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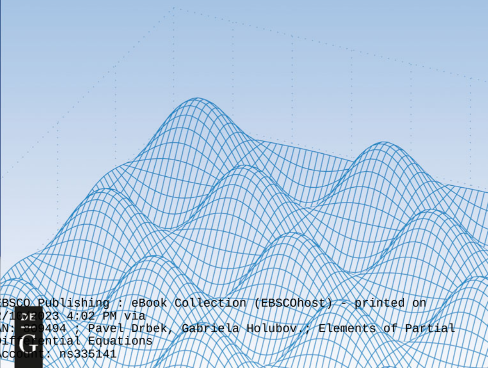
TEXTBOOK

Pavel Drábek, Gabriela Holubová

ELEMENTS OF PARTIAL DIFFERENTIAL EQUATIONS

2ND EDITION



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Preface

The changes we made in this second edition of our book are inspired by our teaching experience from the PDE class. Besides minor revisions, we also added some new material and slightly reorganized its exposition. We quote from the Preface to the first edition in order to clarify the original motivation behind the publication of this book:

"Nowadays, there are hundreds of books (textbooks as well as monographs) devoted to partial differential equations which represent one of the most powerful tools of mathematical modeling of real-world problems. These books contain an enormous amount of material. This is, on the one hand, an advantage since due to this fact we can solve plenty of complicated problems. However, on the other hand, the existence of such an extensive literature complicates the orientation in this subject for the beginners. It is difficult for them to distinguish an important fact from a marginal information, to decide what to study first and what to postpone for later time.

Our book is addressed not only to students who intend to specialize in mathematics during their further studies, but also to students of engineering, economy and applied sciences. To understand our book, only basic facts from calculus and linear ordinary differential equations of the first and second orders are needed. We try to present *the first introduction to PDEs* and that is why our text is on a very elementary level. Our aim is to enable the reader to understand what the partial differential equation is, where it comes from and why it must be solved. We also want the reader to understand the basic principles which are valid for particular types of PDEs, and to acquire some classical methods of their solving. We limit ourselves only to the fundamental types of equations and the basic methods. At present, there are many software packages that can be used for solving a lot of special types of partial differential equations and can be very helpful, but these tools are used only as a "black box" without deeper understanding of the general principles and methods.

We would like to point out that *we work with most of the notions only intuitively and avoid intentionally some precise definitions and exact proofs*. We believe that this way of exposition makes the text more readable and the main ideas and methods used become thus more lucid. Hence, the reader who is not a mathematician, will be not disturbed by technical hypotheses, which in concrete models are usually assumed to be satisfied. On the other hand, we trust that a mathematician will fill up these gaps easily or find the answers in more specialized literature. In order to guide the reader, we put the most important equations, formulas and facts in shaded boxes, so that he/she could better distinguish them from intermediate steps. Further, the text contains many solved examples and illustrating figures. At the end of each chapter there are several exercises. More complicated and theoretical ones are accompanied by hints, the exercises based on calculations include the expected solution. We want to emphasize that there may exist many different forms of a solution and the reader can easily check the correctness of his/her results by substituting them in the equation.

In any case, with the limited extent of this text, we do not claim our book to be comprehensive. It is a selection which is subjective, but which – in our opinion – covers the minimum which should be understood by everybody who wants to use or to study the theory of PDEs more deeply. We deal only with the classical methods which are a necessary starting point for further and more advanced studies. We draw from our experience that springs not only from our teaching activities but also from our scientific work.

Finally, we would like to thank our colleagues Petr Girg, Petr Nečesal, Pavel Krejčí, Alois Kufner, Herbert Leinfelder, Luboš Pick, Josef Polák and Robert Plato, editor of De Gruyter, for careful reading of the manuscript and valuable comments. Our special thanks belong to Jiří Jarník for correcting our English."

The exposition of the material in the second edition is as follows. In Chapter 1, we present the basic conservation and constitutive laws. We also derive some basic models. Chapter 2 is devoted to different notions of a solution, to boundary and initial conditions and to the classification of PDEs. Chapter 3 deals with the first-order linear PDEs. Chapters 4–9 study the simplest possible forms of the wave, diffusion and Laplace (Poisson) equations and explain standard methods how to find their solutions. In Chapter 10, we point out the general principles of the above mentioned main three types of second-order linear PDEs. Chapters 11–13 are devoted to the Laplace (Poisson), diffusion and wave equations in higher dimensions. For the sake of brevity, we restrict ourselves to equations in three dimensions with respect to the space variable.

Many standard procedures and methods from our exposition have been widely used by many other authors. Let us mention, e.g., the textbooks and monographs by Strauss [21], Logan [15], Arnold [2], Bassanini and Elcrat [5], Evans [7], Farlow [8], Franců [9], Míka and Kufner [16], Stavroulakis and Tersian [20] and many others. We do not refer to all of them in our text in order not to disturb the flow of ideas.

Some of the exercises were also borrowed from the books by Asmar [3], Barták et al. [4], Keane [12], Logan [15], Snider [19], Stavroulakis and Tersian [20], Strauss [21], Zauderer [24]. We do not list them explicitly as in the first edition. Many of them appear in several books parallelly and it is thus difficult to hunt out the original source.

We would like to thank our colleagues Martina Langerová, Sarath Sasi, Anoop Thazhe Veetil, Kalappattil Lakshmi and Athiyanathum Poytl Reshma for careful reading of the final version of this manuscript and for their valuable suggestions. We also thank the editorial staff of De Gruyter for the agreeable collaboration.

Pilsen, January 2014 Pavel Drábek and Gabriela Holubová

Contents

Chapter 1

Motivation, Derivation of Basic Mathematical Models

The beginning and development of the theory of partial differential equations were connected with physical sciences and with the effort to describe some physical processes and phenomena in the language of mathematics as precisely (and simply) as possible. With the invasion of new branches of science, this mathematical tool found its usage also outside physics. The width and complexity of problems studied gave rise to a new branch called *mathematical modeling*. The theory of partial differential equations was set apart as a separate scientific discipline. However, studying partial differential equations still stays closely connected with the description – modeling – of physical or other phenomena.

In this introductory chapter, we outline the derivation of several basic mathematical models, which we will deal with in more detail in the further text.

1.1 Conservation Laws

In our text, the notion of a *mathematical model* is understood as a mathematical problem whose solution describes the behavior of the studied system. In general, a mathematical model is a simplified mathematical description of a real-world problem. In our case, we will deal with models described by partial differential equations, that is, differential equations with two or more independent variables.

Studying natural, technical, economical, biological, chemical and even social processes, we observe two main tendencies: the tendency to achieve a certain *balance* between causes and consequences, or the tendency to break this balance. Thus, as a starting point for the derivation of many mathematical models, we usually use some law or principle that expresses such a balance between the so called *state quantities* and *flow quantities* and their spatial and time changes.

Let us consider a medium (body, liquid, gas, solid substance, etc.) that fills a domain

$$\Omega \subset \mathbb{R}^N.$$

Here N denotes the spatial dimension. In real situations, usually, $N = 3$, in simplified models, $N = 2$ or $N = 1$. We denote by

$$u = u(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega,\ t \in [0, T) \subset [0, +\infty)$$

the state function (scalar, vector or tensor) of the substance considered at a point $\boldsymbol{x}$ and time t. In further considerations, we assume u to be a scalar function. The flow function (vector function, in general) of the same substance will be denoted by

$$\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega,\ t \in [0, T) \subset [0, +\infty).$$

The density of sources at a point $\boldsymbol{x}$ and time t is usually described by a scalar function

$$f = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega,\ t \in [0, T) \subset [0, +\infty).$$

Let $\Omega_B \subset \Omega$ be an arbitrary inner subdomain (the so called *balance domain*) of Ω. The integral

$$U(\Omega_B, t) = \int\limits_{\Omega_B} u(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}$$

represents the total amount of the considered quantity u in the balance domain Ω_B at time t. The integral

$$U(\Omega_B, t_1, t_2) = \int\limits_{t_1}^{t_2} \int\limits_{\Omega_B} u(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

then represents the total amount of the quantity in Ω_B and in the time interval $[t_1, t_2] \subset [0, T)$. (The set $\Omega_B \times [t_1, t_2]$ is called the space-time balance domain.)

In particular, if the state function $u(\boldsymbol{x}, t)$ corresponds to the mass density $\varrho(\boldsymbol{x}, t)$, then the integral

$$m(\Omega_B, t) = \int\limits_{\Omega_B} \varrho(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}$$

represents the mass of the substance in the balance domain Ω_B at time t, and the value

$$m(\Omega_B, t_1, t_2) = \int\limits_{t_1}^{t_2} \int\limits_{\Omega_B} \varrho(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

corresponds to the total mass of the substance contained in Ω_B during the time interval $[t_1, t_2]$.

If we denote by $\partial\Omega_B$ the boundary of the balance domain Ω_B, then the boundary integral (surface integral in $\mathbb{R}^3$, curve integral in $\mathbb{R}^2$)

$$\Phi(\partial\Omega_B, t) = \int\limits_{\partial\Omega_B} \boldsymbol{\phi}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S$$

represents the amount of the quantity “flowing through” the boundary $\partial\Omega_B$ in the direction of the outer normal $\boldsymbol{n}$ at time t. Similarly,

$$\Phi(\partial\Omega_B, t_1, t_2) = \int_{t_1}^{t_2} \int_{\partial\Omega_B} \boldsymbol{\phi}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) \,\mathrm{d}S \,\mathrm{d}t$$

corresponds to the total amount of the quantity flowing through $\partial\Omega_B$ in the direction of the outer normal $\boldsymbol{n}$ during the time interval $[t_1, t_2]$.

As we have stated above, the distribution of sources is usually described by a function $f = f(\boldsymbol{x}, t)$ corresponding to the source density at a point $\boldsymbol{x}$ and time t. The integral

$$F(\Omega_B, t_1, t_2) = \int_{t_1}^{t_2} \int_{\Omega_B} f(\boldsymbol{x}, t) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t$$

then represents the total source production in Ω_B during the time interval $[t_1, t_2]$.

1.1.1 Evolution Conservation Law

If the time evolution of the system has to be taken into account, we speak about an *evolution* process. To derive a balance principle for such a process, we choose an arbitrary balance domain $\Omega_B \subset \Omega$ and an arbitrary time interval $[t_1, t_2] \subset [0, +\infty)$. For simplicity, we consider a scalar state function $u = u(\boldsymbol{x}, t)$, a vector flow function $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x}, t)$, and a scalar source function $f = f(\boldsymbol{x}, t)$.

The basic balance law says that the change of the total amount of the quantity u contained in Ω_B between times t_1 and t_2 must be equal to the total amount flowing across the boundary $\partial\Omega_B$ from time t_1 to t_2, and to the increase (or decrease) of the quantity produced by sources (or sinks) inside Ω_B during the time interval $[t_1, t_2]$. In the language of mathematics, we write it as

$$\begin{aligned} &\int_{\Omega_B} u(\boldsymbol{x}, t_2) \,\mathrm{d}\boldsymbol{x} - \int_{\Omega_B} u(\boldsymbol{x}, t_1) \,\mathrm{d}\boldsymbol{x} \\ &\qquad = -\int_{t_1}^{t_2} \int_{\partial\Omega_B} \boldsymbol{\phi}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) \,\mathrm{d}S \,\mathrm{d}t + \int_{t_1}^{t_2} \int_{\Omega_B} f(\boldsymbol{x}, t) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t. \end{aligned} \tag{1.1}$$

The minus sign in front of the first term on the right-hand side corresponds to the fact that the flux is understood as positive in the outward direction. Equation (1.1) represents the *evolution conservation law in its integral* (*global*) *form*.

If we assume u to have continuous partial derivative with respect to t, we can write the difference $u(\boldsymbol{x}, t_2) - u(\boldsymbol{x}, t_1)$ as $\int_{t_1}^{t_2} \frac{\partial}{\partial t} u(\boldsymbol{x}, t)\,\mathrm{d}t = \int_{t_1}^{t_2} u_t(\boldsymbol{x}, t)\,\mathrm{d}t$ and change the order of integration on the left-hand side of (1.1):

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\Omega_B} u_t(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \qquad (1.2)\\ &\quad = -\int_{t_1}^{t_2} \int_{\partial\Omega_B} \boldsymbol{\phi}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S\,\mathrm{d}t + \int_{t_1}^{t_2} \int_{\Omega_B} f(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \end{aligned}$$

Since the time interval $[t_1, t_2]$ has been chosen arbitrarily, we can come (under the assumption of continuity in the time variable of all functions involved) to the expression

$$\int_{\Omega_B} u_t(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega_B} \boldsymbol{\phi}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S + \int_{\Omega_B} f(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}. \qquad (1.3)$$

Now, if we assume $\boldsymbol{\phi}$ to be continuously differentiable in the spatial variables, we can use the Divergence Theorem, according to which we can write

$$\int_{\partial\Omega_B} \boldsymbol{\phi}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S = \int_{\Omega_B} \operatorname{div} \boldsymbol{\phi}(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}.$$

If we substitute this relation into (1.3), we come to

$$\int_{\Omega_B} u_t(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_B} \big(-\operatorname{div} \boldsymbol{\phi}(\boldsymbol{x}, t) + f(\boldsymbol{x}, t) \big)\,\mathrm{d}\boldsymbol{x}. \qquad (1.4)$$

Since the balance domain Ω_B has been chosen arbitrarily, the equality must hold directly for the integrands in (1.4) (under the assumption of continuity of all involved functions in their spatial variables). Hence, we obtain the evolution conservation law in the so called *local* (*differential*) form

$$u_t(\boldsymbol{x}, t) + \operatorname{div} \boldsymbol{\phi}(\boldsymbol{x}, t) = f(\boldsymbol{x}, t). \qquad (1.5)$$

The relation (1.5) represents a single equation for two unknown functions u and $\boldsymbol{\phi}$. The sources f are usually given, however, they can depend on $\boldsymbol{x}, t$ also via the quantity u, that is, we can have $f = f(\boldsymbol{x}, t, u(\boldsymbol{x}, t))$.

1.1.2 Stationary Conservation Law

Sometimes we are not interested in the time evolution of the system considered. In such cases we study the *stationary state*, the *steady state* or *stationary behavior* of the system. It means that we suppose all quantities to be time-independent (they have zero time derivatives). In such cases, we use simplified versions of conservation laws. In particular, the global form of the stationary conservation law is given by

$$\int\limits_{\partial\Omega_B} \boldsymbol{\phi}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S = \int\limits_{\Omega_B} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{1.6}$$

and its local version takes the form

$$\operatorname{div} \boldsymbol{\phi}(\boldsymbol{x}) = f(\boldsymbol{x}) \tag{1.7}$$

(under the assumptions of continuity of all functions involved and their spatial derivatives).

1.1.3 Conservation Law in One Dimension

In some situations, we can assume that all significant changes proceed only in one direction (for instance, in modeling the convection in a wide tube, when we are not interested in the situation near the tube walls; or, conversely, in modeling the behavior of a thin string or a thin bar with constant cross-section). In such cases, we can reduce our model to one spatial dimension. Since the corresponding one-dimensional basic conservation law differs in some minor points from the general one (e.g., ϕ is now a scalar function), we state it here explicitly.

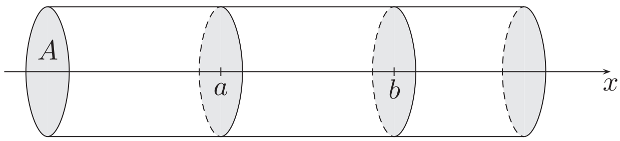


Figure 1.1. An isolated tube with cross-section A; the quantities considered change only in the direction of the x-axis.

Let us consider a tube with constant cross-section A, an arbitrary (but fixed) segment $a \le x \le b$, a time interval $[t_1, t_2]$, and a quantity with density u. The conservation law says that the change of quantity in the spatial segment $[a, b]$ between times t_1 and t_2 equals the total flow incoming through the point $x = a$ minus the total flow outgoing through the point $x = b$, plus the contribution of sources acting in $[a, b]$ during the time interval $[t_1, t_2]$:

$$\int_a^b u(x, t_2)\,\mathrm{d}x - \int_a^b u(x, t_1)\,\mathrm{d}x = \int_{t_1}^{t_2} \big(\phi(a, t) - \phi(b, t)\big)\,\mathrm{d}t + \int_{t_1}^{t_2}\int_a^b f(x, t)\,\mathrm{d}x\,\mathrm{d}t. \tag{1.8}$$

This equation represents the *one-dimensional conservation law in its integral* (*global*) *form* (cf. (1.1)). If the functions u and ϕ are smooth enough, we can proceed similarly as in the general multidimensional case, and obtain the differential formulation.

To be specific, if u has a continuous partial derivative with respect to t and ϕ has a continuous partial derivative with respect to x, equation (1.8) reduces to the form

$$\int_{t_1}^{t_2}\int_a^b \big(u_t(x, t) + \phi_x(x, t) - f(x, t)\big)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since the intervals $[a, b]$ and $[t_1, t_2]$ have been chosen arbitrarily, the integrand must be identically equal to zero, thus

$$u_t(x, t) + \phi_x(x, t) = f(x, t). \tag{1.9}$$

Equation (1.9) is a local version of (1.8) and expresses the *one-dimensional conservation law in its differential* (*local*) *form* (cf. (1.5)).

If we model a one-dimensional stationary phenomenon, we use the stationary version of the previous conservation law, that is

$$\phi_x(x) = f(x), \tag{1.10}$$

which is actually an ordinary differential equation for the unknown function ϕ.

1.2 Constitutive Laws

As we have mentioned above, the conservation law represents one equation for two unknown functions: the state quantity u and flow quantity $\boldsymbol{\phi}$. To create a solvable mathematical model, we need another relation between these functions. In particular processes and phenomena, the state and flow functions (quantities) have their concrete terms and notation. For example, for the description of thermodynamic processes, we usually use some of the following quantities:

state quantities:	density	flow quantities:	velocity
	pressure		momentum
	temperature		tension
	entropy		heat flux

The problem of mutual dependence or independence of these quantities is very complicated and it is connected with the choice of the mathematical model. Relations between the state quantity and the relevant flow quantity are usually based on the generalization of experimental observations and depend on the properties of the particular medium or material. They are usually called *constitutive laws* or *material relations*.

A typical example of the constitutive law from the elasticity theory is *Hook's law*, which states that, for relatively small deformations of an object, the displacement or size of the deformation (state quantity) is directly proportional to the deforming force or tension (flow quantity).

1.3 Basic Models

In this chapter, we derive some simple mathematical models of several fundamental physical processes: drift of a contaminant by a flowing liquid, diffusion in a narrow tube and a three-dimensional container, heat conduction in a bar and a three-dimensional body, and vibrations of a thin string and a two-dimensional membrane. Via mathematical description of their evolution and stationary behavior, we obtain transport, diffusion, wave and Laplace equations, which represent the fundamental linear partial differential equations. Their properties and methods of their solving form the basis of the classical theory of partial differential equations.

1.3.1 Convection and Transport Equation

The one-dimensional evolution model, where the flux density ϕ is proportional to the quantity u:

$$\phi = cu$$

with a constant c, describes, for instance, a drift of a substance in a tube with a flowing liquid. Here, the state quantity u represents concentration of the drifted substance (say contaminant) and the parameter c corresponds to the velocity of the flowing liquid. The model does not include diffusion. Substituting this relation into the local conservation law (1.9) and considering a constant velocity $c > 0$ and zero forces f, we obtain the so called *transport equation* (for the unknown concentration u)

$$u_t + cu_x = 0. \tag{1.11}$$

As we will show later, the solution of (1.11) is a function

$$u(x,t) = F(x - ct), \tag{1.12}$$

where F is an arbitrary differentiable function. Such a solution is called the *right traveling wave*, since the graph of the function $F(x - ct)$ at a given time t is the graph of the function $F(x)$ shifted to the right by the value ct. Thus, with growing time, the profile $F(x)$ is moving without changes to the right at the speed c (see Figure 1.2).

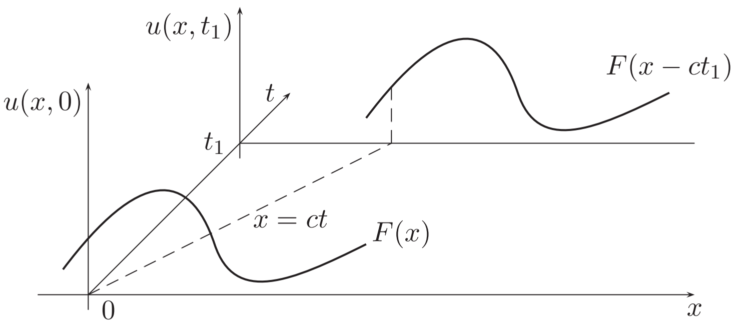


Figure 1.2. Traveling wave.

If the velocity parameter c is negative, which means that the liquid and so the drifted substance flow to the *left* at speed $|c|$, the solution $u(x,t) = F(x - ct)$ is called *left traveling wave*.

If the flux density ϕ is a nonlinear function of the quantity u, then the conservation law (in case of $f \equiv 0$) has the form

$$u_t + (\phi(u))_x = u_t + \phi'(u)u_x = 0. \tag{1.13}$$

This relation models nonlinear transport, which, from the point of further analysis, is more complicated.

Transport with Decay. The particle decay (for example, radioactive decay of nuclei) can be described by the decay equation

$$u_t = -\lambda u,$$

where u is the number of nondecayed particles (nuclei) at time t and λ is a decay constant. The behavior of the radioactive chemical substance drifting in a tube at speed c can be modeled by the equation

$$u_t + cu_x = -\lambda u. \tag{1.14}$$

In this case we have again $\phi = cu$ and $f = -\lambda u$ represents the sources due to the decay.

1.3.2 Diffusion in One Dimension

Let us study the behavior of a gas in a one-dimensional tube. We denote its concentration at a point x and time t by $u = u(x,t)$ (the state function) and the corresponding flux density by $\phi = \phi(x,t)$ (the flow function). If we do not admit any sources ($f = 0$), then these two quantities obey the one-dimensional conservation law (1.9)

$$u_t + \phi_x = 0.$$

Experiments show that the molecules of the gas move from the higher concentration area to the lower concentration area, and that the higher the concentration gradient, the greater is the flux density. The simplest relation (the constitutive law) that corresponds to these assumptions is the linear dependence

$$\phi = -ku_x, \tag{1.15}$$

where k is a constant of proportionality. The minus sign ensures that if $u_x < 0$, then ϕ is positive and the flow moves "to the right". Equation (1.15) is called *Fick's Law of diffusion* and k is the *diffusion constant.* If we insert (1.15) into the conservation law, we obtain the one-dimensional *diffusion equation* (for the unknown concentration u)

$$u_t - ku_{xx} = 0. \tag{1.16}$$

Transport with Diffusion. If we want to include the transport with diffusion into the model, then the flux density must satisfy the constitutive law

$$\phi = cu - ku_x,$$

and using the conservation law we obtain the equation

$$u_t + cu_x - ku_{xx} = 0. \tag{1.17}$$

In this way we can describe, for instance, the density distribution of some chemical which is drifted by a flowing liquid at a speed c and, at the same time, diffuses into this liquid with a diffusion constant k.

1.3.3 Heat Equation in One Dimension

The assumptions we have used in the derivation of the diffusion equation can also be applied in modeling the heat flow. Let us consider a one-dimensional bar with constant mass density ρ and constant specific heat capacity c. If we denote the thermodynamic temperature at a point x and time t by $u = u(x,t)$, then the quantity $\rho cu(x,t)$ represents the volume density of internal heat energy. In this case, the conservation law expresses the balance between the internal energy ρcu and the heat flux ϕ:

$$(\rho cu)_t(x,t) + \phi_x(x,t) = 0 \tag{1.18}$$

(for simplicity, we admit no sources). The constitutive law connecting the density of the heat flux ϕ and the temperature u is *Fourier's heat law* which says that the density of the heat flux is directly proportional to the temperature gradient with a negative constant of proportionality:

$$\phi = -Ku_x.$$

The constant K represents the heat (or thermal) conductivity. Fourier's law is an equivalent of Fick's law: heat flows from warmer places of the domain to colder places. If we substitute for the heat flux back into (1.18), we obtain

$$u_t - ku_{xx} = 0, \tag{1.19}$$

which is again a diffusion equation in one dimension. Here, the constant $k = \frac{K}{\rho c}$ is called the thermal diffusivity. Both phenomena, the heat flow and the diffusion, can be thus modeled by the same equation.

1.3.4 Heat Equation in Three Dimensions

Deriving the heat flow equation in higher (in this case three) dimensions, we proceed in a way similar as in the one-dimensional case. Let Ω be a domain in $\mathbb{R}^3$ and let $u = u(x,y,z,t)$ denote the temperature at time t and a point

$(x, y, z) \in \Omega$. We assume that the material which fills the domain is homogeneous and characterized by a constant mass density ρ and a constant specific heat capacity c. The internal energy at the point (x, y, z) and time t corresponds to the quantity $c\rho u(x, y, z, t)$, the heat flux is a vector function $\boldsymbol{\phi} = \boldsymbol{\phi}(x, y, z, t)$, and the heat sources are described by a scalar function $f = f(x, y, z, t)$. The conservation law of heat energy (in its differential form) expresses the balance between these quantities in the following way:

$$c\rho u_t + \operatorname{div} \boldsymbol{\phi} = f \tag{1.20}$$

for all $(x, y, z) \in \Omega$, $t > 0$.

The constitutive law is, in this case, a three-dimensional version of Fourier's heat law:

$$\boldsymbol{\phi} = -K \operatorname{grad} u.$$

As in the one-dimensional case, this law says that the heat flux is proportional to the temperature gradient with a negative constant of proportionality (the heat flows from warmer places to colder areas). If we realize that

$$\Delta u = \operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = u_{xx} + u_{yy} + u_{zz},$$

where Δ denotes the Laplace operator and ∇ the gradient, we obtain, after substituting into (1.20), the final form of the (nonhomogeneous) heat equation in three dimensions

$$u_t - k\Delta u = \frac{1}{c\rho} f. \tag{1.21}$$

This equation describes also the behavior of a diffusing substance in a three-dimensional domain, which is why we call it a (nonhomogeneous) *diffusion equation*.

Similarly, we would obtain the same expression also in the case of a two-dimensional problem. The Laplace operator has then the form $\Delta u = u_{xx} + u_{yy}$.

1.3.5 String Vibrations and Wave Equation in One Dimension

In the previous sections, the transport and diffusion equations were derived by a standard scheme: we substituted a particular constitutive law into the basic conservation law in its local form and obtained the corresponding model. Now, we proceed in a little different way to derive another fundamental equation – the *wave equation* – that describes one of the most frequent phenomena in nature, the wave motion (let us recall electromagnetic waves, surface waves, or acoustic waves).

Let us consider a flexible string of length l and assume that only small vibrations in the vertical direction (in the vertical plane) occur. The displacement at a point x and time t will be denoted by a continuously differentiable function $u(x,t)$. The properties of the string are described by continuous functions $\rho(x,t)$ and $T(x,t)$ which represent the mass density and the inner tension of the string at a point x and time t. We assume that the tension T always acts in the direction tangent to the string profile at a point x. Now, let us consider an arbitrary but fixed string segment between points $x = a$, $x = b$ (see Figure 1.3). The angle formed by the tangent at a given point and the horizontal line will be a continuous function denoted by $\varphi(x,t)$. Let us notice that the relation

$$\tan \varphi(x,t) = u_x(x,t) \tag{1.22}$$

holds.

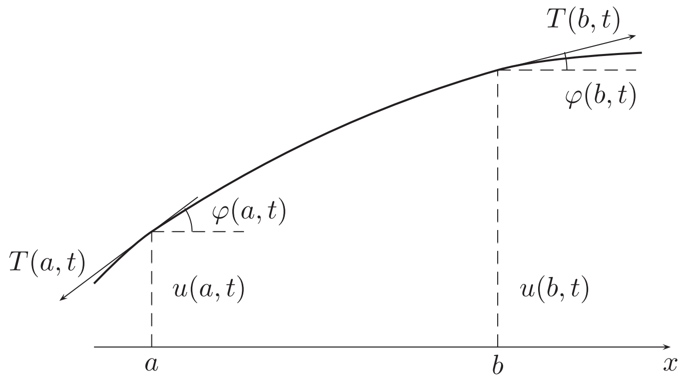


Figure 1.3. String segment.

To derive an equation describing the motion of the string, we use Newton's Second Law of Motion, which implies that the rate of change of the momentum in a given segment is equal to the acting force. We will assume that the only force which acts on the string segment is the tension caused by the neighboring parts of the string (gravity and damping are for now neglected). Since there is no movement in the horizontal direction, the following relation must hold:

$$T(b,t)\cos\varphi(b,t) - T(a,t)\cos\varphi(a,t) = 0. \tag{1.23}$$

Since $[a,b]$ has been chosen arbitrarily, the quantity $T(x,t)\cos\varphi(x,t)$ does not depend on x.

In the vertical direction, the law of motion implies the kinetic equation for the displacement u:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)u_t(x,t)\sqrt{1+u_x(x,t)^2}\mathrm{d}x \tag{1.24}$$
$$= T(b,t)\sin\varphi(b,t) - T(a,t)\sin\varphi(a,t).$$

(The expression $\sqrt{1+u_x(x,t)^2}\,\mathrm{d}x$ represents the arc length element.) The reader should notice that this relation corresponds to the one-dimensional version of conservation law (1.3) with momentum as the quantity considered. To be able to pass to the local relation, we utilize another conservation law – the mass conservation law. It says that the rate of change of the total mass of a given segment is zero. (Here, the quantity considered is the mass, the flow as well as the sources are zero.) This, in other words, means that the mass of a given segment at an arbitrary time t must be equal to the mass of the same segment at time $t = 0$. If we denote $\rho_0(x) = \rho(x,0)$ and suppose that at the beginning, at time $t = 0$, the string is in its equilibrium state $u(x,0) \equiv u_0 = \text{const.}$, we obtain the equality

$$\int_a^b \rho(x,t)\sqrt{1+u_x(x,t)^2}\,\mathrm{d}x = \int_a^b \rho_0(x)\,\mathrm{d}x.$$

However, the interval $[a,b]$ has been chosen arbitrarily, thus the relation

$$\rho(x,t)\sqrt{1+u_x(x,t)^2} = \rho_0(x) \tag{1.25}$$

must hold for any x and t.

Now, we can return to the kinetic equation (1.24). If we use (1.25) and change the order of differentiation and integration (which requires again some smoothness assumptions), we obtain

$$\int_a^b \rho_0(x)u_{tt}(x,t)\,\mathrm{d}x = T(b,t)\sin\varphi(b,t) - T(a,t)\sin\varphi(a,t). \tag{1.26}$$

Since $T(x,t)\cos\varphi(x,t)$ does not depend on x as claimed above, we may denote $\tau(t) = T(x,t)\cos\varphi(x,t)$, and employing (1.22), we can rewrite the right-hand side of (1.26) in the following way:

$$T(b,t)\cos\varphi(b,t)\tan\varphi(b,t) - T(a,t)\cos\varphi(a,t)\tan\varphi(a,t)$$
$$= \tau(t)(u_x(b,t) - u_x(a,t)) = \tau(t)\int_a^b u_{xx}(x,t)\,\mathrm{d}x.$$

Hence, we obtain

$$\int_a^b \rho_0(x)u_{tt}(x,t)\,\mathrm{d}x = \tau(t)\int_a^b u_{xx}(x,t)\,\mathrm{d}x.$$

This relation must hold again for an arbitrary interval $[a,b]$, thus we get the differential formulation

$$\rho_0(x)u_{tt}(x,t) = \tau(t)u_{xx}(x,t).$$

In the special case $\rho_0(x) \equiv \rho_0$, $\tau(t) \equiv \tau_0$, and if we denote $c = \sqrt{\tau_0/\rho_0}$, we arrive at the fundamental equation of mathematical modeling, which describes string vibrations in one dimension:

$$u_{tt} = c^2 u_{xx}. \tag{1.27}$$

Equation (1.27) is called the *wave equation* and the constant $c > 0$ expresses the *speed of wave propagation*.

The basic wave equation can be modified in various ways.

(i) If the presence of *external damping* (for example, resistance of the surrounding medium) is taken into account, the equation is enriched by a term proportional to the velocity u_t:

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

with the *damping* coefficient $r > 0$.

(ii) If the model considers an *elastic force* of the string, it takes the form

$$u_{tt} - c^2 u_{xx} + k u = 0$$

with the *stiffness* coefficient $k > 0$.

(iii) If we want to include the presence of an *external force* in the model described by a function $f(x,t)$, we obtain a *nonhomogeneous* wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t).$$

(iv) All these effects can appear simultaneously. Then we obtain an equation of the form

$$u_{tt} - c^2 u_{xx} + r u_t + k u = f(x,t).$$

1.3.6 Wave Equation in Two Dimensions – Vibrating Membrane

A two-dimensional analogue of the oscillating string is a vibrating membrane fastened on a fixed frame. Let us consider again only vertical oscillations and denote the displacement at a point (x, y) and time t by $u(x, y, t)$. Let us choose an arbitrary fixed subdomain Ω of the membrane (see Figure 1.4) and apply Newton's Second Law of Motion and the mass conservation law. We proceed in a way similar to that used in one dimension (see Section 1.3.5).

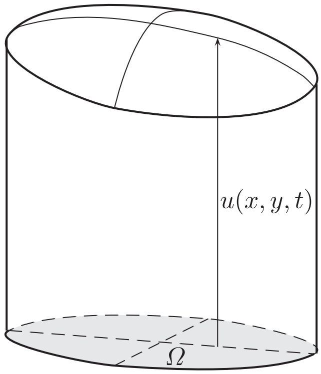


Figure 1.4. Vibrating membrane over the subdomain Ω.

The application of both the above mentioned conservation laws leads to the equality

$$\iint\limits_{\Omega} \rho_0(x, y) u_{tt}(x, y, t)\, \mathrm{d}x\, \mathrm{d}y = \int\limits_{\partial\Omega} T(x, y, t) \frac{\partial u}{\partial n}\, \mathrm{d}s \tag{1.28}$$

(cf. relation (1.26) for the one-dimensional string). Here $\partial\Omega$ denotes the boundary of the domain Ω and we assume that the normal exists at every point of $\partial\Omega$. The function $T = T(x, y, t)$ represents the inner tension of the membrane and $\rho_0 = \rho_0(x, y)$ denotes the mass density distribution of the membrane at time $t = 0$. Since we do not consider any motion in the horizontal direction, we can assume $T(x, y, t)$ independent of x and y, that is, $T(x, y, t) = \tau(t)$. If we use Green's Theorem, we can transform the curve integral on the right-hand side of equation (1.28) to the surface integral:

$$\int_{\partial\Omega} \tau(t)\frac{\partial u}{\partial n}\,\mathrm{d}s = \iint_{\Omega} \operatorname{div}\big(\tau(t)\ \operatorname{grad} u(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y.$$

Consequently,

$$\iint_{\Omega} \rho_0(x,y)u_{tt}(x,y,t)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\Omega} \tau(t)\underbrace{\operatorname{div}\big(\operatorname{grad} u(x,y)\big)}_{\Delta u}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\Delta u = u_{xx} + u_{yy}$ is the Laplace operator in two dimensions. Since the domain Ω has been chosen arbitrarily, we can come (imposing some smoothness assumptions on u) to the differential (local) formulation. Moreover, if we consider the mass density ρ_0 as well as the tension τ to be constant, that is, $\rho_0(x,y) = \rho_0$, $\tau(t) = \tau_0$, and if we denote $c = \sqrt{\tau_0/\rho_0}$, we obtain the relation

$$u_{tt} = c^2\Delta u. \tag{1.29}$$

It is obvious that a similar process leads to a similar relation in the three-dimensional case, where, however, $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplace operator in three dimensions. From the physical point of view, models of this type can describe vibrations in an elastic body, propagation of sound waves in the air, propagation of seismic waves in the earth crust, electromagnetic waves, etc.

Equation (1.29) is called the wave equation in two or three spatial variables, respectively.

1.3.7 Laplace and Poisson Equations – Steady States

Studying dynamical models, we are often interested only in the behavior in the *steady* (*stationary*, or *equilibrium*) *state*, that is, in the state when the solution does not depend on time ($u_t = u_{tt} = 0$). In such a case, the (in general, multidimensional) diffusion equation $u_t = k\Delta u$ as well as the wave equation $u_{tt} = c^2\Delta u$ are reduced to the *Laplace equation*

$$\Delta u = 0, \tag{1.30}$$

which in two dimensions takes the form $u_{xx} + u_{yy} = 0$. Solutions of the Laplace equation are the so called *harmonic functions*.

Let us consider a plane body that is heated in an oven. We assume that the temperature in the oven is not the same everywhere (it is not spatially constant). After a certain time, the temperature in the body achieves the steady state which will be described by a harmonic function $u(x, y)$. In the case that the temperature in the oven is spatially constant, the steady state corresponds also to $u(x, y) = \text{const}$. In one-dimensional case, we can imagine a laterally isolated rod in which the heat exchange with the neighborhood acts only at its ends. The function u describing the temperature in the rod then depends only on x. The Laplace equation has thus the form

$$u_{xx} = 0$$

and its solution is any linear function $u(x) = c_1 x + c_2$. In higher dimensions, the situation is much more interesting.

The steady state can be studied also in the case when the model includes time-independent sources. The nonhomogeneous analogue of the Laplace equation with a given function f is the *Poisson equation*

$$\Delta u = f. \tag{1.31}$$

Besides the stationary diffusion and wave processes, the Laplace and Poisson equations appear, for instance, in the following models.

Electrostatics. The Maxwell equations

$$\operatorname{rot} \boldsymbol{E} = 0, \quad \operatorname{div} \boldsymbol{E} = \frac{\rho}{\varepsilon}$$

describe an electrostatic vector field of intensity $\boldsymbol{E}$ in a medium of constant permittivity ε; ρ represents the volume density of the electric charge. The first equation implies the existence of the so called *electric potential*, which is a scalar function ϕ satisfying the relation $\boldsymbol{E} = -\operatorname{grad} \phi$. If we substitute ϕ into the latter equation, we obtain

$$\Delta \phi = \operatorname{div}(\operatorname{grad} \phi) = -\operatorname{div} \boldsymbol{E} = -\frac{\rho}{\varepsilon},$$

which is the Poisson equation with the right-hand side $f = -\frac{\rho}{\varepsilon}$.

Steady Flow. Let us assume that we model an irrotational flow described by the equation $\operatorname{rot} \boldsymbol{v} = 0$, where $\boldsymbol{v}$ is the flow speed (independent of time). This equation implies the existence of a scalar function ϕ (the so called *velocity potential*) satisfying $\boldsymbol{v} = -\operatorname{grad} \phi$. Moreover, let the flowing liquid be incompressible (for example, water) and let the flow be solenoidal (without sources and sinks). Then $\operatorname{div} \boldsymbol{v} = 0$. If we substitute here the potential ϕ, we can write $-\Delta \phi = -\operatorname{div}(\operatorname{grad} \phi) = \operatorname{div} \boldsymbol{v} = 0$. Thus, we obtain the Laplace equation $\Delta \phi = 0$.

Holomorphic Function of One Complex Variable. Let us denote by $z = x + \mathrm{i}y$ a complex variable and by

$$f(z) = u(z) + \mathrm{i}v(z) = u(x + \mathrm{i}y) + \mathrm{i}v(x + \mathrm{i}y),$$

a complex function of a variable z. Functions u and v are real functions of a complex variable z and represent the real and imaginary part of the function f. Since the Gauss complex plane can be identified with $\mathbb{R}^2$, we can view $u(z) = u(x, y)$, $v(z) = v(x, y)$ as functions of two independent real variables x and y. Theory of functions of a complex variable says that a holomorphic function f on a domain Ω (it means a complex function f that has a derivative $f'(z)$ for every $z \in \Omega$) can be expanded locally into a power series with a center $z_0 \in \Omega$. If $z_0 = 0$, then this expansion of f takes the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n,$$

where a_n are complex constants. If we substitute for f and z, we obtain

$$u(x, y) + \mathrm{i}v(x, y) = \sum_{n=0}^{+\infty} a_n (x + \mathrm{i}y)^n.$$

Formal differentiation of this series (the reader is asked to verify it) leads to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

which are the so-called *Cauchy-Riemann* conditions for differentiability of a complex function of a complex variable. By further differentiation we find out that

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy},$$

and thus $\Delta u = 0$. Similarly, $\Delta v = 0$. These formal calculations illustrate that both the real and imaginary parts of a holomorphic function are harmonic functions.

1.4 Exercises

1. How would you change the derivation of the conservation law (1.9) in the case of the tube with a variable cross-section $A = A(x)$?

2. Derive the model of the vibrating string including the gravitation force:

$$u_{tt} = c^2 u_{xx} - g.$$

Here g is the constant representing the gravitational acceleration. Notice that gravity acts at every point of the string in the vertical direction, that is, it appears as the source term in the momentum conservation law.

3. Derive a *damped wave equation*

$$u_{tt} = c^2 u_{xx} - k u_t$$

describing vibrations of the string whose vertical motion is decelerated by a damping force proportional to the string velocity. Here, k is a damping coefficient.

4. Consider heat conduction in a rod with perfect lateral insulation, no internal heat sources, and specific heat, mass density and thermal conductivity as functions of x, that is, $c(x)$, $\rho(x)$ and $K(x)$. Start with the energy conservation law and derive a new form of the heat equation.

5. Consider a rod with perfect lateral insulation, but with the cross-sectional area dependent on x, that is, $A(x)$. Use the conservation law derived in Exercise 1 and Fourier's heat law (as a constitutive law) to obtain the corresponding heat equation.

6. Consider $u = u(x,t)$ to be the density of cars at a point x and time t on a one-way road without any ramps and exits, and $\phi = \phi(x,t)$ to be the flux of the cars. Observe the following facts. If there is no car on the road, the flux is zero. There exists a critical value of the car density, say u_j, for which the jam occurs and the flux is zero again. And there must exist an optimal value u_m, $0 < u_m < u_j$, for which the flux is maximal. Try to formulate a simple constitutive law relating u and ϕ and derive a basic (nonlinear) traffic model.

   [The traffic constitutive law can have the form $\phi = u(u_j - u)$ and then the traffic model is $u_t + u_j u_x - 2uu_x = 0$.]

7. Derive a nonlinear equation describing the behavior of bacteria in a one-dimensional tube under the assumption that the population growth obeys the logistic law $ru(1-u/K)$. Here $u = u(x,t)$ denotes the concentration of the bacteria, r is a growth constant and K represents the carrying capacity. Use the basic conservation law with $\phi = -Du_x$ (the bacteria are diffusing inside the tube with the diffusion constant D) and $f = ru(1-u/K)$ (the sources reflect the reproduction of the population).

   [The resulting model $u_t - Du_{xx} = ru(1-u/K)$ is known as *Fisher's equation*.]

8. Derive the so called *Burgers equation*

$$u_t - Du_{xx} + uu_x = 0,$$

which describes the coupling between nonlinear convection and diffusion in fluid mechanics. Use the one-dimensional conservation law (1.9) with no sources and the constitutive relation $\phi = -Du_x + \frac{1}{2}u^2$. Here, the first term represents the diffusion process, the latter term corresponds to the nonlinear transport (or convection).

9. Show that the Burgers equation $u_t - Du_{xx} + uu_x = 0$ can be transformed into the diffusion equation $\varphi_t - D\varphi_{xx} = 0$ using the *Cole-Hopf transform* $u = \psi_x$, $\psi = -2D \ln \varphi$.

10. Model an electric cable which is not well insulated from the ground, so that leakage occurs along its entire length. Consider that the voltage $V(x,t)$ and current $I(x,t)$ satisfy the relations $V_x = -LI_t - RI$ and $I_x = -CV_t - GV$, where L is the inductance, R is the resistance, C is the capacitance, and G is the leakage to the ground. Show that V and I both satisfy the *telegraph equation*
$$u_{xx} = LCu_{tt} + (RC + LG)u_t + RGu.$$

Chapter 2

Classification, Types of Equations, Boundary and Initial Conditions

One of the main goals of the theory of partial differential equations is to express the unknown function of several independent variables from an identity where this function appears together with its partial derivatives. In the sequel, we keep the following notation: t denotes the time variable, $x, y, z, \dots$ stand for the spatial variables. The *general partial differential equation* (PDE) for an unknown function $u = u(x, y, z, t)$ in 3D can be written as

$$F(x, y, z, t, u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, u_{xz}, u_{xt}, \dots) = 0,$$

where $(x, y, z) \in \Omega \subset \mathbb{R}^3$, $t \in I$, Ω is a given domain in $\mathbb{R}^3$ and $I \subset \mathbb{R}$ is a time interval. If $\boldsymbol{F}$ is a vector-valued function, $\boldsymbol{F} = (F_1, \dots, F_m)$, and we look for several unknown functions $u = u(x, y, z, t)$, $v = v(x, y, z, t)$, $\dots$, then

$$\boldsymbol{F}(x, y, z, t, u, u_x, \dots, v, v_x, \dots) = \mathbf{0}$$

is a *system of partial differential equations*. It is clear that these relations can be, in general, very complicated, and only some of their particular cases can be successfully studied by a mathematical theory. That is why it is important to know how to recognize these types of equations and to distinguish them.

2.1 Basic Types of Equations

Partial differential equations can be classified from various points of view. If time t is one of the independent variables of the searched-for function, we speak about *evolution* equations. If it is not the case (the equation contains only spatial independent variables), we speak about *stationary* equations. The highest order of the derivative of the unknown function in the equation determines the *order* of the equation. If the equation consists only of a linear combination of u and its derivatives (for example, it does not contain products as uu_x, $u_x u_{xy}$, etc.), then it is called a *linear* equation. Otherwise, it is a *nonlinear* equation. A linear equation can be written symbolically by means of a linear differential operator L, i.e., the operator with the property

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v),$$

where α, β are real constants and u, v are real functions. The equation

$$L(u) = 0$$

is called *homogeneous*, the equation

$$L(u) = f,$$

where f is a given function, is called *nonhomogeneous*. The function f represents the "right-hand side" of the equation.

According to the above-mentioned aspects, we can classify the following equations:

1. The transport equation in one spatial variable:

$$u_t + u_x = 0$$

is evolution, of the first order, linear with $L(u) = u_t + u_x$, and homogeneous.

2. The Laplace equation in three spatial variables:

$$u_{xx} + u_{yy} + u_{zz} = 0$$

is stationary, of the second order, linear with $L(u) = \Delta u = u_{xx} + u_{yy} + u_{zz}$, and homogeneous.

3. The Poisson equation in two spatial variables:

$$u_{xx} + u_{yy} = f,$$

where $f = f(x, y)$ is a given function, is stationary, of the second order, linear with $L(u) = \Delta u = u_{xx} + u_{yy}$, and nonhomogeneous.

4. The wave equation with interaction in one spatial variable:

$$u_{tt} - u_{xx} + u^3 = 0$$

is evolution, of the second order, and nonlinear. The interaction is represented by the term u^3.

5. The diffusion equation in one spatial variable:

$$u_t - u_{xx} = f$$

is evolution, of the second order, linear with $L(u) = u_t - u_{xx}$, and nonhomogeneous.

6. The equation of the vibrating beam:

$$u_{tt} + u_{xxxx} = 0$$

is evolution, of the fourth order, linear with $L(u) = u_{tt} + u_{xxxx}$, and homogeneous.

7. The Schrödinger equation (a special case):

$$u_t - \mathrm{i}u_{xx} = 0$$

is evolution, of the second order, linear with $L(u) = u_t - \mathrm{i}u_{xx}$, and homogeneous (here i is the imaginary unit: $\mathrm{i}^2 = -1$).

8. The equation of a disperse wave:

$$u_t + uu_x + u_{xxx} = 0$$

is evolution, of the third order, and nonlinear.

2.2 Classical, General, Generic and Particular Solutions

A function u is called a *solution* of a partial differential equation if, when substituted (together with its partial derivatives) into the equation, the latter becomes an identity. It means that the function u must have all derivatives appearing in the equation. Usually, we require even more. If k is the order of the given partial differential equation, then by its solution we understand a function of the class C^k satisfying the equation at each point. In such a case, we speak about the *classical solution* of a PDE. If we solve, for example, the diffusion equation in one spatial variable, that is,

$$u_t = ku_{xx},$$

which is of the second order, then its classical solution will be a C^2-function, i.e., a function having continuous partial derivatives up to the second order at all points (x, t) considered. We thus require the existence and continuity of derivatives u_{tt} and u_{xt} that do not occur in the equation at all!

As in ordinary differential equations (ODEs), solutions of partial differential equation are not determined uniquely. Concerning ODEs, we speak about the so called *general solution*, which includes *arbitrary constants* and their number is given by the order of the equation. In the case of PDEs, the situation can be more interesting. We illustrate this fact by the examples below (cf. [21] and [6]).

Example 2.1. Let us search for a function of two variables $u = u(x, y)$ satisfying the equation

$$u_{xx} = 0. \tag{2.1}$$

This problem can be solved by direct integration of equation (2.1). Since we integrate with respect to x, the integration "constant" can depend, in general, on "parameter" y. From (2.1) it follows that

$$u_x(x, y) = f(y)$$

and, by further integration,

$$u(x, y) = f(y)x + g(y).$$

Thus, we have obtained a solution of (2.1) for arbitrary functions f and g of the variable y. However, if we want to speak about the classical solution, the functions f and g must be twice continuously differentiable. ◇

Example 2.2. Let us search for a function $u = u(x, y, z)$ satisfying the equation

$$u_{yy} + u = 0. \tag{2.2}$$

Similarly to the case of the ODE for the unknown function $v = v(t)$,

$$v'' + v = 0,$$

when the general solution is a function $v(t) = A \cos t + B \sin t$ with arbitrary constants $A, B \in \mathbb{R}$ (the reader is asked to explain why), the classical solution of equation (2.2) has the form

$$u(x, y, z) = f(x, z) \cos y + g(x, z) \sin y,$$

where f and g are arbitrary twice continuously differentiable functions of the variables x and z. ◇

Example 2.3. Let us search for a function $u = u(x, y)$ satisfying the equation

$$u_{xy} = 0. \tag{2.3}$$

Integrating (2.3) with respect to y, we obtain

$$u_x = f(x)$$

(f is an arbitrary "constant" depending on "parameter" x). Further integration with respect to x then leads to

$$u(x, y) = F(x) + G(y),$$

where $F' = f$. Functions F and G are again arbitrary. If we look for the classical solution u, then both F and G must be again twice continuously differentiable. Notice that if $u \in C^2$, then its second partial derivatives are exchangeable. Hence we can integrate (2.3) first with respect to x and then with respect to y with the same result. ◇

Based on the previous examples, we could conclude that the solution of any partial differential equation depends on *arbitrary functions* (by analogy with arbitrary constants in the case of ODEs). Moreover, if n denotes the number of independent variables in the equation, then these functions depend on $(n-1)$ variables and their number is given by the order of the equation. This rule, however, is disproved by the following example.

Example 2.4. Let us search for a function $u = u(x, y)$ satisfying the equation

$$(u_{xx})^2 + (u_{yy})^2 = 0. \tag{2.4}$$

Equation (2.4) says that the sum of quadratic terms $(u_{xx})^2$ and $(u_{yy})^2$ has to be zero. It means that $u_{xx} = 0$ as well as $u_{yy} = 0$. The first equality implies

$$u(x, y) = f_1(y)x + f_2(y),$$

where f_1, f_2 are arbitrary functions, whereas the latter equality gives us

$$u(x, y) = g_1(x)y + g_2(x),$$

where again g_1, g_2 are arbitrary functions. Both these formulas must hold simultaneously, that is, the equality

$$f_1(y)x + f_2(y) = g_1(x)y + g_2(x)$$

must hold for any x and y. Since the left-hand side is linear in x and the right-hand side is linear in y, it follows that f_i, g_i, $i = 1, 2$, are linear functions. It follows from here that

$$u(x, y) = axy + bx + cy + d,$$

where a, b, c, d are arbitrary real numbers. The solution thus depends on four arbitrary constants instead of two arbitrary function of one variable! ◇

Based on the above examples we distinguish the so called *general solution* and the *generic solution*.

By the *general solution* of a partial differential equation we understand a set of all solutions of the given equation. Very often, the general solution can be described by a formula including arbitrary functions or constants and their particular choice leads to one *particular solution* of the given equation.

By the *generic solution* of a partial differential equation of order k with n independent variables we understand a function depending on k arbitrary C^k functions of $(n-1)$ variables.

In many cases, both notions coincide. However, in many other cases, they differ substantially. As we see later, in the example of the diffusion equation, there are also PDEs, for which we are not able to find the general or generic solution at all.

2.3 Boundary and Initial Conditions

As in ODEs, a single PDE does not provide sufficient information to enable us to determine its solution uniquely. For the unique determination of a solution, we need further information. In the case of *stationary equations*, it is usually *boundary conditions* which, together with the equation, form a *boundary value problem*. For example,

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in B(\mathbf{0}, 1) = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 < 1\}, \\ u(x, y) = 0, & (x, y) \in \partial B(\mathbf{0}, 1) = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 = 1\} \end{cases}$$

forms a homogeneous Dirichlet boundary value problem for the Laplace equation on a unit disc. If, in general, Ω is a bounded domain in $\mathbb{R}^3$, we distinguish the following basic types of (linear) boundary conditions:

(i) the *Dirichlet* boundary condition:

$$u(x, y, z) = g(x, y, z), \quad (x, y, z) \in \partial\Omega,$$

(ii) the *Neumann* boundary condition:

$$\frac{\partial u}{\partial n}(x, y, z) = g(x, y, z), \quad (x, y, z) \in \partial\Omega,$$

(iii) the *Robin* (sometimes called also *Newton*) boundary condition:

$$A\frac{\partial u}{\partial n}(x, y, z) + Bu(x, y, z) = g(x, y, z), \quad (x, y, z) \in \partial\Omega,$$

where $\frac{\partial u}{\partial n}$ denotes the derivative with respect to the outer normal $\boldsymbol{n}$ to the boundary (surface) of the domain Ω; $A, B, \ A^2 + B^2 \neq 0$, are given constants.

If on various parts of the boundary $\partial\Omega$ different types of boundary conditions are given, we speak about a problem with *mixed* boundary conditions. In the case $g \equiv 0$, the boundary conditions are called *homogeneous*, otherwise they are *nonhomogeneous*. In one dimension, that is in the case of problems on the interval $\Omega = (a, b)$, the boundary $\partial\Omega$ consists of two points $x = a$, $x = b$. Then, for example, the nonhomogeneous Neumann boundary conditions reduce to

$$-u_x(a) = g_1, \quad u_x(b) = g_2.$$

On an unbounded domain, for example, on the interval $\Omega = (0, +\infty)$, where it is not possible to speak about a value of the given function at the point "infinity", the homogeneous Dirichlet boundary condition has the form

$$u(0) = 0, \quad \lim_{x \to +\infty} u(x) = 0.$$

As the differential equations themselves, the boundary conditions have also their physical interpretation.

Vibrating string. Vibrations of a string of length l, which is fixed in a zero position in both ends, are described by a one-dimensional wave equation with homogeneous Dirichlet boundary conditions $u(0, t) = u(l, t) = 0$. On the contrary, free ends of a string are described by homogeneous Neumann boundary conditions $u_x(0, t) = u_x(l, t) = 0$ (the tension at the end points is zero). Robin boundary condition could describe the end of a string fastened to a spring (obeying Hook's law) that pulls it back to the equilibrium position.

Diffusion. If the diffusing substance is closed in a container Ω so that nothing can get out or penetrate inside, its flux across the container boundary is zero and hence (from Fick's law) $\partial u/\partial n = 0$ on $\partial\Omega$, which is homogeneous Neumann boundary condition. If the container is constructed in such a way that any substance reaching the boundary flows immediately out, we have $u = 0$ on $\partial\Omega$, i.e., the homogeneous Dirichlet boundary condition.

Heat flow. The heat flow process is again described by the diffusion equation with temperature as the searched quantity $u(x, y, z, t)$. If the object Ω, where the heat flows, is perfectly isolated, the heat flux across the boundary is zero and we obtain homogeneous Neumann boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$. If the whole object Ω is immersed into a reservoir with a given temperature $g(t)$ and the heat conductivity is ideal (i.e., infinite), we have the Dirichlet boundary condition $u = g(t)$ on the boundary $\partial\Omega$. Robin boundary conditions describe the heat transfer between the object Ω and the surrounding media. For example, if we model a homogeneous bar of length l, which is laterally isolated and whose end $x = l$ is immersed into a water with temperature $g(t)$, then the heat exchange between the bar end and the water occurs and can be described by Newton's Law of Cooling

$$\frac{\partial u}{\partial x}(l, t) = -a\big(u(l, t) - g(t)\big).$$

Here, $a > 0$ is the heat transfer coefficient. For this reason, the Robin conditions are also known as Newton conditions.

In the case of *evolution equations*, along with boundary conditions we also deal with *initial conditions.* We then speak about an *initial boundary value* problem. For example,

$$\begin{cases} u_{tt} = u_{xx}, \quad t \in (0, +\infty),\ x \in (0, 1), \\ u(0, t) = u(1, t) = 0, u(x, 0) = \varphi(x),\ u_t(x, 0) = \psi(x) \end{cases}$$

forms an initial boundary value problem for the one-dimensional wave equation. Here the boundary conditions are the homogeneous Dirichlet ones. The function φ denotes the *initial displacement* and ψ stands for the *initial velocity* at a given point x. The derivative u_t at time $t = 0$ is understood as the derivative from the right. If we look for the classical solution, the functions φ and ψ are supposed to be continuous and also the function u is continuous (even the partial derivatives of the second order are continuous). That is why the boundary and initial conditions must satisfy the *compatibility conditions*

$$\varphi(0) = \varphi(1) = 0.$$

By a *solution* (*classical solution*) of the initial boundary (or boundary) value problem, we understand a function differentiable up to the order of the equation which satisfies the equation as well as the boundary and initial conditions pointwise. These requirements can be too strong and thus the notion of a solution of a PDE (or of a system of PDEs) is often understood in another (generalized) sense. In our text, we confine ourselves mainly to searching for classical solutions. However, in several examples in later chapters we handle more general situations and we will notify the reader about that.

2.4 Well-Posed and Ill-Posed Problems

Another notion which we introduce in this part, is that of a *well-posed* boundary (or initial boundary) value problem. The problem is called *well-posed* if the following three conditions are satisfied:

(i) a solution of the problem *exists*;

(ii) the solution of the problem is determined *uniquely*;

(iii) the solution of the problem is *stable* with respect to the *given data*, which means that a “small change” of initial or boundary conditions, right-hand side (or other problem data) causes only a “small change” of the solution.

The last condition concerns especially models of physical problems, since the given data can never be measured with absolute accuracy. However, the question left in the definition of stability is what does "very small" or "small" change mean. The answer depends on the particular problem and, at this moment, we put up with only an intuitive understanding of this notion.

The contrary of a well-posed problem is the *ill-posed* problem, i.e., a problem which does not satisfy at least one of the three previous requirements. If the solution exists but the uniqueness is not ensured, the problem can be underdetermined. Conversely, if the solution does not exist, it can be an overdetermined problem. An underdetermined problem, overdetermined problem, as well as unstable problem can, however, make real sense. Further, it is worth mentioning that the notion of a well-posed problem is closely connected to the definition of a solution. As we will see later, the wave equation with non-smooth initial conditions is, in the sense of the classical solution defined above, an ill-posed problem, since its classical solution does not exist. However, if we consider the solution in a more general sense, the problem becomes well-posed, the generalized solution exists, it is unique and stable with respect to "small" changes of given data.

2.5 Classification of Linear Equations of the Second Order

In this section we state the classification of the basic types of PDEs of the second order that can be found most often in practical models. We start with equations with two independent variables.

The basic types of linear evolution equations of the second order are the *wave equation* (in one spatial variable):

$$u_{tt} - u_{xx} = 0 \quad (c = 1),$$

which is of *hyperbolic type*, and the *diffusion equation* (in one spatial variable):

$$u_t - u_{xx} = 0 \quad (k = 1),$$

which is of *parabolic type*. The basic type of the linear stationary equation of the second order (in two spatial variables) is the *Laplace equation*:

$$u_{xx} + u_{yy} = 0,$$

which is of *elliptic type*.

Formal analogues of these PDEs are equations of conics in the plane: the equation of a hyperbola, $t^2 - x^2 = 1$, the equation of a parabola, $t - x^2 = 1$, and the equation of an ellipse (here we mention its special case – a circle), $x^2 + y^2 = 1$.

Let us consider a general linear homogeneous PDE of the second order

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0 \tag{2.5}$$

with two independent variables x, y and with six real coefficients that can depend on x and y. Let us denote by

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

the matrix formed by the coefficients of the partial derivatives of the second order. It is possible to show that there exists a linear transformation of variables x, y, which reduces equation (2.5) to one of the following forms. In this respect, an important role is played by the determinant $\det \boldsymbol{A}$ of the matrix $\boldsymbol{A}$.

(i) *Elliptic form:* If $\det \boldsymbol{A} > 0$, that is $a_{11}a_{22} > a_{12}^2$, the equation is reducible to the form

$$u_{xx} + u_{yy} + \cdots = 0,$$

where the dots represent terms with derivatives of lower orders.

(ii) *Hyperbolic form:* If $\det \boldsymbol{A} < 0$, that is $a_{11}a_{22} < a_{12}^2$, the equation is reducible to the form

$$u_{xx} - u_{yy} + \cdots = 0.$$

The dots stand for the terms with derivatives of lower orders.

(iii) *Parabolic form:* If $\det \boldsymbol{A} = 0$, that is $a_{11}a_{22} = a_{12}^2$, the equation is reducible to the form

$$u_{xx} + \cdots = 0, \quad (\text{or } u_{yy} + \cdots = 0),$$

unless $a_{11} = a_{12} = a_{22} = 0$. Here, again, the dots represent terms with derivatives of lower orders.

Finding the corresponding transformation relations and reducing the equation is based on the same idea as the analysis of conics in analytic geometry. For simplicity, let us consider only the principal terms in the equation, that is, let $a_1 = a_2 = a_0 = 0$, and let us normalize the equation by $a_{11} = 1$. If, moreover, we denote $\partial_x = \partial/\partial x$, $\partial_y^2 = \partial^2/\partial y^2$, etc., we can write equation (2.5) as

$$(\partial_x^2 + 2a_{12}\partial_x\partial_y + a_{22}\partial_y^2)u = 0$$

and, by formally completing the square, we convert it to

$$(\partial_x + a_{12}\partial_y)^2 u + (a_{22} - a_{12}^2)\partial_y^2 u = 0. \tag{2.6}$$

Further, let us consider the elliptic case $a_{22} > a_{12}^2$, and denote $b = (a_{22} - a_{12}^2)^{1/2}$, which means $b \in \mathbb{R}$. We introduce new independent variables ξ and η by

$$x = \xi, \quad y = a_{12}\xi + b\eta.$$

The transformed derivatives assume the form

$$\partial_\xi = \partial_x + a_{12}\partial_y, \quad \partial_\eta = b\partial_y$$

(you can prove it using the chain rule), and equation (2.6) becomes

$$\partial_\xi^2 u + \partial_\eta^2 u = 0$$

or, equivalently,

$$u_{\xi\xi} + u_{\eta\eta} = 0.$$

In the remaining two cases, we would proceed analogously (in the parabolic case we have $b = 0$, and in the hyperbolic case, $b = \mathrm{i}(a_{12}^2 - a_{22})^{1/2} \in \mathbb{C}$).

Example 2.5. Let us determine types of the following equations:

$$\begin{aligned} &\text{(a)} \qquad u_{xx} - 3u_{xy} = 0, \\ &\text{(b)} \qquad 3u_{xx} - 6u_{xy} + 3u_{yy} + u_x = 0, \\ &\text{(c)} \qquad 2u_{xx} + 2u_{xy} + 3u_{yy} = 0. \end{aligned}$$

In terms of the previous explanation, we decide according to the sign of $\det \boldsymbol{A} = a_{11}a_{22} - a_{12}^2$. Thus, in case (a), we obtain $\det \boldsymbol{A} = -9/4 < 0$ and the equation is of hyperbolic type. In case (b), we have $\det \boldsymbol{A} = 0$, and thus the equation is of parabolic type. In case (c), we have $\det \boldsymbol{A} = 5 > 0$, and the equation is of elliptic type. $\diamond$

If $\boldsymbol{A}$ is a function of x and y (i.e., the equation has non-constant coefficients), then the type of the equation may be different in different parts of the xy-plane. See the following two examples.

Example 2.6. Let us find regions of the xy-plane where the equation

$$xu_{xx} - u_{xy} + yu_{yy} = 0$$

is of elliptic, hyperbolic, or parabolic type, respectively.

In this case the coefficients depend on x and y and we obtain $\det \boldsymbol{A} = a_{11}a_{22} - a_{12}^2 = xy - \frac{1}{4}$. The equation is thus of parabolic type on the hyperbola $xy = \frac{1}{4}$, of elliptic type in two convex regions $xy > \frac{1}{4}$, and of hyperbolic type in the connected region $xy < \frac{1}{4}$. The reader is invited to sketch a picture of corresponding regions. $\diamond$

Example 2.7. Again, let us find regions of the xy-plane where the equation

$$-x^2 u_{xx} + 2xyu_{xy} + (1+y)u_{yy} = 0$$

is of elliptic, hyperbolic, or parabolic type, respectively.

This time we have $\det \boldsymbol{A} = a_{11}a_{22} - a_{12}^2 = -x^2(1+y) - x^2y^2$. The equation is thus of hyperbolic type in the whole plane except the axis y, where it is of parabolic type. $\diamond$

In a similar way as above we can classify linear PDEs of the second order with an arbitrary finite number of variables $N \geq 3$. The coefficient matrix $\boldsymbol{A}$ is then of type $N \times N$. The type of the equation is related to definiteness of the matrix $\boldsymbol{A}$ and can be determined by signs of its eigenvalues:

(i) the equation is of *elliptic* type, if the eigenvalues of $\boldsymbol{A}$ are all positive or all negative (i.e., $\boldsymbol{A}$ is positive or negative definite);

(ii) the equation is of *parabolic* type, if $\boldsymbol{A}$ has exactly one zero eigenvalue and all the other eigenvalues have the same sign (i.e., $\boldsymbol{A}$ is a special case of a positive or negative semidefinite matrix);

(iii) the equation is of *hyperbolic* type, if $\boldsymbol{A}$ has only one negative eigenvalue and all the others are positive, or $\boldsymbol{A}$ has only one positive eigenvalue and all the others are negative (i.e., $\boldsymbol{A}$ is a special case of an indefinite matrix);

(iv) the equation is of *ultrahyperbolic* type, if $\boldsymbol{A}$ has more than one positive eigenvalue and more than one negative eigenvalue, and no zero eigenvalues (i.e., $\boldsymbol{A}$ is indefinite).

Notice that the matrix $\boldsymbol{A}$ is symmetric, since we consider exchangeable second partial derivatives, and thus all its eigenvalues have to be real.

2.6 Exercises

1. Determine which of the following operators are linear.

   (a) $u \mapsto y\,u_x + u_y$,

(b) $u \mapsto uu_x + u_y$,

(c) $u \mapsto u_x^3 + u_y$,

(d) $u \mapsto u_x + u_y + x + y$,

(e) $u \mapsto (x^2 + y^2)(\sin y)\, u_x + x^3\, u_{yxy} + \big(\arccos(xy)\big)u$.

[a,d,e]

2. In the following equations, determine their order and whether they are nonlinear, linear nonhomogeneous, or linear homogeneous. Explain your reasoning.

(a) $u_t - 3u_{xx} + 5 = 0$,

(b) $u_t - u_{xx} + xt^3\, u = 0$,

(c) $u_t + u_{xxt} + u^2 u_x = 0$,

(d) $u_{tt} - 4u_{xx} + x^4 = 0$,

(e) $\mathrm{i}u_t - u_{xx} + x^3 = 0$,

(f) $u_x(1 + u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2} = 0$,

(g) $\mathrm{e}^x u_x + u_y = 0$,

(h) $u_t + u_{xxxx} + \sqrt[3]{1 + u} = 0$.

(i) $u_{xx} + \mathrm{e}^t u_{tt} = u \cos x$.

[linear: a,b,d,e,g,i]

3. Verify that the function $u(x, y, z) = f(x)g(y)h(z)$ is a generic solution of the equation $u^2 u_{xyz} = u_x u_y u_z$ for arbitrary (differentiable) functions f, g and h of a single real variable.

4. Show that the nonlinear equation $u_t = u_x^2 + u_{xx}$ can be transformed to the diffusion equation $u_t = u_{xx}$ by using the transformation of the dependent variable $w = \mathrm{e}^u$ (i.e., introducing a new unknown function w).

5. What are the types of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$,

[hyperbolic]

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.

[parabolic]

6. Classify the equation

$$u_{xx} + 2ku_{xt} + k^2 u_{tt} = 0, \quad k \neq 0.$$

Use the transformation $\xi = x + bt$, $\tau = x + dt$ of the independent variables with unknown coefficients b and d such that the equation is reduced to the form $u_{\xi\xi} = 0$. Find a generic solution of the original equation.

[equation of parabolic type; $u(x,t) = f(x - \frac{t}{k})x + g(x - \frac{t}{k})$]

7. Classify the equation
$$xu_{xx} - 4u_{xt} = 0$$
in the domain $x > 0$. Solve this equation by a *nonlinear* substitution $\tau = t$, $\xi = t + 4\ln x$.

[equation of hyperbolic type; $u = \mathrm{e}^{-t/4} f(t + 4\ln x) + g(t)$]

8. Show that the equation
$$u_{tt} - c^2 u_{xx} + au_t + bu_x + du = f(x,t)$$
can be transformed to the form
$$w_{\xi\tau} + kw = g(\xi,\tau), \quad w = w(\xi,\tau)$$
by substitutions $\xi = x - ct$, $\tau = x + ct$ and $u = w\mathrm{e}^{\alpha\xi+\beta\tau}$ for a suitable choice of constants α and β.

[$\alpha = \frac{b+ac}{4c^2}$, $\beta = \frac{b-ac}{4c^2}$]

9. Classify the equation
$$u_{xx} - 6u_{xy} + 12u_{yy} = 0.$$
Find a transformation of independent variables which converts it into the Laplace equation.

[equation of elliptic type; $\xi = x$, $\eta = \sqrt{3}x + \frac{1}{\sqrt{3}}y$]

10. Determine in which regions of the xy-plane the following equations are elliptic, hyperbolic, or parabolic.

   (a) $2u_{xx} + 4u_{yy} + 4u_{xy} - u = 0$,

   (b) $u_{xx} + 2y\,u_{xy} + u_{yy} + u = 0$,

   (c) $\sin(xy)u_{xx} - 6u_{xy} + u_{yy} + u_y = 0$,

   (d) $u_{xx} - \cos(x)u_{xy} + u_{yy} + u_y - u_x + 5u = 0$,

   (e) $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$,

   (f) $\mathrm{e}^{2x}u_{xx} + 2\mathrm{e}^{x+y}u_{xy} + \mathrm{e}^{2y}u_{yy} + (\mathrm{e}^{2y} - \mathrm{e}^{x+y})u_y = 0$,

   (g) $u_{xx} - 2\sin x\,u_{xy} - \cos^2 x\,u_{yy} - \cos x\,u_y = 0$,

(h) $e^{xy}u_{xx} + (\cosh x)u_{yy} + u_x - u = 0$,

(i) $\big(\log(1 + x^2 + y^2)\big)u_{xx} - \big(2 + \cos x\big)u_{yy} = 0$.

11. Try to find PDEs whose general (and generic) solutions are of the form

(a) $u(x, y) = \varphi(x + y) + \psi(x - 2y)$,

(b) $u(x, y) = x\varphi(x + y) + y\psi(x + y)$,

(c) $u(x, y) = \frac{1}{x}(\varphi(x - y) + \psi(x + y))$.

Here φ, ψ are arbitrary differentiable functions.

[(a) $2u_{xx} - u_{xy} - u_{yy} = 0$, (b) $u_{xx} - 2u_{xy} + u_{yy} = 0$, (c) $x(u_{xx} - u_{yy}) + 2u_x = 0$.]

12. Consider the *Tricomi equation*

$$yu_{xx} + u_{yy} = 0.$$

Show that this equation is

(a) elliptic for $y > 0$ and can be reduced to

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta}u_\eta = 0$$

using the transformation $\xi = x$, $\eta = \frac{2}{3}y^{3/2}$;

(b) hyperbolic for $y < 0$ and can be reduced to

$$u_{\xi\eta} - \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta) = 0$$

using the transformation $\xi = x - \frac{2}{3}(-y)^{3/2}$, $\eta = x + \frac{2}{3}(-y)^{3/2}$.

13. Show that the equation

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0$$

can be reduced to the (canonical) form $u_{\xi\eta} = 0$ in the region where it is of hyperbolic type. Use this result to show that in the hyperbolic region it has the general (and generic) solution

$$u(x, y) = f(x + 2\sqrt{-y}) + g(x - 2\sqrt{-y}),$$

where f and g are arbitrary functions.

14. Determine whether the following three-dimensional equations are of elliptic, hyperbolic, ultrahyperbolic or parabolic type (determine the eigenvalues of the corresponding matrix $\boldsymbol{A}$):

(a) $u_{xx} + 2u_{yz} + (\cos x)u_z - e^{y^2} u = \cosh z$,

(b) $u_{xx} + 2u_{xy} + u_{yy} + 2u_{zz} - (1 + xy)u = 0$,

(c) $7u_{xx} - 10u_{xy} - 22u_{yz} + 7u_{yy} - 16u_{xz} - 5u_{zz} = 0$,

(d) $e^z u_{xy} - u_{xx} = \log(x^2 + y^2 + z^2 + 1)$.

15. Determine the regions of the xyz-space where

$$u_{xx} - 2x^2 u_{xz} + u_{yy} + u_{zz} = 0$$

is of hyperbolic, ultrahyperbolic, elliptic, or parabolic type.

Chapter 3

Linear Partial Differential Equations of the First Order

A general linear partial differential equations of the first order for a function $u = u(x, y)$ of two independent variables x and y has a form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y), \tag{3.1}$$

where coefficients a, b, c and the right-hand side f are, in general, functions mapping (x, y) from a set $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}$. A simple example is the *transport equation*, which has been derived in Section 1.3.1:

$$u_t + cu_x = 0. \tag{3.2}$$

It describes, for instance, the drift of a substance in a tube with a flowing liquid. The quantity u represents the concentration of the drifted substance (a contaminant) and the parameter c corresponds to the constant velocity of the flowing liquid. Let us recall that this simple model does not consider diffusion of the contaminant into the liquid.

If we model the behavior of a radioactive chemical drifted by a liquid at constant speed c and we include the particle decay into the transport model, we obtain the *transport equation with decay* (see Section 1.3.1):

$$u_t + cu_x + \lambda u = 0. \tag{3.3}$$

In this case, u denotes the number of nondecayed particles (nuclei) at time t and point x and λ represents a decay constant.

In the next sections, we present several ways to find solutions of the transport equation (3.2), the transport equation with decay (3.3), or the general linear equation of the first order (3.1), respectively.

3.1 Equations with Constant Coefficients

Let us consider a linear PDE of the first order with two independent variables and with *constant coefficients* in the form

$$au_x + bu_y = 0. \tag{3.4}$$

Here $u = u(x, y)$ is the unknown function and a, b are constants such that $a^2 + b^2 > 0$ (they are not both equal to zero). Equation (3.4) is a special case of the general equation (3.1) with the choice $a(x, y) \equiv a$, $b(x, y) \equiv b$, $c(x, y) \equiv f(x, y) \equiv 0$. Its solving can be approached from various points of view. Below, we present three basic methods.

3.1.1 Geometric Interpretation – Method of Characteristics

Let us denote $\boldsymbol{v} = (a, b)$, $\nabla u = \operatorname{grad} u = (u_x, u_y)$. The left-hand side of equation (3.4) can be then considered as a scalar product

$$au_x + bu_y = \boldsymbol{v} \cdot \nabla u = \frac{\partial u}{\partial \boldsymbol{v}}$$

and equation (3.4) can be interpreted in the following way: "the derivative of the function u in the direction of the vector $\boldsymbol{v}$ is equal to zero," or "the value of the function u does not change (is constant) in the direction of the vector $\boldsymbol{v}$." In other words, u is constant on every line with the directional vector $\boldsymbol{v}$ (warning: this constant differs, in general, on different lines!). Thus,

$$u(x, y) = f(c) = f(bx - ay), \tag{3.5}$$

since the function $u(x, y)$ assumes the value $f(c)$ (and it is thus constant) on the given line $bx - ay = c$. Here f is an arbitrary differentiable real function. Lines described by $bx - ay = c$, $c \in \mathbb{R}$, are called the *characteristic lines*, or the *characteristics* of equation (3.4) (see Figure 3.1).

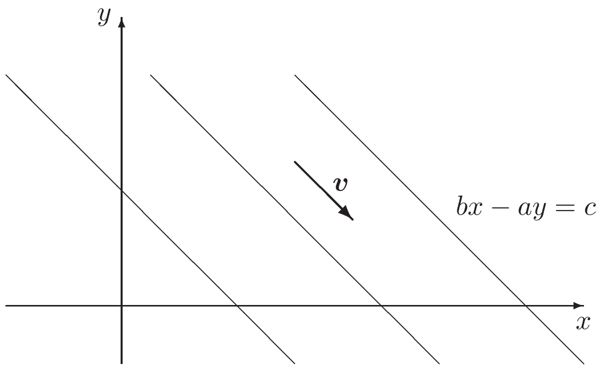


Figure 3.1. Characteristic lines.

Expression (3.5) represents the general (and generic, as well) solution of equation (3.4). As we have mentioned in Section 1.3.1, a solution in this form is called a *right* (or left) *traveling wave*, since the profile of function f is just shifted to the right (or to the left) along characteristics. To determine the particular form of the solution, we have to add an *initial* or *boundary* condition. We illustrate the detailed process in the following example.

Example 3.1. Let us solve the equation

$$2u_x - 3u_y = 0$$

with the boundary condition $u(0, y) = y^2$.

On the basis of the previous text and relation (3.5), we can write the general solution in the form

$$u(x, y) = f(-3x - 2y),$$

where f is an arbitrary differentiable function. If we use the condition

$$y^2 = u(0, y) = f(-2y)$$

and substitute $w = -2y$, we obtain

$$f(w) = \frac{w^2}{4}.$$

Hence,

$$u(x, y) = \frac{(3x + 2y)^2}{4}$$

(see Figure 3.2).

Finally, we should verify that our function u is indeed a solution to the equation. Since

$$u_x = \frac{3}{2}(3x + 2y), \quad u_y = 3x + 2y,$$

after substituting them into the equation we find out that the left-hand side is equal to the right-hand side:

$$2u_x - 3u_y = 3(3x + 2y) - 3(3x + 2y) = 0,$$

and also $u(0, y) = y^2$. ◇

Remark 3.2. In Example 3.1, we have solved linear equation (3.4) with constant coefficients $a \neq 0$, $b \neq 0$ together with the boundary condition given on one of the coordinate axes, that is,

$$u(0, y) = g(y), \tag{3.6}$$

where g is a given function. In such a case, it is not difficult to prove that the corresponding solution is determined *uniquely.* Indeed, if this is not true and

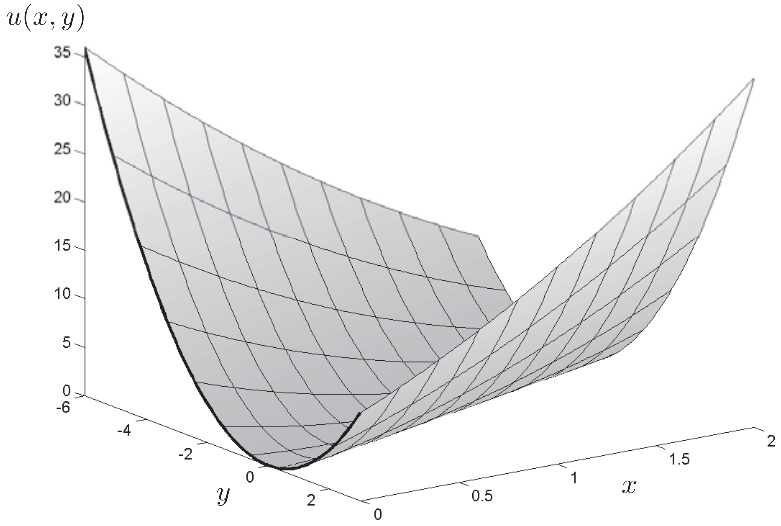


Figure 3.2. Solution from Example 3.1.

there are two solutions $u_1 = u_1(x, y)$ and $u_2 = u_2(x, y)$ satisfying equation (3.4) as well as condition (3.6), then their difference $w = u_1 - u_2$ must solve (3.4) with the boundary condition $w(0, y) = 0$. Hence, we obtain $w(x, y) = f(bx - ay)$ and $0 = w(0, y) = f(-ay)$. This, however, implies $f \equiv 0$, and thus $w(x, y) \equiv 0$. So, u_1 and u_2 coincide.

Figure 3.3 depicts the function $u(x, t) = \mathrm{e}^{-(3t+x)^2}$ which solves the equation

$$u_t - 3u_x = 0$$

with the condition $u(x, 0) = \mathrm{e}^{-x^2}$. Similarly, Figure 3.4 shows the graph of the function $u(x, t) = -\sin(3t - x)$ which solves the equation

$$u_t + 3u_x = 0$$

with the condition $u(x, 0) = \sin x$. The reader is invited to find solutions of both problems, and to notice how the boundary conditions "propagate" along the characteristics.

A disadvantage of the geometric method of characteristics is that it is not applicable for solving more general problems, for instance, equations with nonzero right-hand side. For this reason, we introduce other methods for solving linear PDEs of the first order based on the transformation of coordinates.

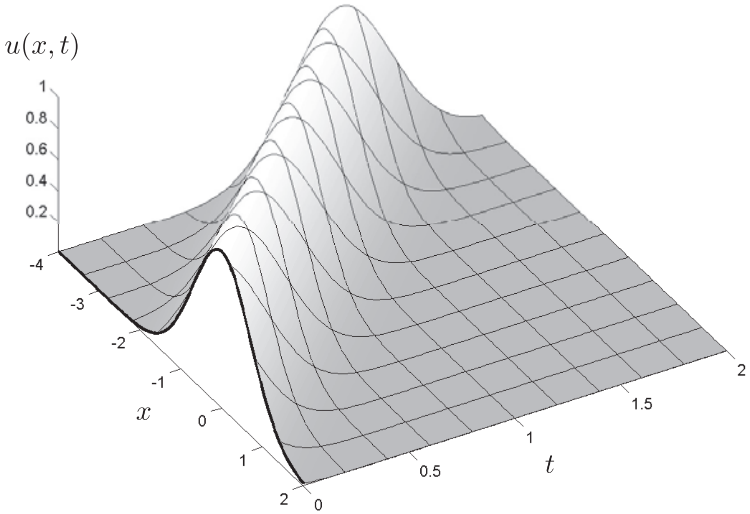


Figure 3.3. Solution of $u_t - 3u_x = 0,\ u(x,0) = \mathrm{e}^{-x^2}$.

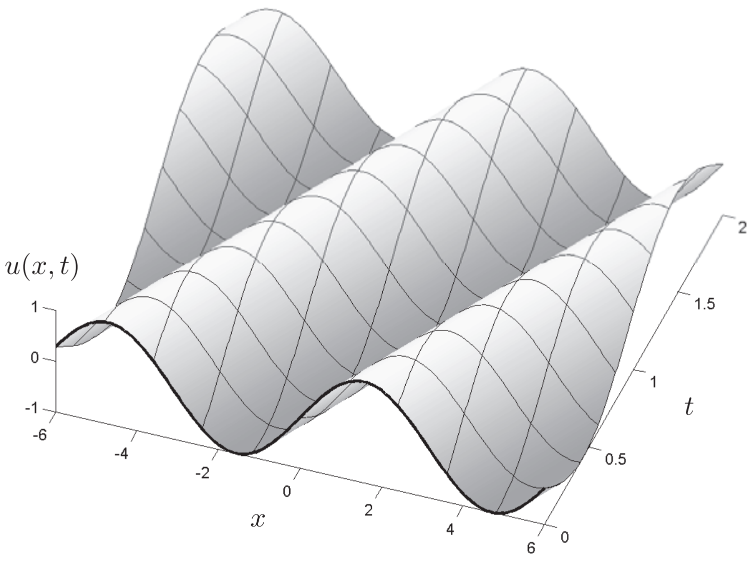


Figure 3.4. Solution of $u_t + 3u_x = 0,\ u(x,0) = \sin x$.

3.1.2 Coordinate Method

Again, let us consider the linear equation (3.4) with constant coefficients, and this time, let us introduce a new rectangular coordinate system by

$$\xi = bx - ay, \quad \eta = ax + by, \tag{3.7}$$

see Figure 3.5.

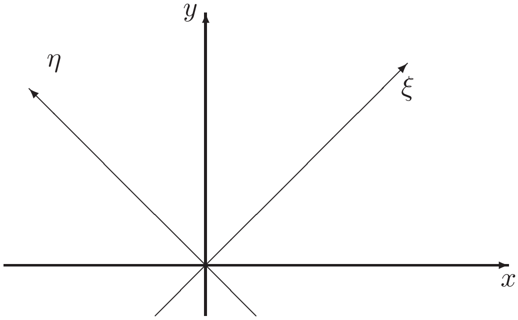


Figure 3.5. Transformation of the coordinate system.

According to the chain rule for the derivative of a composite function, we have

$$\begin{aligned} u_x = \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = bu_\xi + au_\eta, \\ u_y = \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = -au_\xi + bu_\eta. \end{aligned}$$

Equation (3.4) can be then written in the form

$$abu_\xi + a^2 u_\eta - abu_\xi + b^2 u_\eta = 0,$$

i.e.,

$$\underbrace{(a^2 + b^2)}_{\neq 0} u_\eta = 0,$$

whence

$$u_\eta = 0.$$

Thus, it follows that

$$u(\xi, \eta) = f(\xi),$$

where f is an arbitrary differentiable function. After passing to the original coordinates, we get the already known solution in the form of a traveling wave

$$u(x, y) = f(bx - ay).$$

3.1.3 Method of Characteristic Coordinates

Another method, which can be applied to find the solution of (3.4), is again based on the change of coordinates. This time, however, we do not rotate the coordinate system, but we apply the "moving" coordinate system. That is, we introduce new independent variables called *characteristic coordinates*:

$$\xi = bx - ay, \quad \tau = y. \tag{3.8}$$

The variable ξ can be understood as the coordinate propagating together with the signal, whereas the latter variable (often representing time) stays without any change. Using the chain rule we easily derive

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\tau \tau_x = b u_\xi, \\ u_y &= u_\xi \xi_y + u_\tau \tau_y = -a u_\xi + u_\tau. \end{aligned}$$

Substituting into the original relation, we obtain

$$abu_\xi - abu_\xi + bu_\tau = bu_\tau = 0.$$

And again, we come to the same conclusion

$$u = f(\xi) = f(bx - ay),$$

where f is an arbitrary differentiable function.

The coordinate method as well as the method of characteristic coordinates can be applied to more general linear PDEs of the first order in the form

$$au_x + bu_y + c(x, y)u = f(x, y), \tag{3.9}$$

where coefficients a, b are real numbers and c, f are real functions of two real variables (x, y). Transformation (3.7) converts (3.9) into

$$(a^2 + b^2)u_\eta + c(\xi, \eta)u = f(\xi, \eta),$$

while transformation (3.8) leads to

$$bu_\tau + c(\xi, \tau)u = f(\xi, \tau).$$

Both equations can be understood as an ODE with a parameter and solved by standard methods. (We point out that we have kept the same notation for functions c and f after transformation of their independent variables, that is, we write $c(\xi, \eta)$ for $c(x(\xi, \eta), y(\xi, \eta))$ and similarly in other cases.)

Example 3.3. Let us find all solutions of the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2. \tag{3.10}$$

We apply the coordinate method since by introducing new variables

$$\xi = bx - ay = 2x - y, \quad \eta = ax + by = x + 2y,$$

equation (3.10) comes to a simpler form

$$5u_\eta + \xi u = \xi\eta.$$

This is nothing but a linear nonhomogeneous ODE of the first order with the variable η and the parameter ξ. First, we find the general solution of the homogeneous equation

$$u_H(\xi, \eta) = f(\xi)\mathrm{e}^{-\frac{1}{5}\xi\eta},$$

where f is an arbitrary differentiable function. Then, using variation of parameters, we determine a particular solution of the nonhomogeneous equation

$$u_P(\xi, \eta) = \eta - \frac{5}{\xi}.$$

The general solution of the nonhomogeneous equation is then the sum of functions u_H and u_P:

$$u(\xi, \eta) = \eta - \frac{5}{\xi} + f(\xi)\mathrm{e}^{-\frac{1}{5}\xi\eta}$$

and, after passing to the original variables x, y, we obtain

$$u(x, y) = x + 2y - \frac{5}{2x - y} + f(2x - y)\mathrm{e}^{-\frac{1}{5}(2x^2+3xy-2y^2)}, \qquad y \neq 2x.$$

If $y = 2x$, equation (3.10) reduces to $u_x + 2u_y = 0$, and thus $u(x, y) = c$ on the line $y = 2x$, where $c \in \mathbb{R}$ is a constant.

The reader is invited to carry out the above steps in detail and to verify the correctness of the solution. ◇

Example 3.4. Let us find the general solution of the transport equation

$$u_x + u_t - u = t.$$

This time, we introduce the characteristic coordinates by the relations

$$\xi = x - t, \quad \tau = t$$

and, according to the instructions above, we transform the equation to the form

$$u_\tau - u = \tau.$$

Now, we treat this equation as an ODE with the variable τ and the parameter ξ. The general solution of the corresponding homogeneous equation assumes the form

$$u_H(\xi, \tau) = g(\xi)\mathrm{e}^{\tau}$$

with an arbitrary differentiable function g, while a particular solution of the nonhomogeneous equation can be written as

$$u_P(\xi, \tau) = -(1 + \tau).$$

The general solution of the nonhomogeneous equation is then the sum $u = u_H + u_P$. Finally, we return to the original variables x and t:

$$u(x, t) = -(1 + t) + g(x - t)\mathrm{e}^{t}.$$

The reader is asked to carry out all the above steps in detail and to check the correctness of the solution. ◇

3.2 Equations with Non-Constant Coefficients

3.2.1 Method of Characteristics

The method of characteristics, based on the geometric interpretation, can be used also in the case of equations with non-constant coefficients $a(x, y)$ and $b(x, y)$. The difference consists in the fact that the characteristics are no more straight lines but general curves. We start with a simple example.

Example 3.5 (Strauss [21]). Let us solve the equation

$$u_x + yu_y = 0. \tag{3.11}$$

Here we have $a = 1$, $b = y$ and introduce a variable vector $\boldsymbol{v} = (1, y)$. The second component of the vector $\boldsymbol{v}$ now depends on the variable y. The set of all vectors $\boldsymbol{v}$ in the xy plane can be depicted as in Figure 3.6.

As in the case of equations with constant coefficients, equation (3.11) can be written as

$$\boldsymbol{v} \cdot \nabla u = \frac{\partial u}{\partial \boldsymbol{v}} = 0,$$

that is, the unknown function u does not change (it is constant) along characteristics determined by the directional vector $\boldsymbol{v}$. The characteristics are thus curves for which $\boldsymbol{v}$ is the tangent vector, i.e., their tangents at a given point have the slope

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

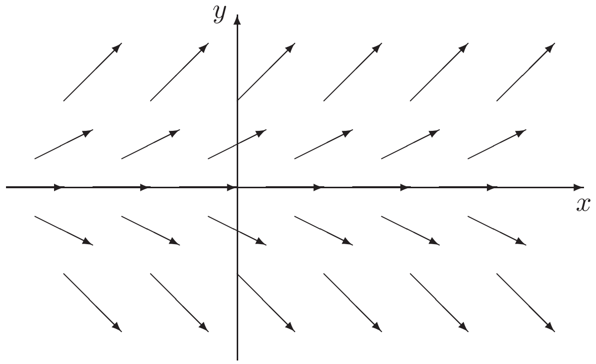


Figure 3.6. Vector field $\boldsymbol{v} = (1, y)$.

By solving this ODE we obtain the characteristics

$$y = c\,\mathrm{e}^x, \quad c \in \mathbb{R}.$$

We easily verify that the solution u is a constant along these characteristics. Indeed, let $c \in \mathbb{R}$ be an arbitrary constant. Then

$$\frac{\partial}{\partial x} u(x, c\mathrm{e}^x) = u_x + u_y \underbrace{c\mathrm{e}^x}_{y} = 0,$$

since u solves equation (3.11); further, we have

$$\frac{\partial}{\partial y} u(x, c\mathrm{e}^x) = u_x \cdot 0 + u_y \cdot 0 = 0,$$

since none of the variables depend on y. The solution u is thus constant on every curve $y = c\mathrm{e}^x$. The choice of the constant c then determines a particular curve. The variables x and y along this curve are linked by the relation

$$y\mathrm{e}^{-x} = c.$$

Thus, for an arbitrary differentiable function $f = f(z)$, the solution u can be written in the form

$$u(x, y) = f(c) = f(y\mathrm{e}^{-x}).$$

◇

If we deal, in general, with the equation

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = 0, \tag{3.12}$$

we proceed quite analogously. Now, $\boldsymbol{v}(x,y) = (a(x,y), b(x,y))$ and characteristics are curves given by the solution of the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y)}{a(x,y)},$$

under the assumption $a(x,y) \neq 0$. Let this solution take the form

$$h(x,y) = c,$$

where c is a real constant. Then the solution of equation (3.12) is an arbitrary differentiable function f of the argument $c = h(x,y)$, that is,

$$u(x,y) = f(c) = f(h(x,y)).$$

Example 3.6 (Strauss [21]). Let us find all solutions of the equation

$$u_x + 2xy^2 u_y = 0.$$

Here $\boldsymbol{v} = (1, 2xy^2)$ and the characteristics are the curves which solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2.$$

We easily find out (for instance, by separation of variables) that the functions

$$y = -\frac{1}{x^2 + c}, \quad c \in \mathbb{R},$$

and also

$$y \equiv 0$$

describe the required characteristic curves. See Figure 3.7. Hence, we can express

$$c = -x^2 - \frac{1}{y},$$

and the solutions can be written as

$$u(x,y) = \begin{cases} f\left(x^2 + \dfrac{1}{y}\right) & \text{for } y \neq 0, \\ \text{const.} & \text{for } y = 0. \end{cases}$$

In general, f can be an arbitrary differentiable function. However, in such a case, we have to understand the solution in the generalized sense. If we require u to be the classical solution, we would have to add more assumptions on f to ensure the continuity of u and its first derivatives at $y = 0$! $\diamond$

As in the case of equations with constant coefficients, the geometric method cannot be applied for solving more general problems than (3.12). For this purpose, we again use methods based on the transformation of the coordinate system. However, the geometric method provided us a hint about the type of transformation to use.

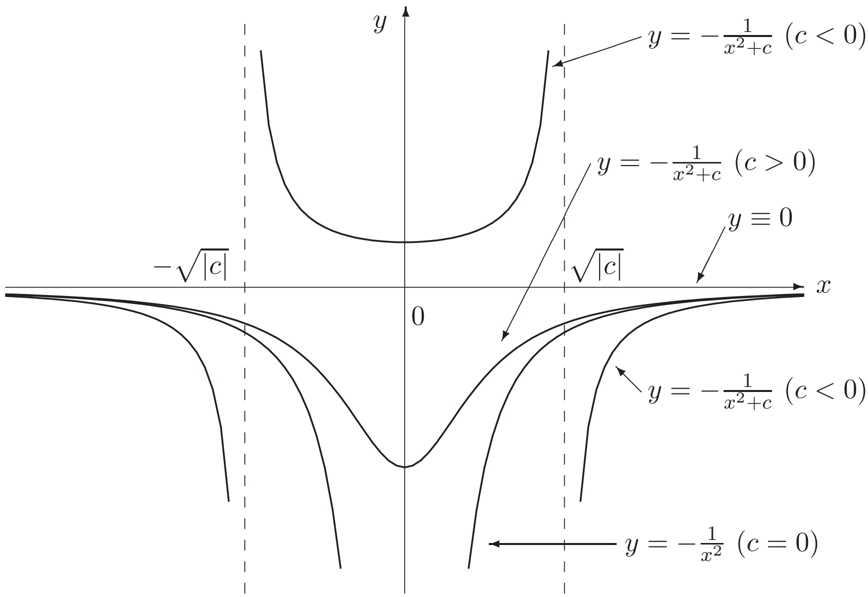


Figure 3.7. Characteristics of the equation $u_x + 2xy^2u_y = 0$.

3.2.2 Method of Characteristic Coordinates

In Section 3.1.3 we introduced the method of characteristic coordinates for equations of the first order with constant coefficients based on the idea that one of the original coordinates is left without any change, whereas the second coordinate moves along the characteristics. That is, we have considered a transformation $\xi = bx - ay$, $\tau = y$, where $bx - ay =$ const. was just the analytic expression of the characteristics corresponding to the given equation. Now, let us consider a general equation (3.1), i.e.,

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

with variable coefficients and let us proceed analogously. We introduce a new coordinate system so that one independent variable again "travels" along the characteristics. Due to the previous geometric method, we already know that characteristics are, in this case, curves described by the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x,y)}{a(x,y)}. \tag{3.13}$$

If we are able to describe the solution of this equation by the analytic relation

$h(x, y) = \text{const.}$, we can introduce a new coordinate system by

$$\xi = h(x, y), \quad \tau = y.$$

Using the chain rule, we can write

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\tau \tau_x = h_x u_\xi, \\ u_y &= u_\xi \xi_y + u_\tau \tau_y = h_y u_\xi + u_\tau. \end{aligned}$$

Since the relation $h(x, y) = \text{const.}$ describes the characteristics, we have (using (3.13))

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} h(x, y(x)) = h_x + h_y \frac{\mathrm{d}y}{\mathrm{d}x} = h_x + h_y \frac{b(x, y)}{a(x, y)},$$

and hence, $a(x, y)h_x + b(x, y)h_y = 0$ at any point $(x, y(x))$. We thus obtain

$$a(x, y)u_x + b(x, y)u_y = \underbrace{(a(x, y)h_x + b(x, y)h_y)}_{=0} u_\xi + b(x, y)u_\tau,$$

and the original equation (3.1) reduces to the form

$$b(\xi, \tau)u_\tau + c(\xi, \tau)u = f(\xi, \tau),$$

which can be treated as an ODE with the variable τ and the parameter ξ.

Example 3.7. Let us find all solutions of the nonhomogeneous equation

$$u_x + y u_y = y \mathrm{e}^y. \tag{3.14}$$

First of all, we determine the characteristics of the equation. As in Example 3.5, these are given by the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

solutions of which are functions $y(x) = c\mathrm{e}^x$ with $c \in \mathbb{R}$. The set of characteristics can be thus described by $y\mathrm{e}^{-x} = c$, $c \in \mathbb{R}$. Now, we introduce a new coordinate system

$$\xi = y\mathrm{e}^{-x}, \qquad \tau = y.$$

The new variable ξ is chosen so that it moves along the characteristics, while the latter variable stays without change. The partial derivatives satisfy

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\tau \tau_x = -y\mathrm{e}^{-x} u_\xi, \\ u_y &= u_\xi \xi_y + u_\tau \tau_y = \mathrm{e}^{-x} u_\xi + u_\tau \end{aligned}$$

and – after substituting into (3.14) – we obtain

$$\tau u_\tau = \tau \mathrm{e}^\tau \qquad \text{or} \qquad u_\tau = \mathrm{e}^\tau.$$

Direct integration leads to the solution of this equation in the form

$$u(\xi, \tau) = \mathrm{e}^\tau + g(\xi).$$

Passing to the original variables, we obtain the final form of the required solution

$$u(x, y) = \mathrm{e}^y + g(y\mathrm{e}^{-x}) \tag{3.15}$$

with g being an arbitrary differentiable function. The reader is asked to verify the correctness of the solution. $\diamond$

3.3 Problems with Side Conditions

In Section 3.1.1 (Example 3.1 and Remark 3.2) we have dealt with an equation accompanied by a boundary condition given on one of the coordinate axes. Let us go back to equation (3.14) from Example 3.7, whose general solution is given by formula (3.15), and let us add, one by one, the following conditions:

$$u(0, y) = \sin y, \qquad u(x, 0) = \sin x, \qquad u(x, 0) = 10.$$

(i) In the first case, we obtain

$$\sin y = u(0, y) = \mathrm{e}^y + g(y),$$

thus $g(y) = \sin y - \mathrm{e}^y$. Equation (3.14) together with the condition $u(0, y) = \sin y$ has a *unique solution* given by the formula

$$u(x, y) = \mathrm{e}^y + \sin(y\mathrm{e}^{-x}) - \mathrm{e}^{y\mathrm{e}^{-x}}.$$

(ii) In the second case, the boundary condition implies the equality

$$\sin x = u(x, 0) = 1 + g(0),$$

which cannot be satisfied and equation (3.14) together with the condition $u(x, 0) = \sin x$ has *no solution.*

(iii) In the third case, we require the equality

$$10 = u(x, 0) = 1 + g(0),$$

which holds true for any function g assuming the value 9 in the origin. Thus, equation (3.14) together with the condition $u(x, 0) = 10$ has *infinitely many solutions.*

Let us notice that in the first case, the condition is assigned on the y-axis, which intersects all characteristics of equation (3.14) just once and under a non-zero angle (*transversally*), while in two other cases, the condition is imposed on the x-axis, which is in fact one of the characteristics!

The boundary condition can be imposed not only on one of the coordinate axes, but we can prescribe values of the solution along a general *curve* γ given by parametric relations

$$\gamma : \quad x = x_0(s), \quad y = y_0(s), \quad s \in I,$$

where $I \subset \mathbb{R}$ is a given interval. In such a case, we usually speak about the so called *side condition* and it takes the form

$$u(x,y) = u_0(s) \quad \text{for } (x,y) \in \gamma,$$

where u_0 is a given function of one real variable (which is actually the parameter of curve γ). For simplicity, we restrict ourselves to *regular* curves in further text. The following assertion is a sufficient condition for unique solvability of a boundary value problem for the equation of the first order with such a general side condition.

Theorem 3.8. *Let us consider a linear PDE of the first order in the form*

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y),$$

where C^1 functions a, b, c, f are defined on a domain $\Omega \subset \mathbb{R}^2$, with a side condition $u = u_0(s)$, $u_0 \in C^1$, imposed on a regular curve

$$\gamma : \begin{cases} x &= x_0(s), \\ y &= y_0(s), \end{cases} \qquad s \in I.$$

If the condition

$$\frac{\mathrm{d}x_0}{\mathrm{d}s} b(x_0(s), y_0(s)) - \frac{\mathrm{d}y_0}{\mathrm{d}s} a(x_0(s), y_0(s)) \neq 0 \quad \forall s \in I, \tag{3.16}$$

holds true, then there exists a unique solution $u = u(x,y)$ of the given equation defined on a neighborhood of the curve γ and satisfying the side condition $u(x_0(s), y_0(s)) = u_0(s)$, $s \in I$.

Remark 3.9. Condition (3.16) says that the vector (a,b) is not a tangent vector of the curve γ at any point $(x_0, y_0) \in \gamma$. It means that γ intersects the characteristics of the given equation transversally. Notice as well that Theorem 3.8 ensures the existence of a unique solution only *locally*.

Example 3.10. Let us solve the equation

$$u_x + u_y = 0$$

with the side condition $u(2s, s) = \mathrm{e}^{-s^2}$, $s \in \mathbb{R}$.

The equation has constant coefficients $a = 1$, $b = 1$ and its characteristics are thus straight lines $y - x = c$, $c \in \mathbb{R}$. Note that $x_0(s) = 2s$, $y_0(s) = s$, $s \in \mathbb{R}$, i.e., γ is the line $y = x/2$. Before we start to solve the problem, we verify condition (3.16):

$$\frac{\mathrm{d}x_0}{\mathrm{d}s} b - \frac{\mathrm{d}y_0}{\mathrm{d}s} a \ = 2 - 1 \neq 0 \quad \forall s \in \mathbb{R}.$$

It corresponds to the fact that the curve γ intersects all characteristics just once and transversally. (The reader is recommended to draw the characteristics as well as the curve γ.) According to Theorem 3.8, the unique solvability of our problem is guaranteed. The general (as well as generic) solution can be written in the form

$$u(x, y) = f(y - x),$$

where f is an arbitrary differentiable function. The particular form of f can be gained using the prescribed side condition. The equalities

$$e^{-s^2} = u(2s, s) = f(s - 2s) = f(-s)$$

must hold for an arbitrary $s \in \mathbb{R}$ and hence the required solution is a function

$$u(x, y) = \mathrm{e}^{-(y-x)^2}.$$

Its graph is depicted in Figure 3.8. ◇

Example 3.11. Let us solve the equation

$$u_x + u_y = 0$$

with the side condition $u(\cos s, \sin s) = s$, $s \in [0, 2\pi)$.

This time, we consider the same equation as in the previous example, but the side condition is imposed on a unit circle

$$\gamma : \begin{cases} x &= \cos s, \\ y &= \sin s, \end{cases} \qquad s \in [0, 2\pi)\,.$$

Condition (3.16) provides us the requirement

$$\frac{\mathrm{d}x_0}{\mathrm{d}s} b - \frac{\mathrm{d}y_0}{\mathrm{d}s} a \ = -\sin s - \cos s \neq 0,$$

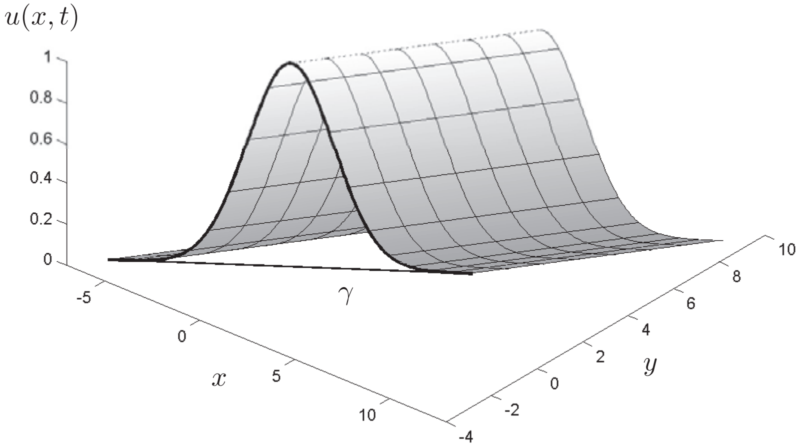


Figure 3.8. Solution of the problem $u_x + u_y = 0,\ u(s, 2s) = \mathrm{e}^{-s^2}$.

which is not satisfied at points $s = \frac{3}{4}\pi$ and $s = \frac{7}{4}\pi$. If we want to ensure the unique solvability of our problem, we have to impose the side condition only on such a part of the unit circle that does not contain these points. Let us consider, for instance, the arc

$$\gamma : \left\{ \begin{array}{lcl} x & = & \cos s, \\ y & = & \sin s, \end{array} \right. \qquad s \in \left[0, \frac{\pi}{2}\right].$$

The general solution of the equation considered is an arbitrary differentiable function $u = f(y - x)$. Substituting into the side condition, we obtain

$$s = u(\cos s, \sin s) = f(\sin s - \cos s), \quad s \in \left[0, \frac{\pi}{2}\right].$$

If we introduce a new variable $w = \sin s - \cos s$ and apply standard trigonometric identities, we easily express

$$s = \frac{\pi}{4} + \arcsin \frac{w}{\sqrt{2}}, \quad w \in [-1, 1].$$

Thus, $f(w) = \frac{\pi}{4} + \arcsin \frac{w}{\sqrt{2}}$ and the solution of our problem with the side condition on a quarter-circle can be written in the form

$$u(x, y) = \frac{\pi}{4} + \arcsin \frac{y - x}{\sqrt{2}}, \quad -1 \leq y - x \leq 1.$$

The graph of the solution is depicted in Figure 3.9. Notice that the solution is determined only in the strip $-1 \le y - x \le 1$, that is, in the domain obtained by moving γ along the characteristics. ◇

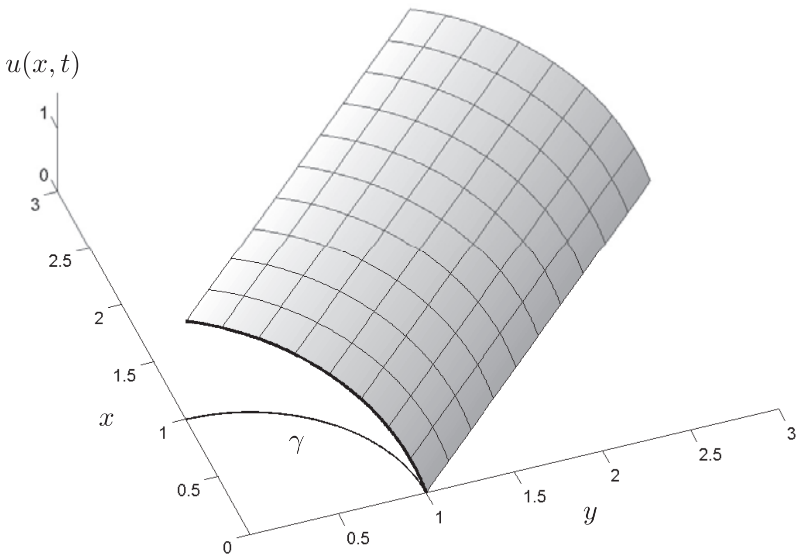


Figure 3.9. Solution of the problem $u_x + u_y = 0$, $u(\cos s, \sin s) = s$, $s \in [0, \pi/2]$.

Example 3.12. Let us solve the equation

$$u_x + yu_y = 0$$

with the side condition $u(s, s^3) = e^{-s^2}$, $s \in I$, where I is an appropriately chosen interval.

In this case, we have coefficients $a = 1$, $b = y$ and the side condition is prescribed along the curve

$$\gamma : \begin{cases} x = s, \\ y = s^3, \end{cases} \qquad s \in I,$$

or, equivalently, $\gamma :\ y = x^3$, $x \in I$. Condition (3.16) has the following form:

$$\frac{\mathrm{d}x_0}{\mathrm{d}s} b(x_0, y_0) - \frac{\mathrm{d}y_0}{\mathrm{d}s} a(x_0, y_0) = s^3 - 3s^2 \neq 0,$$

which is not satisfied for $s = 0$ and $s = 3$. Notice that γ and the corresponding characteristic have indeed the same tangent line at points $(0, 0)$ and $(3, 27)$.

If we want to ensure the unique solvability of our problem, we must avoid these points. Let us consider, for instance, $s \in I = (0, 3)$. Due to Example 3.5 we already know that the characteristics of the equation $u_x + yu_y = 0$ are curves described by

$$y\mathrm{e}^{-x} = c, \quad c \in \mathbb{R},$$

and the general solution of our problem takes the form

$$u(x, y) = f(y\mathrm{e}^{-x})$$

with an arbitrary differentiable function f. Applying the side condition, we obtain the equality

$$e^{-s^2} = u(s, s^3) = f(s^3\mathrm{e}^{-s}), \quad s \in (0, 3),$$

that could theoretically provide us the formula for f and hence the final form of the solution. Unlike the previous examples, we are not able to specify f in a simple way. In this and similar cases, the explicit analytic expression of the solution can be difficult or even impossible to obtain. ◇

3.4 Solution in Parametric Form

Example 3.12 illustrates the fact that sometimes it is difficult (or even impossible) to gain the analytic description of the solution of a boundary value problem for the equation of the first order. In this section, we show that the parametric form is much more convenient and – in a certain sense – even more natural. Again, let us consider a general linear PDE of the first order in two independent variables

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

with a side condition $u(x, y) = u_0(s)$ imposed on a curve

$$\gamma : \left\{ \begin{array}{lcl} x & = & x_0(s), \\ y & = & y_0(s), \end{array} \right. \quad s \in I.$$

The curve γ is already described in a parametric form with a parameter s. To search the solution $u = u(x, y)$ in the parametric form means to look for the expression

$$\left\{ \begin{array}{lcl} x & = & x(t, s), \\ y & = & y(t, s), \\ u & = & u(t, s), \end{array} \right. \quad s \in I, \; t \in \mathbb{R},$$

so that functions $x = x(t, s)$, $y = y(t, s)$, $u = u(t, s)$ satisfy both the equation and the side condition. The parameter t will express the motion along the characteristics and it is usual to choose $t = 0$ on γ (see Figure 3.10). The parametric form of the characteristics can be obtained from the system of two ODEs:

$$\begin{cases} \dfrac{\partial}{\partial t} x(t,s) = a(x(t,s), y(t,s)), \\ \dfrac{\partial}{\partial t} y(t,s) = b(x(t,s), y(t,s)) \end{cases}$$

with the initial conditions $x(0, s) = x_0(s)$, $y(0, s) = y_0(s)$. Further, we have

$$a(x,y)u_x + b(x,y)u_y = \frac{\partial x}{\partial t}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial u}{\partial y} = \frac{\mathrm{d}u}{\mathrm{d}t}$$

and the original PDE reduces to

$$u_t + c(x(t,s), y(t,s))u = f(x(t,s), y(t,s)),$$

which can be dealt as an ODE with the variable t and parameter s. Finally, we add the initial condition $u(0, s) = u_0(s)$. We illustrate all these steps once more on Example 3.12.

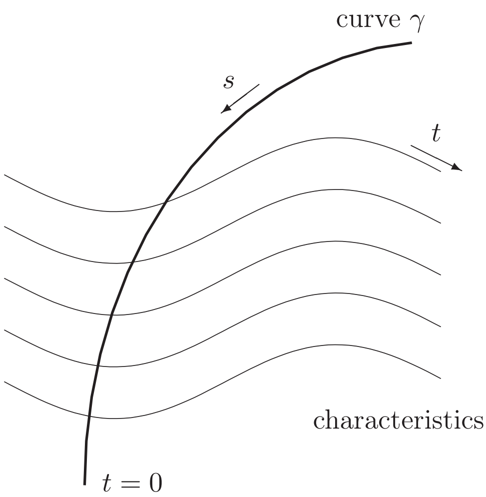


Figure 3.10. Characteristics and curve γ of the transport problem.

Example 3.13. Let us search for the solution of problem from Example 3.12 in a parametric form. First of all, we find the parametric expression of the characteristics. It means to solve the system

$$\begin{cases} \dfrac{\partial}{\partial t}x(t,s) = 1, \\ \dfrac{\partial}{\partial t}y(t,s) = y(t,s) \end{cases}$$

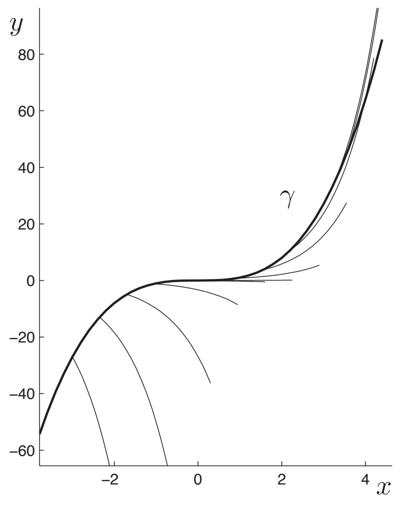
with the initial condition $x(0,s) = s$, $y(0,s) = s^3$. The corresponding general solution takes the form:

$$x(t,s) = t + g_1(s), \quad y(t,s) = g_2(s)\mathrm{e}^t, \quad s \in I,\ t \in \mathbb{R},$$

where g_1 and g_2 are arbitrary functions. Using the initial conditions, we obtain the parametric description of characteristics:

$$x(t,s) = t + s, \quad y(t,s) = s^3\mathrm{e}^t, \quad s \in I,\ t \in \mathbb{R}.$$

Figure 3.11 illustrates the curve γ and characteristics for the choice $I = (-3,4)$ and $t \in (0,2)$. Let us notice two problematic points $(0,0)$ and $(3,27)$, we have already spotted in Example 3.12.



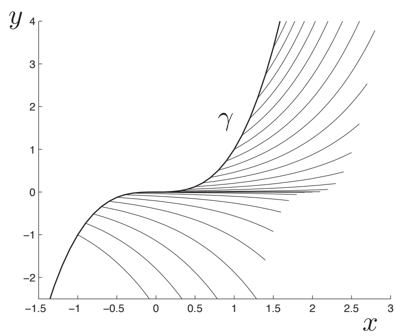


Figure 3.11. Curve $\gamma : x = s,\ y = s^3$ and characteristics of the equation $u_x + yu_y = 0$. Left: $s \in (-3,4)$, right: detail around the origin.

Further, we have $u_x + yu_y = x_t u_x + y_t u_y = u_t$ and the original PDE reduces to the simple form

$$u_t = 0,$$

whose solution is an arbitrary constant with respect to the variable t, that is, $u(t, s) = f(s)$. If we apply the initial condition $u(0, s) = \mathrm{e}^{-s^2}$, we obtain trivially $f(s) = \mathrm{e}^{-s^2}$. The final parametric description of the solution is given by the trio of relations

$$\left\{\begin{array}{lcl} x(t,s) & = & t+s, \\ y(t,s) & = & s^3\mathrm{e}^t, \\ u(t,s) & = & \mathrm{e}^{-s^2}, \end{array}\right. \qquad s \in I,\ t \in \mathbb{R}.$$

(Compare this approach with the one in Example 3.12 and verify correctness of the solution.) Figure 3.12 illustrates the solution for the choice $I = (-1, 2)$ and $t \in (0, 2)$. ◇

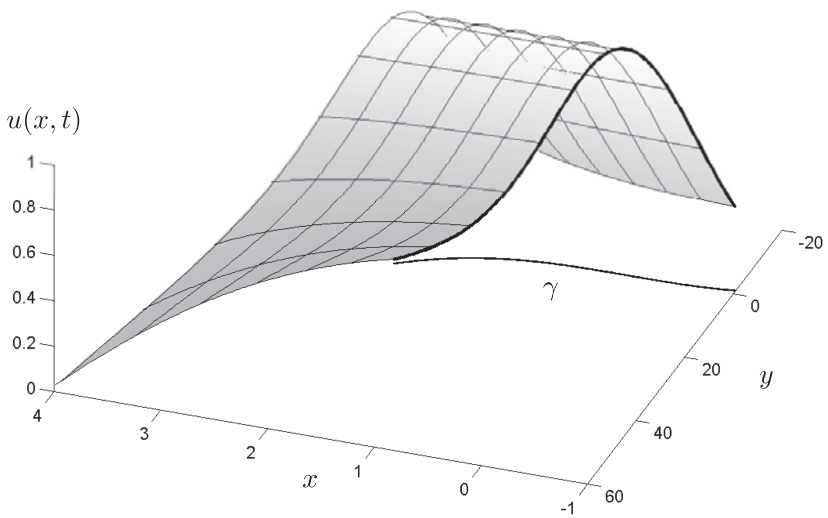


Figure 3.12. Solution of the problem $u_x + yu_y = 0$ with the condition $u(s, s^3) = \mathrm{e}^{-s^2}$.

Remark 3.14. Condition (3.16) is a sufficient condition, but not a necessary one. It means that its violation does not necessarily imply the non-existence or non-uniqueness of the solution. Have a look once more at the previous

Example 3.13. Although the characteristics does not intersect the curve γ transversally at the point $(0,0)$, they form a "fan" (see Figure 3.11 right) and the solution exists and is given uniquely in the neighborhood of this point. On the contrary, around the latter suspicious point $(3,27)$, problems really occur, the characteristics "flip" over the curve γ, they cross each other and the problem does not have a unique solution if $3 \in I$. The parametric expression of the solution has the disadvantage that its formula does not reveal these troubles at first sight. However, upon closer examination, we find out that u cannot be expressed as a function of x, y around the value $s = 3$. The reader is recommended to draw the graph of the solution, for instance, for the choice $I = (2,4)$.

Remark 3.15. The method of searching solutions in the parametric form can be easily generalized into higher dimensions. For example, if we solve the problem

$$a(x,y,z)u_x + b(x,y,z)u_y + c(x,y,z)u_z + d(x,y,z)u = f(x,y,z)$$

with three independent variables x, y, z, then the characteristics are curves, whose parametric description is obtained by solving the system

$$\begin{cases} \dfrac{\partial}{\partial t}x(t,s,r) = a(x(t,s,r), y(t,s,r), z(t,s,r)), \\ \dfrac{\partial}{\partial t}y(t,s,r) = b(x(t,s,r), y(t,s,r), z(t,s,r)), \\ \dfrac{\partial}{\partial t}z(t,s,r) = c(x(t,s,r), y(t,s,r), z(t,s,r)), \end{cases}$$

where the variable t represents the motion along the characteristics and s, r are – at this point – free parameters. The equation itself is then reduced into the form

$$u_t + c(x(t,s,r), y(t,s,r), z(t,s,r))u = f(x(t,s,r), y(t,s,r), z(t,s,r))$$

and it is further solvable by ODE techniques. If we want to add a side condition $u(0,s,r) = u_0(s,r)$, it is necessary to impose it on a *surface* σ:

$$\sigma : \begin{cases} x &= x_0(s,r), \\ y &= y_0(s,r), \qquad s \in I,\ r \in J, \\ z &= z_0(s,r), \end{cases}$$

which intersects again all characteristics just once and under a non-zero angle.

3.5 Exercises

1. Solve the equation $u_t - 3u_x = 0$ with the initial condition $u(x, 0) = \mathrm{e}^{-x^2}$ (see Figure 3.3).

$$[u(x,t) = \mathrm{e}^{-(x+3t)^2}]$$

2. Solve the equation $u_t + 3u_x = 0$ with the initial condition $u(x, 0) = \sin x$ (see Figure 3.4).

$$[u(x,t) = \sin(x - 3t)]$$

3. Solve the equation $3u_y + u_{xy} = 0$ using substitution $v = u_y$.

$$[u(x,y) = \mathrm{e}^{-3x} g(y) + f(x)]$$

4. Solve the linear equation $(1 + x^2)u_x + u_y = 0$. Draw some characteristics.

$$[u(x,y) = f(y - \mathrm{arctan} x)]$$

5. Solve the equation $\sqrt{1 - x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.

$$[u(x,y) = y - \arcsin x]$$

6. Using the coordinate method, solve the equation $au_x + bu_y + cu = 0$.

$$[u(x,y) = \mathrm{e}^{-\frac{c}{a^2+b^2}} f(bx - ay)]$$

7. Using the coordinate method, solve the equation $u_x + u_y + u = \mathrm{e}^{x+2y}$ with the initial condition $u(x, 0) = 0$.

$$[u(x,y) = \tfrac{1}{4}\left(\mathrm{e}^{x+2y} - \mathrm{e}^{x-2y}\right)]$$

8. Solve the equation $u_t + au_x = x^2 t + 1$, where a is a constant, with the initial condition $u(x, 0) = x + 2$.

$$[u(x,t) = x - at + 2 + t + \tfrac{x^2t^2}{2} - \tfrac{1}{3}axt^3 + \tfrac{1}{12}a^2t^4]$$

9. Solve the equation $u_t + t^\alpha u_x = 0$, where $\alpha > -1$ is a constant, with the initial condition $u(x, 0) = \varphi(x)$.

$$[u(x,t) = \varphi\left(x - \tfrac{t^{\alpha+1}}{\alpha+1}\right)]$$

10. Solve the equation $u_t + xt\, u_x = x^2$ with the initial condition $u(x, 0) = \varphi(x)$.

$$[u(x,t) = \varphi(x\mathrm{e}^{-1/2t^2}) + x^2\mathrm{e}^{-t^2}\int_0^t \mathrm{e}^{s^2}\,\mathrm{d}s]$$

11. Find the general solution to the transport equation with decay $u_t + cu_x = -\lambda u$ using the transformation of independent variables

$$\xi = x - ct, \quad \tau = t.$$

$$[u(x,t) = \mathrm{e}^{-\lambda t} f(x - ct)]$$

12. Show that the decay term in the transport equation with decay

$$u_t + cu_x = -\lambda u$$

can be eliminated by the substitution $w = u\mathrm{e}^{\lambda t}$.

13. Solve the Cauchy problem

$$\begin{cases} u_t + u_x - 3u = t, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = x^2, & x \in \mathbb{R}. \end{cases}$$

$$\left[u(x,t) = -\tfrac{1}{3}t - \tfrac{1}{9} + \mathrm{e}^{3t}\left((x-t)^2 + \tfrac{1}{9}\right)\right]$$

14. Solve the transport equation with the convective term

$$u_t + 2u_x = -3u$$

under the condition $u(x,0) = \frac{1}{1+x^2}$.

[The solution takes the form $u(x,t) = \frac{\mathrm{e}^{-3t}}{1+4t^2-4tx+x^2}$ and is depicted in Figure 3.13. Notice the influence of the convective term $3u$ on the solution on various time levels.]

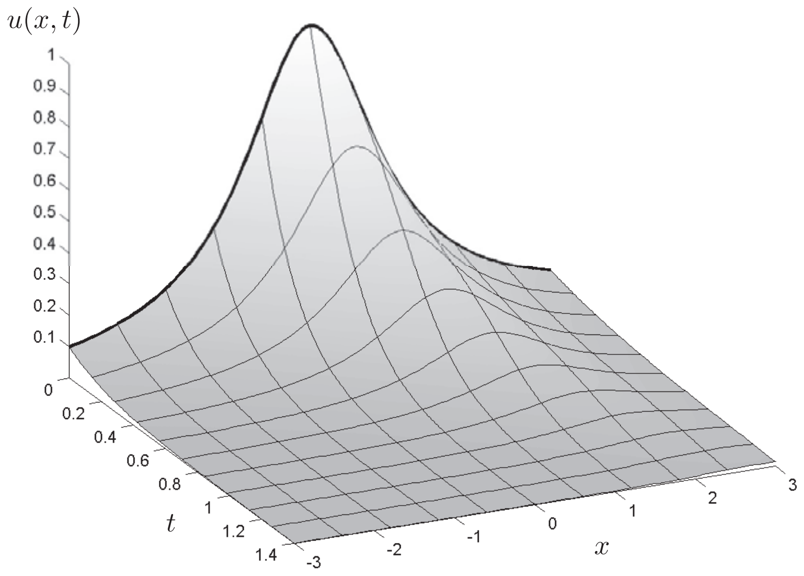


Figure 3.13. Solution of the problem $u_t + 2u_x = -3u,\ u(x,0) = 1/(1+x^2)$.

15. Solve the initial boundary value problem

$$\begin{cases} u_t + cu_x = -\lambda u, & x, t > 0, \\ u(x,0) = 0, \quad x > 0, & u(0,t) = g(t), \quad t > 0. \end{cases}$$

Consider separately the cases $x > ct$ and $x < ct$. The boundary condition takes effect in the domain $x < ct$, whereas the initial condition influences the solution only in the domain $x > ct$.

$[u(x,t) = g(t - \frac{x}{c})\mathrm{e}^{-\frac{\lambda}{c}x} \quad \text{for } x - ct < 0, \quad u(x,t) = 0 \quad \text{for } x - ct > 0]$

16. Find an implicit formula for the solution $u = u(x,t)$ of the initial value problem for the equation of transport reaction

$$\begin{cases} u_t + vu_x = -\frac{\alpha u}{\beta + u}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Here v, α, β are positive constants. Show that u can be always expressed in terms of x and t from the implicit formula.

$[\beta \ln |u(x,t)| + u(x,t) = -\alpha t + f(x - vt) + \beta \ln |f(x - vt)|]$

17. Find general solutions of the following equations.

(a) $u_x + x^2 u_y = 0$,

(b) $u_x + \sin x\, u_y = 0$,

(c) $x\, u_x + y\, u_y = 0$,

(d) $\mathrm{e}^{x^2} u_x + x\, u_y = 0$,

(e) $x\, u_x + y\, u_y = x^n$.

18. Solve the linear equation

$$x\, u_x - y\, u_y + y^2 u = y^2, \quad x, y \neq 0.$$

$[u(x,y) = f(xy)\mathrm{e}^{y^2/2} + 1]$

19. Consider the equation $y\, u_x - x\, u_y = 0$. Find curves and side conditions along these curves for which this problem has a unique solution, no solution, or infinitely many solutions.

[a) $u_0(x,0) = x^2$, b) $u_0(x,y) = y$ on $x^2 + y^2 = 1$, c) $u_0(x,y) = 1$ on $x^2 + y^2 = 1$]

20. Consider the quasi-linear equation $u_y + a(u)u_x = 0$ with the initial condition $u(x,0) = h(x)$. Show that its solution can be given implicitly by $u = h(x - a(u)y)$. What are the characteristics? What happens if $a(h(x))$ is an increasing function?

21. Consider the equation

$$u_y = \left(\frac{y}{x}u\right)_x. \tag{3.17}$$

Show that

(a) the general solution of (3.17) is given by $u = xf(x^2+y^2)$;

(b) the function

$$I(x,y) = \frac{x}{y}\int_0^{+\infty} \mathrm{e}^{-y\sqrt{1+t^2}} \cos xt \,\mathrm{d}t$$

satisfies equation (3.17);

(c) the following identity is satisfied:

$$\int_0^{+\infty} \mathrm{e}^{-y\sqrt{1+t^2}} \cos xt \,\mathrm{d}t = \frac{y}{\sqrt{x^2+y^2}} \int_0^{+\infty} \mathrm{e}^{-\sqrt{(1+t^2)(x^2+y^2)}} \mathrm{d}t, \quad y > 0.$$

22. Show that the initial value problem

$$u_t + u_x = 0, \quad u(x,t) = x \ \text{ on } x^2+t^2 = 1$$

has no solution. However, if the initial data are given only over the semicircle that lies in the half-plane $x + t \leq 0$, a solution exists but is not differentiable along the characteristics coming from the two end points of the semicircle.

23. Show that the initial value problem

$$(t-x)u_x - (t+x)u_t = 0, \quad u(x,0) = f(x), \quad x > 0,$$

has no solution in general. Draw the characteristics.

24. Show that the equation $a(x)u_x + b(t)u_t = 0$ has the general solution $u(x,t) = F(A(x) - B(t))$, where $A'(x) = 1/a(x)$ and $B'(t) = 1/b(t)$.

25. Show that the equation $a(t)u_x + b(x)u_t = 0$ has the general solution $u(x,t) = F(B(x) - A(t))$, where $B'(x) = b(x)$ and $A'(t) = a(t)$.

26. Show that the problem

$$u_t + u_x = x, \quad u(x,x) = 1$$

has no solution, and explain why.

27. Consider the semilinear equation

$$a(x, y, u(x, y))\, u_x + b(x, y, u(x, y))\, u_y = c(x, y, u(x, y))$$

and show that the method of characteristics yields

$$\frac{\mathrm{d}x}{a(x, y, u(x, y))} = \frac{\mathrm{d}y}{b(x, y, u(x, y))} = \frac{\mathrm{d}u(x, y)}{c(x, y, u(x, y))}.$$

[Hint: The differential equation can be understood as the scalar product of vectors $(a(x, y, u), b(x, y, u), c(x, y, u))$ and $(u_x, u_y, -1)$, where the last one represents the normal vector to the solution surface $u = u(x, y)$ in the Euclidean space (x, y, u).]

Chapter 4

Wave Equation in One Spatial Variable – Cauchy Problem in $\mathbb{R}$

4.1 General Solution of the Wave Equation

Let us consider the wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{4.1}$$

which we have derived in Section 1.3.5, and look for its general solution. For illustration, we can imagine an "infinitely long" string. We will present two methods for finding the general solution of equation (4.1). Both methods are standard and the reader can find them in many other textbooks.

4.1.1 Transformation to System of Two First Order Equations

Equation (4.1) can be formally rewritten as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0. \tag{4.2}$$

If we introduce a new function v by the relation $v = u_t + cu_x$, we transform the original equation (4.1) into a system of two equations of the first order

$$\left\{\begin{array}{rcl} v_t - cv_x & = & 0, \\ u_t + cu_x & = & v. \end{array}\right. \tag{4.3}$$

Both the equations are now solvable by methods introduced in Section 3.1. The first equality implies

$$v(x,t) = h(x + ct),$$

where h is an arbitrary differentiable function. We substitute for v into the latter equation of system (4.3) and obtain

$$u_t + cu_x = h(x + ct),$$

which represents a transport equation with constant coefficients and non-zero right-hand side. The solution of the corresponding homogeneous equation has the form

$$u_H(x,t) = g(x - ct). \tag{4.4}$$

Since the right-hand side of the equation is formed by an arbitrary differentiable function of the argument $x + ct$, the particular solution (which reflects the influence of the right-hand side) must be also an arbitrary differentiable function of the same argument, thus

$$u_P(x,t) = f(x+ct). \tag{4.5}$$

The general (and generic) solution of the wave equation is then the sum of solutions (4.4) and (4.5):

$$\boxed{u(x,t) = f(x+ct) + g(x-ct).} \tag{4.6}$$

4.1.2 Method of Characteristics

The second way of derivation of the general solution of the wave equation on the real line consists of introducing special coordinates

$$\xi = x + ct, \quad \eta = x - ct.$$

According to the chain rule, we have

$$\partial_x = \partial_\xi + \partial_\eta, \qquad \partial_t = c\partial_\xi - c\partial_\eta$$

and thus

$$\begin{aligned} \partial_t - c\partial_x &= -2c\partial_\eta, \\ \partial_t + c\partial_x &= 2c\partial_\xi. \end{aligned}$$

The reader is invited to verify these formulas. After substituting into the expression (4.2), we obtain a transformed equation

$$-4c^2\partial_\eta\partial_\xi u = 0$$

or, equivalently,

$$u_{\xi\eta} = 0.$$

Its solution has been found in Chapter 2 (see Example 2.3), namely

$$u(\xi,\eta) = f(\xi) + g(\eta),$$

where f and g are again arbitrary differentiable functions. If we go back to the original variables x and t, we obtain the foregoing general solution of the one-dimensional wave equation in the form (4.6).

As we can see, the solution is the sum of two traveling waves, the left one and the right one, which move at the speed $c > 0$. Lines $x + ct =$ const., $x - ct =$ const., along which the traveling waves propagate, are called the *characteristics of the wave equation.*

4.2 Cauchy Problem on the Real Line

If we now consider an initial value (Cauchy) problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), \end{cases} \tag{4.7}$$

the particular forms of functions f and g from the general solution (4.6) can be determined in terms of the given functions φ and ψ, which describe the *initial displacement* and the *initial velocity* of the searched wave.

If we start with the formula (4.6), then the following equalities hold for $t = 0$:

$$\varphi(x) = f(x) + g(x), \qquad \psi(x) = cf'(x) - cg'(x).$$

The first equality implies (assuming that all indicated derivatives exist)

$$\varphi'(x) = f'(x) + g'(x),$$

which, in combination with the latter equality, gives

$$\begin{aligned} f'(x) &= \frac{1}{2}\varphi'(x) + \frac{1}{2c}\psi(x), \\ g'(x) &= \frac{1}{2}\varphi'(x) - \frac{1}{2c}\psi(x) \end{aligned}$$

and, after integration,

$$\begin{aligned} f(x) &= \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(\tau)\,\mathrm{d}\tau + A, \\ g(x) &= \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(\tau)\,\mathrm{d}\tau + B, \end{aligned}$$

where A, B are integration constants. The condition $u(x,0) = \varphi(x)$, however, implies $A + B = 0$. After substituting the previous relations into the general expression of solution (4.6), we obtain the solution of the Cauchy problem for the wave equation in one dimension:

$$u(x,t) = \frac{1}{2}\big(\varphi(x+ct) + \varphi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)\mathrm{d}\tau. \tag{4.8}$$

This formula was derived by d'Alembert in 1746. The first term on the right-hand side expresses the influence of the initial displacement: the initial wave is divided into two parts, the former proceeding in the direction of the negative x half-axis at a speed c, and the latter proceeding in the direction of the positive x half-axis at the same speed c. The integral on the right-hand side expresses the influence of the initial velocity. The solution expressed by d'Alembert's formula is determined uniquely (see Exercise 6 in Section 10.9).

The following assertion is the basic existence and uniqueness result for the wave equation.

Theorem 4.1. *Let $\varphi \in C^2$, $\psi \in C^1$ on the entire real line $\mathbb{R}$. The Cauchy problem* (4.7) *for the wave equation on the real line with the initial displacement $\varphi(x)$ and the initial velocity $\psi(x)$ has a unique classical solution $u \in C^2$ given by d'Alembert's formula* (4.8).

Example 4.2. Let us find a solution of the wave equation on the real line, if the initial displacement is zero and the initial velocity is given by $\sin x$.

This problem can be written as a Cauchy problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0,\ u_t(x,0) = \sin x. \end{cases} \tag{4.9}$$

After substituting the initial conditions into d'Alembert's formula (4.8), we obtain

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin\tau \, \mathrm{d}\tau = \frac{1}{2c}\big(\cos(x-ct) - \cos(x+ct)\big)$$

and, applying the trigonometric formula

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2},$$

the solution $u(x,t)$ can be written in the form

$$u(x,t) = \frac{1}{c}\sin ct \, \sin x. \tag{4.10}$$

$\diamond$

Let us note the following special feature of solution (4.10): the zero points of u lie at the points $x = k\pi$, $k \in \mathbb{Z}$, for arbitrary $t \geq 0$. They do not "travel" along the x-axis with growing time. Solutions of the wave equation with the above mentioned property are called *standing waves* (see Figure 4.1).

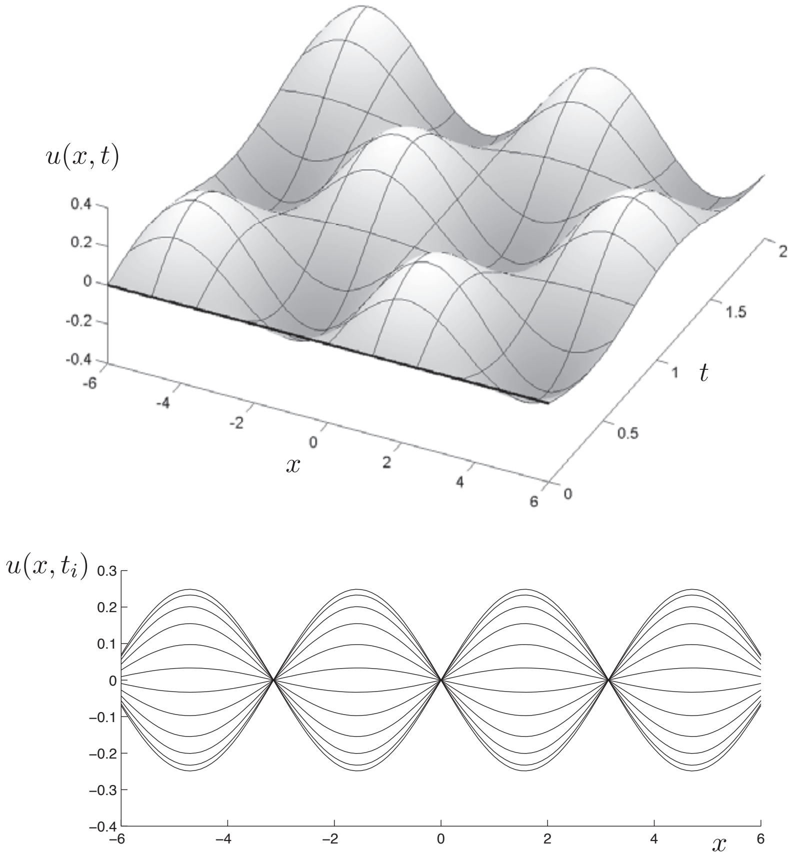


Figure 4.1. Standing waves – a solution of the initial value problem (4.9) with $c = 4$.

Remark 4.3. In fact, a more general assertion than Theorem 4.1 holds:

> The initial value problem (4.7) for the wave equation has a unique classical solution *if and only if* $\varphi \in C^2$ and $\psi \in C^1$.

These assumptions on initial conditions are, however, strongly restrictive and very often contradict the practical problems that necessarily have to be solved by methods of mathematical modeling. These problems, if they are understood in the sense of a classical solution, are ill-posed. This fact caused big difficulties to mathematicians in the eighteenth century and it took a long time before they came to a more general notion of a solution (weak solution, very weak solution, generalized solution, strong solution, etc.).

In this text, we will not deal with these questions in detail. We restrict ourselves only to the statement that formula (4.8), expressing a solution of the initial value problem for the wave equation in explicit form, makes sense also for much more general initial conditions than $\varphi \in C^2$, $\psi \in C^1$. The solution u can be then viewed, for instance, as a function which satisfies the differential equation only at those points where the corresponding partial derivatives u_{tt} and u_{xx} exist. On the other hand, the set of points where these partial derivatives do not exist (and hence the equation itself does not make sense) cannot be "too large".

If φ is a C^2 function and ψ is a C^1 function on $\mathbb{R}$ with the exception of a finite number of points (the so called *singular points* or *singularities*) then d'Alembert's formula (4.8) makes sense, the partial derivatives of u exist and are continuous with the exception of a finite number of lines in the xt plane and equation (4.1) holds at every point which does not belong to these lines. In such a way, we will understand also solutions of the following Cauchy problems. However, the reader should notice that even more general functions φ and ψ can be considered (for example, locally integrable) and the corresponding solution of (4.1) makes sense if it is understood in a more general sense. Existence and uniqueness results can be still proved in such a more general setting.

Example 4.4 (Strauss [21]). Let us solve the wave equation with the initial displacement

$$\varphi(x) = \begin{cases} b - \dfrac{b}{a}|x| & \text{for } |x| \le a, \\ 0 & \text{for } |x| > a \end{cases}$$

and with zero initial velocity

$$\psi(x) \equiv 0.$$

This problem describes the behavior of an infinitely long string which at time $t = 0$ is displaced by "three fingers" and then released. The three points $x \in \{-a, 0, a\}$ represent singularities of the initial displacement φ. According to d'Alembert's formula (4.8), the corresponding solution has the form

$$u(x,t) = \frac{1}{2}\big(\varphi(x - ct) + \varphi(x + ct)\big).$$

It is a sum of two "triangle functions" which diverge with increasing time. The shape of the solution on particular time levels is sketched in Figure 4.2, the whole graph of the function $u(x,t)$ is illustrated by Figure 4.3 (for the values $c = 2$, $b = 1$, $a = 2$). The reader should observe lines in the xt plane where the partial derivatives of u do not exist and equation (4.1) makes no sense. ◇

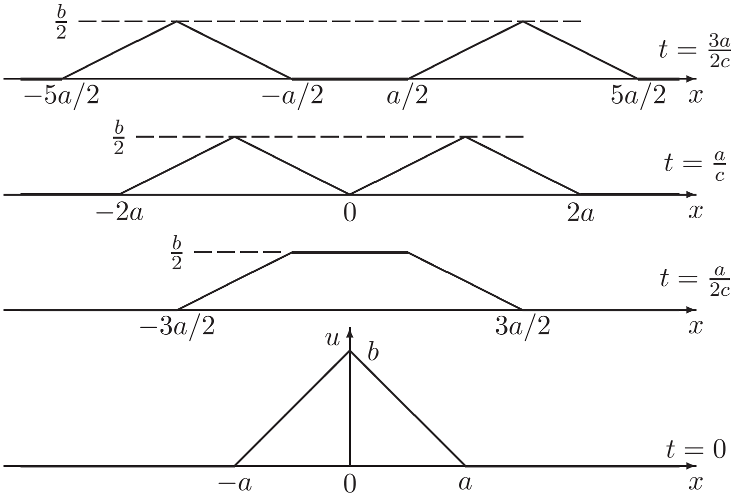


Figure 4.2. Solution of Example 4.4 on particular time levels.

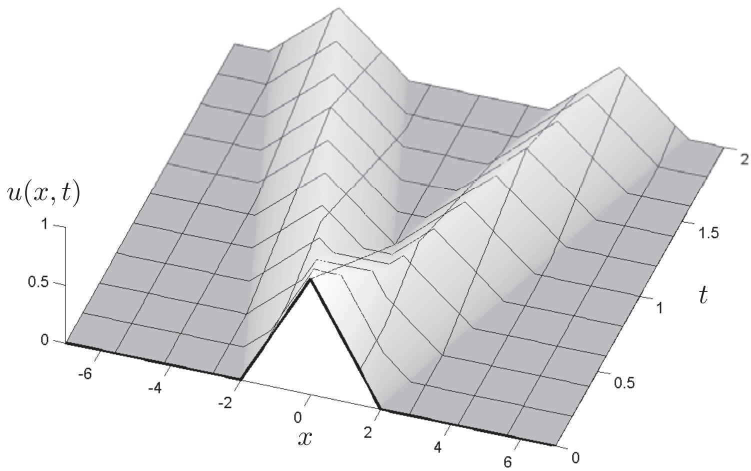


Figure 4.3. Graph of solution from Example 4.4 for $c = 2$, $b = 1$, $a = 2$.

Example 4.5 (Strauss [21]). Let us solve the wave equation with zero initial displacement $\varphi(x) \equiv 0$ and with the initial velocity

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \le a, \\ 0 & \text{for } |x| > a. \end{cases}$$

This problem can be regarded as a simplified model of the behavior of an infinitely long string after a stroke by a hammer of width $2a$. Here, the two points $x \in \{-a, a\}$ represent singularities of the initial velocity ψ. D'Alembert's formula implies

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\tau)\,\mathrm{d}\tau = \frac{1}{2c} \times \text{ length of interval } \{(-a,a) \cap (x-ct, x+ct)\}.$$

The shape of the solution on particular time levels is sketched in Figure 4.4, the whole graph of the function $u(x,t)$ is illustrated by Figure 4.5, where the values of parameters are chosen as $c = 2.3$ and $a = 1.3$. The reader is asked again to pay attention to the lines in the xt plane where the partial derivatives of u do not exist and equation (4.1) makes no sense. ◇

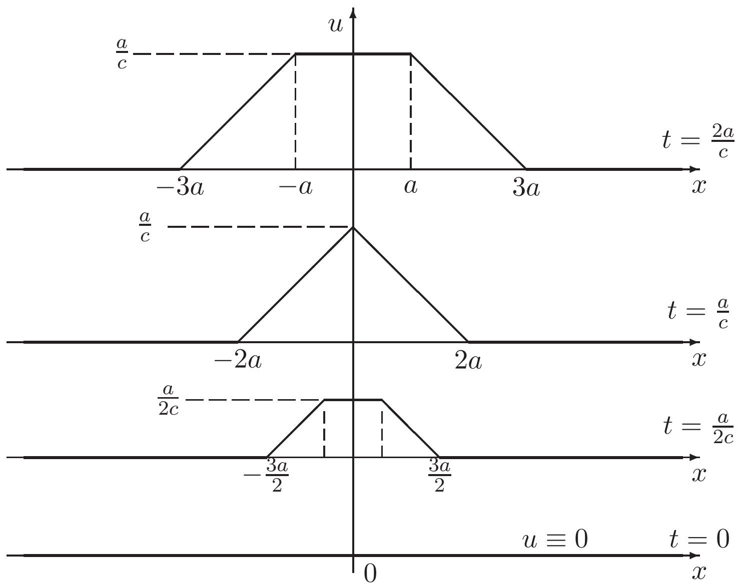


Figure 4.4. Solution of Example 4.5 on particular time levels.

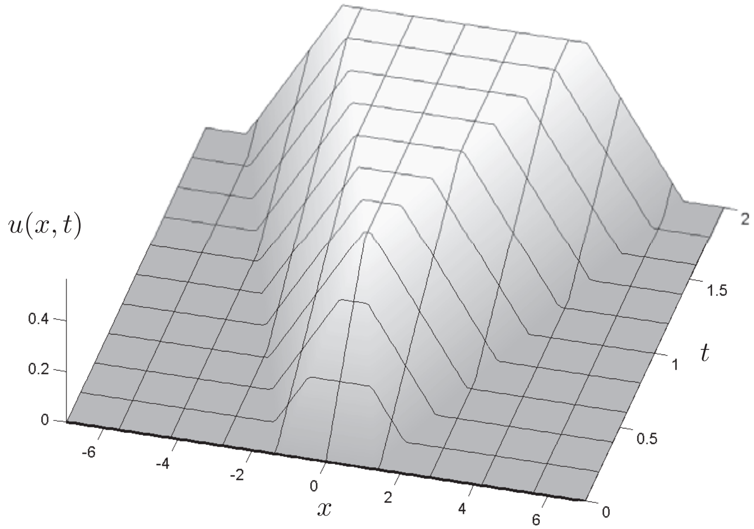


Figure 4.5. Graph of solution from Example 4.5 for $c = 2.3$, $a = 1.3$.

4.3 Principle of Causality

Let us investigate the solution of the initial value problem for the wave equation on the real line in more detail. We find out that the initial condition at the point $(x_0, 0)$ can "spread" only to that part of the xt plane which lies between the lines with equations $x \pm ct = x_0$ (the characteristics passing through the point $(x_0, 0)$). See Figure 4.6. The sector with these boundary points is called the *domain of influence* of the point $(x_0, 0)$.

In particular, this means that the initial conditions with the property

$$\varphi(x) = \psi(x) \equiv 0 \quad \text{for } |x| > R$$

result in the solution which is identically zero "to the right" of the line $x - ct = R$ and "to the left" of the line $x + ct = -R$ (see Figure 4.7).

The opposite (dual) view of the above situation is the following: let us choose an arbitrary point (x, t) and ask what values of the initial conditions on the x-axis (for $t = 0$) can influence the value of the solution at a point (x, t). The above-mentioned information implies that these are just the values $\varphi(x - ct)$, $\varphi(x + ct)$ and the values $\psi(x)$ for x from the interval between $x - ct$ and $x + ct$ (see Figure 4.8).

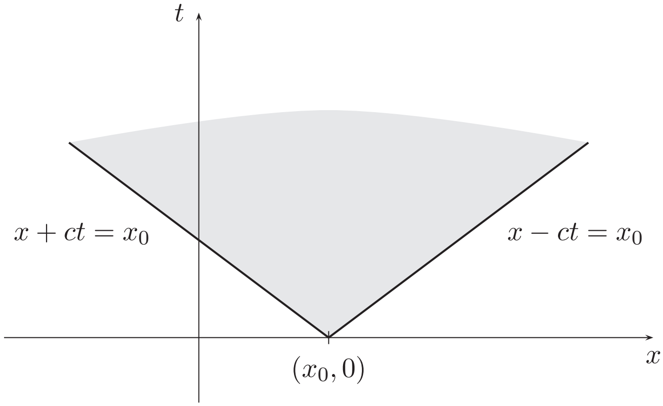


Figure 4.6. Domain of influence of the point $(x_0, 0)$ at time $t \geq 0$.

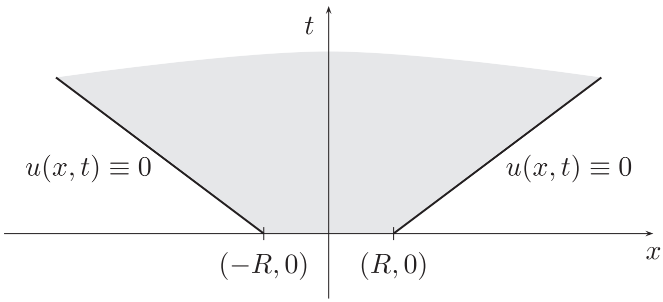


Figure 4.7. Domain of influence of the interval $(-R, R)$ at time $t \geq 0$.

The triangle $\triangle_{xt}$ with vertices at the points $(x - ct, 0)$, $(x + ct, 0)$ and (x, t) is called the *domain of dependence* (or the *characteristic triangle*) of the point (x, t).

4.4 Wave Equation with Sources

Let us now consider the Cauchy problem for the wave equation with a non-zero right-hand side

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases} \tag{4.11}$$

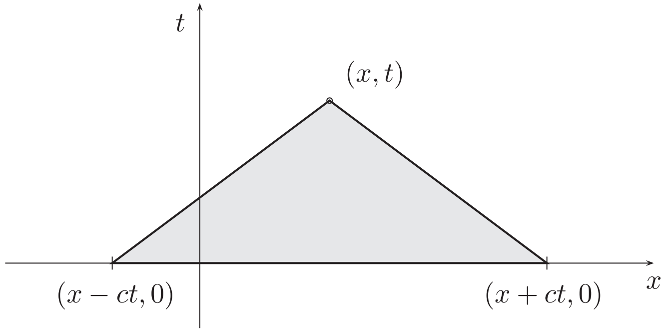


Figure 4.8. Domain of dependence (characteristic triangle) of the point (x,t).

The following existence and uniqueness result generalizes Theorem 4.1.

Theorem 4.6. *Let $\varphi \in C^2$, $\psi \in C^1$, $f \in C^1$. The initial value problem* (4.11) *has a unique classical solution which has the form*

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big(\varphi(x+ct)+\varphi(x-ct)\big) \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y + \frac{1}{2c}\iint_{\triangle} f(y,s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{4.12}$$

The symbol $\triangle = \triangle_{xt}$ in the last integral represents the characteristic triangle (see Figure 4.8), that is,

$$\iint_{\triangle} f(y,s)\,\mathrm{d}y\,\mathrm{d}s = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Notice that the influence of the external force f on the behavior of the modeled string is given only by the integration of function f over the whole domain of dependence of the point (x,t) up to the time $t=0$. Hence, the principle of causality holds again.

Remark 4.7. The reader can easily check that the function given by (4.12) is indeed a classical solution of problem (4.11). (See Exercise 12 in Section 4.5.) However, the reader should notice that the classical solution exists under the more general assumption $f \in C$.

There are several ways how to derive formula (4.12). One of the possibilities is based on the application of the method of characteristics, and another one uses the transformation of the wave equation to the system of two transport equations. The latter derivation is the scope of Exercise 19 in Section 4.5. In what follows, we focus our attention on other two standard approaches that can be found in several textbooks and have not been mentioned yet, namely, the use of *Green's Theorem* and application of the *Operator Method* (cf. Strauss [21]).

4.4.1 Use of Green's Theorem

For simplicity, let us consider a fixed point (x_0, t_0) and assume that u is a classical solution of (4.11). We integrate the wave equation over the domain of dependence of the point (x_0, t_0), that is, over the characteristic triangle $\triangle$:

$$\iint\limits_{\triangle} f \,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{\triangle} (u_{tt} - c^2 u_{xx})\,\mathrm{d}x\,\mathrm{d}t.$$

Now, we apply Green's Theorem to the right-hand side. It reads:

$$\iint\limits_{\triangle} (P_x - Q_t)\,\mathrm{d}x\,\mathrm{d}t = \int\limits_{\partial\triangle} P\,\mathrm{d}t + Q\,\mathrm{d}x$$

for arbitrary continuously differentiable functions P, Q. The curve integral over the boundary $\partial\triangle$ of the domain $\triangle$ is considered in the positive direction, that is in the counterclockwise direction. In our case, we set $P = -c^2 u_x$, $Q = -u_t$. If we denote the particular sides of the characteristic triangle by L_0, L_1, L_2 (see Figure 4.9), we obtain

$$\iint\limits_{\triangle} f\,\mathrm{d}x\,\mathrm{d}t = \int\limits_{L_0\cup L_1\cup L_2} -c^2 u_x\,\mathrm{d}t - u_t\,\mathrm{d}x,$$

which can be written as a sum of three curve integrals over the corresponding straight line segments.

On the side L_0, we have $t = 0$, $\mathrm{d}t = 0$ and $u_t(x, 0) = \psi(x)$, thus

$$\int\limits_{L_0} -c^2 u_x\,\mathrm{d}t - u_t\,\mathrm{d}x = -\int\limits_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,\mathrm{d}x.$$

On L_1, we have $x + ct = x_0 + ct_0$ and thus $\mathrm{d}x + c\,\mathrm{d}t = 0$. Hence, we obtain

$$\int\limits_{L_1} -c^2 u_x\,\mathrm{d}t - u_t\,\mathrm{d}x = c\int\limits_{L_1} u_x\,\mathrm{d}x + u_t\,\mathrm{d}t = c\int\limits_{L_1} \mathrm{d}u = cu(x_0, t_0) - c\varphi(x_0 + ct_0),$$

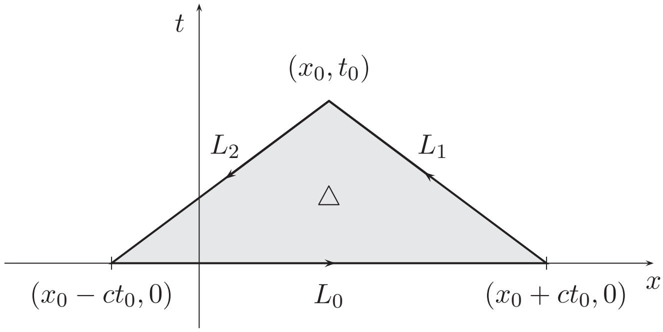


Figure 4.9. Characteristic triangle of the point (x_0, t_0).

where $\mathrm{d}u$ denotes the total differential of function $u = u(x,t)$. Similarly, for L_2, where $\mathrm{d}x - c\,\mathrm{d}t = 0$, we express

$$\int\limits_{L_2} -c^2 u_x \,\mathrm{d}t - u_t \,\mathrm{d}x = -c\int\limits_{L_2} u_x \,\mathrm{d}x + u_t \,\mathrm{d}t = -c\int\limits_{L_2} \mathrm{d}u = -c\varphi(x_0 - ct_0) + cu(x_0, t_0).$$

Combining these three partial results, we obtain

$$\iint\limits_{\triangle} f \,\mathrm{d}x\,\mathrm{d}t = 2cu(x_0, t_0) - c\big(\varphi(x_0 + ct_0) + \varphi(x_0 - ct_0)\big) - \int\limits_{x_0 - ct_0}^{x_0 + ct_0} \psi(x)\,\mathrm{d}x$$

wherefrom the required form of the solution u at the point (x_0, t_0) follows.

4.4.2 Operator Method

This time we try to derive the solution of the initial value problem for the nonhomogeneous wave equation on the basis of an analogue of the solution of the ODE

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + A^2 v(t) = f(t) \tag{4.13}$$

with initial conditions

$$v(0) = \varphi, \qquad \frac{\mathrm{d}v}{\mathrm{d}t}(0) = \psi,$$

where φ and ψ are real numbers. Applying the variation of constants formula, the solution of equation (4.13) for a constant $A \neq 0$ can be written in the form

$$v(t) = S'(t)\varphi + S(t)\psi + \int\limits_0^t S(t-s) f(s)\,\mathrm{d}s, \tag{4.14}$$

where

$$S(t) = \frac{1}{A}\sin At, \qquad S'(t) = \cos At. \tag{4.15}$$

Thus, in the case $\varphi = 0$, $f = 0$, the solution reduces to $v(t) = S(t)\psi$.

Now, we turn back to our wave equation. We have derived that the solution of the homogeneous equation for $\varphi(x) \equiv 0$, $f(x,t) \equiv 0$ can be written in the form

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y.$$

If we define the *source operator* $\mathcal{S}(t)$ by

$$\mathcal{S}(t)\psi(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y, \tag{4.16}$$

we can write

$$u(x,t) = \mathcal{S}(t)\psi(x).$$

Analogously to the first term on the right-hand side of relation (4.14), we could expect the reaction on the non-zero initial displacement in the form $\frac{\partial}{\partial t}\mathcal{S}(t)\varphi(x)$. Indeed, we have

$$\frac{\partial}{\partial t}\mathcal{S}(t)\varphi(x) = \frac{\partial}{\partial t}\frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(y)\,\mathrm{d}y = \frac{1}{2c}\big(c\varphi(x+ct) - (-c)\varphi(x-ct)\big), \tag{4.17}$$

which corresponds to d'Alembert's formula (4.8).

Now, let us consider only the influence of the right-hand side. To this end, put $\varphi = \psi = 0$. If we use again the analogue of the solution of the ODE (4.14), we write the corresponding solution of the wave equation in the form

$$u(x,t) = \int_0^t \mathcal{S}(t-s)f(x,s)\,\mathrm{d}s;$$

thus, using the definition of $\mathcal{S}(t)$ in (4.16), we conclude that

$$u(x,t) = \int_0^t \left(\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,\mathrm{d}y\right)\mathrm{d}s = \frac{1}{2c}\iint_{\triangle} f(y,s)\,\mathrm{d}y\,\mathrm{d}s. \tag{4.18}$$

Putting together (4.16)–(4.18), we arrive at (4.12). Using the operator method, we "guessed" the solution and now we should verify it (which is the purpose of Exercise 12 in Section 4.5).

The approach based on the idea that the knowledge of a solution of the homogeneous equation can be used for the derivation of a solution of the nonhomogeneous equation is, in connection with the wave equation, called *Duhamel's principle*.

4.5 Exercises

1. Verify that the function

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)\,\mathrm{d}\xi$$

solves the wave equation $u_{tt} = c^2 u_{xx}$, where c is a constant and g is a continuously differentiable function. Use the rule for the derivative of the integral with respect to parameters x and t occurring in the limits of integration.

2. A linear approximation of the one-dimensional isotropic flow of an ideal gas is given by

$$u_t + \rho_x = 0, \quad u_x + c^2 \rho_t = 0,$$

where $u = u(x,t)$ is the velocity of the gas and $\rho = \rho(x,t)$ is its density. Show that u and ρ satisfy the wave equation.

3. Deriving the general solution (4.6) of the wave equation, we have used the fact that the linear wave operator $L = (\partial_t)^2 - c^2(\partial_x)^2$ is *reducible* (or *factorable*), that is, it can be written as a product of linear first-order operators: $L = L_1 L_2$. Using the same idea, find the general solutions of the following equations.

   (a) $u_{xx} + u_x = u_{yy} + u_y$.
   [$L_1 = \partial_x - \partial_y$, $L_2 = \partial_x + \partial_y + 1$. The general solution can be written as $u(x,y) = \varphi(x+y) + \mathrm{e}^{-x}\psi(x-y)$ or as $u(x,y) = \varphi(x+y) + \mathrm{e}^{-y}h(x-y)$, where φ, ψ and h are arbitrary differentiable functions.]

   (b) $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$.
   [$L_1 = 3\partial_x + \partial_y$, $L_2 = \partial_x + 3\partial_y$; $u(x,y) = \varphi(3x-y) + \psi(x-3y)$ with arbitrary functions φ, ψ]

4. Solve the Cauchy problem $u_{tt} = c^2 u_{xx},\ u(x,0) = \mathrm{e}^x,\ u_t(x,0) = \sin x$.

[$u(x,t) = \frac{1}{2}(\mathrm{e}^{x+ct} + \mathrm{e}^{x-ct}) - \frac{1}{2c}(\cos(x+ct) - \cos(x-ct))$]

5. Solve the Cauchy problem $u_{tt} = c^2 u_{xx},\ u(x,0) = \ln(1+x^2),\ u_t(x,0) = 4+x$.

[$u(x,t) = \ln\sqrt{(1+(x+ct)^2)(1+(x-ct)^2)} + t(4+x)$]

6. Solve the Cauchy problem $u_{tt} - 3u_{xt} - 4u_{xx} = 0,\ u(x,0) = x^2,\ u_t(x,0) = \mathrm{e}^x$. Proceed in the same way as when deriving the general solution of the wave equation.

   $[u(x,t) = x^2 + 4t^2 + \frac{\mathrm{e}^{x+4t} - \mathrm{e}^{x-t}}{5}]$

7. Solve the Cauchy problem $u_{tt} - u_{xx} = 0,\ u(x,0) = 0,\ u_t(x,0) = -2x\mathrm{e}^{-x^2}$.

   $[u(x,t) = \frac{1}{2}(\mathrm{e}^{-(x+t)^2} - \mathrm{e}^{-(x-t)^2})]$

8. Solve the Cauchy problem $u_{tt} - u_{xx} = 0,\ u(x,0) = 0,\ u_t(x,0) = \frac{x}{(1+x^2)^2}$.

   $[u(x,t) = \frac{1}{4}(\frac{1}{1+(x-t)^2} - \frac{1}{1+(x+t)^2})]$

9. Solve the Cauchy problem $u_{tt} - u_{xx} = 0$ for

$$u(x,0) = \begin{cases} \mathrm{e}^{-x}, & |x| < 1, \\ 0, & |x| > 1, \end{cases} \qquad u_t(x,0) = 0.$$

10. Solve the Cauchy problem $u_{tt} - u_{xx} = 0$ for

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} \mathrm{e}^{-x}, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

11. Prove that the function

$$u(x,t) = \frac{1}{2}\left[\mathrm{e}^{-(x-2t)^2} + \mathrm{e}^{-(x+2t)^2}\right]$$

(see Figure 4.10) solves the Cauchy problem

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \mathrm{e}^{-x^2},\ u_t(x,0) = 0. \end{cases} \tag{4.19}$$

12. By a simple substitution, verify that the function

$$u(x,t) = \frac{1}{2c} \iint\limits_{\triangle} f(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

solves the nonhomogeneous wave equation $u_{tt} - c^2 u_{xx} = f$ (cf. Theorem 4.6). Explain why we need the assumption $f \in C^1$.

13. Solve the Cauchy problem $u_{tt} = c^2 u_{xx} + xt,\ u(x,0) = 0,\ u_t(x,0) = 0$.

   $[u(x,t) = \frac{xt^3}{6}]$

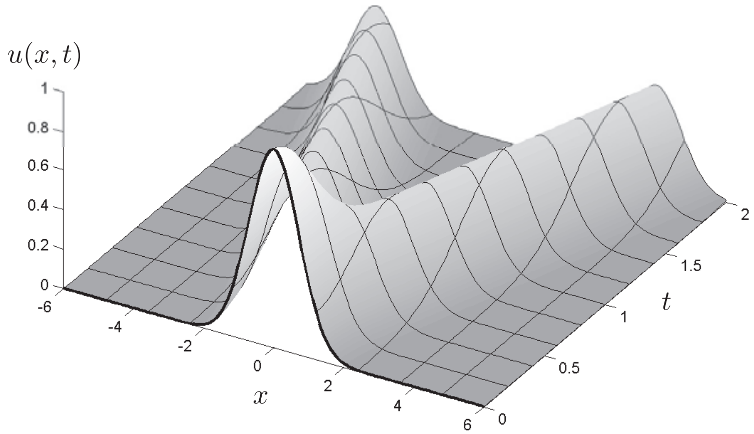


Figure 4.10. Solution of problem (4.19).

14. Solve the Cauchy problem $u_{tt} = c^2 u_{xx} + \mathrm{e}^{at},\ u(x,0) = 0,\ u_t(x,0) = 0.$

$$\left[u(x,t) = \tfrac{1}{a^2}(\mathrm{e}^{at} - at - 1)\right]$$

15. Solve the Cauchy problem $u_{tt} = c^2 u_{xx} + \cos x,\ u(x,0) = \sin x,\ u_t(x,0) = 1 + x.$

$$\left[u(x,t) = \cos ct \sin x + (1+x)t + \tfrac{\cos x}{c^2} - \tfrac{\cos x \cos ct}{c^2}\right]$$

16. Solve the Cauchy problem $u_{tt} - u_{xx} = \mathrm{e}^{x-t},\ u(x,0) = 0,\ u_t(x,0) = 0.$

$$\left[u(x,t) = \tfrac{1}{4}(\mathrm{e}^{x+t} - \mathrm{e}^{x-t}) - \tfrac{1}{2}t\mathrm{e}^{x-t}\right]$$

17. Solve the Cauchy problem $u_{tt} - u_{xx} = \sin x,\ u(x,0) = \cos x,\ u_t(x,0) = x.$

$$\left[u(x,t) = \cos x\, \cos t + xt + \sin x - \sin x\, \cos t\right]$$

18. Solve the Cauchy problem $u_{tt} - u_{xx} = x^2,\ u(x,0) = \cos x,\ u_t(x,0) = 0.$

$$\left[u(x,t) = \cos x\, \cos t + \tfrac{x^2 t^2}{2} + \tfrac{t^4}{12}\right]$$

19. Derive the solution of the nonhomogeneous wave equation in another possible way:

(a) Rewrite the equation to the system

$$u_t + cu_x = v, \quad v_t - cv_x = f.$$

(b) In the case of the former equation, find a solution u dependent on v in the form

$$u(x,t) = \int_0^t v(x - ct + cs, s)\,\mathrm{d}s.$$

(c) Similarly, solve the latter equation, i.e., find v dependent on f.

(d) Insert the result of part (c) into the result of part (b).

20. Consider the *telegraph equation* $u_{xx} - \frac{1}{c^2}u_{tt} + \alpha u_t + \beta u = 0$ and put $v = u$, $w = u_x$ and $z = u_t$. Show that v, w and z satisfy the following system of three equations:

$$\begin{aligned}
&v_t - z = 0,\\
&w_t - z_x = 0,\\
&z_t - c^2(w_x + \alpha z + \beta v) = 0.
\end{aligned}$$

Chapter 5

Diffusion Equation in One Spatial Variable – Cauchy Problem in $\mathbb{R}$

5.1 Cauchy Problem on the Real Line

Let us consider a Cauchy problem for the diffusion equation

$$\begin{cases} u_t = ku_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{5.1}$$

From the physical point of view, this problem describes diffusion in an infinitely long tube or heat propagation in an infinitely long bar. In the former case, the function $\varphi(x)$ describes the initial concentration of the diffusing substance, whereas in the latter case, it represents the initial distribution of temperature in the bar.

Since the general solution is not known for the diffusion equation, we proceed in a completely different way than we did in the case of the wave equation. We start with solving problem (5.1) with a special "unit step" initial condition $\varphi(x)$. More precisely, we solve the problem

$$\begin{cases} w_t = kw_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ w(x,0) = 0 \quad \text{for } x < 0; \quad w(x,0) = w_0 \equiv 1 \quad \text{for } x > 0. \end{cases} \tag{5.2}$$

To derive a solution of this special problem, we use the fact that any physical law can be transferred into a *dimensionless form*. In other words, if we consider an equation linking physical quantities $q_1, \dots, q_m$ of certain dimensions (time, length, mass, etc.), we can find an equivalent relation with dimensionless quantities derived from $q_1, \dots, q_m$. This process is known as *Buckingham Π Theorem* (see, e.g., [14]) and we illustrate it by a simple example. Let us imagine an object that was thrown upright at time $t = 0$ at speed v. The height h of the object at time t is given by the formula

$$h = -\frac{1}{2}gt^2 + vt.$$

The constant g represents the gravitational acceleration. The quantities used here are h, t, v and g with dimensions of length, time, length per time, and length per time-squared, respectively. This law can be written equivalently also as

$$\frac{h}{vt} = -\frac{1}{2}\left(\frac{gt}{v}\right) + 1.$$

If we denote

$$P_1 = \frac{h}{vt} \quad \text{and} \quad P_2 = \frac{gt}{v},$$

then P_1, P_2 are quantities without dimensions and the original equation has the form

$$P_1 = -\frac{1}{2}P_2 + 1.$$

A similar process can be applied also in the case of our special problem (5.2). The quantities considered are x, t, w, w_0, k, which have – for the heat transfer model – dimensions of length, time, temperature, again temperature, and length-squared per time, respectively. It is clear that w/w_0 is a dimensionless quantity. The only other dimensionless quantity derived from the remaining parameters is $x/\sqrt{4kt}$ (constant 4 is here only for simplification of further relations). We can thus expect the solution of (5.2) to have the form of a combination of these dimensionless variables, that is

$$\frac{w}{w_0} = f\left(\frac{x}{\sqrt{4kt}}\right),$$

where f is for now an unknown function that has to be determined. We recall that $w_0 \equiv 1$. Now, let us introduce a substitution

$$w = f(z), \quad z = \frac{x}{\sqrt{4kt}}$$

and put it into the equation of problem (5.2). According to the chain rule, we find

$$\begin{aligned} w_t &= f'(z)z_t = -\frac{1}{2}\frac{x}{\sqrt{4kt^3}}f'(z), \\ w_x &= f'(z)z_x = \frac{1}{\sqrt{4kt}}f'(z), \quad w_{xx} = \frac{\partial}{\partial x}w_x = \frac{1}{4kt}f''(z). \end{aligned}$$

If we substitute these expressions into (5.2) and simplify, we obtain an ODE

$$f''(z) + 2zf'(z) = 0$$

for an unknown function $f(z)$. We easily derive

$$f(z) = c_1 \int_0^z \mathrm{e}^{-s^2}\,\mathrm{d}s + c_2,$$

where c_1, c_2 are integration constants. (The reader is asked to do it in detail.) Thus we obtain a solution of the Cauchy problem (5.2) in the form

$$w(x,t) = c_1 \int_0^{x/\sqrt{4kt}} \mathrm{e}^{-s^2}\,\mathrm{d}s + c_2.$$

To determine the constants c_1, c_2, we use the initial condition. Let us consider a fixed negative x and pass to the limit for $t \to 0+$; then

$$0 = w(x,0) = c_1 \int_0^{-\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s + c_2.$$

Conversely, for a fixed positive x and $t \to 0+$ we have

$$1 = w(x,0) = c_1 \int_0^{+\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s + c_2.$$

Since

$$\int_0^{+\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi}}{2},$$

we easily determine $c_1 = 1/\sqrt{\pi}$, $c_2 = 1/2$. Hence we obtain a formula for the solution of problem (5.2):

$$w(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} \mathrm{e}^{-s^2}\,\mathrm{d}s. \tag{5.3}$$

Using the so called *error function*

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-s^2}\,\mathrm{d}s,$$

solution (5.3) can be written in an equivalent form

$$w(x,t) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right). \tag{5.4}$$

Several time levels of solution (5.4) are depicted in Figure 5.1.

Now, we come to the second step of derivation of a solution of the general Cauchy problem (5.1). Later (see Chapter 9), we will support our considerations by arguments based on the Fourier transform. For now, however, we put up with an intuitive approach based on physical reasoning. First, let us notice that if w solves the diffusion equation, the partial derivatives w_{xt}, w_{xxx} exist and $w_{xt} = w_{tx}$, then w_x also solves the same equation, namely,

$$0 = (w_t - kw_{xx})_x = (w_x)_t - k(w_x)_{xx}.$$

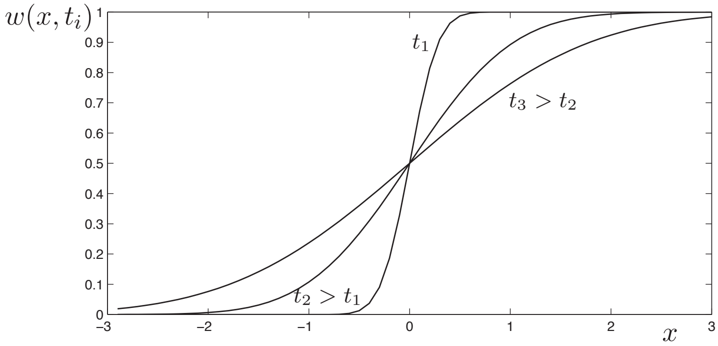


Figure 5.1. Temperature profile on several time levels for a step initial temperature.

Thus, the function

$$G(x,t) = w_x(x,t),$$

where $w(x,t)$ is given by formula (5.3), must solve the diffusion equation as well. By direct differentiation with respect to x we obtain

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}}\mathrm{e}^{-x^2/(4kt)}. \tag{5.5}$$

The function G is called the *heat* (*diffusion*) *kernel* or the *fundamental solution* of the diffusion equation (sometimes we can also meet the terms Green's function, source function, Gaussian, or propagator). Its graph for any fixed $t > 0$ is a "bell-shaped" curve (see Figure 5.2), which has the property that the area below for each t is equal to one:

$$\int_{-\infty}^{+\infty} G(x,t)\,\mathrm{d}x = 1, \quad t > 0.$$

For $t \to 0+$, $G(x,t)$ "approaches" the so called *Dirac distribution* $\delta(x)$.

Remark 5.1. Let us remark that the Dirac distribution can be understood intuitively as a "generalized function" which achieves an infinite value at point 0, is equal to zero at the other points, and $\int_{-\infty}^{+\infty} \delta(x)\,\mathrm{d}x = 1$. (Note that the integral has to be understood in a more general sense than the Riemann integral!) The problem of correct definition of the Dirac distribution and the word "approaches" is a matter of the theory of distributions and goes beyond the scope of this text.

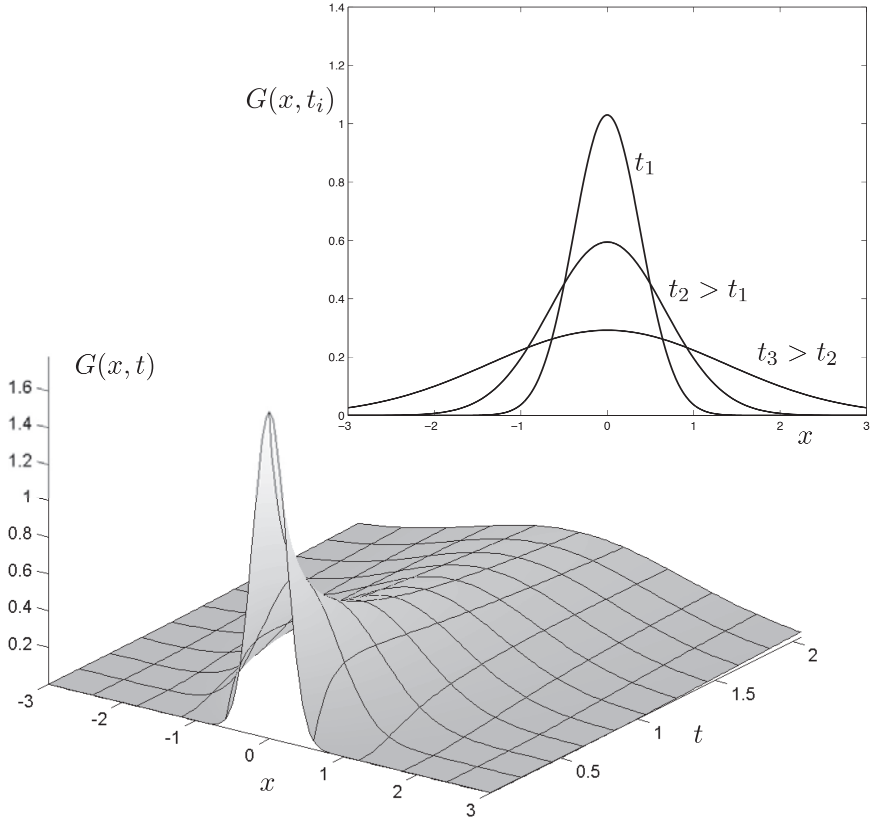


Figure 5.2. Fundamental solution of the diffusion equation (here, with the choice $k = 0.5$).

From the physical point of view, the function $G(x, t)$ describes the distribution of temperature as a reaction to the initial unit point source of heat at the point $x = 0$. Further, we observe that the diffusion equation is invariant with respect to translation. Thus, the shifted diffusion kernel $G(x - y, t)$ also solves the diffusion equation and represents a reaction to the initial unit point source of heat at a fixed, but arbitrary point y. If the initial source is not unit, but has a magnitude $\varphi(y)$, then its contribution at a point x and time t is given by the function $\varphi(y)G(x - y, t)$. The area below the temperature curve is then equal to $\varphi(y)$, where y is the point where the source is located.

Let us suppose now that the initial temperature φ in problem (5.1) represents a continuous distribution of heat sources $\varphi(y)$ at points $y \in \mathbb{R}$. Then we obtain the resulting distribution of temperature as a "sum" of all reactions $\varphi(y)G(x-y,t)$ to particular sources $\varphi(y)$ at all points y. That is,

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{+\infty} \varphi(y) G(x-y,t)\,\mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} \varphi(y) \mathrm{e}^{-\frac{(x-y)^2}{4kt}}\,\mathrm{d}y.
\end{aligned}
\tag{5.6}
$$

We can say that the solution of the Cauchy problem for the diffusion equation is a *convolution* of the corresponding initial condition $\varphi(x)$ and the diffusion kernel $G(x,t)$, i.e.,

$$u(x,t) = (G * \varphi)(x,t).$$

We have just derived intuitively the following basic existence result for the diffusion equation.

Theorem 5.2. *Let φ be a bounded continuous function on $\mathbb{R}$. The Cauchy problem* (5.1) *for the diffusion equation on the real line has a classical solution given by formula* (5.6).

Remark 5.3. It can be shown that the function $u(x,t)$ given by formula (5.6) solves problem (5.1) also in the case that φ is only *piecewise continuous*. Then, at the points of discontinuity, the solution converges for $t \to 0+$ to the arithmetical average of the left and right limits of the function φ, that is,

$$u(x,t) \to \frac{1}{2}\big(\varphi(x-) + \varphi(x+)\big).$$

The decaying character of $G(x-y,t)$ as $|y| \to +\infty$ and x is fixed allows to show that the integral in (5.6) is finite (and expresses the solution of the Cauchy problem (5.1)) also for certain unbounded initial conditions φ.

Concerning the uniqueness, it can be proved that there exists only one *bounded* solution of the Cauchy problem (5.1). In general, without any conditions at infinity, the uniqueness does not hold true (see, e.g., [11]).

Now, let us mention some fundamental properties of the solution of the Cauchy problem for the diffusion equation. Relation (5.6) has an integral form and it can be seen that it cannot be expressed analytically (that is, it cannot be written in terms of elementary functions) for majority of initial conditions.

If we are interested in the form of a solution in a particular case, we have to integrate numerically.

Further, let us notice that, for $t > 0$, the solution $u(x,t)$ is non-zero at an arbitrary point x even if the initial condition φ is non-zero only on a "small interval". It would mean that the *heat propagates at infinite speed*, and also the *diffusion has infinite speed*. But this phenomenon does not correspond to reality and reflects the fact that the diffusion equation is only an approximate model of the real process. On the other hand, for small t, the influence of the initial distribution trails away quickly with growing distance. Thus, we can say the model is precise enough to be applicable from the practical point of view.

Another property of solution (5.6) is its smoothness. Regardless of the smoothness of the function φ, the solution u is of the class C^∞ for $t > 0$ (that is, infinitely times continuously differentiable in both variables).

Example 5.4. Let us solve the problem

$$\begin{cases} u_t = ku_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x) = \begin{cases} 1 & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases} \end{cases} \tag{5.7}$$

It is a Cauchy problem for the diffusion equation with a piecewise continuous (possibly *non-smooth*) initial condition. The solution can be determined by substituting the initial condition into formula (5.6):

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-1}^{1} \mathrm{e}^{-\frac{(x-y)^2}{4kt}}\, \mathrm{d}y.$$

If we introduce $p = (x-y)/\sqrt{4kt}$, we obtain the expression

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{4kt}}}^{\frac{x+1}{\sqrt{4kt}}} \mathrm{e}^{-p^2}\, \mathrm{d}p$$

or

$$u(x,t) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{x+1}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}}\right)\right).$$

◇

Remark 5.5. The graph of the solution from Example 5.4 is sketched in Figure 5.3 (for $k = 2$). Let us mention its basic features. The initial distribution of temperature was a piecewise continuous function, however, it is immediately completely smoothened. After an arbitrarily small time, the solution is non-zero on the whole real line, although the initial condition is non-zero only on the interval $(-1, 1)$. Further, it is evident that the solution achieves its maximal value at time $t = 0$ and, with growing time, it is being "spread".

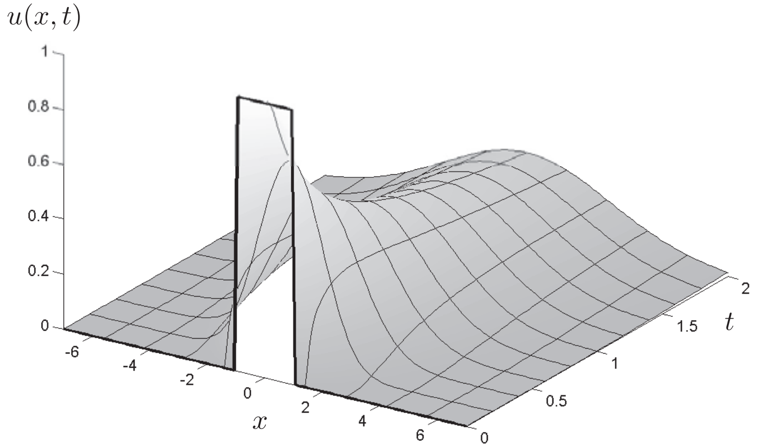


Figure 5.3. Solution of Example 5.4 with $k = 2$.

Example 5.6 (Stavroulakis, Tersian [20]). Let us solve the problem

$$\begin{cases} u_t = k u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \mathrm{e}^{-x}. \end{cases} \tag{5.8}$$

Let us observe that the given *initial condition is not bounded* on $\mathbb{R}$ (cf. Remark 5.3). If we use formula (5.6), we obtain the solution in the form

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y)^2}{4kt}} \mathrm{e}^{-y}\, \mathrm{d}y. \tag{5.9}$$

The integral on the right-hand side can be calculated and we obtain

$$u(x,t) = \mathrm{e}^{kt-x} \tag{5.10}$$

(cf. Exercise 2 in Section 5.3). In this case, the solution does not decrease with growing time, but it propagates in the direction of the positive half-axis x. ◇

5.2 Diffusion Equation with Sources

Let us consider the Cauchy problem for the diffusion equation with nonzero right-hand side

$$\begin{cases} u_t - ku_{xx} = f(x,t), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{5.11}$$

The following basic existence result holds true.

Theorem 5.7. *Let $f = f(x,t)$ and $\varphi = \varphi(x)$ be bounded and continuous functions. The Cauchy problem for the nonhomogeneous diffusion equation* (5.11) *has a classical solution given by the formula*

$$u(x,t) = \int_{-\infty}^{+\infty} G(x-y,t)\varphi(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{+\infty} G(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s, \tag{5.12}$$

where G is the diffusion kernel.

Idea of proof. First, we derive formula (5.12) using the *operator method* (see Section 4.4) based on the analogue with the solution of ODE

$$\frac{\mathrm{d}v}{\mathrm{d}t} + Av(t) = f(t), \quad v(0) = \varphi, \tag{5.13}$$

where A is a constant and $\varphi \in \mathbb{R}$. We easily find out that the corresponding solution has the form

$$v(t) = S(t)\varphi + \int_0^t S(t-s)f(s)\,\mathrm{d}s, \tag{5.14}$$

where $S(t) = \mathrm{e}^{-tA}$.

Now, we turn back to the original diffusion problem (5.11). The solution of the homogeneous diffusion equation can be written in the form

$$u(x,t) = \int_{-\infty}^{+\infty} G(x-y,t)\varphi(y)\,\mathrm{d}y.$$

Similar to Section 4.4, we set

$$u(x,t) = \mathcal{S}(t)\varphi(x), \quad \text{i.e.,} \quad \mathcal{S}(t)\varphi(x) = \int_{-\infty}^{+\infty} G(x-y,t)\varphi(y)\,\mathrm{d}y. \tag{5.15}$$

The operator $\mathcal{S}(t)$, called the *source operator*, transforms the function φ into a solution of the homogeneous diffusion equation and hence is an obvious analogue of the function $S(t)$. If we use this analogue, we can expect that the solution of the nonhomogeneous diffusion equation will have the form (in accordance with relation (5.14))

$$u(x,t) = \mathcal{S}(t)\varphi(x) + \int_0^t \mathcal{S}(t-s)f(x,s)\,\mathrm{d}s,$$

which is (after substituting for $\mathcal{S}(t)$) the derived formula (5.12):

$$u(x,t) = \int_{-\infty}^{+\infty} G(x-y,t)\varphi(y)\,\mathrm{d}y + \int_0^t \int_{-\infty}^{+\infty} G(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Now, the only point left is to verify that (5.12) really solves problem (5.11). For simplicity, let us assume $\varphi(x) \equiv 0$ and consider only the influence of the right-hand side f. First, we verify that the solution fulfills the equation. We use the rule for differentiation of the integral with respect to a parameter t, thus obtaining

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{+\infty} G(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^t \int_{-\infty}^{+\infty} \frac{\partial G}{\partial t}(x-y,t-s)f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\qquad + \lim_{s\to t}\int_{-\infty}^{+\infty} G(x-y,t-s)f(y,s)\,\mathrm{d}y. \end{aligned}$$

The reader should notice that the exchange of differentiation and integration is not always possible, in particular, we have to be careful of the singularities of the function $G(x-y,t-s)$ at the time $t=s$! If, in the first integral, we use the fact that G solves the diffusion equation and substitute $s = t - \epsilon$ in the second term, we obtain

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \int_0^t \int_{-\infty}^{+\infty} k \frac{\partial^2 G}{\partial x^2}(x-y, t-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} G(x-y,\epsilon) f(y, t-\epsilon)\,\mathrm{d}y.\end{aligned} \tag{5.16}$$

Now, we can change the order of integration and differentiation in the first term on the right-hand side. Since for $\epsilon \to 0$ the function $G(x-y,\epsilon)$ converges to the Dirac distribution at the point x (as follows from the theory of distributions, see, e.g., [10]) and $f(y, t-\epsilon)$ converges to $f(y,t)$ (due to the continuity of f), we can write

$$\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} G(x-y,\epsilon) f(y,t-\epsilon)\,\mathrm{d}y = \int_{-\infty}^{+\infty} \delta(x-y) f(y,t)\,\mathrm{d}y = f(x,t).$$

Thus, equality (5.16) takes the form

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= k \frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{+\infty} G(x-y, t-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s + f(x,t) \\ &= k \frac{\partial^2 u}{\partial x^2} + f(x,t),\end{aligned}$$

which is exactly the nonhomogeneous diffusion equation of problem (5.11).

Further, we have to verify the initial condition. Due to the properties of the diffusion kernel G, the first term in (5.12) converges for $t \to 0+$ to the initial condition $\varphi(x)$. The second term is an integral from 0 to 0, thus

$$\lim_{t \to 0+} u(x,t) = \varphi(x) + \int_0^0 \cdots = \varphi(x),$$

which we wanted to prove. □

If we substitute the concrete form of diffusion kernel (5.5) into expression (5.12), we obtain the solution of the Cauchy problem for the nonhomogeneous diffusion equation (5.11) in the form

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,\mathrm{d}y \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \mathrm{e}^{-\frac{(x-y)^2}{4k(t-s)}} f(y,s)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned} \tag{5.17}
$$

Remark 5.8. The reader should notice that above we have used formally some assertions like the derivative of the integral with respect to the parameter, passage to the limit under the integral sign, properties of the Dirac distribution, etc., without checking carefully the assumptions. Similarly to the case of Theorem 5.2, the existence result in Theorem 5.7 still holds if φ and f are more general functions.

5.3 Exercises

1. Verify that the function

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \mathrm{e}^{-x^2/4kt}$$

solves the diffusion equation $u_t = ku_{xx}$ on the domain $-\infty < x < +\infty$, $t > 0$. Observe how the diffusion parameter k influences the solution.

2. Show that the solution from Example 5.6 given by (5.9) assumes the simple form of (5.10).

Here, use $\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y)^2}{4kt}} \mathrm{e}^{-y}\,\mathrm{d}y = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(y-x+2kt)^2}{4kt}} \mathrm{e}^{kt-x}\,\mathrm{d}y$ and the substitution $s = \frac{y-x+2kt}{\sqrt{4kt}}$.

3. For which values of a and b is the function $u(x,t) = \mathrm{e}^{at} \sin bx$ a solution of the diffusion equation $u_t - u_{xx} = 0$?

$[a + b^2 = 0]$

4. Suppose $|\varphi(x)| \le M$ for all $x \in \mathbb{R}$ (M is a positive constant). Use the fact that $|\int f| \le \int |f|$ and show that the solution of the Cauchy problem (5.1) for the diffusion equation satisfies $|u(x,t)| \le M$ for all $x \in \mathbb{R}$, $t > 0$.

5. Verify that

$$\int_{-\infty}^{+\infty} G(x,t)\,\mathrm{d}x = 1, \quad t > 0.$$

6. Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition

$$\varphi(x) = 1 \text{ for } x > 0, \quad \varphi(x) = 3 \text{ for } x < 0.$$

Write the solution using the error function $\operatorname{erf}(x)$.

$[u(x,t) = 2 - \operatorname{erf}(\frac{x}{\sqrt{4kt}})]$

7. Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $\varphi(x) = \mathrm{e}^{3x}$.

$[u(x,t) = \mathrm{e}^{3x+9kt}]$

8. Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition

$$\varphi(x) = \mathrm{e}^{-x} \text{ for } x > 0, \quad \varphi(x) = 0 \text{ for } x < 0.$$

$[u(x,t) = \frac{1}{2}\mathrm{e}^{kt-x}(1 - \operatorname{erf}(\frac{2kt-x}{\sqrt{4kt}}))]$

9. Solve the diffusion equation $u_t = u_{xx}$ with the initial condition

$$\varphi(x) = \begin{cases} 1-x, & 0 \le x \le 1, \\ 1+x, & -1 \le x \le 0, \\ 0, & |x| \ge 1. \end{cases}$$

Show that $u(x,t) \to 0$ as $t \to +\infty$ for every x.

10. Using the substitution $u(x,t) = \mathrm{e}^{-bt}v(x,t)$, solve the diffusion equation

$$u_t - ku_{xx} + bu = 0, \quad u(x,0) = \varphi(x).$$

Here, b is a positive constant representing dissipation.

11. Using the substitution $y = x - Vt$, solve the heat equation

$$u_t - ku_{xx} + Vu_x = 0, \quad u(x,0) = \varphi(x).$$

Here, V is a positive constant representing convection.

$[u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int\limits_{-\infty}^{+\infty} \mathrm{e}^{-(x-Vt-z)^2/(4kt)}\varphi(z)\,\mathrm{d}z]$

12. Show that the equation $u_t = k(t)u_{xx}$ can be transformed into a diffusion equation by changing the time variable t into

$$\tau = \int\limits_0^t k(\eta)\,\mathrm{d}\eta.$$

Similarly, show that the equation $u_t = ku_{xx} - b(t)u_x$ can be transformed into a diffusion equation by changing the spatial variable x into

$$\xi = x - \int_0^t b(\eta)\,\mathrm{d}\eta.$$

13. Find the solution of the problem

$$u_t - ku_{xx} = \sin x,\ x \in \mathbb{R},\ t > 0, \quad u(x,0) = 0.$$

$[u(x,t) = \frac{1}{k}(1 - \mathrm{e}^{-kt})\sin x]$

14. Show that the transport equation with diffusion and decay

$$u_t = ku_{xx} - cu_x - \lambda u$$

can be transformed into a diffusion equation by a substitution

$$u(x,t) = w(x,t)\mathrm{e}^{\alpha x - \beta t}$$

with $\alpha = \frac{c}{2k}$ and $\beta = \lambda + \frac{c^2}{4k}$.

Chapter 6

Laplace and Poisson Equations in Two Dimensions

In the previous chapters we have met the basic representatives of hyperbolic equations (the wave equation) and of parabolic equations (the diffusion equation). This chapter is devoted to the simplest elliptic equation in two dimensions, that is, the *Laplace equation*

$$\Delta u = u_{xx} + u_{yy} = 0. \tag{6.1}$$

As we have stated in Section 1.3.7, the Laplace equation can be understood as a stationary diffusion or stationary wave and its solutions are so called *harmonic functions*. A nonhomogeneous analogue of the Laplace equation with a given function f is so called *Poisson equation*

$$\Delta u = f. \tag{6.2}$$

6.1 Invariance of the Laplace Operator

The Laplace operator Δ (also called the *Laplacian*) is invariant with respect to any transformation consisting of translations and rotations.

The *translation* in plane by a vector (a, b) is given by the transformation

$$x' = x + a, \qquad y' = y + b.$$

Obviously, $u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$.

The *rotation* in plane through an angle α is given by the transformation

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha, \\ y' &= -x\sin\alpha + y\cos\alpha. \end{aligned}$$

Using the chain rule, we derive

$$\begin{aligned} u_x &= u_{x'}\cos\alpha - u_{y'}\sin\alpha, \\ u_y &= u_{x'}\sin\alpha + u_{y'}\cos\alpha, \\ u_{xx} &= u_{x'x'}\cos^2\alpha - 2u_{x'y'}\sin\alpha\cos\alpha + u_{y'y'}\sin^2\alpha, \\ u_{yy} &= u_{x'x'}\sin^2\alpha + 2u_{x'y'}\sin\alpha\cos\alpha + u_{y'y'}\cos^2\alpha, \end{aligned}$$

and, summing up, we obtain

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}.$$

For these properties, the Laplace operator is used in modeling of *isotropic* physical phenomena.

6.2 Transformation of the Laplace Operator into Polar Coordinates

The invariance with respect to rotation suggests that the Laplace operator could assume a simple form in *polar coordinates* (Figure 6.1), in particular, in the radially symmetric case. The transformation formulas between the Cartesian and polar coordinates have the form

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and the corresponding Jacobi matrix $\boldsymbol{J}$ and its inverse $\boldsymbol{J}^{-1}$ are

$$\boldsymbol{J} = \begin{pmatrix} x_r & y_r \\ x_\theta & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix},$$

$$\boldsymbol{J}^{-1} = \begin{pmatrix} r_x & \theta_x \\ r_y & \theta_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\dfrac{\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{pmatrix}.$$

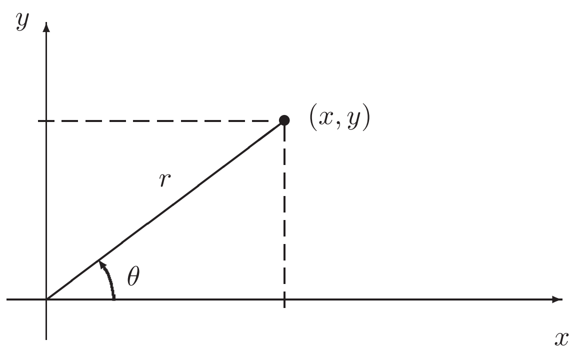


Figure 6.1. Polar coordinates r and θ.

Written formally in symbols, we easily find out by differentiation that

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$
$$= \cos^2\theta\frac{\partial^2}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r},$$
$$\frac{\partial^2}{\partial y^2} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$
$$= \sin^2\theta\frac{\partial^2}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}.$$

Summing these operators, we obtain

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{6.3}$$

6.3 Solutions of Laplace and Poisson Equations in $\mathbb{R}^2$

6.3.1 Laplace Equation

Using the similarity to the wave equation (4.1), we can find a "general solution" of the Laplace equation in the xy-plane. Indeed, the wave and Laplace equations are formally identical provided we set the speed of wave propagation c to be equal to the imaginary unit $\mathrm{i} = \sqrt{-1}$. Thus, according to (4.6), we can conclude that any function of the form

$$u(x, y) = f(x + \mathrm{i}y) + g(x - \mathrm{i}y)$$

solves the Laplace equation $\Delta u = 0$ in two dimensions. Here f and g are arbitrary differentiable functions of a complex variable. Since $x - \mathrm{i}y$ is the complex conjugate number to $x + \mathrm{i}y$, the general solution of the Laplace equation can be written simply as

$$u(x, y) = f(x + \mathrm{i}y).$$

However, further analysis is a subject of the theory of complex functions and exceeds the scope of this text.

In the radially symmetric case (when the functions considered do not depend on the angle θ), the Laplace equation in polar coordinates reduces to the ODE

$$u_{rr} + \frac{1}{r}u_r = 0.$$

Multiplying by $r > 0$, we obtain the equation

$$ru_{rr} + u_r = 0,$$

which is equivalent to

$$(ru_r)_r = 0.$$

This is an ODE that is easy to solve by direct integration:

$$ru_r = c_1$$

and

$$u(r) = c_1 \ln r + c_2. \tag{6.4}$$

Thus, the radially symmetric harmonic functions in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ are the constants and the logarithm. In particular, the logarithm will play an important role in subsequent chapters.

6.3.2 Poisson Equation

The same approach can also be applied to the Poisson equation. Considering again the radially symmetric case (when $f = f(\sqrt{x^2 + y^2})$), this equation reduces to the ODE

$$u_{rr} + \frac{1}{r}u_r = f(r),$$

which is equivalent to

$$ru_{rr} + u_r = r\,f(r) \quad \text{or} \quad (ru_r)_r = r\,f(r).$$

Again, by direct integration, we obtain

$$ru_r = c_1 + \int\limits_0^r s\,f(s)\,\mathrm{d}s$$

and, finally,

$$u(r) = c_1 \ln r + c_2 + \int\limits_0^r \frac{1}{\sigma} \int\limits_0^\sigma s\,f(s)\,\mathrm{d}s\,\mathrm{d}\sigma. \tag{6.5}$$

6.4 Exercises

1. Verify that the given functions solve the two-dimensional Laplace equation.

   (a) $u = x + y$.

   (b) $u = x^2 - y^2$.

   (c) $u = \frac{x}{x^2+y^2}$.

   (d) $u = \ln(x^2 + y^2)$.

   (e) $u = \ln \sqrt{x^2 + y^2}$.

   (f) $u = \mathrm{e}^y \cos x$.

2. Decide whether the following functions satisfy the Laplace equation.

   (a) $u = \frac{y}{x^2+y^2}$.

   (b) $u = \frac{1}{\sqrt{x^2+y^2}}$.

   (c) $u = \arctan(\frac{y}{x})$.

   (d) $u = \arctan(\frac{y}{x})\frac{y}{x^2+y^2}$.

   (e) $u = xy$.

   [yes for a,c,e]

3. Show that if u and v are harmonic and α and β are (real) numbers, then $\alpha u + \beta v$ is harmonic.

4. Give an example of two harmonic functions u and v such that uv is not harmonic.

5. Show that if u and u^2 are both harmonic, then u must be constant.

6. Show that if u, v and u^2+v^2 are harmonic, then u and v must be constants.

7. If $u(x, y)$ is a solution of the Laplace equation, prove that any partial derivative of $u(x, y)$ with respect to one or more Cartesian coordinates (for example, u_x, u_{xx}, u_{xy}) is also a solution.

8. Consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in } \mathbb{R} \times (0, +\infty), \\ u(x, 0) = 0, \quad u_y(x, 0) = \frac{\cos nx}{n^2}. \end{cases}$$

   Show that $u_n(x, y) = \frac{1}{n^3} \sinh ny \cos nx$ is the solution, but

$$\lim_{n\to+\infty} \sup_{(x,y)\in\mathbb{R}\times[0,+\infty)} |u_n(x, y)| \neq 0.$$

9. Functions z^2, z^3, e^z, $\ln z$ of a complex variable $z = x + \mathrm{i}y$ are analytic. Rewrite them in the following way:

$$\begin{aligned} z^2 &= (x^2 - y^2) + \mathrm{i}(2xy), \\ z^3 &= (x^3 - 3xy^2) + \mathrm{i}(3x^2y - y^3), \\ \mathrm{e}^z &= (\mathrm{e}^x \cos y) + \mathrm{i}(\mathrm{e}^x \sin y), \\ \ln z &= (\ln \sqrt{x^2 + y^2}) + \mathrm{i}(\arg z) = \ln r + \mathrm{i}\theta, \end{aligned}$$

and show that all of them (as functions of x and y) satisfy the Laplace equation.

10. Show that the function $u(x, y) = \arctan(y/x)$ satisfies the Laplace equation $u_{xx} + u_{yy} = 0$ for $y > 0$. Using this fact, try to find a solution of the Laplace equation on the domain $y > 0$, that satisfies boundary conditions $u(x, 0) = 1$ for $x > 0$ and $u(x, 0) = -1$ for $x < 0$.

11. Show that $\mathrm{e}^{-\xi y} \sin(\xi x)$, $x \in \mathbb{R}$, $y > 0$ is a solution of the Laplace equation for an arbitrary value of the parameter ξ. Prove that the function

$$u(x, y) = \int_0^{+\infty} c(\xi)\mathrm{e}^{-\xi y} \sin(\xi x)\, \mathrm{d}\xi$$

solves the same equation for an arbitrary function $c(\xi)$ that is bounded and continuous on $[0, +\infty)$. (Assumptions on the function c allow to differentiate under the integral.)

12. Find a radially symmetric solution of the equation $u_{xx} + u_{yy} = 1$ in the disc $x^2 + y^2 < a^2$ with $u(x, y)$ vanishing on the boundary $x^2 + y^2 = a^2$.

$$\left[u(r) = \tfrac{1}{4}(r^2 - a^2)\right]$$

13. Find a radially symmetric solution of the equation $u_{xx} + u_{yy} = 1$ in the annulus $a^2 < x^2 + y^2 < b^2$ with $u(x, y)$ vanishing on both boundary circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$.

$$\left[u(r) = \tfrac{r^2}{4} + \tfrac{b^2 - a^2}{4 \ln \frac{a}{b}} \ln r - \tfrac{b^2 \ln a \ - a^2 \ln b}{4 \ln \frac{a}{b}}\right]$$

14. (a) Show that if $v(x, y)$ is a harmonic function, then $u(x, y) = v(x^2 - y^2, 2xy)$ is also a harmonic function.

    (b) Using transformation into polar coordinates, show that the transformation $(x, y) \mapsto (x^2 - y^2, 2xy)$ maps the first quadrant onto the half-plane $\{y > 0\}$.

Chapter 7

Solutions of Initial Boundary Value Problems for Evolution Equations

7.1 Initial Boundary Value Problems on Half-Line

Let us start with the solutions of the diffusion and wave equations on the whole real line. Notice that if the initial condition for the diffusion equation is an even or odd function, then the solution of the Cauchy problem is also an even or odd function, respectively. The same holds in the case of the Cauchy problem for the wave equation too. (The reader is asked to prove both cases.) We will use this observation in solving the initial boundary value problems for the diffusion and wave equations on the half-line with homogeneous boundary conditions.

7.1.1 Diffusion and Heat Flow on Half-Line

First, let us consider the initial boundary value problem for the heat equation

$$\begin{cases} u_t = ku_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = \varphi(x), \end{cases} \tag{7.1}$$

which describes the temperature distribution in the half-infinite bar with heat insulated lateral surface. The Dirichlet boundary condition corresponds to the fact that the end $x = 0$ is kept at the zero temperature. We will solve this problem using the so called *reflection method*, which is based on the idea of extending the problem to the whole real line in such a way that the boundary condition $u(0,t) = 0$ is fulfilled by itself. In our case, that is, in the case of the homogeneous Dirichlet boundary condition, this means to use the *odd extension* of the initial condition $\varphi(x)$. We define

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x > 0, \\ -\varphi(-x), & x < 0, \end{cases} \qquad \tilde{\varphi}(0) = 0. \tag{7.2}$$

Since an odd initial condition corresponds to an odd solution, we obtain $u(0,t) = 0$ automatically for all $t > 0$ (see Figure 7.1). Let us consider the extended problem

$$\begin{cases} v_t = kv_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ v(x,0) = \tilde{\varphi}(x), \end{cases} \tag{7.3}$$

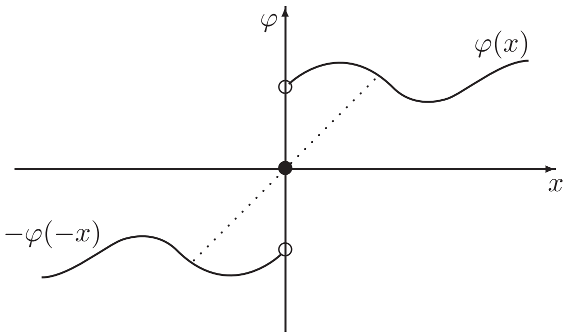


Figure 7.1. The odd extension.

the solution of which can be written in the form

$$v(x,t) = \int_{-\infty}^{+\infty} G(x-y,t)\tilde{\varphi}(y)\,\mathrm{d}y,$$

where G is the diffusion kernel (5.5). If we split the integral into two parts ($y > 0$ and $y < 0$), we obtain

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{0} G(x-y,t)\tilde{\varphi}(y)\,\mathrm{d}y + \int_{0}^{+\infty} G(x-y,t)\tilde{\varphi}(y)\,\mathrm{d}y \\ &= -\int_{-\infty}^{0} G(x-y,t)\varphi(-y)\,\mathrm{d}y + \int_{0}^{+\infty} G(x-y,t)\varphi(y)\,\mathrm{d}y \\ &= -\int_{0}^{+\infty} G(x+y,t)\varphi(y)\,\mathrm{d}y + \int_{0}^{+\infty} G(x-y,t)\varphi(y)\,\mathrm{d}y \\ &= \int_{0}^{+\infty} \big(G(x-y,t) - G(x+y,t)\big)\varphi(y)\,\mathrm{d}y. \end{aligned}$$

The solution u of the original problem (7.1) is then the restriction of the function v to $x > 0$, i.e.

$$u(x,t) = \int_{0}^{+\infty} \big(G(x-y,t) - G(x+y,t)\big)\varphi(y)\,\mathrm{d}y, \qquad x > 0,\ t > 0. \qquad (7.4)$$

Example 7.1. The reader is asked to verify that the function

$$u(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right),$$

illustrated in Figure 7.2 for $k = 1$, solves the initial value problem

$$\begin{cases} u_t = ku_{xx}, & x > 0,\ t > 0, \\ u(x,0) = 1, & u(0,t) = 0. \end{cases} \tag{7.5}$$

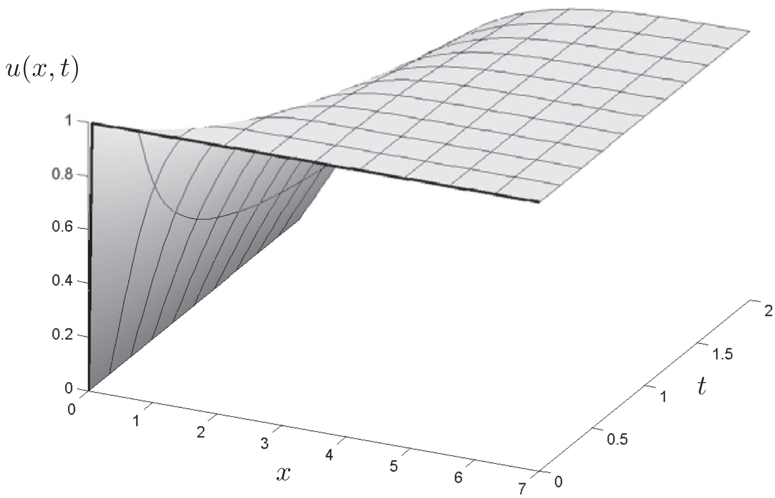


Figure 7.2. Solution of problem (7.5) with $k = 1$.

◇

7.1.2 Wave on the Half-Line

The wave equation on the half-line can be solved in the same way. Let us consider a half-infinite string ($x > 0$) whose end $x = 0$ is fixed. The corresponding Cauchy problem takes the form

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = \varphi(x), & u_t(x,0) = \psi(x). \end{cases} \tag{7.6}$$

We again use the method of the *odd extension* of both initial conditions to the whole real line. We introduce

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x > 0, \\ -\varphi(-x), & x < 0, \end{cases} \qquad \tilde{\psi}(x) = \begin{cases} \psi(x), & x > 0, \\ -\psi(-x), & x < 0, \end{cases} \qquad \begin{matrix} \tilde{\varphi}(0) = 0, \\ \tilde{\psi}(0) = 0, \end{matrix}$$

and consider the extended problem

$$\begin{cases} v_{tt} = c^2 v_{xx}, & x \in \mathbb{R},\ t > 0, \\ v(x,0) = \tilde{\varphi}(x), & v_t(x,0) = \tilde{\psi}(x). \end{cases} \tag{7.7}$$

Its solution is given by d'Alembert's formula

$$v(x,t) = \frac{1}{2}\big(\tilde{\varphi}(x+ct) + \tilde{\varphi}(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(y)\,\mathrm{d}y.$$

The solution u of the original problem (7.6) will be obtained as the restriction of the function v to $x > 0$.

Let us consider first the region $x > ct$. In the case of points (x,t) from this area, the whole bases of their domains of influence lie in the interval $(0,+\infty)$, where $\tilde{\varphi}(x) = \varphi(x)$, $\tilde{\psi}(x) = \psi(x)$. Thus, the solution here is given by the "usual" relation

$$u(x,t) = \frac{1}{2}\big(\varphi(x+ct) + \varphi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y, \quad x > ct. \tag{7.8}$$

However, in the region $0 < x < ct$ we have $\tilde{\varphi}(x-ct) = -\varphi(ct-x)$ and thus

$$u(x,t) = \frac{1}{2}\big(\varphi(x+ct) - \varphi(ct-x)\big) + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,\mathrm{d}y + \frac{1}{2c}\int_{x-ct}^{0} \big(-\psi(-y)\big)\,\mathrm{d}y.$$

If we pass in the last integral from the variable y to $-y$, we obtain the solution in the form

$$u(x,t) = \frac{1}{2}\big(\varphi(ct+x) - \varphi(ct-x)\big) + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,\mathrm{d}y, \quad 0 < x < ct. \tag{7.9}$$

The complete solution of problem (7.6) is given by formulas (7.8) and (7.9).

This result can be interpreted in the following graphical way. In the xt plane, we draw the backward characteristics from the point (x,t). If (x,t) lies in the region $x < ct$, one of the characteristics crosses the t axis earlier than it touches the x axis. Relation (7.9) says that there occurs a reflection at the end $x = 0$ and the solution depends on the values of the function φ at the points $ct \pm x$ and on the values of the function ψ on the short interval between these points. Other values of the function ψ have no influence on the solution at the point (x,t) (see Figure 7.3).

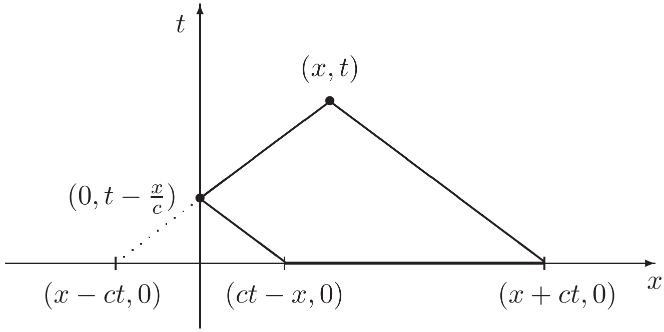


Figure 7.3. Reflection method for the wave equation.

In the case of problems with the homogeneous Neumann boundary condition (see Section 2.3), we would proceed analogously. We would use, however, the method of *even extension*, which uses the fact that even initial conditions correspond to even solutions. The latter then fulfil the condition $u_x(0,t) = 0$ at the point $x = 0$ automatically. The derivation of the corresponding solution formulas is left to the reader.

Example 7.2. Let us consider the initial boundary value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = \mathrm{e}^{-(x-3)^2}, \quad u_t(x,0) = 0. \end{cases} \tag{7.10}$$

Its solution is sketched in Figure 7.4 (here, $c = 2$). Notice the reflection of the initial wave on the boundary line $x = 0$. ◇

Example 7.3. Let us consider the initial boundary value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = \sin x. \end{cases} \tag{7.11}$$

By direct substitution into formulas (7.8) and (7.9) we find that $u(x,t) = \frac{1}{c}\sin x \, \sin ct$ in both regions $0 < x < ct$ and $x > ct$. The graph of the solution is shown in Figure 7.5 for the choice $c = 4$. (We have seen the same problem on the real line in Chapter 4, see Example 4.2 and Figure 4.1. We recall that the solution was called the *standing wave*, for its properties.) ◇

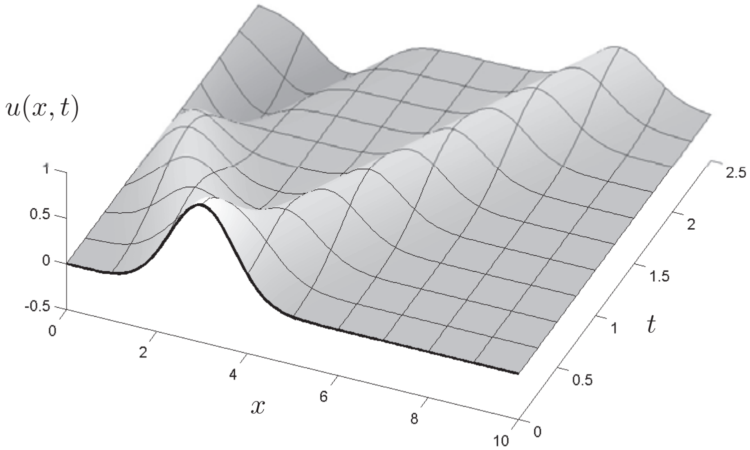


Figure 7.4. Solution of problem (7.10) for $c = 2$.

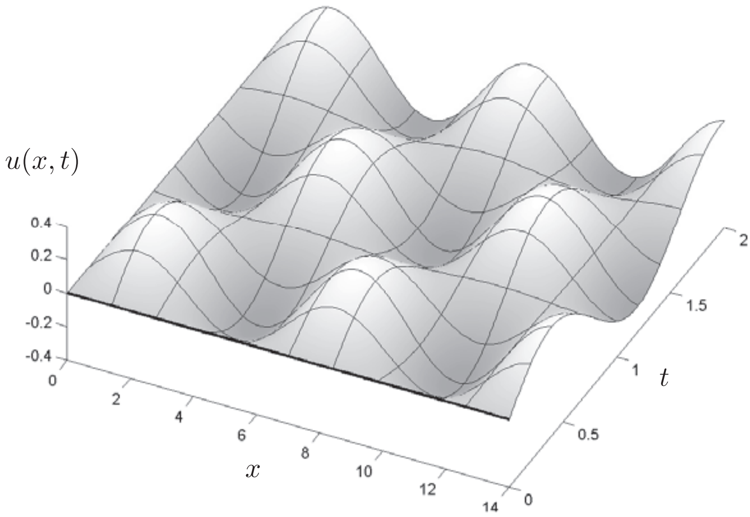


Figure 7.5. Solution of problem (7.11) for $c = 4$.

7.1.3 Problems with Nonhomogeneous Boundary Condition

Let us consider an initial boundary value problem for the diffusion equation on the half-line with a *nonhomogeneous Dirichlet* boundary condition

$$\begin{cases} u_t - ku_{xx} = 0, & x > 0,\ t > 0, \\ u(0,t) = h(t), \\ u(x,0) = \varphi(x). \end{cases} \tag{7.12}$$

In this case we introduce a new function

$$v(x,t) = u(x,t) - h(t),$$

which then solves the problem

$$\begin{cases} v_t - kv_{xx} = -h'(t), & x > 0,\ t > 0, \\ v(0,t) = 0, \\ v(x,0) = \varphi(x) - h(0). \end{cases}$$

We have thus transferred the influence of the boundary condition to the right-hand side of the equation. This means that we solve a *nonhomogeneous* diffusion equation with a *homogeneous* boundary condition. To find a solution of this transformed problem, we can use again the method of odd extension. If the compatibility condition ($\varphi(0) = h(0)$) is not satisfied, we obtain a "generalized" solution which is not continuous at the point $(0,0)$ (however, it is continuous everywhere else).

In the case of a *nonhomogeneous Neumann* boundary condition $u_x(0,t) = h(t)$, we would use the transformation $v(x,t) = u(x,t) - xh(t)$, which results in the nonhomogeneous right-hand side $-xh'(t)$, but it ensures the homogeneity of the boundary condition $v_x(0,t) = 0$. The transformed problem is then solved by the method of even extension. The reader is invited to go through the details.

7.2 Initial Boundary Value Problem on Finite Interval, Fourier Method

In this section, we deal with initial boundary value problems for the wave and diffusion equations on finite intervals. Boundary conditions are now given on both ends of the interval considered. Solving these problems can be approached in various ways. One – apparent – way is to apply the *reflection method* (which was discussed in the previous section) on both ends of the interval. In the case of homogeneous Dirichlet boundary conditions, it means to use the odd extension of the initial conditions with respect to both the end-points and, further, to extend all functions periodically to the whole real line. That is, instead of the original initial condition $\varphi = \varphi(x)$, $x \in (0, l)$, we consider the extended function

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \in (0, l), \\ -\varphi(-x), & x \in (-l, 0), \\ 2l\text{-periodic} & \text{elsewhere.} \end{cases}$$

(In the case of Neumann boundary conditions we use the even extension.) Then we can use the formulas for the solution of the particular problems on the real line. However, after substituting back for the "tilde" initial conditions, the resulting formulas become very complicated!

For example, considering the problem for the *wave equation*, we can obtain the value of the solution at a point (x, t) in the following way: We draw the backwards characteristics from the point (x, t) and reflect them whenever they hit the lines $x = 0$, $x = l$, until we reach the zero time level $t = 0$ (see Figure 7.6 (a)). Thus we obtain a couple of points x_1 and x_2. The solution is then determined by the initial displacements at these points, by the initial velocity in the interval (x_1, x_2), and also by the number of reflections. In general, the characteristic lines divide the space-time domain $(0, l) \times (0, +\infty)$ into "diamond-shaped" areas, and the solution is given by different formulas on each of these areas (see Figure 7.6 (b)).

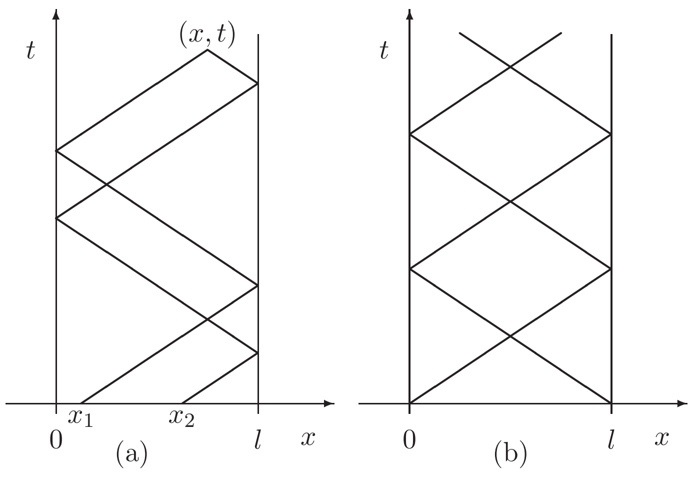


Figure 7.6. Reflection method for the wave equation on finite interval.

In the sequel, we focus on the explanation of another standard method, the so called *Fourier method* (also called the *method of separation of variables*). Its application in solving the initial boundary value problems for the wave equation led to the systematic investigation of trigonometric series (much later called Fourier series).

7.2.1 Dirichlet Boundary Conditions, Wave Equation

First, we will consider the initial boundary value problem that describes vibrations of a string of finite length l, whose end points are fixed in the "zero position", and whose initial displacement and initial velocity are given by functions $\varphi(x)$ and $\psi(x)$, respectively:

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{cases} \tag{7.13}$$

We start with the assumption that there exists a solution of the wave equation in the form

$$u(x,t) = X(x)T(t),$$

where $X = X(x)$ and $T = T(t)$ are real functions of one real variable, their second derivatives exist and are continuous. Variables x and t are thus separated from each other. If we now insert this solution back into the equation, we obtain

$$XT'' = c^2 X'' T$$

and, after dividing by the term $-c^2 XT$ (under the assumption $XT \neq 0$),

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)}.$$

This relation says that the function $-\frac{T''}{c^2 T}$, which depends only on the variable t, is equal to the function $-\frac{X''}{X}$, which depends only on the spatial variable x. This equality must hold for all $t > 0$ and $x \in (0,l)$, and thus

$$-\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda,$$

where λ is a (so far unknown) *constant*. The original PDE is thus transformed into two *separated* ODEs for unknown functions $X(x)$ and $T(t)$:

$$X''(x) + \lambda X(x) = 0, \tag{7.14}$$
$$T''(t) + c^2 \lambda T(t) = 0. \tag{7.15}$$

Further, we are given homogeneous Dirichlet boundary conditions

$$X(0) = X(l) = 0, \tag{7.16}$$

since the setting of problem (7.13) implies $u(0,t) = X(0)T(t) = 0$, $u(l,t) = X(l)T(t) = 0$ for all $t > 0$. First, we will solve the boundary value problem (7.14), (7.16). Since we are evidently not interested in the trivial solution $X(x) \equiv 0$, we exclude the case $\lambda \leq 0$. If $\lambda > 0$, equation (7.14) yields the solution in the form

$$X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x$$

and the boundary conditions (7.16) imply

$$X(0) = C = 0 \qquad \text{and} \qquad X(l) = D\sin\sqrt{\lambda}l = 0.$$

The nontrivial solution can be obtained only in the case

$$\sin\sqrt{\lambda}l = 0,$$

that is,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad n \in \mathbb{N}. \tag{7.17}$$

Every such λ_n then corresponds to a solution

$$X_n(x) = D_n \sin\frac{n\pi x}{l}, \qquad n \in \mathbb{N}, \tag{7.18}$$

where D_n are arbitrary constants. Now, we go back to equation (7.15). For $\lambda = \lambda_n$ its solution assumes the form

$$T_n(t) = A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}, \qquad n \in \mathbb{N}, \tag{7.19}$$

where A_n and B_n are again arbitrary constants. The original PDE of problem (7.13) is then solved by each of functions

$$u_n(x,t) = \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}, \qquad n \in \mathbb{N},$$

which already satisfy the prescribed homogeneous boundary conditions. Let us notice that, instead of A_nD_n and B_nD_n, we write only A_n and B_n, since all these real constants are arbitrary. Since the problem is linear, an *arbitrary finite sum* of the form

$$u(x,t) = \sum_{n=1}^{N}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \tag{7.20}$$

is also a solution. The function given by formula (7.20) satisfies the initial conditions provided

$$\varphi(x) = \sum_{n=1}^{N} A_n \sin\frac{n\pi x}{l}, \tag{7.21}$$

$$\psi(x) = \sum_{n=1}^{N} \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}. \tag{7.22}$$

For arbitrary initial data in this form, problem (7.13) is uniquely solvable and the corresponding solution is given by relation (7.20).

Obviously, conditions (7.21), (7.22) are very restrictive and their satisfaction is hard to ensure. For this reason, we look for a solution of problem (7.13) in the form of an *infinite sum* and we express it in the form of a *Fourier series*

$$u(x,t) = \sum_{n=1}^{+\infty} \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l} \right) \sin\frac{n\pi x}{l}. \tag{7.23}$$

The constants A_n and B_n (or, more precisely, $\frac{n\pi c}{l}B_n$) are then given as the Fourier coefficients of sine expansions of the functions $\varphi(x)$, $\psi(x)$, thus

$$\varphi(x) = \sum_{n=1}^{+\infty} A_n \sin\frac{n\pi x}{l},$$

$$\psi(x) = \sum_{n=1}^{+\infty} \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}.$$

In other words, the solution of the initial boundary value problem for the wave equation can be expressed at every time t in the form of a Fourier sine series in the variable x, provided we are able to express the initial conditions $\varphi(x)$, $\psi(x)$ in this way. It can be seen that such an expansion makes sense for a sufficiently wide class of functions. In such cases, we use orthogonality of the functions $\sin\frac{n\pi x}{l}$, $n = 1, 2, \ldots$, to calculate the coefficients A_n, B_n. We obtain the following expressions:

$$A_n = \frac{2}{l}\int_0^l \varphi(x) \sin\frac{n\pi x}{l}\,\mathrm{d}x,$$

$$B_n = \frac{l}{n\pi c}\frac{2}{l}\int_0^l \psi(x) \sin\frac{n\pi x}{l}\,\mathrm{d}x = \frac{2}{n\pi c}\int_0^l \psi(x) \sin\frac{n\pi x}{l}\,\mathrm{d}x.$$

Remark 7.4. To ensure that this solution of problem (7.13) is mathematically correct, it is necessary to prove that the series (7.23) converges and that it really solves (7.13). It depends on the properties of the functions φ, ψ and on the kind of the required convergence whether, for example, we can exchange the order of the derivative and the infinite sum. These issues are by no means trivial and it is the role of the Theory of Fourier Series to provide the answers to these delicate questions (see, e.g., [13], [18]). They lie beyond the scope of this text and we will not deal with them. In what follows, we automatically assume that the formal calculations may be performed.

Remark 7.5. Coefficients in front of the time variable in the arguments of trigonometric functions in expression (7.23) (that is, the values $n\pi c/l$) are called *frequencies*. If we go back to the string which is described by problem (7.13), the corresponding frequencies take the form

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad \text{for } n = 1, 2, 3, \dots$$

The "fundamental" tone of the string corresponds to the least of these values: $\pi\sqrt{T}/(l\sqrt{\rho})$. Higher (aliquot) tones are then exactly the integer multiples of this basic tone. The discovery that musical tones can be described in this easy mathematical way was made by Euler in 1749.

Example 7.6. Let us solve the initial boundary value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad x \in (0, \pi),\ t > 0, \\ u(0, t) = u(\pi, t) = 0, \\ u(x, 0) = \sin 2x, \quad u_t(x, 0) = 0. \end{cases} \tag{7.24}$$

Using the above relations, we can write the solution as

$$u(x, t) = \sum_{n=1}^{+\infty} (A_n \cos nct + B_n \sin nct) \sin nx.$$

The zero initial velocity implies zero "sine coefficients" $B_n = 0$ for all $n \in \mathbb{N}$. The initial displacement determines the "cosine coefficients" A_n:

$$u(x, 0) = \sin 2x = \sum_{n=1}^{+\infty} A_n \sin nx.$$

Since the functions $\sin nx$ form a complete orthogonal set on $(0, \pi)$, we easily obtain

$$A_2 = 1, \quad A_n = 0 \quad \text{for } n \in \mathbb{N} \setminus \{2\}.$$

Hence, we can conclude that the solution of (7.24) reduces to

$$u(x,t) = \cos 2ct\ \sin 2x.$$

◇

Example 7.7. Similarly to the previous example, we can show that the function

$$u(x,t) = \sum_{n=1}^{+\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \tag{7.25}$$

with

$$A_n = \frac{2}{l} \int_0^l \mathrm{e}^{-(x-l/2)^2} \sin \frac{n\pi x}{l}\, \mathrm{d}x$$

solves the initial boundary value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad x \in (0,l),\ t > 0, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \mathrm{e}^{-(x-l/2)^2}, \quad u_t(x,0) = 0. \end{cases} \tag{7.26}$$

Figure 7.7 sketches a partial sum of the series (7.25) up to the term $n = 15$ with values $c = 6$ and $l = 8$, which is the approximate solution of problem (7.26).

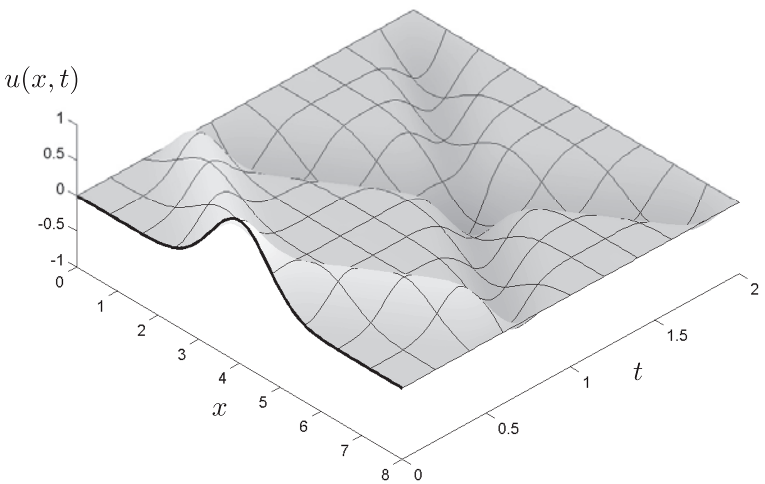


Figure 7.7. Graphic illustration of the solution of problem (7.26).

◇

7.2.2 Dirichlet Boundary Conditions, Diffusion Equation

Now let us consider an initial boundary value problem for the heat equation

$$\begin{cases} u_t = ku_{xx}, \quad 0 < x < l,\ t > 0, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x), \end{cases} \tag{7.27}$$

which can model a thin bar whose ends are kept at a constant temperature $u = 0$. The distribution of the temperature in the bar at time $t = 0$ is given by a function $\varphi = \varphi(x)$. To find the distribution of the temperature at time $t > 0$ means to find a solution $u = u(x,t)$ of problem (7.27). The same problem describes the diffusion process of a substance in a tube which is constructed in such a way that any substance that reaches the ends of the tube flows immediately out.

Let us proceed in the same way as in the previous example. First, we look for such a solution of the equation that satisfies only the homogeneous boundary conditions and has the form

$$u(x,t) = X(x)T(t).$$

After substitution into the heat equation and a simple rearrangement, we obtain

$$-\frac{T'(t)}{kT(t)} = -\frac{X''(x)}{X(x)} = \lambda,$$

where λ is a constant. Thus, we have transformed the original equation into a couple of ODEs

$$T' + \lambda kT = 0, \tag{7.28}$$

$$X'' + \lambda X = 0. \tag{7.29}$$

Now, we add the homogeneous boundary conditions $X(0) = X(l) = 0$ to equation (7.29) and look for such values of λ for which this problem has a nontrivial solution. Just as in the case of the wave equation, we obtain

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad n \in \mathbb{N}, \tag{7.30}$$

and the corresponding solutions have the form

$$X_n(x) = D_n \sin\frac{n\pi x}{l}, \qquad n \in \mathbb{N}, \tag{7.31}$$

where D_n are arbitrary constants. If we go back to equation (7.28) and substitute $\lambda = \lambda_n$, we obtain

$$T_n(t) = A_n \mathrm{e}^{-(n\pi/l)^2 kt}, \quad n \in \mathbb{N}. \tag{7.32}$$

The solution u of the original problem (7.27) can be then expressed in the form of an infinite Fourier series

$$u(x,t) = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l} \tag{7.33}$$

under the assumption that the initial condition is also expandable into the corresponding series, that is,

$$\varphi(x) = \sum_{n=1}^{+\infty} A_n \sin \frac{n\pi x}{l}. \tag{7.34}$$

From the physical point of view, expression (7.33) says that, with growing time, heat is dissipated by the ends of the bar and the temperature in the whole bar decreases to zero.

Example 7.8 (Logan [15]). Let $\varphi(x) = 10x^3(l - x)$, $x \in (0, l)$, in (7.27). Then Figure 7.8 sketches a partial sum of the series (7.33) up to the term $n = 42$ with values $l = 1$ and $k = 1$.

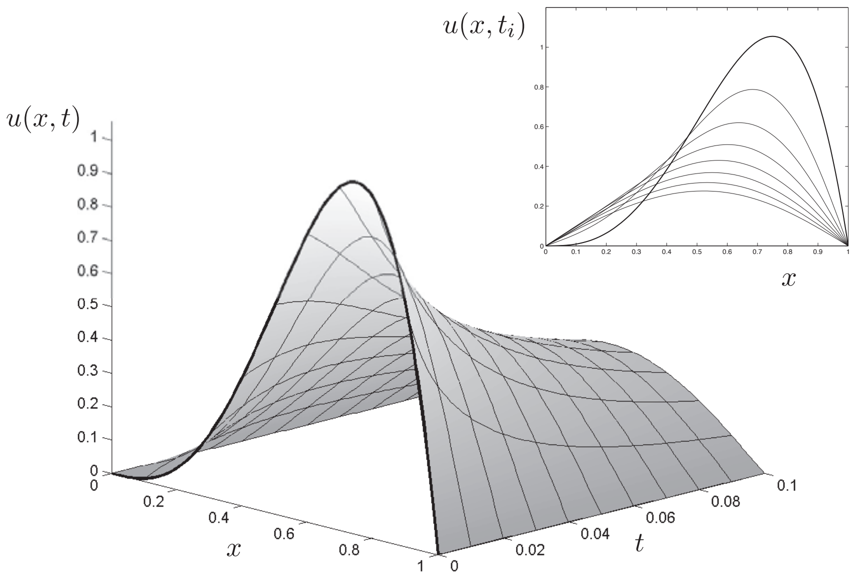


Figure 7.8. Graphic illustration of the solution of Example 7.8.

◇

7.2.3 Neumann Boundary Conditions

The same method can be used also in the case of homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(l,t) = 0.$$

In the case of the wave equation, these conditions correspond to a string with free ends. If we model the diffusion process, then they describe a tube with isolated ends (nothing can penetrate in or out and the flow across the boundary is zero). Similarly, when modeling the heat flow, the homogeneous Neumann conditions represent a totally isolated tube (again, the heat flux across the boundary is zero).

Let us consider a problem for the wave or diffusion equation in the interval $(0,l)$. Separation of variables leads this time to the ODE

$$X'' + \lambda X = 0, \quad X'(0) = X'(l) = 0,$$

which has a nontrivial solution for $\lambda > 0$ and for $\lambda = 0$. In particular, we obtain

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n \in \mathbb{N} \cup \{0\},$$

$$X_n(x) = C_n \cos\frac{n\pi x}{l}, \quad n \in \mathbb{N} \cup \{0\}.$$

The initial boundary value problem for the diffusion equation with Neumann boundary conditions has then a solution in the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-(n\pi/l)^2 kt} \cos\frac{n\pi x}{l}, \tag{7.35}$$

provided the initial condition is expandable into the Fourier cosine series

$$\varphi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{+\infty} A_n \cos\frac{n\pi x}{l}.$$

In the case of an initial boundary value problem for the wave equation with homogeneous Neumann boundary conditions on $(0,l)$ we obtain

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{+\infty} \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right) \cos\frac{n\pi x}{l}, \tag{7.36}$$

where the initial data have to satisfy

$$\begin{aligned}\varphi(x) &= \frac{1}{2}A_0 + \sum_{n=1}^{+\infty} A_n \cos\frac{n\pi x}{l},\\ \psi(x) &= \frac{1}{2}B_0 + \sum_{n=1}^{+\infty} \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}.\end{aligned}$$

In both cases, the reader is asked to carry out detailed calculations.

Example 7.9. Let A_n be the cosine Fourier coefficients of the function $\varphi(x) = \frac{1}{2} - 200x^4(l-x)^4$ in the interval $(0,l)$. Then the function

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-(n\pi/l)^2 kt} \cos\frac{n\pi x}{l} \tag{7.37}$$

solves the initial boundary value problem

$$\begin{cases} u_t = ku_{xx}, \quad x \in (0,l),\ t > 0, \\ u_x(0,t) = u_x(l,t) = 0, \\ u(x,0) = \frac{1}{2} - 200x^4(l-x)^4. \end{cases} \tag{7.38}$$

Figure 7.9 sketches a partial sum of the series (7.37) up to the term $n = 15$ with values $l = 1$ and $k = 1$.

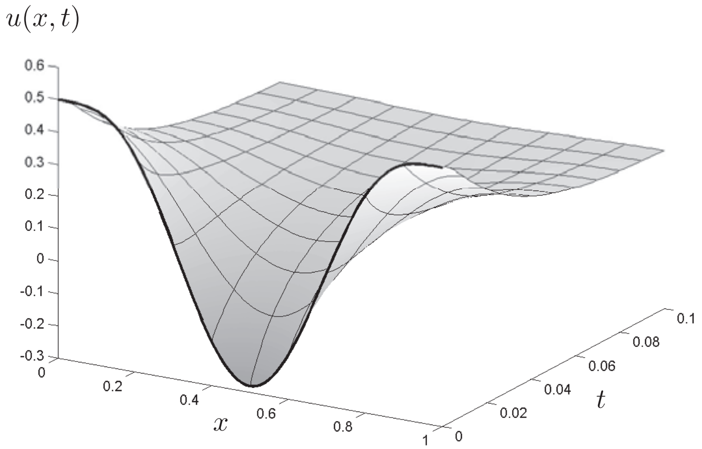


Figure 7.9. Graphic illustration of the solution of problem (7.38).

◇

7.2.4 Robin Boundary Conditions

Together with the wave equation, the Robin boundary conditions describe a string whose ends are held by springs (obeying Hooke's law) which pull it back to the equilibrium. In the case of the heat flow in a bar, these boundary conditions model the heat transfer between the bar ends and the surrounding media.

Let us consider the following modeling problem illustrated in Figure 7.10. Let us take a vertical bar of unit length, whose upper end is kept at zero temperature while the lower end is immersed into a reservoir with water of zero temperature. The heat convection proceeding between the lower end and water is described by the law $u_x(1,t) = -hu(1,t)$. The constant $h > 0$ corresponds to the heat transfer coefficient. Let us suppose that the initial temperature of the bar is given by a function $u(x,0) = x$. To look for the distribution of the temperature in the bar means to solve the initial boundary value problem for the heat equation with mixed boundary conditions (Dirichlet and Robin boundary conditions):

$$\begin{cases} u_t = ku_{xx}, \quad 0 < x < 1, \ t > 0, \\ u(0,t) = 0, \\ u_x(1,t) + hu(1,t) = 0, \\ u(x,0) = x. \end{cases} \tag{7.39}$$

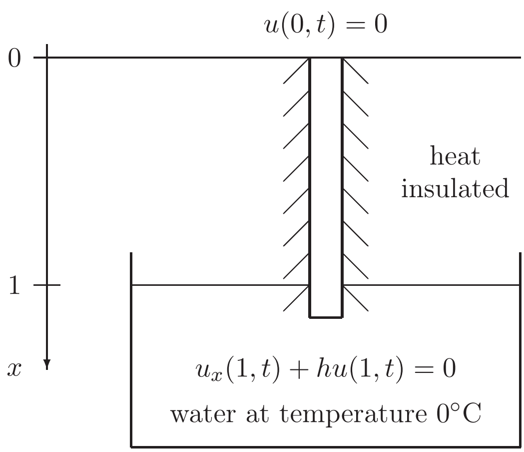


Figure 7.10. Schematic illustration of problem (7.39).

Searching for the solution, we proceed in the same way as in the previous examples. In the first step, we separate the variables, that means, we consider a solution in the form $u(x,t) = X(x)T(t)$, and after substitution into the equation, we obtain

$$-\frac{T'}{kT} = -\frac{X''}{X} = \lambda,$$

where λ is a so far unknown constant. Thus we have transformed the original equation into a couple of independent ODEs

$$T' + \lambda k T = 0, \tag{7.40}$$

$$X'' + \lambda X = 0. \tag{7.41}$$

Moreover, the function $X(x)$ must satisfy the boundary conditions

$$X(0) = 0, \quad X'(1) + hX(1) = 0.$$

By a simple discussion, we exclude the values $\lambda \leq 0$ that lead only to the trivial solution $X(x) \equiv 0$. Thus, let us consider $\lambda = \mu^2 > 0$. Then

$$X(x) = C \cos \mu x + D \sin \mu x,$$

and after substituting into the boundary conditions we obtain

$$C = 0, \qquad D\mu \cos \mu + hD \sin \mu = 0.$$

Since we look for the nontrivial solution, i.e. $D \neq 0$, the last equality can be written in the form

$$\tan \mu = -\frac{\mu}{h}. \tag{7.42}$$

To find the roots of the transcendent equation (7.42) means to find the intersections of the graphs of functions $\tan \mu$ and $-\frac{\mu}{h}$ (see Figure 7.11).

It is evident that there exists an infinite sequence of positive values μ_n, $n \in \mathbb{N}$, such that the corresponding solutions assume the form

$$X_n(x) = D_n \sin \mu_n x.$$

The following table specifies the first five approximate values μ_n for $h = 1$:

n	1	2	3	4	5
μ_n	2.02	4.91	7.98	11.08	14.20

If we go back to equation (7.40), we obtain

$$T_n(t) = A_n \mathrm{e}^{-k\mu_n^2 t}$$

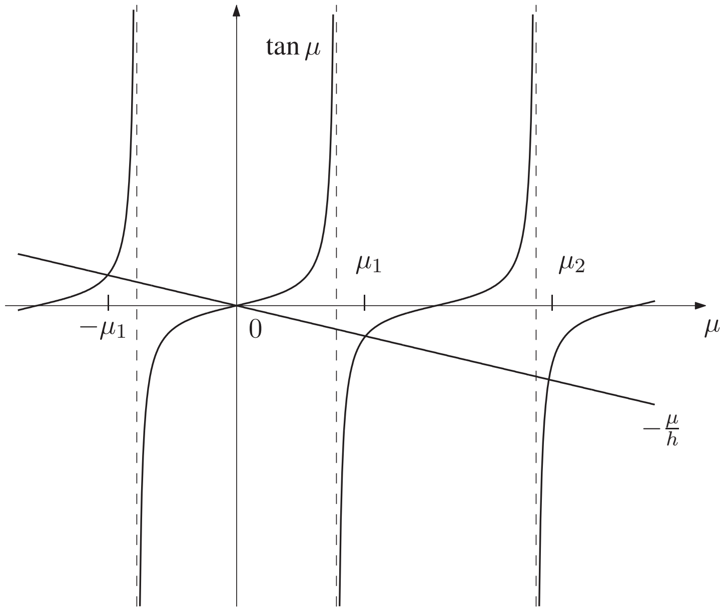


Figure 7.11. Intersections of graphs of functions $\tan\mu$ and $-\frac{\mu}{h}$.

and hence

$$u_n(x,t) = A_n \mathrm{e}^{-k\mu_n^2 t} \sin \mu_n x.$$

The final result of the original problem is then searched in the form of the Fourier series

$$u(x,t) = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-k\mu_n^2 t} \sin \mu_n x. \tag{7.43}$$

The coefficients A_n shall be determined in such a way that the initial condition holds, that is,

$$u(x,0) = \sum_{n=1}^{+\infty} A_n \sin \mu_n x = x.$$

If we multiply this relation by a function $\sin \mu_m x$ and integrate from 0 to 1, we arrive at

$$\int_0^1 x \sin \mu_m x \,\mathrm{d}x = \sum_{n=1}^{+\infty} A_n \int_0^1 \sin \mu_n x \sin \mu_m x \,\mathrm{d}x.$$

Since

$$\int_0^1 \sin \mu_n x \sin \mu_m x \,\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ \dfrac{\mu_n - \sin \mu_n \cos \mu_n}{2\mu_n}, & n = m, \end{cases}$$

we obtain, after simplification,

$$A_n = \frac{2\mu_n}{\mu_n - \sin \mu_n \cos \mu_n} \int_0^1 x \sin \mu_n x \,\mathrm{d}x.$$

Figure 7.12 sketches a partial sum of the series (7.43) up to the term $n = 55$ with values $h = 1$ and $k = 1$.

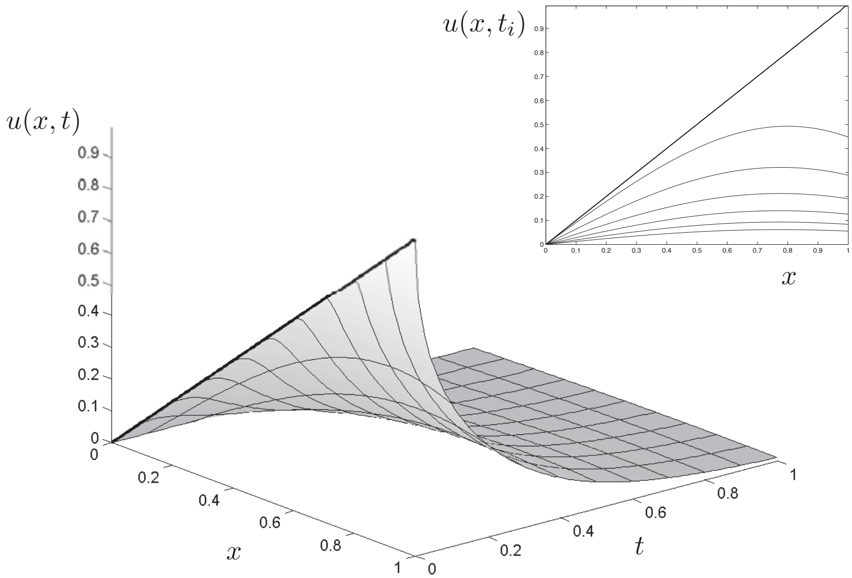


Figure 7.12. Graphic illustration of the solution $u(x,t)$ of problem (7.39) for $h = 1$, $k = 1$.

Remark 7.10. In the previous examples we have seen special types of the boundary value problem for the second order ODE:

$$\begin{cases} -y'' = \lambda y, \quad 0 < x < l, \\ \alpha_0 y(0) + \beta_0 y'(0) = 0, \\ \alpha_1 y(l) + \beta_1 y'(l) = 0. \end{cases} \tag{7.44}$$

We say that any value of the parameter $\lambda \in \mathbb{R}$ for which problem (7.44) has a nontrivial solution, is an *eigenvalue*. The corresponding nontrivial solution is called an *eigenfunction* related to the eigenvalue λ. In the Fourier method, we used some special properties of eigenvalues λ_n and eigenfunctions $y_n(x)$ of (7.44), namely that $y_n(x)$ form a *complete orthogonal set.* It means

$$\int_0^l y_n(x) y_m(x)\,\mathrm{d}x = 0$$

provided $y_m(x)$ and $y_n(x)$ are two eigenfunctions corresponding to two different eigenvalues λ_m and λ_n. Moreover, many functions defined on $(0, l)$ are expandable into Fourier series with respect to the eigenfunctions y_n:

$$f(x) = \sum_{n=1}^{+\infty} F_n y_n(x),$$

where F_n are the Fourier coefficients defined by the relation

$$F_n = \frac{\int_0^l f(x) y_n(x)\,\mathrm{d}x}{\int_0^l y_n^2(x)\,\mathrm{d}x}.$$

It can be seen that these properties are typical not only for sines and cosines, which solve problem (7.44), but also for more general functions which arise as solutions of the so called *Sturm-Liouville boundary value problem.* The reader can find basic facts of Sturm-Liouville theory in Appendix A.

7.2.5 Principle of the Fourier Method

The principle of the Fourier method for initial boundary value problems on a finite interval with a homogeneous equation and homogeneous boundary conditions can be summarized into the following steps:

(i) We search for the solution in the separated form $u(x, t) = X(x)T(t)$.

(ii) We transform the PDE into a couple of ODEs for unknown functions $X(x)$ and $T(t)$.

(iii) Considering the ODE for $X(x)$ with homogeneous boundary conditions we find the eigenvalues λ_n and the eigenfunctions $X_n(x)$ of the corresponding boundary value problem.

(iv) We substitute the eigenvalues λ_n into the ODE for the unknown function $T(t)$ and find its general solution.

(v) We write the solution of the original PDE in the form of an infinite Fourier series $u(x,t) = \sum_{n=1}^{+\infty} X_n(x)T_n(t)$.

(vi) We expand the initial conditions into Fourier series with respect to the system of orthogonal eigenfunctions $X_n(x)$.

(vii) Comparing the expansions of the initial conditions and the solution $u(x,t)$, we calculate the remaining coefficients.

The above mentioned technique is formal and the precise justification of particular steps requires much of mathematical calculus that lies beyond the scope of this text. The interested reader can find the details, e.g., in [24].

7.3 Fourier Method for Nonhomogeneous Problems

7.3.1 Nonhomogeneous Equation

In the case of linear nonhomogeneous equations we usually solve first the homogenous equation and then we use the knowledge of its solution to find a particular solution of nonhomogeneous equation, which reflects the influence of the right-hand side (let us recall, e.g., the method of variation of parameters for ODEs). In the Fourier method for initial boundary value problems for nonhomogeneous PDEs we use the analogue of this approach. As usual, we search the solution in the separated form $u(x,t) = T(t)X(x)$ and, consequently, in the form of the Fourier series

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t)X_n(x).$$

Functions $X_n(x)$ will be again obtained as eigenfunctions of the corresponding homogeneous problem, whereas functions $T_n(t)$ will respect the influence of the right-hand side. We illustrate particular steps on a simple example of the diffusion equation with homogeneous Dirichlet boundary conditions.

Let us solve the initial boundary value problem

$$\begin{cases} u_t - ku_{xx} = f(x,t), & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{7.45}$$

First of all, we determine the eigenvalues λ_n and the system of eigenfunctions $X_n(x)$, $n \in \mathbb{N}$. We obtain them in the same way as in the previous sections, by solving the corresponding homogeneous problem

$$\begin{cases} u_t - ku_{xx} = 0, & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \varphi(x). \end{cases}$$

Let us recall that, in this case, we have

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \sin n\pi x, \quad n \in \mathbb{N}.$$

Now, we expand all data of the original problem (7.45) into a Fourier series with respect to the eigenfunctions $X_n(x)$. That is, for a fixed $t > 0$, we write the right-hand side $f(x,t)$ as

$$f(x,t) = \sum_{n=1}^{+\infty} f_n(t) \sin n\pi x,$$

where the components $f_n(t)$ are the Fourier sine coefficients of $f(x,t)$:

$$f_n(t) = 2 \int_0^1 f(x,t) \sin n\pi x \, \mathrm{d}x.$$

Similarly, we expand the initial condition to

$$\varphi(x) = \sum_{n=1}^{+\infty} \varphi_n \sin n\pi x$$

with coefficients

$$\varphi_n = 2 \int_0^1 \varphi(x) \sin n\pi x \, \mathrm{d}x.$$

Now, we search for the solution of problem (7.45) in the form of a series

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t) \sin n\pi x.$$

Substituting all the above expansions into the equation of (7.45), we obtain

$$\sum_{n=1}^{+\infty} T_n'(t)\sin n\pi x + k\sum_{n=1}^{+\infty}(n\pi)^2 T_n(t)\sin n\pi x = \sum_{n=1}^{+\infty} f_n(t)\sin n\pi x. \tag{7.46}$$

Due to the completeness of the system of functions $\sin n\pi x$, (7.46) is equivalent to the system of ODEs

$$T_n' + k(n\pi)^2 T_n = f_n(t), \quad n \in \mathbb{N}.$$

To fulfil the initial condition

$$u(x,0) = \sum_{n=1}^{+\infty} T_n(0)\sin n\pi x = \sum_{n=1}^{+\infty} \varphi_n \sin n\pi x = \varphi(x),$$

all functions $T_n(t)$ must satisfy $T_n(0) = \varphi_n$. Hence, we easily obtain

$$T_n(t) = \varphi_n \mathrm{e}^{-k(n\pi)^2 t} + \int_0^t \mathrm{e}^{-k(n\pi)^2(t-\tau)} f_n(\tau)\,\mathrm{d}\tau.$$

The resulting solution of problem (7.45) then assumes the form

$$u(x,t) = \sum_{n=1}^{+\infty} \varphi_n \mathrm{e}^{-k(n\pi)^2 t}\sin n\pi x + \sum_{n=1}^{+\infty} \sin n\pi x \int_0^t \mathrm{e}^{-k(n\pi)^2(t-\tau)} f_n(\tau)\,\mathrm{d}\tau.$$

The first sum on the right-hand side represents the solution corresponding to the homogeneous problem with the given initial condition, whereas the other one describes the influence of the right-hand side.

7.3.2 Nonhomogeneous Boundary Conditions and Their Transformation

As we could see in the previous paragraphs, the assumption of homogeneity of boundary conditions is essential for applicability of the Fourier method. That is why we will now study the problem of transforming the initial boundary value problem with nonhomogeneous boundary conditions to a problem with homogeneous conditions.

In the following example we illustrate the simplest model situation.

Example 7.11. Let the heat-insulated bar of length l have its ends kept at *constant* temperatures g_0 and g_1. The initial temperature distribution is given by a function $\varphi = \varphi(x)$. The development of the temperature $u(x,t)$ in the bar is thus the solution of the initial boundary value problem

$$\begin{cases} u_t - ku_{xx} = 0, \quad 0 < x < l, \ t > 0, \\ u(0,t) = g_0, \quad u(l,t) = g_1, \\ u(x,0) = \varphi(x). \end{cases} \tag{7.47}$$

Physical intuition leads us to the hypothesis that, for $t \to +\infty$, the distribution of temperature $u(x,t)$ in the bar converges to the linear function $w(x) = g_0(1 - \frac{x}{l}) + g_1\frac{x}{l}$. It is thus reasonable to assume that the solution of problem (7.47) will have the form

$$u(x,t) = g_0\left(1 - \frac{x}{l}\right) + g_1\frac{x}{l} + U(x,t). \tag{7.48}$$

Here the term $w(x) = g_0(1 - \frac{x}{l}) + g_1\frac{x}{l}$ represents the stationary part (it does not depend on time and satisfies the equation $w_t = kw_{xx}$ and the boundary conditions $w(0) = g_0$, $w(1) = g_1$). The term $U(x,t)$ represents the time-dependent part, which converges to zero for $t \to +\infty$. Due to the fact that the stationary part $w(x)$ is uniquely determined by the constants g_0, g_1, we can – instead of the function $u(x,t)$ – look directly for the unknown function $U(x,t)$. If we insert expression (7.48) into (7.47), we find out that $U(x,t)$ solves the *homogeneous* initial boundary value problem

$$\begin{cases} U_t - kU_{xx} = 0, \quad 0 < x < l, \ t > 0, \\ U(0,t) = 0, \quad U(l,t) = 0, \\ U(x,0) = \varphi(x) - \big(g_0 + \frac{x}{l}(g_1 - g_0)\big), \end{cases}$$

which can be solved by the standard Fourier method. ◇

In practice, however, we have more often to deal with boundary conditions that *depend on time*. We again illustrate their transformation to the homogeneous boundary conditions through an example.

Example 7.12. Let us consider the initial boundary value problem

$$\begin{cases} u_t - ku_{xx} = 0, \quad 0 < x < l, \ t > 0, \\ u(0,t) = g_1(t), \\ u_x(l,t) + hu(l,t) = g_2(t), \\ u(x,0) = \varphi(x). \end{cases} \tag{7.49}$$

We look for a solution of the form

$$u(x,t) = A(t)\left(1 - \frac{x}{l}\right) + B(t)\frac{x}{l} + U(x,t),$$

where functions $A(t)$ and $B(t)$ will be chosen so that the "quasi-stationary" part $w(x,t) = A(t)\left(1-\frac{x}{l}\right) + B(t)\frac{x}{l}$ will satisfy the boundary conditions of problem (7.49). The function $U(x,t)$ must then fulfil the homogeneous boundary conditions. If we substitute the function $w(x,t)$ into the boundary conditions

$$\begin{aligned}
&w(0,t) \equiv A(t) = g_1(t),\\
&w_x(l,t) + hw(l,t) \equiv -\frac{A(t)}{l} + \frac{B(t)}{l} + hB(t) = g_2(t),
\end{aligned}$$

we obtain

$$A(t) = g_1(t), \quad B(t) = \frac{g_1(t) + lg_2(t)}{1+lh},$$

and thus

$$u(x,t) = g_1(t)\left(1 - \frac{x}{l}\right) + \frac{g_1(t) + lg_2(t)}{1+lh}\frac{x}{l} + U(x,t).$$

Substituting this expression into (7.49), we easily find out that $U(x,t)$ must solve the initial boundary value problem

$$\begin{cases}
U_t - kU_{xx} = -w_t, \quad 0 < x < l,\ t > 0,\\
U(0,t) = 0,\\
U_x(l,t) + hU(l,t) = 0,\\
U(x,0) = \varphi(x) - w(x,0).
\end{cases}$$

In this case we have transformed the original problem with a homogeneous equation and nonhomogeneous boundary conditions into a problem with nonzero right-hand side but with homogeneous boundary conditions, which can be solved by the Fourier method. ◇

7.4 Transformation to Simpler Problems

The goal of this section is to point out some transformations that can lead to simpler PDEs.

7.4.1 Lateral Heat Transfer in Bar

Let us consider the initial boundary value problem

$$\begin{cases}
u_t - ku_{xx} + qu = 0, \quad 0 < x < 1,\ t > 0,\\
u(0,t) = u(1,t) = 0,\\
u(x,0) = \varphi(x).
\end{cases} \tag{7.50}$$

This problem describes heat conduction in the bar, where the heat is transferred to the surroundings by the bar surface. The heat-transfer coefficient is denoted by q. We look for a substitution that would simplify the PDE of problem (7.50). We will base our considerations on the physical properties of the model. The temperature $u(x,t)$ develops at every point x_0 in terms of the following two phenomena:

1. the diffusion of the heat along the bar (described by the term $-ku_{xx}$),
2. the heat transfer by the lateral bar surface (described by the term qu).

Let us assume that there is no diffusion along the bar (that is, $k = 0$). Then the development of temperature at every point of the bar is given by

$$u(x_0,t) = u(x_0,0)\mathrm{e}^{-qt}$$

(since for $k = 0$ the function $u(x_0,t)$ solves the ODE $u_t + qu = 0$ with the initial condition $u = u(x_0,0)$). Making use of this fact, we try to express the solution of the initial boundary value problem (7.50) (now, with $k \neq 0$) in the form

$$u(x,t) = \mathrm{e}^{-qt}w(x,t), \tag{7.51}$$

where $w = w(x,t)$ is an unknown function that should describe the heat transfer caused only by the diffusion process. If we substitute (7.51) into (7.50), we obtain the following problem for the required function $w(x,t)$:

$$\begin{cases} w_t - kw_{xx} = 0, \quad 0 < x < 1,\ t > 0, \\ w(0,t) = w(1,t) = 0, \\ w(x,0) = \varphi(x). \end{cases}$$

This is nothing but the classical homogeneous problem for the heat equation, the solution of which is already known to the reader.

7.4.2 Problem with Convective Term

The PDE

$$u_t - ku_{xx} + cu_x = 0 \tag{7.52}$$

describes the so called *convective diffusion*, where c is a constant representing the propagation speed of the medium (see Section 1.3.2). Equation (7.52) can be transformed into the standard diffusion equation by the substitution

$$u(x,t) = \mathrm{e}^{\frac{c}{2k}(x - \frac{ct}{2})}w(x,t).$$

(The reader is asked to verify it.) The exponential term in this case reflects the motion of the medium, $w(x,t)$ corresponds only to the diffusion process.

7.5 Exercises

1. Prove that if $f(x) \in C^2([0, +\infty))$, then its odd extension $\tilde{f}(x)$ is of the class $C^2(\mathbb{R})$ if and only if $f(0) = f''(0) = 0$.

2. Using the method of even extension, derive the formula for the solution of the diffusion equation on the half-line with homogeneous Neumann boundary condition at $x = 0$. Consider the general initial condition $u(x, 0) = \varphi(x)$.

$$[u(x,t) = \int\limits_0^{+\infty} \varphi(y)(G(x+y,t) + G(x-y,t))\mathrm{d}y]$$

3. Using the method of even extension, derive the formula for the solution of the wave equation on the half-line with homogeneous Neumann boundary condition at $x = 0$. Consider the general initial conditions $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$.

$$[u(x,t) = \tfrac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \tfrac{1}{2c}\int\limits_{x-ct}^{x+ct} \psi(\tau)\mathrm{d}\tau \qquad \text{for } x > ct,$$

$$u(x,t) = \tfrac{1}{2}(\varphi(ct+x) + \varphi(ct-x)) + \tfrac{1}{2c}\int\limits_0^{ct+x} \psi(\tau)\mathrm{d}\tau + \tfrac{1}{2c}\int\limits_0^{ct-x} \psi(\tau)\mathrm{d}\tau$$

$$\text{for } 0 < x < ct]$$

4. Find a solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = 1,\ u_t(x,0) = 0. \end{cases}$$

Sketch the graph of the solution on several time levels.

$$[u(x,t) = 1 \text{ for } x > t,\ u(x,t) = 0 \text{ for } 0 < x < t]$$

5. Find a solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, & x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = x\mathrm{e}^{-x},\ u_t(x,0) = 0. \end{cases}$$

Sketch the graph of the solution on several time levels. Notice the wave reflection on the boundary.

$$[u(x,t) = \tfrac{1}{2}(x+t)\mathrm{e}^{-x-t} + \tfrac{1}{2}(x-t)\mathrm{e}^{-x+t} \qquad \text{for } x > t,$$

$$u(x,t) = \tfrac{1}{2}(t+x)\mathrm{e}^{-t-x} - \tfrac{1}{2}(t-x)\mathrm{e}^{-t+x} \qquad \text{for } 0 < x < t]$$

6. Find a solution of the problem

$$\begin{cases} u_{tt} = u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = 0, \end{cases}$$

with

$$\varphi(x) = \begin{cases} \cos^3 x, & x \in (\frac{3\pi}{2}, \frac{5\pi}{2}), \\ 0, & x \in (0, +\infty) \setminus (\frac{3\pi}{2}, \frac{5\pi}{2}). \end{cases}$$

Sketch the graph of the solution on several time levels.

7. Find a solution of the problem

$$\begin{cases} u_t = k u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = 1, \\ u(x,0) = 0. \end{cases}$$

[$u(x,t) = 0$ for $x > t$, $u(x,t) = 1$ for $0 < x < t$]

8. The heat flow in a metal rod with an inner heat source is described by the problem

$$\begin{cases} u_t = k u_{xx} + 1, \quad 0 < x < l,\ t > 0, \\ u(0,t) = 0, \quad u(1,t) = 1. \end{cases}$$

What will be the temperature of the rod in the steady state that will be achieved after a sufficiently long time? (Realize that in the steady state u depends only on x.) Does the absence of an initial condition cause any trouble?

[$u(x) = -\frac{x^2}{2k} + (1 + \frac{1}{2k})x$]

9. Consider the case that the heat leaks from the rod over its lateral surface at a speed proportional to its temperature u. The corresponding problem has the form

$$\begin{cases} u_t = k u_{xx} - au, \quad 0 < x < l,\ t > 0, \\ u(0,t) = 0, \quad u(l,t) = 1. \end{cases}$$

Draw the temperature distribution in the steady state and discuss how the heat flows in the rod and across its boundary.

10. Bacteria in a one-dimensional medium (a tube of a unit cross-section, length l, closed on both ends) breed according to the logistic law $ru(l - u/K)$, where r is a growth constant, K is the capacity of the medium, and $u = u(x,t)$ denotes the density of bacteria per unit length. At the beginning, the density is given by $u = ax(l - x)$. At time $t > 0$, the bacteria also diffuse with the diffusion constant D. Formulate the initial boundary value

problem describing their density. What will be the density distribution if we wait long enough? Sketch intuitively several profiles illustrating the density evolution in time. Consider cases $al^2 < 4K$ and $al^2 > 4K$ separately.

11. Consider a bar of length l which is insulated in such a way that there is no exchange of heat with the surrounding medium. Show that the average temperature

$$\frac{1}{l}\int_0^l u(x,t)\,\mathrm{d}x$$

is constant with respect to time t.

[Hint: Integrate the corresponding series term by term.]

12. Solve the problem describing the motion of a string of unit length

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

for the data given below. Illustrate the string motion by a graphic representation of a partial sum of the resulting series for various values t. Comparing the graph for $t = 0$ and the graph of the function $\varphi(x)$, decide whether the number of terms in the sum is sufficient.

(a) $\varphi(x) = 0.05\sin\pi x$, $\psi(x) = 0$, $c = 1/\pi$.

[$u(x,t) = 0.05\sin\pi x\ \cos t$]

(b) $\varphi(x) = \sin\pi x\cos\pi x$, $\psi(x) = 0$, $c = 1/\pi$.

(c) $\varphi(x) = \sin\pi x + 3\sin 2\pi x - \sin 5\pi x$, $\psi(x) = 0$, $c = 1$.

[$u(x,t) = \sin\pi x\ \cos\pi t + 3\sin 2\pi x\ \cos 2\pi t - \sin 5\pi x\ \cos 5\pi t$]

(d) $\varphi(x) = \sin\pi x + 0.5\sin 3\pi x + 3\sin 7\pi x$, $\psi(x) = \sin 2\pi x$, $c = 1$.

(e) $\psi(x) = 0$, $c = 4$,

$$\varphi(x) = \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2(1-x), & 1/2 < x \le 1. \end{cases}$$

[$u(x,t) = \sum_{k=0}^{+\infty} \frac{8(-1)^k}{\pi^2(2k+1)^2}\sin(2k+1)\pi x\ \cos 4(2k+1)\pi t$]

(f) $\psi(x) = 2$, $c = 1/\pi$,

$$\varphi(x) = \begin{cases} 0, & 0 \le x \le 1/3, \\ 1/30(x - 1/3), & 1/3 \le x \le 2/3, \\ 1/30(1-x), & 2/3 < x \le 1. \end{cases}$$

(g) $\psi(x) = 1$, $c = 4$,

$$\varphi(x) = \begin{cases} 4x, & 0 \le x \le 1/4, \\ 1, & 1/4 \le x \le 3/4, \\ 4(1-x), & 3/4 < x \le 1. \end{cases}$$

$$[u(x,t) = \sum_{n=1}^{+\infty} \tfrac{8}{\pi^2 n^2}(\sin(n\pi/4) + \sin(3n\pi/4)) \sin n\pi x \, \cos 4n\pi t$$
$$+ \sum_{k=0}^{+\infty} \tfrac{1}{\pi^2(2k+1)^2} \sin(2k+1)\pi x \, \sin 4(2k+1)\pi t]$$

(h) $\varphi(x) = x \sin \pi x$, $\psi(x) = 0$, $c = 1/\pi$.

(i) $\varphi(x) = x(1-x)$, $\psi(x) = \sin \pi x$, $c = 1$.

$$[u(x,t) = \sum_{k=1}^{+\infty} \tfrac{8}{\pi^3(2k+1)^3} \sin(2k+1)\pi x \, \cos(2k+1)\pi t + \tfrac{1}{\pi} \sin \pi x \, \sin \pi t]$$

(j) $\psi(x) = 0$, $c = 1$,

$$\varphi(x) = \begin{cases} 4x, & 0 \le x \le 1/4, \\ -4(x-1/2), & 1/4 \le x \le 3/4, \\ 4(x-1), & 3/4 < x \le 1. \end{cases}$$

13. Solve the wave equation on the interval $(0, 4\pi)$ with $c = 1$, homogeneous Dirichlet boundary conditions, zero initial velocity $\psi = 0$ and the initial displacement given by

$$\varphi(x) = \begin{cases} \cos^3 x, & x \in [\frac{3\pi}{2}, \frac{5\pi}{2}], \\ 0, & x \in [0, 4\pi] \setminus [\frac{3\pi}{2}, \frac{5\pi}{2}]. \end{cases}$$

Plot the graph of the solution on several time levels.

14. Solve the wave equation on the interval $(0, 4\pi)$ with $c = 1$, homogeneous Neumann boundary conditions, zero initial velocity $\psi = 0$ and the initial displacement given by

$$\varphi(x) = \begin{cases} \cos^3 x, & x \in [\frac{3\pi}{2}, \frac{5\pi}{2}], \\ 0, & x \in [0, 4\pi] \setminus [\frac{3\pi}{2}, \frac{5\pi}{2}]. \end{cases}$$

Plot the graph of the solution on several time levels.

15. Solve the following initial boundary value problem for the wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad 0 < x < 3, \ t > 0, \\ u(0,t) = 0, \quad u(3,t) = 0, \\ u(x,0) = 1 - \cos \frac{\pi x}{3}, \\ u_t(x,0) = 0. \end{cases}$$

16. Solve the following initial boundary value problem for the wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \ t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = x\cos \pi x, \\ u_t(x,0) = 1. \end{cases}$$

17. A string of length 2π with fixed ends is excited by the impact of a hammer which gives it the initial velocity

$$\psi(x) = \begin{cases} 100, & \frac{\pi}{2} < x < \frac{3\pi}{2}, \\ 0, & x \in [0, 2\pi] \setminus (\frac{\pi}{2}, \frac{3\pi}{2}). \end{cases}$$

Find the string vibrations provided the initial displacement was zero.

18. A uniform string with a fixed end at 0 and free end at 2π has the initial displacement

$$\varphi(x) = \begin{cases} -x, & 0 \le x < \frac{3\pi}{2}, \\ 3x - 6\pi, & \frac{3\pi}{2} \le x \le 2\pi, \end{cases}$$

and zero initial velocity. Assume that the string is vibrating in the medium that resists the vibrations. The resistance is proportional to the velocity with the constant of proportionality 0.03. Formulate the corresponding model and find the solution.

19. Solve the problem

$$\begin{cases} u_{tt} + u_t = u_{xx}, \quad 0 < x < \pi, \ t > 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = \sin x, \ u_t(x,0) = 0. \end{cases}$$

$$[u(x,t) = \mathrm{e}^{-t/2}(\cos \tfrac{\sqrt{3}}{2}t + \tfrac{1}{\sqrt{3}} \sin \tfrac{\sqrt{3}}{2}t) \sin x]$$

20. Solve the problem

$$\begin{cases} u_{tt} + u_t = u_{xx}, \quad 0 < x < \pi, \ t > 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = x\sin x, \ u_t(x,0) = 0. \end{cases}$$

$$[u(x,t) = \tfrac{\pi}{2}\mathrm{e}^{-t/2}(\cos \tfrac{\sqrt{3}}{2}t + \tfrac{1}{\sqrt{3}} \sin \tfrac{\sqrt{3}}{2}t) \sin x$$

$$-\tfrac{16}{\pi}\mathrm{e}^{-t/2} \sum_{k=1}^{+\infty} \tfrac{k}{(4k^2-1)^2}\big(\cos \sqrt{4k^2 - \tfrac{1}{4}}t + \tfrac{1}{2\sqrt{4k^2-1/4}} \sin \sqrt{4k^2 - \tfrac{1}{4}}t\big) \sin 2kx]$$

21. Solve the problem

$$\begin{cases} u_{tt} + 3u_t = u_{xx}, \quad 0 < x < \pi,\ t > 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = 10. \end{cases}$$

Illustrate by a graph the fact that the solution decreases to zero for t going to infinity.

$$[u(x,t) = \tfrac{16\sqrt{5}}{\pi} \mathrm{e}^{-3t/2} \sin x \, \sinh \tfrac{\sqrt{5}}{2} t$$
$$+ \tfrac{40}{\pi} \mathrm{e}^{-3t/2} \sum_{k=1}^{+\infty} \tfrac{1}{(2k+1)\sqrt{(2k+1)^2 - 9/4}} \sin(2k+1)x \, \sin \sqrt{(2k+1)^2 - 9/4}\, t]$$

22. Solve the problem

$$\begin{cases} u_t = k u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x) \end{cases}$$

for the following data.

(a) $l = \pi,\ k = 1,\ \varphi(x) = 78.$

$$[u(x,t) = \tfrac{312}{\pi} \sum_{k=0}^{+\infty} \tfrac{1}{2k+1} \mathrm{e}^{-(2k+1)^2 t} \sin(2k+1)x]$$

(b) $l = \pi,\ k = 1,\ \varphi(x) = 30 \sin x.$

(c) $l = \pi,\ k = 1,\ \varphi(x) = \begin{cases} 33x, & 0 < x \le \pi/2, \\ 33(\pi - x), & \pi/2 < x < \pi. \end{cases}$

$$[u(x,t) = \tfrac{132}{\pi} \sum_{k=0}^{+\infty} \tfrac{(-1)^k}{(2k+1)^2} \mathrm{e}^{-(2k+1)^2 t} \sin(2k+1)x]$$

(d) $l = \pi,\ k = 1,\ \varphi(x) = \begin{cases} 100, & 0 < x \le \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$

(e) $l = 1,\ k = 1,\ \varphi(x) = x.$

$$[u(x,t) = \tfrac{2}{\pi} \sum_{n=1}^{+\infty} \tfrac{(-1)^{n+1}}{n} \mathrm{e}^{-n^2 \pi^2 t} \sin n\pi x]$$

(f) $l = 1,\ k = 1,\ \varphi(x) = \mathrm{e}^{-x}.$

23. Draw the temperature distribution for various $t > 0$ for the values from Exercise 22(a). Estimate how long it takes until the maximal temperature decreases to 50°C.

24. Solve the problem

$$\begin{cases} u_t = k u_{xx}, \quad 0 < x < l,\ t > 0, \\ u_x(0,t) = u_x(l,t) = 0, \\ u(x,0) = \varphi(x) \end{cases}$$

for the following data.

(a) $l = \pi$, $k = 1$, $\varphi(x) = 100$.
[In this case, the answer can be guessed by physical intuition: $u(x,t) = 100$.]

(b) $l = \pi$, $k = 1$, $\varphi(x) = x$.

(c) $l = \pi$, $k = 1$, $\varphi(x) = \begin{cases} 100x, & 0 < x \le \pi/2, \\ 100(\pi - x), & \pi/2 < x < \pi. \end{cases}$

$$[u(x,t) = 25\pi - \tfrac{200}{\pi} \sum_{n=0}^{+\infty} \tfrac{1}{(2n+1)^2} \mathrm{e}^{-4(2n+1)^2 t} \cos 2(2n+1)x]$$

(d) $l = 1$, $k = 1$, $\varphi(x) = \begin{cases} 100, & 0 < x \le 1/2, \\ 0, & 1/2 < x < \pi. \end{cases}$

(e) $l = 1$, $k = 1$, $\varphi(x) = \cos \pi x$.

$$[u(x,t) = \mathrm{e}^{-\pi^2 t} \cos \pi x]$$

(f) $l = 1$, $k = 1$, $\varphi(x) = \sin \pi x$.

25. Solve the following initial boundary value problem for the diffusion equation:

$$\begin{cases} u_t - u_{xx} = 0, \quad 0 < x < 2,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(2,t) = 0, \\ u(x,0) = \varphi(x), \end{cases}$$

where

$$\varphi(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2. \end{cases}$$

26. Solve the following initial boundary value problem for the diffusion equation:

$$\begin{cases} u_t - u_{xx} + 2u = 0, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = \cos x. \end{cases}$$

27. Solve the nonhomogeneous initial boundary value problem

$$\begin{cases} u_t = k u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(0,t) = T_1,\ u(l,t) = T_2, \quad t > 0, \\ u(x,0) = \varphi(x), \quad 0 < x < l \end{cases}$$

for the following data:

(a) $T_1 = 100$, $T_2 = 0$, $\varphi(x) = 30 \sin(\pi x)$, $l = 1$, $k = 1$.

$$[u(x,t) = 100(1-x) + 30\mathrm{e}^{-\pi^2 t} \sin \pi x - \tfrac{200}{\pi} \sum_{n=1}^{+\infty} \tfrac{1}{n} \mathrm{e}^{-n^2\pi^2 t} \sin n\pi x]$$

(b) $T_1 = 100$, $T_2 = 100$, $\varphi(x) = 50x(1-x)$, $l = 1$, $k = 1$.

(c) $T_1 = 100$, $T_2 = 50$, $\varphi(x) = \begin{cases} 33x, & 0 < x \le \pi/2, \\ 33(\pi - x), & \pi/2 < x < \pi, \end{cases}$

$l = \pi$, $k = 1$.

$$[u(x,t) = 100 - \tfrac{50x}{\pi} + \tfrac{132}{\pi}\sum_{k=0}^{+\infty} \tfrac{(-1)^k}{(2k+1)^2} \mathrm{e}^{-(2k+1)^2 t} \sin(2k+1)x$$
$$- \tfrac{100}{\pi}\sum_{n=1}^{+\infty} \tfrac{2-(-1)^n}{n} \mathrm{e}^{-n^2 t} \sin nx]$$

(d) $T_1 = 0$, $T_2 = 100$, $\varphi(x) = \begin{cases} 100, & 0 < x \le \pi/2, \\ 0, & \pi/2 < x < \pi, \end{cases}$

$l = \pi$, $k = 1$.

28. Solve the following initial boundary value problem for the diffusion equation:
$$\begin{cases} u_t - u_{xx} = 0, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 2, \quad u(1,t) = 6, \\ u(x,0) = \sin 2\pi x + 4x. \end{cases}$$

29. Solve the following initial boundary value problem for the nonhomogeneous wave equation:
$$\begin{cases} u_{tt} - 4u_{xx} = 2\sin \pi x, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 1, \quad u(1,t) = 1, \\ u(x,0) = 0, \\ u_t(x,0) = 0. \end{cases}$$

30. Solve the wave equation
$$u_{tt} - c^2 u_{xx} = A \sin \omega t, \quad 0 < x < l,\ t > 0,$$
with zero initial and boundary conditions. For which ω does the solution grow in time (the so called *resonance* occurs)?

31. Consider heat flow in a thin circular ring of unit radius that is insulated along its lateral surface. The temperature distribution in the ring can be described by the standard one-dimensional diffusion equation, where x represents the arc length along the ring. The shape of the domain causes that we have to consider periodic boundary conditions
$$u(-\pi,t) = u(\pi,t), \quad u_x(-\pi,t) = u_x(\pi,t).$$
Solve this problem for a general initial condition $u(x,0) = \varphi(x)$, $x \in (-\pi,\pi)$.

32. Separate the following PDEs into appropriate ODEs:

    (a) $u_t = ku_{xx} + u$,

    (b) $u_t = ku_{xx} - mu_x + u$,

    (c) $u_t = (k(x)u_x)_x + u$.

33. Determine if the following PDEs are separable. If so, separate them into appropriate ODEs. If not, explain why.

    (a) $u_{tt} = c^2 u_{xx} + u$,

    (b) $u_{tt} = c^2 u_{xx} - mu_x + u$,

    (c) $c(x)\rho(x)u_{tt} = (T(x)u_x)_x - u_t + u$.

34. Solve the following problem:

$$\begin{cases} u_{tt} = u_{xx} - u_t + u_x, \quad 0 < x < \frac{3\pi}{2},\ t > 0, \\ u(0,t) = 0, \quad u(3\pi/2, t) = 0, \\ u(x,0) = \cos x, \\ u_t(x,0) = x^2 - (3\pi/2)^2. \end{cases}$$

    Explain its physical meaning.

35. Solve the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} + 2u_t = \sin^3 x, \quad 0 < x < \pi,\ t > 0, \\ u(0,t) = 0, \quad u(\pi, t) = 0, \\ u(x,0) = \sin x, \\ u_t(x,0) = 0. \end{cases}$$

    Here, use the identity $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.

Chapter 8

Solutions of Boundary Value Problems for Stationary Equations

In this chapter we consider two-dimensional boundary value problems for the Laplace (or Poisson) equation. The basic mathematical problem is to solve these equations on a given domain (open and connected set) $\Omega \subset \mathbb{R}^2$ with given conditions on the boundary $\partial\Omega$:

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = h_1 & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial n} = h_2 & \text{on } \Gamma_2, \\ \dfrac{\partial u}{\partial n} + au = h_3 & \text{on } \Gamma_3, \end{cases}$$

where f and h_i, $i = 1, 2, 3$, are given functions, a is a given constant, and $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial\Omega$. In particular cases, some of the boundary segments can be empty.

On a rectangle (or on a strip, on a half-plane), the solution of the Laplace equation can be found using the separation of variables (the Fourier method). The general scheme is the same as in the case of evolution equations.

(i) The solution of PDE is searched in a separated form.

(ii) We take into account the homogeneous boundary conditions and obtain the eigenvalues of the problem. It is in this step that the geometry of the rectangle is very important.

(iii) The solution is written in the form of a series.

(iv) We include the nonhomogeneous boundary conditions.

There are several special domains which can be transformed to a rectangle. For example, this is the case with the disc or its suitable parts if we use the transformation into polar coordinates.

8.1 Laplace Equation on Rectangle

Let us consider the Laplace equation $u_{xx} + u_{yy} = 0$ on a rectangle $R = \{0 < x < a,\ 0 < y < b\}$ with the boundary conditions illustrated in Figure 8.1.

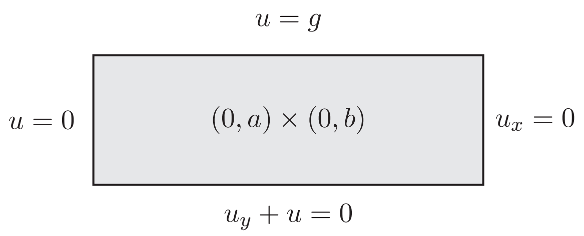


Figure 8.1. The rectangle R and boundary conditions of (8.1).

We thus solve the problem

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in } R, \\ u(0, y) = u_x(a, y) = 0, \\ u_y(x, 0) + u(x, 0) = 0, \\ u(x, b) = g(x). \end{cases} \tag{8.1}$$

In the first step, we look for the solution in a separated form: $u(x, y) = X(x)Y(y)$, $X \neq 0$, $Y \neq 0$. Substituting into the equation and dividing by XY, we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

There must exist a constant λ such that $X'' + \lambda X = 0$ for $0 < x < a$, and $Y'' - \lambda Y = 0$ for $0 < y < b$. Moreover, the function X must fulfil the homogeneous boundary conditions $X(0) = X'(a) = 0$. By simple analysis, we find out that a nontrivial solution $X = X(x)$ exists only for

$$\lambda_n = \beta_n^2 = \left((2n - 1)\frac{\pi}{2a}\right)^2, \qquad n \in \mathbb{N}, \tag{8.2}$$

and the corresponding solutions are

$$X_n(x) = C_n \sin \beta_n x. \tag{8.3}$$

Now, we return to the variable y and solve the problem

$$Y'' - \beta_n^2 Y = 0, \quad Y'(0) + Y(0) = 0$$

(the *nonhomogeneous* boundary condition for $y = b$ will be considered in the last step). Since all $\lambda_n = \beta_n^2$ are positive, we obtain Y in the form

$$Y(y) = A_n \cosh \beta_n y + B_n \sinh \beta_n y.$$

Further, we have $0 = Y'(0) + Y(0) = B_n\beta_n + A_n$. Without loss of generality, we can put $B_n = -1$ for all $n \in \mathbb{N}$, and thus $A_n = \beta_n$. Hence, we obtain

$$Y_n(y) = \beta_n \cosh \beta_n y - \sinh \beta_n y.$$

The series

$$u(x, y) = \sum_{n=1}^{+\infty} C_n \sin \beta_n x(\beta_n \cosh \beta_n y - \sinh \beta_n y) \tag{8.4}$$

then represents the harmonic function on the rectangle R that satisfies the homogeneous boundary conditions $u(0, y) = 0$, $u_x(a, y) = 0$ for $y \in (0, b)$, and $u_y(x, 0) + u(x, 0) = 0$ for $x \in (0, a)$. It remains to deal with the boundary condition $u(x, b) = g(x)$. In order to satisfy it, we must ensure that

$$g(x) = \sum_{n=1}^{+\infty} C_n(\beta_n \cosh \beta_n b - \sinh \beta_n b) \sin \beta_n x \tag{8.5}$$

for all $x \in (0, a)$. Here we assume that $(\beta_n \cosh \beta_n b - \sinh \beta_n b) \neq 0$ for all $n \in \mathbb{N}$. Then expression (8.5) is nothing but the Fourier series of the function g with respect to the system of eigenfunctions $\sin \beta_n x$. Hence, we obtain formulas for the remaining unknown coefficients C_n:

$$C_n = \frac{2}{a}(\beta_n \cosh \beta_n b - \sinh \beta_n b)^{-1} \int_0^a g(x) \sin \beta_n x \, \mathrm{d}x. \tag{8.6}$$

If $(\beta_n \cosh \beta_n b - \sinh \beta_n b) = 0$ for some $n \in \mathbb{N}$, then in general the boundary condition $g(x)$ cannot be expressed as in (8.5). In that case, problem (8.1) can have either no solution, or infinitely many solutions depending on the relation between $g(x)$ and the other data. This means that (8.1) is an ill-posed problem.

Remark 8.1 (Nonhomogeneous Boundary Conditions). Let us consider again the Laplace equation on a rectangle, but this time let all four boundary conditions be *nonhomogeneous*. (It does not matter which types of boundary conditions (Dirichlet, Neumann, or Robin) are given on particular sides.) The previous example has shown the advantage of the situation when only one boundary condition is nonhomogeneous and all the other three conditions are homogeneous. Using the linearity of the problem, we can decompose the totally nonhomogeneous problem into four partially homogeneous problems which are easy to solve. Schematically, we illustrate the decomposition in Figure 8.2.

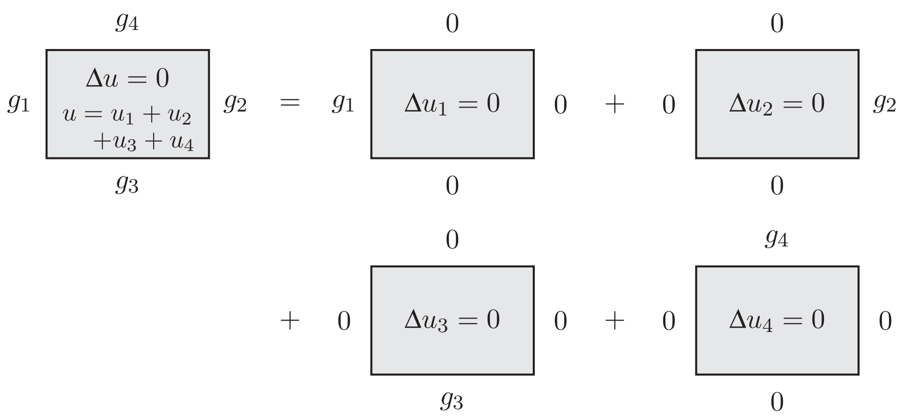


Figure 8.2. Decomposition of the nonhomogeneous boundary value problem for the Laplace equation on a rectangle.

8.2 Laplace Equation on Disc

A much more interesting but classical example deals with the *Dirichlet problem for the Laplace equation on a disc.* The rotational invariance of the Laplace operator Δ indicates that the disc is a natural shape for harmonic functions in the plane. So, let us consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } x^2 + y^2 < a^2, \\ u = h(\theta) & \text{for } x^2 + y^2 = a^2. \end{cases} \tag{8.7}$$

We solve the equation on the disc D with the center at the origin and radius a. The boundary condition $h(\theta)$ is given on the circle ∂D. Note that θ is the polar coordinate denoting the central angle which is formed by the radius vector of the point (x, y) and the positive half-axis x.

We again use the method of separation of variables, but this time in polar coordinates: $u = R(r)\Theta(\theta)$. If we use the transformation formula (6.3), we rewrite the equation into the form

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$

Dividing by $R\Theta$ (under the assumption $R\Theta \neq 0$) and multiplying by r^2, we obtain two equations

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0, && (8.8) \\ r^2 R'' + rR' - \lambda R &= 0, && (8.9) \end{aligned}$$

where λ is so far unknown constant. Both these ODEs are easily solvable. We only have to add the appropriate boundary conditions.

For $\Theta(\theta)$, it is natural to introduce periodic boundary conditions:

$$\Theta(\theta + 2\pi) = \Theta(\theta) \qquad \text{for } \theta \in \mathbb{R}$$

(that is, $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$). Hence, we obtain

$$\lambda_n = n^2 \quad \text{and} \quad \Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \quad n \in \mathbb{N} \cup \{0\}.$$

The equation for the function R is of Euler type and its solution must be in the form $R(r) = r^\alpha$. Since $\lambda = n^2$, the corresponding characteristic equation is

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0,$$

and thus $\alpha = \pm n$. For $n \in \mathbb{N}$, we obtain $R_n(r) = \tilde{A}_n r^n + \tilde{B}_n r^{-n}$ and the solution u_n can be written as

$$u_n = \left(\tilde{A}_n r^n + \frac{\tilde{B}_n}{r^n}\right)(A_n \cos n\theta + B_n \sin n\theta). \tag{8.10}$$

For $\lambda = 0$ ($n = 0$), the functions $R = 1$ and $R = \ln r$ form the couple of linearly independent solutions of equation (8.9). The corresponding u_0 thus assumes the form

$$u_0 = \tilde{A}_0 + \tilde{B}_0 \ln r. \tag{8.11}$$

For physical reasons, functions u_n and u_0 must be bounded on the whole disc D (this means also at the origin $r = 0$) and thus, in all the cases, we put $\tilde{B}_n = 0$, $n \in \mathbb{N} \cup \{0\}$. We sum up the remaining solutions and write the resulting function u as an infinite series

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{+\infty} r^n (A_n \cos n\theta + B_n \sin n\theta). \tag{8.12}$$

Now, we take into account the nonhomogeneous boundary condition on the boundary $r = a$. It is fulfilled provided the function h is expandable into the Fourier series

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{+\infty} a^n (A_n \cos n\theta + B_n \sin n\theta).$$

Hence, we easily derive

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi \, \mathrm{d}\phi, \qquad n \in \mathbb{N} \cup \{0\}, \tag{8.13}$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi \, \mathrm{d}\phi, \qquad n \in \mathbb{N}. \tag{8.14}$$

8.3 Poisson Formula

The previous example has an interesting consequence: the sum of the series (8.12) can be expressed explicitly by an integral formula. If we put (8.13) and (8.14) into (8.12), we obtain

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi} \int_0^{2\pi} h(\phi) \, \mathrm{d}\phi + \sum_{n=1}^{+\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} h(\phi) \left(\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\right) \mathrm{d}\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} h(\phi) \left(1 + 2 \sum_{n=1}^{+\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right) \mathrm{d}\phi. \end{aligned}$$

If we express the cosine function using the complex exponential (that is, $\cos t = \frac{1}{2}(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})$), we can rewrite the expression in the brackets in the following way:

$$1 + 2 \sum_{n=1}^{+\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = 1 + \sum_{n=1}^{+\infty} \left(\frac{r}{a}\right)^n \mathrm{e}^{\mathrm{i}n(\theta-\phi)} + \sum_{n=1}^{+\infty} \left(\frac{r}{a}\right)^n \mathrm{e}^{-\mathrm{i}n(\theta-\phi)}.$$

The series in this formulation are geometric series with quotients $q = \frac{r}{a}\mathrm{e}^{\pm \mathrm{i}(\theta-\phi)}$ which, for $r < a$, satisfy the condition $|q| < 1$. Thus, we obtain

$$\begin{aligned} 1 + 2 \sum_{n=1}^{+\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) &= 1 + \frac{r\mathrm{e}^{\mathrm{i}(\theta-\phi)}}{a - r\mathrm{e}^{\mathrm{i}(\theta-\phi)}} + \frac{r\mathrm{e}^{-\mathrm{i}(\theta-\phi)}}{a - r\mathrm{e}^{-\mathrm{i}(\theta-\phi)}} \\ &= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}. \end{aligned}$$

Hence, substituting back into the integral, we arrive at the solution of the original problem (8.7) in the form

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \, \mathrm{d}\phi, \tag{8.15}$$

which is called the *Poisson formula* in polar coordinates. The expression (8.15) implies that the harmonic function inside the circle can be described by its boundary values only.

Now, we go back to the Cartesian coordinates. We denote by $\boldsymbol{x}$ a point inside the circle with polar coordinates (r, θ), and by $\boldsymbol{x}'$ a point on the boundary with polar coordinates (a, ϕ). Then $r = |\boldsymbol{x}|$, $a = |\boldsymbol{x}'|$ and for $|\boldsymbol{x} - \boldsymbol{x}'|$ we have

$$|\boldsymbol{x} - \boldsymbol{x}'|^2 = a^2 + r^2 - 2ar\cos(\theta - \phi).$$

(The reader is asked to draw a picture.) An element of the arc length is, in this case, $\mathrm{d}s = a\,\mathrm{d}\phi$. The Poisson formula in polar coordinates (8.15) can be then rewritten into the Cartesian coordinates

$$u(\boldsymbol{x}) = \frac{a^2 - |\boldsymbol{x}|^2}{2\pi a} \int\limits_{|\boldsymbol{x}'|=a} \frac{u(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^2}\,\mathrm{d}s \tag{8.16}$$

for $\boldsymbol{x} \in D$. Here $u(\boldsymbol{x}') = h(\phi)$ and the integral is considered with respect to the arc length over the whole circumference.

The above calculations are summarized in the following assertion.

Theorem 8.2. *Let $h(\phi)$ be a continuous function on a circle ∂D. Then the Poisson formula* (8.16) *describes the unique harmonic function on the disc D with the property*

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}'} u(\boldsymbol{x}) = h(\phi),$$

where ϕ is the angle corresponding to the point $\boldsymbol{x}' \in \partial D$.

Some important consequences of the Poisson Formula are summarized in Section 10.7.

8.4 Exercises

In the following exercises, r and θ denote the polar coordinates.

1. Solve the equation $u_{xx} + u_{yy} = 1$ in the disc $\{r < a\}$ with the boundary condition $u(x, y) = 0$ on the boundary $r = a$.

   $[u(r) = \frac{1}{4}(r^2 - a^2)]$

2. Solve the equation $u_{xx} + u_{yy} = 1$ in the annulus $\{a < r < b\}$ with the boundary condition $u(x, y) = 0$ on both boundary circles $r = a$, $r = b$.

   $[u(r) = \frac{r^2}{4} + \frac{b^2 - a^2}{4\ln\frac{a}{b}} \ln r - \frac{b^2 \ln a - a^2 \ln b}{4 \ln \frac{a}{b}}]$

3. Solve the equation $u_{xx} + u_{yy} = 0$ in the rectangle $\{0 < x < a,\ 0 < y < b\}$ with the boundary conditions

$$\begin{aligned} u_x(0, y) &= -a, \quad u_x(a, y) = 0, \\ u_y(x, 0) &= b, \qquad u_y(x, b) = 0. \end{aligned}$$

Search for the solution in the form of a quadratic polynomial in x and y.

[$u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + c$, where c is an arbitrary constant]

4. Find a harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi,\ 0 < y < \pi\}$ with the boundary conditions

$$\begin{aligned} &u_y(x, 0) = u_y(x, \pi) = 0, \\ &u(0, y) = 0, \quad u(\pi, y) = \cos^2 y = \frac{1}{2}(1 + \cos 2y). \end{aligned}$$

5. Find a harmonic function $u(x, y)$ in the square $D = \{0 < x < 1,\ 0 < y < 1\}$ with the boundary conditions

$$\begin{aligned} &u(x, 0) = x, \quad u(x, 1) = 0, \\ &u_x(0, y) = 0, \quad u_x(1, y) = y^2. \end{aligned}$$

6. Let u be a harmonic function in the disc $D = \{r < 2\}$ and $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the concrete form of the solution, determine the value of u at the origin.

[$u(0, 0) = 1$]

7. Solve the equation $u_{xx} + u_{yy} = 0$ in the disc $\{r < a\}$ with the boundary condition $u = 1 + 3\sin\theta$ for $r = a$.

[$u(r, \theta) = 1 + 3\frac{r}{a}\sin\theta$]

8. Solve the equation $u_{xx} + u_{yy} = 0$ in the disc $\{r < a\}$ with the boundary condition $u = \sin^3\theta$ for $r = a$. Here, use the identity $\sin^3\theta = 3\sin\theta - 4\sin 3\theta$.

[$u(r, \theta) = 3\frac{r}{a}\sin\theta - 4(\frac{r}{a})^3\sin 3\theta$]

9. Solve the equation $u_{xx} + u_{yy} = 0$ in the domain $\{r > a\}$ (that is, in the *exterior of the disc*) with the boundary condition $u = 1 + 3\sin\theta$ on the boundary $r = a$ and with the condition that the solution u is bounded for $r \to +\infty$.

[$u(r, \theta) = 1 + 3\frac{a}{r}\sin\theta$]

10. Solve the equation $u_{xx} + u_{yy} = 0$ in the disc $\{r < a\}$ with the boundary condition
$$\frac{\partial u}{\partial r} - hu = f(\theta),$$
where $f(\theta)$ is an arbitrary function. Write the solution using the Fourier coefficients of the function f.

11. Derive the Poisson formula for the exterior of the disc in $\mathbb{R}^2$.

12. Find a steady-state temperature distribution inside the annulus $\{1 < r < 2\}$ the outer edge of which ($r = 2$) is heat insulated and the inner edge ($r = 1$) is kept at the temperature described by $\sin^2\theta$.

$[u(r,\theta) = \frac{1}{2}(1 - \frac{\ln r}{\ln 2}) + (\frac{r^2}{30} - \frac{8}{15r^2})\cos 2\theta]$

13. Find a harmonic function u in the semi-disc $\{r < 1,\ 0 < \theta < \pi\}$ satisfying the conditions
$$u(r,0) = u(r,\pi) = 0, \quad u(1,\theta) = \pi\sin\theta - \sin 2\theta.$$

14. Solve the equation $u_{xx} + u_{yy} = 0$ in the disc sector $\{r < a,\ 0 < \theta < \beta\}$ with the boundary conditions
$$u(a,\theta) = \theta, \quad u(r,0) = 0, \quad u(r,\beta) = \beta.$$
Search for a function independent of r.

15. Solve the equation $u_{xx} + u_{yy} = 0$ in the quarter-disc $\{x^2 + y^2 < a^2,\ x > 0,\ y > 0\}$ with the boundary conditions
$$u(0,y) = u(x,0) = 0, \quad \frac{\partial u}{\partial r} = 1 \text{ for } r = a.$$
Find the solution in the form of an infinite series and write the first two nonzero terms explicitly.

[first two terms: $\frac{r^2}{2a}\sin 2\theta + \frac{r^4}{4a^3}\sin 4\theta$]

16. Solve the equation $u_{xx} + u_{yy} = 0$ in the domain $\{\alpha < \theta < \beta,\ a < r < b\}$ with the boundary conditions $u = 0$ on both sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.

17. Solve the boundary value problem for the Laplace equation in the square $K = \{(x,y);\ 0 < x < \pi,\ 0 < y < \pi\}$ for the following data:

(a) $u_y(x,0) = u_y(x,\pi) = u_x(0,y) = 0, \quad u_x(\pi,y) = \cos 3y;$

$[u(x,y) = \frac{\cosh 3x}{3\sinh 3\pi}\cos 3y]$

(b) $u(0,y) = u_y(x,0) + u(x,0) = u_x(\pi,y) = 0, \quad u_x(x,\pi) = \sin\frac{3x}{2}.$

$$\left[u(x,y) = \frac{3\cosh(3y/2) - 2\sinh(3y/2)}{3\cosh(3\pi/2) - 2\sinh(3\pi/2)} \sin\frac{3x}{2}\right]$$

18. Solve the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } x^2 + y^2 < 1, \\ u(x,y) = x^4 - y^3 & \text{on } x^2 + y^2 = 1. \end{cases}$$

$$\left[u(r,\theta) = \frac{3}{8} - \frac{3}{4} r\sin\theta + \frac{r^2}{2}\cos 2\theta + \frac{r^3}{4}\sin 3\theta + \frac{r^4}{8}\cos 4\theta\right]$$

19. Solve the Poisson equation $u_{xx} + u_{yy} = f(x,y)$ in the unit square $\{0 < x < 1, 0 < y < 1\}$ for the following data.

(a) $f(x,y) = x$, $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$.

$$\left[u(x,y) = \frac{8}{\pi^4} \sum_{k=0}^{+\infty} \sum_{m=1}^{+\infty} \frac{(-1)^m}{(m^2 + (2k+1)^2) m (2k+1)} \sin m\pi x \, \sin(2k+1)\pi y\right]$$

(b) $f(x,y) = \sin\pi x$, $u(x,0) = u(0,y) = u(1,y) = 0$, $u(x,1) = x$.

$$\left[u(x,y) = \frac{-4\sin\pi x}{\pi^3} \sum_{k=0}^{+\infty} \frac{\sin(2k+1)\pi y}{(1+(2k+1)^2)(2k+1)} + \frac{2}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} \sin n\pi x \, \sinh n\pi y}{n \sinh n\pi}\right]$$

(c) $f(x,y) = xy$, $u(x,0) = u(0,y) = u(1,y) = 0$, $u(x,1) = x$.

20. Solve the equation $u_{xx} + u_{yy} = 3u - 1$ inside the unit square $\{0 < x < 1, 0 < y < 1\}$ with u vanishing on the boundary.

$$\left[u(x,y) = \frac{16}{\pi^2} \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{\sin(2k+1)\pi x \, \sin(2l+1)\pi y}{(2l+1)(2k+1)(3 + \pi^2((2l+1)^2 + (2k+1)^2))}\right]$$

Chapter 9

Methods of Integral Transforms

In this chapter we introduce another class of methods that can be used for solving the initial value or initial boundary value problems for the evolution equations. These are the so called *methods of integral transforms*. The fundamental ones are the Laplace and the Fourier transforms.

9.1 Laplace Transform

The reader could probably have seen the Laplace transform when solving linear ODEs with constant coefficients, where they are used to transform ODEs to algebraic equations. This idea can be easily extended to PDEs, where the transformation decreases of the number of independent variables. PDEs in two variables are thus reduced to ODEs.

Let $u = u(t)$ be a *piecewise continuous* function on $[0, +\infty)$ that "does not grow too fast". Let us assume, for example, that u is of *exponential order*, which means that $|u(t)| \le c\mathrm{e}^{at}$ for t large enough, where $a, c > 0$ are appropriate constants. The Laplace transform of the function u is then defined by the formula

$$(\mathcal{L}u)(s) \equiv U(s) = \int_0^{+\infty} u(t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{9.1}$$

Here U and s are the *transformed variables*, U is the dependent one, s is the independent one, and U is defined for $s > a$ with $a > 0$ depending on $u(t)$. The function U is called the *Laplace image* of the function u, which is then called the *original*. The Laplace transform is a linear mapping, that is,

$$\mathcal{L}(c_1u + c_2v) = c_1\mathcal{L}u + c_2\mathcal{L}v,$$

where c_1, c_2 are arbitrary constants. If we know the Laplace image $U(s)$, then the original $u(t)$ can be obtained by the *inverse Laplace transform* of the image $U(s)$: $\mathcal{L}^{-1}U = u$. Some of Laplace images and their originals can be found in tables, or the transformation can be done using various software packages.

An important property of the Laplace transform, as well as of other integral transforms, is the fact that it turns differential operators in originals into multiplication operators in images. The following formulas hold:

$$(\mathcal{L}u')(s) = sU(s) - u(0), \tag{9.2}$$
$$(\mathcal{L}u^{(n)})(s) = s^nU(s) - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - u^{(n-1)}(0), \tag{9.3}$$

if the derivatives considered are transformable (that is, piecewise continuous functions of exponential order). To be precise, we should write $\lim_{t\to 0+} u(t)$, $\lim_{t\to 0+} u'(t), \ldots$ instead of $u(0)$, $u'(0), \ldots$. However, without loss of generality, we can assume that the function u and its derivatives are continuous from the right at 0. Relations (9.2), (9.3) can be easily derived directly from the definition using integration by parts (the reader is asked to do it in detail.). Applying the Laplace transform to a linear ODE with constant coefficients, we obtain a linear algebraic equation for the unknown function $U(s)$. After solving it, we find the original function $u(t)$ by the inverse transform.

The same idea can be exploited also when solving PDEs for functions of two variables, say $u = u(x,t)$. The transformation will be done with respect to the time variable $t \geq 0$, and the spatial variable x will be treated as a parameter unaffected by this transform. The reason is the fact that the definition of the Laplace transform requires the transformed independent variable from the interval $[0, +\infty)$. In particular, we define the Laplace transform of a function $u(x,t)$ by the formula

$$(\mathcal{L}u)(x,s) \equiv U(x,s) = \int_0^{+\infty} u(x,t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{9.4}$$

The time derivatives are transformed in the same way as in the case of functions of one variable, that is, for example,

$$(\mathcal{L}u_t)(x,s) = sU(x,s) - u(x,0).$$

The spatial derivatives remain unchanged, that is,

$$(\mathcal{L}u_x)(x,s) = \int_0^{+\infty} \frac{\partial}{\partial x}u(x,t)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{\partial}{\partial x}\int_0^{+\infty} u(x,t)\mathrm{e}^{-st}\,\mathrm{d}t = U_x(x,s).$$

Thus, applying the Laplace transform to a PDE in two variables x and t, we obtain an ODE in the variable x and with the parameter s.

Example 9.1 (Diffusion with Constant Boundary Condition). Using the Laplace transform, we solve the following initial boundary value problem for the diffusion equation. Let $u = u(x,t)$ denote the concentration of a chemical contaminant dissolved in a liquid on a half-infinite domain $x > 0$. Let us assume that, at time $t = 0$, the concentration is zero. On the boundary $x = 0$, constant unit concentration of the contaminant is kept for $t > 0$. Assuming the unit diffusion constant, the behavior of the system is described by a mathematical model

$$\begin{cases} u_t - u_{xx} = 0, \quad x > 0,\ t > 0, \\ u(x,0) = 0, \\ u(0,t) = 1, \quad u(x,t) \text{ bounded.} \end{cases} \tag{9.5}$$

Here the boundedness assumption is related to the physical properties of the model and its solution.

If we apply the Laplace transform to both sides of the equation, we obtain the following relation for the image U:

$$sU(x,s) - U_{xx}(x,s) = 0.$$

This is an ODE with respect to the variable x and with real positive parameter s. Its general solution has the form

$$U(x,s) = a(s)\mathrm{e}^{-\sqrt{s}x} + b(s)\mathrm{e}^{\sqrt{s}x}.$$

Since we require the solution u to be bounded in both variables x and t, the image U must be bounded in x as well. Thus, $b(s)$ must vanish, and hence

$$U(x,s) = a(s)\mathrm{e}^{-\sqrt{s}x}.$$

Now, we apply the Laplace transform to the boundary condition obtaining $U(0,s) = \mathcal{L}(1) = 1/s$. It implies $a(s) = 1/s$ and the transformed solution has the form

$$U(x,s) = \frac{1}{s}\mathrm{e}^{-\sqrt{s}x}.$$

Using the tables of the Laplace transform or some of the software packages, we easily find out that

$$u(x,t) = \operatorname{erfc}\left(\frac{x}{\sqrt{4t}}\right),$$

where erfc is the function defined by the relation

$$\operatorname{erfc}(y) = 1 - \frac{2}{\sqrt{\pi}}\int_0^y \mathrm{e}^{-r^2}\,\mathrm{d}r = 1 - \operatorname{erf}(y).$$

◇

In the previous example, we were able to find the original $u(x,t)$ to the Laplace image $U(x,s)$ using tables or software packages. There exists a general formula for inverse Laplace transform, which is based on theory of functions of complex variables (see, e.g., [23]). However, it has a theoretical character, and from the practical point of view, it is used very rarely. In most cases, it is more or less useless.

In some cases, instead of above mentioned inverse formula, we can exploit another useful tool, which is stated in the *Convolution Theorem* given below.

Theorem 9.2. *Let u and v be piecewise continuous functions on the interval $(0,+\infty)$, both of exponential order, and let $U = \mathcal{L}u$, $V = \mathcal{L}v$ be their Laplace images. Let us denote by*

$$(u * v)(t) = \int_0^t u(t-\tau)v(\tau)\,\mathrm{d}\tau$$

the convolution of functions u and v (which is also of exponential order). Then

$$\mathcal{L}(u * v)(s) = (\mathcal{L}u)(s)\ (\mathcal{L}v)(s) = U(s)V(s).$$

Remark 9.3. In particular, it follows from Theorem 9.2 that

$$u * v = \mathcal{L}^{-1}(\mathcal{L}u\ \mathcal{L}v).$$

Notice that the Laplace transform is additive, however, it is not multiplicative!

Example 9.4 (Diffusion with Non-Constant Boundary Condition). Let us consider the same situation as in the previous example with the only change – the boundary condition will be a time-dependent function:

$$\begin{cases} u_t - u_{xx} = 0, & x > 0,\ t > 0, \\ u(x,0) = 0, & \\ u(0,t) = f(t), & u(x,t) \text{ bounded}. \end{cases} \tag{9.6}$$

Applying the Laplace transform to the equation, we obtain again the ODE

$$sU(x,s) - U_{xx}(x,s) = 0,$$

the solution of which has the form

$$U(x,s) = a(s)\mathrm{e}^{-\sqrt{s}x}.$$

Here we have used the boundedness assumption. The transformation of the boundary condition (we assume that the Laplace transform of f does exist) leads to the relation $U(0,s) = F(s)$ with $F = \mathcal{L}f$. Hence $a(s) = F(s)$ and the solution in images takes the form

$$U(x,s) = F(s)\mathrm{e}^{-\sqrt{s}x}.$$

The Convolution Theorem and Remark 9.3 now imply

$$u = \mathcal{L}^{-1}U = \mathcal{L}^{-1}F * \mathcal{L}^{-1}\left(\mathrm{e}^{-\sqrt{s}x}\right) = f * \mathcal{L}^{-1}\left(\mathrm{e}^{-\sqrt{s}x}\right).$$

If we exploit the knowledge of the transform relation

$$\mathcal{L}^{-1}\left(\mathrm{e}^{-\sqrt{s}x}\right) = \frac{x}{\sqrt{4\pi t^3}}\mathrm{e}^{-x^2/4t},$$

we obtain the solution of the original problem in the form

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}}\mathrm{e}^{-x^2/4(t-\tau)}f(\tau)\,\mathrm{d}\tau.$$

◇

Example 9.5 (Forced Vibrations of "Half-Infinite String"). Let us consider a "half-infinite string" that has one end fixed at the origin and that lies motionless at time $t = 0$. The string is set in motion by acting of a force $f(t)$. The string behavior is then modeled by the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(t), \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = u_t(x,0) = 0, \quad u(x,t) \text{ bounded.} \end{cases} \tag{9.7}$$

We transform both sides of the equation with respect to the time variable and use the initial condition. Thus we obtain

$$-c^2 U_{xx}(x,s) + s^2 U(x,s) = F(s).$$

This is an ODE in the x-variable with constant coefficients and a non-zero right-hand side. Its solution is the sum of the solution U_H of the homogeneous equation: $U_H(x,s) = A(s)\mathrm{e}^{-sx/c} + B(s)\mathrm{e}^{sx/c}$, and the particular solution U_P of the nonhomogeneous equation: $U_P(x,s) = F(s)/s^2$. Hence

$$U(x,s) = A(s)\mathrm{e}^{-\frac{s}{c}x} + B(s)\mathrm{e}^{\frac{s}{c}x} + \frac{F(s)}{s^2}.$$

Since we require the original solution $u(x,t)$ to be bounded, the transformed solution $U(x,s)$ must be bounded for $x > 0$, $s > 0$ as well, and thus $B(s) = 0$. The transformed boundary condition implies $A(s) = -F(s)/s^2$ and thus

$$U(x,s) = F(s)\frac{1 - \mathrm{e}^{-\frac{s}{c}x}}{s^2}.$$

To obtain the inverse Laplace transform, we use the Convolution Theorem and the relations

$$\mathcal{L}^{-1}(1/s^2) = t, \quad \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-sx/c}}{s^2}\right) = \left(t - \frac{x}{c}\right)\mathcal{H}\left(t - \frac{x}{c}\right),$$

where $\mathcal{H}$ is the Heaviside step function, that is, $\mathcal{H}(t) = 0$ for $t \le 0$, $\mathcal{H}(t) = 1$ for $t > 0$. The solution of the original problem then has the form

$$u(x,t) = f(t) * \mathcal{L}^{-1}\left(\frac{1 - \mathrm{e}^{-\frac{s}{c}x}}{s^2}\right) = f(t) * \left(t - \left(t - \frac{x}{c}\right)\mathcal{H}\left(t - \frac{x}{c}\right)\right)$$

or

$$u(x,t) = \int_0^t f(t-\tau)\left(\tau - \left(\tau - \frac{x}{c}\right)\mathcal{H}\left(\tau - \frac{x}{c}\right)\right)\,\mathrm{d}\tau.$$

◇

The following example illustrates one particular interesting case of Example 9.5.

Example 9.6 (String Vibrations due to Gravitational Acceleration). If the only acting external force in Example 9.5 is the gravitational acceleration g, we solve the wave equation in the form

$$u_{tt} - c^2 u_{xx} = -g.$$

Under the same initial and boundary conditions as in the previous example (that is, $u(x,0) = u_t(x,0) = 0$ for $x > 0$, and $u(0,t) = 0$ for $t > 0$), the solution assumes the form

$$\begin{aligned} u(x,t) &= -g\int_0^t \left(\tau - \left(\tau - \frac{x}{c}\right)\mathcal{H}\left(\tau - \frac{x}{c}\right)\right)\,\mathrm{d}\tau \\ &= -g\left(\frac{1}{2}t^2 - \int_0^t \left(\tau - \frac{x}{c}\right)\mathcal{H}\left(\tau - \frac{x}{c}\right)\,\mathrm{d}\tau\right). \end{aligned}$$

By simple calculation, we express the integral on the right-hand side and obtain the final formulation of the solution:

$$u(x,t) = \begin{cases} -\dfrac{g}{2}\left(t^2 - \left(t - \dfrac{x}{c}\right)^2\right) & \text{for } 0 < x < ct, \\ -\dfrac{gt^2}{2} & \text{for } x > ct. \end{cases}$$

Figure 9.1 shows the solution on several time levels. This example models a half-infinite string with one fixed end, which falls from the zero (horizontal) position due to the gravitation. Recalling that the position of the free-falling mass point is described by the function $-gt^2/2$, we see that, for x bigger than ct, the string falls freely. The remaining part of the string ($x < ct$) falls more slowly due to the fixed end. Notice that this effect propagates from the point $x = 0$ to the right at the speed equal to the constant c (it propagates along the characteristic $x - ct = 0$).

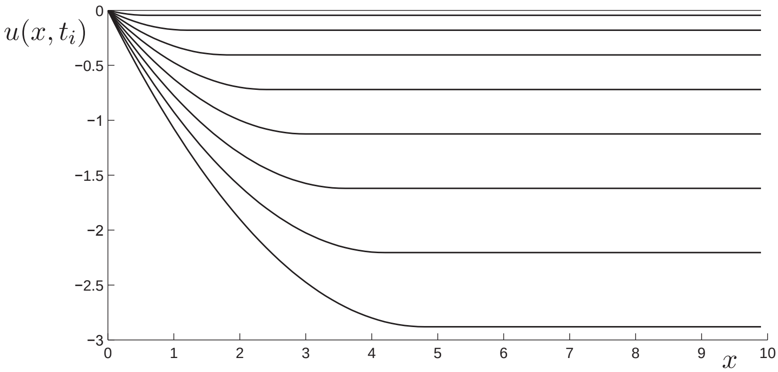


Figure 9.1. A string falling due to the gravitation.

◇

9.2 Fourier Transform

The Fourier transform is another integral transform with properties similar to the Laplace transform. Since it again turns differentiation of the originals into multiplication of the images, it is a useful tool in solving differential equations. Contrary to the Laplace transform, which usually uses the time variable, the Fourier transform is applied to the spatial variable on the whole real line.

First, we start with functions of one spatial variable. The Fourier transform of a function $u = u(x)$, $x \in \mathbb{R}$, is a mapping defined by the formula

$$(\mathcal{F}u)(\xi) \equiv \hat{u}(\xi) = \int_{-\infty}^{+\infty} u(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x. \tag{9.8}$$

If $|u|$ is integrable in $\mathbb{R}$, that is, $\int_{-\infty}^{+\infty} |u|\,\mathrm{d}x < +\infty$, then $\hat{u}$ exists. However, the theory of the Fourier transform usually works with a smaller set of functions.

We define the so called *Schwartz space* $\mathscr{S}$ as the space of functions on $\mathbb{R}$ that have continuous derivatives of all orders and that, together with their derivatives, decrease to zero for $x \to \pm + \infty$ more rapidly than $|x|^{-n}$ for an arbitrary $n \in \mathbb{N}$. It means

$$\mathscr{S} = \left\{ u \in C^{\infty};\ \exists M \in \mathbb{R},\ \left| \frac{\mathrm{d}^k u}{\mathrm{d}x^k} \right| \leq \frac{M}{|x|^n} \text{ for } |x| \to +\infty,\ k \in \mathbb{N} \cup \{0\};\ n \in \mathbb{N} \right\}.$$

It can be shown that, if $u \in \mathscr{S}$, then $\hat{u} \in \mathscr{S}$, and vice versa. We say that the Schwartz space $\mathscr{S}$ is closed with respect to the Fourier transform.

It is important to mention that there exists no established convention how to define the Fourier transform. In literature, we can find an equivalent of the definition (9.8) with the constant $1/\sqrt{2\pi}$ or $1/(2\pi)$ in front of the integral. There also exist definitions with positive sign in the exponent. The reader should keep this fact in mind while working with various sources or using the transformation tables.

The fundamental formula of the Fourier transform is that for the image of the k^{th} derivative $u^{(k)}$:

$$(\mathcal{F}u^{(k)})(\xi) = (i\xi)^k \hat{u}(\xi), \quad u \in \mathscr{S}. \tag{9.9}$$

The derivation of this formula is based on integration by parts where all "boundary values" vanish due to zero values of the function and its derivatives at infinity. In the case of functions of two variables, say $u = u(x,t)$, the variable t plays the role of a parameter and we define

$$(\mathcal{F}u)(\xi,t) \equiv \hat{u}(\xi,t) = \int_{-\infty}^{+\infty} u(x,t)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x. \tag{9.10}$$

The derivatives with respect to the spatial variable are transformed analogously as in (9.9), the derivatives with respect to the time variable t stay unchanged; thus, for instance,

$$\begin{aligned}
(\mathcal{F}u_x)(\xi,t) &= (\mathrm{i}\xi)\hat{u}(\xi,t), \\
(\mathcal{F}u_{xx})(\xi,t) &= (\mathrm{i}\xi)^2\hat{u}(\xi,t), \\
(\mathcal{F}u_t)(\xi,t) &= \hat{u}_t(\xi,t).
\end{aligned}$$

The PDE in two variables x,t passes under the Fourier transform to an ODE in the t-variable. By solving it, we obtain the transformed function (the image) $\hat{u}$ which can be converted to the original function u by the *inverse Fourier transform*

$$(\mathcal{F}^{-1}\hat{u})(x,t) \equiv u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}(\xi,t)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi. \tag{9.11}$$

In comparison with the inverse Laplace transform, where the general inverse formula is quite complicated, this relation is very simple. Nevertheless, it is convenient to use the transformation tables or some software packages when solving particular problems.

It is again important to recall that in the case of modified definition relation (9.10), it is necessary to modify the inverse relation (9.11) as well.

As in the case of the Laplace transform, the Convolution Theorem holds true for the Fourier transform and is directly applicable for solving differential equations. However, the convolution of two functions u and v is now defined in the following way:

$$(u*v)(x) = \int_{-\infty}^{+\infty} u(x-y)v(y)\,\mathrm{d}y.$$

Theorem 9.7. *If $u, v \in \mathscr{S}$, then*

$$\mathcal{F}(u*v)(\xi) = \hat{u}(\xi)\hat{v}(\xi).$$

Example 9.8 (Cauchy Problem for Diffusion Equation). Now we use the Fourier transform for derivation of the solution of the Cauchy problem for the diffusion equation which we treated in Chapter 5. Thus, let us consider the problem

$$\begin{cases} u_t - ku_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{9.12}$$

Let us assume $\varphi \in \mathscr{S}$. Using the Fourier transform, we reduce the diffusion equation to the form

$$\hat{u}_t = -\xi^2 k\hat{u},$$

which is an ODE in the t-variable for the required function $\hat{u}(\xi,t)$ with the parameter ξ. Its solution is

$$\hat{u}(\xi,t) = C\mathrm{e}^{-\xi^2 kt}.$$

The initial condition implies $\hat{u}(\xi,0) = \hat{\varphi}(\xi)$ and thus $C = \hat{\varphi}(\xi)$. The solution in images then assumes the form

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi)\mathrm{e}^{-\xi^2 kt}.$$

If we use the transformation relation

$$\mathcal{F}\left(\frac{1}{\sqrt{4\pi kt}}\mathrm{e}^{-x^2/(4kt)}\right) = \mathrm{e}^{-\xi^2 kt}$$

and the Convolution Theorem, we obtain the solution of the Cauchy problem (9.12) in the form

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}}\mathrm{e}^{-(x-y)^2/(4kt)}\varphi(y)\,\mathrm{d}y, \tag{9.13}$$

which is exactly formula (5.6) derived in Chapter 5. ◇

Remark 9.9. When using the Fourier transform, we have obtained the solution (9.13) under the assumption that the initial condition φ belongs to the Schwartz space. However, once the solution is derived, we can try to show that it exists even under weaker assumptions on the function φ. It can be proved, for instance, that the function u in (9.13) solves problem (9.12) provided φ is a continuous and bounded function on $\mathbb{R}$.

Example 9.10 (Cauchy Problem for Wave Equation). Let us solve the Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x), & u_t(x,0) = \psi(x). \end{cases} \tag{9.14}$$

We apply again the Fourier transform with respect to the spatial variable to the equation and both initial conditions. Thus we obtain the transformed problem

$$\begin{cases} \hat{u}_{tt}(\xi,t) + c^2\xi^2\hat{u}(\xi,t) = 0, \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi). \end{cases}$$

Its solution is the function

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi)\cos c\xi t + \frac{1}{c\xi}\hat{\psi}(\xi)\sin c\xi t.$$

The solution of the original problem is then found by the inverse Fourier transform:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\hat{\varphi}(\xi)\cos c\xi t + \frac{1}{c\xi}\hat{\psi}(\xi)\sin c\xi t\right)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi. \tag{9.15}$$

This integral expression, where the Fourier transforms of the initial conditions occur, is not very transparent. Nevertheless, it can be converted to d'Alembert's formula (4.8) derived in Chapter 4. Indeed, substituting the complex representation of the sine and cosine functions into (9.15), we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2}\hat{\varphi}(\xi)\left(\mathrm{e}^{\mathrm{i}c\xi t}+\mathrm{e}^{-\mathrm{i}c\xi t}\right)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi \\ &\quad +\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2\mathrm{i}c\xi}\hat{\psi}(\xi)\left(\mathrm{e}^{\mathrm{i}c\xi t}-\mathrm{e}^{-\mathrm{i}c\xi t}\right)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi. \end{aligned} \tag{9.16}$$

The first integral on the right-hand side can be written as

$$\frac{1}{4\pi}\int_{-\infty}^{+\infty}\left(\hat{\varphi}(\xi)\,\mathrm{e}^{\mathrm{i}(x+ct)\xi}+\hat{\varphi}(\xi)\,\mathrm{e}^{\mathrm{i}(x-ct)\xi}\right)\,\mathrm{d}\xi,$$

which is (using the definition of the inverse Fourier transform (9.11)) exactly the first term in d'Alembert's formula

$$\frac{1}{2}\left(\varphi(x+ct)+\varphi(x-ct)\right).$$

Similarly, the second integral term in (9.16) equals

$$\frac{1}{4\pi c}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}\xi}\hat{\psi}(\xi)\,\left(\mathrm{e}^{\mathrm{i}(x+ct)\xi}-\mathrm{e}^{\mathrm{i}(x-ct)\xi}\right)\,\mathrm{d}\xi=\frac{1}{4\pi c}\int_{-\infty}^{+\infty}\hat{\psi}(\xi)\int_{x-ct}^{x+ct}\mathrm{e}^{\mathrm{i}y\xi}\,\mathrm{d}y\,\mathrm{d}\xi.$$

Changing the order of integration and using again the inverse Fourier transform, we obtain the second term in d'Alembert's formula

$$\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,\mathrm{d}y.$$

◇

Remark 9.11. In some cases, the methods of integral transforms are applicable also to equations with non-constant coefficients. Let us consider, for example, the Cauchy problem for the transport equation

$$\begin{cases} tu_x+u_t=0, \quad x\in\mathbb{R},\ t>0, \\ u(x,0)=f(x). \end{cases}$$

Since the varying coefficient is – in this case – the time variable t, we use the Fourier transform with time playing the role of a parameter. We have

$$\mathcal{F}(tu_x)=t\mathcal{F}(u_x)=\mathrm{i}\xi t\hat{u}.$$

Transforming the equation and the initial conditions, we obtain

$$\mathrm{i}\xi t\hat{u} + \hat{u}_t = 0, \quad \hat{u}(\xi, 0) = \hat{f}(\xi),$$

and hence

$$\hat{u}(\xi, t) = \hat{f}(\xi)\mathrm{e}^{-\mathrm{i}\frac{t^2}{2}\xi}.$$

By the inverse Fourier transform (e.g., using the transformation formulas), we obtain the solution of the original equation in the form

$$u(x, t) = f\left(x - \frac{t^2}{2}\right).$$

Remark 9.12 (Laplace and Poisson Equations). The Laplace and Poisson equations can also be solved, in some cases, by the method of integral transforms.

As an example, let us consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ y > 0, \\ u(x, 0) = f(x), \\ u(x, y) \text{ bounded for } y \to +\infty. \end{cases}$$

We will search for a solution using the Fourier transform with respect to x. Its application to our problem leads to the equation

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0,$$

whose general solution is $\hat{u}(\xi, y) = a(\xi)\mathrm{e}^{-\xi y} + b(\xi)\mathrm{e}^{\xi y}$ for arbitrary functions $a,\ b$. The boundedness assumption implies

$$\begin{aligned} b(\xi) = 0 \qquad &\text{for } \xi > 0, \\ a(\xi) = 0 \qquad &\text{for } \xi < 0. \end{aligned}$$

Hence, $\hat{u}(\xi, y) = c(\xi)\mathrm{e}^{-|\xi|y}$ with c being an arbitrary function. If we take into account the boundary condition, we derive $c(\xi) = \hat{f}(\xi)$ and thus

$$\hat{u}(\xi, y) = \mathrm{e}^{-|\xi|y}\hat{f}(\xi).$$

The inverse transformation leads to the solution of the original problem in the form of a convolution:

$$u(x, y) = \left(\frac{y}{\pi}\frac{1}{x^2 + y^2}\right) * f = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{f(\tau)\,\mathrm{d}\tau}{(x - \tau)^2 + y^2}.$$

The reader should notice that in this convolution y is just a parameter.

9.3 Exercises

1. Derive the following transform relations:

   (a) $\mathcal{L}\{1\} = \dfrac{1}{s}, \quad s > 0,$

   (b) $\mathcal{L}\{t\} = \dfrac{1}{s^2},$

   (c) $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n \in \mathbb{N},\ s > 0,$

   (d) $\mathcal{L}\{\mathrm{e}^{at}\} = \dfrac{1}{s-a}, \quad s > a,$

   (e) $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}, \quad s > 0,$

   (f) $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}, \quad s > 0.$

2. Derive the following basic properties of the Laplace transform (here $U = \mathcal{L}\{u\}$):

   (a) $\mathcal{L}\{t^n u(t)\} = (-1)^n U^{(n)}(s),$

   (b) $\mathcal{L}\{\mathrm{e}^{at} u(t)\} = U(s-a),$

   (c) $\mathcal{L}\{\int_0^t u(\tau)\mathrm{d}\tau\} = \dfrac{1}{s}U(s), \quad s > 0,$

   (d) $\mathcal{L}\{\dfrac{1}{t}u(t)\} = \displaystyle\int_s^{+\infty} U(\sigma)\mathrm{d}\sigma,$

   (e) $\mathcal{L}\{u(ct)\} = \dfrac{1}{c}U(\dfrac{s}{c}), \quad c > 0.$

3. Using substitution and Fubini's Theorem, prove the formulas in Theorems 9.2 and 9.7.

In the following exercises we suppose that all the solutions we search for are bounded.

4. Using the Laplace transform method, solve the following initial boundary value problems. Simplify the results as much as possible.

   (a)
$$\begin{cases} u_t = u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = 70, \\ u(x,0) = 0. \end{cases}$$

   $[u(x,t) = 70\,\mathrm{erfc}\,(\frac{x}{\sqrt{4t}})]$

(b)
$$\begin{cases} u_{tt} = u_{xx} + t, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = 0. \end{cases}$$

$[u(x,t) = \frac{1}{3!}t^3 - \frac{1}{3!}\mathcal{H}(t-x)(t-x)^3]$

(c)
$$\begin{cases} u_{tt} = u_{xx} + \mathrm{e}^{-t}, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = 0. \end{cases}$$

(d)
$$\begin{cases} u_{tt} = u_{xx} - g, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = 1. \end{cases}$$

$[u(x,t) = t - (t-x)\mathcal{H}(t-x) - \frac{g}{2}(t^2 - (t-x)^2\mathcal{H}(t-x))]$

(e)
$$\begin{cases} u_{tt} = u_{xx} + t^2, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = 0. \end{cases}$$

(f)
$$\begin{cases} u_{tt} = u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = \sin t, \\ u(x,0) = 0,\ u_t(x,0) = 1. \end{cases}$$

$[u(x,t) = t + \sin(t-x)\mathcal{H}(t-x) - (t-x)\mathcal{H}(t-x)]$

(g)
$$\begin{cases} u_{tt} = u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = 1. \end{cases}$$

5. Using the Laplace transform method, solve the initial boundary value problem

$$\begin{cases} u_{tt} = u_{xx} + \sin \pi x, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0,\ u(1,t) = 0, \\ u(x,0) = 0,\ u_t(x,0) = 0. \end{cases}$$

6. Show that the solution of the initial boundary value problem

$$\begin{cases} u_t = ku_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = T_0, \\ u(x,0) = T_1 \end{cases}$$

is given by

$$u(x,t) = (T_0 - T_1)\mathrm{erfc}\left(\frac{x}{\sqrt{4kt}}\right) + T_1 = (T_0 - T_1)\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) + T_0.$$

7. Use the Laplace transform to solve the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} + \cos\omega t \sin\pi x, & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_t(x,0) = 0. \end{cases}$$

Assume that $\omega > 0$ and be careful of the case $\omega = ct$. Check your answer by direct differentiation.

8. Prove the following transform relations (here $\hat{u}(\xi) = \mathcal{F}\{u(x)\}$):

(a) $u(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a, \end{cases} \qquad \hat{u}(\xi) = 2\dfrac{\sin a\xi}{\xi},$

(b) $u(x) = \begin{cases} 1 - \frac{|x|}{a}, & |x| < a, \\ 0, & |x| > a, \end{cases} \qquad \hat{u}(\xi) = 4\dfrac{\sin^2(a\xi/2)}{a\xi^2},$

(c) $u(x) = \dfrac{1}{x^2 + a^2},\ a > 0, \quad \hat{u}(\xi) = \dfrac{\pi e^{-a\xi}}{a},$

(d) $u(x) = e^{-ax^2},\ a > 0, \quad \hat{u}(\xi) = \dfrac{\sqrt{\pi}}{\sqrt{a}} e^{-\xi^2/4a},$

(e) $u(x) = \dfrac{\sin ax}{x},\ a > 0, \quad \hat{u}(\xi) = \begin{cases} \pi, & |\xi| < a, \\ \frac{\pi}{2}, & |\xi| = a, \\ 0, & |\xi| > a. \end{cases}$

9. Derive the following basic properties of Fourier transform (here $\hat{u} = \mathcal{F}\{u\}$):

(a) $\mathcal{F}\{x^n u(x)\} = i^n \hat{u}^{(n)}(\xi)$,

(b) $\mathcal{F}\{e^{iax} u(x)\} = \hat{u}(\xi - a)$,

(c) $\mathcal{F}\{u(x - a)\} = e^{-ia\xi}\hat{u}(\xi)$,

(d) $\mathcal{F}\{u(ax)\} = \dfrac{1}{|a|}\hat{u}(\dfrac{\xi}{a}), \quad a \neq 0.$

10. Using the Fourier transform method, solve the following Cauchy problems.

(a) $$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \dfrac{1}{1 + x^2},\ u_t(x,0) = 0. \end{cases}$$

$$\left[u(x,t) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|\xi|} \cos\xi t\, e^{i\xi x} d\xi\right]$$

(b) $$\begin{cases} u_t = \frac{1}{100} u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x), \end{cases}$$

where $\varphi(x) = 100$ for $x \in (-1, 1)$ and $\varphi(x) = 0$ elsewhere.

(c) $$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \sqrt{\dfrac{2}{\pi}} \dfrac{\sin x}{x},\ u_t(x,0) = 0. \end{cases}$$

$$[u(x,t) = \tfrac{50}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} \tfrac{1}{1+\xi^2} \mathrm{e}^{-(x-\xi)^2/4t} \mathrm{d}\xi]$$

(d) $$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x), \end{cases}$$

where $\varphi(x) = 1 - \dfrac{|x|}{2}$ for $x \in (-2,2)$ and $\varphi(x) = 0$ elsewhere.

(e) $$\begin{cases} u_t = \mathrm{e}^{-t} u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = 100. \end{cases}$$

11. Using the Fourier transform, solve the linearized Korteweg-deVries equation

$$u_t = u_{xxx}, \quad x \in \mathbb{R},\ t > 0,$$

subject to the initial condition

$$u(x,0) = \mathrm{e}^{-x^2/2}.$$

12. Using the Fourier transform, solve the Cauchy problem

$$\begin{cases} u_{tt} = a^2 u_{txx} - b u_{xxxx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x), \\ u_t(x,0) = \psi(x). \end{cases}$$

13. Using the Fourier transform, solve the heat equation with a convection term

$$u_t = k u_{xx} + \mu u_x, \quad x \in \mathbb{R},\ t > 0,$$

with an initial condition $u(x,0) = \varphi(x)$, assuming that $u(x,t)$ is bounded and $k > 0$.

$$[u(x,t) = \tfrac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} \varphi(y) \mathrm{e}^{-(\mu t + x - y)^2/(4kt)} \mathrm{d}y]$$

14. Use the Fourier transform in the x variable to find the harmonic function in the half-plane $y > 0$ that satisfies the Neumann condition $\frac{\partial u}{\partial y} = h(x)$ on the boundary $y = 0$.

15. Use the Fourier transform to solve the Laplace equation $u_{xx} + u_{yy} = 0$ in the infinite strip $\{x \in \mathbb{R},\ 0 < y < 1\}$, together with the conditions $u(x,0) = 0$ and $u(x,1) = f(x)$.

$$[u(x,y) = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \tfrac{1}{\pi \sinh k} f(\xi) \sinh ky \, \cos(kx - k\xi) \, \mathrm{d}\xi \, \mathrm{d}k]$$

Chapter 10

General Principles

In this chapter we summarize the main qualitative properties of the PDEs we dealt with in the previous chapters.

10.1 Principle of Causality (Wave Equation)

From d'Alembert's formula and Chapter 4, we already know the following.

> The values of the initial displacement φ and the initial velocity ψ at a point x_0 influence the solution of the wave equation only in the *domain of influence*, which is a sector determined by the characteristic lines $x \pm ct = x_0$ (see Figure 10.1).
>
> Conversely, a solution at a point (x, t) is influenced only by the values from the *domain of dependence*, which is formed by the *characteristic triangle* with vertices $(x - ct, 0)$, $(x + ct, 0)$ and (x, t) (see Figure 10.1).

However, these properties follow directly from the wave equation itself and the knowledge of the formula for the solution is not needed. To prove it, we proceed in the following way.

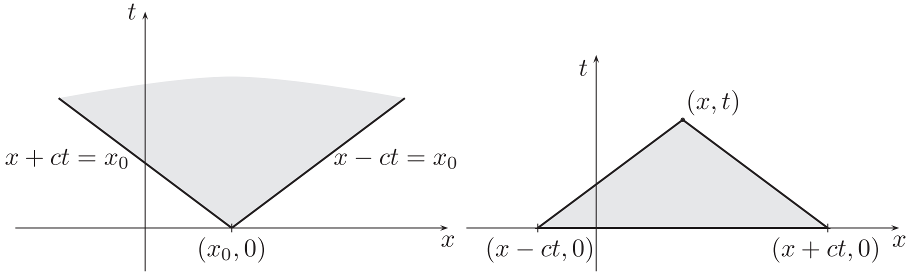


Figure 10.1. Domain of influence of the point $(x_0, 0)$ and domain of dependence of the point (x, t).

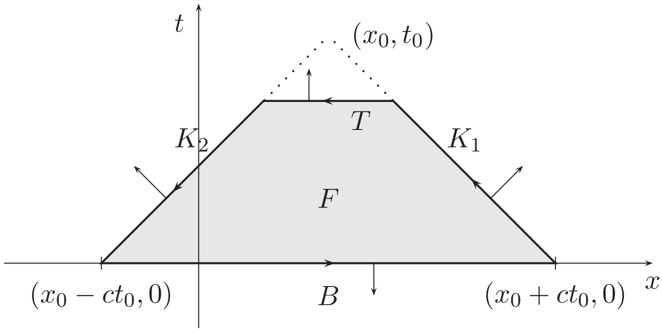


Figure 10.2. Trapezoid of characteristic triangle.

We start with the wave equation $u_{tt} - c^2 u_{xx} = 0$ and multiply it by u_t. The resulting identity can be written as

$$\begin{aligned} 0 &= u_{tt}u_t - c^2 u_{xx}u_t \\ &= \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2 u_x^2\right)_t - c^2(u_t u_x)_x \\ &= (\partial_x, \partial_t) \cdot \left(-c^2 u_t u_x,\ \frac{1}{2}u_t^2 + \frac{1}{2}c^2 u_x^2\right). \end{aligned} \tag{10.1}$$

Notice that the last scalar product is a two-dimensional divergence of the vector $\boldsymbol{f} = (-c^2 u_t u_x,\ \frac{1}{2}u_t^2 + \frac{1}{2}c^2 u_x^2)$. Now, we integrate (10.1) over a trapezoid F, which is part of the characteristic triangle (see Figure 10.2). If we use Green's Theorem, which can be written as

$$\iint\limits_F \operatorname{div} \boldsymbol{f}\, \mathrm{d}x\, \mathrm{d}t = \int\limits_{\partial F} \boldsymbol{f} \cdot \boldsymbol{n}\, \mathrm{d}s$$

(cf. its other version used in Section 4.4), we obtain

$$\int\limits_{\partial F} \left((-c^2 u_t u_x)n_1 + \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2 u_x^2\right) n_2\right) \mathrm{d}s = 0. \tag{10.2}$$

Here $\boldsymbol{n} = (n_1, n_2)$ is an outer normal vector to ∂F. The boundary ∂F consists of "top" T, "bottom" B, and "sides" $K = K_1 \cup K_2$. Thus, the integral in (10.2) splits into four parts

$$\int_{\partial F} = \int_T + \int_B + \int_{K_1} + \int_{K_2} = 0.$$

Now, we consider each part separately. On the top T, the normal vector is $\boldsymbol{n} = (0, 1)$ and thus

$$\int_T \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}s = \int_T \frac{1}{2}(u_t^2 + c^2 u_x^2) \, \mathrm{d}s.$$

On the bottom B, we have $\boldsymbol{n} = (0, -1)$ and thus

$$\int_B \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}s = \int_B -\frac{1}{2}(u_t^2 + c^2 u_x^2) \, \mathrm{d}s = \int_B -\frac{1}{2}(\psi^2 + c^2 \varphi_x^2) \, \mathrm{d}s.$$

On the side K_1, there is $\boldsymbol{n} = \frac{1}{\sqrt{1+c^2}}(1, c)$ and

$$\begin{aligned} \int_{K_1} \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}s &= \frac{1}{\sqrt{1+c^2}} \int_{K_1} \left(c\frac{1}{2}(u_t^2 + c^2 u_x^2) - c^2 u_t u_x \right) \mathrm{d}s \\ &= \frac{c}{2\sqrt{1+c^2}} \int_{K_1} (u_t - c u_x)^2 \, \mathrm{d}s \geq 0. \end{aligned}$$

Similarly, on K_2, we have $\boldsymbol{n} = \frac{1}{\sqrt{1+c^2}}(-1, c)$ and

$$\begin{aligned} \int_{K_1} \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}s &= \frac{1}{\sqrt{1+c^2}} \int_{K_1} \left(c\frac{1}{2}(u_t^2 + c^2 u_x^2) + c^2 u_t u_x \right) \mathrm{d}s \\ &= \frac{c}{2\sqrt{1+c^2}} \int_{K_1} (u_t + c u_x)^2 \, \mathrm{d}s \geq 0. \end{aligned}$$

Putting all these partial results together, we can conclude that

$$\int_T \frac{1}{2}(u_t^2 + c^2 u_x^2) \, \mathrm{d}s - \int_B \frac{1}{2}(\psi^2 + c^2 \varphi_x^2) \, \mathrm{d}s \leq 0$$

or, equivalently,

$$\int_T (u_t^2 + c^2 u_x^2) \, \mathrm{d}s \leq \int_B (\psi^2 + c^2 \varphi_x^2) \, \mathrm{d}s. \tag{10.3}$$

If we now assume that both the functions φ and ψ are zero on B, inequality (10.3) implies $u_t^2 + c^2 u_x^2 = 0$ on T, and thus $u_t \equiv u_x \equiv 0$ on T. Moreover, since this result holds true for a trapezoid of any height, we obtain that u_t and u_x are zero (and thus u is constant) in the whole characteristic triangle. And since we have assumed $u \equiv 0$ on B, we can conclude that $u \equiv 0$ in the whole triangle.

This result also implies uniqueness: if we take two solutions u_1 and u_2 with the same initial conditions on B, i.e., in the interval $(x_0 - ct_0, x_0 + ct_0)$, then $u_1 \equiv u_2$ in the whole characteristic triangle.

The principle of causality can be obtained also by other (and probably simpler) methods. However, the advantage of this approach is its applicability in any dimension (see Section 13.4). Notice that the analogue of Green's Theorem in two-dimensional case is the so called Divergence Theorem in higher dimensions.

10.2 Energy Conservation Law (Wave Equation)

Let us start with the infinitely long string described by the equation

$$\rho u_{tt}(x,t) = T u_{xx}(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{10.4}$$

where ρ, T are constants (we assume the displacements to be small enough). Moreover, let us consider for simplicity $\varphi \equiv \psi \equiv 0$ for $|x| > R$, $R > 0$ large enough. We suppose that the Cauchy problem has a classical solution.

The kinetic energy of the string takes the form

$$E_k(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \rho u_t^2(x,t)\,\mathrm{d}x$$

(cf. the well-known formula $E_k = \frac{1}{2}mv^2$ for the kinetic energy of the mass point). The continuity assumption on u_{tt} implies

$$\frac{\mathrm{d}E_k(t)}{\mathrm{d}t} = \rho \int_{-\infty}^{+\infty} u_t(x,t) u_{tt}(x,t)\,\mathrm{d}x,$$

and, after substituting for u_{tt} from equation (10.4), we obtain

$$\begin{aligned} \frac{\mathrm{d}E_k(t)}{\mathrm{d}t} &= T \int_{-\infty}^{+\infty} u_t(x,t) u_{xx}(x,t)\,\mathrm{d}x \qquad (10.5) \\ &= \Big[T u_t(x,t) u_x(x,t)\Big]_{x=-\infty}^{x=+\infty} - T \int_{-\infty}^{+\infty} u_{tx}(x,t) u_x(x,t)\,\mathrm{d}x. \end{aligned}$$

Since

$$\Big[T u_t(x,t) u_x(x,t)\Big]_{x=-\infty}^{x=+\infty} = 0$$

(cf. the assumptions $\varphi \equiv \psi \equiv 0$ for $|x|$ large enough and the principle of causality) and

$$u_{tx}(x,t)u_x(x,t) = \left(\frac{1}{2}u_x^2(x,t)\right)_t,$$

we obtain, after substituting into (10.5),

$$\frac{\mathrm{d}E_k(t)}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{+\infty} \frac{1}{2}Tu_x^2(x,t)\,\mathrm{d}x. \tag{10.6}$$

The potential energy of the string can be expressed as

$$E_p(t) = \frac{1}{2}T\int\limits_{-\infty}^{+\infty} u_x^2(x,t)\,\mathrm{d}x \tag{10.7}$$

(see Remark 10.1 below). Relations (10.6) and (10.7) imply

$$\frac{\mathrm{d}E_k(t)}{\mathrm{d}t} = -\frac{\mathrm{d}E_p(t)}{\mathrm{d}t}.$$

Since the total string energy is

$$E(t) = E_k(t) + E_p(t),$$

we obtain

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = 0,$$

which is – in the language of mathematics – the *energy conservation law*. In other words, the total string energy

$$\boxed{E(t) = \frac{1}{2}\int\limits_{-\infty}^{+\infty} \left(\rho u_t^2(x,t) + Tu_x^2(x,t)\right)\mathrm{d}x \equiv E}$$

is constant with respect to t!

Remark 10.1. Formula (10.7) can be derived, for instance, in the following way. The potential energy represents the product of the force and the extension caused by this force (cf. the well-known formula $E_p = mgh$ for the potential energy of the mass point). In our case, the acting force is represented by the tension T. The extension h of the string of length l is the difference between the arc length (of deflected string) s and the original length l, thus

$$h(t) = s(t) - l = \int\limits_0^l \sqrt{1 + u_x^2(x,t)}\,\mathrm{d}x - l.$$

If we replace the square root on the right-hand side by the first two terms of its Taylor expansion, we obtain

$$h(t) \approx \int_0^l \left(1 + \frac{1}{2}u_x^2(x,t)\right) \mathrm{d}x - l = \frac{1}{2}\int_0^l u_x^2(x,t)\,\mathrm{d}x.$$

The potential energy is then $E_p(t) = \dfrac{1}{2}T\displaystyle\int_0^l u_x^2(x,t)\,\mathrm{d}x$. For the string of infinite length, we obtain expression (10.7).

Example 10.2. Let us determine the total energy of the infinitely long string, if the initial velocity at time $t = 0$ is zero and the initial displacement is given by the function

$$\varphi(x) = \begin{cases} b - \frac{b}{a}|x| & \text{for } |x| \le a, \\ 0 & \text{for } |x| > a. \end{cases}$$

Since the total string energy does not depend on time, we have

$$E = E(t) = E(0).$$

So, we need not find the solution at arbitrary time t, since the initial condition is sufficient for determination of the total energy. Zero initial velocity implies zero kinetic energy at time $t = 0$, thus

$$E_k(0) = 0.$$

The potential energy is expressed by relation (10.7), i.e.,

$$E_p(0) = \frac{1}{2}\int_{-\infty}^{+\infty} T\varphi_x^2(x)\,\mathrm{d}x = \frac{1}{2}\int_{-a}^{a} T\varphi_x^2(x)\,\mathrm{d}x = \frac{1}{2}\int_{-a}^{a} T\frac{b^2}{a^2}\,\mathrm{d}x = \frac{b^2}{a}T.$$

The total energy is then the sum of the potential and kinetic energies:

$$E = E(0) = \frac{b^2}{a}T.$$

◇

10.3 Ill-Posed Problem (Diffusion Equation for Negative t)

First, let us consider the following initial value problem for a "special variant of the diffusion equation":

$$u_t = -u_{xx}, \quad u(x,0) = 1, \quad x \in \mathbb{R},\ t > 0. \tag{10.8}$$

Here, the diffusion coefficient k is equal to -1! Obviously, $u(x,t) \equiv 1$ is the solution. On the other hand, we can easily verify that the function

$$u_n(x,t) = 1 + \frac{1}{n} \sin nx \; \mathrm{e}^{n^2 t}$$

solves the initial value problem

$$u_t = -u_{xx}, \quad u(x,0) = 1 + \frac{1}{n} \sin nx, \quad x \in \mathbb{R},\ t < 0, \tag{10.9}$$

for an arbitrary $n \in \mathbb{N}$. The initial conditions in problems (10.8) and (10.9) differ only in the term $\frac{1}{n} \sin nx$, which converges to zero uniformly for $n \to +\infty$. However, the difference of the corresponding solutions is

$$\frac{1}{n} \sin nx \; \mathrm{e}^{n^2 t},$$

which for any fixed x (except integer multiples of π) goes to infinity for $n \to +\infty$. It means that the *stability of the constant solution* $u(x,t) \equiv 1$ *fails* for the diffusion equation with a negative coefficient and such a problem is ill-posed.

Now, let us consider the initial value problem for the standard diffusion equation ($k > 0$) with negative time

$$u_t = k u_{xx}, \quad u(x,0) = \varphi(x), \quad x \in \mathbb{R},\ t < 0. \tag{10.10}$$

Using the substitution

$$w(x,t) = u(x,-t),$$

$$w_t = -u_t, \quad w_{xx} = u_{xx},$$

we obtain the problem

$$w_t = -k w_{xx}, \quad w(x,0) = \varphi(x), \quad x \in \mathbb{R},\ t > 0. \tag{10.11}$$

However, we have shown above that such a problem (for $k = 1$) is ill-posed. So, problem (10.10) is ill-posed as well.

This phenomenon has also its physical explanation:

> Diffusion, heat flow, the so called Brownian motion, etc. are *irreversible processes* and return in time leads to *chaos*. On the other hand, the wave motion is a reversible process and the wave equation for $t < 0$ is well-posed.

10.4 Maximum Principle (Heat Equation)

Let us now leave the problems on the whole real line and consider the heat flow in a finite bar of length l, whose ends are kept at temperatures $g(t)$ and $h(t)$, respectively, at time t, while the initial distribution of temperature at time $t = 0$ is given by a continuous function $\varphi(x)$. We have thus an initial boundary value problem

$$\begin{cases} u_t = ku_{xx}, \quad x \in (0,l),\ t > 0, \\ u(x,0) = \varphi(x), \\ u(0,t) = g(t),\ u(l,t) = h(t). \end{cases} \tag{10.12}$$

We introduce a notion of a space-time cylinder $\Re$, by which we understand, in our case, a rectangle in the xt-plane whose one vertex is placed at the origin $(0,0)$, and two sides lie on coordinate axes (see Figure 10.3). The length of side on the x-axis is l, the length of side on the t-axis is T. By the bottom of the cylinder $\Re$ we understand the horizontal side lying on the x-axis, by the cylinder jacket we understand both lateral sides. The upper horizontal line is then called the top of the cylinder $\Re$. The reason why we use these terms is that the maximum principle can be derived in the same way also for the diffusion equation in more spatial variables (where the idea of the cylinder is more realistic).

We will prove the following assertion.

Theorem 10.3 (Maximum Principle). *Let $u = u(x,t)$ be a classical solution of problem* (10.12)*. Then u achieves its extremal values (minimal as well as maximal) on the bottom or jacket of the space-time cylinder $\Re$.*

Actually, a stronger assertion holds true (see, e.g., Protter, Weinberger [17]):

The values of the function u inside the cylinder and on the top are *strictly* less (or greater) than the maximum (or minimum, respectively) on the rest of the boundary of the cylinder $\Re$ (unless the function $u(x,t)$ is constant).

Proof of Theorem 10.3. The proof will be done for the maximum value. In the case of the minimum value, we would proceed analogously (using the fact that $\min u = -\max(-u)$).

The idea of the proof uses the fact that the first partial derivatives of the function must be zero and the second derivatives must be non-positive at the *inner* maximum point. If we could exclude the case $u_{xx} = 0$, we would obtain

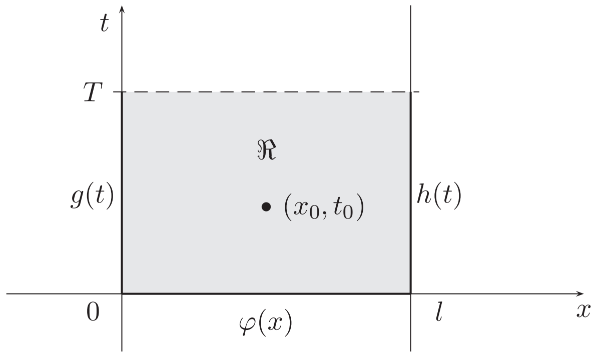


Figure 10.3. Space-time cylinder $\Re$.

$u_{xx} < 0$ together with $u_t = 0$ and hence $u_{xx} \neq u_t$. This contradiction with the diffusion equation would imply that the maximum point must lie somewhere on the boundary. However, since we are not able to exclude the case $u_{xx} = 0$, we must proceed in a more careful way.

Let us denote by M the maximal value of the function $u(x,t)$ on the sides $t = 0$, $x = 0$, $x = l$, and let us put $v(x,t) = u(x,t) + \epsilon x^2$, where ϵ is a positive constant. Our goal is to show that $v(x,t) \leq M + \epsilon l^2$ on the whole cylinder $\Re$.

The definition of the function v implies that the inequality $v(x,t) \leq M + \epsilon l^2$ is satisfied on the boundary lines $t = 0$, $x = 0$ and $x = l$. Further, the so called *diffusion inequality* $v_t(x,t) - kv_{xx}(x,t) < 0$ holds for all $(x,t) \in \Re$. Indeed,

$$\begin{aligned} v_t(x,t) - kv_{xx}(x,t) &= u_t(x,t) - k(u(x,t) + \epsilon x^2)_{xx} \\ &= u_t(x,t) - ku_{xx}(x,t) - 2\epsilon k \\ &= -2\epsilon k < 0. \end{aligned}$$

Since v is a continuous function and $\Re$ is a bounded closed set, the maximum point (x_0, t_0) of the function v must exist on $\Re$. First, let us suppose that this point lies *inside* the cylinder $\Re$ $(0 < x_0 < l,\ 0 < t_0 < T)$. Now, however, $v_t = 0$, $v_{xx} \leq 0$ must hold at (x_0, t_0), which contradicts the diffusion inequality. Then, let (x_0, t_0) lie on the upper side of the space-time cylinder $\Re$, that is, $t_0 = T$, $0 < x_0 < l$. Thus $v_x(x_0, t_0) = 0$, $v_{xx}(x_0, t_0) \leq 0$ and $v_t(x_0, t_0) \geq 0$, which again contradicts the diffusion inequality. Consequently, the maximum point of the function v must be achieved on the rest of the boundary: on the lines $t = 0$, $x = 0$, $x = l$. Here, however, we have the inequality $v(x,t) \leq M + \epsilon l^2$. These facts imply that the relation $v(x,t) \leq M + \epsilon l^2$ holds for all $(x,t) \in \Re$. If we substitute for v, we obtain $u(x,t) \leq M + \epsilon(l^2 - x^2)$. Since $\epsilon > 0$ has been chosen completely arbitrarily, it follows that

$$u(x,t) \le M$$

for any $(x,t) \in \Re$, which we wanted to prove. □

Remark 10.4. We have seen a variant of Maximum Principle also in the case of the diffusion equation on the whole real line. The maximal as well as minimal values of the solution were achieved at time $t = 0$ and, with the growing time, the solution values were "spread" and tended to some value between those extremes (see Example 5.4).

Corollary 10.5 (Uniqueness). *The initial boundary value problem for the diffusion equation*

$$\begin{cases} u_t - ku_{xx} = f(x,t), \quad x \in (0,l),\ t > 0, \\ u(x,0) = \varphi(x), \\ u(0,t) = g(t),\ u(l,t) = h(t), \end{cases} \tag{10.13}$$

has at most one classical solution.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two classical solutions of problem (10.13). Let us denote $w = u_1 - u_2$. Then the function w satisfies

$$w_t - kw_{xx} = 0, \quad w(x,0) = 0, \quad w(0,t) = 0,\ w(l,t) = 0.$$

According to the maximum principle, the function w achieves its maximum on the sides $t = 0$, $x = 0$, $x = l$, where it is, however, equal to zero. Thus $w(x,t) \le 0$. The same argument for the minimum value yields $w(x,t) \ge 0$. Hence, we obtain $w(x,t) \equiv 0$, and thus $u_1(x,t) \equiv u_2(x,t)$. □

Corollary 10.6 (Uniform stability). *Let u_1, u_2 be two classical solutions of the initial boundary value problem* (10.13) *corresponding to two initial conditions φ_1, φ_2. Then*

$$\max_{x\in[0,l]} |u_1(x,t) - u_2(x,t)| \le \max_{x\in[0,l]} |\varphi_1(x) - \varphi_2(x)|$$

for each $t > 0$. In particular, the classical solution of (10.13) *is stable with respect to small perturbations of the initial condition.*

This statement says that a "small" change in the initial condition results in a "small" change in the solution at arbitrary time.

Proof. Let the solution $u_1(x,t)$ corresponds to the initial condition $\varphi_1(x)$, the solution $u_2(x,t)$ corresponds to the initial condition $\varphi_2(x)$; boundary conditions as well as the right hand side are the same in both cases. Again, let us denote by $w = u_1 - u_2$ the difference of the two solutions. The function w solves the problem

$$w_t - kw_{xx} = 0, \quad w(x,0) = \varphi_1(x) - \varphi_2(x), \quad w(0,t) = 0,\ w(l,t) = 0.$$

The maximum principle then implies

$$w(x,t) = u_1(x,t) - u_2(x,t) \le \max_{x\in[0,l]} (\varphi_1 - \varphi_2) \le \max_{x\in[0,l]} |\varphi_1 - \varphi_2|.$$

Similarly, according to the "minimum principle" (i.e., the maximum principle applied to $-w$),

$$w(x,t) = u_1(x,t) - u_2(x,t) \ge \min_{x\in[0,l]} (\varphi_1 - \varphi_2) \ge -\max_{x\in[0,l]} |\varphi_1 - \varphi_2|.$$

Consequently,

$$\max_{x\in[0,l]} |u_1(x,t) - u_2(x,t)| \le \max_{x\in[0,l]} |\varphi_1(x) - \varphi_2(x)|$$

for each $t > 0$. $\square$

10.5 Energy Method (Diffusion Equation)

We will show another way to prove uniqueness of the classical solution of problem (10.13) and its stability (now, however, with respect to a more general norm). The technique of the proof is called the *energy method.*

Let us consider again two solutions $u_1(x,t)$, $u_2(x,t)$ of problem (10.13) and their difference $w(x,t)$. The function w satisfies the homogeneous diffusion equation with homogeneous boundary conditions $w(0,t) = w(l,t) = 0$. If we multiply the diffusion equation in w by the function w itself, we obtain

$$0 = (w_t - kw_{xx})w = \left(\frac{1}{2}w^2\right)_t + (-kw_x w)_x + kw_x^2.$$

Integrating over the interval $0 < x < l$, we get

$$0 = \int_0^l \left(\frac{1}{2}w^2(x,t)\right)_t \mathrm{d}x - kw_x(x,t)w(x,t)\Big|_{x=0}^{x=l} + k\int_0^l w_x^2(x,t)\,\mathrm{d}x.$$

The second term on the right-hand side vanishes due to the zero boundary conditions. Changing the order of time differentiation and integration in the first term, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \frac{1}{2}w^2(x,t)\,\mathrm{d}x = -k\int_0^l w_x^2(x,t)\,\mathrm{d}x \le 0.$$

But this means that the integral depending on the parameter t, $\int_0^l w^2(x,t)\,\mathrm{d}x$, is – as a function of time t – decreasing, and thus

$$\int_0^l w^2(x,t)\,\mathrm{d}x \le \int_0^l w^2(x,0)\,\mathrm{d}x \tag{10.14}$$

for $t \ge 0$. In the case that both solutions $u_1(x,t)$, $u_2(x,t)$ correspond to the same initial condition, we obtain $w(x,0) = 0$, and hence $\int_0^l w^2(x,t)\,\mathrm{d}x = 0$ for all $t > 0$. This means that $w \equiv 0$ and hence $u_1 \equiv u_2$ for all $t \ge 0$. In other words, we obtain again uniqueness of the solution of the initial boundary value problem (10.13).

If the solutions $u_1(x,t)$, $u_2(x,t)$ correspond to different initial conditions $\varphi_1(x)$, $\varphi_2(x)$, respectively, then $w(x,0) = \varphi_1(x) - \varphi_2(x)$ and relation (10.14) becomes

$$\int_0^l \big(u_1(x,t) - u_2(x,t)\big)^2\,\mathrm{d}x \le \int_0^l \big(\varphi_1(x) - \varphi_2(x)\big)^2\,\mathrm{d}x,$$

which expresses *stability of the solution with respect to the initial condition in the L^2-norm.*

10.6 Maximum Principle (Laplace Equation)

One of the fundamental properties of all harmonic functions is the *Maximum Principle.*

Theorem 10.7 (Maximum Principle). *Let Ω be a bounded domain (i.e., an open and connected set) in $\mathbb{R}^2$. Let $u(x,y)$ be a harmonic function in Ω (that is, $\Delta u = 0$ in Ω) which is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$. Then the maximal and minimal values of the function u are achieved on the boundary $\partial\Omega$.*

As in the diffusion equation case, we can formulate a stronger assertion. We will state it later and use the Poisson formula for its proof, see Section 10.7. For now, however, we put up with the weaker version formulated in Theorem 10.7 and give its elementary proof, which is somewhat similar to that of Theorem 10.3.

Proof of Theorem 10.7. Let us denote $\boldsymbol{x} = (x, y)$ and $|\boldsymbol{x}| = (x^2 + y^2)^{1/2}$. We introduce $v(\boldsymbol{x}) = u(\boldsymbol{x}) + \epsilon|\boldsymbol{x}|^2$, where ϵ is an arbitrarily small positive constant. We have

$$\Delta v = \Delta u + \epsilon\Delta(x^2 + y^2) = 0 + 4\epsilon > 0$$

in the whole domain Ω. If the function v achieved its maximum inside Ω, the inequality $\Delta v = v_{xx} + v_{yy} \leq 0$ would have to hold at such a point, but this contradicts the previous inequality. Thus the maximum of v must be achieved at a point on the boundary – let us denote this point $\boldsymbol{x}_0 \in \partial\Omega$. Then, for all $\boldsymbol{x} \in \Omega$, we obtain

$$u(\boldsymbol{x}) \leq v(\boldsymbol{x}) \leq v(\boldsymbol{x}_0) = u(\boldsymbol{x}_0) + \epsilon|\boldsymbol{x}_0|^2 \leq \max_{\boldsymbol{y}\in\partial\Omega} u(\boldsymbol{y}) + \epsilon|\boldsymbol{x}_0|^2.$$

Since ϵ has been chosen arbitrary, we have

$$u(\boldsymbol{x}) \leq \max_{\boldsymbol{y}\in\partial\Omega} u(\boldsymbol{y}) \qquad \forall \boldsymbol{x} \in \overline{\Omega} = \Omega \cup \partial\Omega.$$

For the case of minimum, we proceed analogously. □

Theorem 10.8 (Uniqueness of the solution of the Dirichlet problem). *The solution of the Dirichlet problem for the Poisson equation in the domain Ω is uniquely determined.*

Proof. Let us assume that

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \quad u = h & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta v = f & \text{in } \Omega, \\ \quad v = h & \text{on } \partial\Omega. \end{cases}$$

If we denote $w = u - v$, we obtain $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$. The Maximum Principle (Theorem 10.7) implies

$$0 = \min_{\boldsymbol{y}\in\overline{\Omega}} w(\boldsymbol{y}) \leq w(\boldsymbol{x}) \leq \max_{\boldsymbol{y}\in\overline{\Omega}} w(\boldsymbol{y}) = 0 \qquad \text{for all } \boldsymbol{x} \in \Omega.$$

However, this means that $w \equiv 0$ and thus $u \equiv v$. □

10.7 Consequences of Poisson Formula (Laplace Equation)

An important result following from the Poisson formula is the so called *Mean Value Property Theorem.*

Theorem 10.9 (Mean Value Property). *Let u be a harmonic function on a disc D, continuous in its closure $\overline{D}$. Then the value of the function u at the center of D is equal to the mean value of u on the circle ∂D.*

Proof. We shift the coordinate system so that the origin $\mathbf{0}$ is placed at the center of the disc (this can be done because the Laplace operator is invariant with respect to translations). We substitute $\boldsymbol{x} = \mathbf{0}$ into the Poisson formula (8.16) obtaining

$$u(\mathbf{0}) = \frac{a^2}{2\pi a} \int\limits_{|\boldsymbol{x}'|=a} \frac{u(\boldsymbol{x}')}{a^2}\,\mathrm{d}s = \frac{1}{2\pi a} \int\limits_{\partial D} u(\boldsymbol{x}')\,\mathrm{d}s.$$

This is exactly the integral mean value of the function u over the circle $|\boldsymbol{x}'| = a$. □

Another important consequence of the Poisson formula is the strong version of the Maximum Principle.

Theorem 10.10 (Strong Maximum Principle). *Let u be a harmonic function in the domain $\Omega \subset \mathbb{R}^2$, continuous on $\overline{\Omega}$. Then either u is constant in the entire closure $\overline{\Omega}$, or u achieves its maximal (minimal) value only on the boundary $\partial\Omega$ (i.e., never inside Ω).*

Idea of proof. Let us denote by $\boldsymbol{x}_M$ the point where u achieves its maximal value M on the closure of the domain Ω (its existence follows from the continuity of u on $\overline{\Omega}$ and from the Weierstrass Theorem). We shall show that $\boldsymbol{x}_M$ cannot lie inside Ω unless u is constant.

Let $\boldsymbol{x}_M \in \Omega$. Let us consider a circle centered at the point $\boldsymbol{x}_M$ which is the boundary of a circular neighborhood of $\boldsymbol{x}_M$ entirely contained in Ω. According to the Mean Value Theorem, $u(\boldsymbol{x}_M)$ equals the mean value of u over the circle. Since the mean value of the function cannot be greater than its maximum, we obtain the inequality

$$M = u(\boldsymbol{x}_M) = \text{mean value over the circle} \leq M,$$

and thus all values of u on the circle must be equal to M as well. Moreover, the same holds for an arbitrary circle with the center $\boldsymbol{x}_M$ and smaller radius. Thus $u(\boldsymbol{x}) = M$ for all $\boldsymbol{x}$ from the original circular neighborhood. Now, we can imagine that we cover the whole domain Ω by circular neighborhoods (see Figure 10.4). Since Ω is connected, we obtain $u(\boldsymbol{x}) \equiv M$ on the whole domain Ω, and thus u is constant. □

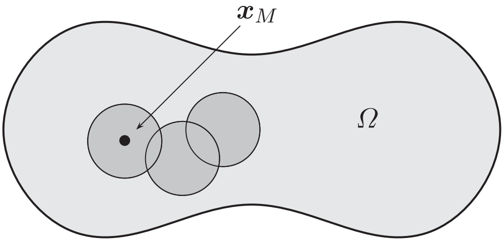


Figure 10.4. Covering of Ω by circular neighborhoods.

The last consequence of the Poisson formula which we state here is the following differentiability assertion.

Theorem 10.11 (Differentiability). *Let u be a harmonic function on an open set $\Omega \subset \mathbb{R}^2$. Then $u(\boldsymbol{x}) = u(x, y)$ has continuous partial derivatives of all orders in Ω.*

This property of harmonic functions is – in a certain sense – similar to the property that we have seen when studying the diffusion equation (see Chapter 5).

Idea of proof. First, let us consider an open disc D with the center at the origin. In the Poisson formula (8.16), the integrand is a function having partial derivatives of arbitrary orders for all $\boldsymbol{x} \in D$. Notice that $\boldsymbol{x}' \in \partial D$ and thus $\boldsymbol{x} \neq \boldsymbol{x}'$. Since we can change the order of integration and differentiation, the function u has partial derivatives of all orders in D as well.

Now, let us denote by D a circular neighborhood of the point $\boldsymbol{x}_0 \in \Omega$ which is wholly contained in Ω. Using the substitution $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$, we translate the center of D to the origin. It then follows from above that u is differentiable in D. Since $\boldsymbol{x}_0 \in \Omega$ is arbitrary, u is differentiable at all points of Ω. □

Remark 10.12 (Laplace Equation in Finite Differences). Let us have a look at the Laplace equation from the "numerical" point of view. Let us consider a point (x, y) and its neighbors $(x \pm h, y)$, $(x, y \pm h)$, where $h > 0$ is small enough (see Figure 10.5).

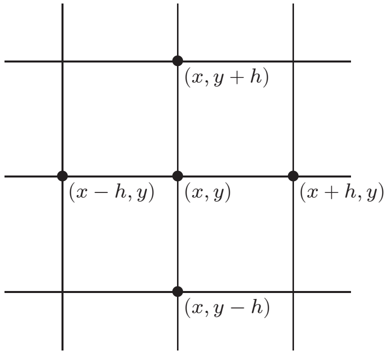


Figure 10.5. Point (x, y) and its "neighbors".

The Taylor expansion yields

$$\begin{aligned} u(x-h,y) &= u(x,y) - hu_x(x,y) + \frac{1}{2}h^2 u_{xx}(x,y) + O(h^3), \\ u(x+h,y) &= u(x,y) + hu_x(x,y) + \frac{1}{2}h^2 u_{xx}(x,y) + O(h^3) \end{aligned}$$

and, after summation, we can express the second derivative in the form

$$u_{xx}(x,y) = \frac{1}{h^2}\big(u(x-h,y) - 2u(x,y) + u(x+h,y)\big) + O(h^2).$$

Similarly,

$$u_{yy}(x,y) = \frac{1}{h^2}\big(u(x,y-h) - 2u(x,y) + u(x,y+h)\big) + O(h^2).$$

Notice that the second derivatives are expressed by central differences used, for instance, in the grid method. If we substitute in the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ and neglect the terms of higher orders, we obtain an approximate value of the function u at the point (x, y) as

$$u(x,y) \approx \frac{1}{4}\big(u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h)\big).$$

However, it means that the value $u(x, y)$ is approximately the arithmetic average of the surrounding values. This arithmetic average can be neither greater nor less than all the surrounding values. Thus, even here, we meet a certain numerical analogue of the Maximum Principle and the Mean Value Theorem.

10.8 Comparison of Wave, Diffusion and Laplace Equations

The above mentioned principles and properties of solutions of initial and boundary value problems yield the following fundamental comparison of all three types of linear equations of the second order (cf. Strauss [21]):

Property	**Wave**	**Diffusion**	**Laplace**
Speed of propagation	Finite ($\leq c$)	Infinite	Zero
Singularities	Propagate along characteristics at speed c	Disappear immediately (solutions are regular)	Solutions are regular
Well-posedness	Yes for $t > 0$ Yes for $t < 0$	Yes for $t > 0$ No for $t < 0$	Yes
Maximum principle	No	Yes	Yes
Energy (for IVPs)	Energy does not decrease (is constant)	Energy decreases (if φ is integrable)	Steady state

10.9 Exercises

1. Show that the wave equation has the following invariance properties:

   (a) Any shifted solution $u(x - y, t)$, where y is fixed, is also a solution.

   (b) Any derivative of the solution (e.g., u_x) is also a solution.

   (c) Dilated solution $u(ax, at)$, where $a > 0$, is also a solution.

2. For a solution $u(x, t)$ of the wave equation (10.4) with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.

   (a) Show that $e_t = p_x$ and $p_t = e_x$.

   (b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.

3. Let $u(x,t)$ solve the wave equation $u_{tt} = u_{xx}$. Prove that the identity

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h)$$

holds for all x, t, k and h. Draw the characteristic parallelogram with vertices formed by the arguments in the previous relation.

4. Consider a *damped* infinite string described by the equation $u_{tt} - c^2 u_{xx} + r u_t = 0$, and show that its total energy decreases.

5. Consider two Cauchy problems for the wave equation with different initial data:

$$\begin{cases} u^i_{tt} = c^2 u^i_{xx}, \quad x \in \mathbb{R},\ 0 < t < T, \\ u^i(x,0) = \varphi^i(x), \quad u^i_t(x,0) = \psi^i(x) \end{cases}$$

for $i = 1, 2$, where φ^1, φ^2, ψ^1, ψ^2 are given functions. If

$$|\varphi^1(x) - \varphi^2(x)| \le \delta_1, \quad |\psi^1(x) - \psi^2(x)| \le \delta_2$$

for all $x \in \mathbb{R}$, show that $|u^1(x,t) - u^2(x,t)| \le \delta_1 + \delta_2 T$ for all $x \in \mathbb{R}$, $0 < t < T$. What does it mean with regard to stability?

6. Using the energy conservation law for the wave equation, prove that the initial value problem

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{aligned}$$

has a unique solution.

7. Consider the diffusion equation on the real line. Using the Maximum Principle, show that an odd (even) initial condition leads to an odd (even) solution.

8. Consider a solution of the diffusion equation $u_t = u_{xx},\ 0 \le x \le l,\ t \ge 0$.

   (a) Let $M(T)$ be the maximum of the function $u(x,t)$ on the rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Is $M(T)$ decreasing or as a increasing function of T?

   (b) Let $m(T)$ be the minimum of the function $u(x,t)$ on the rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Is $m(T)$ decreasing or increasing as a function of T?

9. Consider the diffusion equation $u_t = u_{xx}$ on the interval $(0,1)$ with boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = 1 - x^2$. Notice that the initial condition does not satisfy the boundary condition on the left end, however, the solution satisfies it at arbitrary time $t > 0$.

(a) Show that $u(x,t) > 0$ at all inner points $0 < x < 1$, $0 < t < +\infty$.

(b) Let, for all $t > 0$, $\mu(t)$ represent the maximum of the function $u(x,t)$ on $0 \le x \le 1$. Show that $\mu(t)$ is a non-increasing function of t.

[Hint: Suppose the maximum to be achieved at a point $X(t)$, i.e., $\mu(t) = u(X(t),t)$. Differentiate $\mu(t)$ (under the assumption that $X(t)$ is a differentiable function).]

(c) Sketch the solution on several time levels.

10. Consider the diffusion equation $u_t = u_{xx}$ on the interval $(0,1)$ with boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = 4x(1-x)$.

    (a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

    (b) Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.

    (c) Using the energy method (see Section 10.5), show that $\int_0^1 u^2(x,t)\,\mathrm{d}x$ is a strictly decreasing function of t.

11. The aim of this exercise is to show that the maximum principle does not hold true for the equation $u_t = xu_{xx}$, which has a variable coefficient.

    (a) Verify that the function $u(x,t) = -2xt - x^2$ is a solution. Find its maximum on the rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.

    (b) Where exactly does our proof of the maximum principle fail in the case of this equation?

12. Consider a heat problem with an internal heat source

$$\begin{cases} u_t = u_{xx} + 2(t+1)x + x(1-x), \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = x(1-x). \end{cases}$$

    Show that the maximum principle does not hold true:

    (a) Verify that $u(x,t) = (t+1)x(1-x)$ is a solution.

    (b) What are the maximum value M and the minimum value m of the initial and boundary data?

    (c) Show that, for some $t > 0$, the temperature distribution exceeds M at certain points of the bar.

13. Prove the comparison principle for the diffusion equation: If u and v are two solutions and $u \le v$ for $t = 0$, for $x = 0$ and $x = l$, then $u \le v$ for $0 \le t < +\infty$, $0 \le x \le l$.

14. (a) More generally, if $u_t - ku_{xx} = f$, $v_t - kv_{xx} = g$, $f \le g$ and $u \le v$ for $x = 0$, $x = l$ and $t = 0$, then $u \le v$ for $0 \le t < +\infty$, $0 \le x \le l$. Prove it.

    (b) Let $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi$, $0 < t < +\infty$. Further, let $v(0,t) \ge 0$, $v(\pi,t) \ge 0$ and $v(x,0) \ge \sin x$. Exploit part (a) for proving that $v(x,t) \ge (1 - \mathrm{e}^{-t}) \sin x$.

15. Consider the diffusion equation on $(0,l)$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(l,t) + a_l u(l,t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to a decrease in $\int_0^l u^2(x,t)\mathrm{d}x$. (Part of the energy is lost at the boundary, so the boundary conditions are called *radiating* or *dissipative*.)

16. Let $u(x,t)$ solve the wave equation on the whole real line, and let its second derivatives be bounded. Define

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \mathrm{e}^{-s^2c^2/(4kt)} u(x,s)\,\mathrm{d}s.$$

    (a) Show that $v(x,t)$ solves the diffusion equation.

    (b) Show that $\lim\limits_{t\to 0} v(x,t) = u(x,0)$.

    Notice that here we show the direct relation between the wave and diffusion equations.

    [Hint: (a) Write the formula in the form $v(x,t) = \int_{-\infty}^{+\infty} H(s,t)u(x,s)\,\mathrm{d}s$, where $H(x,t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x,t)$.
    (b) Use the fact that $H(s,t)$ is a fundamental solution of the diffusion equation with the spatial variable s.]

17. Show that there is no maximum principle for the wave equation.

18. Let u be a harmonic function in the disc $D = \{r < 2\}$ and let $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the concrete form of the solution, answer the following questions:

    (a) What is the maximal value of u on $\overline{D}$?

    (b) What is the value of u at the origin?

    [(a) 4, (b) 1]

19. Prove uniqueness of the Dirichlet problem $\Delta u = f$ in D, $u = g$ on ∂D by the energy method.

20. Let Ω be a bounded open set and consider the Neumann problem

$$\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega.$$

Show that any two solutions differ by a constant.

21. Let Ω be a bounded open set. Show that the Neumann problem

$$\Delta u + \alpha u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega$$

has at most one solution if $\alpha < 0$ in Ω.

22. Prove that the function $u(x,y) = \frac{1-x^2-y^2}{x^2+(y-1)^2}$ is harmonic in $\mathbb{R}^2 \setminus \{0,1\}$. Find the maximum M and the minimum m of function $u(x,y)$ in the disc $D_\rho = \{x^2+y^2 \le \rho^2\}$, $\rho < 1$, and show that $Mm = 1$. Plot the graph of $u(x,y)$, where $(x,y) \in D_{0.9}$, using polar coordinates.

Chapter 11

Laplace and Poisson equations in Higher Dimensions

In this chapter we treat the Laplace operator and harmonic functions in $\mathbb{R}^3$. Unlike in the two-dimensional case, we cannot rely on methods based on direct computation of the solution, since the situation is much more complicated. That is why we try to obtain as much information as possible about the solution and its properties from the equation itself. In particular, we focus on those aspects that differ from the two-dimensional case. Features that do not depend on the dimension are stated without details.

11.1 Invariance of the Laplace Operator and its Transformation into Spherical Coordinates

The Laplace operator is invariant with respect to translations and rotations in three as well as in all higher dimensions. Let us recall that, using the matrix notation, rotation in $\mathbb{R}^3$ is given by the transformation formula

$$\boldsymbol{x}' = \boldsymbol{B}\boldsymbol{x},$$

where $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{B} = (b_{ij})$, $i, j = 1, 2, 3$, is an *orthogonal matrix* (that is, $\boldsymbol{B}\boldsymbol{B}^t = \boldsymbol{B}^t\boldsymbol{B} = \boldsymbol{I}$). Using the chain rule, we derive

$$\begin{aligned}
u_x &= b_{11}u_{x'} + b_{21}u_{y'} + b_{31}u_{z'},\\
u_y &= b_{12}u_{x'} + b_{22}u_{y'} + b_{32}u_{z'},\\
u_z &= b_{13}u_{x'} + b_{23}u_{y'} + b_{33}u_{z'}.
\end{aligned}$$

Further, for u_{xx} we have

$$\begin{aligned}
u_{xx} &= b_{11}^2 u_{x'x'} + b_{11}b_{21}u_{x'y'} + b_{11}b_{31}u_{x'z'} + b_{21}b_{11}u_{y'x'} + b_{21}^2 u_{y'y'}\\
&\quad + b_{21}b_{31}u_{y'z'} + b_{31}b_{11}u_{z'x'} + b_{31}b_{21}u_{z'y'} + b_{31}^2 u_{z'z'}
\end{aligned}$$

and similar formulas for u_{yy} and u_{zz}. Summing up and assuming the symmetry of second (mixed) partial derivatives, we obtain

$$\begin{aligned}
u_{xx} + u_{yy} + u_{zz} &= \underbrace{(b_{11}^2 + b_{12}^2 + b_{13}^2)}_{=1} u_{x'x'}\\
&\quad + 2\underbrace{(b_{11}b_{21} + b_{12}b_{22} + b_{13}b_{23})}_{=0} u_{x'y'} + 2\underbrace{(b_{11}b_{31} + b_{12}b_{32} + b_{13}b_{33})}_{=0} u_{x'z'}\\
&\quad + \cdots = u_{x'x'} + u_{y'y'} + u_{z'z'}
\end{aligned}$$

due to the orthogonality of the matrix $\boldsymbol{B}$. The reader is invited to carry out all the above calculations in detail.

The rotational invariance of the Laplace operator implies that, in radially symmetric cases, the transformation into *spherical coordinates* (r, θ, ϕ) could bring a significant simplification.

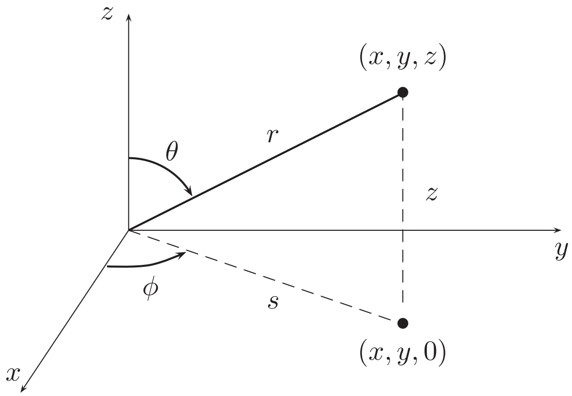


Figure 11.1. Spherical coordinates.

We will use the notations (see Figure 11.1)

$$\begin{aligned}
&r = \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2},\\
&s = \sqrt{x^2 + y^2},\\
&x = s\cos\phi, \qquad z = r\cos\theta,\\
&y = s\sin\phi, \qquad s = r\sin\theta,
\end{aligned}$$

and the knowledge of the transformation of the Laplace operator into the polar coordinates introduced in Section 6.2:

$$\begin{aligned}
u_{zz} + u_{ss} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta},\\
u_{xx} + u_{yy} &= u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.
\end{aligned}$$

Summing up and canceling u_{ss}, we obtain

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

In the last term we insert $s^2 = r^2 \sin^2\theta$. Moreover, u_s can be written as

$$u_s = \frac{\partial u}{\partial s} = u_r \frac{\partial r}{\partial s} + u_\theta \frac{\partial \theta}{\partial s} + u_\phi \frac{\partial \phi}{\partial s}.$$

Evidently, $\frac{\partial\phi}{\partial s} = 0$. If we use, e.g., the inverse Jacobi matrix of the transformation into polar coordinates (cf. Section 6.1 with $r = r$, $\theta = \theta$ and $s = y$), we obtain

$$\frac{\partial r}{\partial s} = \sin\theta = \frac{s}{r}, \qquad \frac{\partial \theta}{\partial s} = \frac{\cos\theta}{r},$$

and thus

$$u_s = u_r \frac{s}{r} + u_\theta \frac{\cos\theta}{r}.$$

Hence, we easily derive

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{\cos\theta}{r^2 \sin\theta} u_\theta + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r^2 \sin^2\theta} u_{\phi\phi}.$$

Written in symbols, we obtain the following analogue of the "two-dimensional formula" (6.3):

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \end{aligned} \tag{11.1}$$

In the radially symmetric situation, that is, when u does not depend on the angles ϕ and θ, the Laplace equation in spherical coordinates reduces to the ODE

$$\Delta u = u_{rr} + \frac{2}{r} u_r = 0,$$

which can be, after multiplying by r^2, written as

$$(r^2 u_r)_r = 0.$$

Hence, by simple integration, we obtain $u_r = c_1/r^2$ and $u = -c_1 r^{-1} + c_2$. The fundamental harmonic function in three dimensions is thus the function

$$u(r) = \frac{1}{r} = (x^2 + y^2 + z^2)^{-\frac{1}{2}},$$

which can be taken as the analogue of the two-dimensional harmonic function $\ln(x^2 + y^2)^{1/2}$, which we discussed in Chapter 6. In *electrostatics*, for instance, $u(\boldsymbol{x}) = r^{-1}$ represents the *electrostatic potential* at point $\boldsymbol{x}$ with the radius r, which corresponds to the unit charge placed at the origin.

11.2 Green's First Identity

In the sequel, we focus on the three-dimensional case; however, all statements remain valid even in the two-dimensional case or, generally, in any dimension.

According to the product rule, we write

$$\begin{aligned}(vu_x)_x &= v_x u_x + v u_{xx},\\ (vu_y)_y &= v_y u_y + v u_{yy},\\ (vu_z)_z &= v_z u_z + v u_{zz}.\end{aligned}$$

Summing up all these three equations, we obtain

$$\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v\Delta u.$$

If we integrate this relation and use the Divergence Theorem (see page 4) for the left-hand side, we obtain *Green's first identity*

$$\iint\limits_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}S = \iiint\limits_{\Omega} \nabla v \cdot \nabla u \,\mathrm{d}\boldsymbol{x} + \iiint\limits_{\Omega} v\Delta u\,\mathrm{d}\boldsymbol{x}, \tag{11.2}$$

where $\partial u/\partial n = \boldsymbol{n}\cdot\nabla u$ is the derivative with respect to the outer normal to the boundary of the domain Ω. The identity (11.2) can be interpreted as a three-dimensional version of integration by parts and has a number of consequences.

11.3 Properties of Harmonic Functions

The fundamental property of harmonic functions is the *Weak Maximum Principle*. This property, together with the proof technique, does not depend on the dimension. We recommend the reader to study Theorem 10.7 and its proof in order to realize its applicability in higher dimensions. Also the consequence concerning *uniqueness of the solution of the Dirichlet problem* (see Theorem 10.8) can be stated without any changes.

11.3.1 Mean Value Property and Strong Maximum Principle

One of the properties of harmonic functions that follows from Green's first identity is the three-dimensional version of the *Mean Value Property*. Let us recall its two-dimensional variant in Chapter 6 (Theorem 10.9).

Theorem 11.1 (Mean Value Property). *The average value of any harmonic function in a domain $\Omega \subset \mathbb{R}^3$ over any sphere which lies in Ω is equal to its value at the center of the sphere.*

Proof. Following, e.g., Strauss [21] or Zauderer [24], let us consider a ball $B(\mathbf{0}, a) = \{|\boldsymbol{x}| < a\} \subset \Omega$ with radius a centered at the origin (recall the invariance of the Laplace operator with respect to translations). Further, let $\Delta u = 0$ in Ω, $\overline{B(\mathbf{0}, a)} \subset \Omega$. For the sphere, the outer normal $\boldsymbol{n}$ has the direction of the radius vector, thus

$$\begin{aligned}\frac{\partial u}{\partial n} &= \boldsymbol{n} \cdot \nabla u = \frac{\boldsymbol{x}}{r} \cdot \nabla u = \frac{x}{r} u_x + \frac{y}{r} u_y + \frac{z}{r} u_z \\ &= \frac{\partial x}{\partial r} u_x + \frac{\partial y}{\partial r} u_y + \frac{\partial z}{\partial r} u_z = \frac{\partial u}{\partial r},\end{aligned}$$

where $r = (x^2 + y^2 + z^2)^{1/2} = |\boldsymbol{x}|$ is the spherical coordinate (the distance of the point (x, y, z) from the center $\mathbf{0}$ of the sphere). If we use Green's first identity for the ball $B(\mathbf{0}, a)$ with the choice $v \equiv 1$, we obtain

$$\iint\limits_{\partial B(\mathbf{0},a)} \frac{\partial u}{\partial r} \, \mathrm{d}S = \iiint\limits_{B(\mathbf{0},a)} \Delta u \, \mathrm{d}\boldsymbol{x} = 0.$$

We rewrite the integral on the left-hand side into spherical coordinates (r, θ, ϕ), that is,

$$\int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) a^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = 0$$

(on the sphere $\partial B(\mathbf{0}, a)$, we have $r = a$), and divide the equality by the constant $4\pi a^2$, which is the measure of $\partial B(\mathbf{0}, a)$ (that is, the surface of the ball $B(\mathbf{0}, a)$). This result holds true for all $a > 0$, thus we can replace a with the variable r. Moreover, if we change the order of integration and differentiation (which is possible under certain assumptions on u), we obtain

$$\frac{\partial}{\partial r} \left(\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \right) = 0. \tag{11.3}$$

However, it means that the expression

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi,$$

which represents the average value of u on the sphere $\{|x| = r\}$, is *independent of the radius* r. In particular, for $r \to 0$ we have

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(\mathbf{0}) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = u(\mathbf{0}). \tag{11.4}$$

Relations (11.3) and (11.4) imply the following result for any $a > 0$:

$$u(\mathbf{0}) = \frac{1}{\operatorname{meas} \partial B(\mathbf{0}, a)} \iint_{\partial B(\mathbf{0},a)} u \, \mathrm{d}S. \tag{11.5}$$

Thus, the proof of the Mean Value Property in three dimensions is completed. (Observe that the idea of the proof can be applied generally in any dimension.)

□

Like in two dimensions (see Theorem 10.10), a direct consequence of the Mean Value Property is the *strong version of the Maximum Principle.*

Theorem 11.2 (Strong Maximum Principle). *Let Ω be an arbitrary domain in $\mathbb{R}^3$. A non-constant harmonic function in Ω, continuous in $\overline{\Omega}$, cannot achieve its maximum (minimum) inside Ω, but only on the boundary $\partial\Omega$.*

The proof follows along the same lines as in Theorem 10.10, so we do not repeat it here.

11.3.2 Dirichlet Principle

Another important theorem that follows from Green's first identity and which has also a physical motivation, is the *Dirichlet Principle.*

Theorem 11.3 (Dirichlet Principle). *Let $u(\boldsymbol{x})$ be the harmonic function on a domain Ω satisfying the Dirichlet boundary condition*

$$u(\boldsymbol{x}) = h(\boldsymbol{x}) \quad \text{on } \partial\Omega. \tag{11.6}$$

Let $w(\boldsymbol{x})$ be an arbitrary continuously differentiable function on $\overline{\Omega}$ satisfying (11.6)*. Then*

$$E(w) \geq E(u),$$

where E denotes the energy defined by the formula

$$E(w) = \frac{1}{2} \iiint\limits_{\Omega} |\nabla w|^2 \, \mathrm{d}\boldsymbol{x}. \tag{11.7}$$

In other words, the Dirichlet Principle says that, among all functions satisfying the boundary condition (11.6), the harmonic function corresponds to the state with the lowest energy. Expression (11.7) represents just the *potential energy* – there is no motion and thus the kinetic energy is zero. One of the fundamental physical principles says that any system tends to keep the state with the lowest energy. Harmonic functions thus describe the most frequent "ground (quiescent) states".

Proof of Theorem 11.3. Let us denote $w = u + v$ and substitute in the formula for the energy (11.7) in the following way:

$$\begin{aligned} E(w) &= \frac{1}{2} \iiint\limits_{\Omega} |\nabla(u+v)|^2 \, \mathrm{d}\boldsymbol{x} \\ &= E(u) + \iiint\limits_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} + E(v). \end{aligned}$$

To the middle term we apply Green's first identity and use the fact that $v = 0$ on $\partial\Omega$ (both u and w satisfy the same Dirichlet boundary condition) and $\Delta u = 0$ in Ω. Consequently, $\iiint_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = 0$ and

$$E(w) = E(u) + E(v).$$

Since, evidently, $E(v) \geq 0$, we obtain $E(w) \geq E(u)$ and the Dirichlet Principle is proved. □

11.3.3 Uniqueness of Solution of Dirichlet Problem

A direct consequence of the Maximum Principle is the uniqueness of the solution of the Dirichlet problem for the Poisson equation. We refer the reader to Theorem 10.8 and its proof that can be applied in any dimension. Here we present another proof based on the so called *energy method*, which involves the application of Green's first identity.

Let us consider the Dirichlet problem $\Delta u = f$ in the domain Ω, $u = h$ on the boundary $\partial\Omega$, and assume that there are two solutions u_1, u_2. We denote their difference by $u = u_1 - u_2$. The function u is harmonic in Ω, vanishing on the boundary $\partial\Omega$. Now, we use Green's first identity (11.2) for $v = u$. Since u is a harmonic function ($\Delta u = 0$ in Ω), we obtain

$$\iint\limits_{\partial\Omega} u \frac{\partial u}{\partial n} \, \mathrm{d}S = \iiint\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}\boldsymbol{x}.$$

Since $u = 0$ on the boundary $\partial\Omega$, the left-hand side is equal to zero. This yields

$$\iiint\limits_{\Omega} |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} = 0,$$

which implies $|\nabla u| = 0$ in Ω. This means that the function u is constant in the domain Ω. But since it vanishes on the boundary $\partial\Omega$, we obtain $u(\boldsymbol{x}) \equiv 0$ in Ω, and thus $u_1(\boldsymbol{x}) \equiv u_2(\boldsymbol{x})$ in Ω.

In a similar way we can prove that the solution of the *Neumann problem* is determined *uniquely up to a constant* (see Exercise 1 in Section 11.7).

11.3.4 Necessary Condition for the Solvability of Neumann Problem

If we use a special choice $v \equiv 1$, Green's first identity reads

$$\iint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}S = \iiint\limits_{\Omega} \Delta u \, \mathrm{d}\boldsymbol{x}. \tag{11.8}$$

Let us consider the Neumann problem in the domain Ω

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = h & \text{on } \partial\Omega \end{cases} \tag{11.9}$$

and let us substitute for $\frac{\partial u}{\partial n}$ and Δu into the relation (11.8). We obtain a necessary condition for the solvability of (11.9) in the form

$$\boxed{\iint\limits_{\partial\Omega} h \, \mathrm{d}S = \iiint\limits_{\Omega} f \, \mathrm{d}\boldsymbol{x}.} \tag{11.10}$$

It means that the solution of the Neumann problem can exist only if the input data (functions f and h) are not completely arbitrary but satisfy condition (11.10). In fact, it can be proved that condition (11.10) is also sufficient for problem (11.9) to have a solution. Thus, from the point of solvability, the Neumann problem is not well-posed. Concerning uniqueness, we find this problem ill-posed as well (if there exists a solution to (11.9) then adding an arbitrary constant to this solution, we obtain another solution of (11.9)). Nevertheless, the Neumann boundary value problem makes reasonable sense and occurs very often in applications.

11.4 Green's Second Identity and Representation Formula

If we apply Green's first identity to the pair of functions (u, v), and then to the pair (v, u), and subtract the two equations, we obtain the relation

$$\iiint_{\Omega} (u\Delta v - v\Delta u)\,\mathrm{d}\boldsymbol{x} = \iint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}S, \tag{11.11}$$

which is known as *Green's second identity.*

An important consequence of Green's second identity is the so called *representation formula.* It says that the value of a harmonic function at any point of a domain Ω can be expressed using only its values on the boundary $\partial\Omega$.

Theorem 11.4 (Representation Formula). *Let $\boldsymbol{x}_0 \in \Omega \subset \mathbb{R}^3$. The value of any harmonic function on a domain Ω, continuous on $\overline{\Omega}$, can be expressed by*

$$u(\boldsymbol{x}_0) = \frac{1}{4\pi} \iint_{\partial\Omega} \left(-u(\boldsymbol{x}) \frac{\partial}{\partial n} \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|} \right) + \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_0|} \frac{\partial u}{\partial n}(\boldsymbol{x}) \right) \mathrm{d}S. \tag{11.12}$$

Observe that relation (11.12) contains the fundamental radially symmetric harmonic function $r^{-1} = |\boldsymbol{x} - \boldsymbol{x}_0|^{-1}$ that we have already discussed in the previous sections of this chapter (here it is shifted by the vector $\boldsymbol{x}_0$).

Proof. Relation (11.12) is a special case of Green's second identity for the choice

$$v(\boldsymbol{x}) = \frac{-1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|}.$$

This function is, however, *unbounded* at $\boldsymbol{x}_0$, thus we cannot use Green's second identity on the whole domain Ω. Let us denote by Ω_ε the domain $\Omega \setminus B(\boldsymbol{x}_0, \varepsilon)$, where $B(\boldsymbol{x}_0, \varepsilon) \subset \Omega$ is the ball centered at the point $\boldsymbol{x}_0$ with radius ε. This domain is now admissible for the application of Green's second identity.

For simplicity, let us shift the point $\boldsymbol{x}_0$ to the origin (we recall again the invariance of the Laplace operator with respect to translations). Hence, $v(\boldsymbol{x}) = -1/(4\pi r)$, where $r = |\boldsymbol{x}| = (x^2+y^2+z^2)^{1/2}$. If we use the fact that $\Delta u = \Delta v = 0$ in Ω_ε, Green's second identity implies

$$-\frac{1}{4\pi}\iint\limits_{\partial\Omega_\varepsilon}\left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{\partial u}{\partial n}\frac{1}{r}\right)\mathrm{d}S = 0.$$

The boundary $\partial\Omega_\varepsilon$ consists of two parts: the boundary of the original domain $\partial\Omega$ and the sphere $\partial B(\boldsymbol{x}_0,\varepsilon)$. Moreover, on this sphere we have $\partial/\partial n = -\partial/\partial r$. The surface integral above is thus decomposed into two parts and the equality can be written as

$$-\frac{1}{4\pi}\iint\limits_{\partial\Omega}\left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{\partial u}{\partial n}\frac{1}{r}\right)\mathrm{d}S \tag{11.13}$$

$$= -\frac{1}{4\pi}\iint\limits_{r=\varepsilon}\left(u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{\partial u}{\partial r}\frac{1}{r}\right)\mathrm{d}S.$$

This equality must hold true for any (small) $\varepsilon > 0$. Concerning the sphere $|x| = r = \varepsilon$, we have

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} = -\frac{1}{\varepsilon^2}.$$

The right-hand side of relation (11.13) can be thus rewritten in the form

$$\frac{1}{4\pi\varepsilon^2}\iint\limits_{r=\varepsilon}u\,\mathrm{d}S + \frac{1}{4\pi\varepsilon}\iint\limits_{r=\varepsilon}\frac{\partial u}{\partial r}\,\mathrm{d}S = \overline{u} + \varepsilon\overline{\frac{\partial u}{\partial r}},$$

where $\overline{u}$ denotes the integral average value of the function $u(\boldsymbol{x})$ on the sphere $|\boldsymbol{x}| = r = \varepsilon$, and $\overline{\partial u/\partial r}$ represents the average value of $\partial u/\partial n$ on this sphere. If we now pass to the limit for $\varepsilon \to 0$, we obtain

$$\overline{u} + \varepsilon\overline{\frac{\partial u}{\partial r}} \longrightarrow u(\mathbf{0}) + 0 \times \frac{\partial u}{\partial r}(\mathbf{0}) = u(\mathbf{0})$$

(note that the function u is continuous and $\partial u/\partial r$ is bounded). Hence, from relation (11.13), we easily get formula (11.12). (The reader is asked to give the reasons.) □

Remark 11.5. In the same way we can obtain the representation formula in any dimension. The concrete form of this formula in N dimensions depends on the corresponding fundamental radially symmetric harmonic function which for $N \geq 3$ has the form r^{-N+2}, and for $N = 2$ is equal to $\ln r$ (see Section 6.3). In particular, in *two dimensions*, the representation formula reads

$$u(\boldsymbol{x}_0) = \frac{1}{2\pi} \int\limits_{\partial\Omega} \left(u(\boldsymbol{x}) \frac{\partial}{\partial n} (\ln |\boldsymbol{x} - \boldsymbol{x}_0|) - \frac{\partial u}{\partial n}(\boldsymbol{x}) \ln |\boldsymbol{x} - \boldsymbol{x}_0| \right) \mathrm{d}s,$$

where $\Delta u = 0$ in the plane domain Ω and $\boldsymbol{x}_0 \in \Omega$. We integrate here along the curve $\partial\Omega$, and $\mathrm{d}s$ denotes an element of the arc length of this curve.

11.5 Boundary Value Problems and Green's Function

The main disadvantage of the representation formula (11.12) is that it contains the boundary values of both the functions u and $\frac{\partial u}{\partial n}$. But solving the standard boundary value problems, we are usually given either the Dirichlet boundary condition or the Neumann boundary condition, not both at the same time! The representation formula is based on two properties of the function $v(\boldsymbol{x}) = -1/(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)$: it is a harmonic function except the point $\boldsymbol{x}_0$, and the singularity at this point has a "proper" form. Our goal is to modify this function in such a way that we could eliminate one term in formula (11.12). The modified function will be called *Green's function* corresponding to the domain Ω.

Definition 11.6. *Green's function* $G(\boldsymbol{x}, \boldsymbol{x}_0)$ corresponding to the Laplace operator, the homogeneous Dirichlet boundary condition, a domain Ω and a point $\boldsymbol{x}_0 \in \Omega$ is a function defined by the following properties:

(i) $G(\boldsymbol{x}, \boldsymbol{x}_0)$ has continuous second partial derivatives with respect to $\boldsymbol{x}$ and $\Delta G = 0$ in Ω except the point $\boldsymbol{x} = \boldsymbol{x}_0$ (here, the Laplace operator is considered with respect to $\boldsymbol{x}$, while $\boldsymbol{x}_0$ is a parameter).

(ii) $G(\boldsymbol{x}, \boldsymbol{x}_0) = 0$ for $\boldsymbol{x} \in \partial\Omega$.

(iii) The function $G(\boldsymbol{x}, \boldsymbol{x}_0) + 1/(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)$ is finite at the point $\boldsymbol{x}_0$, has continuous partial derivatives of the second order in the whole domain Ω, and is harmonic.

It can be proved that Green's function exists and is determined uniquely (the uniqueness proof is based on the Maximum Principle, or on the theorem on the unique solvability of the Dirichlet problem; we leave the details to the reader – see Exercise 2 in Section 11.7).

Theorem 11.7. *If $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is Green's function, then the solution of the Dirichlet problem for the Laplace equation can be expressed by*

$$u(\boldsymbol{x}_0) = \iint\limits_{\partial\Omega} u(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial n} \,\mathrm{d}S. \tag{11.14}$$

Proof. The representation formula implies

$$u(\boldsymbol{x}_0) = \iint\limits_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v \right) \mathrm{d}S, \tag{11.15}$$

where again $v(\boldsymbol{x}) = -(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)^{-1}$. Now, we define $H(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{x}_0) - v(\boldsymbol{x})$. According to the property (iii) of Definition 11.6, the function $H(\boldsymbol{x})$ is harmonic on the whole domain Ω. We can thus apply Green's second identity to the couple $u(\boldsymbol{x})$, $H(\boldsymbol{x})$:

$$0 = \iint\limits_{\partial\Omega} \left(u\frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} H \right) \mathrm{d}S. \tag{11.16}$$

Summing (11.15) and (11.16), we obtain

$$u(\boldsymbol{x}_0) = \iint\limits_{\partial\Omega} \left(u\frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G \right) \mathrm{d}S.$$

Moreover, according to (ii), Green's function satisfies $G = 0$ on the boundary $\partial\Omega$. This directly implies (11.14). □

Remark 11.8. Green's function is symmetric, that is,

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = G(\boldsymbol{x}_0, \boldsymbol{x}) \quad \text{for } \boldsymbol{x} \neq \boldsymbol{x}_0.$$

In electrostatics, $G(\boldsymbol{x}, \boldsymbol{x}_0)$ represents the electric potential inside a closed conductive surface $S = \partial\Omega$ induced by a charge placed at the point $\boldsymbol{x}_0$. The symmetry of Green's function is then known as the *Reciprocity Principle*, according to which the source placed at $\boldsymbol{x}_0$ causes the same effect at the point $\boldsymbol{x}$ as the source at $\boldsymbol{x}$ causes at the point $\boldsymbol{x}_0$.

Green's function can be used also for solving the *Poisson equation*.

Theorem 11.9. *The Dirichlet boundary value problem for the Poisson equation*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases} \tag{11.17}$$

has a unique solution given by the formula

$$u(\boldsymbol{x}_0) = \iint\limits_{\partial\Omega} h(\boldsymbol{x}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial n}\, \mathrm{d}S + \iiint\limits_{\Omega} f(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{x}_0)\, \mathrm{d}\boldsymbol{x} \tag{11.18}$$

for any $\boldsymbol{x}_0 \in \Omega$.

It is straightforward to verify that $u = u(\boldsymbol{x})$ given by (11.18) is a solution of (11.17) – see Exercise 3 in Section 11.7. Uniqueness of the solution was discussed in Section 11.3.

The disadvantage of relations (11.14), (11.18) is the necessity to know the explicit expression of Green's function. This is possible only on domains with special geometry. Two such cases are considered in the forthcoming section (cf. Strauss [21], or Stavroulakis, Tersian [20]).

11.6 Dirichlet Problem on Half-Space and on Ball

The half-space and the ball in $\mathbb{R}^3$ are some of the domains for which Green's function and, consequently, the solution of the corresponding Dirichlet problem can be found explicitly. In both cases we use the so called *reflection method.*

11.6.1 Dirichlet Problem on Half-Space

Although the half-space is an unbounded domain, all assertions stated above – including the notion of Green's function – remain valid, provided we add the "boundary condition at infinity". By this condition, we understand the assumption that functions and their derivatives vanish for $|\boldsymbol{x}| \to +\infty$.

We denote the coordinates of the point $\boldsymbol{x}$ by (x, y, z) as usual. The half-space $\Omega = \{\boldsymbol{x}, z > 0\}$ is the domain lying "above" the xy-plane. To each point $\boldsymbol{x} = (x, y, z) \in \Omega$ there corresponds the *reflected point* $\boldsymbol{x}^* = (x, y, -z)$ that evidently does not lie in Ω (see Figure 11.2).

We already know that the function $1/(4\pi|\boldsymbol{x} - \boldsymbol{x}_0|)$ satisfies the conditions (i) and (iii) imposed on Green's function. We try to modify it so as to ensure the validity of condition (ii).

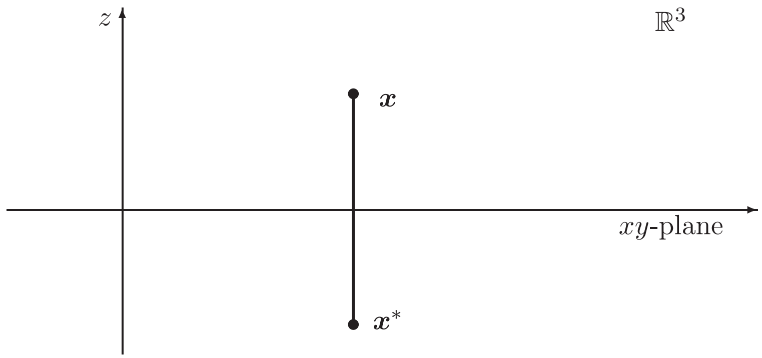


Figure 11.2. Half-space and reflection method.

We claim that Green's function for the half-space Ω has the form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0^*|}. \tag{11.19}$$

Rewritten into coordinates, this becomes

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{x}_0) \quad = \quad & -\frac{1}{4\pi}\left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right)^{-1/2} \\ & \qquad +\frac{1}{4\pi}\left((x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2\right)^{-1/2}. \end{aligned}$$

Observe that the two terms differ only by the element $(z \pm z_0)$. Let us verify that Green's function defined by formula (11.19) has the properties of Green's function stated in Definition 11.6.

(i) Obviously, G is finite and differentiable except at the point $\boldsymbol{x}_0$. Also $\Delta G = 0$.

(ii) Let $\boldsymbol{x} \in \partial\Omega$, that is $z = 0$. Figure 11.3 illustrates that $|\boldsymbol{x} - \boldsymbol{x}_0| = |\boldsymbol{x} - \boldsymbol{x}_0^*|$. Hence, $G(\boldsymbol{x}, \boldsymbol{x}_0) = 0$ on $\partial\Omega$.

(iii) Since the point $\boldsymbol{x}_0^*$ lies outside the domain Ω, the function $-1/(4\pi|\boldsymbol{x} - \boldsymbol{x}_0^*|)$ has no singularities in Ω. The function G has thus a single singularity at the point $\boldsymbol{x}_0$ and this corresponds to the claims imposed on Green's function.

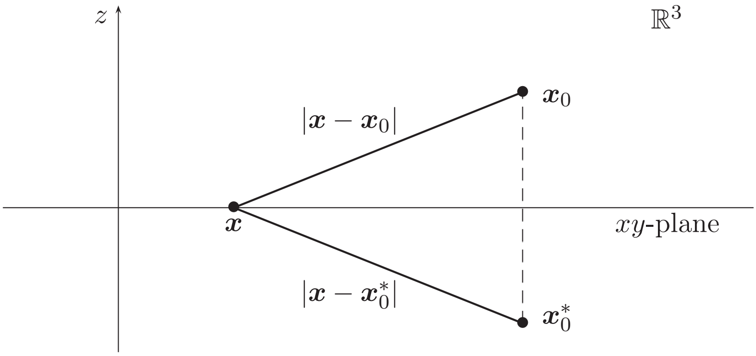


Figure 11.3. Verification of the property (ii) of Green's function.

Thus, we have proved that formula (11.19) determines Green's function corresponding to the half-space Ω.

Now, we can use it for finding the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 \quad \text{for } z > 0, \\ u(x,y,0) = h(x,y). \end{cases} \tag{11.20}$$

We use formula (11.14). Notice that $\partial G/\partial n = -\partial G/\partial z|_{z=0}$, since the outer normal $\boldsymbol{n}$ has the "downward" direction (out of the domain). Further,

$$-\frac{\partial G}{\partial z} = \frac{1}{4\pi}\left(\frac{z+z_0}{|\boldsymbol{x}-\boldsymbol{x}_0^*|^3} - \frac{z-z_0}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}\right) = \frac{1}{2\pi}\frac{z_0}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}$$

for $z = 0$. Hence, by direct substitution, we obtain the solution of problem (11.20) in the form

$$u(x_0,y_0,z_0) = \frac{z_0}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left((x-x_0)^2+(y-y_0)^2+z_0^2\right)^{-3/2} h(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

or, in the vector notation,

$$u(\boldsymbol{x}_0) = \frac{z_0}{2\pi}\iint\limits_{\partial\Omega}\frac{h(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}\,\mathrm{d}S. \tag{11.21}$$

Remark 11.10. We can proceed similarly in any dimension. In particular, let us have a look at the same problem in two dimensions, that is, let us consider the Laplace equation on the "upper half-plane":

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R},\ y > 0, \\ u(x,0) = h(x), & x \in \mathbb{R}. \end{cases} \tag{11.22}$$

The corresponding Green's function has the form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi} \ln |\boldsymbol{x} - \boldsymbol{x}_0| - \frac{1}{2\pi} \ln |\boldsymbol{x} - \boldsymbol{x}_0^*|.$$

Here $\boldsymbol{x} = (x, y)$ and $|\boldsymbol{x} - \boldsymbol{x}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. The solution of the Dirichlet problem (11.22) is then given by

$$u(x_0, y_0) = \frac{y_0}{\pi} \int\limits_{-\infty}^{+\infty} \frac{h(x)}{|\boldsymbol{x} - \boldsymbol{x}_0|^2} \, \mathrm{d}x = \frac{y_0}{\pi} \int\limits_{-\infty}^{+\infty} \frac{h(x)}{(x - x_0)^2 + y_0^2} \, \mathrm{d}x.$$

Notice that problem (11.22) was solved with the same result also in Chapter 9 by the Fourier transform (see Remark 9.12)!

11.6.2 Dirichlet Problem on a Ball

Another domain where we can solve the Dirichlet problem using the explicitly found Green's function, is the ball $\Omega = \{|\boldsymbol{x}| < a\}$ with radius a. Again we use the *reflection method*, this time, however, with respect to the sphere $\{|\boldsymbol{x}| = a\}$ which forms the boundary $\partial\Omega$ (see Figure 11.4). The method is – in this case – called the *spherical inversion*.

Let us consider a fixed point $\boldsymbol{x}_0 \in \Omega$. The *reflected point* $\boldsymbol{x}_0^*$ is determined by the following properties:

(i) $\boldsymbol{x}_0^*$ lies on the straight line passing through $\mathbf{0}$ and $\boldsymbol{x}_0$,

(ii) its distance from the origin is given by the relation $|\boldsymbol{x}_0|\,|\boldsymbol{x}_0^*| = a^2$.

It means that

$$\boldsymbol{x}_0^* = \frac{a^2 \boldsymbol{x}_0}{|\boldsymbol{x}_0|^2}.$$

Let $\boldsymbol{x} \in \Omega$ be an arbitrary point and let us denote $\rho(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_0|$ and $\rho^*(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_0^*|$. Then, for $\boldsymbol{x}_0 \neq \mathbf{0}$, Green's function on the ball Ω is given by

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi\rho} + \frac{a}{|\boldsymbol{x}_0|} \frac{1}{4\pi\rho^*}. \tag{11.23}$$

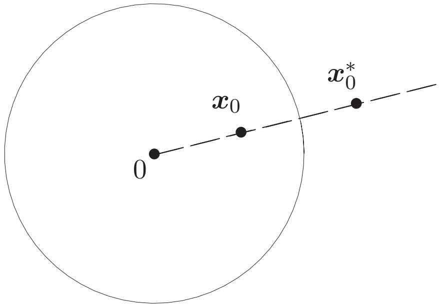


Figure 11.4. Ball Ω and spherical inversion.

We prove this statement by verifying the properties (i), (ii) and (iii) of Definition 11.6. The case $\boldsymbol{x}_0 = \mathbf{0}$ will be treated separately.

First of all, the single singularity of the function G is the point $\boldsymbol{x} = \boldsymbol{x}_0$, since $\boldsymbol{x}_0^*$ lies outside the ball Ω. Functions $1/\rho$ and $1/\rho^*$ are both harmonic in Ω except at the point $\boldsymbol{x}_0$. Conditions (i) and (iii) are thus fulfilled.

For the verification of condition (ii), we show that ρ^* is proportional to ρ for all $\boldsymbol{x}$ lying on the sphere $|\boldsymbol{x}| = a$. The congruent triangles in Figure 11.5 imply

$$\left| \frac{r_0}{a}\boldsymbol{x} - \frac{a}{r_0}\boldsymbol{x}_0 \right| = |\boldsymbol{x} - \boldsymbol{x}_0|, \tag{11.24}$$

where $r_0 = |\boldsymbol{x}_0|$. For the left-hand side of (11.24) we have

$$\frac{r_0}{a}\left| \boldsymbol{x} - \frac{a^2}{r_0^2}\boldsymbol{x}_0 \right| = \frac{r_0}{a}\rho^*.$$

Hence, we obtain

$$\frac{r_0}{a}\rho^* = \rho \quad \text{for} \quad \text{all } |\boldsymbol{x}| = a.$$

However, this means that the function $G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi\rho} + \frac{a}{r_0}\frac{1}{4\pi\rho^*}$ is zero on the sphere $|\boldsymbol{x}| = a$ and condition (ii) is satisfied.

Formula (11.23) can be rewritten to the form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{1}{4\pi|\frac{r_0}{a}\boldsymbol{x} - \frac{a}{r_0}\boldsymbol{x}_0|}. \tag{11.25}$$

In the case $\boldsymbol{x}_0 = \mathbf{0}$, Green's function takes the following form (verify – see Exercise 4 in Section 11.7):

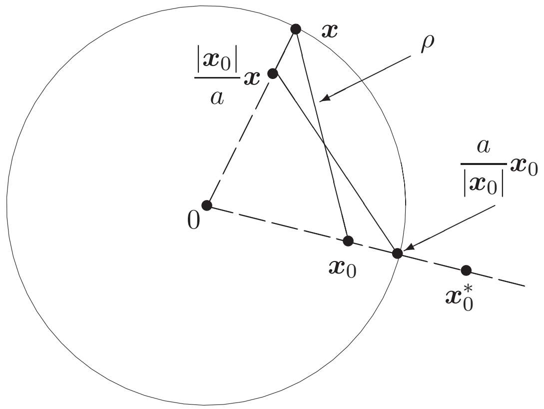


Figure 11.5. Congruent triangles and the proportionality of ρ and ρ^*.

$$G(\boldsymbol{x}, \mathbf{0}) = -\frac{1}{4\pi|\boldsymbol{x}|} + \frac{1}{4\pi a}. \tag{11.26}$$

Now, we use the knowledge of Green's function for finding the solution of the Dirichlet boundary value problem for the Laplace equation in the ball

$$\begin{cases} \Delta u = 0 & \text{for } |\boldsymbol{x}| < a, \\ u = h & \text{for } |\boldsymbol{x}| = a. \end{cases} \tag{11.27}$$

We know from Theorem 11.1 (Mean Value Property) that $u(\mathbf{0})$ is the average value of the function $h(\boldsymbol{x})$ on the sphere $\partial\Omega$. Let us consider only the case $\boldsymbol{x}_0 \neq \mathbf{0}$. Since we want to use the representation formula (11.14) (see Theorem 11.7), we have to determine $\partial G/\partial n$ on $|\boldsymbol{x}| = a$. We start with the relation $\rho^2 = |\boldsymbol{x}-\boldsymbol{x}_0|^2$. Differentiating it with respect to $\boldsymbol{x}$, we obtain $2\rho\nabla\rho = 2(\boldsymbol{x}-\boldsymbol{x}_0)$. Thus, $\nabla\rho = (\boldsymbol{x} - \boldsymbol{x}_0)/\rho$ and $\nabla(\rho^*) = (\boldsymbol{x} - \boldsymbol{x}_0^*)/\rho^*$. Now we determine the gradient of the function G from relation (11.23):

$$\begin{aligned} \nabla G &= \nabla\left(-\frac{1}{4\pi\rho} + \frac{a}{|\boldsymbol{x}_0|}\frac{1}{4\pi\rho^*}\right) \\ &= \frac{\boldsymbol{x} - \boldsymbol{x}_0}{4\pi\rho^3} - \frac{a}{|\boldsymbol{x}_0|}\frac{\boldsymbol{x} - \boldsymbol{x}_0^*}{4\pi\rho^{*3}}. \end{aligned} \tag{11.28}$$

We recall that $\boldsymbol{x}_0^* = (a/r_0)^2\boldsymbol{x}_0$. In the case $|\boldsymbol{x}| = a$, we have shown above that

$\rho^* = (a/r_0)\rho$. If we put these relations into expression (11.28), we obtain

$$\nabla G = \frac{1}{4\pi\rho^3}\left(\boldsymbol{x} - \boldsymbol{x}_0 - \left(\frac{r_0}{a}\right)^2 \boldsymbol{x} + \boldsymbol{x}_0\right)$$

on the sphere $\partial\Omega$, and thus

$$\frac{\partial G}{\partial \boldsymbol{n}} = \frac{\boldsymbol{x}}{a} \cdot \nabla G = \frac{a^2 - r_0^2}{4\pi a\ \rho^3}.$$

Now, substituting into the representation formula (11.14), we obtain the solution of problem (11.27) in the form

$$u(\boldsymbol{x}_0) = \frac{a^2 - |\boldsymbol{x}_0|^2}{4\pi a} \iint\limits_{|\boldsymbol{x}|=a} \frac{h(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}_0|^3}\,\mathrm{d}S. \tag{11.29}$$

This is nothing but the three-dimensional version of the Poisson formula. In the literature, formula (11.29) is often rewritten in spherical coordinates:

$$u(r_0, \theta_0, \phi_0) = \frac{a(a^2 - r_0^2)}{4\pi} \int_0^{2\pi}\int_0^{\pi} \frac{h(\theta,\phi)}{(a^2 + r_0^2 - 2ar_0\cos\psi)^{3/2}} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi,$$

where ψ denotes the angle between the "vectors" $\boldsymbol{x}_0$ and $\boldsymbol{x}$.

Remark 11.11. In the same way we can proceed in two dimensions. Let us consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 < a^2, \\ u(x,y) = h(x,y), & x^2 + y^2 = a^2. \end{cases} \tag{11.30}$$

The corresponding Green's function has the form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi}\ln\rho - \frac{1}{2\pi}\ln\left(\frac{a}{|\boldsymbol{x}_0|}\rho^*\right).$$

The solution of the Dirichlet problem (11.30) is then given by

$$u(\boldsymbol{x}_0) = \frac{a^2 - |\boldsymbol{x}_0|^2}{2\pi a} \int\limits_{|\boldsymbol{x}|=a} \frac{h(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}_0|^2}\,\mathrm{d}s,$$

which is exactly the Poisson formula (8.16) derived in Chapter 6 in a completely different way.

11.7 Exercises

In the following exercises, $r = \sqrt{x^2 + y^2 + z^2}$ denotes one of the spherical coordinates introduced in Section 11.1.

1. Prove that the solution of the Neumann problem for the Poisson equation is determined in the domain Ω uniquely up to a constant.

2. Prove that Green's function corresponding to the Laplace operator, a domain Ω and a point $\boldsymbol{x}_0 \in \Omega$ is determined uniquely.

3. Prove relation (11.18).

4. Verify formula (11.26) for $G(\boldsymbol{x}, \mathbf{0})$.

5. Consider the Dirichlet problem

$$\begin{cases} \Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

   Multiply the equation by the function u and integrate it over the domain Ω. Use Green's first identity to prove that a nontrivial solution $u = u(x, y, z)$ can exist only for λ negative.

6. Find radially symmetric solutions of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. Use the substitution $v = u/r$.

   $[u(x, y, z) = \frac{1}{r}(A\mathrm{e}^{kr} + B\mathrm{e}^{-kr})]$

7. Solve the equation $u_{xx} + u_{yy} + u_{zz} = 0$ in the shell $\{0 < a < r < b\}$ with the boundary conditions $u = A$ for $r = a$ and $u = B$ for $r = b$, where A and B are constants. Search for a radially symmetric solution.

   $[u(x, y, z) = B + (A - B)(\frac{1}{a} - \frac{1}{b})^{-1}(\frac{1}{r} - \frac{1}{b})]$

8. Solve the equation $u_{xx} + u_{yy} + u_{zz} = 1$ in the shell $\{0 < a < r < b\}$ with the condition $u = 0$ on both the outer and the inner boundary.

   $[u(x, y, z) = \frac{1}{6}(r^2 - a^2) - \frac{1}{6}ab(a + b)(\frac{1}{a} - \frac{1}{r})]$

9. Solve the equation $u_{xx} + u_{yy} + u_{zz} = 1$ in the shell $\{0 < a < r < b\}$ with the conditions $u = 0$ for $r = a$ and $\partial u/\partial r = 0$ for $r = b$. Then consider the limit as $a \to 0$ and give reasons for the result.

10. Show that the homogeneous Robin problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + au = 0 \quad \text{on } \partial\Omega,$$

has only the trivial solution $u \equiv 0$. Here Ω is a domain in $\mathbb{R}^3$ and a is a positive constant. Using this result, prove the uniqueness of the boundary value problem

$$\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + au = g \quad \text{on } \partial\Omega.$$

11. Let $\phi(\boldsymbol{x})$ be an arbitrary C^2-function defined on $\mathbb{R}^3$ and nonzero outside some ball. Show that

$$\phi(\mathbf{0}) = -\frac{1}{4\pi} \iiint \frac{1}{|\boldsymbol{x}|} \Delta\phi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Here we integrate over the domain where $\phi(\boldsymbol{x})$ is nonzero.

12. Find Green's function on the half-ball $\Omega = \{x^2 + y^2 + z^2 < a^2,\ z > 0\}$. Consider the solution on the whole ball and use the reflection method similarly to Section 11.6.

[The result is a sum of four terms involving the distances of $\boldsymbol{x}$ to $\boldsymbol{x}_0$, $\boldsymbol{x}_0^*$, $\boldsymbol{x}_0^\#$ and $\boldsymbol{x}_0^{*\#}$, where * denotes reflection across the sphere and $^\#$ denotes reflection across the plane $z = 0$.]

13. Find Green's function on the eighth of the ball $\Omega = \{x^2 + y^2 + z^2 < a^2,\ x > 0,\ y > 0,\ z > 0\}$.

14. In the same way as we have defined Green's function on the domain Ω, we can define the so called *Neumann function* $N(\boldsymbol{x}, \boldsymbol{x}_0)$ with the only difference that the property (ii) is replaced by the Neumann boundary condition

$$\frac{\partial N}{\partial n} = 0 \quad \text{for } \boldsymbol{x} \in \partial\Omega.$$

Formulate and prove the analogue of Theorem 11.7 on the expression for the solution of the Neumann problem using the Neumann function.

15. Solve the Neumann problem on the half-space:

$$\begin{cases} \Delta u = 0 \quad \text{for } z > 0, \\ \frac{\partial u}{\partial z}(x, y, 0) = h(x, y), \quad u \ \text{ bounded at infinity.} \end{cases}$$

Consider the problem for the function $v = \partial u / \partial z$.

$$[u(x, y, z) = C + \int_{-\infty}^{+\infty} h(x - \xi)\, \ln(y^2 + \xi^2)\, \mathrm{d}\xi]$$

16. Consider the four-dimensional Laplace operator $\Delta u = u_{xx} + u_{yy} + u_{zz} + u_{ww}$. Show that its fundamental symmetric solution is $r^{-3/2}$, where we denoted $r^2 = x^2 + y^2 + z^2 + w^2$.

17. Prove the vector form of Green's second identity

$$\iiint\limits_{\Omega}(\boldsymbol{u}\cdot\operatorname{rot}\operatorname{rot}\boldsymbol{v}-\boldsymbol{v}\cdot\operatorname{rot}\operatorname{rot}\boldsymbol{u})\mathrm{d}\boldsymbol{x}=\iint\limits_{\partial\Omega}(\boldsymbol{u}\times\operatorname{rot}\boldsymbol{v}-\boldsymbol{v}\times\operatorname{rot}\boldsymbol{u})\cdot\boldsymbol{n}\,\mathrm{d}S,$$

where $\boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{v}(\boldsymbol{x})$ are smooth vector-valued functions, Ω is a domain with smooth boundary, $\boldsymbol{n}$ is the outward normal vector to $\partial\Omega$, $\boldsymbol{u}\times\boldsymbol{v}$ means the vector product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, and $\operatorname{rot}\boldsymbol{u}=\nabla\times\boldsymbol{u}$ is the rotation of the vector $\boldsymbol{u}$.

18. Prove Green's first identity for the biharmonic operator Δ^2:

$$\iiint\limits_{\Omega} v\Delta^2 u\,\mathrm{d}\boldsymbol{x}=\iiint\limits_{\Omega}\Delta u\Delta v\,\mathrm{d}\boldsymbol{x}-\iint\limits_{\partial\Omega}\Delta u\frac{\partial v}{\partial n}\mathrm{d}S+\iint\limits_{\partial\Omega} v\frac{\partial}{\partial n}(\Delta u)\,\mathrm{d}S.$$

Here $\Delta^2 u=\Delta(\Delta u)=u_{xxxx}+u_{yyyy}+u_{zzzz}+2u_{xxyy}+2u_{yyzz}+2u_{xxzz}$.

19. A function u satisfying $\Delta^2 u=0$ is called biharmonic. Prove the Dirichlet principle for biharmonic functions: "Among all functions v satisfying the boundary conditions

$$v(\boldsymbol{x})=g(\boldsymbol{x}),\quad \frac{\partial v}{\partial n}(\boldsymbol{x})=h(\boldsymbol{x}),\quad \boldsymbol{x}\in\partial\Omega,$$

where $g(\boldsymbol{x}),\ h(\boldsymbol{x})\in C(\partial\Omega)$, the lowest energy

$$E(v)=\frac{1}{2}\iiint\limits_{\Omega}|\Delta v|^2\mathrm{d}\boldsymbol{x}$$

is attained by the biharmonic function."

Chapter 12

Diffusion Equation in Higher Dimensions

In the previous chapters we considered only one-dimensional models of evolution equations. However, majority of physical phenomena occur in the plane, or in the space. Therefore we now focus on the heat (and diffusion) equation in higher dimensions, that is, on the equation

$$u_t - k\Delta u = f, \tag{12.1}$$

where $\Delta u = u_{xx} + u_{yy}$ in the case of the two-dimensional model, or $\Delta u = u_{xx} + u_{yy} + u_{zz}$ in the case of three-dimensional model, respectively.

12.1 Cauchy Problem in $\mathbb{R}^3$

12.1.1 Homogeneous Problem

Let us consider the Cauchy problem

$$\begin{cases} u_t = k\Delta u, \quad \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}). \end{cases} \tag{12.2}$$

As usual, we denote $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$.

We already know from the one-dimensional case (see Chapter 5) that we can express the solution of the Cauchy problem on $\mathbb{R}$ in the integral form

$$u(x,t) = \int_{-\infty}^{+\infty} G(x-y,t)\varphi(y)\,\mathrm{d}y,$$

where φ is the given initial condition and G the so called *fundamental solution* (diffusion kernel)

$$G(x,t) = \frac{1}{2\sqrt{\pi k t}} \mathrm{e}^{-\frac{x^2}{4kt}}.$$

As we will see, the same holds true in higher dimensions. We start with the following observation. Let $u_1(x,t)$, $u_2(y,t)$, $u_3(z,t)$ be solutions of the one-dimensional diffusion equation. Then $u(x,y,z,t) = u_1(x,t)u_2(y,t)u_3(z,t)$ solves the diffusion equation in $\mathbb{R}^3$. We recommend the reader to verify it by direct substitution. It means that also the function

$$\begin{aligned} G_3(\boldsymbol{x},t) &= G(x,t)G(y,t)G(z,t) \\ &= \frac{1}{8\sqrt{(\pi k t)^3}}\mathrm{e}^{-\frac{1}{4kt}(x^2+y^2+z^2)} = \frac{1}{8\sqrt{(\pi k t)^3}}\mathrm{e}^{-\frac{1}{4kt}|\boldsymbol{x}|^2} \end{aligned}$$

solves the diffusion equation $u_t - k\Delta u = 0$ in $\mathbb{R}^3$. Since it satisfies

$$\iiint\limits_{\mathbb{R}^3} G_3(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = 1$$

(the reader is kindly asked to verify this fact using the Fubini Theorem), it is called the *fundamental solution* (or *diffusion kernel*).

Our aim is to show that the solution of the Cauchy problem (12.2) can be written in the form of a convolution of this fundamental solution $G_3(\boldsymbol{x},t)$ and the initial condition $\varphi(\boldsymbol{x})$, that is,

$$u(\boldsymbol{x},t) = \iiint\limits_{\mathbb{R}^3} G_3(\boldsymbol{x}-\boldsymbol{y},t)\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{12.3}$$

Here $\boldsymbol{y} = (\xi,\eta,\theta) \in \mathbb{R}^3$.

We start with a special initial condition with separated variables

$$\varphi(\boldsymbol{x}) = \phi(x)\psi(y)\zeta(z).$$

In this case we have

$$\begin{aligned} u(\boldsymbol{x},t) &= \iiint\limits_{\mathbb{R}^3} G_3(\boldsymbol{x}-\boldsymbol{y},t)\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\ &= \int\limits_{-\infty}^{+\infty} G(x-\xi)\phi(\xi)\,\mathrm{d}\xi \int\limits_{-\infty}^{+\infty} G(y-\eta)\psi(\eta)\,\mathrm{d}\eta \int\limits_{-\infty}^{+\infty} G(z-\theta)\zeta(\theta)\,\mathrm{d}\theta \\ &= u_1(x,t)\,u_2(y,t)\,u_3(z,t), \end{aligned}$$

where u_1, u_2, u_3 are solutions of the one-dimensional diffusion equation. Thus $u(\boldsymbol{x},t)$ must solve the three-dimensional diffusion equation. Moreover,

$$\begin{aligned} \lim_{t\to 0+} u(\boldsymbol{x},t) &= \lim_{t\to 0+} u_1(x,t) \lim_{t\to 0+} u_2(y,t) \lim_{t\to 0+} u_3(z,t) \\ &= \phi(x)\psi(y)\zeta(z) = \varphi(\boldsymbol{x}) \end{aligned}$$

and the initial condition with separated variables is satisfied as well. Due to linearity, the same result must hold true for any finite linear combination of functions with separated variables:

$$\varphi(\boldsymbol{x}) = \sum_{k=1}^{n} c_k\phi_k(x)\psi_k(y)\zeta_k(z). \tag{12.4}$$

It can be shown that any continuous and bounded function on $\mathbb{R}^3$ can be uniformly approximated by functions of type (12.4) on bounded domains. This follows from the properties of Bernstein's polynomials, which go beyond the scope of this book. Nevertheless, this fact is the starting point for the following existence result (see, e.g., Stavroulakis, Tersian [20]).

Theorem 12.1. *Let $\varphi(\boldsymbol{x})$ be a continuous and bounded function on $\mathbb{R}^3$. Then the solution of the Cauchy problem* (12.2) *exists and is given by the formula*

$$u(\boldsymbol{x}, t) = \iiint_{\mathbb{R}^3} G_3(\boldsymbol{x} - \boldsymbol{y}, t)\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Moreover, we have

$$\lim_{t \to 0+} u(\boldsymbol{x}, t) = \varphi(\boldsymbol{x})$$

uniformly on bounded sets of $\mathbb{R}^3$.

Remark 12.2. The same result holds true in any dimension. In particular, the N-dimensional ($N \geq 1$) fundamental solution assumes the form

$$G_N(\boldsymbol{x}, t) = \frac{1}{2^N \sqrt{(\pi k t)^N}} \mathrm{e}^{-\frac{1}{4kt}|\boldsymbol{x}|^2},$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$, $|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, and the solution of the Cauchy problem for a homogeneous diffusion equation on $\mathbb{R}^N$ is given by the formula

$$u(\boldsymbol{x}, t) = \int_{\mathbb{R}^N} G_N(\boldsymbol{x} - \boldsymbol{y}, t)\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

12.1.2 Nonhomogeneous Problem

Using the same approach as in Section 5.2, we can solve a diffusion equation on $\mathbb{R}^3$ with sources:

$$\begin{cases} u_t - k\Delta u = f, & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}). \end{cases} \tag{12.5}$$

Its solution is given by the following formula (we leave the derivation and verification to the reader):

$$u(\boldsymbol{x},t) = \iiint\limits_{\mathbb{R}^3} G_3(\boldsymbol{x}-\boldsymbol{y},t)\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int\limits_0^t \iiint\limits_{\mathbb{R}^3} G_3(\boldsymbol{x}-\boldsymbol{y},t-s) f(\boldsymbol{y},s)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s.$$

12.2 Diffusion on Bounded Domains, Fourier Method

In this section we focus on the diffusion equation on a bounded domain and on its solution. That is, we deal with the problem

$$\begin{cases} u_t(\boldsymbol{x},t) = k\Delta u(\boldsymbol{x},t), & \boldsymbol{x} \in \Omega,\ t > 0, \\ u(\boldsymbol{x},t) = h_1(\boldsymbol{x},t), & \boldsymbol{x} \in \Gamma_1, \\ \frac{\partial u}{\partial n}(\boldsymbol{x},t) = h_2(\boldsymbol{x},t), & \boldsymbol{x} \in \Gamma_2, \\ \frac{\partial u}{\partial n}(\boldsymbol{x},t) + au(\boldsymbol{x},t) = h_3(\boldsymbol{x},t), & \boldsymbol{x} \in \Gamma_3, \\ u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}), \end{cases}$$

where, in general, Ω is a domain in $\mathbb{R}^N$, φ, h_i, $i = 1,2,3$ are given functions, a is a given constant, and $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial\Omega$. From the physical point of view, such a problem describes the heat flow in a body filling the domain Ω with the initial temperature given by $\varphi(\boldsymbol{x})$. On Γ_1 we keep the temperature on the values $h_1(\boldsymbol{x},t)$; on the boundary Γ_2 we consider the heat flux described by $h_2(\boldsymbol{x},t)$; and on the third boundary segment Γ_3 we consider the heat exchange with the surrounding medium described by the heat transfer coefficient a and a function $h_3(\boldsymbol{x},t)$. Usually, $h_3 = aT_0$, where T_0 is the temperature of the surrounding medium.

Similarly, the same problem describes the diffusion process of a gas in the domain Ω. The function $\varphi(\boldsymbol{x})$ represents the initial concentration. The Dirichlet boundary condition on Γ_1 describes the concentration kept on Γ_1, the Neumann boundary condition on Γ_2 determines the flow of the gas across the boundary, and the Robin boundary condition on Γ_3 describes a certain balance of the gas concentration and its flow across the third boundary segment. In special cases, some of the boundary segments can be empty.

One way to solve initial boundary value problems of this type is using the *Fourier Method.* We illustrate this in a simpler situation. The following exposition is very informative and a lot of stated facts would require deep discussion to make them precise. To make the basic idea clear, we present only formal calculations.

12.2.1 Fourier Method

Let us find a solution of the diffusion equation $u_t = k\Delta u$ on a bounded domain Ω with *homogeneous* Dirichlet, Neumann, or Robin boundary conditions on $\partial\Omega$, and a standard initial condition. The idea of the Fourier method is the same as in the one-dimensional case. First of all, we assume the solution in a separated form

$$u(\boldsymbol{x}, t) = V(\boldsymbol{x})T(t)$$

which, substituted into the equation, results in the identities

$$\frac{T'(t)}{kT(t)} = \frac{\Delta V(\boldsymbol{x})}{V(\boldsymbol{x})} = -\lambda,$$

where λ is a constant. Thus, we come to the *eigenvalue problem* for the Laplace operator

$$-\Delta V = \lambda V \quad \text{in } \Omega \tag{12.6}$$

with general homogenous boundary conditions

$$V = 0 \quad \text{on} \quad \Gamma_1, \tag{12.7}$$

$$\frac{\partial V}{\partial n} = 0 \quad \text{on} \quad \Gamma_2, \tag{12.8}$$

$$\frac{\partial V}{\partial n} + a\, V = 0 \quad \text{on} \quad \Gamma_3. \tag{12.9}$$

It can be shown that the boundary value problem (12.6)–(12.9) has an infinite sequence of nonnegative eigenvalues

$$\lambda_n \to +\infty \quad \text{as } n \to +\infty$$

and a corresponding complete system of orthogonal eigenfunctions $V_n(\boldsymbol{x})$. The reader should notice properties of the eigenvalue problem (12.6)–(12.9) similar to those of the Sturm-Liouville problem stated in Appendix A. However, it can be very difficult to find the actual values of λ_n.

Returning to the ODE in time variable

$$T'(t) + k\lambda_n T(t) = 0,$$

we obtain a system of time-dependent functions

$$T_n(t) = A_n \mathrm{e}^{-k\lambda_n t}.$$

Putting these partial results together, we end up with a solution in the form

$$u(\boldsymbol{x}, t) = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-k\lambda_n t} V_n(\boldsymbol{x}) \tag{12.10}$$

which satisfies the given initial condition provided φ is expandable into the Fourier series

$$\varphi(\boldsymbol{x}) = \sum_{n=1}^{+\infty} A_n V_n(\boldsymbol{x})$$

according to the system $\{V_n\}_{n=1}^{+\infty}$. Using the orthogonality of the eigenfunctions $V_n(\boldsymbol{x})$, we obtain the formula for the coefficients A_n:

$$A_n = \frac{\int\limits_{\Omega} \varphi(\boldsymbol{x}) V_n(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}}{\int\limits_{\Omega} V_n^2(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}}.$$

(Since Ω is, in general, a domain in $\mathbb{R}^N$, the integrals above are also N-dimensional!) We illustrate the previous steps by a concrete example.

Example 12.3. Let us find the temperature distribution in the rectangle $\Omega = (0,a) \times (0,b)$ whose boundary is kept on zero temperature; the initial distribution is given by a function $\varphi(x,y)$. That is, we solve the initial boundary value problem

$$\begin{cases} u_t = k(u_{xx} + u_{yy}), \quad (x,y) \in (0,a) \times (0,b),\ t > 0, \\ u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0, \\ u(x,y,0) = \varphi(x,y). \end{cases} \tag{12.11}$$

First of all, we separate the time and space variables:

$$u(x,y,t) = V(x,y)T(t)$$

thus obtaining the system of equations

$$\frac{T'(t)}{kT(t)} = \frac{V_{xx}(x,y) + V_{yy}(x,y)}{V(x,y)} = -\lambda,$$

where λ is a constant. This yields

$$T' + k\lambda T = 0, \quad V_{xx} + V_{yy} + \lambda V = 0.$$

Now, we will have a look at the spatial problem in more detail. Since it is a linear stationary PDE in a rectangle, we can use the separation of variables again. Thus, we look for its solution in the form

$$V(x,y) = X(x)Y(y).$$

After the substitution, we obtain

$$X''Y + XY'' + \lambda XY = 0$$

and hence, dividing by XY,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - \lambda.$$

Since the left-hand side depends only on the x-variable and the right-hand side depends only on the y-variable, we conclude that both sides of the identity must be equal to a constant, say, $-\mu$. Thus, we obtain two separated ODEs

$$X'' + \mu X = 0, \quad Y'' + \nu Y = 0,$$

where $\nu = \lambda - \mu$. To satisfy the homogeneous boundary conditions in (12.11), the functions X and Y must satisfy the conditions

$$X(0) = X(a) = 0, \quad Y(0) = Y(b) = 0.$$

Starting with the problem in the x-variable, we obtain

$$\mu_n = \frac{n^2\pi^2}{a^2}, \quad X_n(x) = \sin\frac{n\pi x}{a}, \quad n \in \mathbb{N}.$$

Similarly, for the problem in the y-variable, we get

$$\nu_m = \frac{m^2\pi^2}{b^2}, \quad Y_m(x) = \sin\frac{m\pi y}{b}, \quad m \in \mathbb{N}.$$

Thus, the eigenvalues of the Laplace operator form a sequence

$$\lambda_{mn} = \mu_n + \nu_m = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \quad m, n \in \mathbb{N},$$

and the corresponding eigenfunctions

$$V_{mn}(x, y) = X_n(x)Y_m(y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}, \quad m, n \in \mathbb{N},$$

form an orthogonal system on $(0, a) \times (0, b)$ (notice the double index!). It can be proved that this system is complete and hence λ_{mn} describe the set of all eigenvalues of the Laplace operator on $(0, a) \times (0, b)$ with homogeneous Dirichlet boundary conditions. Now, we solve the time problems $T' + k\lambda_{mn}T = 0$, that is,

$$T_{mn} = A_{mn}\mathrm{e}^{-k\lambda_{mn}t}.$$

Hence, the solution of the original two-dimensional diffusion equation can be written as

$$u(x, y, t) = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} A_{mn}\mathrm{e}^{-k\lambda_{mn}t}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}. \tag{12.12}$$

Functions of this form already satisfy homogeneous Dirichlet boundary conditions. The remaining unused information is the initial condition. We can conclude that for all initial conditions which are expandable into a series

$$\varphi(x,y) = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}, \tag{12.13}$$

the solution of problem (12.11) is given by formula (12.12). Using the orthogonality of the eigenfunctions, we can determine the constants A_{mn}:

$$A_{mn} = \frac{\int_0^a\int_0^b \varphi(x,y)V_{mn}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^a\int_0^b V_{mn}^2(x,y)\,\mathrm{d}y\,\mathrm{d}x}.$$

◇

Figure 12.1 depicts the solution of problem (12.11) with the constant initial condition $\varphi(\boldsymbol{x}) = 100$ and with the data $a = 1$, $b = 1$, $k = 1$. The first graph corresponds to the approximated initial condition, the other three graphs illustrate the approximated solution at times $t = 0.01$, $t = 0.04$ and $t = 0.09$. We used formulae (12.12) and (12.13) with partial sums up to $m = 15$, $n = 15$.

The same approach can be used also for other types of boundary conditions and for similar problems in higher dimensions.

Example 12.4. Let us solve the initial boundary value problem for the diffusion equation $u_t = k\Delta u$ in the cube $\Omega = \{0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi\}$. This time, consider homogeneous Neumann boundary conditions

$$u_x(0,y,z,t) = u_x(\pi,y,z,t) = 0,$$
$$u_y(x,0,z,t) = u_y(x,\pi,z,t) = 0,$$
$$u_z(x,y,0,t) = u_z(x,y,\pi,t) = 0,$$

and initial condition

$$u(x,y,z,0) = \varphi(x,y,z).$$

We proceed in the same way as in the previous example. First, we separate the time and space variables to obtain

$$T' + k\lambda T = 0, \quad V_{xx} + V_{yy} + V_{zz} + \lambda V = 0.$$

Second, we apply the separation of variables to the spatial problem. Thus, we get three ODEs

$$\begin{aligned} X'' + \mu X &= 0, \\ Y'' + \nu Y &= 0, \\ Z'' + \eta Z &= 0, \end{aligned}$$

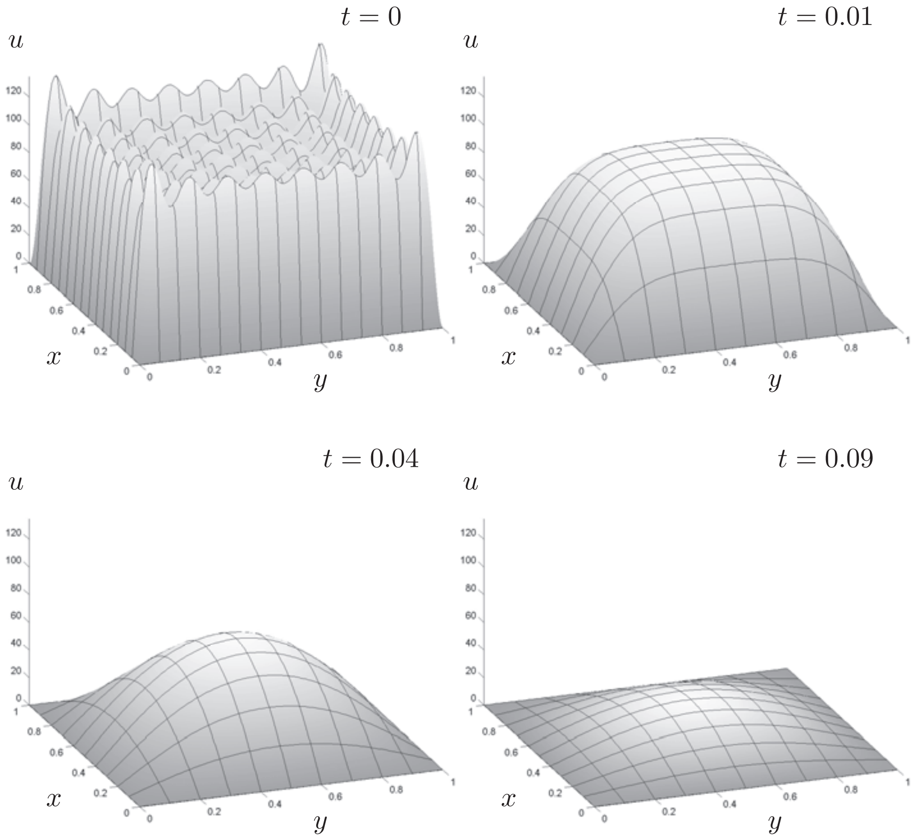


Figure 12.1. Graphic illustration of the solution of the initial boundary value problem (12.11) with constant initial condition on time levels $t = 0$, 0.01, 0.04, 0.09.

where $\lambda = \mu + \nu + \eta$. Adding the boundary conditions $X'(0) = X'(\pi) = Y'(0) = Y'(\pi) = Z'(0) = Z'(\pi) = 0$ and solving the corresponding ODE problems, we obtain

$$\mu_n = n^2, \quad \nu_m = m^2, \quad \eta_l = l^2, \quad l, m, n \in \mathbb{N} \cup \{0\}$$

and

$$X_n(x) = \cos nx, \quad Y_m(y) = \cos my, \quad Z_l(z) = \cos lz, \quad l, m, n \in \mathbb{N} \cup \{0\}.$$

Thus, the eigenvalues of the Laplace operator in the cube $\Omega = \{0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi\}$ with homogeneous Neumann boundary conditions form a sequence

$$\lambda_{lmn} = m^2 + n^2 + l^2, \quad m, n, l \in \mathbb{N} \cup \{0\},$$

with the corresponding system of eigenfunctions

$$V_{lmn} = \cos nx \, \cos my \, \cos lz, \quad m, n, l \in \mathbb{N} \cup \{0\}.$$

Notice that in this case the problem has zero eigenvalue with a constant eigenfunction.

Now, we can continue with the time problem. In the same way as in the previous cases, we obtain

$$T_{lmn}(t) = B_{lmn} \mathrm{e}^{-k\lambda_{lmn} t} \quad \text{for } (m, n, l) \neq (0, 0, 0)$$

and

$$T_{000}(t) = B_{000},$$

and we can conclude that the required solution is given by

$$u(x, y, z, t) = \sum_{l=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} B_{lmn} \mathrm{e}^{-k\lambda_{lmn} t} \cos nx \, \cos my \, \cos lz,$$

where the coefficients B_{lmn} follow from the expansion of the initial condition

$$\varphi(x, y, z) = \sum_{l=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} B_{lmn} \cos nx \, \cos my \, \cos lz.$$

That is,

$$B_{lmn} = \frac{2^3}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \varphi(x, y, z) \, \cos nx \, \cos my \, \cos lz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{12.14}$$

for $n, m, l > 0$; for B_{0mn}, B_{l0n}, or B_{lm0} we have to use one-half of (12.14), for B_{00n}, B_{0m0}, B_{l00} we use one-fourth, and for B_{000} we use one-eighth of (12.14). ◇

As we can see, the geometry (in particular, the rectangularity) of the domain Ω is crucial for easy determination of the eigenvalues and the corresponding eigenfunctions of the problems considered. We already know that other domains which allow the application of the Fourier method, are a disc and a ball (or their suitable parts), since they both become rectangular under the transformation into the polar or spherical coordinates, respectively. Moreover, in the radially symmetric situations, the problems are considerably simplified.

Example 12.5 (Diffusion in Disc). Let us consider the heat problem in the disc

$$\begin{cases} u_t = k\Delta u, & x^2 + y^2 < a^2, \ t > 0, \\ u(x, y, t) = 0, & x^2 + y^2 = a^2, \ t > 0, \\ u(x, y, 0) = \varphi(\sqrt{x^2 + y^2}), & x^2 + y^2 < a^2. \end{cases} \tag{12.15}$$

Since the domain is circular, we will transform the problem using polar coordinates (r, θ). Moreover, the problem data (that is, the boundary and initial conditions) do not depend on the angle θ, thus the solution u is expected to be radially symmetric and we solve the simplified problem in two variables r and t:

$$\begin{cases} u_t = k(u_{rr} + \frac{1}{r}u_r), & 0 < r < a,\ t > 0, \\ u(r,t) = 0, & r = a,\ t > 0, \\ u(r,0) = \varphi(r), & 0 \le r < a. \end{cases}$$

As usual, we separate the variables

$$u(r,t) = R(r)T(t)$$

and obtain

$$\frac{T'(t)}{kT(t)} = \frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} = -\lambda.$$

The spatial ODE is the so called *Bessel equation*

$$R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = 0$$

which has a pair of linearly independent solutions. The first, which is finite at $r = 0$, is the *Bessel function of order zero*

$$R(r) = J_0(\sqrt{\lambda}r) = \sum_{j=0}^{+\infty}(-1)^j \frac{(\sqrt{\lambda}r/2)^{2j}}{(j!)^2}.$$

The second solution of the Bessel equation is infinite at $r = 0$ and thus we are not interested in it. (For more details, see Appendix B.) Further, we have to satisfy the homogeneous boundary condition on the boundary $r = a$, that is,

$$R(a) = J_0(\sqrt{\lambda}a) = 0.$$

Thus we get a sequence of eigenvalues

$$\lambda_n = \frac{1}{a^2}\mu_n^2, \quad n \in \mathbb{N},$$

and corresponding eigenfunctions

$$R_n(r) = J_0(\sqrt{\lambda_n}r) = J_0(\mu_n \frac{r}{a}), \quad n \in \mathbb{N}.$$

Here μ_n are the roots of the Bessel function J_0. (Each Bessel function has an infinite number of positive roots that go to infinity, cf. Appendix B.)

Now, we go back to the time problem, which has the standard solution

$$T_n(t) = A_n \mathrm{e}^{-k\lambda_n t}.$$

The solution of the original problem (12.15) then can be written in the form

$$u(r,t) = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{-k\lambda_n t} J_0(\sqrt{\lambda_n} r),$$

where coefficients A_n are determined by the initial condition

$$\varphi(r) = \sum_{n=1}^{+\infty} A_n J_0(\sqrt{\lambda_n} r).$$

For $\rho = \frac{r}{a} \in [0,1]$ we have

$$\varphi(a\rho) = \sum_{n=1}^{+\infty} A_n J_0(\mu_n \rho)$$

and the properties of the Bessel functions stated at the end of Appendix B imply

$$A_n = \frac{2}{J'^2_0(\mu_n)} \int_0^1 \rho J_0(\mu_n \rho) \varphi(a\rho) \, \mathrm{d}\rho$$

(the reader is kindly asked to carry out detailed calculations). ◇

Figure 12.2 depicts the solution of problem (12.15) for the choice $a = 1$, $k = 1$ and with the initial condition

$$\varphi(x,y) = \varphi(r) = J_0(\mu_1 r) + J_0(\mu_2 r), \tag{12.16}$$

where J_0 is the Bessel function of order zero and μ_1, μ_2 are its first two roots. Notice that, for this data, the solution assumes the form

$$u(x,y,t) = u(r,t) = \mathrm{e}^{-\mu_1^2 t} J_0(\mu_1 r) + \mathrm{e}^{-\mu_2^2 t} J_0(\mu_2 r).$$

The particular graphs in Figure 12.2 correspond to the solution at times $t = 0$, $t = 0.01$, $t = 0.04$ and $t = 0.09$.

12.2.2 Nonhomogeneous Problems

The idea of solving nonhomogeneous problems for the diffusion equation in higher dimensions is exactly the same as in the one-dimensional case. If we solve a *nonhomogeneous equation*, we find the system of eigenfunctions $V_n(\boldsymbol{x})$ corresponding to the homogeneous problem and expand all the problem data (that is, the right-hand side, the initial condition, as well as the searched solution) to Fourier series with respect to this system. Using its completeness

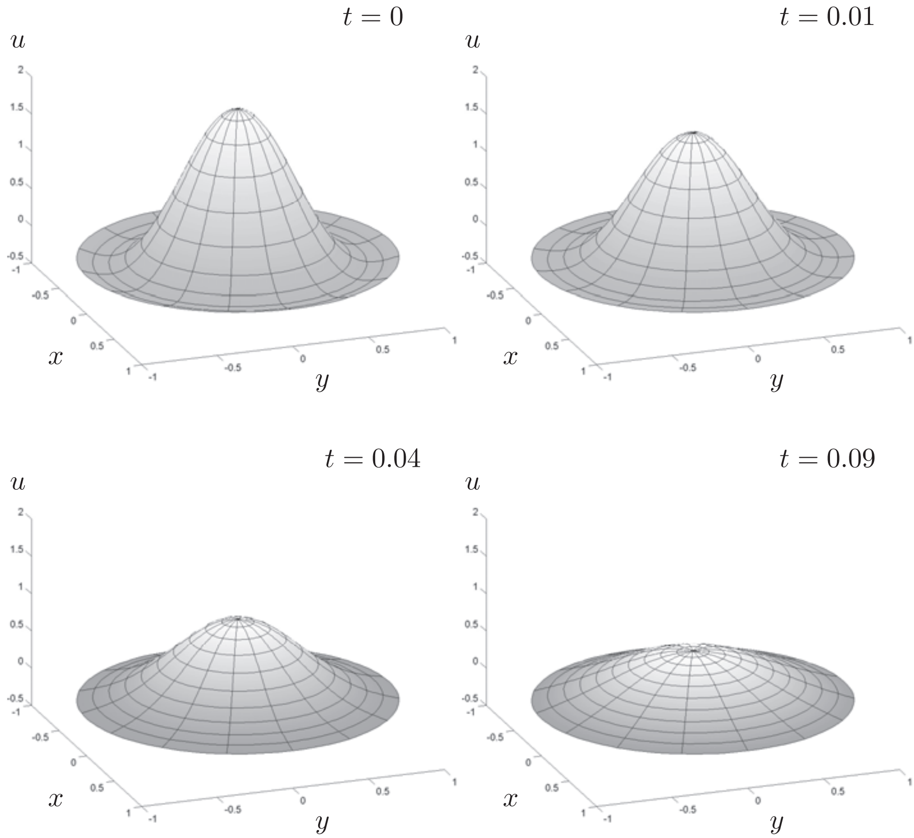


Figure 12.2. Graphic illustration of the solution of the initial boundary value problem (12.15) with initial condition (12.16) on time levels $t = 0$, 0.01, 0.04, 0.09.

and orthogonality, we split the original PDE problem into an infinite system of ODEs in the time variable which are easy to solve.

Problems with *nonhomogeneous boundary conditions* can cause more trouble. The idea is, again, to split the solution into two parts: the first part corresponds to the solution satisfying the equation with homogeneous boundary conditions, while the second "stationary" (or "quasi-stationary") part respects the nonhomogeneous boundary conditions. In one-dimensional cases, we usually "guessed" the stationary part easily (see Section 7.3). In higher dimensions, it means to solve the Laplace equation with given nonhomogeneous boundary conditions, which can be very laborious. Again, the rectangularity of the domain can be essential.

12.3 General Principles for Diffusion Equation

The aim of this section is to recall all basic properties of the diffusion equation which remain unchanged in any dimension.

First of all, the solution formula in Theorem 12.1 implies that diffusion (as well as heat) propagates at *infinite speed*. After any short time, the solution is nonzero everywhere, even if the initial condition was nonzero only on a small domain. As we have already mentioned in Chapter 5, this fact reflects the inaccuracy of the diffusion model. However, the incurred error is very small and the diffusion equation can be used as a good approximation of many real problems.

Another property which occurs in any dimension is the *ill-posedness* of the diffusion problems for $t < 0$. It is not possible to determine the temperature of the body backwards in time, neither to find the original concentration of a diffusing gas, provided we know only the actual state.

A very important property of the diffusion equation on any (bounded or unbounded) domain in any dimension is the *Maximum Principle*. Its strong version says that the maximum and minimum values of the solution are achieved only on the bottom or jacket of the space-time cylinder, unless the solution is constant. Here the bottom of the cylinder is the (in general, N-dimensional) domain Ω at time $t = 0$, and the jacket represents the boundary $\partial\Omega$ at any time $t > 0$! See Figure 12.3.

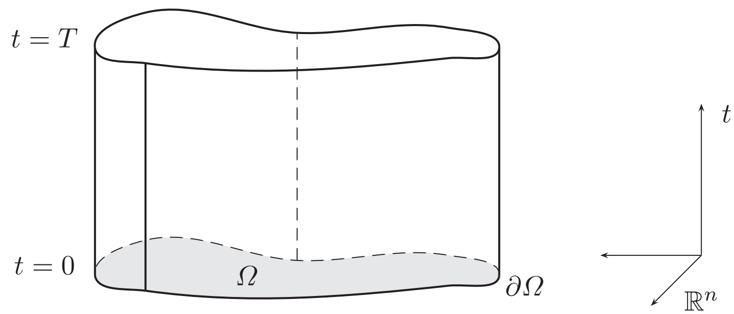


Figure 12.3. Space-time cylinder $\Omega \times [0, T]$.

As in the one-dimensional case, the Maximum Principle has many consequences. The most important ones are the *uniqueness* and *uniform stability* of the solution. Studying the corresponding proofs in Section 10.4, notice their independence of the dimension.

The uniqueness and stability properties can be obtained also via the *energy method*, which can be applied in any dimension. We again refer to Section 10.5.

12.4 Exercises

1. Solve the following problem for the diffusion equation on the whole space:

$$\begin{cases} u_t - \Delta u = 0, & (x,y,z) \in \mathbb{R}^3,\ t > 0, \\ u(x,y,z,0) = x^2yz. \end{cases}$$

$$[u(x,y,z,t) = x^2yz + 2tyz]$$

2. Solve the following problem for the diffusion equation on the whole space:

$$\begin{cases} u_t - \Delta u = 0, & (x,y,z) \in \mathbb{R}^3,\ t > 0, \\ u(x,y,z,0) = x^2yz - xyz^2. \end{cases}$$

$$[u(x,y,z,t) = y(xz - 2t)(x - z)]$$

3. Using the reflection method (method of odd extension), find a formula for the solution of the initial boundary value problem for the diffusion equation in the half-plane

$$\begin{cases} u_t - k\Delta u = 0, & x > 0,\ y \in \mathbb{R},\ t > 0, \\ u(0,y,t) = 0, \\ u(x,y,0) = \varphi(x,y). \end{cases}$$

$$[u(x,y,t) = \int_{-\infty}^{+\infty} \int_0^{+\infty} (G_2(x-\xi, y-\eta, t) - G_2(x+\xi, y-\eta, t))\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta]$$

4. Using the reflection method (method of even extension), find a formula for the solution of the initial boundary value problem for the diffusion equation in the half-space

$$\begin{cases} u_t - k\Delta u = 0, & (x,y) \in \mathbb{R}^2,\ z > 0,\ t > 0, \\ u_z(x,y,0,t) = 0, \\ u(x,y,z,0) = \varphi(x,y,z). \end{cases}$$

$$[u(x,y,z,t) =$$
$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{+\infty} (G_3(x-\xi, y-\eta, z-\theta, t) - G_3(x+\xi, y-\eta, z-\theta, t))\varphi(\xi,\eta,\theta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\theta]$$

5. Solve the diffusion equation $u_t = u_{xx} + u_{yy}$ in the disc $x^2 + y^2 < 1$ with homogeneous Dirichlet boundary condition and with the initial condition $u(x,y,0) = 1 - x^2 - y^2$.

[in polar coordinates: $u(r,t) = 8 \sum_{k=1}^{+\infty} \mathrm{e}^{-\mu_k^2 t} \frac{J_0(\mu_k r)}{\mu_k^3 J_1(\mu_k)}$.]

6. Solve the problem

$$\begin{cases} u_t = a^2(u_{rr} + \frac{1}{r}u_r), \quad 0 < r < R,\ t > 0, \\ u(r,0) = T, \\ \frac{\partial}{\partial r}u(r,t)|_{r=R} = q. \end{cases}$$

[$u(r,t) = T + qR(2\frac{a^2t}{R^2} - \frac{1}{4}(1-2\frac{r^2}{R^2}) - \sum_{n=1}^{+\infty} \frac{2e^{-(a\mu_n/R)^2t}}{\mu_n^2 J_0(\mu_n)} J_0(\frac{\mu_n r}{R})$, where μ_n are positive roots of J_1]

7. Consider the problem of cooling of the ball of radius R with a radiation boundary condition
$$u_r(R,t) = -hu(R,t),$$
where h is a positive constant and $Rh < 1$. Assume that the initial temperature $u(\boldsymbol{x},t) = \varphi(r)$ depends only on the radius r. Solve the radially symmetric diffusion equation using the Fourier method. (The eigenvalues λ_n are obtained as the positive roots of the equation $\tan R\lambda = \frac{R\lambda}{1-Rh}$.)

8. Consider a thin rectangular plate of length a and width b with perfect lateral insulation. Find the distribution of temperature in the plate for the following data: $a = 2\pi$, $b = 4\pi$, $k = 1$, boundary conditions

$$u_x(0,y,t) = 0, \quad u_x(a,y,t) = 0,$$
$$u_y(x,0,t) = 0, \quad u_y(x,b,t) = 0,$$

and the initial condition

$$u(x,y,0) = \cos 3x, \quad 0 \le x \le a,\ 0 \le y \le b.$$

9. Find the distribution of temperature in a semicircular plate ($0 < \theta < \pi$) of radius 1 with the initial condition $u(r,\theta,0) = g(r,\theta)$ and the boundary conditions

(a) $u(r,0,t) = 0, \quad u(r,\pi,t) = 0, \quad u(1,\theta,t) = 0.$

(b) $u_\theta(r,0,t) = 0, \quad u_\theta(r,\pi,t) = 0, \quad u(1,\theta,t) = 0.$

What happens after infinitely long time?

Chapter 13

Wave Equation in Higher Dimensions

13.1 Cauchy Problem in $\mathbb{R}^3$ – Kirchhoff's Formula

Let us consider the Cauchy problem for the wave equation in $\mathbb{R}^3$

$$\begin{cases} u_{tt} = c^2 \Delta u, \quad \boldsymbol{x} = (x, y, z) \in \mathbb{R}^3, \ t > 0, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \quad u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}). \end{cases} \tag{13.1}$$

First of all, we state and derive the explicit formula for its solution.

Theorem 13.1 (Kirchhoff's Formula). *Let $\varphi \in C^3(\mathbb{R}^3)$ and $\psi \in C^2(\mathbb{R}^3)$. The classical solution of the Cauchy problem for the homogeneous wave equation* (13.1) *exists, it is unique and is given by the formula*

$$u(\boldsymbol{x}_0, t) = \frac{1}{4\pi c^2 t} \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \psi(\boldsymbol{x}) \, \mathrm{d}S + \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \varphi(\boldsymbol{x}) \, \mathrm{d}S \right). \tag{13.2}$$

Here the integrals are surface integrals over the sphere with its center at $\boldsymbol{x}_0$ and radius ct. This formula is known as *Kirchhoff's formula* but its author is Poisson. For its derivation, we use the so called *spherical means.*

Let us denote by $\overline{u}(\boldsymbol{x}_0, r, t)$ the mean (average) value of the function $u(\boldsymbol{x}, t)$ over the sphere $|\boldsymbol{x} - \boldsymbol{x}_0| = r$, that is

$$\overline{u}(\boldsymbol{x}_0, r, t) = \frac{1}{4\pi r^2} \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=r} u(\boldsymbol{x}, t) \mathrm{d}S.$$

Using transformation to spherical coordinates, we can write

$$\overline{u}(\boldsymbol{x}_0, r, t) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \varphi, t) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta,$$

where

$$u(r, \theta, \varphi, t) = u(x_0 + r\cos\theta\sin\varphi, y_0 + r\sin\theta\sin\varphi, z_0 + r\cos\varphi, t).$$

Proof of Theorem 13.1 (Derivation of Kirchhoff's Formula). The main idea of the derivation of Kirchhoff's formula consists of two steps. First, we solve the problem (13.1) for the spherical means, and, second, we pass to the solution of the original problem using the relation

$$u(\boldsymbol{x}_0, t) = \lim_{r\to 0} \overline{u}(\boldsymbol{x}_0, r, t). \tag{13.3}$$

Let us start with the following observation: if u satisfies the wave equation, then $\overline{u}$ satisfies it as well. Indeed, the equality $\overline{u_{tt}} = (\overline{u})_{tt}$ is obvious. Further, using spherical coordinates and the rotational invariance of the Laplace operator, we obtain

$$\overline{\Delta u} = \Delta\overline{u} = \overline{u}_{rr} + \frac{2}{r}\overline{u}_r. \tag{13.4}$$

(The direct derivation of (13.4) is required in Exercise 3 in Section 13.6.) Thus, $\overline{u}$ satisfies the equation

$$\overline{u}_{tt} = c^2(\overline{u}_{rr} + \frac{2}{r}\overline{u}_r). \tag{13.5}$$

Now, we introduce the substitution

$$v(r, t) = r\overline{u}(\boldsymbol{x}_0, r, t). \tag{13.6}$$

Since $v_{tt} = r\overline{u}_{tt}$, $v_r = r\overline{u}_r + \overline{u}$ and $v_{rr} = r\overline{u}_{rr} + 2\overline{u}_r$, equation (13.5) reduces to

$$v_{tt} = c^2 v_{rr} \tag{13.7}$$

for $(r, t) \in (0, +\infty) \times (0, +\infty)$. Obviously, we can set

$$v(0, t) = 0. \tag{13.8}$$

Moreover, since u solves the original Cauchy problem (13.1), v must fulfil the initial conditions

$$v(r, 0) = r\overline{\varphi}(\boldsymbol{x}_0, r), \qquad v_t(r, 0) = r\overline{\psi}(\boldsymbol{x}_0, r). \tag{13.9}$$

However, equation (13.7) with the boundary condition (13.8) and initial conditions (13.9) forms a standard one-dimensional problem for the wave equation on the half-line. The solution was found in Section 7.1 and for $0 \le r \le ct$ it can be written in the form

$$v(r, t) = \frac{1}{2}\Big((ct + r)\overline{\varphi}(\boldsymbol{x}_0, ct + r) - (ct - r)\overline{\varphi}(\boldsymbol{x}_0, ct - r)\Big) + \frac{1}{2c}\int_{ct-r}^{ct+r} s\overline{\psi}(\boldsymbol{x}_0, s)\,\mathrm{d}s.$$

If we rewrite the first term on the right-hand side, we obtain an equivalent formula

$$v(r,t) = \frac{1}{2c}\left(\frac{\partial}{\partial t}\int\limits_{ct-r}^{ct+r} s\overline{\varphi}(\boldsymbol{x}_0,s)\,\mathrm{d}s + \int\limits_{ct-r}^{ct+r} s\overline{\psi}(\boldsymbol{x}_0,s)\,\mathrm{d}s\right) \tag{13.10}$$

for $0 \le r \le ct$.

Now we determine the value of $u(\boldsymbol{x}_0,t)$. As we have stated above, we use relation (13.3), that is,

$$\begin{aligned} u(\boldsymbol{x}_0,t) &= \lim_{r\to 0}\overline{u}(\boldsymbol{x}_0,r,t) = \lim_{r\to 0}\frac{v(r,t)}{r} \\ &= \lim_{r\to 0}\frac{v(r,t)-v(0,t)}{r} = \frac{\partial v}{\partial r}(0,t). \end{aligned}$$

We differentiate (13.10) to obtain

$$\begin{aligned} \frac{\partial v}{\partial r} &= \frac{1}{2c}\frac{\partial}{\partial t}\Big((ct+r)\overline{\varphi}(\boldsymbol{x}_0,ct+r) + (ct-r)\overline{\varphi}(\boldsymbol{x}_0,ct-r)\Big) \\ &\quad + \frac{1}{2c}\Big((ct+r)\overline{\psi}(\boldsymbol{x}_0,ct+r) + (ct-r)\overline{\psi}(\boldsymbol{x}_0,ct-r)\Big). \end{aligned}$$

Putting $r=0$, we get

$$\begin{aligned} u(\boldsymbol{x}_0,t) &= \frac{\partial v}{\partial r}(0,t) = \frac{1}{2c}\frac{\partial}{\partial t}\Big((2ct)\overline{\varphi}(\boldsymbol{x}_0,ct)\Big) + \frac{1}{2c}(2ct)\overline{\psi}(\boldsymbol{x}_0,ct) \\ &= \frac{\partial}{\partial t}\Big(t\overline{\varphi}(\boldsymbol{x}_0,ct)\Big) + t\overline{\psi}(\boldsymbol{x}_0,ct) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct}\varphi(\boldsymbol{x})\,\mathrm{d}S\right) + \frac{1}{4\pi c^2 t}\iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct}\psi(\boldsymbol{x})\,\mathrm{d}S, \end{aligned}$$

which is exactly Kirchhoff's formula (13.2).

The uniqueness of the classical solution is a consequence of the linearity of the equation and can be proved easily (see Exercise 13 in Section 13.6). □

Remark 13.2. Unlike the one-dimensional case when the solution given by d'Alembert's formula is as regular as the initial displacement, here the solution is less regular because of the time derivative in Kirchhoff's formula. In general, if $\varphi \in C^{n+1}(\mathbb{R}^3)$ and $\psi \in C^n(\mathbb{R}^3)$, $n \ge 2$, then u is of the class C^n on $\mathbb{R}^3 \times (0,+\infty)$. If φ and ψ are both of class C^2, then the second derivatives of u can be unbounded at some points and the solution is not the classical one. This fact is known as the *focusing effect*.

Huygens' principle. Let us notice that, according to Kirchhoff's formula, the solution of (13.1) at the point $(\boldsymbol{x}_0, t)$ depends only on the values of $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ for $\boldsymbol{x}$ from the *spherical surface* $|\boldsymbol{x} - \boldsymbol{x}_0| = ct$, but it does not depend on the values of the initial data *inside* this sphere. Similarly, using the opposite point of view we conclude that the values of φ and ψ at a point $\boldsymbol{x}_1 \in \mathbb{R}^3$ influence the solution of the three-dimensional wave equation only on the *spherical surface* $|\boldsymbol{x} - \boldsymbol{x}_1| = ct$. This phenomenon is called *Huygens' principle*.

This principle corresponds to the fact that, in the "three-dimensional world", solutions of the wave equation propagate *exactly* at the speed c. For instance, any electromagnetic signal in a vacuum propagates exactly at the speed of light, or any sound is carried through the air exactly at the speed of sound without any "echoes" (assuming no barriers). This means that the listener hears at time t what the speaker said exactly at time $(t - d/c)$ (here d is the distance between the persons), and not a mess of sounds produced at different times.

As we already know from d'Alembert's formula, this principle does not hold true in one dimension, and as we shall see later, neither in two dimensions.

13.2 Cauchy Problem in $\mathbb{R}^2$

Let us consider now the Cauchy problem for the homogeneous wave equation in $\mathbb{R}^2$

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}), & (x, y) \in \mathbb{R}^2,\ t > 0, \\ u(x, y, 0) = \varphi(x, y), & u_t(x, y, 0) = \psi(x, y). \end{cases} \tag{13.11}$$

We can handle it as a "special three-dimensional problem" the solution of which does not depend on the variable z. Then, according to Kirchhoff's formula, the solution $u = u(\boldsymbol{x}_0, t) = u(x_0, y_0, 0, t)$ satisfies

$$u(\boldsymbol{x}_0, t) = \frac{1}{4\pi c^2 t} \iint\limits_{|\boldsymbol{x} - \boldsymbol{x}_0| = ct} \psi(\boldsymbol{x})\, \mathrm{d}S + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \iint\limits_{|\boldsymbol{x} - \boldsymbol{x}_0| = ct} \varphi(\boldsymbol{x})\, \mathrm{d}S \right). \tag{13.12}$$

Here $\boldsymbol{x}_0 = (x_0, y_0, 0)$, $\boldsymbol{x} = (x, y, z)$ and $\varphi(\boldsymbol{x}) = \varphi(x, y)$, $\psi(\boldsymbol{x}) = \psi(x, y)$ for any z. Relation (13.12) really describes the solution of (13.11) (the reader is asked to verify it), but we can obtain a simpler formula.

First of all, both integrals in (13.12) can be written as

$$\iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \cdots = \iint\limits_{S^+} \cdots + \iint\limits_{S^-} \cdots = 2\iint\limits_{S^+} \cdots,$$

where

$$S^+ = \{(x,y,z) \in \mathbb{R}^3;\ z = \sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}\},$$
$$S^- = \{(x,y,z) \in \mathbb{R}^3;\ z = -\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}\}$$

are the upper and lower hemispheres. On the upper hemisphere, we can rewrite the surface element $\mathrm{d}S$ as

$$\begin{aligned} \mathrm{d}S &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sqrt{1 + \left(\frac{-(x-x_0)}{z}\right)^2 + \left(\frac{-(y-y_0)}{z}\right)^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{ct}{z}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{ct}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Thus, formula (13.12) can be simplified to

$$\begin{aligned} u(x_0,y_0,t) &= 2\frac{1}{4\pi c^2 t}\iint\limits_{D} \psi(x,y)\frac{ct}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + 2\frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint\limits_{D} \varphi(x,y)\frac{ct}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}\,\mathrm{d}x\,\mathrm{d}y\right), \end{aligned}$$

where D is the disc $(x-x_0)^2 + (y-y_0)^2 \le c^2t^2$. Thus, we can conclude that the solution of the Cauchy problem (13.11) for the wave equation on $\mathbb{R}^2$ is given by

$$\boxed{\begin{aligned} u(\boldsymbol{x}_0,t) &= \frac{1}{2\pi c}\iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|\le ct} \frac{\psi(\boldsymbol{x})}{\sqrt{c^2t^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2}}\,\mathrm{d}\boldsymbol{x} \\ &\quad + \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|\le ct} \frac{\varphi(\boldsymbol{x})}{\sqrt{c^2t^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2}}\,\mathrm{d}\boldsymbol{x}\right). \end{aligned}} \qquad (13.13)$$

Here we have $\boldsymbol{x} = (x,y)$ and $\boldsymbol{x}_0 = (x_0,y_0)$.

Let us notice the main difference between Kirchhoff's formula (13.2) for the three-dimensional problem and formula (13.13) for the two-dimensional problem. This difference concerns the domain of integration: in the former case, it is just the *spherical surface* $|\boldsymbol{x} - \boldsymbol{x}_0| = ct$, however, in the latter case, we integrate over the *whole disc* $|\boldsymbol{x} - \boldsymbol{x}_0| \leq ct$. It means that Huygens' principle does not hold true in two dimensions! For instance, in the ideal case, waves caused by a pebble thrown onto the water level propagate at a certain speed c. At the same time, every point of the water level once reached by the front wave stays in the wave motion for an infinitely long time. We could see new and new circles appearing on the water-level forever. However, the wave equation is only an approximate model and the real situation is more complicated.

Another – fictitious – example considers life in "Flatland". In such a "world" (which is only two-dimensional), any sound propagates not at the given speed c, but at all speeds less or equal to c, and thus it is heard forever. So the listener hears at one moment a mix of words the speaker has said at different times.

It can be shown that the method of spherical means can be applied in any *odd* dimension greater or equal to three, and thus Huygens' principle holds true there. Conversely, it is false in any *even* dimension.

Example 13.3. A simple example that illustrates the different wave propagation in various dimensions is the "unit hammer blow". Let us solve the problem

$$\begin{cases} u_{tt} = c^2 \Delta u, & \boldsymbol{x} \in \mathbb{R}^N,\ t > 0, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), & u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}) \end{cases}$$

with

$$\varphi(\boldsymbol{x}) \equiv 0, \quad \psi(\boldsymbol{x}) = \begin{cases} 1, & |\boldsymbol{x}| < a, \\ 0, & |\boldsymbol{x}| > a, \end{cases}$$

choosing $N = 1$, 2 and 3. For $N = 1$ the solution is given by d'Alembert's formula, for $N = 2$ we use (13.13), and for $N = 3$ the solution is described by Kirchhoff's formula (13.2). We can observe the following behavior:

$N = 1$: At time t ($> \frac{a}{c}$), the front wave reaches the point $|x| = ct + a$. At the point $|x| = ct - a$, the wave achieves its maximal displacement (equal to $\frac{a}{c}$) and stays constant on the whole interval $|x| < ct - a$. The front wave propagates at speed c, but its influence is evident at all points $|x| < ct + a$. For details, see Example 4.5.

$N = 2$: At time t, the front wave reaches the point $|\boldsymbol{x}| = ct + a$, then the wave achieves its maximum (of order $\frac{1}{\sqrt{t}}$), and, for $|\boldsymbol{x}| \to 0$, it decreases as $\frac{1}{\sqrt{(ct)^2 - |\boldsymbol{x}|^2}}$. The wave has a sharp front, but it has not sharp tail. As in one dimension, the nonzero initial condition at $|\boldsymbol{x}| < a$ results in nonzero displacement at all points $|\boldsymbol{x}| < ct + a$.

$N = 3$: At time t, again, the front wave reaches the point $|\boldsymbol{x}| = ct + a$. The maximal displacement $\frac{a^2}{4c^2 t}$ is achieved at $|\boldsymbol{x}| = ct$, and then the wave decreases again to zero position at $|\boldsymbol{x}| = ct - a$. The whole wave propagates at speed c and does not change its shape – a nonzero initial condition at $|\boldsymbol{x}| < a$ causes a nonzero displacement only at points $ct - a < |\boldsymbol{x}| < ct + a$.

The different behavior in these three cases is sketched in Figure 13.1.

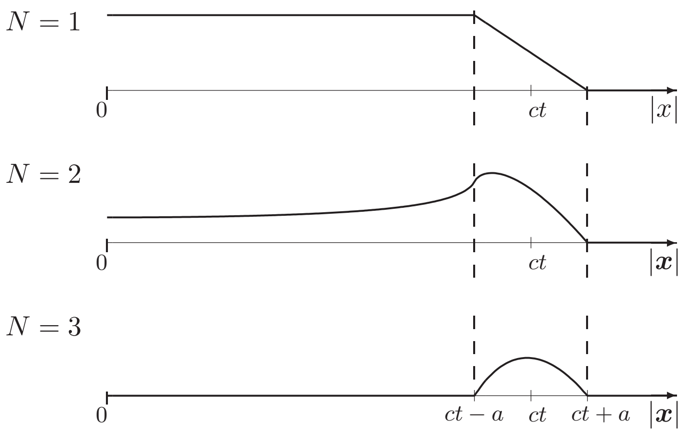


Figure 13.1. “Hammer blow” in one, two and three dimensions.

◇

13.3 Wave with Sources in $\mathbb{R}^3$

Let us consider the non-homogeneous Cauchy problem

$$\begin{cases} u_{tt} - c^2 \Delta u = f(\boldsymbol{x}, t), & \boldsymbol{x} = (x, y, z) \in \mathbb{R}^3, \ t > 0, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \ u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}). \end{cases} \tag{13.14}$$

We will use the *operator method* for its solving. Let us denote by $u_H(\boldsymbol{x}, t)$ the solution of the homogeneous problem (i.e., (13.14) with $f \equiv 0$). We have found in Section 13.1 that such a solution can be written in the form

$$u_H(\boldsymbol{x}_0, t) = (\partial_t \mathcal{S}(t)\varphi)(\boldsymbol{x}_0) + (\mathcal{S}(t)\psi)(\boldsymbol{x}_0),$$

where $\mathcal{S}$ is the so called *source operator* given by the formula

$$(\mathcal{S}(t)\psi)(\boldsymbol{x}_0) = \frac{1}{4\pi c^2 t} \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \psi(\boldsymbol{x})\,\mathrm{d}S. \tag{13.15}$$

The idea of the operator method is exactly the same as in Section 4.4. Again, it can be shown that the influence of the right-hand side f in problem (13.14) can be described by the term

$$u_P(\boldsymbol{x}_0, t) = \int\limits_0^t \mathcal{S}(t-s) f(\boldsymbol{x}_0, s)\,\mathrm{d}s.$$

Hence, after substitution,

$$u_P(\boldsymbol{x}_0, t) = \int\limits_0^t \frac{1}{4\pi c^2 (t-s)} \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=c(t-s)} f(\boldsymbol{x}, s)\,\mathrm{d}S\,\mathrm{d}s,$$

and, using the relation $s = t - \frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}_0|$ on the sphere of integration, we obtain

$$u_P(\boldsymbol{x}_0, t) = \frac{1}{4\pi c} \int\limits_0^t \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=c(t-s)} \frac{f(\boldsymbol{x}, t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}_0|)}{|\boldsymbol{x}-\boldsymbol{x}_0|}\,\mathrm{d}S\,\mathrm{d}s. \tag{13.16}$$

Here the domain of integration is, in fact, the jacket of a four-dimensional space-time cone with its vertex at $(\boldsymbol{x}_0, t)$ and the base formed by the ball $|\boldsymbol{x}-\boldsymbol{x}_0| \le ct$. Thus, we can rewrite the expression in (13.16) into a triple integral obtaining

$$u_P(\boldsymbol{x}_0, t) = \frac{1}{4\pi c} \iiint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|\le ct} \frac{f(\boldsymbol{x}, t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}_0|)}{|\boldsymbol{x}-\boldsymbol{x}_0|}\,\mathrm{d}\boldsymbol{x}. \tag{13.17}$$

(The reader is asked to justify it.) Due to linearity of the equation, the final solution of (13.14) is the sum of u_H and u_P:

$$\begin{aligned} u(\boldsymbol{x}_0, t) \quad = \quad & \frac{1}{4\pi c^2 t} \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \psi(\boldsymbol{x})\,\mathrm{d}S + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \iint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \varphi(\boldsymbol{x})\,\mathrm{d}S \right) \\ & + \frac{1}{4\pi c} \iiint\limits_{|\boldsymbol{x}-\boldsymbol{x}_0|\le ct} \frac{f(\boldsymbol{x}, t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}_0|)}{|\boldsymbol{x}-\boldsymbol{x}_0|}\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Remark 13.4. Let us compare (13.17) with the stationary solution of the same problem, that is, the solution u_{stat} of the Poisson problem

$$-c^2\Delta u = f$$

on the whole space $\mathbb{R}^3$. Using formula (11.18) without the boundary term and with the choice $G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{4\pi c|\boldsymbol{x}-\boldsymbol{x}_0|}$, we obtain

$$u_{\text{stat}}(\boldsymbol{x}_0) = \frac{1}{4\pi c} \iiint\limits_{\mathbb{R}^3} \frac{f(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{x}_0|}\, \mathrm{d}\boldsymbol{x}. \tag{13.18}$$

(The reader is asked to verify that it really solves the Poisson equation on $\mathbb{R}^3$.) As we can see, evolution formula (13.17) differs from the bounded stationary solution (13.18) just at its "retarded" time by the amount $\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}_0|$.

13.4 Characteristics, Singularities, Energy and Principle of Causality

Now we focus on the qualitative properties of the wave equation and its solution. We show how to derive these properties directly from the equation itself but not from the formula expressing the solution.

13.4.1 Characteristics

Like in one dimension, we can introduce the notion of characteristics, but now we speak about *characteristic surfaces.* The fundamental one arises if we rotate a one-dimensional characteristic line $x - x_0 = c(t - t_0)$ around the $t = t_0$ axis. We thus obtain a cone in the four-dimensional space-time (a "hypercone"):

$$|\boldsymbol{x}-\boldsymbol{x}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = c|t-t_0|. \tag{13.19}$$

This set is called the *characteristic cone* or the *light cone* at the point $(\boldsymbol{x}_0, t_0)$. We can imagine it as the union of all (light) rays emanating from the point $(\boldsymbol{x}_0, t_0)$ at the speed c, that is $|\mathrm{d}\boldsymbol{x}/\mathrm{d}t| = c$. For a fixed t, the light cone reduces to a sphere and the light rays are all orthogonal to it (see Figure 13.2 for $\mathbb{R}^2$ illustration).

The body $|\boldsymbol{x}-\boldsymbol{x}_0| \leq c|t-t_0|$ is called the *solid light cone*; it consists of the *future* and *past* half cone. The *past* of the point $(\boldsymbol{x}_0, t_0)$ is formed by all points that have influenced the solution at $(\boldsymbol{x}_0, t_0)$; the *future* of the point $(\boldsymbol{x}_0, t_0)$ contains points that can be affected by the situation at $(\boldsymbol{x}_0, t_0)$, that is points that can be reached by a particle traveling from $(\boldsymbol{x}_0, t_0)$ at a speed less or equal to c.

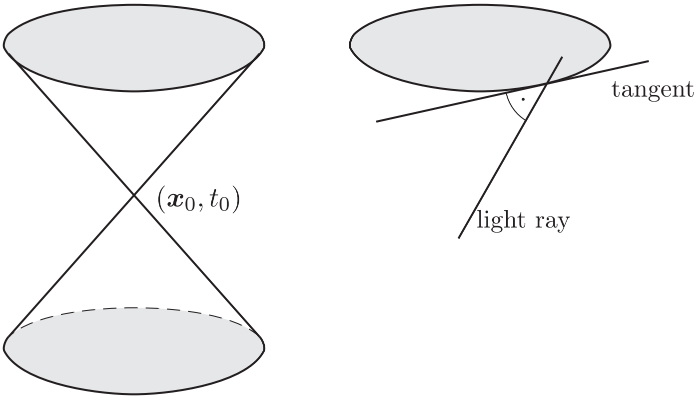


Figure 13.2. Light cone at a point $(\boldsymbol{x}_0, t_0)$, $\boldsymbol{x}_0 \in \mathbb{R}^2$, and orthogonality of light rays to the sphere $|\boldsymbol{x} - \boldsymbol{x}_0| = c|t - t_0|$.

The fundamental property of characteristic surfaces in any dimension is that they are the only surfaces that can carry singularities of the solutions of the wave equation. We only recall that by a singularity we mean a point of discontinuity of the solution or of some of its derivatives.

13.4.2 Energy

Another property of the wave equation that remains valid in the same way as in one dimension is the conservation of energy. Indeed, if we multiply the wave equation by u_t and integrate it over $\mathbb{R}^3$, we obtain

$$\begin{aligned} 0 &= \iiint_{\mathbb{R}^3} (u_{tt} - c^2 \Delta u) u_t \,\mathrm{d}\boldsymbol{x} \qquad (13.20) \\ &= \iiint_{\mathbb{R}^3} \left(\frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2\right)_t \mathrm{d}\boldsymbol{x} - \iiint_{\mathbb{R}^3} c^2 \nabla \cdot (u_t \nabla u) \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

If we rewrite the last integral as

$$\iiint_{\mathbb{R}^3} c^2 \nabla \cdot (u_t \nabla u) \,\mathrm{d}\boldsymbol{x} = \lim_{r \to +\infty} \iiint_{B_r(\mathbf{0})} c^2 \nabla \cdot (u_t \nabla u) \,\mathrm{d}\boldsymbol{x},$$

where $B_r(\mathbf{0})$ is the ball centered at the origin with radius r, and use the Divergence Theorem, we obtain

$$\iiint\limits_{\mathbb{R}^3} c^2\nabla\cdot(u_t\nabla u)\,\mathrm{d}\boldsymbol{x} = \lim_{r\to+\infty}\iint\limits_{\partial B_r(\mathbf{0})} c^2 u_t\nabla u\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}.$$

If we assume that the derivatives of $u(\boldsymbol{x},t)$ tend to zero for $|\boldsymbol{x}|\to+\infty$ quickly enough, then the last integral vanishes. Hence, (13.20) reduces to

$$0 = \iiint\limits_{\mathbb{R}^3}\left(\frac{1}{2}u_t^2+\frac{1}{2}c^2|\nabla u|^2\right)_t\mathrm{d}\boldsymbol{x}.$$

Moreover, if we change the order of integration and time differentiation, we obtain

$$0 = \frac{\partial}{\partial t}\iiint\limits_{\mathbb{R}^3}\left(\frac{1}{2}u_t^2+\frac{1}{2}c^2|\nabla u|^2\right)\mathrm{d}\boldsymbol{x}.$$

Since the term $\iiint\frac{1}{2}u_t^2\,\mathrm{d}\boldsymbol{x}$ corresponds to the *kinetic energy* E_k and the term $\iiint\frac{1}{2}c^2|\nabla u|^2\,\mathrm{d}\boldsymbol{x}$ represents the *potential energy* E_p, we can conclude that the *total energy* $E=E_k+E_p$ is a constant function with respect to time t.

13.4.3 Principle of Causality

We already know (from Huygens' principle and the solution formula) that the solution of the N-dimensional Cauchy problem for the wave equation at a point $(\boldsymbol{x}_0,t_0)$ depends on the values of the initial displacement $\varphi(\boldsymbol{x})$ and the initial velocity $\psi(\boldsymbol{x})$ for $\boldsymbol{x}$ belonging to the sphere $|\boldsymbol{x}-\boldsymbol{x}_0|=ct_0$ if N is odd ($N\geq 3$), and $\boldsymbol{x}$ belonging to the whole ball $|\boldsymbol{x}-\boldsymbol{x}_0|\leq ct_0$ if N is even. However, a similar (though a little bit weaker) information follows directly from the wave equation itself. In particular, we can formulate the so called *principle of causality.*

Theorem 13.5. *The value of $u(\boldsymbol{x}_0,t_0)$ can depend only on the values of $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ for $\boldsymbol{x}$ from the ball $|\boldsymbol{x}-\boldsymbol{x}_0|\leq ct_0$.*

Idea of proof (cf. Strauss [21]). We use the same approach as in one dimension (see Section 10.1). We consider the three-dimensional case; however, the idea is applicable in any dimension. We take the wave equation and multiply it by u_t. After standard calculations and assuming that all derivatives make sense, we obtain

$$\begin{aligned}
0 &= u_{tt}u_t - c^2\Delta u u_t \\
&= \Big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\Big)_t - c^2\nabla\cdot(u_t\nabla u) \\
&= \Big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\Big)_t + (-c^2u_tu_x)_x + (-c^2u_tu_y)_y + (-c^2u_tu_z)_z \\
&= \operatorname{\mathbf{div}} \boldsymbol{f},
\end{aligned}$$

where $\boldsymbol{f}$ is a four-dimensional vector

$$\boldsymbol{f} = \Big(- c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z, \frac{1}{2}(u_t^2 + c^2|\nabla u|^2)\Big).$$

Now, we integrate the equality $\operatorname{\mathbf{div}} \boldsymbol{f} = 0$ over a solid cone frustum F, which is a piece of the solid light cone in the four-dimensional space-time. If we use the four-dimensional Divergence Theorem, we can write

$$\begin{aligned}
0 &= \iiiint\limits_F \operatorname{div}\boldsymbol{f} = \iiint\limits_{\partial F} \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}V \\
&= \iiint\limits_{\partial F} \Big(\frac{1}{2}n_4(u_t^2 + c^2|\nabla u|^2) - n_1(c^2u_tu_x) - n_2(c^2u_tu_y) - n_3(c^2u_tu_z)\Big)\,\mathrm{d}V,
\end{aligned}$$

where ∂F denotes the boundary of F and $\boldsymbol{n} = (n_1, n_2, n_3, n_4)$ is the unit outward normal vector to ∂F with components n_i, $i = 1,\ldots,4$, in directions x, y, z, t. The rest of the proof is the same as in one dimension (cf. Section 10.1). Now ∂F is three-dimensional and consists of the top T, the bottom B and the jacket K (see Figure 13.3 for $\mathbb{R}^2$ illustration). Thus, the integral splits into three parts

$$\iiint\limits_{\partial F} = \iiint\limits_{T} + \iiint\limits_{B} + \iiint\limits_{K} = 0.$$

On the top T, the normal vector has the upward direction $\boldsymbol{n} = (0,0,0,1)$ and the corresponding integral reduces to

$$\iiint\limits_T \Big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\Big)\,\mathrm{d}\boldsymbol{x}.$$

Similarly, on the bottom B, the normal vector has the downward direction, that is $\boldsymbol{n} = (0,0,0,-1)$ and we have

$$\iiint\limits_B -\Big(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\Big)\,\mathrm{d}\boldsymbol{x} = -\iiint\limits_B \Big(\frac{1}{2}\psi^2 + \frac{1}{2}c^2|\nabla\varphi|^2\Big)\,\mathrm{d}\boldsymbol{x}.$$

On the jacket K, we cannot argue so simply, but it can be proved that the corresponding integral is positive or zero (see, e.g., Strauss [21]). Using these facts, we obtain the inequality

$$\iiint_T \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) \mathrm{d}\boldsymbol{x} \leq \iiint_B \left(\frac{1}{2}\psi^2 + \frac{1}{2}c^2|\nabla \varphi|^2\right) \mathrm{d}\boldsymbol{x}. \tag{13.21}$$

Now, let us assume that the functions φ and ψ are zero on B. Inequality (13.21) implies that $\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2 = 0$ on T, and thus $u_t \equiv \nabla u \equiv 0$ on T. Moreover, since this result holds true for a frustum of an arbitrary height, we obtain that u_t and ∇u are zero (and thus u constant) in the entire solid cone. And since $u = 0$ on B, we can conclude $u \equiv 0$ in the entire cone. In particular, this implies that if we take two solutions u_1, u_2 with the same initial conditions on B, then $u_1 \equiv u_2$ in the entire solid cone. □

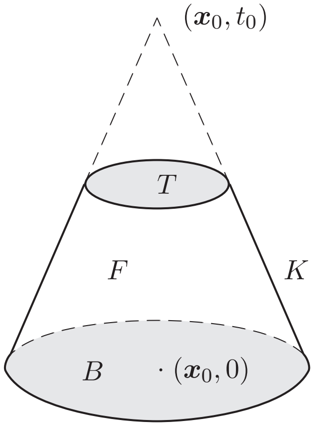


Figure 13.3. Solid cone frustum F.

Remark 13.6. We can state the "converse" assertion to the Principle of Causality: the initial conditions φ, ψ at the point $\boldsymbol{x}_0$ can influence the solution only in the solid light cone with its vertex at $(\boldsymbol{x}_0, 0)$. (Notice that this statement as well as the Principle of Causality hold true even for the nonhomogeneous wave equation.) We can also meet the terminology which we already know from one dimension. The past solid cone is called the *domain of dependence* and the future solid cone is called the *domain of influence* of the point $(\boldsymbol{x}_0, t_0)$.

13.5 Wave on Bounded Domains, Fourier Method

In the rest of this chapter we study initial boundary value problems for the wave equation. In general, we consider the problem

$$\begin{cases} u_{tt}(\boldsymbol{x},t) = c^2\Delta u(\boldsymbol{x},t), & \boldsymbol{x} \in \Omega,\ t > 0, \\ u(\boldsymbol{x},t) = h_1(\boldsymbol{x},t), & \boldsymbol{x} \in \Gamma_1, \\ \frac{\partial u}{\partial n}(\boldsymbol{x},t) = h_2(\boldsymbol{x},t), & \boldsymbol{x} \in \Gamma_2, \\ \frac{\partial u}{\partial n}(\boldsymbol{x},t) + au(\boldsymbol{x},t) = h_3(\boldsymbol{x},t), & \boldsymbol{x} \in \Gamma_3, \\ u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}),\ u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}). \end{cases}$$

As usual, Ω denotes a domain in $\mathbb{R}^N$, φ, ψ, h_i, $i = 1, 2, 3$ are given functions, a is a given constant, and $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial\Omega$.

Fist of all, we recall the physical meaning of the boundary conditions. If we model a vibrating membrane, $u = u(x, y, t)$ corresponds to the displacement of the membrane and the Dirichlet boundary condition on Γ_1 describes the shape of the fixed frame on which the membrane is fastened. If h_1 is not a constant, then the frame is warped. The Neumann boundary condition on Γ_2 determines the "slope" of the membrane on the boundary. In particular, the *homogeneous* Neumann boundary condition (i.e., $h_2 \equiv 0$) corresponds to the "free rim" of the membrane, which is free to flap. The Robin boundary condition on Γ_3 can describe a flexible rim of the membrane.

If we use the three-dimensional wave equation as a model of sound waves in a fluid with $u = u(x, y, z, t)$ being the fluid density, then the most common boundary condition is the homogeneous Neumann boundary condition. It corresponds to the situation when the domain has rigid walls and the fluid cannot penetrate them.

As in the previous chapters, we search for the solution of the initial boundary value problems for the wave equation using the Fourier method. Since the main idea, as well as the basic scheme, coincide completely with those for the case of the diffusion equation, we do not repeat them here in detail and refer the reader to Section 12.2. We confine ourselves only to several examples which illustrate some interesting phenomena or situations which were not treated in the previous chapter.

Example 13.7 (Rectangular Membrane). We start with a simple situation. Let us consider a two-dimensional wave equation describing a vibrating membrane fastened on a rectangular frame which is fixed in zero position. At the beginning, let the membrane be pulled up at the center point and then released. The corresponding model can have the form

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}), & (x, y) \in (0, a) \times (0, b),\ t > 0, \\ u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0, \\ u(x, y, 0) = \varphi(x, y),\ u_t(x, y, 0) = 0, \end{cases} \tag{13.22}$$

where

$$\varphi(x, y) = \begin{cases} xy, & 0 \le x < \frac{a}{2}, \quad 0 \le y < \frac{b}{2}, \\ x(b - y), & 0 \le x < \frac{a}{2}, \quad \frac{b}{2} \le y \le b, \\ (a - x)y, & \frac{a}{2} \le x \le a, \quad 0 \le y < \frac{b}{2}, \\ (a - x)(b - y), & \frac{a}{2} \le x \le a, \quad \frac{b}{2} \le y \le b. \end{cases} \tag{13.23}$$

The shape of the initial displacement is depicted in Figure 13.4 for the data $a = 2$, $b = 3$.

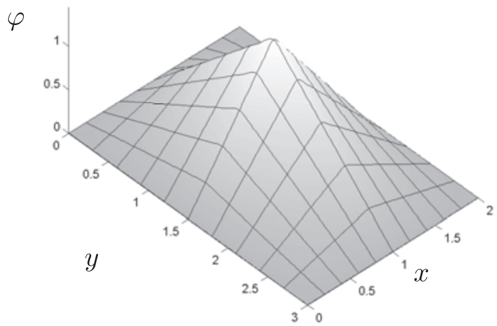


Figure 13.4. Initial condition (13.23) with the choice $a = 2$, $b = 3$.

To solve this problem, we first of all separate the time and space variables: $u(x, y, t) = V(x, y)T(t)$. Substituting into the equation in (13.22), we obtain a couple of separated equations:

$$V_{xx} + V_{yy} + \lambda V = 0, \qquad 0 < x < a,\ 0 < y < b, \tag{13.24}$$

$$T'' + \lambda c^2 T = 0, \qquad t > 0, \tag{13.25}$$

where λ is a constant. Moreover, $V = V(x, y)$ satisfies the homogeneous boundary conditions

$$V(0, y) = V(a, y) = V(x, 0) = V(x, b) = 0, \quad 0 < x < a,\ 0 < y < b. \tag{13.26}$$

As we already know from Example 12.3, problem (13.24), (13.26) can be solved by the Fourier method and it yields the eigenvalues

$$\lambda_{mn} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2, \quad m, n \in \mathbb{N},$$

and the corresponding orthogonal system of eigenfunctions

$$V_{mn}(x,y) = \sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}.$$

If we go back to the time equation (13.25), we obtain

$$T_{mn} = A_{mn}\cos(c\sqrt{\lambda_{mn}}t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}t).$$

(Recall that all the eigenvalues λ_{mn} are positive!) Hence, the solution of the original problem can be written in the form of the double Fourier series

$$u(x,y,t) = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\Big(A_{mn}\cos(c\sqrt{\lambda_{mn}}t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}t)\Big)\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}.$$

This function satisfies the required initial conditions in (13.22) provided these are also expandable into a Fourier series with respect to the system $\{V_{mn}(x,y)\}$. In our case, this means

$$\begin{aligned}\varphi(x,y) &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} A_{mn}\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b},\\ \psi(x,y)\equiv 0 &= \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} c\sqrt{\lambda_{mn}}B_{mn}\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}.\end{aligned}$$

The latter relation implies $B_{mn}=0$ for all $m,n\in\mathbb{N}$. Using the orthogonality of the eigenfunctions, we can determine the coefficients A_{mn} as

$$A_{mn} = \frac{\int_0^a\int_0^b \varphi(x,y)V_{mn}(x,y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^a\int_0^b V_{mn}^2(x,y)\,\mathrm{d}y\,\mathrm{d}x}.$$

Substituting for V_{mn} and for φ from (13.23), we can calculate

$$\begin{aligned}A_{mn} &= \frac{4}{ab}\int_0^a\int_0^b \varphi(x,y)\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{4}{ab}\frac{4a^2b^2}{n^2m^2\pi^4}\sin\frac{n\pi}{2}\sin\frac{m\pi}{2} = \frac{16a\ b}{n^2m^2\pi^4}\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}.\end{aligned}$$

Now we can conclude that the solution of the initial boundary value problem (13.22) can be expressed in the form

$$u(x,y,t) = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} A_{mn}\cos(c\sqrt{\lambda_{mn}}t)\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}.$$

Graph of the solution on several time levels is sketched in Figure 13.5. We have used the data $c = 3$, $a = 2$, $b = 3$ and the partial summation up to $n = m = 25$. The reader is invited to notice the propagation of the singularities and the reflection of the waves on the boundary.

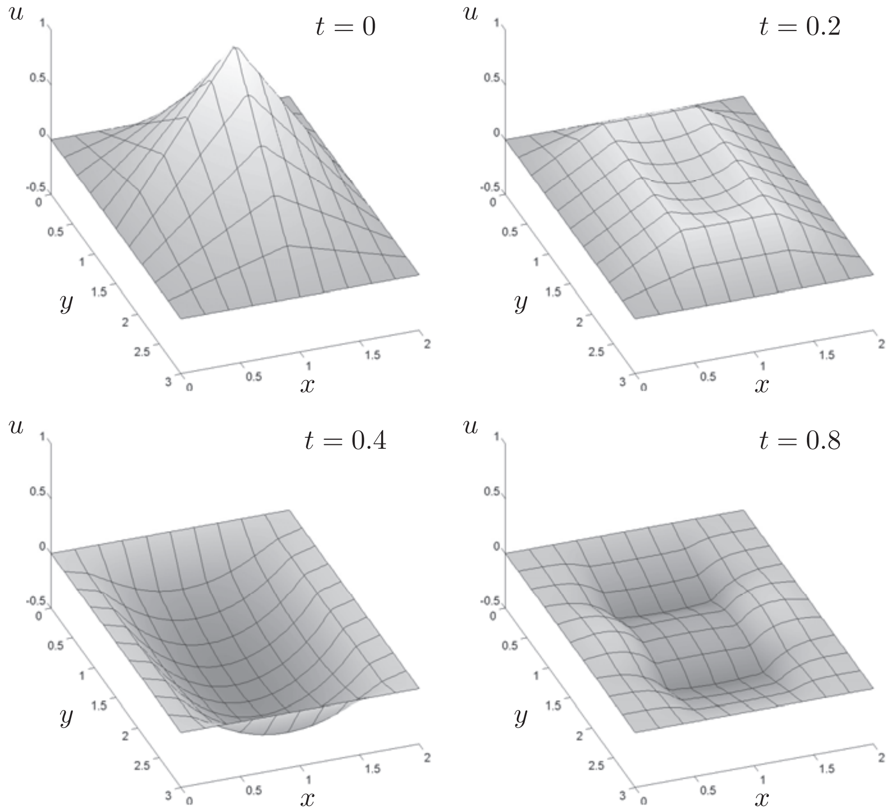


Figure 13.5. Graphic illustration of the solution of the initial boundary value problem (13.22) for the data $c = 3$, $a = 2$, $b = 3$, on time levels $t = 0$, 0.2, 0.4, 0.8.

◇

The other examples deal with the wave equation on circular domains. We consider the radially symmetric as well as non-symmetric cases.

Example 13.8 (Circular Membrane – Symmetric Case). This example is the wave analogue of Example 12.5 for the diffusion equation. This time we solve the problem

$$\begin{cases} u_{tt} = c^2 \Delta u, & x^2+y^2 < a^2,\ t>0, \\ u(x,y,t) = 0, & x^2+y^2 = a^2, \\ u(x,y,0) = \varphi(\sqrt{x^2+y^2}), & u_t(x,y,0) = \psi(\sqrt{x^2+y^2}), \end{cases} \tag{13.27}$$

which can serve as a model of a vibrating circular membrane with the frame fixed in zero position. Since the initial conditions depend only on the radius $r = \sqrt{x^2+y^2}$, we can assume the solution to be radially symmetric, and after transformation into polar coordinates we obtain a simpler problem

$$\begin{cases} u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r), & 0 < r < a,\ t > 0, \\ u(r,t) = 0, & r = a,\ t > 0, \\ u(r,0) = \varphi(r),\ u_t(r,0) = \psi(r), & 0 \le r < a. \end{cases} \tag{13.28}$$

If we repeat the steps of Example 12.5, we obtain the solution in the form

$$u(r,t) = \sum_{n=1}^{+\infty} T_n(t) R_n(r).$$

The system of eigenfunctions R_n is given by

$$R_n(r) = J_0(\sqrt{\lambda_n} r),$$

where J_0 is the Bessel function of the first kind of order zero (see Appendix B); the eigenvalues λ_n are given by

$$\lambda_n = \frac{1}{a^2}\mu_n^2, \quad n \in \mathbb{N},$$

where μ_n are the zeros of J_0. The time functions T_n are now the solutions of the equation

$$T''(t) + c^2 \lambda_n T(t) = 0.$$

Since all the eigenvalues are positive, we can write

$$T_n(t) = A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t).$$

Thus, we can conclude that the solution of (13.28) assumes the form

$$u(r,t) = \sum_{n=1}^{+\infty} \Big(A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)\Big) J_0(\sqrt{\lambda_n} r).$$

The constants A_n, B_n can be determined from the initial conditions provided these are expandable into Fourier series with respect to the system $\{R_n(r)\}$:

$$\begin{aligned} \varphi(r) &= \sum_{n=1}^{+\infty} A_n J_0(\sqrt{\lambda_n} r), \\ \psi(r) &= \sum_{n=1}^{+\infty} c\sqrt{\lambda_n} B_n J_0(\sqrt{\lambda_n} r). \end{aligned}$$

Using the orthogonality of the Bessel functions (see Appendix B), we obtain

$$
\begin{aligned}
A_n &= \frac{2}{J'^2_0(\mu_n)} \int\limits_0^1 \rho J_0(\mu_n \rho)\varphi(a\rho)\,\mathrm{d}\rho, \\
B_n &= \frac{2}{c\sqrt{\lambda_n} J'^2_0(\mu_n)} \int\limits_0^1 \rho J_0(\mu_n \rho)\psi(a\rho)\,\mathrm{d}\rho.
\end{aligned}
$$

In particular, let us take $a = 1$, $c = 1$ and consider the initial data in the form

$$
\begin{aligned}
\varphi(r) &= J_0(\mu_1 r) + J_0(\mu_2 r), \\
\psi(r) &= 0.
\end{aligned}
\tag{13.29}
$$

Then, obviously, $B_n = 0$ for all $n \in \mathbb{N}$, and $A_1 = A_2 = 1$, $A_n = 0$ for $n \geq 3$. The corresponding solution can be then written as

$$u(x, y, t) = u(r, t) = \cos \mu_1 t \, J_0(\mu_1 r) + \cos \mu_2 t \, J_0(\mu_2 r).$$

Figure 13.6 illustrates the initial displacement (13.29). The graph of the function $u = u(x, y, t)$ on several time levels is depicted in Figure 13.7. (We recall that $x = r\cos\theta$, $y = r\sin\theta$, $\theta \in [0, 2\pi]$.)

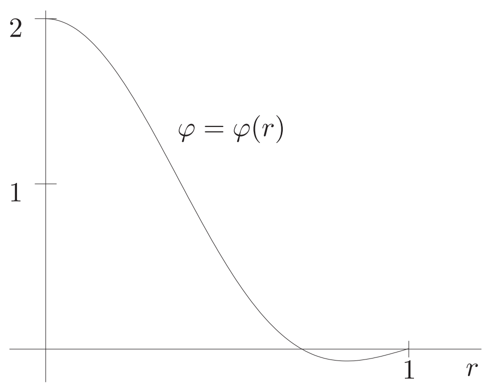


Figure 13.6. The initial displacement (13.29).

◇

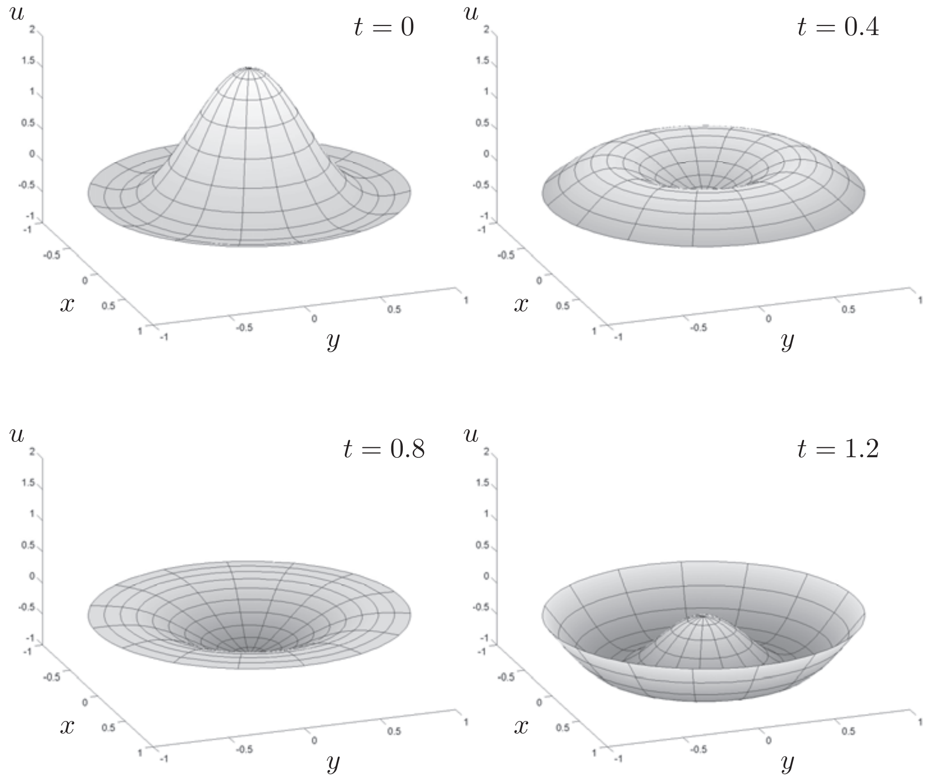


Figure 13.7. Graphic illustration of the solution of the initial boundary value problem (13.27) with initial condition (13.29) on time levels $t = 0$, 0.4, 0.8, 1.2.

Example 13.9 (Circular Membrane – Non-Symmetric Case). Let us consider the same problem as in the previous example, but now without any symmetry. That is, we model a vibrating circular membrane with the frame fixed in zero position; the initial displacement and initial velocity are now general functions $\varphi = \varphi(x, y)$, $\psi = \psi(x, y)$:

$$\begin{cases} u_{tt} = c^2 \Delta u, & x^2 + y^2 < a^2,\ t > 0, \\ u(x, y, t) = 0, & x^2 + y^2 = a^2, \\ u(x, y, 0) = \varphi(x, y), & u_t(x, y, 0) = \psi(x, y). \end{cases} \tag{13.30}$$

As in the previous examples, we find the solution using the Fourier method. Since the domain is circular, we have to transform the problem again into polar coordinates, which provides the required rectangularity. However, now we have to use the general (non-symmetric) transformation formula (6.3) for the Laplace operator. Thus, (13.30) becomes

$$\begin{cases} u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}), & 0 < r < a,\ 0 \le \theta < 2\pi,\ t > 0, \\ u(r,\theta,t) = 0, & r = a,\ 0 \le \theta < 2\pi,\ t > 0, \\ u(r,\theta,0) = \varphi(r,\theta), & \\ u_t(r,\theta,0) = \psi(r,\theta), & 0 \le r < a,\ 0 \le \theta < 2\pi. \end{cases} \tag{13.31}$$

In the first step, we separate the time and space variables:

$$u(r,\theta,t) = V(r,\theta)T(t),$$

and since the spatial problem will be solved again by the Fourier method, we can also separate $V(r,\theta) = R(r)\Theta(\theta)$. Thus, we have

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

and the standard argument leads to

$$\frac{T''}{c^2T} = -\lambda \quad \text{and} \quad \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda.$$

The separation in the latter equation results in

$$\frac{\Theta''}{\Theta} = -\nu \quad \text{and} \quad \lambda r^2 + \frac{r^2R''}{R} + \frac{rR'}{R} = \nu.$$

Obviously, Θ must satisfy the periodic boundary conditions $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$, which implies

$$\nu_n = n^2, \quad \Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \quad n = \mathbb{N} \cup \{0\}.$$

Hence, the radial equation assumes the form

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0, \tag{13.32}$$

which is the *Bessel equation of order n*. As follows from Appendix B, its bounded solutions have the form

$$R(r) = J_n(\sqrt{\lambda}r),$$

where J_n is the *Bessel function of the first kind of order n*. The boundary condition in (13.31) gives $R(a) = 0$, which implies

$$\lambda_{mn} = \frac{1}{a^2}\mu_{mn}^2, \quad n \in \mathbb{N} \cup \{0\},\ m \in \mathbb{N},$$

where μ_{mn} are positive zeros of J_n. Thus, we can write $R(r) = R_{mn}(r) = J_n(\sqrt{\lambda_{mn}}r)$. Inserting $\lambda = \lambda_{mn}$ into the time equation, we obtain

$$T_{mn} = C_{mn} \cos c\sqrt{\lambda_{mn}}t + D_{mn} \sin c\sqrt{\lambda_{mn}}t.$$

Using the expressions for R, Θ, and T, we can conclude that the solution of the wave equation on the disc with homogeneous Dirichlet boundary condition has the form

$$\begin{aligned} u(r,\theta,t) \;&=\; \sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} J_n(\sqrt{\lambda_{mn}}r)(A_{mn}\cos n\theta + B_{mn}\sin n\theta)\cos c\sqrt{\lambda_{mn}}t \\ &\quad + \sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} J_n(\sqrt{\lambda_{mn}}r)(\overline{A}_{mn}\cos n\theta + \overline{B}_{mn}\sin n\theta)\sin c\sqrt{\lambda_{mn}}t. \end{aligned}$$

Here we write A_{mn} instead of A_nC_{mn}, and similarly for B_{mn}, $\overline{A}_{mn}$, $\overline{B}_{mn}$. To determine these coefficients, we use the initial conditions. We illustrate this process on a simple example.

Let us consider problem (13.31) with the initial conditions

$$\begin{cases} \varphi(r,\theta) = (a^2 - r^2)r\sin\theta, \\ \psi(r,\theta) = 0. \end{cases} \tag{13.33}$$

The zero initial velocity implies that all coefficients in the sine series (with respect to time variable) are zero, that is, $\overline{A}_{mn} = \overline{B}_{mn} = 0$ for all $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$. Thus the solution formula reduces to

$$u(r,\theta,t) = \sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} J_n(\sqrt{\lambda_{mn}}r)(A_{mn}\cos n\theta + B_{mn}\sin n\theta)\cos c\sqrt{\lambda_{mn}}t.$$

Setting $t = 0$, we obtain

$$\varphi(r,\theta) = \sum_{n=0}^{+\infty}\sum_{m=1}^{+\infty} J_n(\sqrt{\lambda_{mn}}r)(A_{mn}\cos n\theta + B_{mn}\sin n\theta). \tag{13.34}$$

Notice that this is a Fourier series of the function φ with respect to the system of functions $\{J_n(\sqrt{\lambda_{mn}}r)\cos n\theta,\ J_n(\sqrt{\lambda_{mn}}r)\sin n\theta\}_{mn}$. Let us rewrite (13.34) as

$$\begin{aligned} \varphi(r,\theta) \;&=\; \underbrace{\sum_{m=1}^{+\infty} A_{m0}J_0(\sqrt{\lambda_{m0}}r)}_{=:A_0(r)} + \sum_{n=1}^{+\infty}\underbrace{\left(\sum_{m=1}^{+\infty} A_{mn}J_n(\sqrt{\lambda_{mn}}r)\right)}_{=:A_n(r)}\cos n\theta \\ &\quad + \sum_{n=1}^{+\infty}\underbrace{\left(\sum_{m=1}^{+\infty} B_{mn}J_n(\sqrt{\lambda_{mn}}r)\right)}_{=:B_n(r)}\sin n\theta. \end{aligned}$$

For a fixed r we have

$$
\begin{aligned}
A_0(r) &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(r,\theta)\,\mathrm{d}\theta, \\
A_n(r) &= \frac{1}{\pi}\int_0^{2\pi} \varphi(r,\theta)\cos n\theta\,\mathrm{d}\theta, \\
B_n(r) &= \frac{1}{\pi}\int_0^{2\pi} \varphi(r,\theta)\sin n\theta\,\mathrm{d}\theta.
\end{aligned}
$$

Substituting $\varphi(r,\theta) = (a^2 - r^2)r\sin\theta$ and using the orthogonality of trigonometric functions, we obtain $A_0 = A_n = 0$ for all $n \in \mathbb{N}$, and $B_n = 0$ for $n = 2, 3, \dots$. The only nonzero coefficient is B_1:

$$
B_1 = \sum_{m=1}^{+\infty} B_{m1} J_1(\sqrt{\lambda_{m1}}r) = \frac{1}{\pi}\int_0^{2\pi} (a^2 - r^2)r\sin^2\theta\,\mathrm{d}\theta.
$$

Using the properties of Bessel functions (see Appendix B), we obtain

$$
\begin{aligned}
B_{m1} &= \frac{2}{\pi a^2 J_2^2(\mu_{m1})}\int_0^a\int_0^{2\pi} (a^2 - r^2)r\ \sin^2\theta\, J_1(\mu_{m1}\frac{r}{a})\,r\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{2}{a^2 J_2^2(\mu_{m1})}\int_0^a (a^2 - r^2)r^2\, J_1(\mu_{m1}\frac{r}{a})\,\mathrm{d}r. \qquad (13.35)
\end{aligned}
$$

We recall that $\lambda_{m1} = (\mu_{m1}/a)^2$, and μ_{m1} are positive roots of the Bessel function J_1. Hence, we can conclude that the solution of problem (13.30) or (13.31) with the initial conditions (13.33) is given by

$$
u(r,\theta,t) = \sin\theta \sum_{m=1}^{+\infty} B_{m1}\, J_1\Big(\mu_{m1}\frac{r}{a}\Big)\,\cos\mu_{m1}\frac{ct}{a} \qquad (13.36)
$$

with B_{m1} given by (13.35).

The solution (13.36) on various time levels is plotted in Figure 13.8. Here we have put $a = 1$, $c = 1$, and used the partial sum up to $m = 3$. ◇

In the last example we add another spatial dimension. However, we stick to the simplest, i.e., radially symmetric, case.

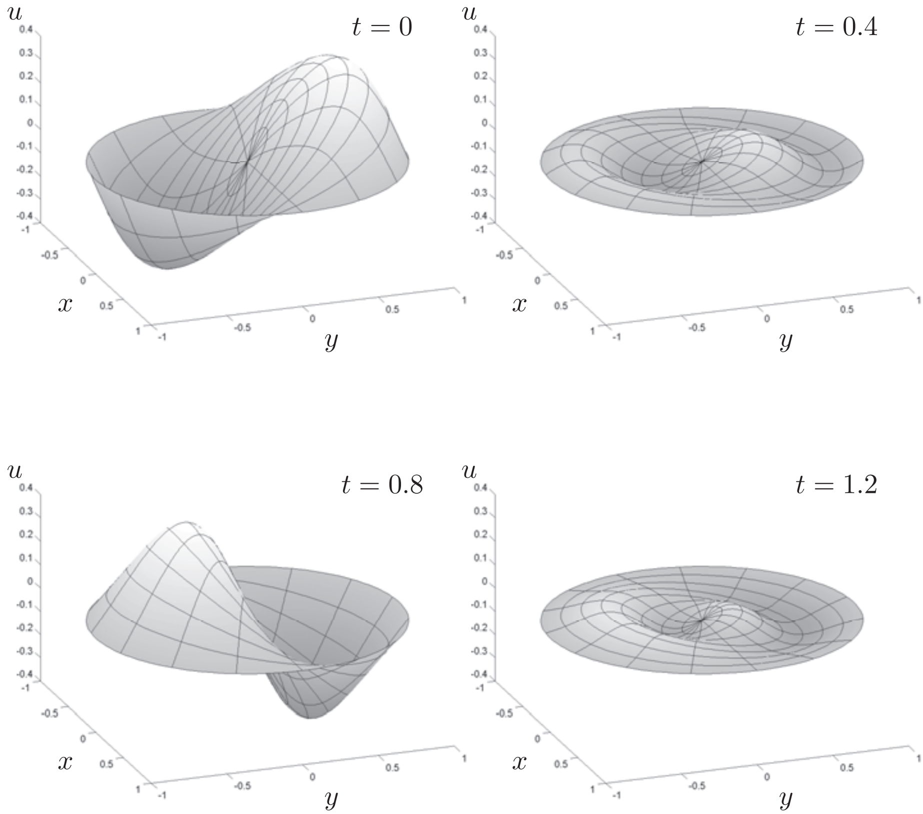


Figure 13.8. Graphic illustration of the solution of the initial boundary value problem (13.30) with initial condition (13.33) on time levels $t = 0$, 0.4, 0.8, 1.2.

Example 13.10 (Vibrations in a Ball – Symmetric Case). Let us consider vibrations in a ball with fixed boundary, and let the initial data depend only on the radius $r = \sqrt{x^2+y^2+z^2}$. That is, we solve the initial boundary value problem for the wave equation with the Dirichlet boundary condition

$$\begin{cases} u_{tt} = c^2\Delta u, & x^2+y^2+z^2 < a^2,\ t>0, \\ u(x,y,z,t) = 0, & x^2+y^2+z^2 = a^2, \\ u(x,y,z,0) = \varphi(\sqrt{x^2+y^2+z^2}), & \\ u_t(x,y,z,0) = \psi(\sqrt{x^2+y^2+z^2}). & \end{cases} \tag{13.37}$$

The geometry of the domain inspires us to transform problem (13.37) into spherical coordinates r, φ, θ. Moreover, since the data do not depend on the angles φ, θ, we can expect the solution to be radially symmetric as well. Thus, the Laplace operator reduces to the simple form $\Delta u = u_{rr} + \frac{2}{r}u_r$, and (13.37) becomes a problem in two variables t and r:

$$\begin{cases} u_{tt} = c^2(u_{rr} + \frac{2}{r}u_r), & 0 < r < a,\ t > 0, \\ u(r,t) = 0, & r = a,\ t > 0, \\ u(r,0) = \varphi(r),\ u_t(r,0) = \psi(r), & 0 \le r < a. \end{cases} \tag{13.38}$$

To solve it, we again use the Fourier method. The separation of variables $u(r,t) = R(r)T(t)$ leads to a pair of ODEs

$$T'' + \lambda c^2 T = 0, \tag{13.39}$$

$$R'' + \frac{2}{r}R' + \lambda R = 0. \tag{13.40}$$

The radial equation can be simplified by introducing a new function $Y(r)$:

$$Y(r) = rR(r).$$

Then (13.40) becomes

$$Y''(r) + \lambda Y(r) = 0$$

and, for $\lambda > 0$, its solutions are $Y(r) = C\cos\sqrt{\lambda}r + D\sin\sqrt{\lambda}r$. Thus, we obtain

$$R(r) = \frac{1}{r}(C\cos\sqrt{\lambda}r + D\sin\sqrt{\lambda}r), \quad 0 < r < a.$$

Further, R must satisfy the boundary conditions

$$R(0) \text{ bounded}, \quad R(a) = 0.$$

Hence $C = 0$, since $\frac{1}{r}\cos\sqrt{\lambda}r$ is unbounded in the neighborhood of $r = 0$. (Remember that $\frac{1}{r}\sin\sqrt{\lambda}r$ is bounded and tends to $\sqrt{\lambda}$ for $r \to 0$.) The latter boundary condition implies

$$D\sin\sqrt{\lambda}a = 0,$$

which gives the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n \in \mathbb{N}$$

and the corresponding system of eigenfunctions

$$R_n(r) = \frac{1}{r}\sin\frac{n\pi r}{a}, \quad n \in \mathbb{N}.$$

The time problem (13.39) has solutions $T_n(t) = A_n \cos c\sqrt{\lambda_n}t + B_n \sin c\sqrt{\lambda_n}t$, and so the radially symmetric solution of the wave equation satisfying the Dirichlet boundary condition can be written as

$$u(r,t) = \sum_{n=1}^{+\infty} \left(A_n \cos \frac{n\pi ct}{a} + B_n \sin \frac{n\pi ct}{a} \right) \frac{1}{r} \sin \frac{n\pi r}{a} \tag{13.41}$$

for $r > 0$. For the evaluation of $u(0,t)$ we use the fact that $\lim_{r \to 0} \frac{1}{r} \sin \frac{n\pi r}{a} = \frac{n\pi}{a}$ and set

$$u(0,t) = \sum_{n=1}^{+\infty} \frac{n\pi}{a} \left(A_n \cos \frac{n\pi ct}{a} + B_n \sin \frac{n\pi ct}{a} \right).$$

To satisfy also the initial conditions, we have to ensure

$$\begin{aligned} \varphi(r) &= \sum_{n=1}^{+\infty} A_n \frac{1}{r} \sin \frac{n\pi r}{a}, \\ \psi(r) &= \sum_{n=1}^{+\infty} \frac{n\pi c}{a} B_n \frac{1}{r} \sin \frac{n\pi r}{a} \end{aligned}$$

for $r > 0$, which is equivalent to

$$\begin{aligned} r\varphi(r) &= \sum_{n=1}^{+\infty} A_n \sin \frac{n\pi r}{a}, \\ r\psi(r) &= \sum_{n=1}^{+\infty} \frac{n\pi c}{a} B_n \sin \frac{n\pi r}{a} \end{aligned}$$

for $r \geq 0$. Using the standard argument, the coefficients A_n, B_n can be written as

$$\begin{aligned} A_n &= \frac{2}{a} \int_0^a r\varphi(r) \sin \frac{n\pi r}{a} \, \mathrm{d}r, \\ B_n &= \frac{2}{n\pi c} \int_0^a r\psi(r) \sin \frac{n\pi r}{a} \, \mathrm{d}r. \end{aligned}$$

Since we cannot easily plot u as a function of all variables x, y, z, t, Figure 13.9 depicts only the values of the solution (13.41) with respect to the radial coordinate r and time t. We have chosen parameters $a = 1$, $c = 1$, zero initial velocity $\psi \equiv 0$, and the initial displacement given by

$$\varphi(r) = J_0(\mu_1 r) + J_0(\mu_2 r). \tag{13.42}$$

(Here J_0 is the Bessel function of the first kind of order zero, see Appendix B.) We have used the partial sum up to $n = 20$. ◇

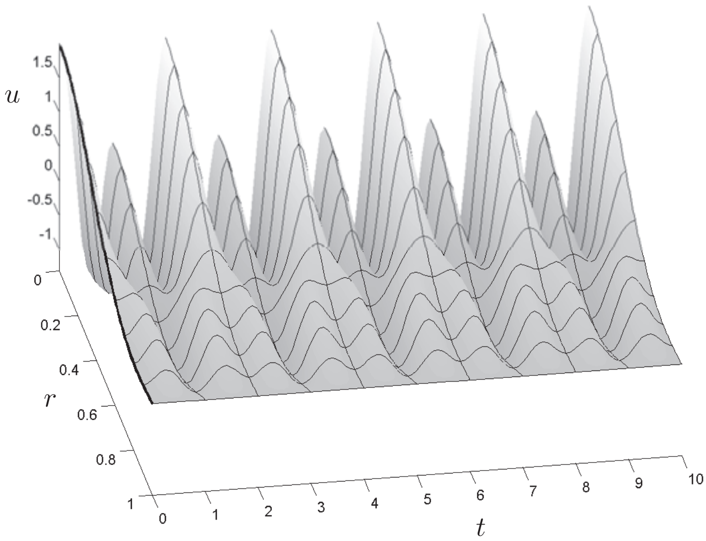


Figure 13.9. Graphic illustration of the solution of the initial boundary value problem (13.39) (symmetric vibrations in a unit ball) with initial condition (13.42) – dependence on r and t.

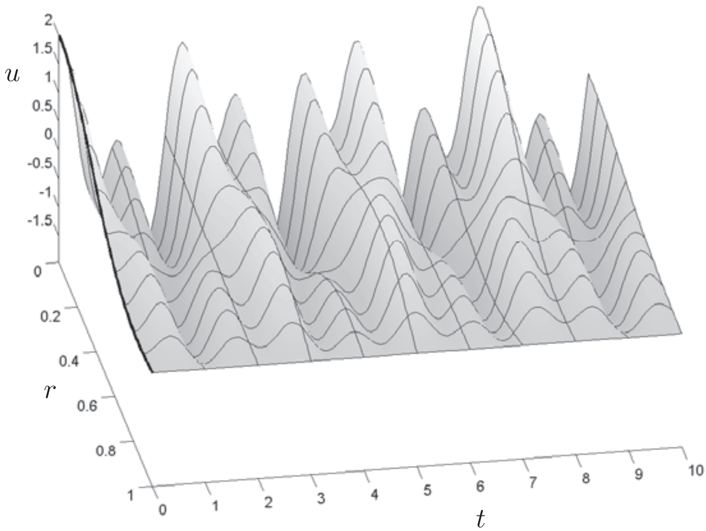


Figure 13.10. Graphic illustration of the solution of the initial boundary value problem (13.27) (symmetric vibrations in a unit disc) with initial condition (13.42) – dependence on r and t.

Remark 13.11. In the previous example, we have chosen the initial displacement (13.42) since the same condition was used in Example 13.8 for the problem of a vibrating membrane (see Figure 13.6). Let us have a detailed look at Figure 13.9 and compare it with Figure 13.10. The former illustrates the radially symmetric vibrations in a unit ball (e.g., sound waves), while the latter depicts radially symmetric vibrations in a unit disc (solved in Example 13.8). In both cases, we have used the same data, but the behavior is very different (notice, e.g., the shape of the propagating waves and the time period).

Another example of different behavior in two and three dimensions is illustrated in Figure 13.11. There we have again depicted the radially symmetric solution of the wave equation in a disc and in a ball. This time we have chosen (in both cases) parameters $a = 10$, $c = 1.5$, zero initial displacement $\varphi \equiv 0$, and the initial velocity given by

$$\psi(r) = 1 \quad \text{for } 0 \le r \le 1, \quad \psi(r) = 0 \quad \text{elsewhere.}$$

(We treated the wave equation with the same initial conditions – but on the whole plane and space – in Example 13.3.) In two dimensions, this corresponds to the situation when we hit the membrane by a unit circular hammer. We can see that the signal propagates along the characteristics, and when it reaches any point, the displacement there never vanishes. On the other hand, in three dimensions, the signal comes and fades away. This corresponds to the fact that, in two dimensions, the (non-reflected) solution at a point $(\boldsymbol{x}_0, t_0)$ is influenced by the initial values from the *whole disc*

$$(x - x_0)^2 + (y - y_0)^2 \le (ct_0)^2,$$

while, in three dimensions, only the initial values from the *spherical surface*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = (ct_0)^2$$

are relevant. Notice also the reflection on the boundary $r = a$ and the effect of the Principle of Causality, which is valid in any dimension.

Remark 13.12. The radial equation (13.40) is a special case (with $n = 0$) of the general equation

$$r^2 R'' + 2rR' + (\lambda r^2 - n(n+1))R = 0, \tag{13.43}$$

which appears in non-symmetric problems in a ball. It can be shown that solutions of (13.43) have the form

$$R(r) = \sqrt{\frac{\pi}{2\sqrt{\lambda} r}} J_{n+\frac{1}{2}}(\sqrt{\lambda} r),$$

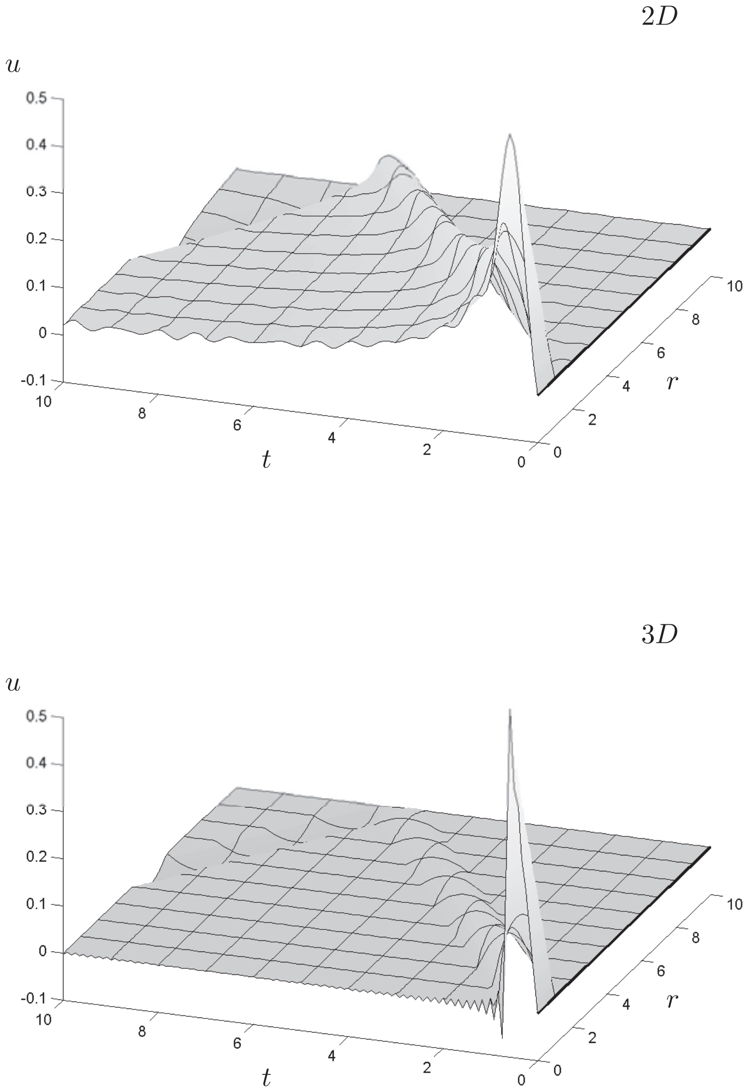


Figure 13.11. Radially symmetric solutions of the Dirichlet problem for the wave equation in a disc (2D) and in a ball (3D) with the initial condition $\psi(r) = 1$ for $0 \leq r \leq 1$ and zero otherwise.

where $J_{n+\frac{1}{2}}$ is the Bessel function of the first kind of (non-integer) order $n+\frac{1}{2}$, see Appendix B. Setting $n = 0$, the solution of (13.40) can be written as

$$R(r) = \sqrt{\frac{\pi}{2\sqrt{\lambda}r}} J_{\frac{1}{2}}(\sqrt{\lambda}r).$$

Using expression (B.5) for Bessel functions, we obtain

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \sum_{k=0}^{+\infty} (-1)^k \frac{(x/2)^{\frac{1}{2}+2k}}{k!\Gamma(\frac{1}{2}+k+1)} \\ &= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sqrt{\frac{2}{\pi x}} \sin x. \end{aligned}$$

Thus, the solution of (13.40) assumes the simple form

$$R(r) = \frac{1}{\sqrt{\lambda}r} \sin \sqrt{\lambda}r,$$

as we have already derived in a different way.

13.6 Exercises

1. Find all three-dimensional plane waves; that is, all solutions of the wave equation in the form $u(\boldsymbol{x}, t) = f(\boldsymbol{k} \cdot \boldsymbol{x} - ct)$, where $\boldsymbol{k}$ is a fixed vector and f is a function of one variable.

   [either $|\boldsymbol{k}| = 1$ or $u(\boldsymbol{x}, t) = a + b(\boldsymbol{k} \cdot \boldsymbol{x} - ct)$, where a, b are arbitrary constants]

2. Verify that $(c^2t^2 - x^2 - y^2 - z^2)^{-1}$ satisfies the wave equation except on the light cone.

3. Prove that $\Delta(\overline{u}) = (\overline{\Delta u}) = \overline{u}_{rr} + \frac{2}{r}\overline{u}_r$ for any function $u = u(x, y, z)$. Here $r = \sqrt{x^2 + y^2 + z^2}$ is the spherical coordinate.

   [Hint: Write Δu in spherical coordinates and show that the angular terms have zero average on spheres centered at the origin.]

4. Using Kirchhoff's formula, solve the wave equation in three dimensions with the initial data $\varphi(x, y, z) \equiv 0$, $\psi(x, y, z) = y$.

   [$u(x, y, z, t) = ty$]

5. Solve the wave equation in three dimensions with the initial data $\varphi(x, y, z) \equiv 0$, $\psi(x, y, z) = x^2 + y^2 + z^2$. Search for a radially symmetric solution and use the substitution $v(r, t) = ru(r, t)$.

6. Solve the wave equation in three dimensions with initial conditions $\varphi(\boldsymbol{x}) \equiv 0$, $\psi(\boldsymbol{x}) = A$ for $|\boldsymbol{x}| < \rho$ and $\psi(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > \rho$, where A is a constant. This problem is an analogue of the hammer blow solved in Section 4.1.

$$[u(\boldsymbol{x},t) = \tfrac{A}{4cr}(\rho - (r-ct)^2) \quad \text{for } |\rho - ct| \le r \le \rho + ct,$$
$$u(\boldsymbol{x},t) = At \quad \text{for } r \le \rho - ct, \text{ and } u(\boldsymbol{x},t) = 0 \text{ elsewhere}]$$

7. Solve the wave equation in three dimensions with initial conditions given by $\varphi(\boldsymbol{x}) = A$ for $|\boldsymbol{x}| < \rho$, $\varphi(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > \rho$ and $\psi(\boldsymbol{x}) \equiv 0$, where A is a constant. Where does the solution have jump discontinuities?

   [Hint: Differentiate the solution from Exercise 6.]

   [$u(\boldsymbol{x},t) = A$ for $r < \rho - ct$, $u(\boldsymbol{x},t) = A(r-ct)/2r$ for $|\rho - ct| < r < \rho + ct$, and $u(\boldsymbol{x},t) = 0$ for $r > \rho + ct$]

8. Use Kirchhoff's formula and the reflection method to solve the wave equation in the half-space $\{(x,y,z,t);\ z > 0\}$ with the Neumann condition $\partial u/\partial z = 0$ on $z = 0$, and with initial conditions $\varphi(x,y,z) \equiv 0$ and arbitrary $\psi(x,y,z)$.

9. Why doesn't the method of spherical means work for two-dimensional waves?

10. Suppose that we do not know d'Alembert's formula and solve the one-dimensional wave equation with the initial data $\varphi(x) \equiv 0$ and arbitrary $\psi(x)$ using the descent method. That is, think of $u(x,t)$ as a solution of the two-dimensional equation independent of the y variable.

11. Consider the wave equation with the condition $\partial u/\partial n + b\,\partial u/\partial t = 0$, $b > 0$, and show that its energy decreases.

12. Consider the equation $u_{tt} - c^2\Delta u + m^2 u = 0$, $m > 0$, known as the *Klein-Gordon equation.* Show that its energy is constant.

13. Prove the uniqueness of the classical solution of the wave equation on $\mathbb{R}^3$. Use the conservation of energy applied to the difference of two solutions.

14. Find the value $u(0,0,0,t)$ of the solution of the wave equation

$$u_{tt} - \Delta u = g$$

   in three spatial variables if

   (a) $\varphi(x,y,z) = f(x^2+y^2+z^2)$, $\psi \equiv 0$, $g \equiv 0$,

   (b) $\varphi \equiv 0$, $\psi(x,y,z) = f(x^2+y^2+z^2)$, $g \equiv 0$,

   (c) $\varphi \equiv 0$, $\psi \equiv 0$, $g(x,y,z,t) = f(x^2+y^2+z^2)$.

[If we denote $v(t) = u(0,0,0,t)$, then we have a) $v(t) = f(c^2t^2) + 2c^2t^2 f'(c^2t^2)$; b) $v(t) = tf(c^2t^2)$; c) $v(t) = \int_0^t (t-\tau) f(c^2(t-\tau)^2)\mathrm{d}\tau$.]

15. Consider the equation

$$u_{tt} = c^2 u_{xx} - bu_t + au_{yy}$$

on the rectangle $(0,a) \times (0,b)$ with boundary conditions

$$u(0,y,t) = 0, \quad u_x(a,y,t) = 0,$$
$$u_y(x,0,t) = 0, \quad u(x,b,t) = 0.$$

Find the corresponding separated ODEs and boundary conditions.

16. Separate the PDE

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) - (u_x + u_y)$$

into the corresponding ODEs.

17. Solve the two-dimensional wave equation on the unit square with the coefficient $c = \frac{1}{\pi}$, homogeneous Dirichlet boundary conditions, and the following initial conditions:

(a) $\varphi(x,y) = \sin 3\pi x \ \sin \pi y, \quad \psi(x,y) = 0,$

$$[u(x,y,t) = \sin 3\pi x \ \sin \pi y \ \cos \sqrt{10}t]$$

(b) $\varphi(x,y) = \sin \pi x \ \sin \pi y, \quad \psi(x,y) = \sin \pi x,$

(c) $\varphi(x,y) = x(1-x)y(1-y), \quad \psi(x,y) = 2 \sin \pi x \ \sin 2\pi y,$

$$[u(x,y,t) = \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{64 \cos \sqrt{(2k+1)^2 + (2l+1)^2}\,t}{\pi^6 (2k+1)^3 (2l+1)^3} \sin(2k+1)\pi x \ \sin(2l+1)\pi y$$
$$+ \frac{2}{\sqrt{5}} \sin \pi x \ \sin 2\pi y \ \sin \sqrt{5}t]$$

(d) $\varphi(x,y) = x(1 - \mathrm{e}^{x-1})y(1-y^2), \quad \psi(x,y) = 0.$

18. Solve the two-dimensional wave equation on a disc of radius a with homogeneous Dirichlet boundary condition. Use the following data:

(a) $a = 2,\ c = 1,\ \varphi(r) = 0,\ \psi(r) = 1,$

$$[u(r,t) = 4 \sum_{n=1}^{+\infty} \frac{J_0(\mu_n r/2)}{\mu_n^2 J_1(\mu_n)} \sin \frac{\mu_n}{2} t]$$

(b) $a = 1,\ c = 10,\ \varphi(r) = 1 - r^2,\ \psi(r) = 1,$

(c) $a = 1,\ c = 1,\ \varphi(r) = 0,\ \psi(r) = J_0(\mu_3 r),$

(d) $a = 1,\ c = 1,\ \varphi(r) = J_0(\mu_3 r),\ \psi(r) = 1 - r^2.$

$$[u(r,t) = J_0(\mu_3 r)\,\cos\mu_3 t + 8\sum_{n=1}^{+\infty}\frac{J_0(\mu_n r)}{\mu_n^4 J_1(\mu_n)}\sin\mu_n t]$$

19. Solve the two-dimensional wave equation on a disc of radius a with homogeneous Dirichlet boundary condition. Use the following data:

(a) $a = 1,\ c = 1,\ \varphi(r,\theta) = (1-r^2)r^2\sin 2\theta,\ \psi(r,\theta) = 0,$

$$[u(r,\theta,t) = 24\sum_{n=1}^{+\infty}\frac{J_2(\mu_{n2} r)}{\mu_{n2}^3 J_3(\mu_{n2})}\sin 2\theta\,\cos\mu_{n2} t]$$

(b) $a = 1,\ c = 1,\ \varphi(r,\theta) = 0,\ \psi(r,\theta) = (1-r^2)r^2\sin 2\theta,$

$$[u(r,\theta,t) = 24\sum_{n=1}^{+\infty}\frac{J_2(\mu_{n2} r)}{\mu_{n2}^4 J_3(\mu_{n2})}\sin 2\theta\,\sin\mu_{n2} t]$$

(c) $a = 1,\ c = 1,\ \varphi(r,\theta) = 1 - r^2,\ \psi(r,\theta) = J_0(r).$

20. Consider a thin rectangular plate of length a and width b and describe its vibrations for the following data: $a = \frac{\pi}{2}$, $b = \pi$, $c = 1$, boundary conditions

$$u(0,y,t) = 0, \quad u_x(a,y,t) = 0,$$
$$u_y(x,0,t) = 0, \quad u(x,b,t) = 0,$$

and initial conditions

$$u(x,y,0) = \begin{cases} y\sin x, & 0 \le x \le a,\ 0 \le y < \frac{b}{2}, \\ (y-b)\sin x, & 0 \le x \le a,\ \frac{b}{2} \le y \le b, \end{cases}$$

and

$$u_t(x,y,0) = \begin{cases} x(\cos y + 1), & 0 \le x < \frac{a}{2},\ 0 \le y \le b, \\ (x-a)(\cos y + 1), & \frac{a}{2} \le x \le a,\ 0 \le y \le b. \end{cases}$$

21. Consider a thin vibrating rectangular membrane of length $\frac{3\pi}{2}$ and width $\frac{\pi}{2}$. Suppose the sides $x = 0$ and $y = 0$ are fixed and the other two sides are free. Given zero initial velocity and the initial displacement $\varphi(x,y) = (\sin x)(\sin y)$, determine the time-dependent solution and plot its graph on several time levels.

22. Solve the problem of a vibrating circular membrane of radius 1 with fixed boundary, zero initial velocity, and the initial displacement described by $f(r)\sin 2\theta$.

23. Solve the problem of a vibrating circular membrane of radius π with free boundary, zero initial velocity, and the initial displacement described by $f(r)\cos\theta$.

24. Consider vertical vibrations of a circular sector $0 < \theta < \frac{\pi}{4}$ with radius 2. Determine the solution if the boundary conditions are

    (a) $u(r, 0, t) = 0, \quad u(r, \frac{\pi}{4}, t) = 0, \quad u(2, \theta, t) = 0.$

    (b) $u_\theta(r, 0, t) = 0, \quad u_\theta(r, \frac{\pi}{4}, t) = 0, \quad u(2, \theta, t) = 0.$

    In both cases, assume zero initial velocity and the initial displacement as a function of the radius and the angle.

25. Solve the problem

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = f(x, y) \sin \omega t, & (x, y) \in \Omega = (0, a) \times (0, b),\ t > 0, \\ u(x, y, t) = 0, & (x, y) \in \partial\Omega, \\ u(x, y, 0) = 0,\ u_t(x, y, 0) = 0. \end{cases}$$

    Consider separately the nonresonance case $\omega \neq \omega_{mn} = c\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$ for all $m, n \in \mathbb{N}$, and the resonance case $\omega = \omega_{m_0 n_0}$ for some (m_0, n_0).

26. Find all solutions of the wave equation of the form $u = \mathrm{e}^{\mathrm{i}\omega t} f(r)$ that are finite at the origin. Here $r = \sqrt{x^2 + y^2}$.

    $[u(r, t) = A\mathrm{e}^{-i\omega t} J_0(\frac{\omega r}{c})]$

Appendix A

Sturm-Liouville Problem

When dealing with the Fourier method we have discussed parametric boundary value problems for the second order ODEs, whose solutions (usually, sines and cosines) form a complete orthogonal system. This fact plays the crucial role in finding the solution of the original PDE problem in the form of an infinite series. These properties are not typical only for sines and cosines, but also for more general functions which arise as solutions of the so called *Sturm-Liouville boundary value problem*

$$\begin{cases} -(p(x)y')' + q(x)y = \lambda r(x)y, \quad a < x < b, \\ \alpha_0 y(a) + \beta_0 y'(a) = 0, \\ \alpha_1 y(b) + \beta_1 y'(b) = 0. \end{cases} \tag{A.1}$$

Here, $\alpha_0^2 + \beta_0^2 > 0$, $\alpha_1^2 + \beta_1^2 > 0$ (i.e., at least one number of each pair is nonzero), and λ is an unknown parameter.

We say that (A.1) forms a *regular Sturm-Liouville problem*, if $[a, b]$ is a closed finite interval and the following *regularity conditions* are fulfilled: $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous real functions on $[a, b]$, and $p(x) > 0$, $r(x) > 0$ for $a \le x \le b$.

Any value of the parameter $\lambda \in \mathbb{R}$ for which the nontrivial solution of problem (A.1) exists is called an *eigenvalue*. The corresponding nontrivial solution is called an *eigenfunction* related to the eigenvalue λ.

Now, we summarize the main important properties of the eigenvalues and eigenfunctions of regular Sturm-Liouville problems:

(i) The eigenvalues of problem (A.1) are all real, and form an increasing infinite sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \to +\infty.$$

(ii) To each λ_n there corresponds a unique (up to a nonzero multiple) eigenfunction $y_n(x)$, which has exactly $n - 1$ zeros in (a, b). (Notice that any multiple of an eigenfunction is also an eigenfunction.) Moreover, between two consecutive zeros of $y_n(x)$ there is exactly one zero of $y_{n+1}(x)$.

(iii) If $y_n(x)$ and $y_m(x)$ are two eigenfunctions corresponding to two different eigenvalues λ_n and λ_m, then

$$\int_a^b r(x)y_n(x)y_m(x)\,\mathrm{d}x = 0$$

(i.e., y_n and y_m are linearly independent and orthogonal with respect to the weight function $r(x)$).

(iv) Any piecewise smooth function defined on (a, b) is expandable into Fourier series with respect to the eigenfunctions y_n, that is

$$f(x) = \sum_{n=1}^{+\infty} F_n y_n(x),$$

where F_n are the Fourier coefficients defined by the relation

$$F_n = \frac{\int_a^b r(x)f(x)y_n(x)\,\mathrm{d}x}{\int_a^b r(x)y_n^2(x)\,\mathrm{d}x}.$$

Moreover, the series converges at $x \in [a, b]$ to $f(x)$ if $f(x)$ is continuous at x, and it converges to $\frac{1}{2}(f(x^+)+f(x^-))$ if $f(x)$ has a jump discontinuity at x (here $f(x^+)$ and $f(x^-)$ are one sided limits at x).

We usually say that the eigenfunctions $y_n(x)$ form a complete orthogonal set.

In some cases and under additional conditions, the above mentioned properties are valid also for the so called *singular Sturm-Liouville problems*, see, e.g., [22]. In our text we deal with only one such case: the parametric Bessel equation (see Examples 12.5, 13.8, 13.9 and Remark B.1 below).

Appendix B

Bessel Functions

In Chapters 12 and 13 we have seen a special case of the so called *Bessel equation of order n*

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{B.1}$$

or, equivalently,

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0, \quad x \neq 0.$$

Here n is a nonnegative constant (not necessarily integer; but for our purposes, we usually consider $n \in \mathbb{N}$). Equation (B.1) is a linear second-order ODE and thus it must have a pair of linearly independent solutions, which can be searched in the form

$$y(x) = \sum_{k=0}^{+\infty} a_k x^{k+\alpha}, \quad a_0 \neq 0.$$

Below, we find concrete values of a_k and α. Substituting back into (B.1), we obtain

$$x^2 \sum_{k=0}^{+\infty}(k+\alpha)(k+\alpha-1)a_k x^{k+\alpha-2} \\ + x\sum_{k=0}^{+\infty}(k+\alpha)a_k x^{k+\alpha-1} + (x^2 - n^2)\sum_{k=0}^{+\infty} a_k x^{k+\alpha} = 0$$

or (after a simplification and canceling the term x^α)

$$\sum_{k=0}^{+\infty}(k+\alpha-n)(k+\alpha+n)a_k x^k + \sum_{k=0}^{+\infty} a_k x^{k+2} = 0.$$

The second sum can be rewritten as

$$\sum_{k=0}^{+\infty} a_k x^{k+2} = \sum_{k=2}^{+\infty} a_{k-2} x^k$$

and we obtain the equation

$$\sum_{k=0}^{+\infty}(k+\alpha-n)(k+\alpha+n)a_k x^k + \sum_{k=2}^{+\infty} a_{k-2}x^k = 0. \tag{B.2}$$

Thus, the coefficients at the particular powers of x must satisfy

$$\begin{aligned}
k=0: &\qquad (\alpha-n)(\alpha+n)a_0 = 0,\\
k=1: &\qquad (1+\alpha-n)(1+\alpha+n)a_1 = 0,\\
k\geq 2: &\qquad (k+\alpha-n)(k+\alpha+n)a_k + a_{k-2} = 0.
\end{aligned}$$

Since we require $a_0 \neq 0$, the first equation implies

$$\alpha = n \quad \text{or} \quad \alpha = -n.$$

The second equation must hold for any nonnegative n, and thus $a_1 = 0$. The third equation leads to the *recursive formula*

$$a_k = \frac{-1}{(k+\alpha-n)(k+\alpha+n)}a_{k-2}. \tag{B.3}$$

Since $a_1 = 0$, it follows that

$$a_3 = a_5 = \cdots = a_{2k+1} = \cdots = 0,$$

and the only nonzero coefficients can be written as

$$\begin{aligned}
a_{2k} &= \frac{-1}{(2k+\alpha-n)(2k+\alpha+n)}a_{2k-2} \\
&= \frac{(-1)^k a_0}{2^{2k}k!(1+n)(2+n)(3+n)\dots(k+n)}.
\end{aligned} \tag{B.4}$$

Thus, making the conventional choice $a_0 = 2^{-n}/n!$ and taking $\alpha = n$, we obtain the first solution of the Bessel equation

$$J_n(x) = \sum_{k=0}^{+\infty}(-1)^k \frac{(x/2)^{n+2k}}{k!(n+k)!}, \tag{B.5}$$

which is called the *Bessel function of the first kind of order* n. (If $n \notin \mathbb{N}$, we have to replace the factorial $(n+k)!$ by the so called Gamma function $\Gamma(n+k+1)$, see, e.g., Abramowitz, Stegun [1].) Several Bessel functions of the first kind are sketched in Figure B.1. Notice that all these functions are finite even at the singular point $x = 0$!

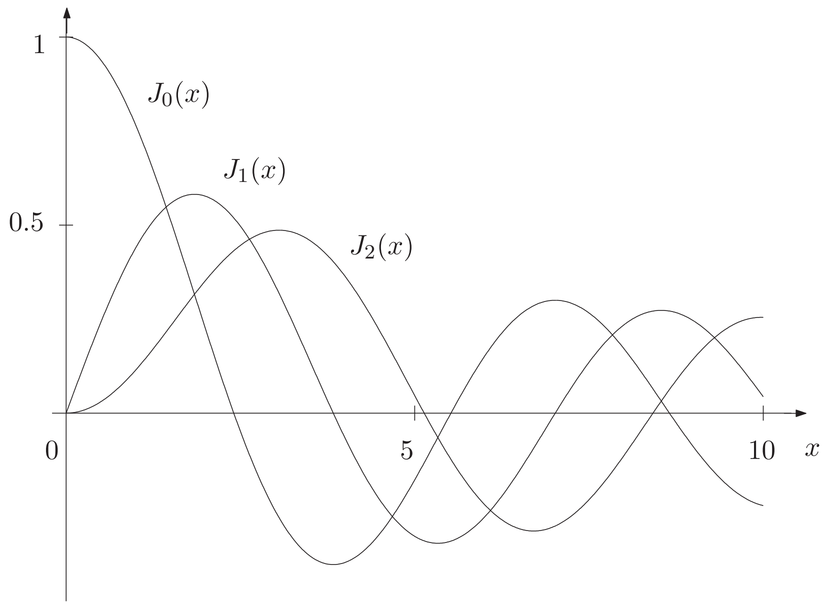


Figure B.1. Bessel functions of the first kind for $n = 0, 1, 2$.

It can be shown that the second linearly independent solution of the Bessel equation has the form

$$Y_n(x) = \lim_{q \to n} \frac{J_q(x) \cos q\pi - J_{-q}(x)}{\sin q\pi}$$

and is known as the *Bessel function of the second kind* (or as the *Neumann* or *Weber function*) *of order* n. This function is unbounded at $x = 0$. In fact, it behaves like $-\frac{1}{x^n}$ near $x = 0$ for $n > 0$. In the case $n = 0$, we can approximate $Y_0(x) \approx \frac{2}{\pi} \ln x$ for $x \to 0$. Several Bessel functions of the second kind are sketched in Figure B.2.

Without proofs, we state here the basic properties of Bessel functions which we have used in this book.

(i) As we can see in Figures B.1, B.2, each Bessel function has a countable number of distinct positive roots μ_k, $k = 1, 2, \ldots$.

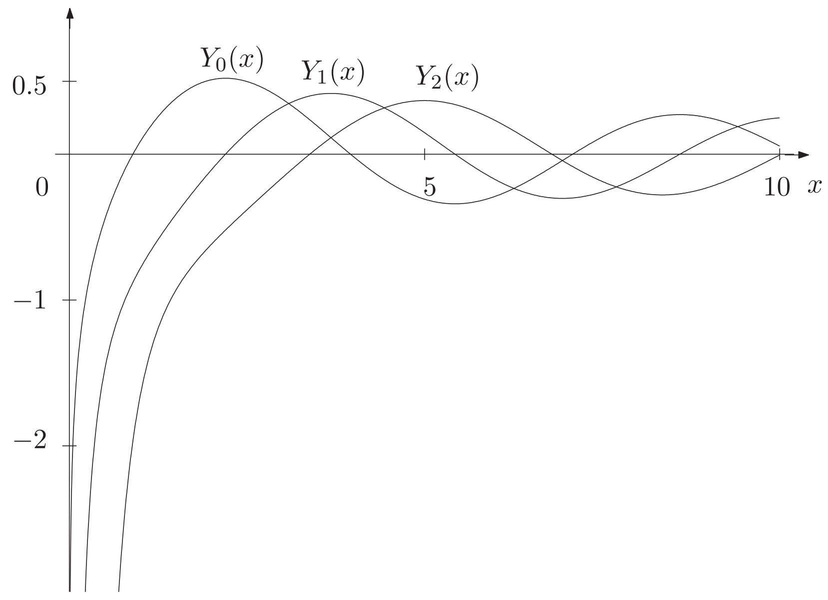


Figure B.2. Bessel functions of the second kind for $n = 0, 1, 2$.

(ii) For any $n \geq 0$, the system of functions $\{\sqrt{x} J_n(\mu_{nk} x)\}_{k=1}^{+\infty}$ is orthogonal on $[0, 1]$:

$$\int_0^1 x J_n(\mu_{nk} x) J_n(\mu_{nj} x) \mathrm{d}x = 0 \quad \text{for } j \neq k$$

and

$$\int_0^1 x J_n^2(\mu_{nk} x) \mathrm{d}x = \frac{1}{2} J_{n+1}^2(\mu_{nk}).$$

Here μ_{nk}, $k = 1, 2, \ldots$, are again the positive roots of $J_n(x)$.

(iii) For any $n \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x^n J_n(x)\big) = x^n J_{n-1}(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\big(x^{-n} J_n(x)\big) = -x^{-n} J_{n+1}(x).$$

In particular, we have $\frac{\mathrm{d}}{\mathrm{d}x} J_0(x) = -J_1(x)$.

Remark B.1. In Examples 12.5, 13.8 and 13.9 we have seen the so called *parametric form of the Bessel equation*

$$x^2y'' + xy' + (\lambda x^2 - n^2)y = 0 \tag{B.6}$$

with the boundary conditions

$$y(0) \text{ finite}, \quad y(a) = 0. \tag{B.7}$$

If μ_{nk} are positive roots of $J_n(x)$, then problem (B.6), (B.7) is nontrivially solvable for the values $\lambda = \lambda_{nk} = \frac{\mu_{nk}^2}{a^2}$, and the corresponding solutions are $J_n(\frac{\mu_{nk}}{a}x)$.

Notice that equation (B.6) can be written as

$$-(xy')' + \frac{n^2}{x}y = \lambda xy,$$

which is (together with the boundary conditions) nothing but the Sturm-Liouville problem on $(0, a)$ with $p(x) = r(x) = x$ and $q(x) = \frac{n^2}{x}$. Since $q(x)$ is not defined and $p(x)$ vanishes at $x = 0$, this problem is a singular one. The Sturm-Liouville theory then implies some of the above mentioned properties of Bessel functions, namely, that the functions $J_n(\frac{\mu_{nk}}{a}x)$ form a complete orthogonal system on $(0, a)$ with respect to the weight function $r(x) = x$.

More properties of Bessel functions can be found, e.g., in [1].

Some Typical Problems Considered in this Book

Transport equation

- transport equation (Chapter 3)

$$u_t + cu_x = 0$$

- general PDE of the first order (Chapter 3)

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

- problem with side condition (Chapter 3)

$$\begin{cases} au_x + bu_y + cu = f, \\ u(x,y) = u_0(s), \quad (x,y) \in \gamma : \begin{cases} x = x_0(s), \\ y = y_0(s), \end{cases} \quad s \in I. \end{cases}$$

Wave equation

- Cauchy problem on $\mathbb{R}$ (Chapters 4, 9)

$$\begin{cases} u_{tt} = c^2 u_{xx} + f, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

- initial boundary value problem in $\mathbb{R}$ (Chapters 7, 9)

$$\begin{cases} u_{tt} = c^2 u_{xx} + f, \quad x \in (0,l),\ t > 0, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

- Cauchy problem in $\mathbb{R}^N$ (Chapter 13)

$$\begin{cases} u_{tt} = c^2 \Delta u + f, \quad \boldsymbol{x} \in \mathbb{R}^N,\ t > 0, \\ u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}),\ u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}) \end{cases}$$

- initial boundary value problem in $\mathbb{R}^N$ (Chapter 13)

$$\begin{cases} u_{tt} = c^2\Delta u + f, & \boldsymbol{x} \in \Omega,\ t > 0, \\ u(\boldsymbol{x}, t) = 0, & \boldsymbol{x} \in \partial\Omega, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}),\ u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}) \end{cases}$$

Diffusion equation

- Cauchy problem on $\mathbb{R}$ (Chapters 5, 9)

$$\begin{cases} u_t = ku_{xx} + f, & x \in \mathbb{R},\ t > 0, \\ u(x, 0) = \varphi(x) \end{cases}$$

- initial boundary value problem in $\mathbb{R}$ (Chapters 7, 9)

$$\begin{cases} u_t = ku_{xx} + f, & x \in (0, l),\ t > 0, \\ u(0, t) = u(l, t) = 0, \\ u(x, 0) = \varphi(x) \end{cases}$$

- Cauchy problem in $\mathbb{R}^N$ (Chapter 12)

$$\begin{cases} u_t = k\Delta u + f, & \boldsymbol{x} \in \mathbb{R}^N,\ t > 0, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}) \end{cases}$$

- initial boundary value problem in $\mathbb{R}^N$ (Chapter 12)

$$\begin{cases} u_t = k\Delta u + f, & \boldsymbol{x} \in \boldsymbol{\Omega},\ t > 0, \\ u(\boldsymbol{x}, t) = 0, & \boldsymbol{x} \in \partial\boldsymbol{\Omega}, \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}) \end{cases}$$

Laplace (Poisson) equation

- in $\mathbb{R}^2$ or $\mathbb{R}^3$ (Chapters 6, 11)

$$\Delta u = f, \quad \boldsymbol{x} \in \mathbb{R}^2 \text{ or } \boldsymbol{x} \in \mathbb{R}^3$$

- boundary value problems (Chapters 8, 11)

$$\begin{cases} \Delta u = f, & \boldsymbol{x} \in \boldsymbol{\Omega}, \\ u = 0, & \boldsymbol{x} \in \partial\boldsymbol{\Omega} \end{cases}$$

Notation

ODE	ordinary differential equation	
PDE	partial differential equation	
$\mathbb{R}$, $\mathbb{R}^N$	the set of real numbers, N-dimensional Euclidean space	p. 1
$\partial\Omega$	boundary of the set Ω	p. 2
$\boldsymbol{a} \cdot \boldsymbol{b}$	scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$	p. 2
$\boldsymbol{n}(\boldsymbol{x})$	outer normal vector at the point $\boldsymbol{x}$	p. 3
$\operatorname{div} \boldsymbol{\phi}(\boldsymbol{x}) \left(= \nabla \cdot \boldsymbol{\phi} = \frac{\partial \phi_1}{\partial x_1} + \cdots + \frac{\partial \phi_N}{\partial x_N}\right)$	divergence of the vector function $\boldsymbol{\phi}$	p. 4
$u_t \left(= \frac{\partial u}{\partial t}\right),\ \phi_x \left(= \frac{\partial \phi}{\partial x}\right),\ u_{tt} \left(= \frac{\partial^2 u}{\partial t^2}\right),\ \phi_{xx} \left(= \frac{\partial^2 \phi}{\partial x^2}\right),\ \ldots$	partial derivatives	p. 6
$\operatorname{grad} u(\boldsymbol{x}) = \nabla u \left(= (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N})\right)$	gradient of the scalar function u	p. 11, 38
$\Delta u(\boldsymbol{x}) \left(= \div(\operatorname{grad} u) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2}\right)$	Laplace operator	p. 11
$\dfrac{\partial u}{\partial n}\,(= \operatorname{grad} u \cdot \boldsymbol{n})$	derivative with respect to outer normal	p. 15
$\operatorname{rot} \boldsymbol{E} \left(= \nabla \times \boldsymbol{E} = (\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}, \frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}, \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y})\right)$	rotation of the vector field $\boldsymbol{E}$	p. 17
i	imaginary unit	p. 18
C^k	the space of k-times continuously differentiable functions	p. 23

$\partial_x = \frac{\partial}{\partial x}, \quad \partial_x^2 = \frac{\partial^2}{\partial x^2}, \ \dots$	partial derivatives	p. 31
$\mathbb{C}$	the set of complex numbers	p. 31
$\operatorname{erf}(z)$	error function	p. 85
$\varphi(x-) = \lim\limits_{t\to x-} \varphi(t), \quad \varphi(x+) = \lim\limits_{t\to x+} \varphi(t)$		p. 88
C^∞	the set of all functions whose partial derivatives of any order are also continuous	p. 89
$\mathbb{N}$	the set of positive integers	p. 112
$\mathcal{L}$	Laplace transform	p. 150
$\mathcal{L}^{-1}$	inverse Laplace transform	p. 150
$\mathcal{F}$	Fourier transform	p. 156
$\mathcal{F}^{-1}$	inverse Fourier transform	p. 157
$u * v$	convolution of functions u and v	p. 153, 158
$\mathscr{S}$	Schwartz set	p. 157
$\Re$	space-time cylinder	p. 173
$L^2(M)$	the space of all functions the second powers of which are integrable on the set M	p. 177
$f(t) = O(t^n)$	the ratio $\frac{f(t)}{t^n}$ is bounded as $t \to 0$	p. 181
$\boldsymbol{B}^t$	transposed matrix to the matrix $\boldsymbol{B}$	p. 187
$\operatorname{meas} \partial B(\mathbf{0}, a)$	measure (surface) of the ball $B(\mathbf{0}, a)$	p. 192
$\|\nabla u\|^2 = \|\operatorname{grad} u\|^2 = u_x^2 + u_y^2 + u_z^2$		p. 193
$\boldsymbol{a} \times \boldsymbol{b}$	vector product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$	p. 208
$\Delta^2 u = \Delta(\Delta u)$	biharmonic operator	p. 208

We keep the same notation for a function u when applying the transformation of its independent variables, i.e., $u = u(x, y)$ and $u = u(r, \theta)$ for transformation from Cartesian into polar coordinates etc.

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