^5+11t^4-15t^3+11t^2-5t+1$		49	0	2	49, 19	r
9_{28}	$(t^2-t+1)(-t^4+4t^3-7t^2+4t-1)$		51	2	2		r
9_{29}	$(t^2-t+1)(-t^4+4t^3-7t^2+4t-1)$		51	2	–		r
9_{30}	$t^6-5t^5+12t^4-17t^3+12t^2-5t+1$		53	0	–		r
9_{31}	$t^6-5t^5+13t^4-17t^3+13t^2-5t+1$		55	2	2	55, 21	r
9_{32}	$t^6-6t^5+14t^4-17t^3+14t^2-6t+1$		59	2	–		n
9_{33}	$t^6-6t^5+14t^4-19t^3+14t^2-6t+1$		61	0	–		n
9_{34}	$t^6-6t^5+16t^4-23t^3+16t^2-6t+1$		69	0	–		r
9_{35}	$7t^2-13t+7$		9, 3	2	2, 3		r
9_{36}	$t^6-5t^5+8t^4-9t^3+8t^2-5t+1$		37	4	–		r
9_{37}	$(t^2-3t+1)(2t^2-5t+2)$		15, 3	0	2		r
9_{38}	$(t^2-t+1)(5t^2-9t+5)$		57	4	–		r
9_{39}	$(t^2-3t+1)(3t^2-5t+3)$		55	2	–		r
9_{40}	$(t^2-t+1)(t^2-3t+1)^2$,	t^2-3t+1	15, 5	2	2, 3		r
9_{41}	$3t^4-12t^3+19t^2-12t+3$		7, 7	0	3		r
$*9_{42}$	$t^4-2t^3+t^2-2t+1$		7	2	–		r
9_{43}	$t^6-3t^5+2t^4-t^3+2t^2-3t+1$		13	4	–		r
9_{44}	$t^4-4t^3+7t^2-4t+1$		17	0	–		r
9_{45}	$t^4-6t^3+9t^2-6t+1$		23	2	–		r
9_{46}	$2t^2-5t+2$		3, 3	0	2		r
9_{47}	$t^6-4t^5+6t^4-5t^3+6t^2-4t+1$		9, 3	2	3		r
9_{48}	$t^4-7t^3+11t^2-7t+1$		9, 3	2	2		r
9_{49}	$3t^4-6t^3+7t^2-6t+3$		5, 5	4	3		r
10_1	$4t^2-9t+4$		17	0	2	17, 2	r
10_2	$t^8-3t^7+3t^6-3t^5+3t^4-3t^3+3t^2-3t+1$		23	6	2	23, 8	r
10_3	$6t^2-13t+6$		25	0	2	25, 6	r
10_4	$3t^4-7t^3+7t^2-7t+3$		27	2	2	27, 7	r
10_5	$(t^2-t+1)(t^6-2t^5+2t^4-t^3+2t^2-2t+1)$		33	4	2	33, 13	r
10_6	$2t^6-6t^5+7t^4-7t^3+7t^2-6t+2$		37	4	2	37, 16	r
10_7	$3t^4-11t^3+15t^2-11t+3$		43	2	2	43, 16	r
10_8	$2t^6-5t^5+5t^4-5t^3+5t^2-5t+2$		29	4	2	29, 6	r
10_9	$(t^2-t+1)(t^6-2t^5+2t^4-3t^3+2t^2-2t+1)$		39	2	2	39, 11	r
10_{10}	$3t^4-11t^3+17t^2-11t+3$		45	0	2	45, 17	r

$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α, β	s
10_{11}	$4t^4 - 11t^3 + 13t^2 - 11t + 4$		43	2	2	43, 13	r
10_{12}	$2t^6 - 6t^5 + 10t^4 - 11t^3 + 10t^2 - 6t + 2$		47	2	2	47, 17	r
10_{13}	$2t^4 - 13t^3 + 23t^2 - 13t + 2$		53	0	2	53, 22	r
10_{14}	$2t^6 - 8t^5 + 12t^4 - 13t^3 + 12t^2 - 8t + 2$		57	4	2	57, 22	r
10_{15}	$2t^6 - 6t^5 + 9t^4 - 9t^3 + 9t^2 - 6t + 2$		43	2	2	43, 19	r
10_{16}	$4t^4 - 12t^3 + 15t^2 - 12t + 4$		47	2	2	47, 14	r
10_{17}	$t^8 - 3t^7 + 5t^6 - 7t^5 + 9t^4 - 7t^3 + 5t^2 - 3t + 1$		41	0	2	41, 9	f
10_{18}	$4t^4 - 14t^3 + 19t^2 - 14t + 4$		55	2	2	55, 23	r
10_{19}	$2t^6 - 7t^5 + 11t^4 - 11t^3 + 11t^2 - 7t + 2$		51	2	2	51, 14	r
10_{20}	$3t^4 - 9t^3 + 11t^2 - 9t + 3$		35	2	2	35, 16	r
10_{21}	$(2t^2 - 5t + 2)(-t^4 + t^3 - t^2 + t - 1)$		45	4	2	45, 16	r
10_{22}	$2t^6 - 6t^5 + 10t^4 - 13t^3 + 10t^2 - 6t + 2$		49	0	2	49, 13	r
10_{23}	$2t^6 - 7t^5 - 13t^4 - 15t^3 + 13t^2 - 7t + 2$		59	2	2	59, 23	r
10_{24}	$4t^4 - 14t^3 + 19t^2 - 14t + 4$		55	2	2	55, 24	r
10_{25}	$2t^6 - 8t^5 + 14t^4 - 17t^3 + 14t^2 - 8t + 2$		65	4	2	65, 24	r
10_{26}	$2t^6 - 7t^5 + 13t^4 - 17t^3 + 13t^2 - 7t + 2$		61	0	2	61, 17	r
10_{27}	$2t^6 - 8t^5 + 16t^4 - 19t^3 + 16t^2 - 8t + 2$		71	2	2	71, 27	r
10_{28}	$4t^4 - 13t^3 + 19t^2 - 13t + 4$		53	0	2	53, 19	r
10_{29}	$t^6 - 7t^5 + 15t^4 - 17t^3 + 15t^2 - 7t + 1$		63	2	2	63, 26	r
10_{30}	$4t^4 - 17t^3 + 25t^2 - 17t + 4$		67	2	2	67, 26	r
10_{31}	$4t^4 - 14t^3 + 21t^2 - 14t + 4$		57	0	2	57, 25	r
10_{32}	$(t^2 - t + 1)(-2t^4 + 6t^3 - 7t^2 + 6t - 2)$		69	0	2	69, 29	r
10_{33}	$4t^4 - 16t^3 + 25t^2 - 16t + 4$		65	0	2	65, 18	f
10_{34}	$3t^4 - 9t^3 + 13t^2 - 9t + 3$		37	0	2	37, 13	r
10_{35}	$2t^4 - 12t^3 + 21t^2 - 12t + 2$		49	0	2	49, 20	r
10_{36}	$3t^4 - 13t^3 + 19t^2 - 13t + 3$		51	2	2	51, 20	r
10_{37}	$4t^4 - 13t^3 + 19t^2 - 13t + 4$		53	0	2	53, 23	f
10_{38}	$4t^4 - 15t^3 + 21t^2 - 15t + 4$		59	2	2	59, 25	r
10_{39}	$2t^6 - 8t^5 + 13t^4 - 15t^3 + 13t^2 - 8t + 2$		61	4	2	61, 22	r
10_{40}	$(t^2 - t + 1)(-2t^4 + 6t^3 - 9t^2 + 6t - 2)$		75	2	2	75, 29	r
10_{41}	$t^6 - 7t^5 + 17t^4 - 21t^3 + 17t^2 - 7t + 1$		71	2	2	71, 26	r
10_{42}	$t^6 - 7t^5 + 19t^4 - 27t^3 + 19t^2 - 7t + 1$		81	0	2	81, 31	r
10_{43}	$t^6 - 7t^5 + 17t^4 - 23t^3 + 17t^2 - 7t + 1$		73	0	2	73, 27	f
10_{44}	$t^6 - 7t^5 + 19t^4 - 25t^3 + 19t^2 - 7t + 1$		79	2	2	79, 30	r
10_{45}	$t^6 - 7t^5 + 21t^4 - 31t^3 + 21t^2 - 7t + 1$		89	0	2	89, 34	f
10_{46}	$t^8 - 3t^7 + 4t^6 - 5t^5 + 5t^4 - 5t^3 + 4t^2 - 3t + 1$		31	6	–		r

$\mathfrak{k}$	$\Delta_1(t)$ $\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α,β	s
10_{47}	$t^8-3t^7+6t^6-7t^5+7t^4-7t^3+6t^2-3t+1$	41	4	–		r
10_{48}	$t^8-3t^7+6t^6-9t^5+11t^4-9t^3+6t^2-3t+1$	49	0	–		r
10_{49}	$3t^6-8t^5+12t^4-13t^3+12t^2-8t+3$	59	6	–		r
10_{50}	$2t^6-7t^5+11t^4-13t^3+11t^2-7t+2$	53	4	–		r
10_{51}	$2t^6-7t^5+15t^4-19t^3+15t^2-7t+2$	67	2	–		r
10_{52}	$2t^6-7t^5+13t^4-15t^3+13t^2-7t+2$	59	2	–		r
10_{53}	$6t^4-18t^3+25t^2-18t+6$	73	4	–		r
10_{54}	$2t^6-6t^5+10t^4-11t^3+10t^2-6t+2$	47	2	–		r
10_{55}	$5t^4-15t^3+21t^2-15t+5$	61	4	–		r
10_{56}	$2t^6-8t^5+14t^4-17t^3+14t^2-8t+2$	65	4	–		r
10_{57}	$2t^6-8t^5+18t^4-23t^3+18t^2-8t+2$	79	2	–		r
10_{58}	$(t^2-3t+1)(3t^2-7t+3)$	65	0	2		r
10_{59}	$(t^2-t+1)(t^2-3t+1)^2$	75	2	–		r
10_{60}	$(t^2-3t+1)(-t^4+4t^3-7t^2+4t-1)$	85	0	2		r
10_{61}	$(t^2-t+1)(-2t^4+3t^3-t^2+3t-2)$	33	4	2		r
10_{62}	$(t^2-t+1)^2(t^4-t^3+t^2-t+1)$	45	4	–		r
10_{63}	$(t^2-t+1)(5t^2-9t+5)$	57	4	2		r
10_{64}	$(t^2-t+1)(t^6-2t^5+3t^4-5t^3+3t^2-2t+1)$	51	2	2		r
10_{65}	$(t^2-t+1)^2(-2t^2+3t-2)$	63	2	–		r
10_{66}	$(t^2-t+1)(-3t^4+6t^3-7t^2+6t-3)$	75	6	2		r
10_{67}	$(2t^2-3t+2)(2t^2-5t+2)$	63	2	2		n
10_{68}	$4t^4-14t^3+21t^2-14t+4$	57	0	2		r
10_{69}	$t^6-7t^5+21t^4-29t^3+21t^2-7t+1$	87	2	2		r
10_{70}	$t^6-7t^5+16t^4-19t^3+16t^2-7t+1$	67	2	–		r
10_{71}	$t^6-7t^5+18t^4-25t^3+18t^2-7t+1$	77	0	–		r
10_{72}	$2t^6-9t^5+16t^4-19t^3+16t^2-9t+2$	73	4	–		r
10_{73}	$t^6-7t^5+20t^4-27t^3+20t^2-7t+1$	83	2	–		r
10_{74}	$(2t^2-3t+2)(2t^2-5t+2)$	21,3	2	2		r
10_{75}	$t^6-7t^5+19t^4-27t^3+19t^2-7t+1$	27,3	0	2		r
10_{76}	$(t^2-t+1)(-2t^4+5t^3-5t^2+5t-2)$	57	4	2		r
10_{77}	$(t^2-t+1)^2(-2t^2+3t-2)$	63	2	–		r
10_{78}	$(t^2-t+1)(-t^4+6t^3-9t^2+6t-1)$	69	4	2		r
10_{79}	$t^8-3t^7+7t^6-12t^5+15t^4-12t^3+7t^2-3t+1$	61	0	–		i
10_{80}	$3t^6-9t^5+15t^4-17t^3+15t^2-9t+3$	71	6	–		n
10_{81}	$t^6-8t^5+20t^4-27t^3+20t^2-8t+1$	85	0	–		i
10_{82}	$(t^2-t+1)^2(t^4-2t^3+t^2-2t+1)$	63	2	–		n

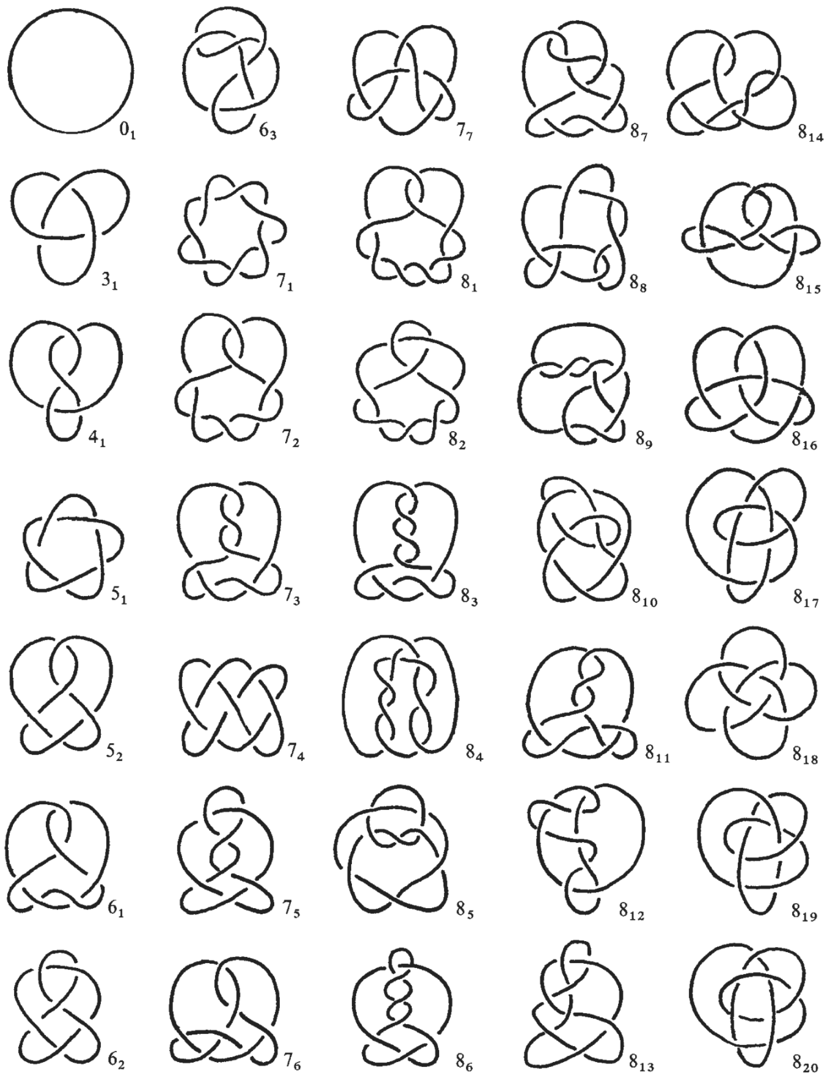
$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α,β	s
10_{83}	$2t^6-9t^5+19t^4-25t^3+19t^2-9t+2$		83	2	–		n
10_{84}	$(t^2-t+1)(-2t^4+7t^3-11t^2+7t-2)$		87	2	–		n
10_{85}	$(t^2-t+1)(t^6-3t^5+4t^4-3t^3+4t^2-3t+1)$		57	4	–		n
10_{86}	$2t^6-9t^5+19t^4-23t^3+19t^2-9t+2$		85	0	–		n
10_{87}	$(t^2-t+1)^2(-2t^2+5t-2)$		81	0	–		n
10_{88}	$t^6-8t^5+24t^4-35t^3+24t^2-8t+1$		101	0	–		i
10_{89}	$t^6-8t^5+24t^4-33t^3+24t^2-8t+1$		99	2	–		r
10_{90}	$2t^6-8t^5+17t^4-23t^3+17t^2-8t+2$		77	0	–		n
10_{91}	$t^8-4t^7+9t^6-14t^5+17t^4-14t^3+9t^2-4t+1$		73	0	–		n
10_{92}	$2t^6-10t^5+20t^4-25t^3+20t^2-10t+2$		89	4	–		n
10_{93}	$2t^6-8t^5+15t^4-17t^3+15t^2-8t+2$		67	2	–		n
10_{94}	$t^8-4t^7+9t^6-14t^5+15t^4-14t^3+9t^2-4t+1$		71	2	–		n
10_{95}	$(2t^2-3t+2)(-t^4+3t^3-5t^2+3t-1)$		91	2	–		n
10_{96}	$t^6-7t^5+22t^4-33t^3+22t^2-7t+1$		93	0	–		r
10_{97}	$5t^4-22t^3+33t^2-22t+5$		87	2	–		r
10_{98}	$(t^2-t+1)^2(-2t^2+5t-2)$,	(t^2-t+1)	27,3	4	2		n
10_{99}	$(t^2-t+1)^4$,	$(t^2-t+1)^2$	9,9	0	–		f
10_{100}	$(t^4-t^3+t^2-t+1)(t^4-3t^3+5t^2-3t+1)$		65	4	–		r
10_{101}	$7t^2-21t^3+29t^2-21t+7$		85	4	–		r
10_{102}	$2t^6-8t^5+16t^4-21t^3+16t^2-8t+2$		73	0	–		n
10_{103}	$(t^2-t+1)(-2t^4+6t^3-9t^2+6t-2)$		15,5	2	–		r
10_{104}	$t^8-4t^7+9t^6-15t^5+19t^4-15t^3+9t^2-4t+1$		77	0	–		r
10_{105}	$t^6-8t^5+22t^4-29t^3+22t^2-8t+1$		91	2	–		r
10_{106}	$(t^2-t+1)(t^6-3t^5+5t^4-7t^3+5t^2-3t+1)$		75	2	–		n
10_{107}	$t^6-8t^5+22t^4-31t^3+22t^2-8t+1$		93	0	–		n
10_{108}	$2t^6-8t^5+14t^4-15t^3+14t^2-8t+2$		63	2	–		r
10_{109}	$t^8-4t^7+10t^6-17t^5+21t^4-17t^3+10t^2-4t+1$		85	0	–		i
10_{110}	$t^6-8t^5+20t^4-25t^3+20t^2-8t+1$		83	2	–		n
10_{111}	$(2t^2-3t+2)(-t^4+3t^3-3t^2+3t-1)$		77	4	–		r
10_{112}	$(t^2-t+1)(t^6-4t^5+6t^4-7t^3+6t^2-4t+1)$		87	2	–		r
10_{113}	$(t^2-t+1)(-2t^4+9t^3-15t^2+9t-2)$		111	2	–		r
10_{114}	$(t^2-t+1)(-2t^4+8t^3-11t^2+8t-2)$		93	0	–		r
10_{115}	$t^6-9t^5+26t^4-37t^3+26t^2-9t+1$		109	0	–		i
10_{116}	$t^8-5t^7+12t^6-19t^5+21t^4-19t^3+12t^2-5t+1$		95	2	–		r
10_{117}	$2t^6-10t^5+24t^4-31t^3+24t^2-10t+2$		103	2	–		n
10_{118}	$t^8-5t^7+12t^6-19t^5+23t^4-19t^3+12t^2-5t+1$		97	0	–		i

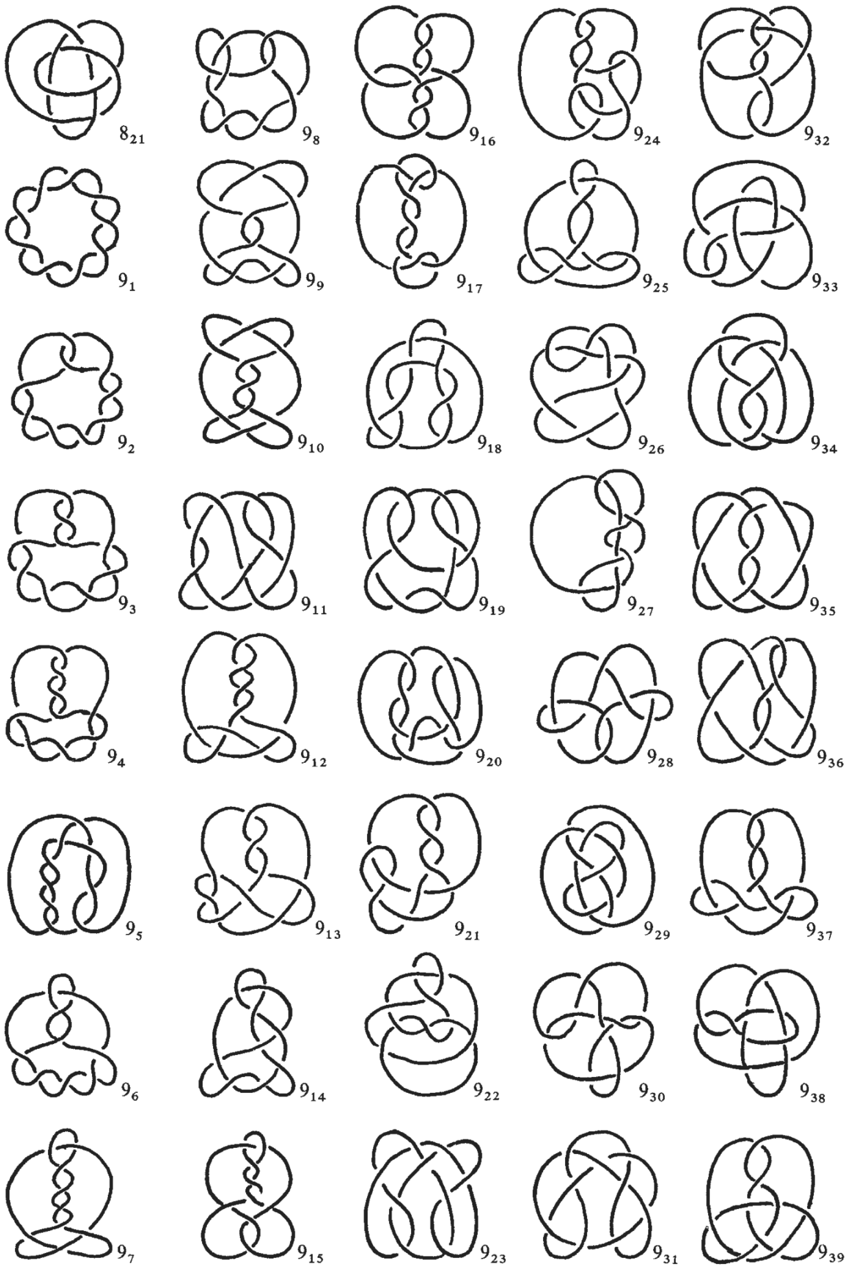
$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α, β	s
10_{119}	$2t^6 - 10t^5 + 23t^4 - 31t^3 + 23t^2 - 10t + 2$		101	0	–		n
10_{120}	$(2t^2 - 3t + 2)(4t^2 - 7t + 4)$		105	4	2		r
10_{121}	$2t^6 - 11t^5 + 27t^4 - 35t^3 + 27t^2 - 11t + 2$		115	2	–		r
10_{122}	$(t^2 - t + 1)(t^2 - 3t + 1)(-2t^2 + 3t - 2)$		105	0	2		r
10_{123}	$(t^4 - 3t^3 + 3t^2 - 3t + 1)^2$,	$t^4 - 3t^3 + 3t^2 - 3t + 1$	11,11	0	5		f
$*10_{124}$	$t^8 - t^7 + t^5 - t^4 + t^3 - t + 1$		1	8	3,5		r
10_{125}	$t^6 - 2t^5 + 2t^4 - t^3 + 2t^2 - 2t + 1$		11	2	–		r
10_{126}	$t^6 - 2t^5 + 4t^4 - 5t^3 + 4t^2 - 2t + 1$		19	2	–		r
10_{127}	$t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 1$		29	4	–		r
10_{128}	$2t^6 - 3t^5 + t^4 + t^3 + t^2 - 3t + 2$		11	6	–		r
10_{129}	$2t^4 - 6t^3 + 9t^2 - 6t + 2$		25	0	–		r
10_{130}	$2t^4 - 4t^3 + 5t^2 - 4t + 2$		17	0	–		r
10_{131}	$2t^4 - 8t^3 + 11t^2 - 8t + 2$		31	2	–		r
10_{132}	$t^4 - t^3 + t^2 - t + 1$		5	0	–		r
10_{133}	$t^4 - 5t^3 + 7t^2 - 5t + 1$		19	2	–		r
10_{134}	$2t^6 - 4t^5 + 4t^4 - 3t^3 + 4t^2 - 4t + 2$		23	6	–		r
10_{135}	$3t^4 - 9t^3 + 13t^2 - 9t + 3$		37	0	–		r
10_{136}	$(t^2 - t + 1)(t^2 - 3t + 1)$		15	2	2		r
10_{137}	$(t^2 - 3t + 1)^2$		25	0	–		r
10_{138}	$(t^2 - 3t + 1)(-t^4 + 2t^3 - t^2 + 2t - 1)$		35	2	2		r
10_{139}	$(t^2 - t + 1)(-t^6 + t^4 - t^3 + t^2 - 1)$		3	6	2		r
10_{140}	$(t^2 - t + 1)^2$		9	0	–		r
10_{141}	$(t^2 - t + 1)(-t^4 + 2t^3 - t^2 + 2t - 1)$		21	0	2		r
10_{142}	$(t^2 - t + 1)(-2t^4 + t^3 + t^2 + t - 2)$		15	6	2		r
10_{143}	$(t^2 - t + 1)^3$		27	2	–		r
10_{144}	$(t^2 - t + 1)(3t^2 - 7t + 3)$		39	2	2		r
10_{145}	$t^4 + t^3 - 3t^2 + t + 1$		3	2	2		r
10_{146}	$2t^4 - 8t^3 + 13t^2 - 8t + 2$		33	0	2		r
10_{147}	$(t^2 - t + 1)(2t^2 - 5t + 2)$		27	2	2		n
10_{148}	$t^6 - 3t^5 + 7t^4 - 9t^3 + 7t^2 - 3t + 1$		31	2	–		n
10_{149}	$t^6 - 5t^5 + 9t^4 - 11t^3 + 9t^2 - 5t + 1$		41	4	–		n
10_{150}	$t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 1$		29	4	–		n
10_{151}	$t^6 - 4t^5 + 10t^4 - 13t^3 + 10t^2 - 4t + 1$		43	2	–		n
10_{152}	$t^8 - t^7 - t^6 + 4t^5 - 5t^4 + 4t^3 - t^2 - t + 1$		11	6	–		r
10_{153}	$t^6 - t^5 - t^4 + 3t^3 - t^2 - t + 1$		1	0	–		n
10_{154}	$t^6 - 4t^4 + 7t^3 - 4t^2 + 1$		13	4	–		r

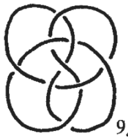
$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	$\lvert\sigma\rvert$	q	α,β	s
10_{155}	$t^6-3t^5+5t^4-7t^3+5t^2-3t+1$		5,5	0	–		r
10_{156}	$t^6-4t^5+8t^4-9t^3+8t^2-4t+1$		35	2	–		r
10_{157}	$t^6-6t^5+11t^4-13t^3+11t^2-6t+1$		7,7	4	–		r
10_{158}	$t^6-4t^5+10t^4-15t^3+10t^2-4t+1$		45	0	–		r
10_{159}	$(t^2-t+1)(-t^4+3t^3-5t^2+3t-1)$		39	2	–		r
10_{160}	$t^6-4t^5+4t^4-3t^3+4t^2-4t+1$		21	4	–		r
10_{161}	$t^6-2t^4+3t^3-2t^2+1$		5	4	–		r
10_{162}	$3t^4-9t^3+11t^2-9t+3$		35	2	–		r
10_{163}	$(t^2-t+1)(-t^4+4t^3-7t^2+4t-1)$		51	2	–		r
10_{164}	$3t^4-11t^3+17t^2-11t+3$		45	0	–		r
10_{165}	$2t^4-10t^3+15t^2-10t+2$		39	2	–		r

Appendix D

Knot projections 0_1–9_{49}




8_{21} 9_8 9_{16} 9_{24} 9_{32}
9_1 9_9 9_{17} 9_{25} 9_{33}
9_2 9_{10} 9_{18} 9_{26} 9_{34}
9_3 9_{11} 9_{19} 9_{27} 9_{35}
9_4 9_{12} 9_{20} 9_{28} 9_{36}
9_5 9_{13} 9_{21} 9_{29} 9_{37}
9_6 9_{14} 9_{22} 9_{30} 9_{38}
9_7 9_{15} 9_{23} 9_{31} 9_{39}



9_{40}

9_{42}

9_{44}

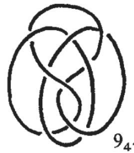
9_{46}

9_{48}

9_{41}

9_{43}

9_{45}



9_{47}

9_{4}

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Author index

Glossary of Symbols

$-\mathfrak{k}$, 16
$\mathfrak{k}^*$, 16
$\widetilde{\mathfrak{k}}$, 21, 335
$\mathfrak{k}_1 \# \mathfrak{k}_2$, 21

$\mathrm{lk}(a_j^-, a_k)$, 114
Λ, 175
$\hat{\Lambda}$, 175
$[\lambda(\zeta)]$, 277
ℓ, 21

$M(t)$, 119
$M(t_1, \dots, t_\mu)$, 149
$M_{a,b}(t)$, 148
$\mathfrak{M}(S^3, \mathfrak{m})$, 232
m, 21

$\nabla_\mathfrak{k}(z)$, 152

Ω, 15
Ω_i, 10
$\Omega_\mathfrak{k}(t)$, 153

$\frac{\partial}{\partial S_i}$, 143
$\pi_n C$, 50
ψ_n, 132
$p_\infty \colon C_\infty \to C$, 112
$\widehat{p}_n \colon \widehat{C} \to S^3$, 133

$q_\mathfrak{k}$, 245

R_n, 133
r_j, 33

$S \times I/h$, 71
S^+, 113
S^-, 113
$S^3_{r/n}(\mathfrak{k})$, 42
$\sum_{i=0}^r c_i u^i$, 125
S, 19
$\sigma(\mathfrak{k})$, 246
$\sigma_\mathfrak{k}$, 264
$\mathfrak{S}_n$, 353

$\mathfrak{t}(a, b)$, 48, 148

$u = t + t^{-1} - 2$, 125

$V(\mathfrak{k})$, 30
v_{jk}, 114

$\mathcal{W}_n$, 353

$(\xi)^{\varphi\psi}$, 144

$\mathfrak{Z}$, 54
$\mathbb{Z}(t)$, 111
Z_k^+, 115
Z_j^-, 115
$\mathbb{Z}\mathfrak{D}$-module, 140
$\mathbb{Z}\mathfrak{Z}$, 158
$\mathbb{Z}\mathfrak{Z}^\mu$, 149, 158
$\mathbb{Z}\mathfrak{G}$, 142
$\mathfrak{Z} \ltimes_\alpha \mathfrak{G}'$, 73
$\mathfrak{Z}_n = \mathfrak{Z}/n\mathfrak{Z}$, 132

Index